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Variable Sampling Interval Control Charts for Monitoring the Ratio of Two Normal Variables

Huu Du Nguyen ^{*1}, Kim Phuc Tran² and Thong Ngee Goh³

¹Division of Artificial Intelligence, Dong A University, 33 Xo Viet Nghe Tinh street, 50206 Hai Chau, Da Nang, Vietnam

²Ecole Nationale Supérieure des Arts et Industries Textiles, GEMTEX Laboratory, BP 30329 59056 Roubaix Cedex 1, France

³Industrial and Systems Engineering Department, National University of Singapore, 21 Lower Kent Ridge Road 119077, Singapore

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Abstract

It is well documented that the distribution of the ratio between two random normal variables is asymmetric. As a consequence, the two-sided Shewhart control chart monitoring this ratio (denoted as Shewhart-RZ chart) has an average run length (*ARL*)–biased property. In order to overcome this drawback, we propose in this paper designing two separated one-sided control charts. Furthermore, we improve the efficiency of the Shewhart-RZ chart by combining the Shewhart chart with a variable sampling interval strategy. The chart performance is then evaluated by using an average time to signal. The numerical results show that our proposed charts outperform the two-sided Shewhart-RZ control chart in detecting process shifts.

Keywords: VSI; Ratio distribution; Markov chain; Statistical Process Control.

1 Introduction

Control charts are considered as a primary tool in statistical process control, which is a set of statistical techniques for monitoring, controlling and improving a process. Being introduced in the year 1926 by Shewhart [1], the Shewhart control chart has been widely used in many industrial areas.

^{*}dunh@donga.edu.vn (corresponding author)

Recently, using control charts to monitor the ratio between two normal distributions is a major concern in a number of studies. This is because in several practical processes, monitoring the change of the ratio between two variables is much more important than considering separately each of them. There are productions that their qualities are determined by the relation between two major ingredients instead of the quantity of each component. Typical examples of these manufacturing environments are food preparation, pharmaceutical and material production industries. The accurate balance of nutrition associated with the relative weights of two ingredients ensures the quality of a food recipe in the food industry. It is even more important to guarantee the accuracy of the proportion ratio of active ingredients in the pharmaceutical industry to make sure of the effectiveness and the safety of drugs. Several common situations where the ratio of two quantities is the characteristic of interest have been discussed broadly in [2, 3]. According to a real example provided in [4], in some battery recycling plants in Italy, the process engineer needs to monitor the ratio of “recyclable batteries” weight and “total batch” weight to quantify the economic loss due to the presence of not recyclable batteries and other materials. For general cases, the authors in [5] and [6] extended their studies to a control chart monitoring compositional data, which are vectors whose strictly positive components represent proportions, percentages, concentrations, or frequencies of some whole.

The statistical properties of a Shewhart chart monitoring the ratio of two normal variables for individual measurements (denoted by Shewhart-RZ control chart) are firstly discussed in [7]. This study is then extended to subgroups consisting of $n > 1$ sample units in [2] where each of these units is allowed to change in size from sample to sample. Despite some of its advantages, the Shewhart control chart is well-known to be inefficient in detecting small or moderate shifts. The reason is that this kind of control chart is based purely on the latest update of process measurements instead of utilizing the information embedded in the entire sequence of measurements. In other words, it has no “memory”. This disadvantage is a basis for the development of other types of control charts with memories. The CUMSUM-RZ type and the EWMA-RZ type of control charts are investigated in [8, 10, 9, 11].

There are also other efficient approaches to improve the sluggishness of the Shewhart chart in detecting small process shifts by considering its parameters. In the standard Shewhart chart, the sampling interval, the sample size, and the control limit are constant. Namely, these parameters are fixed from time to time and from sample to sample. The adaptive strategies are then developed by enabling these parameters to vary. Tagaras [12] presented a comprehensive survey about the adaptive charts, including not only the adaptive design parameters but also their combinations. Among these al-

ternatives strategies, it is called the variable sampling interval (VSI) control chart that allows sampling interval to change at different levels. The control limit in the VSI control charts is divided into three areas, involving central (or “safe”) region, warning region and out-of-control region. The interval between two successive samples is determined based on the position of the first sample. An overview of using VSI charts is discussed by Castagliola et al. [13]. They have also come to the conclusion that the VSI Shewhart chart gives significantly better performance than the fixed sampling interval (FSI) Shewhart chart in general. In particular, for the case of ratio charts, the VSI EWMA control chart investigated in [14] has improved substantially the performance of the EWMA control chart.

Although the new memory-type control charts are particularly efficient, their implementation is not straightforward. This makes them has not been widely accepted by the quality practitioners [15]. From this point of view, the Shewhart type chart has the advantage of being easy to implement. Enhancing the performance of the Shewhart chart but still keeping its simplicity is then a problem of interest. The VSI Shewhart control chart provides a good solution for this problem: it is not only easy to implement but also increases the power of the Shewhart control chart in detecting small to moderate process shifts. In the literature, this type of control chart has not been considered yet to monitor the ratio of two random variables. The goal of this paper is then to design a Shewhart control chart using a VSI feature for monitoring the ratio of two normal variables, say VSI Shewhart-RZ chart. Furthermore, because of the ARL -biased property of a two-sided Shewhart chart presented in [2], we also design again Shewhart-RZ chart but for two one-sided control charts. The numerical results show that the two one-sided Shewhart charts not only overcome the problem of ARL -biased property but also lead to better performance compared to one two-sided Shewhart chart; and in turn, the two one-sided VSI Shewhart charts considerably outperform those the two one-sided Shewhart charts. The efficiency of the proposed charts is evaluated by using the average time to signal (ATS) and standard-deviation time to signal (SDTS). Numerical analysis for both deterministic and uniformly distributed shift sizes is also considered.

The remainder of this study is organized as follows. In Section 2, we recall a brief review of the distribution of two random normal variables. The implementation of the two one-sided Shewhart-RZ charts is presented in Section 3. Section 4 is devoted to the design of the VSI Shewhart-RZ charts while the numerical analysis is displayed in Section 5. Section 6 presents an illustrative example. Finally, some concluding remarks are given Section 7.

2 The distribution of the ratio Z

Let X and Y be two components of a bivariate normal random vector \mathbf{W} , namely $\mathbf{W} = (X, Y)^T \sim N(\boldsymbol{\mu}_{\mathbf{W}}, \boldsymbol{\Sigma}_{\mathbf{W}})$ with mean vector

$$\boldsymbol{\mu}_{\mathbf{W}} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \quad (1)$$

and variance-covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{W}} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \quad (2)$$

where ρ is the correlation coefficient between X and Y . By this notation, the coefficients of variation, the standard-deviation ratio of X and Y are $\gamma_X = \frac{\sigma_X}{\mu_X}$, $\gamma_Y = \frac{\sigma_Y}{\mu_Y}$ and $\omega = \frac{\sigma_X}{\sigma_Y}$, respectively.

In the literature, the distribution of the ratio of X to Y , $Z = \frac{X}{Y}$, is carried out in a large number of studies, for example, in [16, 17, 18, 19]. From these studies, it is complicated to express the probability density function of Z while it is impossible to derive an exact expression of its cumulative distribution. According to Celano and Castagliola [2], an approximation for the probability density function (*p.d.f*), the cumulative distribution function (*c.d.f*) and the inverse distribution function (*i.d.f*) of Z can be obtained as

$$F_Z(z|\gamma_X, \gamma_Y, \omega, \rho) \simeq \Phi\left(\frac{A}{B}\right), \quad (3)$$

$$f_Z(z|\gamma_X, \gamma_Y, \omega, \rho) \simeq \left(\frac{1}{B\gamma_Y} - \frac{(z - \rho\omega)A}{B^3}\right) \times \phi\left(\frac{A}{B}\right), \quad (4)$$

and

$$F_Z^{-1}(p|\gamma_X, \gamma_Y, \omega, \rho) \simeq \begin{cases} \frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{2C_1} & \text{if } p \in (0, 0.5], \\ \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1} & \text{if } p \in [0.5, 1), \end{cases} \quad (5)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the *c.d.f* and the *p.d.f* of the standard normal distribution; A, B, C_1, C_2 and C_3 are the function of $z, \gamma_X, \gamma_Y, \omega$ and ρ with

$$\begin{aligned} A &= \frac{z}{\gamma_Y} - \frac{\omega}{\gamma_X}, \\ B &= \sqrt{\omega^2 - 2\rho\omega z + z^2}, \\ C_1 &= \frac{1}{\gamma_Y^2} - (\Phi^{-1}(p))^2, \\ C_2 &= 2\omega \left(\rho(\Phi^{-1}(p))^2 - \frac{1}{\gamma_X\gamma_Y} \right), \\ C_3 &= \omega^2 \left(\frac{1}{\gamma_X^2} - (\Phi^{-1}(p))^2 \right). \end{aligned}$$

Figure 1: The density function of the ratio of two random normal variables according to the approximation formula (4).

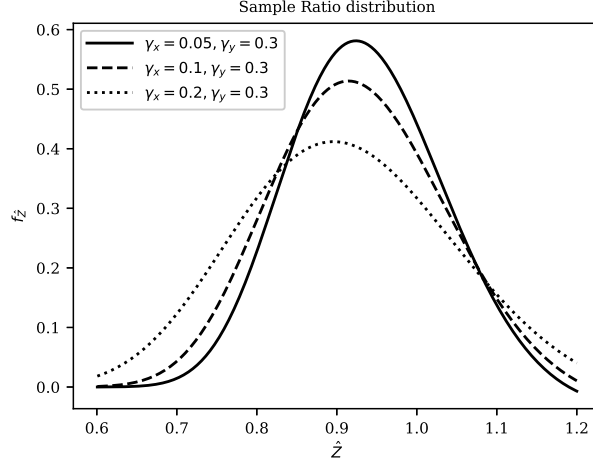


Figure 1 displays a graph of density functions of the ratio of two normal variables for different values of $\gamma_X, \gamma_Y, \omega = \gamma_X/\gamma_Y$ and $\rho = 0.4$.

3 Implementation of the two one-sided Shewhart-RZ control chart

Suppose that two random variables X and Y of interest are correlated with an in-control coefficient of correlation ρ_0 . For monitoring the ratio $Z = \frac{X}{Y}$, a set of n independent samples $\{\mathbf{W}_{i,1}, \mathbf{W}_{i,2}, \dots, \mathbf{W}_{i,n}\}$ at each sampling period $i = 1, 2, \dots$ is collected, where each $\mathbf{W}_{i,j} = (X_{i,j}, Y_{i,j})^T, j = 1, \dots, n$, is a bivariate normal random vector with mean vector

$$\boldsymbol{\mu}_{\mathbf{W},i} = \begin{pmatrix} \mu_{X,i} \\ \mu_{Y,i} \end{pmatrix}, \quad (6)$$

and variance-covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{W},i} = \begin{pmatrix} \sigma_{X,i}^2 & \rho_0 \sigma_{X,i} \sigma_{Y,i} \\ \rho_0 \sigma_{X,i} \sigma_{Y,i} & \sigma_{Y,i}^2 \end{pmatrix}. \quad (7)$$

Let γ_X and γ_Y be the coefficient of variation (CV) of X and Y . It is important to consider that the values of $\boldsymbol{\mu}_{\mathbf{W},i}$ and $\boldsymbol{\Sigma}_{\mathbf{W},i}$ may change from one subgroup to another. Assume that each independent subgroup $\{\mathbf{W}_{i,1}, \dots, \mathbf{W}_{i,n}\}$ has a linear relationship $\sigma_{X,i} = \gamma_X \times \mu_{X,i}$ and $\sigma_{Y,i} = \gamma_Y \times \mu_{Y,i}$. This implies that the values of $\mu_{X,i}, \mu_{Y,i}, \sigma_{X,i}$ and $\sigma_{Y,i}$ may change, but the CVs γ_X and

γ_Y must be constant from sample to sample. We suggest to monitor the statistic

$$\hat{Z}_i = \frac{\hat{\mu}_{X,i}}{\hat{\mu}_{Y,i}} = \frac{\bar{X}_i}{\bar{Y}_i} = \frac{\sum_{j=1}^n X_{i,j}}{\sum_{j=1}^n Y_{i,j}}, i = 1, 2, \dots \quad (8)$$

Since $\bar{X}_i \sim N(\mu_{X,i}, \frac{\sigma_{X,i}}{\sqrt{n}})$ and $\bar{Y}_i \sim N(\mu_{Y,i}, \frac{\sigma_{Y,i}}{\sqrt{n}})$, the CV of \bar{X}_i and \bar{Y}_i are equal to

$$\gamma_{\bar{X}} = \frac{\sigma_{X,i}}{\mu_{X,i}\sqrt{n}} = \frac{\gamma_X}{\sqrt{n}}, \quad (9)$$

$$\gamma_{\bar{Y}} = \frac{\sigma_{Y,i}}{\mu_{Y,i}\sqrt{n}} = \frac{\gamma_Y}{\sqrt{n}}, \quad (10)$$

and, at each inspection i , the standard-deviation ratio $\omega_i = \frac{\sigma_{X,i}}{\sigma_{Y,i}}$ is equal to

$$\omega_i = \frac{\mu_{X,i}}{\mu_{Y,i}} \times \frac{\gamma_X}{\gamma_Y}, i = 1, 2, \dots \quad (11)$$

When the process is in-control, we assume the ratio $\frac{\mu_{X,i}}{\mu_{Y,i}}$ is equal to the in-control value z_0 , $i = 1, 2, \dots$. Therefore, the standard-deviation ratio is

$$\omega_i = \omega_0 = z_0 \times \frac{\gamma_X}{\gamma_Y}, i = 1, 2, \dots \quad (12)$$

where ω_0 is the in-control standard-deviation ratio. The *c.d.f* and the *i.d.f* of \hat{Z}_i now can be deduced from the *c.d.f* in (3) and the *i.d.f* in (5), which are

$$F_{\hat{Z}_i}(z|n, \gamma_X, \gamma_Y, \omega_0, \rho_0) = F_Z\left(z \left| \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{z_0\gamma_X}{\gamma_Y}, \rho_0 \right.\right), \quad (13)$$

$$F_{\hat{Z}_i}^{-1}(p|n, \gamma_X, \gamma_Y, \omega_0, \rho_0) = F_Z^{-1}\left(p \left| \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{z_0\gamma_X}{\gamma_Y}, \rho_0 \right.\right). \quad (14)$$

Since the distribution of Z is asymmetrical, designing a single two-sided chart to track both an increase and a decrease in the ratio of two random normal variables will more likely result in an *ARL*-biased control chart. The *ARL*-biased results can be seen in [13] as the authors used a two-sided VSI Shewhart control chart in monitoring the coefficient of variation. In [13], the values of *ATS* corresponding to $\tau < 1$ (where τ stands for the shift size) are so large that the authors willingly omitted decrease case in their proposed chart. The same results are also witnessed in [2] in which the author designed a phase *II* control chart for monitoring the ratio of two normal variables using the Shewhart control chart. Many situations of process parameters lead to the result that the out-of-control value ARL_1 is greater than the in-control value of ARL_0 . Clearly, the two-sided chart for monitoring variables with skewed distribution is not a competitive alternative. To avoid this *ARL*-biased property, we propose in this section two

separated one-sided Shewhart charts, involving an upper-sided chart, which detects an increase in the ratio (denoted as “upward Shewhart-RZ chart”), and a lower-sided chart, which detects a decrease in the ratio (denoted as “downward Shewhart-RZ chart”).

Typically, the control limit in the Shewhart scheme is set such that type I error rate is equal to α_0 , which corresponds to an in-control average run length $ARL_0 = 1/\alpha_0$. The control limit for both an upward chart and a downward chart is then defined as follows.

- For the downward Shewhart-RZ chart, the lower control limit (LCL) is equal to

$$LCL^- = F_{\hat{Z}_i}^{-1}(\alpha_0 | n, \gamma_X, \gamma_Y, \omega_0, \rho_0), \quad (15)$$

where $F_{\hat{Z}_i}^{-1}(\cdot | n, \gamma_X, \gamma_Y, \omega_0, \rho_0)$ is the *i.d.f* of \hat{Z}_i defined in (14). When $\hat{Z}_i < LCL^-$, the process is considered to be out-of-control; the assignable causes might exist and they should be removed to put the process back to normal operation.

- For the upward Shewhart-RZ chart, the upper control limit (UCL) is equal to

$$UCL^+ = F_{\hat{Z}_i}^{-1}(1 - \alpha_0 | n, \gamma_X, \gamma_Y, \omega_0, \rho_0). \quad (16)$$

When $\hat{Z}_i > UCL^+$, the process is considered as out-of-control; appropriate actions need to be taken to remove the assignable causes if they do exist.

According to a discussion in [2], the control limits can be rewritten in the following form:

$$\begin{aligned} LCL^- &= z_0 \times K_D, \\ UCL^+ &= z_0 \times K_U, \end{aligned} \quad (17)$$

where

$$\begin{aligned} K_D &= F_Z^{-1}(\alpha_0 | \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\gamma_X}{\gamma_Y}, \rho_0), \\ K_U &= F_Z^{-1}(1 - \alpha_0 | \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\gamma_X}{\gamma_Y}, \rho_0). \end{aligned} \quad (18)$$

Suppose that the existence of an assignable cause shifts the in-control ratio z_0 to out-of-control ratio, say $z_1 = \tau \times z_0$, where $\tau > 0$ represents the shift size. The value $\tau < 1$ corresponds to a decrease of the in-control ratio z_0 while the value $\tau > 1$ corresponds to an increase of z_0 . Also, when the process is shifted to the out-of-control condition, the coefficient of correlation between X and Y might be shifted from $\rho = \rho_0$ to $\rho = \rho_1$. We now evaluate

the performance of Shewhart-RZ chart by using a traditional measure, the out-of-control average run length (ARL_1). Let p denote the probability of detecting a process shift, then

- for the downward chart,

$$\begin{aligned}
p &= Pr(\hat{Z}_i < LCL^-) \\
&= F_{\hat{Z}_i}(LCL^- | n, \gamma_X, \gamma_Y, \omega_1, \rho_1) \\
&= F_Z(LCL^- | \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau^{z_0}\gamma_X}{\gamma_Y}, \rho_1), \tag{19}
\end{aligned}$$

- for the upward chart,

$$\begin{aligned}
p &= Pr(\hat{Z}_i > UCL^+) \\
&= 1 - F_{\hat{Z}_i}(UCL^+ | n, \gamma_X, \gamma_Y, \omega_1, \rho_1) \\
&= 1 - F_Z(UCL^+ | \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau^{z_0}\gamma_X}{\gamma_Y}, \rho_1). \tag{20}
\end{aligned}$$

In the Shewhart control chart, the run length obeys a geometric distribution, leading to the following formulas of the out-of-control ARL_1 and $SDRL_1$:

$$ARL_1 = \frac{1}{p}, \tag{21}$$

and

$$SDRL_1 = \frac{\sqrt{1-p}}{p}. \tag{22}$$

4 Implementation of the VSI Shewhart-RZ control chart

It is well-known that the Shewhart chart is only efficient in detecting large process shifts. For small or moderate shifts, it takes a long time to detect. In order to overcome this problem, we suggest in this section combining the Shewhart-RZ control chart with the VSI feature. That is to say, the sampling intervals are enabled to vary at different levels rather than being fixed. The time between two successive samples \hat{Z}_{i+1} and \hat{Z}_i is defined based on the value of the current sample \hat{Z}_i . The control interval is split up into three regions: the central (“safe”) region, the warning region and the out-of-control region. This sub-division is performed by introducing a warning control limit, which is a lower warning limit (LWL) for the downward chart and an upper warning limit (UWL) for the upward chart. Similar to those control charts designed in Section 3, we define two one-sided VSI Shewhart-RZ control charts as follows.

- A lower-sided chart for detecting a decrease in \hat{Z}_i with a lower control limit $LCL^- = z_0 \times K_D$ and a lower warning limit $LWL^- = z_0 \times W_D$, where $K_D > 0$ and $W_D > K_D$ are the control limit and the warning limit parameters.
- An upper-sided chart for detecting an increase in \hat{Z}_i with an upper control limit $UCL^+ = z_0 \times K_U$ and an upper warning limit $UWL^+ = z_0 \times W_U$, where $W_U > 0$ and $K_U > W_U$ are the control limit and the warning limit parameters.

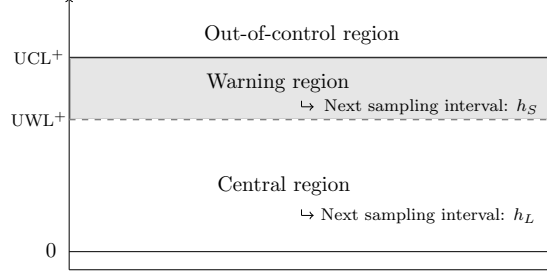
It should be considered that X and Y represent two components in a mixture and their values are measured directly from the product. Therefore, it is practical to assume that z_0 is positive and we have $UWL^+ < UCL^+$ and $LCL^- < LWL^-$.

According to a recommendation in [20], the most improvement in the detection power of the VSI type chart compared to a fixed sampling interval (FSI) type chart can be obtained by using only two sampling intervals. The choice of using only two sampling intervals also reduces the complexity of VSI schemes. Therefore, we assume that the sampling interval in VSI Shewhart chart is only picked from a set of two values, say h_S (for shorter time interval) and h_L (for longer time interval) where $h_S < h_L$. The VSI strategy works as follows:

- If the current sample point falls into the safe region, namely $\hat{Z}_i > LWL^-$ (downward chart) or $\hat{Z}_i < UWL^+$ (upward chart), it is said that the process is still “in-control” and the next sample is collected after a longer interval h_L .
- If the current sample point is located in the warning region, namely $\hat{Z}_i \in [LCL^-, LWL^-]$ (downward chart) or $\hat{Z}_i \in [UWL^+, UCL^+]$ (upward chart), the process seems to be put in risk, and the next sample should be taken after a shorter interval h_S , to quickly detect assignable causes when they do occur.
- Otherwise, the process is considered to be out-of-control and actions should be taken to locate and remove assignable causes.

Figure 2 illustrates a graphical view of the operation of the upward VSI Shewhart-RZ chart. The ARL is a typical measure to evaluate the performance of the FSI charts; however, for the VSI charts, it is not a reasonable choice due to varying of the sampling interval. Instead, the ATS is an appropriate alternative as discussed in [21]. Moreover, to make a fair comparison, it is necessary to maintain the equality of the average sampling interval (ASI) in both FSI and VSI Shewhart-RZ control chart. In the FSI version, the sampling interval h is a constant while in the VSI version, the ASI is

Figure 2: Three regions and operation rules of the upward VSI Shewhart-RZ chart.



computed by the following formula

$$ASI = E(h) = \frac{h_S p_S + h_L p_L}{1 - q}, \quad (23)$$

where p_S, p_L and q are the probabilities that a sample point drops into the safe, warning and out-of-control region, respectively. Similar to the notation in Section 3, let τ stands for the shift size. By its definition, the formulas for p_S, p_L and q are

- for the downward chart,

$$\begin{aligned} p_S &= P(\hat{Z}_i \geq LWL^-) \\ &= 1 - F_Z(LWL^- \mid \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho), \end{aligned} \quad (24)$$

$$\begin{aligned} p_L &= P(LCL^- \leq \hat{Z}_i \leq LWL^-) \\ &= F_Z(LWL^- \mid \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho) - F_Z(LCL^- \mid \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho), \end{aligned} \quad (25)$$

$$q = P(\hat{Z}_i < LCL^-) = 1 - p_S - p_L; \quad (26)$$

- for the upward chart,

$$\begin{aligned} p_S &= P(\hat{Z}_i \leq UWL^+) \\ &= F_Z(UWL^+ \mid \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho), \end{aligned} \quad (27)$$

$$\begin{aligned} p_L &= P(UWL^+ \leq \hat{Z}_i \leq UCL^+) \\ &= F_Z(UCL^+ \mid \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho) - F_Z(UWL^+ \mid \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho), \end{aligned} \quad (28)$$

$$q = P(\hat{Z}_i > UCL^+) = 1 - p_S - p_L, \quad (29)$$

in which the *c.d.f* $F_Z(\cdot | \frac{\gamma_X}{\sqrt{n}}, \frac{\gamma_Y}{\sqrt{n}}, \frac{\tau z_0 \gamma_X}{\gamma_Y}, \rho)$ is defined in (3). Based on the values of p_S, p_L and q , the *ATS* and *SDTS* are calculated by (see [22]):

$$ATS = \frac{h_S p_S + h_L p_L}{q(1-q)}, \quad (30)$$

and

$$SDTS = \sqrt{\frac{h_S^2 p_S + h_L^2 p_L}{q(1-q)} + \frac{(1-2q)(h_S p_S + h_L p_L)^2}{q^2(1-q)^2}}. \quad (31)$$

Let ATS_0 and ASI_0 denote the in-control values of the *ATS* and the *ASI*. When the process is out-of-control, they are changed to ATS_1 and ASI_1 . In the designing procedure of the VSI Shewhart-RZ charts, a fair comparison with the FSI Shewhart charts is performed by imposing constraints on the values of both ATS_0 and ASI_0 . Since the sampling interval in the FSI charts is a constant, one can suppose that $h = 1$ without loss of generality. This assumption immediately leads to the in-control value $ASI_0 = 1$ and $ATS_0 = ARL_0$ for both two types of chart. That is to say, the VSI Shewhart-RZ chart parameters (h_S, h_L, K_D, W_D) or (h_S, h_L, K_U, W_U) must be chosen to satisfy the following equations:

- for the downward chart,

$$\begin{cases} ATS(n, h_S, h_L, K_D, W_D, \rho, \gamma_X, \gamma_Y, \tau = 1) = ATS_0 \\ ASI_0 = 1 \end{cases} ; \quad (32)$$

- for the upward chart,

$$\begin{cases} ATS(n, h_S, h_L, K_U, W_U, \rho, \gamma_X, \gamma_Y, \tau = 1) = ATS_0 \\ ASI_0 = 1 \end{cases} . \quad (33)$$

5 Numerical analysis

In this section we explore the statistical performance of both two one-sided Shewhart-RZ control charts and two one-sided VSI Shewhart-RZ control charts for different out-of-control conditions, assuming that $z_0 = 1$. For the former, the control limits LCL^- or UCL^+ are chosen such that the desired false alarm probability $\alpha_0 = 0.005$, corresponding to fixed value of the in-control $ARL_0 = 200$. Based on these values of the control limits, the out-of-control values ARL_1 are computed. Similarly, for the later we set $ATS_0 = 200$. However, finding the chart parameters and the ATS_1 in the later requires more. In particular, it is needed to fix the values of sampling intervals (h_S, h_L) . We use the same combinations $(h_S, h_L) = \{(0.5, 1.5), (0.3, 1.7), (0.1, 1.9), (0.1, 1.1), (0.1, 1.3), (0.1, 1.5), (0.1, 4.0)\}$ as recommended in [22] and adopted in [13]. Many scenarios of other process parameters are also considered as follows.

- $(\gamma_X, \gamma_Y) = (0.01, 0.01), (\gamma_X, \gamma_Y) = (0.2, 0.2)$.
- $\rho_0 \in \{0.0, \pm 0.4, \pm 0.8\}$.
- $\tau \in \{0.95, 0.98, 0.99, 1.01, 1.02, 1.05\}$.
- $n \in \{1, 5, 10, 15\}$.

For the sake of brevity, we choose to present the numerical illustrations for the case $\gamma_X = \gamma_Y$ and $\rho_1 = \rho_0$. Similar tables for other investigated scenarios are not presented but are available upon request from authors. The obtained control limits and warning limits are displayed in Table 1 and Table 2. A number of remarks can be drawn from these two tables as follows.

- The control limits are the same for both FSI type and VSI type of Shewhart charts no matter the values of process parameters are. That means the control limits in the VSI Shewhart chart are independent of the sampling intervals (h_S, h_L). This result can be seen theoretically from the formulas of the control limits in the designing procedure of both two types of control charts.
- The control limits vary with n and ρ , given (γ_X, γ_Y) . To be more specific, the LCL^- increases while the UCL^+ decreases when either n or ρ increase. For example, when $(\gamma_X, \gamma_Y) = (0.01, 0.01)$, we have $(K_D, K_U) = (0.9523, 1.0501)$ for the case $n = 1, \rho = -0.8$ in Table 1 and $(K_D, K_U) = (0.9911, 1.0090)$ for the case $n = 10, \rho = 0.4$ in Table 2. This trend is consistent with that in the two-sided Shewhart-RZ chart suggested by Celano and Castagliola [2].
- Given n and ρ , the values of (K_D, K_U) depends strongly on (γ_X, γ_Y) . The rise of γ_X and γ_Y reduces the value of K_D but prolongs K_U . For example, with the same $n = 15, \rho = 0$ in Table 2, $(K_D, K_U) = (0.9906, 1.0095)$ when $\gamma_X = \gamma_Y = 0.01$ but $(K_D, K_U) = (0.8274, 1.2087)$ when $\gamma_X = \gamma_Y = 0.2$.

Based on the warning limits and the control limits, we can calculate the measures of $(ARL_1, SDRL_1)$ for the FSI charts and $(ATS_1, SDTS_1, ASI_1)$ for the VSI charts, leading to the understanding of how the performance of proposed control charts vary with process parameters. These results are exhibited in Tables 3-10. The FSI Shewhart-RZ chart parameters are put in the column titled *FSI* in each table. Some remarks from the obtained results can be summarized as follows.

- In general, the two one-sided Shewhart-RZ charts proposed in this study outperform substantially that the two-sided Shewhart-RZ chart investigated in [2] as they are able to detect process shift in both decrease and increase sides more quickly. For example, with $n =$

1, $\gamma_X = \gamma_Y = 0.2$, $\rho = 0.4$, and $\tau = 0.98$, we have $ARL_1 = 196.9$ in Table II of the two-sided Shewhart-RZ chart in [2], which is larger than $ARL_1 = 167.6$ in Table 4 of the downward chart in this study.

- In general, the VSI Shewhart-RZ control charts enhance significantly the performance of the FSI Shewhart-RZ control chart. The values of ARL_1 are always larger than those the values of ATS_1 regardless of the value (h_S, h_L) . For example, with $n = 5$, $\gamma_X = \gamma_Y = 0.2$, $\rho = -0.4$ and $\tau = 1.01$ in Table 6, we have $ARL_1 = 167.2$, which is larger than $ATS_1 = 159.3$ when $(h_S, h_L) = (0.1, 1.9)$ and $ATS_1 = 161.1$ when $(h_S, h_L) = (0.3, 1.7)$.
- The performance of VSI Shewhart-RZ control chart is greatly influenced by (γ_X, γ_Y) . The smaller values of (γ_X, γ_Y) are, the faster the control charts are in detecting the ratio shifts. For example, when $n = 10$, $\rho = -0.8$, $\tau = 0.95$, $(h_S, h_L) = (0.5, 1.5)$ and $(\gamma_X, \gamma_Y) = (0.01, 0.01)$, one can see $ATS_1 = 0.5$ and very small value of $SDTS_1$ in Table 7, while $ATS_1 = 54.2$ and $SDTS_1 = 53.9$ in Table 8.
- The correlation ρ between two random variables has a considerable impact on the control charts. In particular, the increase of ρ (from -0.8 to 0.8 in this study) makes the efficiency of the control charts better. For example, in Table 10 we have $ATS_1 = 28.7$ for $\rho = -0.4$ and $ATS_1 = 11.0$ for $\rho = 0.4$ in the case $n = 15$, $\gamma_X = \gamma_Y = 0.2$, $\tau = 1.05$ and $(h_S, h_L) = (0.1, 1.9)$.
- Given h_S , the increase of h_L results in the decrease of ATS_1 and ASI_1 . That is to say, the longer is the sampling interval in safe region, the better is the efficiency of the VSI Shewhart-RZ charts and the shorter is the average sampling interval. For example, with the same values of $n = 10$, $\rho = 0$, $h_S = 0.1$, $\gamma_X = \gamma_Y = 0.2$ and $\tau = 0.95$ in Table 8, we have $(ATS_1, ASI_1) = (39.9, 0.8703)$ for $h_L = 1.1$ while $(ATS_1, ASI_1) = (21.9, 0.4788)$ for $h_L = 4.0$. Similarly, if we fix the value of h_L and let h_S decrease, the values of both ATS_1 and ASI_1 also decline. Take a look at Table 8 with the same row mentioned above, we have $(ATS_1, ASI_1) = (36.1, 0.7874)$ for $(h_S, h_L) = (0.5, 1.5)$ and $(ATS_1, ASI_1) = (31.8, 0.6933)$ for $(h_S, h_L) = (0.1, 1.5)$, for instance. For both mentioned cases, the rise of h_L (fix h_S) or the decline of h_S (fix h_L) lead to a larger interval $[h_S, h_L]$. That means the width of the interval $[h_S, h_L]$ has an effect on both ATS_1 and ASI_1 in the sense: the larger width of $[h_S, h_L]$ brings out the better performance and the smaller average sampling interval. This result is really noticeable from an economic perspective. If one only pays attention to raise the efficiency of proposed charts, it is recommended to select couples (h_S, h_L) over a range as large as possible. This result is consistent

with the finding in [13]. However, the smaller ASI_1 corresponds to a faster sampling, or equivalently more load and more cost. Therefore, from an economic point of view, it is better to choose the couple in a balanced way so that it leads to the highest performance but also an acceptable cost.

Evaluating the power of control charts through the ARL_1 (FSI chart) or the ATS_1 (VSI charts) requires us to predefine a specific shift size τ . This prediction can not be accurate in practice since the quality practitioners do not always have enough information to pick up a specific value of τ . Instead, they may have some useful preferences about the range of τ . In this situation, the expected average run length ($EARL$) or expected average time to signal ($EATS$) should be used. These measures can be computed by

$$EARL = \int_{\Omega} f_{\tau}(\tau) ARL d\tau, \quad (34)$$

$$EATS = \int_{\Omega} f_{\tau}(\tau) ATS d\tau, \quad (35)$$

where $f_{\tau}(\tau)$ is a *p.d.f* of the shift size τ and Ω is a prediction range of τ . Without any information about the shape of $f_{\tau}(\tau)$, one can use a uniform distribution over Ω as suggested in [2, 23]. Tables 11-12 present the values of both $EARL_1$ and $EATS_1$ for $\Omega = [0.9, 1)$ (decreasing shift) and $\Omega = (1, 1.1]$ (increasing shift). In general, the trends shown by $EARL_1$ and $EATS_1$ are obviously analogous to those results corresponding to the case of deterministic shift sizes discussed above. Furthermore, some more remarks can be obtained as follows.

- With small values of $(\gamma_X = \gamma_Y)$, the performance of the control chart is quite symmetric. For example, in Table 12, when $\gamma_X = \gamma_Y = 0.01$, $n = 10$, $\rho = 0.0$, and $(h_S, h_L) = (0.3, 1.7)$, we have $EARL_1 = 1.2$ and $EATS_1 = 0.2$ for both the increasing and decreasing cases. With large values of (γ_X, γ_Y) , the performance of the downward chart is somewhat better than that of the upward chart. For example, $\gamma_X = \gamma_Y = 0.2$, $n = 5$, $\rho = -0.8$, and $(h_S, h_L) = (0.1, 1.9)$ in Table 11, we have $EARL_1 = 92.8$ ($EATS_1 = 76.2$) for the downward chart and $EARL_1 = 96.3$ ($EATS_1 = 79.7$) for the upward chart. This can be explained by the asymmetry of the distribution of \hat{Z}_i .
- In comparison with the two-sided Shewhart-RZ chart in [2], our proposed charts overcome the ARL -biased property. In some conditions such as when the shift size or the sample size is small or when X or Y has a high coefficient of variation, the results in [2] showed that the ARL_1 is larger than the ARL_0 . This does not happen in our study based on two one-sided control charts. The specific examples are not presented here for purpose of brevity, but are available from authors.

Table 1: The control limits of the FSI Shewhart-RZ chart, the warning limits and the control limits of the VSI Shewhart-RZ charts for $n = \{1, 5\}$ and different values of other parameters. The upper row in each horizontal block is for the downward charts, the bottom row is for the upward charts. The values of (h_S, h_L) are in the first row of the table.

| n | ρ | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|-----|-------------------------|------------------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| | | $\gamma_X = \gamma_Y = 0.01$ | | | | | | |
| 1 | -0.8 | 0.9523 (1.0001, 0.9523) | (1.0001, 0.9523) | (0.9764, 0.9523) | (0.9875, 0.9523) | (0.9932, 0.9523) | (1.0001, 0.9523) | (1.0141, 0.9523) |
| | | 1.0501 (0.9999, 1.0501) | (0.9999, 1.0501) | (1.0241, 1.0501) | (1.0127, 1.0501) | (1.0068, 1.0501) | (0.9999, 1.0501) | (0.9861, 1.0501) |
| | -0.4 | 0.9578 (1.0001, 0.9578) | (1.0001, 0.9578) | (0.9792, 0.9578) | (0.9890, 0.9578) | (0.9940, 0.9578) | (1.0001, 0.9578) | (1.0125, 0.9578) |
| | | 1.0441 (0.9999, 1.0441) | (0.9999, 1.0441) | (1.0212, 1.0441) | (1.0112, 1.0441) | (1.0060, 1.0441) | (0.9999, 1.0441) | (0.9877, 1.0441) |
| | 0.0 | 0.9642 (1.0001, 0.9642) | (1.0001, 0.9642) | (0.9824, 0.9642) | (0.9907, 0.9642) | (0.9950, 0.9642) | (1.0001, 0.9642) | (1.0105, 0.9642) |
| | 1.0371 (0.9999, 1.0371) | (0.9999, 1.0371) | (1.0179, 1.0371) | (1.0094, 1.0371) | (1.0051, 1.0371) | (0.9999, 1.0371) | (0.9896, 1.0371) | |
| 1 | 0.4 | 0.9722 (1.0001, 0.9722) | (1.0001, 0.9722) | (0.9863, 0.9722) | (0.9928, 0.9722) | (0.9961, 0.9722) | (1.0001, 0.9722) | (1.0081, 0.9722) |
| | | 1.0286 (0.9999, 1.0286) | (0.9999, 1.0286) | (1.0139, 1.0286) | (1.0073, 1.0286) | (1.0039, 1.0286) | (0.9999, 1.0286) | (0.9919, 1.0286) |
| | 0.8 | 0.9838 (1.0000, 0.9838) | (1.0000, 0.9838) | (0.9921, 0.9838) | (0.9958, 0.9838) | (0.9977, 0.9838) | (1.0000, 0.9838) | (1.0047, 0.9838) |
| | | 1.0164 (1.0000, 1.0164) | (1.0000, 1.0164) | (1.0080, 1.0164) | (1.0042, 1.0164) | (1.0023, 1.0164) | (1.0000, 1.0164) | (0.9953, 1.0164) |
| | | $\gamma_X = \gamma_Y = 0.2$ | | | | | | |
| 1 | -0.8 | 0.3375 (1.0024, 0.3375) | (1.0024, 0.3375) | (0.6140, 0.3375) | (0.7764, 0.3375) | (0.8729, 0.3375) | (1.0024, 0.3375) | (1.3272, 0.3375) |
| | | 2.9631 (0.9976, 2.9631) | (0.9976, 2.9631) | (1.6286, 2.9631) | (1.2880, 2.9631) | (1.1456, 2.9631) | (0.9976, 2.9631) | (0.7535, 2.9631) |
| | -0.4 | 0.3800 (1.0021, 0.3800) | (1.0021, 0.3800) | (0.6498, 0.3800) | (0.7999, 0.3800) | (0.8870, 0.3800) | (1.0021, 0.3800) | (1.2838, 0.3800) |
| | | 2.6316 (0.9979, 2.6316) | (0.9979, 2.6316) | (1.5389, 2.6316) | (1.2502, 2.6316) | (1.1274, 2.6316) | (0.9979, 2.6316) | (0.7790, 2.6316) |
| | 0.0 | 0.4377 (1.0018, 0.4377) | (1.0018, 0.4377) | (0.6941, 0.4377) | (0.8279, 0.4377) | (0.9036, 0.4377) | (1.0018, 0.4377) | (1.2353, 0.4377) |
| | 2.2849 (0.9982, 2.2849) | (0.9982, 2.2849) | (1.4406, 2.2849) | (1.2078, 2.2849) | (1.1066, 2.2849) | (0.9982, 2.2849) | (0.8096, 2.2849) | |
| 1 | 0.4 | 0.5236 (1.0014, 0.5236) | (1.0014, 0.5236) | (0.7532, 0.5236) | (0.8638, 0.5236) | (0.9245, 0.5236) | (1.0014, 0.5236) | (1.1779, 0.5236) |
| | | 1.9100 (0.9986, 1.9100) | (0.9986, 1.9100) | (1.3277, 1.9100) | (1.1576, 1.9100) | (1.0817, 1.9100) | (0.9986, 1.9100) | (0.8489, 1.9100) |
| | 0.8 | 0.6853 (1.0008, 0.6853) | (1.0008, 0.6853) | (0.8487, 0.6853) | (0.9189, 0.6853) | (0.9557, 0.6853) | (1.0008, 0.6853) | (1.0992, 0.6853) |
| | | 1.4592 (0.9992, 1.4592) | (0.9992, 1.4592) | (1.1782, 1.4592) | (1.0882, 1.4592) | (1.0464, 1.4592) | (0.9992, 1.4592) | (0.9097, 1.4592) |
| | | $\gamma_X = \gamma_Y = 0.01$ | | | | | | |
| 5 | -0.8 | 0.9784 (1.0001, 0.9784) | (1.0001, 0.9784) | (0.9894, 0.9784) | (0.9944, 0.9784) | (0.9970, 0.9784) | (1.0001, 0.9784) | (1.0063, 0.9784) |
| | | 1.0221 (0.9999, 1.0221) | (0.9999, 1.0221) | (1.0107, 1.0221) | (1.0056, 1.0221) | (1.0030, 1.0221) | (0.9999, 1.0221) | (0.9937, 1.0221) |
| | -0.4 | 0.9809 (1.0000, 0.9809) | (1.0000, 0.9809) | (0.9906, 0.9809) | (0.9951, 0.9809) | (0.9973, 0.9809) | (1.0000, 0.9809) | (1.0056, 0.9809) |
| | | 1.0195 (1.0000, 1.0195) | (1.0000, 1.0195) | (1.0094, 1.0195) | (1.0050, 1.0195) | (1.0027, 1.0195) | (1.0000, 1.0195) | (0.9945, 1.0195) |
| | 0.0 | 0.9838 (1.0000, 0.9838) | (1.0000, 0.9838) | (0.9921, 0.9838) | (0.9958, 0.9838) | (0.9977, 0.9838) | (1.0000, 0.9838) | (1.0047, 0.9838) |
| | 1.0164 (1.0000, 1.0164) | (1.0000, 1.0164) | (1.0080, 1.0164) | (1.0042, 1.0164) | (1.0023, 1.0164) | (1.0000, 1.0164) | (0.9953, 1.0164) | |
| 5 | 0.4 | 0.9875 (1.0000, 0.9875) | (1.0000, 0.9875) | (0.9939, 0.9875) | (0.9968, 0.9875) | (0.9983, 0.9875) | (1.0000, 0.9875) | (1.0036, 0.9875) |
| | | 1.0127 (1.0000, 1.0127) | (1.0000, 1.0127) | (1.0062, 1.0127) | (1.0033, 1.0127) | (1.0018, 1.0127) | (1.0000, 1.0127) | (0.9964, 1.0127) |
| | 0.8 | 0.9927 (1.0000, 0.9927) | (1.0000, 0.9927) | (0.9965, 0.9927) | (0.9981, 0.9927) | (0.9990, 0.9927) | (1.0000, 0.9927) | (1.0021, 0.9927) |
| | | 1.0073 (1.0000, 1.0073) | (1.0000, 1.0073) | (1.0036, 1.0073) | (1.0019, 1.0073) | (1.0010, 1.0073) | (1.0000, 1.0073) | (0.9979, 1.0073) |
| | | $\gamma_X = \gamma_Y = 0.2$ | | | | | | |
| 5 | -0.8 | 0.6405 (1.0011, 0.6405) | (1.0011, 0.6405) | (0.8072, 0.6405) | (0.8935, 0.6405) | (0.9411, 0.6405) | (1.0011, 0.6405) | (1.1340, 0.6405) |
| | | 1.5613 (0.9989, 1.5613) | (0.9989, 1.5613) | (1.2388, 1.5613) | (1.1192, 1.5613) | (1.0626, 1.5613) | (0.9989, 1.5613) | (0.8818, 1.5613) |
| | -0.4 | 0.6746 (1.0009, 0.6746) | (1.0009, 0.6746) | (0.8278, 0.6746) | (0.9055, 0.6746) | (0.9479, 0.6746) | (1.0009, 0.6746) | (1.1173, 0.6746) |
| | | 1.4823 (0.9991, 1.4823) | (0.9991, 1.4823) | (1.2080, 1.4823) | (1.1044, 1.4823) | (1.0550, 1.4823) | (0.9991, 1.4823) | (0.8950, 1.4823) |
| | 0.0 | 0.7166 (1.0008, 0.7166) | (1.0008, 0.7166) | (0.8524, 0.7166) | (0.9195, 0.7166) | (0.9558, 0.7166) | (1.0008, 0.7166) | (1.0983, 0.7166) |
| | 1.3955 (0.9992, 1.3955) | (0.9992, 1.3955) | (1.1732, 1.3955) | (1.0876, 1.3955) | (1.0463, 1.3955) | (0.9992, 1.3955) | (0.9105, 1.3955) | |
| 5 | 0.4 | 0.7721 (1.0006, 0.7721) | (1.0006, 0.7721) | (0.8836, 0.7721) | (0.9370, 0.7721) | (0.9656, 0.7721) | (1.0006, 0.7721) | (1.0754, 0.7721) |
| | | 1.2952 (0.9994, 1.2952) | (0.9994, 1.2952) | (1.1318, 1.2952) | (1.0672, 1.2952) | (1.0357, 1.2952) | (0.9994, 1.2952) | (0.9299, 1.2952) |
| | 0.8 | 0.8611 (1.0004, 0.8611) | (1.0004, 0.8611) | (0.9310, 0.8611) | (0.9631, 0.8611) | (0.9800, 0.8611) | (1.0004, 0.8611) | (1.0428, 0.8611) |
| | | 1.1614 (0.9996, 1.1614) | (0.9996, 1.1614) | (1.0741, 1.1614) | (1.0383, 1.1614) | (1.0204, 1.1614) | (0.9996, 1.1614) | (0.9589, 1.1614) |
| | | $\gamma_X = \gamma_Y = 0.2$ | | | | | | |

Table 2: The control limits of the FSI Shewhart-RZ chart, the warning limits and the control limits of the VSI Shewhart-RZ charts for $n = \{10, 15\}$ and different values of other parameters. The upper row in each horizontal block is for the downward charts, the bottom row is for the upward charts. The values of (h_S, h_L) are in the first row of the table.

| n | ρ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|------------------------------|--------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\gamma_X = \gamma_Y = 0.01$ | | | | | | | | | |
| 10 | -0.8 | 0.9847 | (1.0000, 0.9847) | (1.0000, 0.9847) | (0.9925, 0.9847) | (0.9960, 0.9847) | (0.9979, 0.9847) | (1.0000, 0.9847) | (1.0045, 0.9847) |
| | | 1.0156 | (1.0000, 1.0156) | (1.0000, 1.0156) | (1.0076, 1.0156) | (1.0040, 1.0156) | (1.0021, 1.0156) | (1.0000, 1.0156) | (0.9956, 1.0156) |
| | -0.4 | 0.9865 | (1.0000, 0.9865) | (1.0000, 0.9865) | (0.9934, 0.9865) | (0.9965, 0.9865) | (0.9981, 0.9865) | (1.0000, 0.9865) | (1.0039, 0.9865) |
| | | 1.0137 | (1.0000, 1.0137) | (1.0000, 1.0137) | (1.0067, 1.0137) | (1.0035, 1.0137) | (1.0019, 1.0137) | (1.0000, 1.0137) | (0.9961, 1.0137) |
| | 0.0 | 0.9885 | (1.0000, 0.9885) | (1.0000, 0.9885) | (0.9944, 0.9885) | (0.9970, 0.9885) | (0.9984, 0.9885) | (1.0000, 0.9885) | (1.0033, 0.9885) |
| | | 1.0116 | (1.0000, 1.0116) | (1.0000, 1.0116) | (1.0056, 1.0116) | (1.0030, 1.0116) | (1.0016, 1.0116) | (1.0000, 1.0116) | (0.9967, 1.0116) |
| 0.4 | 0.9911 | (1.0000, 0.9911) | (1.0000, 0.9911) | (0.9957, 0.9911) | (0.9977, 0.9911) | (0.9988, 0.9911) | (1.0000, 0.9911) | (1.0026, 0.9911) | |
| | 1.0090 | (1.0000, 1.0090) | (1.0000, 1.0090) | (1.0044, 1.0090) | (1.0023, 1.0090) | (1.0012, 1.0090) | (1.0000, 1.0090) | (0.9974, 1.0090) | |
| -0.8 | 0.9949 | (1.0000, 0.9949) | (1.0000, 0.9949) | (0.9975, 0.9949) | (0.9987, 0.9949) | (0.9993, 0.9949) | (1.0000, 0.9949) | (1.0015, 0.9949) | |
| | 1.0052 | (1.0000, 1.0052) | (1.0000, 1.0052) | (1.0025, 1.0052) | (1.0013, 1.0052) | (1.0007, 1.0052) | (1.0000, 1.0052) | (0.9985, 1.0052) | |
| $\gamma_X = \gamma_Y = 0.2$ | | | | | | | | | |
| 10 | -0.8 | 0.7320 | (1.0008, 0.7320) | (1.0008, 0.7320) | (0.8598, 0.7320) | (0.9235, 0.7320) | (0.9580, 0.7320) | (1.0008, 0.7320) | (1.0930, 0.7320) |
| | | 1.3662 | (0.9992, 1.3662) | (0.9992, 1.3662) | (1.1631, 1.3662) | (1.0828, 1.3662) | (1.0438, 1.3662) | (0.9992, 1.3662) | (0.9150, 1.3662) |
| | -0.4 | 0.7593 | (1.0007, 0.7593) | (1.0007, 0.7593) | (0.8752, 0.7593) | (0.9322, 0.7593) | (0.9629, 0.7593) | (1.0007, 0.7593) | (1.0815, 0.7593) |
| | | 1.3171 | (0.9993, 1.3171) | (0.9993, 1.3171) | (1.1426, 1.3171) | (1.0727, 1.3171) | (1.0386, 1.3171) | (0.9993, 1.3171) | (0.9246, 1.3171) |
| | 0.0 | 0.7922 | (1.0006, 0.7922) | (1.0006, 0.7922) | (0.8934, 0.7922) | (0.9424, 0.7922) | (0.9685, 0.7922) | (1.0006, 0.7922) | (1.0685, 0.7922) |
| | | 1.2624 | (0.9994, 1.2624) | (0.9994, 1.2624) | (1.1193, 1.2624) | (1.0611, 1.2624) | (1.0325, 1.2624) | (0.9994, 1.2624) | (0.9359, 1.2624) |
| 0.4 | 0.8347 | (1.0004, 0.8347) | (1.0004, 0.8347) | (0.9164, 0.8347) | (0.9551, 0.8347) | (0.9755, 0.8347) | (1.0004, 0.8347) | (1.0527, 0.8347) | |
| | 1.1980 | (0.9996, 1.1980) | (0.9996, 1.1980) | (1.0912, 1.1980) | (1.0470, 1.1980) | (1.0251, 1.1980) | (0.9996, 1.1980) | (0.9500, 1.1980) | |
| -0.8 | 0.9009 | (1.0003, 0.9009) | (1.0003, 0.9009) | (0.9508, 0.9009) | (0.9738, 0.9009) | (0.9858, 0.9009) | (1.0003, 0.9009) | (1.0301, 0.9009) | |
| | 1.1100 | (0.9997, 1.1100) | (0.9997, 1.1100) | (1.0517, 1.1100) | (1.0269, 1.1100) | (1.0144, 1.1100) | (0.9997, 1.1100) | (0.9708, 1.1100) | |
| $\gamma_X = \gamma_Y = 0.01$ | | | | | | | | | |
| 15 | -0.8 | 0.9875 | (1.0000, 0.9875) | (1.0000, 0.9875) | (0.9939, 0.9875) | (0.9968, 0.9875) | (0.9983, 0.9875) | (1.0000, 0.9875) | (1.0036, 0.9875) |
| | | 1.0127 | (1.0000, 1.0127) | (1.0000, 1.0127) | (1.0062, 1.0127) | (1.0033, 1.0127) | (1.0018, 1.0127) | (1.0000, 1.0127) | (0.9964, 1.0127) |
| | -0.4 | 0.9889 | (1.0000, 0.9889) | (1.0000, 0.9889) | (0.9946, 0.9889) | (0.9971, 0.9889) | (0.9985, 0.9889) | (1.0000, 0.9889) | (1.0032, 0.9889) |
| | | 1.0112 | (1.0000, 1.0112) | (1.0000, 1.0112) | (1.0054, 1.0112) | (1.0029, 1.0112) | (1.0015, 1.0112) | (1.0000, 1.0112) | (0.9968, 1.0112) |
| | 0.0 | 0.9906 | (1.0000, 0.9906) | (1.0000, 0.9906) | (0.9954, 0.9906) | (0.9976, 0.9906) | (0.9987, 0.9906) | (1.0000, 0.9906) | (1.0027, 0.9906) |
| | | 1.0095 | (1.0000, 1.0095) | (1.0000, 1.0095) | (1.0046, 1.0095) | (1.0024, 1.0095) | (1.0013, 1.0095) | (1.0000, 1.0095) | (0.9973, 1.0095) |
| 0.4 | 0.9927 | (1.0000, 0.9927) | (1.0000, 0.9927) | (0.9965, 0.9927) | (0.9981, 0.9927) | (0.9990, 0.9927) | (1.0000, 0.9927) | (1.0021, 0.9927) | |
| | 1.0073 | (1.0000, 1.0073) | (1.0000, 1.0073) | (1.0036, 1.0073) | (1.0019, 1.0073) | (1.0010, 1.0073) | (1.0000, 1.0073) | (0.9979, 1.0073) | |
| 0.8 | 0.9958 | (1.0000, 0.9958) | (1.0000, 0.9958) | (0.9980, 0.9958) | (0.9989, 0.9958) | (0.9994, 0.9958) | (1.0000, 0.9958) | (1.0012, 0.9958) | |
| | 1.0042 | (1.0000, 1.0042) | (1.0000, 1.0042) | (1.0021, 1.0042) | (1.0011, 1.0042) | (1.0006, 1.0042) | (1.0000, 1.0042) | (0.9988, 1.0042) | |
| $\gamma_X = \gamma_Y = 0.2$ | | | | | | | | | |
| 15 | -0.8 | 0.7757 | (1.0006, 0.7757) | (1.0006, 0.7757) | (0.8840, 0.7757) | (0.9371, 0.7757) | (0.9656, 0.7757) | (1.0006, 0.7757) | (1.0752, 0.7757) |
| | | 1.2891 | (0.9994, 1.2891) | (0.9994, 1.2891) | (1.1312, 1.2891) | (1.0671, 1.2891) | (1.0357, 1.2891) | (0.9994, 1.2891) | (0.9300, 1.2891) |
| | -0.4 | 0.7992 | (1.0005, 0.7992) | (1.0005, 0.7992) | (0.8970, 0.7992) | (0.9443, 0.7992) | (0.9696, 0.7992) | (1.0005, 0.7992) | (1.0661, 0.7992) |
| | | 1.2512 | (0.9995, 1.2512) | (0.9995, 1.2512) | (1.1149, 1.2512) | (1.0590, 1.2512) | (1.0314, 1.2512) | (0.9995, 1.2512) | (0.9380, 1.2512) |
| | 0.0 | 0.8274 | (1.0005, 0.8274) | (1.0005, 0.8274) | (0.9122, 0.8274) | (0.9527, 0.8274) | (0.9742, 0.8274) | (1.0005, 0.8274) | (1.0556, 0.8274) |
| | | 1.2087 | (0.9995, 1.2087) | (0.9995, 1.2087) | (1.0963, 1.2087) | (1.0496, 1.2087) | (1.0265, 1.2087) | (0.9995, 1.2087) | (0.9474, 1.2087) |
| 0.4 | 0.8634 | (1.0004, 0.8634) | (1.0004, 0.8634) | (0.9313, 0.8634) | (0.9632, 0.8634) | (0.9800, 0.8634) | (1.0004, 0.8634) | (1.0428, 0.8634) | |
| | 1.1582 | (0.9996, 1.1582) | (0.9996, 1.1582) | (1.0738, 1.1582) | (1.0382, 1.1582) | (1.0204, 1.1582) | (0.9996, 1.1582) | (0.9590, 1.1582) | |
| 0.8 | 0.9186 | (1.0002, 0.9186) | (1.0002, 0.9186) | (0.9597, 0.9186) | (0.9786, 0.9186) | (0.9884, 0.9186) | (1.0002, 0.9186) | (1.0245, 0.9186) | |
| | 1.0886 | (0.9998, 1.0886) | (0.9998, 1.0886) | (1.0420, 1.0886) | (1.0219, 1.0886) | (1.0117, 1.0886) | (0.9998, 1.0886) | (0.9761, 1.0886) | |

Table 3: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = 0.01$, $n = 1$ and different values of other parameters.

| ρ | τ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|--------|------------|--------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| -0.8 | 0.95 | (1.8, 1.2) | (0.9, 0.6, 0.5075) | (0.6, 0.4, 0.3105) | (0.5, 0.6, 0.2646) | (0.3, 0.4, 0.1552) | (0.2, 0.3, 0.1296) | (0.2, 0.3, 0.1135) | (0.2, 0.2, 0.1025) |
| | 0.98 | (15.3, 14.8) | (10.0, 9.8, 0.6520) | (7.9, 7.8, 0.5128) | (11.0, 10.8, 0.7162) | (8.3, 8.3, 0.5414) | (7.0, 7.2, 0.4590) | (5.7, 6.1, 0.3736) | (3.8, 4.7, 0.2484) |
| | 0.99 | (49.1, 48.6) | (39.4, 39.1, 0.8021) | (35.5, 35.4, 0.7230) | (43.3, 43.0, 0.8822) | (38.2, 38.0, 0.7773) | (35.2, 35.2, 0.7169) | (31.6, 31.8, 0.6439) | (24.9, 26.0, 0.5064) |
| | 1.01 | (49.8, 49.2) | (40.0, 39.7, 0.8039) | (36.1, 36.0, 0.7255) | (44.0, 43.6, 0.8837) | (38.8, 38.6, 0.7797) | (35.8, 35.8, 0.7197) | (32.2, 32.4, 0.6471) | (25.4, 26.5, 0.5101) |
| | 1.02 | (16.0, 15.4) | (10.5, 10.2, 0.6567) | (8.3, 8.3, 0.5194) | (11.5, 11.3, 0.7233) | (8.8, 8.8, 0.5502) | (7.5, 7.6, 0.4679) | (6.1, 6.5, 0.3821) | (4.1, 5.0, 0.2550) |
| 1.05 | (2.0, 1.4) | (1.0, 0.7, 0.5099) | (0.6, 0.5, 0.3139) | (0.6, 0.7, 0.2878) | (0.3, 0.5, 0.1675) | (0.3, 0.4, 0.1375) | (0.2, 0.3, 0.1179) | (0.2, 0.2, 0.1036) | |
| -0.4 | 0.95 | (1.5, 0.8) | (0.7, 0.4, 0.5034) | (0.4, 0.3, 0.3048) | (0.3, 0.4, 0.2128) | (0.2, 0.3, 0.1313) | (0.2, 0.2, 0.1152) | (0.2, 0.2, 0.1062) | (0.1, 0.1, 0.1009) |
| | 0.98 | (11.7, 11.2) | (7.3, 7.1, 0.6230) | (5.5, 5.5, 0.4722) | (7.8, 7.7, 0.6681) | (5.7, 5.7, 0.4845) | (4.7, 4.9, 0.4027) | (3.8, 4.1, 0.3214) | (2.5, 3.2, 0.2098) |
| | 0.99 | (41.5, 41.0) | (32.3, 32.0, 0.7787) | (28.6, 28.6, 0.6902) | (35.8, 35.5, 0.8624) | (30.9, 30.8, 0.7453) | (28.2, 28.2, 0.6795) | (25.0, 25.2, 0.6016) | (19.1, 20.2, 0.4597) |
| | 1.01 | (42.1, 41.6) | (32.8, 32.6, 0.7806) | (29.2, 29.1, 0.6929) | (36.4, 36.0, 0.8641) | (31.5, 31.3, 0.7480) | (28.7, 28.7, 0.6826) | (25.5, 25.7, 0.6051) | (19.5, 20.6, 0.4635) |
| | 1.02 | (12.2, 11.7) | (7.7, 7.4, 0.6275) | (5.8, 5.8, 0.4785) | (8.3, 8.1, 0.6762) | (6.0, 6.1, 0.4938) | (5.0, 5.2, 0.4117) | (4.0, 4.4, 0.3296) | (2.6, 3.4, 0.2156) |
| 1.05 | (1.6, 1.0) | (0.8, 0.5, 0.5047) | (0.5, 0.3, 0.3066) | (0.4, 0.5, 0.2322) | (0.2, 0.3, 0.1397) | (0.2, 0.2, 0.1201) | (0.2, 0.2, 0.1085) | (0.2, 0.1, 0.1014) | |
| 0.0 | 0.95 | (1.2, 0.4) | (0.6, 0.2, 0.5010) | (0.4, 0.1, 0.3013) | (0.2, 0.3, 0.1606) | (0.1, 0.1, 0.1124) | (0.1, 0.1, 0.1052) | (0.1, 0.1, 0.1017) | (0.1, 0.1, 0.1002) |
| | 0.98 | (8.0, 7.5) | (4.7, 4.4, 0.5865) | (3.4, 3.3, 0.4212) | (4.7, 4.6, 0.5936) | (3.2, 3.4, 0.4045) | (2.6, 2.8, 0.3275) | (2.0, 2.4, 0.2558) | (1.3, 1.9, 0.1672) |
| | 0.99 | (32.2, 31.7) | (24.0, 23.7, 0.7443) | (20.7, 20.6, 0.6420) | (26.7, 26.4, 0.8300) | (22.4, 22.3, 0.6956) | (20.1, 20.1, 0.6230) | (17.4, 17.7, 0.5397) | (12.7, 13.8, 0.3956) |
| | 1.01 | (32.7, 32.2) | (24.4, 24.2, 0.7464) | (21.1, 21.1, 0.6450) | (27.2, 26.9, 0.8322) | (22.9, 22.8, 0.6987) | (20.5, 20.6, 0.6266) | (17.8, 18.1, 0.5436) | (13.1, 14.2, 0.3994) |
| | 1.02 | (8.4, 7.8) | (4.9, 4.7, 0.5907) | (3.6, 3.5, 0.4269) | (5.0, 4.9, 0.6030) | (3.5, 3.6, 0.4141) | (2.8, 3.0, 0.3363) | (2.2, 2.5, 0.2632) | (1.4, 2.0, 0.1717) |
| 1.05 | (1.2, 0.5) | (0.6, 0.3, 0.5014) | (0.4, 0.2, 0.3020) | (0.2, 0.3, 0.1740) | (0.1, 0.2, 0.1167) | (0.1, 0.1, 0.1073) | (0.1, 0.1, 0.1026) | (0.1, 0.1, 0.1003) | |
| 0.4 | 0.95 | (1.0, 0.1) | (0.5, 0.1, 0.5001) | (0.3, 0.0, 0.3001) | (0.1, 0.1, 0.1175) | (0.1, 0.1, 0.1020) | (0.1, 0.0, 0.1006) | (0.1, 0.0, 0.1001) | (0.1, 0.0, 0.1000) |
| | 0.98 | (4.3, 3.8) | (2.3, 2.1, 0.5418) | (1.5, 1.5, 0.3586) | (2.0, 2.0, 0.4624) | (1.2, 1.4, 0.2855) | (1.0, 1.2, 0.2250) | (0.8, 1.0, 0.1753) | (0.5, 0.8, 0.1248) |
| | 0.99 | (20.6, 20.1) | (14.1, 13.9, 0.6869) | (11.6, 11.5, 0.5617) | (15.8, 15.5, 0.7650) | (12.4, 12.4, 0.6038) | (10.8, 10.9, 0.5234) | (9.0, 9.3, 0.4364) | (6.2, 7.2, 0.2997) |
| | 1.01 | (21.0, 20.5) | (14.5, 14.2, 0.6892) | (11.9, 11.8, 0.5649) | (16.1, 15.9, 0.7679) | (12.8, 12.7, 0.6078) | (11.1, 11.2, 0.5276) | (9.2, 9.6, 0.4406) | (6.4, 7.4, 0.3033) |
| | 1.02 | (4.5, 4.0) | (2.5, 2.2, 0.5447) | (1.6, 1.6, 0.3626) | (2.1, 2.2, 0.4732) | (1.3, 1.5, 0.2943) | (1.1, 1.3, 0.2321) | (0.8, 1.1, 0.1805) | (0.6, 0.9, 0.1272) |
| 1.05 | (1.0, 0.2) | (0.5, 0.1, 0.5001) | (0.3, 0.1, 0.3002) | (0.1, 0.2, 0.1230) | (0.1, 0.1, 0.1030) | (0.1, 0.0, 0.1010) | (0.1, 0.0, 0.1002) | (0.1, 0.0, 0.1000) | |
| 0.8 | 0.95 | (1.0, 0.0) | (0.5, 0.0, 0.5000) | (0.3, 0.0, 0.3000) | (0.1, 0.0, 0.1002) | (0.1, 0.0, 0.1000) | (0.1, 0.0, 0.1000) | (0.1, 0.0, 0.1000) | (0.1, 0.0, 0.1000) |
| | 0.98 | (1.4, 0.7) | (0.7, 0.4, 0.5026) | (0.4, 0.2, 0.3036) | (0.3, 0.4, 0.1981) | (0.2, 0.2, 0.1254) | (0.2, 0.2, 0.1119) | (0.1, 0.1, 0.1046) | (0.1, 0.1, 0.1006) |
| | 0.99 | (6.2, 5.7) | (3.5, 3.3, 0.5660) | (2.4, 2.4, 0.3924) | (3.3, 3.3, 0.5410) | (2.2, 2.3, 0.3536) | (1.7, 2.0, 0.2821) | (1.4, 1.7, 0.2188) | (0.9, 1.3, 0.1462) |
| | 1.01 | (6.3, 5.8) | (3.6, 3.4, 0.5678) | (2.5, 2.5, 0.3950) | (3.5, 3.4, 0.5461) | (2.3, 2.4, 0.3583) | (1.8, 2.0, 0.2863) | (1.4, 1.7, 0.2221) | (0.9, 1.4, 0.1480) |
| | 1.02 | (1.4, 0.8) | (0.7, 0.4, 0.5030) | (0.4, 0.2, 0.3041) | (0.3, 0.4, 0.2051) | (0.2, 0.2, 0.1282) | (0.2, 0.2, 0.1134) | (0.1, 0.1, 0.1053) | (0.1, 0.1, 0.1007) |
| 1.05 | (1.0, 0.0) | (0.5, 0.0, 0.5000) | (0.3, 0.0, 0.3000) | (0.1, 0.0, 0.1001) | (0.1, 0.0, 0.1001) | (0.1, 0.0, 0.1001) | (0.1, 0.0, 0.1001) | (0.1, 0.0, 0.1001) | |

Table 4: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.2$, $n = 1$ and different values of other parameters.

| ρ | τ | FSI | $(h_S, h_L) = (0.5, 1.5)$ | $(h_S, h_L) = (0.3, 1.7)$ | $(h_S, h_L) = (0.1, 1.1)$ | $(h_S, h_L) = (0.1, 1.3)$ | $(h_S, h_L) = (0.1, 1.5)$ | $(h_S, h_L) = (0.1, 1.9)$ | $(h_S, h_L) = (0.1, 4.0)$ |
|--------|---------------|---------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| -0.8 | 0.95 | (150.7,150.2) | (142.7,142.0) | (139.5,139.3) | (147.2,146.7) | (142.9,142.5) | (140.0,139.8) | (136.3,136.3) | (128.5,129.5) |
| | 0.98 | (178.9,178.4) | (175.1,174.8) | (173.6,173.4) | (177.3,176.9) | (175.3,175.0) | (173.9,173.7) | (172.1,172.1) | (168.2,169.0) |
| | 0.99 | (189.2,188.7) | (187.2,186.9) | (186.4,186.2) | (188.4,187.9) | (187.3,187.0) | (186.6,186.4) | (185.6,185.6) | (183.5,184.4) |
| | 1.01 | (189.3,188.8) | (187.3,187.0) | (186.6,186.3) | (188.5,188.0) | (187.5,187.1) | (186.7,186.5) | (185.8,185.7) | (183.6,184.5) |
| | 1.02 | (179.3,178.8) | (175.6,175.2) | (174.1,173.9) | (177.7,177.3) | (175.8,175.4) | (174.4,174.2) | (172.6,172.6) | (168.7,169.6) |
| 1.05 | (152.8,152.3) | (145.1,144.7) | (145.1,144.8) | (149.4,149.0) | (145.2,144.9) | (142.5,142.3) | (138.9,138.9) | (131.3,132.3) | |
| -0.4 | 0.95 | (146.4,145.9) | (137.5,137.2) | (134.0,133.8) | (142.4,142.0) | (137.7,137.4) | (134.5,134.3) | (130.5,130.5) | (122.2,123.1) |
| | 0.98 | (176.8,176.3) | (172.6,172.3) | (170.9,170.7) | (175.1,174.6) | (172.8,172.5) | (171.3,171.1) | (169.3,169.2) | (164.9,165.8) |
| | 0.99 | (188.1,187.6) | (185.9,185.5) | (185.0,184.8) | (187.2,186.8) | (186.0,185.7) | (185.2,184.9) | (184.1,184.0) | (181.7,182.6) |
| | 1.01 | (188.2,187.7) | (186.0,185.7) | (185.1,184.9) | (187.3,186.9) | (186.2,185.8) | (185.3,185.1) | (184.3,184.2) | (181.9,182.8) |
| | 1.02 | (177.3,176.8) | (173.1,172.8) | (171.5,171.2) | (175.5,175.1) | (173.3,173.0) | (171.8,171.6) | (169.8,169.8) | (165.5,166.4) |
| 1.05 | (148.6,148.1) | (140.1,139.7) | (136.7,136.5) | (144.8,144.4) | (140.2,139.9) | (137.2,137.0) | (133.3,133.3) | (125.2,126.1) | |
| 0.0 | 0.95 | (139.7,139.2) | (129.7,129.4) | (125.7,125.5) | (135.2,134.7) | (129.8,129.5) | (126.3,126.1) | (121.7,121.8) | (112.7,113.7) |
| | 0.98 | (173.6,173.1) | (168.7,168.4) | (166.8,166.5) | (171.6,171.1) | (168.9,168.6) | (167.2,166.9) | (164.8,164.8) | (159.8,160.7) |
| | 0.99 | (186.4,185.9) | (183.8,183.4) | (182.7,182.5) | (185.3,184.9) | (183.9,183.6) | (183.0,182.7) | (181.7,181.6) | (178.9,179.8) |
| | 1.01 | (186.5,186.0) | (184.0,183.6) | (182.9,182.7) | (185.5,185.0) | (184.1,183.7) | (183.2,182.9) | (181.9,181.8) | (179.1,180.0) |
| | 1.02 | (174.1,173.6) | (169.3,168.9) | (167.4,167.1) | (172.1,171.6) | (169.5,169.2) | (167.8,167.5) | (165.4,165.4) | (160.5,161.4) |
| 1.05 | (142.1,141.6) | (132.5,132.2) | (128.6,128.4) | (137.8,137.4) | (132.6,132.3) | (129.2,129.0) | (124.8,124.8) | (116.0,116.9) | |
| 0.4 | 0.95 | (127.5,127.0) | (115.8,115.5) | (111.1,111.0) | (122.1,121.6) | (115.7,115.4) | (111.6,111.4) | (106.5,106.5) | (96.6,97.6) |
| | 0.98 | (167.6,167.1) | (161.4,161.1) | (159.0,158.8) | (165.0,164.5) | (161.7,161.3) | (159.5,159.2) | (156.6,156.5) | (150.5,151.4) |
| | 0.99 | (183.2,182.7) | (179.8,179.5) | (178.5,178.3) | (181.8,181.3) | (180.0,179.7) | (178.8,178.5) | (177.2,177.1) | (173.7,174.6) |
| | 1.01 | (183.3,182.8) | (180.0,179.7) | (178.7,178.5) | (182.0,181.5) | (180.2,179.8) | (179.0,178.7) | (177.4,177.3) | (174.0,174.8) |
| | 1.02 | (168.1,167.6) | (162.1,161.8) | (159.7,159.5) | (165.6,165.2) | (162.4,162.0) | (160.2,160.0) | (157.3,157.3) | (151.4,152.3) |
| 1.05 | (130.4,129.8) | (119.0,118.6) | (114.4,114.3) | (125.1,124.7) | (118.9,118.6) | (114.9,114.7) | (109.9,109.9) | (100.1,101.1) | |
| 0.8 | 0.95 | (93.8,93.3) | (79.2,78.9) | (73.4,73.3) | (86.2,85.8) | (78.0,77.8) | (73.2,73.2) | (67.5,67.7) | (57.6,58.7) |
| | 0.98 | (148.6,148.1) | (139.2,138.9) | (135.5,135.3) | (144.5,144.0) | (139.4,139.1) | (136.1,135.9) | (131.8,131.8) | (123.2,124.2) |
| | 0.99 | (172.6,172.1) | (167.1,166.8) | (165.0,164.7) | (170.3,169.9) | (167.4,167.0) | (165.4,165.2) | (162.8,162.8) | (157.4,158.3) |
| | 1.01 | (172.8,172.3) | (167.4,167.1) | (165.3,165.1) | (170.6,170.1) | (167.7,167.3) | (165.7,165.5) | (163.1,163.1) | (157.8,158.7) |
| | 1.02 | (149.5,149.0) | (140.2,139.9) | (136.6,136.4) | (145.4,145.0) | (140.4,140.1) | (137.1,136.9) | (132.9,132.9) | (124.4,125.4) |
| 1.05 | (97.3,96.8) | (82.9,82.6) | (77.1,77.0) | (89.9,89.5) | (81.9,81.6) | (77.1,77.0) | (71.3,71.5) | (61.2,62.3) | |

Table 5: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.01$, $n = 5$ and different values of other parameters.

| ρ | τ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|--------|--------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| -0.8 | 0.95 | (1.0, 0.0) | (0.5, 0.0, 0.5000) | (0.3, 0.0, 0.3000) | (0.1, 0.1, 0.1032) | (0.1, 0.0, 0.1002) | (0.1, 0.0, 0.1000) | (0.1, 0.0, 0.1000) | (0.1, 0.0, 0.1000) |
| | 0.98 | (2.4, 1.8) | (1.2, 0.9, 0.5147) | (0.8, 0.6, 0.3206) | (0.8, 0.9, 0.3258) | (0.4, 0.6, 0.1891) | (0.4, 0.5, 0.1522) | (0.3, 0.4, 0.1265) | (0.3, 0.3, 0.1061) |
| | 0.99 | (12.2, 11.7) | (7.6, 7.4, 0.6273) | (5.8, 5.8, 0.4783) | (8.2, 8.1, 0.6759) | (6.0, 6.1, 0.4934) | (5.0, 5.2, 0.4113) | (4.0, 4.4, 0.3292) | (2.6, 3.4, 0.2153) |
| | 1.01 | (12.5, 11.9) | (7.8, 7.6, 0.6296) | (6.0, 6.0, 0.4815) | (8.5, 8.3, 0.6799) | (6.2, 6.3, 0.4980) | (5.2, 5.4, 0.4158) | (4.2, 4.5, 0.3333) | (2.7, 3.5, 0.2183) |
| | 1.02 | (2.5, 1.9) | (1.3, 1.0, 0.5162) | (0.8, 0.7, 0.3227) | (0.8, 0.9, 0.3361) | (0.5, 0.6, 0.1954) | (0.4, 0.5, 0.1566) | (0.3, 0.4, 0.1291) | (0.3, 0.3, 0.1069) |
| 1.05 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.1, 0.1047) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | |
| -0.4 | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.101) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.8, 1.2) | (0.9, 0.6, 0.5076) | (0.6, 0.4, 0.3106) | (0.5, 0.6, 0.2652) | (0.3, 0.4, 0.1555) | (0.2, 0.3, 0.1298) | (0.2, 0.3, 0.1136) | (0.2, 0.2, 0.1025) |
| | 0.99 | (9.2, 8.7) | (5.5, 5.3, 0.5994) | (4.0, 4.0, 0.4392) | (5.7, 5.6, 0.6223) | (4.0, 4.1, 0.4341) | (3.3, 3.5, 0.3548) | (2.6, 2.9, 0.2790) | (1.7, 2.3, 0.1815) |
| | 1.01 | (9.4, 8.9) | (5.7, 5.4, 0.6016) | (4.2, 4.1, 0.4422) | (5.9, 5.8, 0.6268) | (4.1, 4.2, 0.4389) | (3.4, 3.6, 0.3592) | (2.7, 3.0, 0.2829) | (1.7, 2.4, 0.1840) |
| | 1.02 | (1.9, 1.3) | (1.0, 0.7, 0.5085) | (0.6, 0.4, 0.3119) | (0.5, 0.6, 0.2743) | (0.3, 0.4, 0.1602) | (0.3, 0.3, 0.1328) | (0.2, 0.3, 0.1152) | (0.2, 0.2, 0.1029) |
| 1.05 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.101) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | |
| 0.0 | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.00, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.4, 0.7) | (0.7, 0.4, 0.5026) | (0.4, 0.2, 0.3036) | (0.3, 0.4, 0.1981) | (0.2, 0.2, 0.1254) | (0.2, 0.2, 0.1119) | (0.1, 0.1, 0.1046) | (0.1, 0.1, 0.1006) |
| | 0.99 | (6.2, 5.7) | (3.5, 3.3, 0.5660) | (2.4, 2.4, 0.3924) | (3.3, 3.3, 0.5410) | (2.2, 2.3, 0.3536) | (1.7, 2.0, 0.2822) | (1.4, 1.7, 0.2188) | (0.9, 1.3, 0.1462) |
| | 1.01 | (6.3, 5.8) | (3.6, 3.4, 0.5678) | (2.5, 2.5, 0.3950) | (3.5, 3.4, 0.5461) | (2.3, 2.4, 0.3583) | (1.8, 2.0, 0.2863) | (1.4, 1.7, 0.2221) | (0.9, 1.4, 0.1480) |
| | 1.02 | (1.4, 0.8) | (0.7, 0.4, 0.5029) | (0.4, 0.2, 0.3041) | (0.3, 0.4, 0.2051) | (0.2, 0.2, 0.1282) | (0.2, 0.2, 0.1134) | (0.1, 0.1, 0.1053) | (0.1, 0.1, 0.1007) |
| 1.05 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | |
| 0.4 | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.1, 0.3) | (0.5, 0.1, 0.5003) | (0.3, 0.1, 0.3004) | (0.1, 0.2, 0.1340) | (0.1, 0.1, 0.1053) | (0.1, 0.1, 0.1019) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.99 | (3.3, 2.8) | (1.8, 1.5, 0.5283) | (1.1, 1.0, 0.3396) | (1.3, 1.4, 0.4046) | (0.8, 1.0, 0.2413) | (0.6, 0.8, 0.1903) | (0.5, 0.7, 0.1509) | (0.4, 0.5, 0.1146) |
| | 1.01 | (3.4, 2.9) | (1.8, 1.6, 0.5294) | (1.2, 1.1, 0.3412) | (1.4, 1.5, 0.4101) | (0.8, 1.0, 0.2453) | (0.7, 0.9, 0.1933) | (0.5, 0.7, 0.1530) | (0.4, 0.6, 0.1154) |
| | 1.02 | (1.1, 0.3) | (0.5, 0.1, 0.5004) | (0.3, 0.1, 0.30) | (0.1, 0.2, 0.1375) | (0.1, 0.1, 0.1061) | (0.1, 0.1, 0.1023) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| 1.05 | (1.0, 0.001) | (0.5, 0.001, 0.49) | (0.3, 0.001, 0.29) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | |
| 0.8 | 0.95 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.98 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.0, 0.1000) | (0.1, 0.0, 0.1000) |
| | 0.99 | (1.2, 0.5) | (0.6, 0.2, 0.5011) | (0.4, 0.2, 0.3016) | (0.2, 0.3, 0.1659) | (0.1, 0.2, 0.1141) | (0.1, 0.1, 0.1060) | (0.1, 0.1, 0.1020) | (0.1, 0.1, 0.1002) |
| | 1.01 | (1.2, 0.5) | (0.6, 0.3, 0.5012) | (0.4, 0.2, 0.3017) | (0.2, 0.3, 0.1685) | (0.1, 0.2, 0.1149) | (0.1, 0.1, 0.1064) | (0.1, 0.1, 0.1022) | (0.1, 0.1, 0.1002) |
| | 1.02 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| 1.05 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | |

Table 6: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.2$, $n = 5$ and different values of other parameters.

| ρ | τ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|--------|--------------|----------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| -0.8 | 0.95 | (90.5, 90.0) | (79.9, 79.6, 0.8831) | (75.7, 75.5, 0.8364) | (85.1, 84.7, 0.9401) | (79.4, 79.2, 0.8778) | (75.9, 75.8, 0.8391) | (71.4, 71.5, 0.7896) | (62.2, 63.3, 0.6880) |
| | 0.98 | (145.2, 144.7) | (138.4, 138.1, 0.9533) | (135.7, 135.5, 0.9346) | (142.1, 141.7, 0.9786) | (138.5, 138.2, 0.9540) | (136.2, 135.9, 0.9378) | (133.0, 133.0, 0.9159) | (125.9, 126.8, 0.8670) |
| | 0.99 | (170.3, 169.8) | (166.4, 166.0, 0.9767) | (164.8, 164.6, 0.9674) | (168.6, 168.1, 0.9897) | (166.5, 166.2, 0.9775) | (165.1, 164.9, 0.9693) | (163.2, 163.1, 0.9580) | (158.8, 159.7, 0.9322) |
| | 1.01 | (170.6, 170.1) | (166.7, 166.3, 0.9769) | (165.1, 164.9, 0.9677) | (168.9, 168.4, 0.9898) | (166.8, 166.5, 0.9778) | (165.4, 165.2, 0.9696) | (163.5, 163.5, 0.9585) | (159.2, 160.1, 0.9329) |
| | 1.02 | (146.1, 145.6) | (139.4, 139.1, 0.9542) | (136.7, 136.5, 0.9359) | (143.1, 142.6, 0.9791) | (139.5, 139.2, 0.9550) | (137.2, 137.0, 0.9390) | (134.1, 134.0, 0.9175) | (127.0, 128.0, 0.8695) |
| 1.05 | (93.9, 93.4) | (83.5, 83.1, 0.8886) | (79.3, 79.1, 0.8441) | (88.6, 88.2, 0.9434) | (83.0, 82.8, 0.8842) | (79.6, 79.4, 0.8471) | (75.1, 75.2, 0.7995) | (65.9, 66.9, 0.7012) | |
| -0.4 | 0.95 | (82.1, 81.6) | (71.3, 71.0, 0.8681) | (66.9, 66.8, 0.8153) | (76.4, 76.0, 0.8602) | (70.6, 70.4, 0.8602) | (67.1, 67.0, 0.8171) | (62.6, 62.7, 0.7625) | (53.6, 54.7, 0.6528) |
| | 0.98 | (139.5, 139.0) | (132.1, 131.7, 0.9470) | (129.1, 128.9, 0.9259) | (136.1, 135.6, 0.9756) | (132.2, 131.9, 0.9477) | (129.6, 129.4, 0.9293) | (126.2, 126.2, 0.9047) | (118.6, 119.5, 0.8502) |
| | 0.99 | (166.9, 166.4) | (162.5, 162.1, 0.9736) | (160.7, 160.5, 0.9630) | (165.0, 164.5, 0.9883) | (162.6, 162.3, 0.9745) | (161.1, 160.8, 0.9651) | (159.0, 158.9, 0.9524) | (154.1, 155.0, 0.9234) |
| | 1.01 | (167.2, 166.7) | (162.8, 162.5, 0.9738) | (161.1, 160.9, 0.9634) | (165.3, 164.8, 0.9884) | (163.0, 162.6, 0.9747) | (161.4, 161.2, 0.9655) | (159.3, 159.3, 0.9529) | (154.5, 155.4, 0.9241) |
| | 1.02 | (140.5, 140.0) | (133.2, 132.8, 0.9481) | (130.2, 130.0, 0.9273) | (137.1, 136.7, 0.9761) | (133.3, 132.9, 0.9487) | (130.7, 130.5, 0.9307) | (127.3, 127.3, 0.9066) | (119.8, 120.8, 0.8530) |
| 1.05 | (85.6, 85.1) | (74.8, 74.5, 0.8743) | (70.5, 70.4, 0.8240) | (80.0, 79.6, 0.9346) | (74.2, 74.0, 0.8675) | (70.7, 70.6, 0.8262) | (66.2, 66.3, 0.7737) | (57.1, 58.2, 0.6672) | |
| 0.0 | 0.95 | (70.7, 70.2) | (59.8, 59.5, 0.8452) | (55.4, 55.3, 0.7833) | (64.7, 64.3, 0.9154) | (58.9, 58.7, 0.8326) | (55.4, 55.3, 0.7829) | (51.0, 51.1, 0.7214) | (42.5, 43.6, 0.6013) |
| | 0.98 | (131.0, 130.5) | (122.8, 122.5, 0.9374) | (119.5, 119.3, 0.9124) | (127.2, 126.8, 0.9707) | (122.8, 122.5, 0.9377) | (120.0, 119.8, 0.9161) | (116.3, 116.3, 0.8874) | (108.0, 109.0, 0.8247) |
| | 0.99 | (161.7, 161.2) | (156.6, 156.3, 0.9687) | (154.6, 154.4, 0.9562) | (159.5, 159.0, 0.9861) | (156.8, 156.5, 0.9697) | (155.0, 154.8, 0.9587) | (152.6, 152.6, 0.9437) | (147.1, 148.0, 0.9098) |
| | 1.01 | (162.0, 161.5) | (157.0, 156.7, 0.9691) | (155.0, 154.8, 0.9567) | (159.8, 159.4, 0.9862) | (157.2, 156.8, 0.9700) | (155.4, 155.2, 0.9591) | (153.0, 153.0, 0.9443) | (147.6, 148.5, 0.9107) |
| | 1.02 | (132.1, 131.6) | (124.0, 123.7, 0.9387) | (120.7, 120.6, 0.9141) | (128.3, 127.9, 0.9714) | (124.0, 123.7, 0.9389) | (121.2, 121.0, 0.9177) | (117.5, 117.5, 0.8896) | (109.4, 110.3, 0.8279) |
| 1.05 | (74.2, 73.7) | (63.2, 62.9, 0.8524) | (58.8, 58.7, 0.7933) | (68.3, 67.9, 0.9203) | (62.4, 62.2, 0.8413) | (58.9, 58.8, 0.7937) | (54.5, 54.6, 0.7342) | (45.8, 46.9, 0.6172) | |
| 0.4 | 0.95 | (53.7, 53.2) | (43.1, 42.9, 0.8040) | (38.9, 38.9, 0.7256) | (47.5, 47.1, 0.8845) | (41.8, 41.7, 0.7796) | (38.6, 38.6, 0.7193) | (34.7, 34.9, 0.6472) | (27.6, 28.7, 0.5142) |
| | 0.98 | (116.5, 116.0) | (107.1, 106.8, 0.9195) | (103.4, 103.2, 0.8873) | (112.0, 111.6, 0.9612) | (107.0, 106.7, 0.9185) | (103.8, 103.7, 0.8911) | (99.6, 99.7, 0.8551) | (90.7, 91.7, 0.7782) |
| | 0.99 | (152.3, 151.8) | (146.2, 145.8, 0.9597) | (143.7, 143.5, 0.9436) | (149.6, 149.1, 0.9818) | (146.3, 146.0, 0.9606) | (144.2, 144.0, 0.9465) | (141.3, 141.3, 0.9274) | (134.8, 135.7, 0.8847) |
| | 1.01 | (152.7, 152.2) | (146.6, 146.3, 0.9601) | (144.2, 144.0, 0.9441) | (150.0, 149.6, 0.9820) | (146.8, 146.5, 0.9610) | (144.7, 144.4, 0.9470) | (141.8, 141.7, 0.9281) | (135.3, 136.2, 0.8858) |
| | 1.02 | (117.7, 117.2) | (108.5, 108.1, 0.9211) | (104.7, 104.6, 0.8895) | (113.3, 112.9, 0.9621) | (108.4, 108.1, 0.9202) | (105.2, 105.0, 0.8933) | (101.0, 101.0, 0.8579) | (92.1, 93.1, 0.7822) |
| 1.05 | (57.0, 56.5) | (46.3, 46.0, 0.8127) | (42.0, 42.0, 0.7378) | (50.8, 50.4, 0.8914) | (45.1, 44.9, 0.7911) | (41.8, 41.7, 0.7330) | (37.8, 38.0, 0.6629) | (30.3, 31.4, 0.5320) | |
| 0.8 | 0.95 | (23.3, 22.8) | (16.1, 15.8, 0.6895) | (13.2, 13.2, 0.5653) | (17.9, 17.7, 0.7682) | (14.1, 14.1, 0.6066) | (12.3, 12.4, 0.5267) | (10.3, 10.6, 0.4411) | (7.2, 8.2, 0.3095) |
| | 0.98 | (80.2, 79.7) | (69.2, 68.9, 0.8627) | (64.8, 64.7, 0.8078) | (74.4, 74.0, 0.9273) | (68.5, 68.3, 0.8539) | (64.9, 64.8, 0.8091) | (60.4, 60.5, 0.7529) | (51.4, 52.5, 0.6412) |
| | 0.99 | (125.7, 125.2) | (116.9, 116.6, 0.9305) | (113.4, 113.2, 0.9027) | (121.5, 121.1, 0.9672) | (116.9, 116.6, 0.9303) | (113.9, 113.7, 0.9064) | (109.9, 109.9, 0.8748) | (101.4, 102.3, 0.8066) |
| | 1.01 | (126.2, 125.7) | (117.5, 117.2, 0.9312) | (114.1, 113.9, 0.9036) | (122.1, 121.7, 0.9675) | (117.5, 117.2, 0.9310) | (114.5, 114.3, 0.9074) | (110.6, 110.6, 0.8761) | (102.0, 103.0, 0.8084) |
| | 1.02 | (81.6, 81.1) | (70.6, 70.3, 0.8653) | (66.2, 66.1, 0.8115) | (75.8, 75.4, 0.9290) | (70.0, 69.7, 0.8570) | (66.4, 66.3, 0.8130) | (61.8, 62.0, 0.7576) | (52.8, 53.9, 0.6471) |
| 1.05 | (25.5, 25.0) | (17.9, 17.7, 0.7011) | (14.9, 14.8, 0.5815) | (20.0, 19.7, 0.7826) | (16.0, 16.0, 0.6261) | (14.0, 14.1, 0.5473) | (11.8, 12.1, 0.4619) | (8.4, 9.4, 0.3277) | |

Table 7: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.01$, $n = 10$ and different values of other parameters.

| ρ | τ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|--------|--------|--------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| -0.8 | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.3, 0.6) | (0.6, 0.3, 0.5017) | (0.4, 0.2, 0.3024) | (0.2, 0.3, 0.1811) | (0.2, 0.2, 0.1192) | (0.1, 0.1, 0.1085) | (0.1, 0.1, 0.1031) | (0.1, 0.1, 0.1004) |
| | 0.99 | (5.4, 4.9) | (3.0, 2.8, 0.5568) | (2.1, 2.0, 0.3795) | (2.8, 2.8, 0.5138) | (1.8, 2.0, 0.3289) | (1.4, 1.7, 0.2609) | (1.1, 1.4, 0.2022) | (0.7, 1.1, 0.1376) |
| | 1.01 | (5.6, 5.0) | (3.1, 2.9, 0.5585) | (2.1, 2.1, 0.3819) | (2.9, 2.9, 0.5190) | (1.9, 2.0, 0.3336) | (1.5, 1.7, 0.2649) | (1.1, 1.4, 0.2053) | (0.8, 1.2, 0.1391) |
| | 1.02 | (1.3, 0.6) | (0.7, 0.3, 0.5020) | (0.4, 0.2, 0.3028) | (0.2, 0.3, 0.1874) | (0.2, 0.2, 0.1214) | (0.1, 0.1, 0.1097) | (0.1, 0.1, 0.1036) | (0.1, 0.1, 0.1004) |
| -0.4 | 1.05 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.1, 0.4) | (0.6, 0.2, 0.5006) | (0.3, 0.1, 0.3009) | (0.2, 0.2, 0.1486) | (0.1, 0.1, 0.1090) | (0.1, 0.1, 0.1035) | (0.1, 0.1, 0.1011) | (0.1, 0.0, 0.1001) |
| | 0.99 | (4.0, 3.5) | (2.2, 1.9, 0.5378) | (1.4, 1.3, 0.3529) | (1.8, 1.8, 0.4465) | (1.1, 1.3, 0.2728) | (0.9, 1.1, 0.2148) | (0.7, 0.9, 0.1680) | (0.5, 0.7, 0.1216) |
| | 1.01 | (4.1, 3.6) | (2.2, 2.0, 0.5391) | (1.5, 1.4, 0.3548) | (1.9, 1.9, 0.4519) | (1.1, 1.3, 0.2771) | (0.9, 1.1, 0.2182) | (0.7, 0.9, 0.1704) | (0.5, 0.7, 0.1226) |
| 0.0 | 1.02 | (1.1, 0.4) | (0.6, 0.2, 0.5007) | (0.3, 0.1, 0.3010) | (0.2, 0.2, 0.1531) | (0.1, 0.1, 0.1102) | (0.1, 0.1, 0.1041) | (0.1, 0.1, 0.1013) | (0.1, 0.001, 0.10) |
| | 1.05 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.95 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.98 | (1.0, 0.2) | (0.5, 0.1, 0.5001) | (0.3, 0.1, 0.3002) | (0.1, 0.1, 0.1213) | (0.1, 0.1, 0.1027) | (0.1, 0.001, 0.1009) | (0.1, 0.001, 0.1002) | (0.1, 0.001, 0.10) |
| | 0.99 | (2.7, 2.1) | (1.4, 1.1, 0.5193) | (0.9, 0.8, 0.3270) | (1.0, 1.0, 0.3558) | (0.6, 0.7, 0.2079) | (0.4, 0.6, 0.1654) | (0.4, 0.5, 0.1347) | (0.3, 0.4, 0.1087) |
| 0.4 | 1.01 | (2.8, 2.2) | (1.4, 1.2, 0.5201) | (0.9, 0.8, 0.3282) | (1.0, 1.1, 0.3611) | (0.6, 0.7, 0.2113) | (0.5, 0.6, 0.1679) | (0.4, 0.5, 0.1363) | (0.3, 0.4, 0.1093) |
| | 1.02 | (1.0, 0.2) | (0.5, 0.1, 0.5001) | (0.3, 0.1, 0.3002) | (0.1, 0.2, 0.1237) | (0.1, 0.1, 0.1031) | (0.1, 0.0, 0.1010) | (0.1, 0.001, 0.1003) | (0.1, 0.001, 0.10) |
| | 1.05 | (1.0, 0.0) | (0.01, 0.0, 0.001) | (0.01, 0.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.95 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.98 | (1.0, 0.001) | (0.5, 0.001, 0.5000) | (0.3, 0.001, 0.30) | (0.1, 0.1, 0.1042) | (0.1, 0.0, 0.1003) | (0.1, 0.0, 0.1001) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| 0.8 | 0.99 | (1.6, 1.0) | (0.8, 0.5, 0.5049) | (0.5, 0.3, 0.3068) | (0.4, 0.5, 0.2342) | (0.2, 0.3, 0.1406) | (0.2, 0.2, 0.1206) | (0.2, 0.2, 0.1088) | (0.2, 0.1, 0.1014) |
| | 1.01 | (1.6, 1.0) | (0.8, 0.5, 0.5052) | (0.5, 0.3, 0.3073) | (0.4, 0.5, 0.2383) | (0.2, 0.3, 0.1425) | (0.2, 0.3, 0.1217) | (0.2, 0.2, 0.1094) | (0.2, 0.1, 0.1015) |
| | 1.02 | (1.0, 0.001) | (0.5, 0.001, 0.5000) | (0.3, 0.001, 0.3000) | (0.1, 0.1, 0.1049) | (0.1, 0.0, 0.1003) | (0.1, 0.001, 0.1001) | (0.1, 0.001, 0.1000) | (0.1, 0.001, 0.10) |
| | 1.05 | (1.0, 0.0) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.95 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| 0.8 | 0.98 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.99 | (1.0, 0.1) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.1, 0.1115) | (0.1, 0.001, 0.1011) | (0.1, 0.001, 0.1003) | (0.1, 0.001, 0.10) | (0.1, 0.0, 0.10) |
| | 1.01 | (1.0, 0.1) | (0.5, 0.0, 0.5000) | (0.3, 0.001, 0.3001) | (0.1, 0.1, 0.1122) | (0.1, 0.001, 0.1012) | (0.1, 0.001, 0.1003) | (0.1, 0.001, 0.1001) | (0.1, 0.001, 0.10) |
| | 1.02 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 1.05 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |

Table 8: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.2$, $n = 10$ and different values of other parameters.

| ρ | τ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|--------|--------|----------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| -0.8 | 0.95 | (64.7, 64.2) | (54.2, 53.9, 0.8375) | (50.0, 49.9, 0.7725) | (58.9, 58.5, 0.9095) | (53.3, 53.1, 0.8230) | (49.9, 49.9, 0.7714) | (45.8, 46.0, 0.7075) | (37.7, 38.8, 0.5826) |
| | 0.98 | (126.0, 125.5) | (117.7, 117.4, 0.9341) | (114.4, 114.2, 0.9078) | (122.1, 121.7, 0.9688) | (117.7, 117.4, 0.9341) | (114.9, 114.7, 0.9115) | (111.1, 111.1, 0.8814) | (102.7, 103.7, 0.8153) |
| | 0.99 | (158.5, 158.0) | (153.3, 152.9, 0.9671) | (151.2, 151.0, 0.9539) | (156.1, 155.7, 0.9852) | (153.4, 153.1, 0.9680) | (151.6, 151.3, 0.9564) | (149.1, 149.1, 0.9408) | (143.4, 144.3, 0.9048) |
| | 1.01 | (158.8, 158.3) | (153.7, 153.3, 0.9674) | (151.6, 151.4, 0.9544) | (156.5, 156.1, 0.9854) | (153.8, 153.5, 0.9683) | (152.0, 151.8, 0.9569) | (149.5, 149.5, 0.9413) | (143.9, 144.8, 0.9058) |
| | 1.02 | (127.1, 126.6) | (118.9, 118.6, 0.9354) | (115.7, 115.5, 0.9096) | (123.3, 122.8, 0.9695) | (118.9, 118.6, 0.9355) | (116.1, 115.9, 0.9133) | (112.4, 112.4, 0.8838) | (104.1, 105.1, 0.8187) |
| -0.4 | 0.95 | (68.2, 67.7) | (57.6, 57.3, 0.8450) | (53.4, 53.3, 0.7829) | (62.4, 62.0, 0.9147) | (56.7, 56.5, 0.8322) | (53.4, 53.3, 0.7826) | (49.1, 49.3, 0.7209) | (40.8, 41.9, 0.5990) |
| | 0.98 | (118.8, 118.3) | (110.0, 109.6, 0.9254) | (106.4, 106.2, 0.8956) | (114.6, 114.1, 0.9642) | (109.9, 109.6, 0.9248) | (106.9, 106.7, 0.8994) | (102.9, 102.9, 0.8658) | (94.2, 95.2, 0.7926) |
| | 0.99 | (153.8, 153.3) | (148.1, 147.7, 0.9627) | (145.8, 145.5, 0.9478) | (151.2, 150.8, 0.9831) | (148.2, 147.9, 0.9636) | (146.2, 146.0, 0.9505) | (143.5, 143.4, 0.9328) | (137.3, 138.2, 0.8926) |
| | 1.01 | (154.2, 153.7) | (148.5, 148.1, 0.9631) | (146.2, 146.0, 0.9483) | (151.6, 151.2, 0.9833) | (148.6, 148.3, 0.9639) | (146.6, 146.4, 0.9510) | (143.9, 143.9, 0.9335) | (137.8, 138.7, 0.8936) |
| | 1.02 | (120.0, 119.5) | (111.2, 110.9, 0.9269) | (107.7, 107.5, 0.8976) | (115.8, 115.4, 0.9650) | (111.2, 110.9, 0.9264) | (108.2, 108.0, 0.9014) | (104.2, 104.2, 0.8684) | (95.6, 96.6, 0.7963) |
| 0.0 | 0.95 | (59.8, 59.3) | (49.4, 49.1, 0.8256) | (45.2, 45.1, 0.7559) | (53.8, 53.5, 0.9008) | (48.3, 48.1, 0.8080) | (45.0, 45.0, 0.7533) | (41.0, 41.2, 0.6862) | (33.3, 34.4, 0.5571) |
| | 0.98 | (108.5, 108.0) | (99.0, 98.6, 0.9120) | (95.1, 95.0, 0.8768) | (103.8, 103.4, 0.9568) | (98.8, 98.5, 0.9103) | (95.5, 95.4, 0.8805) | (91.3, 91.4, 0.8416) | (82.3, 83.3, 0.7583) |
| | 0.99 | (146.8, 146.3) | (140.3, 140.0, 0.9559) | (137.7, 137.5, 0.9382) | (143.8, 143.4, 0.9798) | (140.4, 140.1, 0.9567) | (138.2, 138.0, 0.9413) | (135.1, 135.1, 0.9206) | (128.3, 129.2, 0.8739) |
| | 1.01 | (147.2, 146.7) | (140.8, 140.5, 0.9563) | (138.2, 138.0, 0.9389) | (144.3, 143.9, 0.9801) | (140.9, 140.6, 0.9571) | (138.7, 138.5, 0.9419) | (135.7, 135.6, 0.9214) | (128.8, 129.8, 0.8751) |
| | 1.02 | (109.8, 109.3) | (100.3, 100.0, 0.9137) | (96.5, 96.4, 0.8792) | (105.1, 104.7, 0.9578) | (100.1, 99.9, 0.9122) | (96.9, 96.8, 0.8830) | (92.7, 92.8, 0.8447) | (83.7, 84.7, 0.7626) |
| 0.4 | 0.95 | (48.9, 48.4) | (39.0, 38.7, 0.7967) | (35.0, 34.9, 0.7154) | (43.0, 42.6, 0.8781) | (37.7, 37.5, 0.7699) | (34.7, 34.6, 0.7081) | (31.0, 31.2, 0.6341) | (24.3, 25.5, 0.4973) |
| | 0.98 | (91.7, 91.2) | (81.3, 81.0, 0.8871) | (77.2, 77.0, 0.8420) | (86.4, 86.0, 0.9423) | (80.9, 80.6, 0.8824) | (77.5, 77.3, 0.8450) | (73.0, 73.1, 0.7969) | (63.9, 64.9, 0.6970) |
| | 0.99 | (134.5, 134.0) | (126.9, 126.6, 0.9432) | (123.8, 123.6, 0.9204) | (131.0, 130.6, 0.9735) | (127.0, 126.6, 0.9436) | (124.3, 124.1, 0.9239) | (120.8, 120.8, 0.8977) | (112.9, 113.9, 0.8394) |
| | 1.01 | (135.1, 134.6) | (127.5, 127.1, 0.9437) | (124.4, 124.2, 0.9212) | (131.5, 131.1, 0.9738) | (127.5, 127.2, 0.9442) | (124.9, 124.7, 0.9247) | (121.4, 121.4, 0.8987) | (113.6, 114.5, 0.8409) |
| | 1.02 | (93.0, 92.5) | (82.7, 82.4, 0.8893) | (78.6, 78.5, 0.8450) | (87.8, 87.4, 0.9436) | (82.3, 82.1, 0.8849) | (78.9, 78.8, 0.8481) | (74.5, 74.6, 0.8008) | (65.3, 66.4, 0.7022) |
| 0.8 | 0.95 | (34.1, 33.6) | (25.4, 25.2, 0.7461) | (22.0, 21.9, 0.6446) | (28.4, 28.1, 0.8322) | (23.8, 23.7, 0.6979) | (21.3, 21.4, 0.6257) | (18.5, 18.8, 0.5430) | (13.7, 14.8, 0.4013) |
| | 0.98 | (10.8, 10.2) | (6.6, 6.3, 0.6097) | (4.9, 4.9, 0.4536) | (6.9, 6.8, 0.6415) | (4.9, 5.0, 0.4547) | (4.0, 4.2, 0.3747) | (3.2, 3.6, 0.2974) | (2.1, 2.8, 0.1961) |
| | 0.99 | (54.2, 53.7) | (43.9, 43.6, 0.8104) | (39.8, 39.7, 0.7345) | (48.2, 47.8, 0.8892) | (42.7, 42.5, 0.7881) | (39.5, 39.5, 0.7296) | (35.7, 35.9, 0.6587) | (28.5, 29.6, 0.5254) |
| | 1.01 | (101.9, 101.4) | (91.9, 91.6, 0.9023) | (87.9, 87.8, 0.8633) | (96.9, 96.5, 0.9514) | (91.6, 91.4, 0.8996) | (88.3, 88.2, 0.8668) | (84.0, 84.0, 0.8242) | (74.8, 75.8, 0.7342) |
| | 1.02 | (102.5, 102.0) | (92.6, 92.3, 0.9033) | (88.7, 88.5, 0.8646) | (97.6, 97.2, 0.9519) | (92.4, 92.1, 0.9007) | (89.0, 88.9, 0.8682) | (84.7, 84.7, 0.8259) | (75.5, 76.6, 0.7366) |
| 1.05 | 1.01 | (55.5, 55.0) | (45.2, 44.9, 0.8138) | (41.0, 40.9, 0.7393) | (49.5, 49.1, 0.8919) | (44.0, 43.8, 0.7926) | (40.8, 40.7, 0.7349) | (36.9, 37.1, 0.6649) | (29.5, 30.7, 0.5326) |
| | 1.02 | (12.0, 11.5) | (7.5, 7.2, 0.6208) | (5.6, 5.6, 0.4691) | (8.0, 7.8, 0.6629) | (5.8, 5.8, 0.4783) | (4.8, 5.0, 0.3972) | (3.8, 4.2, 0.3174) | (2.5, 3.3, 0.2098) |

Table 9: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.01$, $n = 15$ and different values of other parameters.

| ρ | τ | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|--------|--------|--------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| -0.8 | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.1, 0.3) | (0.5, 0.1, 0.5003) | (0.3, 0.1, 0.3004) | (0.1, 0.2, 0.1340) | (0.1, 0.1, 0.1053) | (0.1, 0.1, 0.1019) | (0.1, 0.001, 0.1005) | (0.1, 0.001, 0.10) |
| | 0.99 | (3.3, 2.8) | (1.8, 1.5, 0.5283) | (1.1, 1.0, 0.3396) | (1.3, 1.4, 0.4047) | (0.8, 1.0, 0.2413) | (0.6, 0.8, 0.1903) | (0.5, 0.7, 0.1509) | (0.4, 0.5, 0.1146) |
| | 1.01 | (3.4, 2.9) | (1.8, 1.6, 0.5294) | (1.2, 1.1, 0.3412) | (1.4, 1.5, 0.4101) | (0.8, 1.0, 0.2453) | (0.7, 0.9, 0.1933) | (0.5, 0.7, 0.1530) | (0.4, 0.6, 0.1154) |
| | 1.02 | (1.1, 0.3) | (0.5, 0.1, 0.5004) | (0.3, 0.1, 0.3005) | (0.1, 0.2, 0.1375) | (0.1, 0.1, 0.1061) | (0.1, 0.1, 0.1022) | (0.1, 0.0, 0.1007) | (0.1, 0.0, 0.1000) |
| -0.4 | 0.95 | (1.0, 0.001) | (0.5, 0.001, 0.49) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 0.98 | (1.0, 0.001) | (0.5, 0.001, 0.001) | (0.3, 0.001, 0.001) | (0.1, 0.001, 0.001) | (0.1, 0.001, 0.001) | (0.1, 0.001, 0.001) | (0.1, 0.001, 0.001) | (0.1, 0.001, 0.001) |
| | 0.99 | (2.5, 1.9) | (1.3, 1.0, 0.5164) | (0.8, 0.7, 0.3230) | (0.8, 0.9, 0.3378) | (0.5, 0.6, 0.1965) | (0.4, 0.5, 0.1573) | (0.3, 0.4, 0.1296) | (0.3, 0.3, 0.1071) |
| | 1.01 | (2.5, 2.0) | (1.3, 1.0, 0.5172) | (0.8, 0.7, 0.3241) | (0.9, 1.0, 0.3430) | (0.5, 0.7, 0.1997) | (0.4, 0.6, 0.1596) | (0.3, 0.5, 0.1310) | (0.3, 0.3, 0.1075) |
| | 1.02 | (1.0, 0.2) | (0.5, 0.1, 0.5001) | (0.3, 0.0, 0.3001) | (0.1, 0.1, 0.1196) | (0.1, 0.1, 0.1024) | (0.1, 0.0, 0.1007) | (0.1, 0.0, 0.1002) | (0.1, 0.0, 0.1000) |
| 0.0 | 0.95 | (1.0, 0.0) | (0.01, 0.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.98 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.99 | (1.8, 1.2) | (0.9, 0.6, 0.5068) | (0.5, 0.4, 0.3095) | (0.5, 0.6, 0.2566) | (0.3, 0.4, 0.1511) | (0.2, 0.3, 0.1271) | (0.2, 0.2, 0.1122) | (0.2, 0.2, 0.1022) |
| | 1.01 | (1.8, 1.2) | (0.9, 0.6, 0.5072) | (0.6, 0.4, 0.3100) | (0.5, 0.6, 0.2610) | (0.3, 0.4, 0.1533) | (0.2, 0.3, 0.1284) | (0.2, 0.2, 0.1129) | (0.2, 0.2, 0.1023) |
| | 1.02 | (1.0, 0.001) | (0.5, 0.0, 0.5000) | (0.3, 0.0, 0.3000) | (0.1, 0.1, 0.1070) | (0.1, 0.0, 0.1005) | (0.1, 0.0, 0.1001) | (0.1, 0.0, 0.1000) | (0.1, 0.001, 0.10) |
| 0.4 | 0.95 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.0, 0.001) | (0.01, 0.0, 0.001) |
| | 0.98 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.99 | (1.2, 0.5) | (0.6, 0.2, 0.5011) | (0.4, 0.2, 0.3016) | (0.2, 0.3, 0.1658) | (0.1, 0.2, 0.1141) | (0.1, 0.1, 0.1059) | (0.1, 0.1, 0.1020) | (0.1, 0.1, 0.1002) |
| | 1.01 | (1.2, 0.5) | (0.6, 0.3, 0.5012) | (0.4, 0.2, 0.3017) | (0.2, 0.3, 0.1685) | (0.1, 0.2, 0.1149) | (0.1, 0.1, 0.1064) | (0.1, 0.1, 0.1022) | (0.1, 0.1, 0.1002) |
| | 1.02 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.001, 0.1010) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| 0.8 | 0.95 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.98 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 0.99 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |
| | 1.01 | (1.0, 0.001) | (0.5, 0.001, 0.50) | (0.3, 0.001, 0.30) | (0.1, 0.1, 0.1028) | (0.1, 0.001, 0.1001) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) | (0.1, 0.001, 0.10) |
| | 1.02 | (1.0, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) | (0.01, 0.001, 0.001) |

Table 10: The out-of-control ($ARL_1, SDRRL_1$) of the Shewhart-RZ charts (column FSI), the out-of-control ($ATS_1, SDTS_1, ASI_1$) of the VSI Shewhart-RZ charts for $\gamma_X = \gamma_Y = 0.2$, $n = 15$ and different values of other parameters.

| ρ | τ | FSI | $(h_S, h_L) = (0.5, 1.5)$ | $(h_S, h_L) = (0.3, 1.7)$ | $(h_S, h_L) = (0.1, 1.1)$ | $(h_S, h_L) = (0.1, 1.3)$ | $(h_S, h_L) = (0.1, 1.5)$ | $(h_S, h_L) = (0.1, 1.9)$ | $(h_S, h_L) = (0.1, 4.0)$ |
|--------|--------------|----------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| -0.8 | 0.95 | (50.8, 50.3) | (40.8, 40.6, 0.8042) | (36.9, 36.8, 0.7258) | (44.9, 44.6, 0.8841) | (39.6, 39.5, 0.7800) | (36.6, 36.5, 0.7199) | (32.9, 33.1, 0.6475) | (26.0, 27.1, 0.5116) |
| | 0.98 | (113.4, 112.9) | (104.3, 104.0, 0.9196) | (100.6, 100.5, 0.8874) | (109.0, 108.6, 0.9609) | (104.2, 103.9, 0.9185) | (101.1, 100.9, 0.8912) | (97.0, 97.0, 0.8552) | (88.1, 89.1, 0.7772) |
| | 0.99 | (150.1, 149.6) | (144.1, 143.7, 0.9597) | (141.7, 141.5, 0.9436) | (147.4, 147.0, 0.9817) | (144.2, 143.9, 0.9606) | (142.1, 141.9, 0.9465) | (139.3, 139.2, 0.9275) | (132.8, 133.7, 0.8843) |
| | 1.01 | (150.6, 150.1) | (144.6, 144.2, 0.9601) | (142.2, 141.9, 0.9442) | (147.8, 147.4, 0.9818) | (144.7, 144.4, 0.9610) | (142.6, 142.4, 0.9470) | (139.8, 139.7, 0.9282) | (133.3, 134.2, 0.8854) |
| | 1.02 | (114.6, 114.1) | (105.6, 105.3, 0.9211) | (102.0, 101.8, 0.8896) | (110.3, 109.8, 0.9618) | (105.5, 105.2, 0.9202) | (102.4, 102.2, 0.8934) | (98.4, 98.4, 0.8581) | (89.6, 90.6, 0.7812) |
| 1.05 | (54.0, 53.5) | (43.9, 43.6, 0.8129) | (39.9, 39.8, 0.7380) | (48.1, 47.8, 0.8910) | (42.7, 42.6, 0.7915) | (39.6, 39.6, 0.7336) | (35.8, 36.0, 0.6632) | (28.6, 29.7, 0.5295) | |
| -0.4 | 0.95 | (43.1, 42.6) | (33.7, 33.4, 0.7809) | (29.9, 29.8, 0.6932) | (37.3, 36.9, 0.8645) | (32.3, 32.1, 0.7482) | (29.4, 29.4, 0.6829) | (26.1, 26.3, 0.6056) | (20.1, 21.2, 0.4652) |
| | 0.98 | (105.5, 105.0) | (95.9, 95.6, 0.9090) | (92.1, 91.9, 0.8726) | (100.8, 100.3, 0.9551) | (95.7, 95.4, 0.9070) | (92.4, 92.3, 0.8763) | (88.2, 88.3, 0.8362) | (79.2, 80.2, 0.7504) |
| | 0.99 | (144.7, 144.2) | (138.1, 137.7, 0.9544) | (135.4, 135.2, 0.9361) | (141.6, 141.2, 0.9790) | (138.2, 137.8, 0.9551) | (135.9, 135.7, 0.9392) | (132.8, 132.8, 0.9179) | (125.8, 126.7, 0.8695) |
| | 1.01 | (145.1, 144.6) | (138.6, 138.2, 0.9548) | (135.9, 135.7, 0.9367) | (142.1, 141.7, 0.9793) | (138.7, 138.4, 0.9556) | (136.4, 136.2, 0.9398) | (133.3, 133.3, 0.9187) | (126.4, 127.3, 0.8707) |
| | 1.02 | (106.8, 106.3) | (97.3, 96.9, 0.9108) | (93.5, 93.3, 0.8751) | (102.1, 101.7, 0.9561) | (97.1, 96.8, 0.9089) | (93.9, 93.7, 0.8788) | (89.6, 89.7, 0.8394) | (80.6, 81.6, 0.7548) |
| 1.05 | (46.1, 45.6) | (36.5, 36.2, 0.7904) | (32.6, 32.5, 0.7065) | (40.3, 39.9, 0.8727) | (35.1, 35.0, 0.7614) | (32.2, 32.2, 0.6981) | (28.7, 29.0, 0.6227) | (22.3, 23.5, 0.4839) | |
| 0.0 | 0.95 | (33.7, 33.2) | (25.1, 24.9, 0.7467) | (21.7, 21.7, 0.6454) | (28.0, 27.7, 0.8326) | (23.5, 23.4, 0.6990) | (21.1, 21.2, 0.6268) | (18.3, 18.6, 0.5441) | (13.5, 14.6, 0.4014) |
| | 0.98 | (94.4, 93.9) | (84.3, 84.0, 0.8928) | (80.2, 80.1, 0.8499) | (89.3, 88.9, 0.9456) | (83.9, 83.6, 0.8888) | (80.5, 80.4, 0.8531) | (76.2, 76.3, 0.8070) | (67.0, 68.1, 0.7102) |
| | 0.99 | (136.6, 136.1) | (129.2, 128.9, 0.9461) | (126.3, 126.1, 0.9245) | (133.2, 132.7, 0.9749) | (129.3, 129.0, 0.9466) | (126.7, 126.5, 0.9279) | (123.3, 123.3, 0.9029) | (115.7, 116.6, 0.8470) |
| | 1.01 | (137.1, 136.6) | (129.8, 129.4, 0.9466) | (126.8, 126.6, 0.9252) | (133.7, 133.3, 0.9752) | (129.8, 129.5, 0.9471) | (127.3, 127.1, 0.9286) | (123.9, 123.9, 0.9039) | (116.3, 117.3, 0.8484) |
| | 1.02 | (95.8, 95.3) | (85.7, 85.4, 0.8949) | (81.7, 81.5, 0.8528) | (90.7, 90.3, 0.9469) | (85.3, 85.1, 0.8912) | (82.0, 81.8, 0.8561) | (77.6, 77.7, 0.8107) | (68.5, 69.5, 0.7153) |
| 1.05 | (36.4, 35.9) | (27.5, 27.3, 0.7572) | (24.0, 24.0, 0.6601) | (30.7, 30.4, 0.8429) | (26.0, 25.9, 0.7145) | (23.4, 23.5, 0.6443) | (20.5, 20.7, 0.5630) | (15.3, 16.4, 0.4205) | |
| 0.4 | 0.95 | (21.7, 21.2) | (15.0, 14.7, 0.6896) | (12.3, 12.3, 0.5654) | (16.7, 16.4, 0.7683) | (13.2, 13.2, 0.6079) | (11.5, 11.6, 0.5278) | (9.6, 9.9, 0.4413) | (6.6, 7.6, 0.3056) |
| | 0.98 | (76.9, 76.4) | (66.4, 66.1, 0.8629) | (62.1, 62.0, 0.8080) | (71.3, 70.9, 0.9269) | (65.7, 65.4, 0.8540) | (62.2, 62.1, 0.8095) | (57.9, 58.0, 0.7532) | (49.2, 50.3, 0.6394) |
| | 0.99 | (122.7, 122.2) | (114.2, 113.8, 0.9305) | (110.8, 110.6, 0.9028) | (118.6, 118.2, 0.9669) | (114.1, 113.8, 0.9303) | (111.2, 111.0, 0.9065) | (107.4, 107.4, 0.8750) | (98.9, 99.8, 0.8057) |
| | 1.01 | (123.3, 122.8) | (114.8, 114.5, 0.9312) | (111.4, 111.2, 0.9037) | (119.2, 118.8, 0.9673) | (114.8, 114.5, 0.9310) | (111.9, 111.7, 0.9075) | (108.0, 108.0, 0.8762) | (99.5, 100.5, 0.8075) |
| | 1.02 | (78.3, 77.8) | (67.8, 67.5, 0.8655) | (63.5, 63.4, 0.8117) | (72.7, 72.3, 0.9286) | (67.1, 66.9, 0.8571) | (63.7, 63.6, 0.8133) | (59.3, 59.4, 0.7579) | (50.5, 51.6, 0.6454) |
| 1.05 | (23.8, 23.3) | (16.7, 16.5, 0.7012) | (13.8, 13.8, 0.5816) | (18.6, 18.4, 0.7827) | (14.9, 14.9, 0.6273) | (13.1, 13.2, 0.5484) | (11.0, 11.3, 0.4621) | (7.7, 8.7, 0.3239) | |
| 0.8 | 0.95 | (6.6, 6.0) | (3.7, 3.5, 0.5684) | (2.6, 2.6, 0.3958) | (3.6, 3.5, 0.5456) | (2.4, 2.5, 0.3582) | (1.9, 2.1, 0.2867) | (1.5, 1.8, 0.2232) | (1.0, 1.4, 0.1499) |
| | 0.98 | (40.9, 40.4) | (31.6, 31.4, 0.7728) | (27.9, 27.8, 0.6819) | (35.1, 34.8, 0.8574) | (30.2, 30.0, 0.7369) | (27.4, 27.4, 0.6698) | (24.2, 24.4, 0.5911) | (18.4, 19.5, 0.4500) |
| | 0.99 | (87.3, 86.8) | (77.0, 76.6, 0.8811) | (72.8, 72.7, 0.8336) | (82.0, 81.6, 0.9386) | (76.5, 76.2, 0.8755) | (73.0, 72.9, 0.8363) | (68.6, 68.7, 0.7861) | (59.6, 60.6, 0.6822) |
| | 1.01 | (88.0, 87.5) | (77.7, 77.4, 0.8823) | (73.5, 73.4, 0.8323) | (82.7, 82.3, 0.9393) | (77.2, 76.9, 0.8768) | (73.8, 73.6, 0.8379) | (69.4, 69.5, 0.7881) | (60.3, 61.4, 0.6849) |
| | 1.02 | (42.1, 41.6) | (32.7, 32.4, 0.7767) | (28.9, 28.9, 0.6874) | (36.2, 35.9, 0.8609) | (31.3, 31.1, 0.7424) | (28.5, 28.5, 0.6762) | (25.2, 25.4, 0.5981) | (19.3, 20.4, 0.4576) |
| 1.05 | (7.4, 6.9) | (4.3, 4.0, 0.5779) | (3.0, 3.0, 0.4090) | (4.2, 4.2, 0.5707) | (2.8, 3.0, 0.3820) | (2.3, 2.5, 0.3077) | (1.8, 2.1, 0.2402) | (1.2, 1.7, 0.1595) | |

6 Illustrative example

In this section, we present an example of implementation of the upward VSI Shewhart-RZ control chart using the dataset proposed by Celano et al. [7]. This is a muesli brand made of a number of ingredients in which the nutrition facts requirements are assured by the use of equal weights of “pumpkin seeds” and “flaxseeds”. That is, their nominal proportions to the total weight of box content are both fixed at $p_p = p_f = 0.1$. Several brand boxes with different dimensions are manufactured to meet the market needs. Let $\mu_{p,i}$ and $\mu_{f,i}$ denote the mean weights for “pumpkin seeds” and “flaxseeds”. For all the boxes dimension, the manufacturers want to monitor the regular intervals $i = 1, 2, \dots$ to check deviations from the in-control ratio $z_0 = \frac{\mu_{p,i}}{\mu_{f,i}} = 1$. During the mixing process, the in-control ratio can be undesirably shifted because of problems occurring at the dosing machine. Suppose that the sample size $n = 5$ boxes have been picked every 30 minutes. One can calculate the sample average weights $\bar{W}_{p,i} = \frac{1}{n} \sum_{j=1}^n W_{p,i,j}$ and $\bar{W}_{f,i} = \frac{1}{n} \sum_{j=1}^n W_{f,i,j}$ for pumpkin seeds and flaxseeds. The ratio $\hat{Z}_i = \frac{\bar{W}_{p,i}}{\bar{W}_{f,i}}$ is then computed and plotted on the VSI Shewhart-RZ control chart. The dataset in [7] said that $\gamma_p = 0.02$ for pumpkin seeds and $\gamma_f = 0.01$ for flaxseeds, while the in-control correlation coefficient between these two variables is $\rho_0 = 0.8$.

Suppose that the increase of this ratio with the size $\tau = 1.01$ is wished to detect. The sampling interval is chosen such that $(h_S, h_L) = (0.1, 1.1)$. The parameters of $K_U = 1.0153766$, $W_U = 0.9955527$ can be obtained numerically. When $z_0 = 1$, the upper control limit $UCL^+ = 1.0153766$ and the warning control limit is $UWL^+ = 0.9955527$. A set of simulated samples collected from the process, the corresponding box sizes 250–500 gr and the \hat{Z}_i statistics is shown in Table 13. The process is supposed to run in-control up to sample #10. Then, between samples #10 and #11 Celano et al. [7] have simulated the occurrence of an assignable cause shifting z_0 to $z_1 = 1.01 \times z_0$, i.e. a ratio percentage increase equal to 1%. Figure 3 reports the upward VSI Shewhart-RZ control chart. The chart signals the occurrence of the out-of-control condition by the point #11 above the control limit $UCL^+ = 1.0153766$. After removing the assignable causes, the process is enabled to continue after sample #13.

7 Conclusions

In this paper, we have investigated the statistical properties of two one-sided Shewhart control charts monitoring the ratio of two normal variables. From the numerical comparison, it is noticeable that two separated one-sided She-

Table 11: The out-of-control $EARL_1$ of the Shewhart-RZ charts (column FSI), the out-of-control $EATS_1$ of the VSI Shewhart-RZ charts for $n = \{1, 5\}$ and different values of other parameters, (D): Decreasing case, (I): Increasing case.

| n | ρ | Type | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|------------------------------|--------|------|-------|------------|------------|------------|------------|------------|------------|------------|
| $\gamma_X = \gamma_Y = 0.01$ | | | | | | | | | | |
| 1 | -0.8 | (D) | 8.1 | 5.8 | 4.9 | 6.0 | 5.0 | 4.5 | 4.0 | 3.1 |
| | | (I) | 8.3 | 6.0 | 5.0 | 6.2 | 5.2 | 4.7 | 4.1 | 3.2 |
| | -0.4 | (D) | 6.7 | 4.6 | 3.8 | 4.7 | 3.9 | 3.5 | 3.0 | 2.3 |
| | | (I) | 6.8 | 4.8 | 4.0 | 4.9 | 4.0 | 3.6 | 3.1 | 2.4 |
| | 0.0 | (D) | 5.1 | 3.4 | 2.7 | 3.4 | 2.7 | 2.4 | 2.1 | 1.5 |
| | | (I) | 5.2 | 3.5 | 2.8 | 3.5 | 2.8 | 2.5 | 2.1 | 1.6 |
| | 0.4 | (D) | 3.4 | 2.1 | 1.6 | 1.9 | 1.5 | 1.3 | 1.1 | 0.8 |
| | | (I) | 3.5 | 2.1 | 1.6 | 2.0 | 1.5 | 1.3 | 1.1 | 0.8 |
| | 0.8 | (D) | 1.6 | 0.6 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 |
| | | (I) | 1.6 | 0.7 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 |
| $\gamma_X = \gamma_Y = 0.2$ | | | | | | | | | | |
| 1 | -0.8 | (D) | 148.1 | 140.2 | 137.0 | 144.5 | 140.2 | 137.4 | 133.8 | 126.5 |
| | | (I) | 150.9 | 143.2 | 140.2 | 147.4 | 143.3 | 140.6 | 137.1 | 129.9 |
| | -0.4 | (D) | 143.8 | 135.1 | 131.7 | 139.8 | 135.1 | 132.1 | 128.2 | 120.5 |
| | | (I) | 146.7 | 138.4 | 135.0 | 142.9 | 138.4 | 135.4 | 131.7 | 124.1 |
| | 0.0 | (D) | 137.2 | 127.6 | 123.8 | 132.7 | 127.5 | 124.2 | 120.0 | 111.8 |
| | | (I) | 140.4 | 131.1 | 127.3 | 136.0 | 131.0 | 127.7 | 123.6 | 115.5 |
| | 0.4 | (D) | 125.6 | 114.7 | 110.3 | 120.2 | 114.3 | 110.6 | 106.0 | 97.4 |
| | | (I) | 129.1 | 118.4 | 114.1 | 123.9 | 118.0 | 114.4 | 109.8 | 101.2 |
| | 0.8 | (D) | 95.1 | 82.7 | 77.7 | 88.2 | 81.3 | 77.4 | 72.8 | 64.8 |
| | | (I) | 98.9 | 86.3 | 81.3 | 92.0 | 85.0 | 81.0 | 76.3 | 68.0 |
| $\gamma_X = \gamma_Y = 0.01$ | | | | | | | | | | |
| 5 | -0.8 | (D) | 2.3 | 1.2 | 0.8 | 1.0 | 0.7 | 0.6 | 0.5 | 0.3 |
| | | (I) | 2.3 | 1.3 | 0.9 | 1.0 | 0.7 | 0.6 | 0.5 | 0.4 |
| | -0.4 | (D) | 1.9 | 0.9 | 0.6 | 0.7 | 0.5 | 0.4 | 0.3 | 0.2 |
| | | (I) | 1.9 | 1.0 | 0.7 | 0.7 | 0.5 | 0.4 | 0.3 | 0.3 |
| | 0.0 | (D) | 1.6 | 0.6 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 |
| | | (I) | 1.6 | 0.7 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 |
| | 0.4 | (D) | 1.2 | 0.4 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| | | (I) | 1.2 | 0.4 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| | 0.8 | (D) | 1.0 | 0.2 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| | | (I) | 1.0 | 0.2 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| $\gamma_X = \gamma_Y = 0.2$ | | | | | | | | | | |
| 5 | -0.8 | (D) | 92.8 | 83.6 | 79.9 | 87.8 | 83.0 | 80.0 | 76.2 | 68.7 |
| | | (I) | 96.3 | 87.1 | 83.4 | 91.4 | 86.5 | 83.5 | 79.7 | 71.9 |
| | -0.4 | (D) | 85.6 | 76.4 | 72.7 | 80.5 | 75.6 | 72.6 | 69.0 | 61.7 |
| | | (I) | 89.1 | 79.7 | 76.0 | 84.0 | 79.0 | 76.0 | 72.2 | 64.7 |
| | 0.0 | (D) | 76.0 | 66.9 | 63.2 | 70.8 | 65.9 | 63.0 | 59.5 | 52.7 |
| | | (I) | 79.3 | 69.9 | 66.2 | 74.0 | 69.1 | 66.1 | 62.5 | 55.4 |
| | 0.4 | (D) | 61.9 | 53.3 | 49.8 | 56.7 | 52.1 | 49.5 | 46.4 | 40.5 |
| | | (I) | 64.8 | 55.9 | 52.3 | 59.4 | 54.7 | 52.0 | 48.7 | 42.5 |
| | 0.8 | (D) | 36.2 | 29.7 | 27.1 | 31.7 | 28.4 | 26.6 | 24.6 | 20.9 |
| | | (I) | 37.9 | 31.1 | 28.4 | 33.2 | 29.7 | 27.8 | 25.6 | 21.8 |

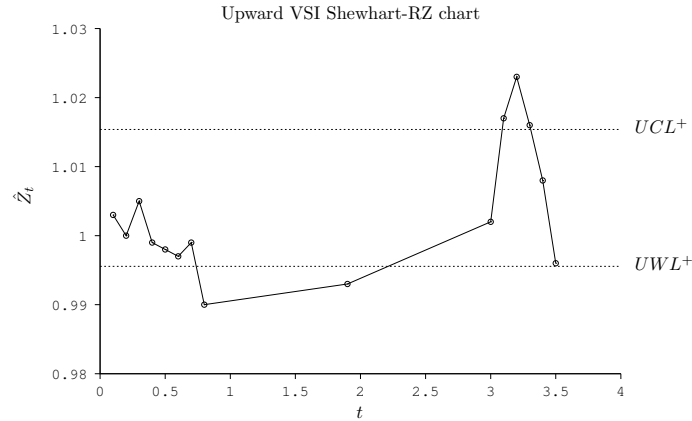
Table 12: The out-of-control $EARL_1$ of the Shewhart-RZ charts (column FSI), the out-of-control $EATS_1$ of the VSI Shewhart-RZ charts for $n = \{10, 15\}$ and different values of other parameters, (D): Decreasing case, (I): Increasing case.

| n | ρ | Type | FSI | (0.5, 1.5) | (0.3, 1.7) | (0.1, 1.1) | (0.1, 1.3) | (0.1, 1.5) | (0.1, 1.9) | (0.1, 4.0) |
|------------------------------|--------|---------|------|------------|------------|------------|------------|------------|------------|------------|
| $\gamma_X = \gamma_Y = 0.01$ | | | | | | | | | | |
| 10 | -0.8 | (D) | 1.5 | 0.6 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 |
| | | (I) | 1.5 | 0.6 | 0.4 | 0.4 | 0.2 | 0.2 | 0.2 | 0.1 |
| | -0.4 | (D) | 1.3 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 |
| | | (I) | 1.3 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 |
| | 0.0 | (D) | 1.2 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| | | (I) | 1.2 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| | 0.4 | (D) | 1.1 | 0.2 | 0.1 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 |
| | | (I) | 1.1 | 0.2 | 0.1 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 |
| | 0.8 | (D) | 1.0 | 0.1 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| | | (I) | 1.0 | 0.1 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| $\gamma_X = \gamma_Y = 0.2$ | | | | | | | | | | |
| 10 | -0.8 | (D) | 71.1 | 62.3 | 58.8 | 66.0 | 61.4 | 58.7 | 55.3 | 48.8 |
| | | (I) | 74.2 | 65.3 | 61.7 | 69.1 | 64.4 | 61.6 | 58.1 | 51.3 |
| | -0.4 | (D) | 64.2 | 55.7 | 52.3 | 59.1 | 54.6 | 52.1 | 48.9 | 42.8 |
| | | (I) | 67.1 | 58.4 | 54.9 | 62.0 | 57.4 | 54.7 | 51.4 | 45.0 |
| | 0.0 | (D) | 55.4 | 47.4 | 44.2 | 50.4 | 46.2 | 43.9 | 41.0 | 35.6 |
| | | (I) | 58.0 | 49.7 | 46.4 | 52.9 | 48.6 | 46.1 | 43.0 | 37.3 |
| | 0.4 | (D) | 43.3 | 36.3 | 33.4 | 38.6 | 35.0 | 33.0 | 30.6 | 26.3 |
| | | (I) | 45.4 | 38.0 | 35.0 | 40.6 | 36.7 | 34.6 | 32.1 | 27.5 |
| | 0.8 | (D) | 23.4 | 18.7 | 16.8 | 19.8 | 17.5 | 16.3 | 14.9 | 12.4 |
| | | (I) | 24.4 | 19.4 | 17.4 | 20.6 | 18.2 | 16.9 | 15.4 | 12.8 |
| $\gamma_X = \gamma_Y = 0.01$ | | | | | | | | | | |
| 15 | -0.8 | (D) | 1.2 | 0.4 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| | | (I) | 1.2 | 0.4 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| | -0.4 | (D) | 1.2 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| | | (I) | 1.2 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| | 0.0 | (D) | 1.1 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 |
| | | (I) | 1.1 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| | 0.4 | (D) | 1.0 | 0.2 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| | | (I) | 1.0 | 0.2 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| | 0.8 | (D) | 1.0 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| | | (I) | 1.0 | 0.1 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| $\gamma_X = \gamma_Y = 0.2$ | | | | | | | | | | |
| 15 | -0.8 | (D) | 59.5 | 51.3 | 48.1 | 54.6 | 50.2 | 47.8 | 44.8 | 39.1 |
| | | (I) | 62.3 | 53.8 | 50.4 | 57.2 | 52.8 | 50.2 | 47.0 | 41.0 |
| | -0.4 | (D) | 53.2 | 45.4 | 42.2 | 48.3 | 44.2 | 41.9 | 39.1 | 33.9 |
| | | (I) | 55.7 | 47.6 | 44.3 | 50.7 | 46.4 | 44.0 | 41.1 | 35.5 |
| | 0.0 | (D) | 45.2 | 38.0 | 35.2 | 40.5 | 36.8 | 34.7 | 32.3 | 27.8 |
| | | (I) | 47.4 | 39.9 | 36.8 | 42.5 | 38.6 | 36.4 | 33.8 | 29.0 |
| | 0.4 | (D) | 34.6 | 28.5 | 26.0 | 30.3 | 27.2 | 25.5 | 23.6 | 20.0 |
| | | (I) | 36.2 | 29.8 | 27.2 | 31.8 | 28.5 | 26.7 | 24.6 | 20.9 |
| | 0.8 | (D) | 17.9 | 14.0 | 12.4 | 14.8 | 12.9 | 11.9 | 10.8 | 8.9 |
| | | (I) | 18.6 | 14.5 | 12.8 | 15.3 | 13.4 | 12.4 | 11.2 | 9.2 |

Table 13: The food industry example data (S.I: sampling interval)

| Sample | S.I | Total time | Box Size | $W_{p,i,j}$ [gr] | | $W_{f,i,j}$ [gr] | | $\bar{W}_{p,i}$ [gr] | | $\bar{W}_{f,i}$ [gr] | | $\hat{Z}_i = \frac{\bar{W}_{p,i}}{\bar{W}_{f,i}}$ |
|--------|-----|------------|----------|------------------|--------|------------------|--------|----------------------|--------|----------------------|--|---|
| | | | | | | | | | | | | |
| 1 | 0.1 | 0.1 | 250 gr | 25.479 | 25.355 | 24.027 | 25.792 | 24.960 | 25.122 | 1.003 | | |
| | | | | 25.218 | 25.171 | 24.684 | 25.052 | 25.107 | 25.046 | | | |
| 2 | 0.1 | 0.2 | 250 gr | 25.359 | 25.172 | 24.508 | 25.292 | 24.449 | 24.956 | 1.003 | | |
| | | | | 25.211 | 25.115 | 24.679 | 24.933 | 24.831 | 24.954 | | | |
| 3 | 0.1 | 0.3 | 250 gr | 24.574 | 24.864 | 25.865 | 25.107 | 24.811 | 25.044 | 1.005 | | |
| | | | | 24.784 | 24.868 | 25.377 | 24.879 | 24.734 | 24.929 | | | |
| 4 | 0.1 | 0.4 | 250 gr | 25.313 | 24.483 | 24.088 | 25.184 | 25.681 | 24.950 | 0.999 | | |
| | | | | 25.338 | 24.859 | 24.305 | 25.115 | 25.251 | 24.974 | | | |
| 5 | 0.1 | 0.5 | 250 gr | 25.557 | 24.959 | 25.023 | 24.482 | 25.531 | 25.111 | 0.998 | | |
| | | | | 25.277 | 25.402 | 25.012 | 24.937 | 25.148 | 25.163 | | | |
| 6 | 0.1 | 0.6 | 250 gr | 24.882 | 24.473 | 24.814 | 25.418 | 24.732 | 24.864 | 0.997 | | |
| | | | | 24.962 | 24.644 | 24.817 | 25.419 | 24.818 | 24.932 | | | |
| 7 | 0.1 | 0.7 | 500 gr | 49.848 | 48.685 | 49.994 | 49.910 | 49.374 | 49.562 | 0.999 | | |
| | | | | 49.993 | 49.128 | 49.830 | 49.566 | 49.422 | 49.588 | | | |
| 8 | 0.1 | 0.8 | 500 gr | 49.668 | 50.338 | 49.149 | 47.807 | 49.064 | 49.205 | 0.990 | | |
| | | | | 49.695 | 50.681 | 49.640 | 48.969 | 49.612 | 49.720 | | | |
| 9 | 1.1 | 1.9 | 500 gr | 51.273 | 48.303 | 48.510 | 50.594 | 48.591 | 49.454 | 0.993 | | |
| | | | | 50.366 | 49.210 | 49.844 | 49.890 | 49.595 | 49.781 | | | |
| 10 | 1.1 | 3.0 | 500 gr | 48.720 | 51.566 | 49.677 | 50.651 | 50.344 | 50.192 | 1.002 | | |
| | | | | 49.721 | 50.215 | 50.178 | 50.324 | 50.071 | 50.102 | | | |
| 11 | 0.1 | 3.1 | 500 gr | 51.372 | 51.700 | 51.000 | 50.886 | 49.641 | 50.920 | 1.017 | | |
| | | | | 50.164 | 50.272 | 49.884 | 50.061 | 49.845 | 50.045 | | | |
| 12 | 0.1 | 3.2 | 500 gr | 52.020 | 53.182 | 51.374 | 51.342 | 48.771 | 51.138 | 1.023 | | |
| | | | | 50.749 | 50.369 | 49.697 | 49.575 | 49.440 | 49.966 | | | |
| 13 | 0.1 | 3.3 | 500 gr | 52.360 | 49.412 | 50.704 | 50.370 | 50.901 | 50.949 | 1.016 | | |
| | | | | 50.047 | 49.981 | 50.297 | 50.408 | 50.026 | 50.152 | | | |
| 14 | 0.1 | 3.4 | 500 gr | 52.498 | 50.447 | 48.713 | 48.574 | 50.275 | 50.101 | 1.008 | | |
| | | | | 50.064 | 50.124 | 49.162 | 48.865 | 50.344 | 49.712 | | | |
| 15 | 0.1 | 3.5 | 250 gr | 25.123 | 24.658 | 24.468 | 25.030 | 25.071 | 24.870 | 0.996 | | |
| | | | | 25.041 | 24.790 | 24.835 | 25.211 | 25.008 | 24.977 | | | |

Figure 3: Upward VSI Shewhart-RZ chart for the food industry example.



whart charts not only outperform the two-sided Shewhart chart in detecting ratio shifts but also overcome the ARL -biased property. In addition, the performance of the Shewhart chart can be considerably enhanced by combining with the VSI feature. Thus, in many situations, these control charts can be applied to avoid the complexity of the use of others without losing the performance. The obtained results also showed that the choice of (h_S, h_L) of sampling intervals has a significant impact on the proposed charts' performance. From the economic point of view, one is suggested to consider both the performance and the average sampling interval in choosing a specific value for this couple.

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