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Hölder-logarithmic stability in Fourier synthesis ^{*}

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Abstract

We prove a Hölder-logarithmic stability estimate for the problem of finding a sufficiently regular compactly supported function v on \mathbb{R}^d from its Fourier transform $\mathcal{F}v$ given on $[-r, r]^d$. This estimate relies on a Hölder stable continuation of $\mathcal{F}v$ from $[-r, r]^d$ to a larger domain. The related reconstruction procedures are based on truncated series of Chebyshev polynomials. We also give an explicit example showing optimality of our stability estimates.

Keywords: ill-posed inverse problems, Hölder-logarithmic stability, exponential instability, analytic continuation, Chebyshev approximation

AMS subject classification: 42A38, 35R30, 49K40

1 Introduction

The Fourier transform \mathcal{F} is defined by

$$\mathcal{F}v(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi x} v(x) dx, \quad \xi \in \mathbb{R}^d,$$

where v is a test function on \mathbb{R}^d and $d \geq 1$. The analysis of this transform is one of the most developed areas of mathematics and has many applications in physics, statistics and

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engineering; see, for example, Bracewell [6]. In particular, it is well known that if v is integrable and compactly supported then $\mathcal{F}v$ is analytic. Thus, the Fourier transform $\mathcal{F}v$ and, consequently, the function v are uniquely determined by the values of $\mathcal{F}v$ within any open non-empty domain. However, in the case of noisy data, the reconstruction can be hard unless the values of $\mathcal{F}v$ are known in a very large domain or v belongs to a specific class of functions (a priori information). In the present paper, we study how much the stability improves with respect to the size of the domain where $\mathcal{F}v$ is given and with respect to the regularity of v .

Specifically, we consider the following problem.

Problem 1.1. Suppose that $v \in \mathcal{L}^1(\mathbb{R}^d)$ is supported in a given compact set in \mathbb{R}^d . The values of $\mathcal{F}v$ are given on $[-r, r]^d$, possibly with some noise. Find v .

Reconstructing a compactly supported function from its partially known Fourier transform or, equivalently, computing the Fourier transform of a band-limited function given within some domain is a classical problem of the Fourier synthesis and image processing; see, for example, [2, 4, 8, 10, 19, 24]. It also arises in studies of inverse scattering problems in the Born approximation. For example, a variant of Problem 1.1 with $\mathcal{F}v$ given on the ball $B_{2\sqrt{E}}$, where $B_r := \{\xi \in \mathbb{R}^d : |\xi| \leq r\}$, can be regarded as a linearized inverse scattering problem for the Schrödinger equation with potential v at fixed positive energy E , for $d \geq 2$, and on the the energy interval $[0, E]$, for $d \geq 1$. More details can be found in the recent paper [23, Section 4]. This version of Problem 1.1 with $\mathcal{F}v$ given on the ball B_{ω_0} also arises, for example, in a multi-frequency inverse source problem for the homogeneous Helmholtz equation with frequencies $\omega \in [0, \omega_0]$; see [3, Section 3] for more details.

We focus on the stability of reconstructions for Problem 1.1. In particular, for a suitable function ϕ such that $\phi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, we show that

$$\|v_1 - v_2\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq \phi \left(\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{L}^\infty([-r, r]^d)} \right), \quad (1.1)$$

under the additional assumption that $v_1 - v_2$ is sufficiently regular (more precisely, $v_1 - v_2$ is from the Sobolev space $\mathcal{H}^m(\mathbb{R}^d)$ for some integer $m > 0$). The function ϕ also depends on a priori information about $v_1 - v_2$. Furthermore, we propose a reconstruction procedure for Problem 1.1 which stability behaviour is consistent with the function ϕ .

It is well known in the community that Problem 1.1 is ill-posed in the sense of Hadamard; see, for example, [2, 19]. For the introduction to the theory of ill-posed problems; see the classical books by Tikhonov, Arsenin [27] and by Lavrent'ev et al. [20]. In fact, one can show that this problem is *exponentially ill-posed* in a similar way to the

results of [14, 18] using the estimates of ϵ -entropy and ϵ -capacity in functional spaces that go back to Kolmogorov and Vitushkin [21]. To completely settle the question, we give an explicit example demonstrating exponential ill-posedness of Problem 1.1 in Section 5 of the present paper. Consequently, a logarithmic bound is the best one could hope to get in (1.1), in general; see Corollary 3.3 and Theorem 5.2.

On the other hand, in the case when r is sufficiently large (such that $\mathcal{F}v$ on $\mathbb{R}^d \setminus [-r, r]^d$ is negligible) one can approximate v in a Lipschitz stable manner by direct computation of *inverse Fourier transform* \mathcal{F}^{-1} :

$$v(x) \approx [\mathcal{F}^{-1}w](x) := \int_{\mathbb{R}^d} w(\xi) e^{-i\xi x} d\xi, \quad (1.2)$$

taking w equal to the given values of $\mathcal{F}v$ in $[-r, r]^d$, and $w \equiv 0$, outside $[-r, r]^d$. However, there remains some error in this approximation even in the absence of noise.

In the present work, we prove a *Hölder-logarithmic stability* estimate for Problem 1.1 tying together the aforementioned two facts; see Theorem 3.1. Furthermore, our estimate illustrates similar stability behaviour in more complicated non-linear inverse problems. In fact, the relationship is closer than a mere illustration for the reason that the monochromatic reconstruction from the scattering amplitude in the Born approximation reduces to Problem 1.1 with $\mathcal{F}v$ given on the ball B_r for some $r > 0$. In connection with this reduction, the logarithmic stability estimates of the present work can be compared with results of [11, 12, 15]. For other known similar stability results in related inverse problems, see, for example, [1, 3, 13, 16, 17, 22, 25] and references therein. However, to our knowledge, Hölder-logarithmic or logarithmic stability estimates for Problem 1.1 are implied by none of the results in the literature before the present work.

The main idea of our stable reconstruction for Problem 1.1 is the following. First, we extrapolate $\mathcal{F}v$ from $[-r, r]^d$ to a larger domain, which size depends on the noise level. Then, we apply the inverse Fourier transform. This leads to our second problem.

Problem 1.2. Suppose that $v \in \mathcal{L}^1(\mathbb{R}^d)$ is supported in a given compact set in \mathbb{R}^d . The values of $\mathcal{F}v$ are given on $[-r, r]^d$, possibly with some noise. Find $\mathcal{F}v$ on $[-R, R]^d$, where $R \geq r$.

Problem 1.2 is equivalent to the band-limited extrapolation (for $d = 1$), which was considered in many works; see, for example, [4, 7, 8, 10, 24]. A more general problem of stable analytic continuation of a complex function was considered in [9, 20, 28, 29]. In particular, [9, Theorem 1.2] or [29, Theorem 1] lead to a *Hölder stability* estimate for

Problem 1.2: for some $0 < \alpha < 1$ and $c_{\alpha,R} > 0$,

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{L^\infty([-R,R]^d)} \leq c_{\alpha,R} \left(\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{L^\infty([-r,r]^d)} \right)^\alpha. \quad (1.3)$$

However, for a fixed α , the factor $c_{\alpha,R}$ in this estimate grows exponentially as R increases, which hinders continuation of $\mathcal{F}v|_{[-r,r]^d}$ to very large domains. This behaviour is natural due to exponential ill-posedness of Problem 1.1.

In this paper, we continue these studies by establishing Hölder stability estimates for a multidimensional extrapolation procedure for Problem 1.2 based on the Chebyshev polynomials; see formula (1.7) and Theorem 2.2. It is essential that we give these estimates with explicit constants. Then, by choosing an appropriate R , we apply this result to Problem 1.1; see formula (1.8), Corollary 2.3, and the proof of Theorem 3.1.

For a fixed $r > 0$, we consider the following family of extrapolation procedures $\mathcal{C}_{R,n}[\cdot]$ depending on two parameters $R \geq r$ and $n \in \mathbb{N} := \{0, 1, \dots\}$. For a function w on $[-r, r]^d$, define

$$\mathcal{C}_{R,n}[w](\xi) := \begin{cases} w(\xi), & \xi \in [-r, r]^d, \\ \sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_n < n}} a_{k_1, \dots, k_n}[w] \prod_{j=1}^d T_{k_j} \left(\frac{\xi_j}{r} \right), & \xi \in [-R, R]^d \setminus [-r, r]^d, \\ 0, & \xi \in \mathbb{R}^d \setminus [-R, R]^d, \end{cases} \quad (1.4)$$

where

$$a_{k_1, \dots, k_n}[w] := \int_{-r}^r \cdots \int_{-r}^r w(\xi) \prod_{j=1}^d \left(\frac{2^{\mathbb{1}[k_j > 0]} T_{k_j} \left(\frac{\xi_j}{r} \right)}{\pi(r^2 - \xi_j^2)^{\frac{1}{2}}} \right) d\xi_1 \cdots d\xi_d \quad (1.5)$$

and $\mathcal{C}_{R,n}[w]$ is taken to be 0 everywhere outside $[-r, r]^d$ in the case when $n = 0$. In the above, $\mathbb{1}[k > 0]$ is the indicator function for $\{k > 0\}$:

$$\mathbb{1}[k > 0] = \begin{cases} 1, & \text{if } k > 0, \\ 0, & \text{otherwise;} \end{cases}$$

and $(T_k)_{k \in \mathbb{N}}$ stand for *the Chebyshev polynomials*, which can be defined by $T_k(t) := \cos(k \arccos(t))$ for $t \in [-1, 1]$ and extended to $|t| > 1$ in a natural way.

Recall that if v is integrable and compactly supported then $\mathcal{F}v$ is analytical in \mathbb{C}^d . It follows that, for all $\xi \in \mathbb{C}^d$,

$$\mathcal{F}v(\xi) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} a_{k_1, \dots, k_d} [\mathcal{F}v|_{[-r,r]^d}] \prod_{j=1}^d T_{k_j} \left(\frac{\xi_j}{r} \right); \quad (1.6)$$

see, for example, Lemma 4.1 and inequality (4.2). We will show that if $w \approx \mathcal{F}v|_{[-r,r]^d}$ then $\mathcal{C}_{R,n}[w](\xi)$ approximates well the series of (1.6) in the region $[-R,R]^d \setminus [-r,r]^d$, provided n is sufficiently large so the tail of the series is negligible, but not very large so the continuation $\mathcal{C}_{R,n}$ is sufficiently stable.

In Section 2 and Section 3, we prove the new stability estimates for Problem 1.2 and Problem 1.1, respectively. We state them in the following forms.

- Let $\|\mathcal{F}v - w\|_{\mathcal{L}^\infty([-r,r]^d)} \leq \delta$. Then, for any $R > r$, there is some $n^* = n^*(\delta, R)$ and $0 < \alpha < 1$ such that,

$$\|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} \leq c_{\alpha,R} \delta^\alpha. \quad (1.7)$$

- In addition, if v is sufficiently regular (more precisely, v is from the Sobolev space $\mathcal{H}^m(\mathbb{R}^d)$ for some integer $m > 0$), then there are some $R(\delta)$ and $n(\delta) = n^*(\delta, R(\delta))$ such that, as $\delta \rightarrow 0$,

$$\|v - \mathcal{F}^{-1}\mathcal{C}_{R(\delta),n(\delta)}[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq \phi(\delta) \rightarrow 0. \quad (1.8)$$

The constant $c_{\alpha,R}$ and the function ϕ depend on a priori information about v . Note that (1.8) and (1.7) imply (1.1) and (1.3), respectively, by setting $v := v_1 - v_2$ and $w := 0$, and using the linearity of the considered problems and the reconstruction procedures.

In Section 4, we prove a technical lemma about $\mathcal{C}_{R,n}$. In Section 5, we give an example demonstrating that our estimates (at least, for Problem 1.1 at fixed r) are essentially optimal; see Theorem 5.2. Finally, in Section 6, we outline several directions for further development of the studies of the present work.

2 Hölder stability in Problem 1.2

In this section, we give stability estimates for the extrapolation procedure $\mathcal{C}_{R,n}$ defined according to (1.4); see Lemma 2.1, Theorem 2.2, and Corollary 2.3.

Assume that the unknown function $v : \mathbb{R}^d \rightarrow \mathbb{C}$ is such that, for some $N, \sigma > 0$,

$$\|v\|_{\mathcal{L}^1(\mathbb{R}^d)} \leq (2\pi)^d N, \quad \text{supp}(v) \subseteq \left\{ x \in \mathbb{R}^d : \sum_{j=1}^d |x_j| \leq \sigma \right\}. \quad (2.1)$$

Assume also that the given data $w : [-r,r]^d \rightarrow \mathbb{C}$ is such that, for some $\delta > 0$,

$$\|w - \mathcal{F}v\|_{\mathcal{L}^\infty([-r,r]^d)} \leq \delta < N, \quad (2.2)$$

where \mathcal{F} is the Fourier transform.

Note that if (2.1) holds then, for any $\xi \in \mathbb{R}^d$,

$$|\mathcal{F}v(\xi)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |v(x)| dx \leq N. \quad (2.3)$$

This explains the condition $\delta < N$ in assumption (2.2). Indeed, if the noise level δ is greater than N then the given data w tells about v as little as the trivial function $w_0 \equiv 0$.

First, we give an estimate for $\mathcal{C}_{R,n}$ for arbitrary R and n .

Lemma 2.1. *Let the assumptions of (2.1) and (2.2) hold for some $N, \delta, r, \sigma > 0$. Then, for any integer $n > 0$ and real $\rho, R > 0$ such that $R \geq r$, $\rho \geq 4R/r$, we have*

$$\|\mathcal{F}v - \mathcal{C}_{R,n}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} \leq \frac{1}{4} \left(4^d \left(\frac{4R}{r} \right)^n \delta + \left(\frac{16}{3} \right)^d N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^n \right).$$

Lemma 2.1 is proved in Section 4. Optimising the parameter n in Lemma 2.1, we obtain the following Hölder stability estimate for Problem 1.2.

Theorem 2.2. *Let the assumptions of (2.1) and (2.2) hold for some $N, \delta, r, \sigma > 0$. Assume that $\rho, R > 0$ are such that $R \geq r$ and $\rho \geq 4R/r$. Then, we have*

$$\|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} \leq \left(\frac{16}{3} \right)^d \frac{R}{r} \left(\frac{N e^{r\sigma\rho}}{\delta} \right)^{\tau(\rho)} \delta,$$

where

$$n^* := \left\lceil \frac{\ln \frac{N}{\delta} + r\sigma\rho}{\ln(3\rho)} \right\rceil \quad \text{and} \quad \tau(\rho) := \frac{\ln \frac{4R}{r}}{\ln(3\rho)}.$$

Proof. Using (2.2), we have that

$$\eta := \frac{\ln \frac{N}{\delta} + r\sigma\rho}{\ln(3\rho)} > 0.$$

By definition, we find that $\eta \leq n^* < \eta + 1$ and $\delta = N e^{r\sigma\rho} (3\rho)^{-\eta}$. Using that $R \geq r$, we get

$$\delta \left(\frac{4R}{r} \right)^{\eta+1} = \frac{4R}{r} N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^\eta \geq 4 N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^\eta.$$

Then, applying Lemma 2.1, we obtain that

$$\begin{aligned} \|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} &\leq \frac{1}{4} \left(4^d \left(\frac{4R}{r} \right)^{n^*} \delta + \left(\frac{16}{3} \right)^d N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^{n^*} \right) \\ &\leq \frac{1}{4} \left(4^d \left(\frac{4R}{r} \right)^{\eta+1} \delta + \left(\frac{16}{3} \right)^d N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^\eta \right) \\ &\leq \left(\frac{16}{3} \right)^d \frac{R}{r} \left(\frac{4R}{r} \right)^\eta \delta \left(\left(\frac{3}{4} \right)^d + \frac{1}{4} \right) \leq \left(\frac{16}{3} \right)^d \frac{R}{r} \left(\frac{4R}{r} \right)^\eta \delta. \end{aligned}$$

Since $\tau(\rho) \ln(3\rho) = \ln \frac{4R}{r}$, we get

$$\left(\frac{4R}{r}\right)^\eta \delta = (3\rho)^{\tau(\rho)\eta} \delta = (Ne^{r\sigma\rho})^{\tau(\rho)} \delta^{1-\tau(\rho)}.$$

Combining the above estimates completes the proof. \square

Next, to achieve optimal stability bounds for Problem 1.1, the parameters R and n in the reconstruction $\mathcal{F}^{-1}\mathcal{C}_{R,n}$ have to be chosen carefully depending on a priori information; see formulas (2.4), (2.5), (2.6) and Corollary 2.3. For any $\tau \in [0, 1]$, let

$$L_\tau(\delta) = L_\tau(N, \delta, r, \sigma) := \max \left\{ 1, \frac{1}{4} \left(\frac{(1-\tau) \ln \frac{N}{\delta}}{r\sigma} \right)^\tau \right\}. \quad (2.4)$$

Here and thereafter, we assume $0 < \delta < N$. Using (1.4), define

$$\mathcal{C}_{\tau,\delta}^* := \mathcal{C}_{R_\tau(\delta), n_\tau(\delta)}, \quad (2.5)$$

where

$$\begin{aligned} R_\tau(\delta) &= R_\tau(N, \delta, r, \sigma) := rL_\tau(\delta), \\ n_\tau(\delta) &= n_\tau(N, \delta, r, \sigma) := \begin{cases} \left\lceil \frac{(2-\tau) \ln \frac{N}{\delta}}{\ln 3 + \frac{1}{\tau} \ln(4L_\tau(\delta))} \right\rceil, & \text{if } \tau > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.6)$$

and $\lceil \cdot \rceil$ denotes the ceiling of a real number.

Theorem 2.2 leads to the following stability estimate for $\mathcal{C}_{\tau,\delta}^*$; which will be crucial for the results of the next section; see Theorem 3.1.

Corollary 2.3. *Let the assumptions of (2.1) and (2.2) hold for some $N, \sigma, r, \delta > 0$. Then, for any $\tau \in [0, 1]$, we have*

$$\|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)} \leq \left(\frac{16}{3}\right)^d N \left(\frac{\delta}{N}\right)^{(1-\tau)^2} L_\tau(\delta),$$

where $L_\tau(\delta)$ and $R_\tau(\delta)$ are defined in (2.4) and (2.6).

Proof. First, we consider the case $L_\tau(\delta) = 1$, for which $R_\tau(\delta) = r$. Recalling from (2.2) that $\delta < N$ and using (1.4) for $R = r$, we find that

$$\|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)} \leq \delta \leq \left(\frac{16}{3}\right)^d N \left(\frac{\delta}{N}\right)^{(1-\tau)^2} L_\tau(\delta).$$

Next, suppose that

$$L_\tau(\delta) = \frac{1}{4} \left(\frac{(1-\tau) \ln \frac{N}{\delta}}{r\sigma} \right)^\tau > 1.$$

This implies that $\tau > 0$. Let $\rho := (4L_\tau(\delta))^{1/\tau}$. Then, we get $e^{r\sigma\rho} = \left(\frac{N}{\delta}\right)^{1-\tau}$ and, by the assumptions,

$$R_\tau(\delta) \geq r \quad \text{and} \quad \rho \geq 4L_\tau(\delta) = 4R_\tau(\delta)/r.$$

Applying Theorem 2.2 and observing that n^* coincides with $n_\tau(\delta)$ defined by (2.6), we get that

$$\|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)} \leq \left(\frac{16}{3}\right)^d L_\tau(\delta) \left(\left(\frac{N}{\delta}\right)^{2-\tau}\right)^{\tau(\rho)} \delta,$$

where $\tau(\rho)$ is defined in Theorem 2.2. Note that $\tau(\rho)$ is different from τ . However, we can replace $\tau(\rho)$ by τ in the estimate above since $\delta < N$ and

$$\tau(\rho) = \frac{\ln(4R_\tau(\delta)/r)}{\ln(3\rho)} = \frac{\ln(4L_\tau(\delta))}{\ln 3 + \frac{1}{\tau} \ln(4L_\tau(\delta))} \leq \tau.$$

The required bound follows. \square

3 Hölder-logarithmic stability in Problem 1.1

To prove our stability estimate for Problem 1.1, in addition to (2.1), we assume also that $v \in \mathcal{H}^m(\mathbb{R}^d)$, where $\mathcal{H}^m(\mathbb{R}^d)$ is the standard Sobolev space of m -times smooth functions in \mathcal{L}^2 on \mathbb{R}^d . Consider the seminorm $|\cdot|_{\mathcal{H}^m(\mathbb{R}^d)}$ in $\mathcal{H}^m(\mathbb{R}^d)$ defined by

$$|v|_{\mathcal{H}^m(\mathbb{R}^d)} := \left(\sum_{j=1}^d \left\| \frac{\partial^m v}{(\partial x_j)^m} \right\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 \right)^{1/2}. \quad (3.1)$$

Theorem 3.1. *Let the assumptions of (2.1) and (2.2) hold for some $N, \delta, r, \sigma > 0$. Assume also that $v \in \mathcal{H}^m(\mathbb{R}^d)$ for some integer $m > 0$ and $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$ for some $\gamma > 0$. Then, for any $\tau \in [0, 1]$, the following holds:*

$$\|v - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq (20\sqrt{r})^d N (L_\tau(\delta))^{d/2+1} \left(\frac{\delta}{N}\right)^{(1-\tau)^2} + \gamma (rL_\tau(\delta))^{-m}. \quad (3.2)$$

The first term of the right-hand side in estimate (3.2) corresponds to the error caused by the Hölder stable continuation of the noisy data w from $[-r, r]^d$ to $[-R_\tau(\delta), R_\tau(\delta)]^d$ and the second (logarithmic) term corresponds to the error caused by ignoring the values of $\mathcal{F}v$ outside $[-R_\tau(\delta), R_\tau(\delta)]^d$; see the proof of Theorem 3.1 for details.

Remark 3.2. Clearly, the stability behaviour in Problem 1.1 should not depend on scaling of functions or arguments. It might be obscure at first sight, but estimate (3.2) is invariant with respect to such scalings. Indeed, for some $\alpha, \beta > 0$, let \tilde{v} be defined by $\tilde{v}(x) := \alpha v(\beta x)$, $x \in \mathbb{R}^d$. The Fourier transforms of v and \tilde{v} satisfy the following relation $\mathcal{F}\tilde{v}(\xi) = \alpha \mathcal{F}v(\beta^{-1}\xi)$ for $\xi \in \mathbb{R}^d$. If $w \approx \mathcal{F}v$ in $[-r, r]^d$ then, equivalently, $\tilde{w} \approx \mathcal{F}\tilde{v}$ in $[-\tilde{r}, \tilde{r}]^d$, where $\tilde{r} = \beta r$ and $\tilde{w}(\xi) := \alpha w(\beta^{-1}\xi)$, $\xi \in \mathbb{R}^d$. The other parameters in (2.1) and (2.2) are modified as follows: $\tilde{N} = \alpha N$, $\tilde{\delta} = \alpha \delta$, and $\tilde{\sigma} = \beta^{-1}\sigma$. Observe that $L_\tau(\delta)$ depends only on $r\sigma$ and N/δ , which are independent of scalings. Finally, we have

$$\begin{aligned} \|\tilde{v} - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[\tilde{w}]\|_{\mathcal{L}^2(\mathbb{R}^d)} &= \alpha\beta^{d/2}\|v - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)}, \\ |\tilde{v}|_{\mathcal{H}^m(\mathbb{R}^d)} &= \alpha\beta^{m+d/2}|v|_{\mathcal{H}^m(\mathbb{R}^d)}. \end{aligned}$$

Thus, both sides of estimate (3.2) get multiplied by the same constant $\alpha\beta^{d/2}$, that is, the statements of Theorem 3.1 for v, w and for \tilde{v}, \tilde{w} are equivalent.

Proof of Theorem 3.1. Let all assumptions of Theorem 3.1 hold. The Parseval-Plancherel identity states that

$$\|u\|_{\mathcal{L}^2(\mathbb{R}^d)} = (2\pi)^{d/2}\|\mathcal{F}u\|_{\mathcal{L}^2(\mathbb{R}^d)} = (2\pi)^{-d/2}\|\mathcal{F}^{-1}u\|_{\mathcal{L}^2(\mathbb{R}^d)}. \quad (3.3)$$

Thus, we get that

$$\|v - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq (2\pi)^{d/2}(I_1 + I_2),$$

where

$$\begin{aligned} I_1 &:= \left(\int_{[-R_\tau(\delta), R_\tau(\delta)]^d} |\mathcal{F}v(\xi) - \mathcal{C}_{\tau,\delta}^*[w](\xi)|^2 d\xi \right)^{1/2}, \\ I_2 &:= \left(\int_{\mathbb{R}^d \setminus [-R_\tau(\delta), R_\tau(\delta)]^d} |\mathcal{F}v(\xi)|^2 d\xi \right)^{1/2}. \end{aligned}$$

Using Corollary 2.3, we get that,

$$\begin{aligned} I_1 &\leq \left(\int_{[-R_\tau(\delta), R_\tau(\delta)]^d} \|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)}^2 d\xi \right)^{1/2} \\ &\leq \left(\frac{16}{3}\right)^d N \left(\frac{\delta}{N}\right)^{(1-\tau)^2} L_\tau(\delta) (2R_\tau(\delta))^{d/2} \\ &\leq \left(20\sqrt{\frac{r}{2\pi}}\right)^d N (L_\tau(\delta))^{d/2+1} \left(\frac{\delta}{N}\right)^{(1-\tau)^2}. \end{aligned}$$

Next, applying (3.3) and recalling the seminorm $|\cdot|_{\mathcal{H}^m(\mathbb{R}^d)}$ defined in (3.1), we find that

$$\sum_{j=1}^d \|\xi_j^m \mathcal{F}v\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \sum_{j=1}^d \left\| \frac{\partial^m v}{(\partial x_j)^m} \right\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 = \frac{|v|_{\mathcal{H}^m(\mathbb{R}^d)}^2}{(2\pi)^d}.$$

Since $\mathbb{R}^d \setminus [-R_\tau(\delta), R_\tau(\delta)]$ is covered by the regions $\Omega_j := \{\xi \in \mathbb{R}^d : |\xi_j| > R_\tau(\delta)\}$, for $j = 1, \dots, d$, we obtain that

$$\begin{aligned} I_2 &\leq \left(\sum_{j=1}^d \int_{|\xi_j| > R_\tau(\delta)} \left| \frac{\xi_j^m \mathcal{F}v(\xi)}{(R_\tau(\delta))^m} \right|^2 d\xi \right)^{1/2} \\ &\leq \left(\sum_{j=1}^d \frac{\|\xi_j^m \mathcal{F}v\|_{\mathcal{L}^2(\mathbb{R}^d)}^2}{(R_\tau(\delta))^{2m}} \right)^{1/2} \leq \frac{|v|_{\mathcal{H}^m(\mathbb{R}^d)}}{(2\pi)^{d/2}} (rL_\tau(\delta))^{-m}. \end{aligned}$$

Combining the above bounds for I_1 and I_2 , we complete the proof of Theorem 3.1. \square

Theorem 3.1 leads to the Hölder-logarithmic stability estimate for $\|v_1 - v_2\|_{\mathcal{L}^2(\mathbb{R}^d)}$ in (1.1), provided $v_1 - v_2$ satisfies assumptions (2.1), (2.2) (for fixed N) and $v_1 - v_2 \in \mathcal{H}^m(\mathbb{R}^d)$, as explained in Section 1 after formula (1.8). Estimate (3.2) with $\tau = 0$ is similar to well-known stability results for approximate reconstruction explained in (1.2). Theorem 3.1 also implies the following corollary.

Corollary 3.3. *Let $v : \mathbb{R}^d \rightarrow \mathbb{C}$ be supported in a compact set $A \subset \mathbb{R}^d$, where $d \geq 1$, and $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$ for some integer $m > 0$ and real $\gamma > 0$. Let B be an open set in \mathbb{R}^d and $\|\mathcal{F}v\|_{\mathcal{L}^\infty(B)} < 1$. Then, for any $0 \leq \mu < m$, there is $c = c(A, B, \gamma, \mu, m) > 0$ such that*

$$\|v\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq c \left(\ln \frac{1}{\|\mathcal{F}v\|_{\mathcal{L}^\infty(B)}} \right)^{-\mu}. \quad (3.4)$$

Proof. Without loss of generality, we can assume that $0 \in B$ by considering $\tilde{v} := ve^{i\xi_0 x}$, for a fixed $\xi_0 \in \mathbb{R}^d$. Then, $[-r, r]^d \subset B$ for a sufficiently small r . Any compact set A lies in $\left\{ x \in \mathbb{R}^d : \sum_{j=1}^d |x_j| \leq \sigma \right\}$ for a sufficiently large σ . Since v is compactly supported, the condition $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$, for $m \geq 1$, implies that the norm $\|v\|_{\mathcal{L}^1(\mathbb{R}^d)}$ is bounded above by $(2\pi)^d N$, where the constant N depends on A , m , γ only. Applying Theorem 3.1 with $\tau := \mu/m$, $w \equiv 0$, $\delta := \|\mathcal{F}v\|_{\mathcal{L}^\infty([-r, r]^d)}$ and observing that the logarithmic term dominates the Hölder term as $\delta \rightarrow 0$ and

$$\|\mathcal{F}v\|_{\mathcal{L}^\infty([-r, r]^d)} \leq \|\mathcal{F}v\|_{\mathcal{L}^\infty(B)} < 1,$$

we complete the proof. \square

In Section 5, we show that the exponent μ in Corollary 3.3 is optimal (or almost optimal for $d = 1$), using an explicit construction. Namely, we prove that, for $d \geq 2$ and any $\mu > m$, there is some v violating (3.4) no matter how large constant c we take. For $d = 1$, the same holds for any $\mu > m + 1/2$. The optimality of the threshold exponent $\mu^* = m$ for the case $d = 1$ remains an open question. Note that our instability examples also show an optimality of (3.2) as a logarithmic stability estimate.

4 Proof of Lemma 2.1

To prove Lemma 2.1, we need the bounds for series of Chebyshev polynomials stated in the following lemma. We will use the standard combinatorial fact that the number of ways to write n as a sum of d nonnegative integers (ordered) equals the binomial coefficient

$$\binom{n+d-1}{n} = \frac{(n+d-1)!}{n!(d-1)!}. \quad (4.1)$$

Lemma 4.1. *Let $\sigma, r, N > 0$ and $R \geq r$. If v satisfies (2.1) then the following holds.*

(a) *For any $\rho \geq 1$, $\xi \in [-R, R]^d$ and $k_1, \dots, k_d \in \mathbb{N}$, we have*

$$\left| a_{k_1, \dots, k_d} [\mathcal{F}v|_{[-r, r]^d}] \prod_{j=1}^d T_{k_j} \left(\frac{\xi_j}{r} \right) \right| \leq 2^d N e^{\frac{1}{2} r \sigma \rho} \left(\frac{2R}{r\rho} \right)^{\sum_{j=1}^d k_j},$$

where \mathcal{F} is the Fourier transform and $a_{k_1, \dots, k_d}[\cdot]$ is defined according to (1.5).

(b) *For any $\rho' \geq 4R/r$, we have*

$$\| \mathcal{F}v - \mathcal{C}_{R, n} [\mathcal{F}v|_{[-r, r]^d}] \|_{\mathcal{L}^\infty([-R, R]^d)} \leq \left(\frac{8}{3} \right)^d N e^{r \sigma \rho'} \binom{n+d-1}{n} \left(\frac{4R}{3r\rho'} \right)^n,$$

where $\mathcal{C}_{R, n}[\cdot]$ is defined according to (1.4).

Proof. For $z_1, \dots, z_d \in \mathbb{C}$, let

$$f(z_1, \dots, z_d) := \mathcal{F}v(r \cos z_1, \dots, r \cos z_d).$$

Observe that, for any $z \in \mathbb{C}$,

$$|\Im(\cos z)| \leq \frac{1}{2} |e^{\Im z} - e^{-\Im z}| \leq \frac{1}{2} e^{|\Im z|},$$

where $\Im z$ denote the imaginary part of z . If $|\Im z_j| \leq \ln \rho$ for all $1 \leq j \leq d$, then, by assumptions, for any $x \in \text{supp}(v)$, we find that

$$\left| \sum_{j=1}^d x_j \Im(\cos z_j) \right| \leq \sum_{j=1}^d |x_j| \rho / 2 \leq \frac{1}{2} \sigma \rho.$$

Therefore,

$$\begin{aligned} |f(z_1, \dots, z_d)| &= \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i \sum_{j=1}^d r x_j \cos z_j} v(x) dx \right| \\ &\leq \frac{1}{(2\pi)^d} \int_{\text{supp}(v)} e^{\frac{1}{2} r \sigma \rho} |v(x)| dx = N e^{\frac{1}{2} r \sigma \rho}. \end{aligned}$$

Observing that f is 2π -periodic even function with respect to each component and recalling definition (1.5) and that $T_k(t) := \cos(k \arccos(t))$ for $t \in [-1, 1]$, we get

$$a_{k_1, \dots, k_d} = \frac{2^{\sum_{j=1}^d \mathbb{1}[k_j > 0]}}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{i \sum_{j=1}^d k_j \varphi_j} f(\varphi_1, \dots, \varphi_d) d\varphi_1 \dots d\varphi_d.$$

Since v is compactly supported, we have that $\mathcal{F}v$ and f are analytic functions in \mathbb{C}^d . Using the Cauchy integral theorem, we estimate

$$\begin{aligned} & \left| \frac{1}{(2\pi)^d} \int_0^{2\pi} e^{i \sum_{j=1}^d k_j \varphi_j} f(\varphi_1, \dots, \varphi_d) d\varphi_1 \dots d\varphi_d \right| \\ &= \left| \frac{1}{(2\pi)^d} \int_{i \ln \rho}^{2\pi + i \ln \rho} \cdots \int_{i \ln \rho}^{2\pi + i \ln \rho} e^{i \sum_{j=1}^d k_j z_j} f(z_1, \dots, z_d) dz_1 \dots dz_d \right| \\ &\leq \frac{1}{(2\pi)^d} \int_{i \ln \rho}^{2\pi + i \ln \rho} \cdots \int_{i \ln \rho}^{2\pi + i \ln \rho} N e^{\frac{1}{2} r \sigma \rho} e^{-\sum_{j=1}^d k_j \ln \rho} dz_1 \dots dz_d \\ &= N e^{\frac{1}{2} r \sigma \rho} \rho^{-\sum_{j=1}^d k_j}. \end{aligned}$$

We complete the proof of part (a), by observing that $|T_k(t)| \leq (2R/r)^k$ for any $|t| \leq R/r$. Indeed, if $|t| \leq 1$ then $|T_k(t)| \leq 1$, otherwise

$$|T_k(t)| = |\cosh(k \operatorname{arccosh}(t))| = \frac{1}{2} \left| (t - \sqrt{t^2 - 1})^k + (t + \sqrt{t^2 - 1})^k \right| \leq (2|t|)^k.$$

For (b), let $\rho := 2\rho'$ and $\lambda := \frac{2R}{r\rho} = \frac{R}{r\rho'} \leq \frac{1}{4}$. Using the Taylor theorem with the remainder in the Lagrange form, we get that, for some $\lambda' \in [0, \lambda]$,

$$\begin{aligned} (1 - \lambda)^{-d} - \sum_{k=0}^{n-1} \binom{k+d-1}{k} \lambda^k &= \binom{n+d-1}{n} (1 - \lambda')^{-d-n} \lambda^n \\ &\leq \binom{n+d-1}{n} \left(\frac{4}{3}\right)^d \left(\frac{4\lambda}{3}\right)^n. \end{aligned}$$

Using (4.1) and part (a), we find that

$$\begin{aligned} \|\mathcal{F}v - \mathcal{C}_{R,n}[\mathcal{F}v]_{[-r,r]^d}\|_{\mathcal{L}^\infty([-R,R]^d)} &\leq \sum_{k=n}^{\infty} \sum_{k_1 + \dots + k_d = k} 2^d N e^{\frac{1}{2} r \sigma \rho} \lambda^k \\ &= 2^d N e^{r \sigma \rho'} \left((1 - \lambda)^{-d} - \sum_{k=0}^{n-1} \binom{k+d-1}{k} \lambda^k \right) \\ &\leq \left(\frac{8}{3}\right)^d N e^{r \sigma \rho'} \binom{n+d-1}{n} \left(\frac{4\lambda}{3}\right)^n. \end{aligned}$$

This completes the proof of Lemma 4.1. \square

Now we are ready to proceed to Lemma 2.1. Recall that $|T_k(t)| \leq 1$, if $|t| \leq 1$. Using (1.5) and the assumptions, we find that, for any $k_1, \dots, k_d \in \mathbb{N}$,

$$\begin{aligned} |a_{k_1, \dots, k_d}[w] - a_{k_1, \dots, k_d}[\mathcal{F}v|_{[-r, r]^d}]| &= |a_{k_1, \dots, k_d}[w - \mathcal{F}v|_{[-r, r]^d}]| \\ &\leq \int_{-r}^r \cdots \int_{-r}^r \delta \prod_{j=1}^d \left(\frac{2^{\mathbb{1}[k_j > 0]} |T_{k_j}(\frac{\xi_j}{r})|}{\pi(r^2 - \xi_j^2)^{\frac{1}{2}}} \right) d\xi_1 \dots d\xi_d \leq 2^d \delta. \end{aligned}$$

Recalling also that $|T_k(t)| \leq (2|t|)^k$ for $|t| \geq 1$, we get

$$\begin{aligned} \|\mathcal{C}_{R, n}[w] - \mathcal{C}_{R, n}[\mathcal{F}v|_{[-r, r]^d}]\|_{\mathcal{L}^\infty([-R, R]^d)} &\leq \sum_{k=0}^{n-1} \sum_{k_1 + \dots + k_d = k} 2^d \delta \left(\frac{2R}{r}\right)^k \\ &= 2^d \delta \sum_{k=0}^{n-1} \binom{k+d-1}{k} \left(\frac{2R}{r}\right)^k \leq 2^d \delta \binom{n+d-1}{n} \left(\frac{2R}{r}\right)^n. \end{aligned}$$

Since $n \geq 1$ and $d \geq 1$, we have that

$$\binom{n+d-1}{n} = \binom{n+d-2}{n-1} + \binom{n+d-2}{n} \leq \binom{n+d-1}{n-1} + \binom{n+d-1}{n+1},$$

where $\binom{n+d-2}{n}$ and $\binom{n+d-1}{n+1}$ are taken to be 0 if $d = 1$. Thus, we get

$$\binom{n+d-1}{n} \leq \frac{1}{2} \sum_{j=0}^{n+d-1} \binom{n+d-1}{j} = 2^{n+d-2}. \quad (4.2)$$

Combining the above and using Lemma 4.1(b), we complete the proof of Lemma 2.1.

5 Exponential ill-posedness of Problem 1.1

In this section, we prove that Problem 1.1 is exponentially ill-posed. For ease of presentation, we employ the asymptotic notations $O(\cdot)$ and $\Omega(\cdot)$ always referring to the passage of the parameter n to infinity. For two sequences of real numbers a_n and b_n , we say $a_n = O(b_n)$ if there exist constants $C > 0$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leq C|b_n|$ for all $n > n_0$. We say $a_n = \Omega(b_n)$ if $a_n > 0$ always and $b_n = O(a_n)$.

First, we consider an explicit function $v_{n, m} : \mathbb{R}^2 \rightarrow \mathbb{C}$ similar to the one given by Mandache [18, Theorem 2]. Let $g \in C^\infty(\mathbb{R})$ be a nontrivial function supported in a compact set of positive real numbers. For example, one can take

$$g(t) := \begin{cases} \exp\left(\frac{1}{(t-1)(t-2)}\right), & \text{if } 1 < t < 2, \\ 0, & \text{otherwise.} \end{cases} \quad (5.1)$$

For integer $n \geq 1$ and $m \geq 0$, let $v_{n,m}$ be defined by

$$v_{n,m}(x_1, x_2) := n^{-m} e^{in\varphi} g(t),$$

where $t \geq 0$, $\varphi \in [0, 2\pi)$, and $(x_1, x_2) = (t \cos \varphi, t \sin \varphi)$. Observe that, as $n \rightarrow \infty$,

$$\|v_{n,m}\|_{\mathcal{L}^2(\mathbb{R}^2)} = \Omega(n^{-m}). \quad (5.2)$$

It is also straightforward that

$$\|v_{n,m}\|_{C^m(\mathbb{R}^2)} = O(1); \quad (5.3)$$

see, for example, the arguments of [18, Theorem 2].

Lemma 5.1. *For any $m \in \mathbb{N}$ and $r > 0$, we have $\|\mathcal{F}[\Re v_{n,m}]\|_{\mathcal{L}^\infty([-r,r]^2)} = O(e^{-n})$.*

Proof. Writing the Fourier transform in the polar coordinates, we find that

$$\mathcal{F}v_{n,m}(\xi) = \frac{n^{-m}}{(2\pi)^2} \int_{\text{supp}(g)} tg(t) \left(\int_0^{2\pi} e^{it|\xi|\cos(\varphi-\varphi_0)} e^{in\varphi} d\varphi \right) dt,$$

where $\xi = (|\xi| \cos \varphi_0, |\xi| \sin \varphi_0)^T \in \mathbb{R}^2$. Using the Cauchy integral theorem, we get that, uniformly over all $\xi \in [-r, r]^2$ and $t \in \text{supp}(g)$,

$$\int_0^{2\pi} e^{it|\xi|\cos(\varphi-\varphi_0)} e^{in\varphi} d\varphi = O\left(\int_{0+i}^{2\pi+i} e^{it|\xi|\cos(z-\varphi_0)} e^{inz} dz\right) = O(e^{-n}).$$

Observing also $\mathcal{F}[\Re v_{n,m}](\xi) = \mathcal{F}v_{n,m}(\xi) + \mathcal{F}v_{n,m}(-\xi)^*$, where z^* denotes the complex conjugate of $z \in \mathbb{C}$, the required bound follows. \square

The following theorem implies that the exponent μ in Corollary 3.3 is optimally bounded above by m (or almost optimally, for $d = 1$) since $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq C\|v\|_{C^m(\mathbb{R}^d)}$ for a compactly supported v , where C depends on $\text{supp}(v)$ only.

Theorem 5.2. *Let $d \geq 1$ and $m \geq 0$ be integers. Let μ be a positive real number satisfying either $\mu > m$ if $d \geq 2$, or $\mu > m + 1/2$ if $d = 1$. Then, for any bounded open set $A \subset \mathbb{R}^d$, compact set $B \subseteq \mathbb{R}^d$, and positive constants γ, c , there exists $v : \mathbb{R}^d \rightarrow \mathbb{R}$ such that:*

$$\begin{aligned} \text{supp}(v) \subseteq A, \quad \|v\|_{C^m(\mathbb{R}^d)} \leq \gamma, \quad \|\mathcal{F}v\|_{\mathcal{L}^\infty(B)} < 1, \\ \|v\|_{\mathcal{L}^2(\mathbb{R}^d)} > c \left(\ln \frac{1}{\|\mathcal{F}v\|_{\mathcal{L}^\infty(B)}} \right)^{-\mu}. \end{aligned} \quad (5.4)$$

Proof. First, we consider the case $d \geq 2$. Define $w_{n,m} : \mathbb{R}^d \rightarrow \mathbb{C}$ by

$$w_{n,m}(x) := \Re v_{n,m}(x_1, x_2) \prod_{j=3}^d g(x_j),$$

where g is given in (5.1). Observe that $w_{n,m} \in C^m(\mathbb{R}^d)$ and is compactly supported. Using (5.3) and taking any $x_0 \in A$ and sufficiently small $\alpha > 0$ and sufficiently big $\beta > 0$, we get that the functions $v_n : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$v_n(x) := \alpha w_{n,m}(\beta(x - x_0))$$

are supported in A and satisfy $\|v_n\|_{C^m(\mathbb{R}^d)} \leq \gamma$ for all $n > 0$. Next, taking r to be sufficiently large and observing from (5.1) that g is supported in $[1, 2]$ and $|g(t)| \leq 1$ for all $t \in \mathbb{R}$, we ensure

$$\|\mathcal{F}v_n\|_{\mathcal{L}^\infty(B)} = O\left(\|\mathcal{F}[\Re v_{n,m}]\|_{\mathcal{L}^\infty([-r, r]^2)}\right).$$

Using (5.2) and Lemma 5.1, we get that, as $n \rightarrow \infty$,

$$\|v_n\|_{\mathcal{L}^2(\mathbb{R}^d)} = \Omega(n^{-m}) \quad \text{and} \quad \|\mathcal{F}v_n\|_{\mathcal{L}^\infty(B)} = O(e^{-n}).$$

Taking $v \equiv v_n$ for sufficiently large n , we get (5.4).

For the case $d = 1$, consider the functions $h_{n,m} : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$h_{n,m}(x) := \int_{\mathbb{R}} \Re v_{2n,m}(t, x) dt = \int_{-2}^2 \Re v_{2n,m}(t, x) dt.$$

From (5.3), we derive that

$$\|h_{n,m}\|_{C^m(\mathbb{R})} = O(1).$$

Using Lemma 5.1, we also find that, for any fixed $r > 0$,

$$\|\mathcal{F}h_{n,m}\|_{\mathcal{L}^\infty([-r, r])} = 2\pi \|\mathcal{F}[\Re v_{2n,m}](0, \cdot)\|_{\mathcal{L}^\infty([-r, r])} = O(e^{-n}).$$

Note that if $|x| \leq (2n)^{-1}$ then, by the definition of $v_{n,m}$,

$$h_{n,m}(x) \geq n^{-m} \left(2 \cos 1 \int_1^2 g(t) dt + O(n^{-1}) \right) = \Omega(n^{-m}).$$

Therefore,

$$\|h_{n,m}\|_{\mathcal{L}^2(\mathbb{R})} \geq \left(n^{-1} \min_{|x| \leq (2n)^{-1}} |h_{n,m}(x)| \right)^{1/2} = \Omega(n^{-m-1/2}).$$

We complete the proof by considering functions of the form $\alpha h_{n,m}(\beta^{-1}(x - x_0))$ and repeating the arguments of the case $d \geq 2$. \square

6 Further developments

Studies of the present work can be developed, in particular, in the following directions. These issues will be addressed in further articles.

6.1 Estimates in \mathcal{H}^s

Theorem 3.1 and Corollary 3.3 admit analogs in the norm $\|\cdot\|_{\mathcal{H}^s(\mathbb{R}^d)}$ in place of $\|\cdot\|_{\mathcal{L}^2(\mathbb{R}^d)}$ in the left-hand sides of (3.2) and (3.4). Recall that, for real s , the Sobolev space $\mathcal{H}^s(\mathbb{R}^d)$ and its norm can be defined by

$$\begin{aligned}\mathcal{H}^s(\mathbb{R}^d) &:= \{u \in \mathcal{L}^2(\mathbb{R}^d) : \mathcal{F}^{-1}(1 + |\xi|^2)^{s/2} \mathcal{F}u \in \mathcal{L}^2(\mathbb{R}^d)\}, \\ \|u\|_{\mathcal{H}^s(\mathbb{R}^d)} &:= \|\mathcal{F}^{-1}(1 + |\xi|^2)^{s/2} \mathcal{F}u\|_{\mathcal{L}^2(\mathbb{R}^d)}.\end{aligned}$$

For example, in a similar way with (3.2) and (3.4), one can estimate $\|v - \mathcal{F}^{-1} \mathcal{C}_{\tau, \delta}^*[w]\|_{\mathcal{H}^s(\mathbb{R}^d)}$ and $\|v\|_{\mathcal{H}^s(\mathbb{R}^d)}$ for arbitrary real s and m such that $s < m$ and $m > -\frac{d}{2}$. In particular, under assumptions (2.1), (2.2), and $\|v\|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$, one can show that

$$\|v - \mathcal{F}^{-1} \mathcal{C}_{\tau, \delta}^*[w]\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq c (\ln(3 + \delta^{-1}))^{-\tau(m-s)} \quad (6.1)$$

for any $\tau \in (0, 1)$ and some constant $c = c(N, \sigma, r, m, s, \gamma, \tau, d) > 0$.

The results of the type (6.5) with $s < 0$ can be used for apodized reconstructions based on $\mathcal{C}_{\tau, \delta}^*$. Indeed, let ϕ be a real-valued compactly supported function on \mathbb{R}^d and $\hat{\phi} = \mathcal{F}\phi$. We assume that ϕ satisfies the following:

$$\int_{\mathbb{R}^d} \phi(x) dx = 1;$$

and, for some $\ell > 0$,

$$N_\ell = N_\ell[\phi] := \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)^{\ell/2} |\hat{\phi}(\xi)| < +\infty.$$

One may also assume, for example, that ϕ is non-negative, spherically symmetric, and non-increasing in $|x|$.

Let $0 < p < \ell$ and $s := p - \ell$. Then, for any $u \in \mathcal{H}^s(\mathbb{R}^d)$, we estimate

$$\begin{aligned}\|\phi * u\|_{\mathcal{H}^p(\mathbb{R}^d)} &= \left\| \mathcal{F}^{-1} \left[(1 + |\xi|^2)^{p/2} \hat{\phi} \hat{u} \right] \right\|_{\mathcal{L}^2(\mathbb{R}^d)} \\ &\leq c(\ell) \left\| \mathcal{F}^{-1} \left[(1 + |\xi|^2)^{s/2} \hat{u} \right] \right\|_{\mathcal{L}^2(\mathbb{R}^d)} = N_\ell \|u\|_{\mathcal{H}^s(\mathbb{R}^d)},\end{aligned} \quad (6.2)$$

where $*$ denote the convolution operator and $\hat{u} := \mathcal{F}u$. Combining (6.5) and (6.2) with $u := v - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]$, we obtain that

$$\|\phi * v - \phi * \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{H}^p(\mathbb{R}^d)} \leq cN_\ell (\ln(3 + \delta^{-1}))^{-\tau(m-s)}, \quad (6.3)$$

under the same assumptions on v , m , $s = p - \ell$ as in (6.5). Estimate (6.3) shows the logarithmic stability in $\mathcal{H}^p(\mathbb{R}^d)$ for the regularized (apodized) reconstruction $\tilde{v} := \phi * v$ from the given data w . For more details about regularized (apodized) reconstructions for problems similar to Problem 1.1, see, for example, [2] and references therein.

6.2 Super-exponential decay

The results of the present works also admit analogs for the case when condition (2.1) is replaced by the condition

$$Q_v(\lambda) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\lambda|x|} |v(x)| dx \leq N \exp(\sigma\lambda^\nu), \quad \text{for all } \lambda \geq 0, \quad (6.4)$$

where constants $N > 0$, $\sigma > 0$ and $\nu \geq 1$ are given a priori. The case $\nu = 1$ corresponds to compactly supported v .

In particular, under assumptions (2.2), (6.4), and $\|v\|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$, one can show that

$$\|v - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq c (\ln(3 + \delta^{-1}))^{-\tau m}, \quad (6.5)$$

for any $\tau \in (0, 1 - \sqrt{1 - (1 - \alpha)\nu^{-1}})$ and some constant $c = c(N, \sigma, \nu, r, m, \gamma, \tau, d) > 0$.

6.3 PSWF approach

Theorem 5.2 and Corollary 3.3 show that the logarithmic estimate (3.4) is optimal with respect to the exponent μ . Thus, estimate (3.2) is also optimal in this logarithmic sense. However, we believe that (3.2) can be improved with respect to dependence on a priori parameters N , r , σ , γ . A natural approach is to employ more advanced basis in place of Chebyshev polynomials and exponents of the inverse Fourier transform. For the case when v is compactly supported like in (2.1), one can use, for example, the basis of the prolate spheroidal wave functions (PSWFs).

The PSWFs $(\Psi_{k,\sigma})_{k \in \mathbb{N}}$ can be defined as the eigenfunctions of the spectral problem

$$\int_{-1}^1 \frac{\sin(\sigma(x-y))}{\pi(x-y)} \psi(y) dy = \lambda \psi(x),$$

where $\psi \in \mathcal{L}^2(\mathbb{R})$ and λ is the spectral parameter. The functions $(\Psi_{k,\sigma})_{k \in \mathbb{N}}$ form simultaneously an orthogonal basis in $\mathcal{L}^2([-1, 1])$ and an orthogonal basis in the subspace of $\mathcal{L}^2(\mathbb{R})$ consisting of the functions whose Fourier transform is supported in the interval $[-\sigma, \sigma]$. For more information about PSWFs, see, for example, [5, 26] and references therein.

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