



HAL
open science

Hölder-logarithmic stability in the Fourier analysis

Mikhail Isaev, Roman Novikov

► **To cite this version:**

Mikhail Isaev, Roman Novikov. Hölder-logarithmic stability in the Fourier analysis. 2020. hal-02562474v1

HAL Id: hal-02562474

<https://hal.science/hal-02562474v1>

Preprint submitted on 4 May 2020 (v1), last revised 11 Nov 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hölder-logarithmic stability in the Fourier analysis*

Mikhail Isaev

School of Mathematics

Monash University

Clayton, VIC, Australia

mikhail.isaev@monash.edu

Roman G. Novikov

CMAP, CNRS, Ecole Polytechnique

Institut Polytechnique de Paris

Palaiseau, France

IEPT RAS, Moscow, Russia

novikov@cmap.polytechnique.fr

Abstract

We prove a Hölder-logarithmic stability estimate for the problem of finding a sufficiently regular compactly supported function v on \mathbb{R}^d from its Fourier transform $\mathcal{F}v$ given on $[-r, r]^d$. This estimate relies on a Hölder stable continuation of $\mathcal{F}v$ from $[-r, r]^d$ to a larger domain. The related reconstruction procedures are based on truncated series of Chebyshev polynomials. We also give an explicit example showing optimality of our stability estimates.

Keywords: ill-posed inverse problems, Hölder-logarithmic stability, exponential instability, analytic continuation, Chebyshev approximation

AMS subject classification: 42A38, 35R30, 49K40

1 Introduction

The Fourier transform \mathcal{F} is defined by

$$\mathcal{F}v(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi x} v(x) dx, \quad \xi \in \mathbb{R}^d,$$

where v is a test function on \mathbb{R}^d and $d \geq 1$. The analysis of this transform is one of the most developed areas of mathematics and has many applications in physics, statistics and

*The first author's research is supported by Australian Research Council Discovery Early Career Researcher Award DE200101045.

engineering; see, for example, Bracewell [4]. In particular, it is well known that if v is integrable and compactly supported then $\mathcal{F}v$ is analytic. Thus, the Fourier transform $\mathcal{F}v$ and, consequently, the function v are uniquely determined by the values of $\mathcal{F}v$ within any open non-empty domain. However, in the case of noisy data, the reconstruction can be hard unless the values of $\mathcal{F}v$ are known in a very large domain or v belongs to a specific class of functions (a priori information). In the present paper we answer how much the stability improves with respect to the size of the domain where $\mathcal{F}v$ is given and with respect to the regularity of v .

Specifically, we consider the following problem.

Problem 1.1. Suppose that $v \in \mathcal{L}^1(\mathbb{R}^d)$ is supported in a given compact set. The values of $\mathcal{F}v$ are given on $[-r, r]^d$, possibly with some noise. Find v .

Reconstructing a compactly supported function from its partially known Fourier transform or, equivalently, computing the Fourier transform of a band-limited function given within some domain is a classical problem of the Fourier analysis; see, for example, Beylkin, Monzón [3] and Papoulis [20]. Candès, Fernandez-Granda [6] and Gerchberg [8] consider this problem in their works on super-resolution in image processing. It also arises in studies of inverse scattering problems in the Born approximation. For example, a variant of Problem 1.1 with $\mathcal{F}v$ given on the ball $\{\xi \in \mathbb{R}^d : |\xi| < 2\sqrt{E}\}$ can be regarded as a linearized inverse scattering problem for the Schrödinger equation with potential v at fixed positive energy E , for $d \geq 2$, and on the energy interval $[0, E]$, for $d \geq 1$. More details can be found in the recent paper by Novikov [19, Section 4].

We focus on the stability of reconstructions for Problem 1.1. In particular, for a suitable function ϕ such that $\phi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, we show that

$$\|v_1 - v_2\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq \phi \left(\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{L}^\infty([-r, r]^d)} \right), \quad (1.1)$$

under the additional assumption that $v_1 - v_2$ is sufficiently regular. Furthermore, we propose a reconstruction procedure for Problem 1.1 which stability behaviour is consistent with the function ϕ .

It is well known in the community that Problem 1.1 is ill-posed in the sense by Hadamard; see Lavrent'ev et al. [16] for the introduction to the theory of ill-posed problems. In fact, one can show that this problem is *exponentially ill-posed* in a similar way to the results by Mandache [15] and Isaev [12] using the estimates of ϵ -entropy and ϵ -capacity in functional spaces that go back to Kolmogorov and Vitushkin [17]. To completely settle the question, we give an explicit example demonstrating exponential ill-posedness of

Problem 1.1 in Section 6 of the present paper. Consequently, a logarithmic bound is the best one could hope to get in (1.1), in general; see Corollary 2.3 and Theorem 6.2.

On the other hand, in the case when r is sufficiently large (such that $\mathcal{F}v$ on $\mathbb{R}^d \setminus [-r, r]^d$ is negligible) one can approximate v in a Lipschitz stable manne by direct computation of *inverse Fourier transform* \mathcal{F}^{-1} :

$$v(x) \approx [\mathcal{F}^{-1}w](x) := \int_{\mathbb{R}^d} w(\xi) e^{-i\xi x} d\xi, \quad (1.2)$$

taking w equal to the given values of $\mathcal{F}v$ in $[-r, r]^d$, and $w \equiv 0$, outside $[-r, r]^d$. However, there remains some error in this approximation even in the absence of noise.

In the present work, we prove a *Hölder-logarithmic stability* estimate for Problem 1.1 tying together the aforementioned two facts; see Theorem 2.1. In particular, we show that ill-posedness of the problem decreases as r grows. Furthermore, our estimate illustrates similar stability behaviour in more complicated non-linear inverse problems. In fact, the relationship is closer than a mere illustration. For example, one can already derive a logarithmic bound in (1.1) from the results on monochromatic inverse scattering by Hähner, Hohage [10] and by Isaev, Novikov [13]. For other known results on logarithmic and Hölder-logarithmic stability in inverse problems, see also Alessandrini [1], Bao et al. [2], Isaev [11], Isakov [14], Novikov [18], Santacesaria [21] and references therein. Despite the huge literature on the topic, an estimate, encapsulating the stability improvement in Problem 1.1 from the logarithmic type to the Hölder type as r grows, is implied by none of the results on related inverse problems we are aware of.

The main idea of our stable reconstruction for Problem 1.1 is the following. First, we continue of $\mathcal{F}v$ from $[-r, r]^d$ to a larger domain, which size depends on the noise level. Then, we apply the inverse Fourier transform. This leads to our second problem.

Problem 1.2. Suppose that $v \in \mathcal{L}^1(\mathbb{R}^d)$ is supported in a given compact set. The values of $\mathcal{F}v$ are given on $[-r, r]^d$, possibly with some noise. Find $\mathcal{F}v$ on $[-R, R]^d$, where $R \geq r$.

Problem 1.2 is equivalent to band-limited extrapolation (for $d = 1$) and has been of interest to a number of different authors: Beylkin, Monzón [3], Cadzow [5], Candès, Fernandez-Granda [6], Gerchberg [8], and Papoulis [20], to name a few. A more general problem of stable analytic continuation of a complex function was considered by Demanet, Townsend [7], Lavrent'ev et al. [16, Chapter 3], Tuan [22], and Vessella [23]. In particular, [7, Theorem 1.2] or [23, Theorem 1] lead to a *Hölder stability* estimate for Problem 1.2: for some $0 < \alpha < 1$ and $c_{\alpha, R} > 0$,

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{L^\infty([-R, R]^d)} \leq c_{\alpha, R} \left(\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{L^\infty([-r, r]^d)} \right)^\alpha. \quad (1.3)$$

However, for a fixed α , the factor $c_{\alpha,R}$ in this estimate grows exponentially as R increases, which hinders continuation of $\mathcal{F}v|_{[-r,r]^d}$ to very large domains. This behaviour is natural due to exponential ill-posedness of Problem 1.1. In this paper, we independently establish estimate (1.3) (under the assumptions that $v_1 - v_2$ is integrable and compactly supported) mainly for the purpose to find a simple explicit expression for the factor in front of the Hölder term; see Theorem 3.2.

For a fixed $r > 0$, we consider the following family of continuations $\mathcal{C}_{R,n}[\cdot]$ depending on two parameters $R \geq r$ and $n \in \mathbb{N} := \{0, 1, \dots\}$. For a function w on $[-r, r]^d$, define

$$\mathcal{C}_{R,n}[w](\xi) := \begin{cases} w(\xi), & \xi \in [-r, r]^d, \\ \sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_d < n}} a_{k_1, \dots, k_d}[w] \prod_{j=1}^d T_{k_j} \left(\frac{\xi_j}{r} \right), & \xi \in [-R, R]^d \setminus [-r, r]^d, \\ 0, & \xi \in \mathbb{R}^d \setminus [-R, R]^d, \end{cases} \quad (1.4)$$

where

$$a_{k_1, \dots, k_d}[w] := \int_{-r}^r \cdots \int_{-r}^r w(\xi) \prod_{j=1}^d \left(\frac{2^{\mathbb{1}[k_j > 0]} T_{k_j} \left(\frac{\xi_j}{r} \right)}{\pi(r^2 - \xi_j^2)^{\frac{1}{2}}} \right) d\xi_1 \cdots d\xi_d \quad (1.5)$$

and $\mathcal{C}_{R,n}[w]$ is taken to be 0 everywhere outside $[-r, r]^d$ in the case when $n = 0$. In the above, $\mathbb{1}[k > 0]$ is the indicator function for $\{k > 0\}$:

$$\mathbb{1}[k > 0] = \begin{cases} 1, & \text{if } k > 0, \\ 0, & \text{otherwise;} \end{cases}$$

and $(T_k)_{k \in \mathbb{N}}$ stand for the Chebyshev polynomials, which can be defined by $T_k(t) := \cos(k \arccos(t))$ for $t \in [-1, 1]$ and extended to $|t| > 1$ in a natural way.

Remark 1.1. In fact, the Chebyshev coefficients $a_{k_1, \dots, k_d}[\cdot]$ can be efficiently computed without evaluating the integral in (1.5), using the fast Fourier cosine transform (FCT) or Clenshaw's method; see, for instance, [9, Section 3.6]. We note also that these computations only require that values of $\mathcal{F}v$ are known at some finite set of points. Thus, our stability estimates for Problems 1.1 and 1.2 can be strengthened by replacing $\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{L^\infty([-r,r]^d)}$ by the maximum of $|\mathcal{F}v_1 - \mathcal{F}v_2|$ over this finite set.

Recall that if v is integrable and compactly supported then $\mathcal{F}v$ is analytical in \mathbb{C}^d . It follows that, for all $\xi \in \mathbb{C}^d$,

$$\mathcal{F}(\xi) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} a_{k_1, \dots, k_d}[\mathcal{F}v|_{[-r,r]^d}] \prod_{j=1}^d T_{k_j} \left(\frac{\xi_j}{r} \right). \quad (1.6)$$

We will show that if $w \approx \mathcal{F}v|_{[-r,r]^d}$ then $\mathcal{C}_{R,n}[w](\xi)$ approximates well the series of (1.6) in the region $[-R, R]^d \setminus [-r, r]^d$, provided n is sufficiently large so the tail of the series is negligible, but not very large so the continuation $\mathcal{C}_{R,n}$ is sufficiently stable.

The main results of the present work are given in detail in Section 2, 3, and 6. Our stability estimates are stated in the following form: if $\|\mathcal{F}v - w\|_{\mathcal{L}^\infty([-r,r]^d)} \leq \delta$ then, for any $R > r$, there is some $n^* = n^*(\delta, R)$ and $0 < \alpha < 1$ such that,

$$\|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} \leq c_{\alpha,R} \delta^\alpha; \quad (1.7)$$

in addition, if v is sufficiently regular then there are some $R(\delta)$ and $n(\delta) = n^*(\delta, R(\delta))$ such that, as $\delta \rightarrow 0$,

$$\|v - \mathcal{F}^{-1}\mathcal{C}_{R(\delta),n(\delta)}[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq \phi(\delta) \rightarrow 0. \quad (1.8)$$

Note that (1.8) and (1.7) imply also (1.1) and (1.3), respectively, by setting $v := v_1 - v_2$ and $w := 0$ and using the linearity of the considered problems and the reconstruction procedures. Finally, the example demonstrating that our stability estimates are essentially optimal is given by Theorem 6.2.

2 Hölder-logarithmic stability in Problem 1.1

In this section, we give, in particular, a Hölder-logarithmic stability estimate for the reconstruction procedure $\mathcal{F}^{-1}\mathcal{C}_{R,n}[\cdot]$ defined using (1.4); see Theorem 2.1. The proof of this result is given in Section 4. It is based on the Hölder stability estimates for the continuation $\mathcal{C}_{R,n}[\cdot]$ obtained in Section 3; see Theorem 3.2 and Corollary 3.3.

All aforementioned results (Theorem 2.1, Theorem 3.2, and Corollary 3.3) share the following assumptions in common: the unknown function $v : \mathbb{R}^d \rightarrow \mathbb{C}$ is such that, for some $N, \sigma > 0$,

$$\|v\|_{\mathcal{L}^1(\mathbb{R}^d)} \leq (2\pi)^d N, \quad \text{supp}(v) \subseteq \left\{ x \in \mathbb{R}^d : \sum_{j=1}^d |x_j| \leq \sigma \right\}; \quad (2.1)$$

and the given data w is such that, for some $\delta, r > 0$,

$$\|w - \mathcal{F}v\|_{\mathcal{L}^\infty([-r,r]^d)} \leq \delta < N, \quad (2.2)$$

where \mathcal{F} is the Fourier transform. Note that if (2.1) holds then, for any $\xi \in \mathbb{R}^d$,

$$|\mathcal{F}v(\xi)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |v(x)| dx \leq N. \quad (2.3)$$

This explains the condition $\delta < N$ in assumption (2.2). Indeed, if the noise level δ is greater than N then the given data w tells about v as little as the trivial function $\tilde{w} \equiv 0$.

To achieve optimal stability bounds, the parameters R and n in the reconstruction $\mathcal{F}^{-1}\mathcal{C}_{R,n}$ have to be chosen carefully depending on N, δ, r, σ . For any $\tau \in [0, 1]$, let

$$L_\tau(\delta) = L_\tau(N, \delta, r, \sigma) := \max \left\{ 1, \frac{1}{4} \left(\frac{(1-\tau) \ln \frac{N}{\delta}}{r\sigma} \right)^\tau \right\}. \quad (2.4)$$

Here and thereafter, we assume $0 < \delta < N$. Using (1.4), define

$$\mathcal{C}_{\tau, \delta}^* := \mathcal{C}_{R_\tau(\delta), n_\tau(\delta)}, \quad (2.5)$$

where

$$\begin{aligned} R_\tau(\delta) &= R_\tau(N, \delta, r, \sigma) := rL_\tau(\delta), \\ n_\tau(\delta) &= n_\tau(N, \delta, r, \sigma) := \begin{cases} \left\lceil \frac{(2-\tau) \ln \frac{N}{\delta}}{\ln 3 + \frac{1}{\tau} \ln(4L_\tau(\delta))} \right\rceil, & \text{if } \tau > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.6)$$

and $\lceil \cdot \rceil$ denotes the ceiling of a real number.

To prove our stability estimate for Problem 1.1, in addition to (2.1), we assume also that $v \in \mathcal{H}^m(\mathbb{R}^d)$, where $\mathcal{H}^m(\mathbb{R}^d)$ is the standard Sobolev space of m -times smooth functions in \mathcal{L}^2 on \mathbb{R}^d . Consider the seminorm $|\cdot|_{\mathcal{H}^m(\mathbb{R}^d)}$ in $\mathcal{H}^m(\mathbb{R}^d)$ defined by

$$|v|_{\mathcal{H}^m(\mathbb{R}^d)} := \left(\sum_{j=1}^d \left\| \frac{\partial^m v}{(\partial x_j)^m} \right\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 \right)^{1/2}. \quad (2.7)$$

Theorem 2.1. *Let the assumptions of (2.1) and (2.2) hold for some $N, \delta, r, \sigma > 0$. Assume also that $v \in \mathcal{H}^m(\mathbb{R}^d)$ for some integer $m > 0$. Then, for any $\tau \in [0, 1]$, the following holds:*

$$\begin{aligned} \|v - \mathcal{F}^{-1}\mathcal{C}_{\tau, \delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} &\leq (20\sqrt{r})^d N (L_\tau(\delta))^{d/2+1} \left(\frac{\delta}{N} \right)^{(1-\tau)^2} \\ &\quad + |v|_{\mathcal{H}^m(\mathbb{R}^d)} (rL_\tau(\delta))^{-m}. \end{aligned} \quad (2.8)$$

The first term of the right-hand side in estimate (2.8) corresponds to the error caused by the Hölder stable continuation of the noisy data w from $[-r, r]^d$ to $[-R_\tau(\delta), R_\tau(\delta)]^d$ and the second (logarithmic) term corresponds to the error caused by ignoring the values of $\mathcal{F}v$ outside $[-R_\tau(\delta), R_\tau(\delta)]^d$; see Section 4 for more details of the proof.

Remark 2.2. Clearly, the stability behaviour in Problem 1.1 should not depend on scaling of functions or arguments. It might be obscure at first sight, but estimate (2.8)

is invariant with respect to such scalings. Indeed, for some $\alpha, \beta > 0$, let \tilde{v} be defined by $\tilde{v}(x) := \alpha v(\beta x)$, $x \in \mathbb{R}^d$. The Fourier transforms of v and \tilde{v} satisfy the following relation $\mathcal{F}\tilde{v}(\xi) = \alpha \mathcal{F}v(\beta^{-1}\xi)$ for $\xi \in \mathbb{R}^d$. If $w \approx \mathcal{F}v$ in $[-r, r]^d$ then, equivalently, $\tilde{w} \approx \mathcal{F}\tilde{v}$ in $[-\tilde{r}, \tilde{r}]^d$, where $\tilde{r} = \beta r$ and $\tilde{w}(\xi) := \alpha w(\beta^{-1}\xi)$, $\xi \in \mathbb{R}^d$. The other parameters in (2.1) and (2.2) are modified as follows: $\tilde{N} = \alpha N$, $\tilde{\delta} = \alpha \delta$, and $\tilde{\sigma} = \beta^{-1}\sigma$. Observe that $L_\tau(\delta)$ depends only on $r\sigma$ and N/δ , which are independent of scalings. Finally, we have

$$\begin{aligned} \|\tilde{v} - \mathcal{F}^{-1}\mathcal{C}_{\tau, \delta}^*[\tilde{w}]\|_{\mathcal{L}^2(\mathbb{R}^d)} &= \alpha \beta^{d/2} \|v - \mathcal{F}^{-1}\mathcal{C}_{\tau, \delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)}, \\ |\tilde{v}|_{\mathcal{H}^m(\mathbb{R}^d)} &= \alpha \beta^{m+d/2} |v|_{\mathcal{H}^m(\mathbb{R}^d)}. \end{aligned}$$

Thus, both sides of estimate (2.8) get multiplied by the same constant $\alpha \beta^{d/2}$, that is, the statements of Theorem 2.1 for v, w and for \tilde{v}, \tilde{w} are equivalent.

Theorem 2.1 leads to the Hölder-logarithmic stability estimate for $\|v_1 - v_2\|_{\mathcal{L}^2(\mathbb{R}^d)}$ in (1.1), provided $v_1 - v_2$ satisfies assumptions (2.1), (2.2) (for fixed N) and $v_1 - v_2 \in \mathcal{H}^m(\mathbb{R}^d)$, as explained in Section 1 after formula (1.8). Estimate (2.8) with $\tau = 0$ is similar to well-known stability results for approximate reconstruction explained in (1.2). Theorem 2.1 also implies the following corollary.

Corollary 2.3. *Let $v : \mathbb{R}^d \rightarrow \mathbb{C}$ be supported in some compact set $A \subset \mathbb{R}^d$, for $d \geq 1$. Assume that $\|\mathcal{F}v\|_{\mathcal{L}^\infty(B)} < 1$ and $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$ for some open set $B \subseteq \mathbb{R}^d$, integer $m > 0$, and real $\gamma > 0$. Then, for any $0 \leq \mu < m$, there is $c = c(A, B, \gamma, \mu, m) > 0$ such that*

$$\|v\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq c \left(\ln \frac{1}{\|\mathcal{F}v\|_{\mathcal{L}^\infty(B)}} \right)^{-\mu}. \quad (2.9)$$

Proof. Without loss of generality, we can assume that $0 \in B$ by considering $\tilde{v} := v e^{i\xi_0 x}$, for a fixed $\xi_0 \in \mathbb{R}^d$. Then, $[-r, r]^d \subset B$ for a sufficiently small r . Any compact set A lies in $\left\{ x \in \mathbb{R}^d : \sum_{j=1}^d |x_j| \leq \sigma \right\}$ for a sufficiently large σ . Since v is compactly supported, the condition $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma$, for $m \geq 1$, implies that the norm $\|v\|_{\mathcal{L}^1(\mathbb{R}^d)}$ is bounded above by $(2\pi)^d N$, where the constant N depends on A, m, γ only. Applying Theorem 2.1 with $\tau := \mu/m$, $w \equiv 0$, $\delta := \|\mathcal{F}v\|_{\mathcal{L}^\infty([-r, r]^d)}$ and observing that the logarithmic term dominates the Hölder term as $\delta \rightarrow 0$ and

$$\|\mathcal{F}v\|_{\mathcal{L}^\infty([-r, r]^d)} \leq \|\mathcal{F}v\|_{\mathcal{L}^\infty(B)} < 1$$

we complete the proof. \square

In Section 6, we show that the exponent $-m$ in the logarithmic term of our estimate (2.8) is optimal (or almost optimal for $d = 1$), using an explicit construction. Namely, we

prove that, for $d \geq 2$ and any $\mu > m$, there is some v violating (2.9) no matter how large constant c we take. For $d = 1$, the same holds for any $\mu > m + 1/2$. The optimality of the exponent $-m$ for the case $d = 1$ remains an open question.

3 Hölder stability in Problem 1.2

In this section, we give stability estimates for continuations $\mathcal{C}_{R,n}$ defined according to (1.4); see Lemma 3.1, Theorem 3.2, and Corollary 3.3. For these estimates, we only need the assumptions of (2.1) and (2.2) to hold.

Lemma 3.1. *Let the assumptions of (2.1) and (2.2) hold for some $N, \delta, r, \sigma > 0$. Then, for any integer $n > 0$ and real $\rho, R > 0$ such that $R \geq r$, $\rho \geq 4R/r$, we have*

$$\|\mathcal{F}v - \mathcal{C}_{R,n}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} \leq \frac{1}{4} \left(4^d \left(\frac{4R}{r} \right)^n \delta + \left(\frac{16}{3} \right)^d N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^n \right).$$

Lemma 3.1 is proved in Section 5. Optimising the parameter n in Lemma 3.1, we obtain the following Hölder stability estimate for Problem 1.2.

Theorem 3.2. *Let the assumptions of (2.1) and (2.2) hold for some $N, \delta, r, \sigma > 0$. Assume that $\rho, R > 0$ are such that $R \geq r$ and $\rho \geq 4R/r$. Then, we have*

$$\|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} \leq \left(\frac{16}{3} \right)^d \frac{R}{r} \left(\frac{N e^{r\sigma\rho}}{\delta} \right)^{\tau(\rho)} \delta,$$

where

$$n^* := \left\lceil \frac{\ln \frac{N}{\delta} + r\sigma\rho}{\ln(3\rho)} \right\rceil \quad \text{and} \quad \tau(\rho) := \frac{\ln \frac{4R}{r}}{\ln(3\rho)}.$$

Proof. Using (2.2), we have that

$$\eta := \frac{\ln \frac{N}{\delta} + r\sigma\rho}{\ln(3\rho)} > 0.$$

By definition, we find that $\eta \leq n^* < \eta + 1$ and $\delta = N e^{r\sigma\rho} (3\rho)^{-\eta}$. Using that $R \geq r$, we get

$$\delta \left(\frac{4R}{r} \right)^{\eta+1} = \frac{4R}{r} N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^\eta \geq 4N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^\eta.$$

Then, applying Lemma 3.1, we obtain that

$$\begin{aligned}
\|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty([-R,R]^d)} &\leq \frac{1}{4} \left(4^d \left(\frac{4R}{r} \right)^{n^*} \delta + \left(\frac{16}{3} \right)^d N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^{n^*} \right) \\
&\leq \frac{1}{4} \left(4^d \left(\frac{4R}{r} \right)^{\eta+1} \delta + \left(\frac{16}{3} \right)^d N e^{r\sigma\rho} \left(\frac{4R}{3r\rho} \right)^\eta \right) \\
&\leq \left(\frac{16}{3} \right)^d \frac{R}{r} \left(\frac{4R}{r} \right)^\eta \delta \left(\left(\frac{3}{4} \right)^d + \frac{1}{4} \right) \leq \left(\frac{16}{3} \right)^d \frac{R}{r} \left(\frac{4R}{r} \right)^\eta \delta.
\end{aligned}$$

Since $\tau(\rho) \ln(3\rho) = \ln \frac{4R}{r}$, we get

$$\left(\frac{4R}{r} \right)^\eta \delta = (3\rho)^{\tau(\rho)\eta} \delta = (N e^{r\sigma\rho})^{\tau(\rho)} \delta^{1-\tau(\rho)}.$$

Combining the above estimates completes the proof. \square

Theorem 3.2 leads to the following stability estimate for the continuation $\mathcal{C}_{\tau,\delta}^*$ used in Theorem 2.1.

Corollary 3.3. *Let the assumptions of (2.1) and (2.2) hold for some $N, \sigma, r, \delta > 0$. Then, for any $\tau \in [0, 1]$, we have*

$$\|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)} \leq \left(\frac{16}{3} \right)^d N \left(\frac{\delta}{N} \right)^{(1-\tau)^2} L_\tau(\delta),$$

where $L_\tau(\delta)$ and $R_\tau(\delta)$ are defined in (2.4) and (2.6).

Proof. First, we consider the case $L_\tau(\delta) = 1$, for which $R_\tau(\delta) = r$. Recalling from (2.2) that $\delta < N$, we find that

$$\|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*w\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)} \leq \delta \leq \left(\frac{16}{3} \right)^d N \left(\frac{\delta}{N} \right)^{(1-\tau)^2} L_\tau(\delta).$$

Next, suppose that

$$L_\tau(\delta) = \frac{1}{4} \left(\frac{(1-\tau) \ln \frac{N}{\delta}}{r\sigma} \right)^\tau > 1.$$

This implies that $\tau > 0$. Let $\rho := (4L_\tau(\delta))^{1/\tau}$. Then, we get $e^{r\sigma\rho} = \left(\frac{N}{\delta} \right)^{1-\tau}$ and, by the assumptions,

$$R_\tau(\delta) \geq r \quad \text{and} \quad \rho \geq 4L_\tau(\delta) = 4R_\tau(\delta)/r.$$

Applying Theorem 3.2 and observing that n^* coincides with $n_\tau(\delta)$ defined by (2.6), we get that

$$\|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)} \leq \left(\frac{16}{3} \right)^d L_\tau(\delta) \left(\left(\frac{N}{\delta} \right)^{2-\tau} \right)^{\tau(\rho)} \delta,$$

where $\tau(\rho)$ is defined in Theorem 3.2. Note that $\tau(\rho)$ is different from τ . However, we can replace $\tau(\rho)$ by τ in the estimate above since $\delta < N$ and

$$\tau(\rho) = \frac{\ln(4R_\tau(\delta)/r)}{\ln(3\rho)} = \frac{\ln(4L_\tau(\delta))}{\ln 3 + \frac{1}{\tau} \ln(4L_\tau(\delta))} \leq \tau.$$

The required bound follows. \square

4 Proof of Theorem 2.1

Let all assumptions of Theorem 2.1 hold. The Parseval-Plancherel identity states that

$$\|u\|_{\mathcal{L}^2(\mathbb{R}^d)} = (2\pi)^{d/2} \|\mathcal{F}u\|_{\mathcal{L}^2(\mathbb{R}^d)} = (2\pi)^{-d/2} \|\mathcal{F}^{-1}u\|_{\mathcal{L}^2(\mathbb{R}^d)}. \quad (4.1)$$

Thus, we get that

$$\|v - \mathcal{F}^{-1}\mathcal{C}_{\tau,\delta}^*[w]\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq (2\pi)^{d/2}(I_1 + I_2),$$

where

$$I_1 := \left(\int_{[-R_\tau(\delta), R_\tau(\delta)]^d} |\mathcal{F}v(\xi) - \mathcal{C}_{\tau,\delta}^*[w](\xi)|^2 d\xi \right)^{1/2},$$

$$I_2 := \left(\int_{\mathbb{R}^d \setminus [-R_\tau(\delta), R_\tau(\delta)]^d} |\mathcal{F}v(\xi)|^2 d\xi \right)^{1/2}.$$

Using Corollary 3.3, we get that,

$$\begin{aligned} I_1 &\leq \left(\int_{[-R_\tau(\delta), R_\tau(\delta)]^d} \|\mathcal{F}v - \mathcal{C}_{\tau,\delta}^*w\|_{\mathcal{L}^\infty([-R_\tau(\delta), R_\tau(\delta)]^d)}^2 d\xi \right)^{1/2} \\ &\leq \left(\frac{16}{3} \right)^d N \left(\frac{\delta}{N} \right)^{(1-\tau)^2} L_\tau(\delta) (2R_\tau(\delta))^{d/2} \\ &\leq \left(20 \sqrt{\frac{r}{2\pi}} \right)^d N (L_\tau(\delta))^{d/2+1} \left(\frac{\delta}{N} \right)^{(1-\tau)^2}. \end{aligned}$$

Next, applying (4.1) and recalling the seminorm $|\cdot|_{\mathcal{H}^m(\mathbb{R}^d)}$ defined in (2.7), we find that

$$\sum_{j=1}^d \|\xi_j^m \mathcal{F}v\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \sum_{j=1}^d \left\| \frac{\partial^m v}{(\partial x_j)^m} \right\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 = \frac{|v|_{\mathcal{H}^m(\mathbb{R}^d)}^2}{(2\pi)^d}.$$

Since $\mathbb{R}^d \setminus [-R_\tau(\delta), R_\tau(\delta)]$ is covered by the regions $\Omega_j := \{\xi \in \mathbb{R}^d : |\xi_j| > R_\tau(\delta)\}$, for $j = 1, \dots, d$, we obtain that

$$\begin{aligned} I_2 &\leq \left(\sum_{j=1}^d \int_{|\xi_j| > R_\tau(\delta)} \left| \frac{\xi_j^m \mathcal{F}v(\xi)}{(R_\tau(\delta))^m} \right|^2 d\xi \right)^{1/2} \\ &\leq \left(\sum_{j=1}^d \frac{\|\xi_j^m \mathcal{F}v\|_{\mathcal{L}^2(\mathbb{R}^d)}^2}{(R_\tau(\delta))^{2m}} \right)^{1/2} \leq \frac{|v|_{\mathcal{H}^m(\mathbb{R}^d)}}{(2\pi)^{d/2}} (rL_\tau(\delta))^{-m} \end{aligned}$$

Combining the above bounds for I_1 and I_2 , we complete the proof of Theorem 2.1.

5 Proof of Lemma 3.1

To prove Lemma 3.1, we need the bounds for series of Chebyshev polynomials stated in the following lemma. We will use the standard combinatorial fact that the number of ways to write n as a sum of d nonnegative integers (ordered) equals the binomial coefficient

$$\binom{n+d-1}{n} = \frac{(n+d-1)!}{n!(d-1)!}. \quad (5.1)$$

Lemma 5.1. *Let $\sigma, r, N > 0$ and $R \geq r$. If v satisfies (2.1) then the following holds.*

(a) *For any $\rho \geq 1$, $\xi \in [-R, R]^d$ and $k_1, \dots, k_d \in \mathbb{N}$, we have*

$$\left| a_{k_1, \dots, k_d} [\mathcal{F}v|_{[-r, r]^d}] \prod_{j=1}^d T_{k_j} \left(\frac{\xi_j}{r} \right) \right| \leq 2^d N e^{\frac{1}{2} r \sigma \rho} \left(\frac{2R}{r\rho} \right)^{\sum_{j=1}^d k_j},$$

where \mathcal{F} is the Fourier transform and $a_{k_1, \dots, k_d}[\cdot]$ is defined according to (1.5).

(b) *For any $\rho' \geq 4R/r$, we have*

$$\| \mathcal{F}v - \mathcal{C}_{R, n} [\mathcal{F}v|_{[-r, r]^d}] \|_{\mathcal{L}^\infty([-R, R]^d)} \leq \left(\frac{8}{3} \right)^d N e^{\frac{5}{4} r \sigma \rho'} \binom{n+d-1}{n} \left(\frac{4R}{3r\rho'} \right)^n,$$

where $\mathcal{C}_{R, n}[\cdot]$ is defined according to (1.4).

Proof. For $z_1, \dots, z_d \in \mathbb{C}$, let

$$f(z_1, \dots, z_d) := \mathcal{F}v(r \cos z_1, \dots, r \cos z_d).$$

Observe that, for any $z \in \mathbb{C}$,

$$|\Im(\cos z)| \leq \frac{1}{2} |e^{\Im z} - e^{-\Im z}| \leq \frac{1}{2} e^{|\Im z|},$$

where $\Im z$ denote the imaginary part of z . If $|\Im z_j| \leq \ln \rho$ for all $1 \leq j \leq d$, then, by assumptions, for any $x \in \text{supp}(v)$, we find that

$$\left| \sum_{j=1}^d x_j \Im(\cos z_j) \right| \leq \sum_{j=1}^d |x_j| \rho / 2 \leq \frac{1}{2} \sigma \rho.$$

Therefore,

$$\begin{aligned} |f(z_1, \dots, z_d)| &= \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i \sum_{j=1}^d r x_j \cos z_j} v(x) dx \right| \\ &\leq \frac{1}{(2\pi)^d} \int_{\text{supp}(v)} e^{\frac{1}{2} r \sigma \rho} |v(x)| dx = N e^{\frac{1}{2} r \sigma \rho}. \end{aligned}$$

Observing that f is 2π -periodic even function with respect to each component and recalling definition (1.5) and that $T_k(t) := \cos(k \arccos(t))$ for $t \in [-1, 1]$, we get

$$a_{k_1, \dots, k_d} = \frac{2^{\sum_{j=1}^d \mathbb{1}[k_j > 0]}}{(2\pi)^d} \int_0^{2\pi} \dots \int_0^{2\pi} e^{i \sum_{j=1}^d k_j \varphi_j} f(\varphi_1, \dots, \varphi_d) d\varphi_1 \dots d\varphi_d.$$

Since v is compactly supported, we have that $\mathcal{F}v$ and f are analytic functions in \mathbb{C}^d . Using the Cauchy integral theorem, we estimate

$$\begin{aligned} &\left| \frac{1}{(2\pi)^d} \int_0^{2\pi} e^{i \sum_{j=1}^d k_j \varphi_j} f(\varphi_1, \dots, \varphi_d) d\varphi_1 \dots d\varphi_d \right| \\ &= \left| \frac{1}{(2\pi)^d} \int_{i \ln \rho}^{2\pi + i \ln \rho} \dots \int_{i \ln \rho}^{2\pi + i \ln \rho} e^{i \sum_{j=1}^d k_j z_j} f(z_1, \dots, z_d) dz_1 \dots dz_d \right| \\ &\leq \frac{1}{(2\pi)^d} \int_{i \ln \rho}^{2\pi + i \ln \rho} \dots \int_{i \ln \rho}^{2\pi + i \ln \rho} N e^{\frac{1}{2} r \sigma \rho} e^{-\sum_{j=1}^d k_j \ln \rho} dz_1 \dots dz_d \\ &= N e^{\frac{1}{2} r \sigma \rho} \rho^{-\sum_{j=1}^d k_j}. \end{aligned}$$

We complete the proof of part (a), by observing that $|T_k(t)| \leq (2R/r)^k$ for any $|t| \leq R/r$. Indeed, if $|t| \leq 1$ then $|T_k(t)| \leq 1$, otherwise

$$|T_k(t)| = |\cosh(k \operatorname{arccosh}(t))| = \frac{1}{2} \left| (t - \sqrt{t^2 - 1})^k + (t + \sqrt{t^2 - 1})^k \right| \leq (2|t|)^k.$$

For (b), let $\rho := 2\rho'$ and $\lambda := \frac{2R}{r\rho} = \frac{R}{r\rho'} \leq \frac{1}{4}$. Using the Taylor theorem with the remainder in the Lagrange form, we get that, for some $\lambda' \in [0, \lambda]$,

$$\begin{aligned} (1 - \lambda)^{-d} - \sum_{k=0}^{n-1} \binom{k+d-1}{k} \lambda^k &= \binom{n+d-1}{n} (1 - \lambda')^{-d-n} \lambda^n \\ &\leq \binom{n+d-1}{n} \left(\frac{4}{3}\right)^d \left(\frac{4\lambda}{3}\right)^n. \end{aligned}$$

Using (5.1) and part (a), we find that

$$\begin{aligned}
\|\mathcal{F}v - \mathcal{C}_{R,n}[\mathcal{F}v|_{[-r,r]^d}]\|_{\mathcal{L}^\infty([-R,R]^d)} &\leq \sum_{k=n}^{\infty} \sum_{k_1+\dots+k_d=k} 2^d N e^{\frac{1}{2}r\sigma\rho} \lambda^k \\
&= 2^d N e^{r\sigma\rho'} \left((1-\lambda)^{-d} - \sum_{k=0}^{n-1} \binom{k+d-1}{k} \lambda^k \right) \\
&\leq \left(\frac{8}{3}\right)^d N e^{r\sigma\rho'} \binom{n+d-1}{n} \left(\frac{4\lambda}{3}\right)^n.
\end{aligned}$$

This completes the proof of Lemma 5.1. \square

Now we are ready to proceed to Lemma 3.1. Recall that $|T_k(t)| \leq 1$, if $|t| \leq 1$. Using (1.5) and the assumptions, we find that, for any $k_1, \dots, k_d \in \mathbb{N}$,

$$\begin{aligned}
|a_{k_1, \dots, k_d}[w] - a_{k_1, \dots, k_d}[\mathcal{F}v|_{[-r,r]^d}]| &= |a_{k_1, \dots, k_d}[w - \mathcal{F}v|_{[-r,r]^d}]| \\
&\leq \int_{-r}^r \dots \int_{-r}^r \delta \prod_{j=1}^d \left(\frac{2^{\mathbb{1}[k_j > 0]} |T_{k_j}\left(\frac{\xi_j}{r}\right)|}{\pi(r^2 - \xi_j^2)^{\frac{1}{2}}} \right) d\xi_1 \dots d\xi_d \leq 2^d \delta.
\end{aligned}$$

Recalling also that $|T_k(t)| \leq (2|t|)^k$ for $|t| \geq 1$, we get

$$\begin{aligned}
\|\mathcal{C}_{R,n}[w] - \mathcal{C}_{R,n}[\mathcal{F}v|_{[-r,r]^d}]\|_{\mathcal{L}^\infty([-R,R]^d)} &\leq \sum_{k=0}^{n-1} \sum_{k_1+\dots+k_d=k} 2^d \delta \left(\frac{2R}{r}\right)^k \\
&= 2^d \delta \sum_{k=0}^{n-1} \binom{k+d-1}{k} \left(\frac{2R}{r}\right)^k \leq 2^d \delta \binom{n+d-1}{n} \left(\frac{2R}{r}\right)^n.
\end{aligned}$$

Since $n \geq 1$ and $d \geq 1$, we have that

$$\binom{n+d-1}{n} = \binom{n+d-2}{n-1} + \binom{n+d-2}{n} \leq \binom{n+d-1}{n-1} + \binom{n+d-1}{n+1},$$

where $\binom{n+d-2}{n}$ and $\binom{n+d-1}{n+1}$ are taken to be 0 if $d = 1$. Thus, we get

$$\binom{n+d-1}{n} \leq \frac{1}{2} \sum_{j=0}^{n+d-1} \binom{n+d-1}{j} = 2^{n+d-2}.$$

Combining the above and using Lemma 5.1(b), we complete the proof of Lemma 3.1.

6 Exponential ill-posedness of Problem 1.1

In this section, we prove that Problem 1.1 is exponentially ill-posed. For ease of presentation, we employ the asymptotic notations $O(\cdot)$ and $\Omega(\cdot)$ always referring to the passage of the parameter n to infinity. For two sequences of real numbers a_n and b_n , we say $a_n = O(b_n)$ if there exist constants $C > 0$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leq C|b_n|$ for all $n > n_0$. We say $a_n = \Omega(b_n)$ if $a_n > 0$ always and $b_n = O(a_n)$.

First, we consider an explicit function $v_{n,m} : \mathbb{R}^2 \rightarrow \mathbb{C}$ similar to the one given by Mandache [15, Theorem 2]. Let $g \in C^\infty(\mathbb{R})$ be a nontrivial function supported in a compact set of positive real numbers. For example, one can take

$$g(t) := \begin{cases} \exp\left(\frac{1}{(t-1)(t-2)}\right), & \text{if } 1 < t < 2, \\ 0, & \text{otherwise.} \end{cases} \quad (6.1)$$

For integer $n \geq 1$ and $m \geq 0$, let $v_{n,m}$ be defined by

$$v_{n,m}(x_1, x_2) := n^{-m} e^{in\varphi} g(t),$$

where $t \geq 0$, $\varphi \in [0, 2\pi)$, and $(x_1, x_2) = (t \cos \varphi, t \sin \varphi)$. Observe that, as $n \rightarrow \infty$,

$$\|v_{n,m}\|_{\mathcal{L}^2(\mathbb{R}^2)} = \Omega(n^{-m}). \quad (6.2)$$

It is also straightforward that

$$\|v_{n,m}\|_{C^m(\mathbb{R}^2)} = O(1); \quad (6.3)$$

see, for example, the arguments of [15, Theorem 2].

Lemma 6.1. *For any $m \in \mathbb{N}$ and $r > 0$, we have $\|\mathcal{F}[\Re v_{n,m}]\|_{\mathcal{L}^\infty([-r,r]^2)} = O(e^{-n})$.*

Proof. Writing the Fourier transform in the polar coordinates, we find that

$$\mathcal{F}v_{n,m}(\xi) = \frac{n^{-m}}{(2\pi)^2} \int_{\text{supp}(g)} tg(t) \left(\int_0^{2\pi} e^{it|\xi| \cos(\varphi-\varphi_0)} e^{in\varphi} d\varphi \right) dt,$$

where $\xi = (|\xi| \cos \varphi_0, |\xi| \sin \varphi_0)^T \in \mathbb{R}^2$. Using the Cauchy integral theorem, we get that, uniformly over all $\xi \in [-r, r]^2$ and $t \in \text{supp}(g)$,

$$\int_0^{2\pi} e^{it|\xi| \cos(\varphi-\varphi_0)} e^{in\varphi} d\varphi = O\left(\int_{0+i}^{2\pi+i} e^{it|\xi| \cos(z-\varphi_0)} e^{inz} dz\right) = O(e^{-n}).$$

Observing also $\mathcal{F}[\Re v_{n,m}](\xi) = \mathcal{F}v_{n,m}(\xi) + \mathcal{F}v_{n,m}(-\xi)^*$, where z^* denotes the complex conjugate of $z \in \mathbb{C}$, the required bound follows. \square

The following theorem implies that the exponent μ in Corollary 2.3 is optimally bounded above by m (or almost optimally, for $d = 1$) since $|v|_{\mathcal{H}^m(\mathbb{R}^d)} \leq C\|v\|_{C^m(\mathbb{R}^d)}$ for a compactly supported v , where C depends on $\text{supp}(v)$ only.

Theorem 6.2. *Let $d \geq 1$ and $m \geq 0$ be integers. Let μ be a positive real number satisfying either $\mu > m$ if $d \geq 2$, or $\mu > m + 1/2$ if $d = 1$. Then, for any open set $A \subset \mathbb{R}^d$, compact set $B \subseteq \mathbb{R}^d$, and positive constants γ, c , there exists $v : \mathbb{R}^d \rightarrow \mathbb{R}$ such that:*

$$\begin{aligned} \text{supp}(v) \subseteq A, \quad \|v\|_{C^m(\mathbb{R}^d)} \leq \gamma, \quad \|\mathcal{F}v\|_{\mathcal{L}^\infty(B)} < 1, \\ \|v\|_{\mathcal{L}^2(\mathbb{R}^d)} > c \left(\ln \frac{1}{\|\mathcal{F}v\|_{\mathcal{L}^\infty(B)}} \right)^{-\mu}. \end{aligned} \quad (6.4)$$

Proof. First, we consider the case $d \geq 2$. Define $w_{n,m} : \mathbb{R}^d \rightarrow \mathbb{C}$ by

$$w_{n,m}(x) := \Re v_{n,m}(x_1, x_2) \prod_{j=3}^d g(x_j),$$

where g is given in (6.1). Observe that $w_{n,m} \in C^m(\mathbb{R}^d)$ and is compactly supported. Using (6.3) and taking any $x_0 \in A$ and sufficiently small $\alpha > 0$ and sufficiently big $\beta > 0$, we get that the functions $v_n : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$v_n(x) := \alpha w_{n,m}(\beta(x - x_0))$$

are supported in A and satisfy $\|v_n\|_{C^m(\mathbb{R}^d)} \leq \gamma$ for all $n > 0$. Next, taking r to be sufficiently large and observing from (6.1) that g is supported in $[1, 2]$ and $|g(t)| \leq 1$ for all $t \in \mathbb{R}$, we ensure

$$\|\mathcal{F}v_n\|_{\mathcal{L}^\infty(B)} = O\left(\|\mathcal{F}[\Re v_{n,m}]\|_{\mathcal{L}^\infty([-r, r]^2)}\right).$$

Using (6.2) and Lemma 6.1, we get that, as $n \rightarrow \infty$,

$$\|v_n\|_{\mathcal{L}^2(\mathbb{R}^d)} = \Omega(n^{-m}) \quad \text{and} \quad \|\mathcal{F}v_n\|_{\mathcal{L}^\infty(B)} = O(e^{-n}).$$

Taking $v \equiv v_n$ for sufficiently large n , we get (6.4).

For the case $d = 1$, consider the functions $h_{n,m} : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$h_{n,m}(x) := \int_{\mathbb{R}} \Re v_{2n,m}(t, x) dt = \int_{-2}^2 \Re v_{2n,m}(t, x) dt.$$

From (6.3), we derive that

$$\|h_{n,m}\|_{C^m(\mathbb{R})} = O(1).$$

Using Lemma 6.1, we also find that, for any fixed $r > 0$,

$$\|\mathcal{F}h_{n,m}\|_{\mathcal{L}^\infty([-r,r])} = 2\pi\|\mathcal{F}[\Re v_{2n,m}](0, \cdot)\|_{\mathcal{L}^\infty([-r,r])} = O(e^{-n}).$$

Note that if $|x| \leq (2n)^{-1}$ then, by the definition of $v_{n,m}$,

$$h_{n,m}(x) \geq n^{-m} \left(2 \cos 1 \int_1^2 g(t) dt + O(n^{-1}) \right) = \Omega(n^{-m}).$$

Therefore,

$$\|h_{n,m}\|_{\mathcal{L}^2(\mathbb{R})} \geq \left(n^{-1} \min_{|x| \leq (2n)^{-1}} |h_{n,m}(x)| \right)^{1/2} = \Omega(n^{-m-1/2}).$$

We complete the proof by considering functions of the form $\alpha h_{n,m}(\beta^{-1}(x - x_0))$ and repeating the arguments of the case $d \geq 2$. \square

References

- [1] G. Alessandrini, Stable determination of conductivity by boundary measurements, *Applicable Analysis*, **27** (1988), 153–172.
- [2] G. Bao, J. Lin, F. Triki, A multi-frequency inverse source problem, *Journal of Differential Equations*, **249** (2010), 3443–3465.
- [3] G. Beylkin, L. Monzón, Nonlinear inversion of a band-limited Fourier transform, *Applied and Computational Harmonic Analysis*, **27**(3) (2009), 351–366.
- [4] R.N. Bracewell, *The Fourier Transform and its Applications*, 3rd edition, McGraw-Hill, Boston (2000), 616pp.
- [5] J. A. Cadzow. An extrapolation procedure for band-limited signals. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, **27**(1) (1979), 4–12.
- [6] E. J. Candès, C. Fernandez-Granda, Towards a mathematical theory of super-resolution, *Communications on Pure and Applied Mathematics*, **67** (2014), 906–956.
- [7] L. Demanet, A. Townsend, Stable Extrapolation of Analytic Functions, *Foundations of Computational Mathematics*, **19** (2019), 297–331.
- [8] R. W. Gerchberg, Superresolution through error energy reduction, *Optica Acta: International Journal of Optics*, **21**(9) (1974), 709–720.

- [9] A. Gil, J. Segura, N.M. Temme, Numerical methods for special functions. *Society for Industrial and Applied Mathematics (SIAM)*, Philadelphia, PA (2007). DOI: 10.1137/1.9780898717822.
- [10] P. Hähner, T. Hohage, New stability estimates for the inverse acoustic inhomogeneous medium problem and applications, *SIAM Journal on Mathematical Analysis*, **33**(3) (2001), 670–685.
- [11] M. Isaev, Energy and regularity dependent stability estimates for near-field inverse scattering in multidimensions, *Journal of Mathematics*, (2013), Article ID 318154, 10 pp.; DOI:10.1155/2013/318154.
- [12] M. Isaev, Exponential instability in the inverse scattering problem on the energy interval, *Functional analysis and its applications*, **47**(3) (2013), 187–194.
- [13] M. Isaev, R.G. Novikov, New global stability estimates for monochromatic inverse acoustic scattering, *SIAM Journal on Mathematical Analysis*, **45**(3) (2013), 1495–1504.
- [14] V. Isakov, Increasing stability for the Schrödinger potential from the Dirichlet-to-Neumann map, *Discrete and Continuous Dynamical Systems*, **4**(3) (2011), 631–640.
- [15] N. Mandache, Exponential instability in an inverse problem for the Schrödinger equation, *Inverse Problems*, **17** (2001), 1435–1444.
- [16] M.M. Lavrent’ev, V.G. Romanov, S.P. Shishatskii, Ill-posed problems of mathematical physics and analysis, Translated from the Russian by J. R. Schulenberger. *Translations of Mathematical Monographs*, **64** (1986). American Mathematical Society, Providence, R.I., vi+290 pp.
- [17] A.N. Kolmogorov, V.M. Tikhomirov, ϵ -entropy and ϵ -capacity in functional spaces, *Uspekhi Matematicheskikh Nauk*, **14**(2) (1959) 3–86 (in Russian). English Translation, *American Mathematical Society Translations: Series 2*, **17** (1961), 277–364.
- [18] R.G. Novikov, New global stability estimates for the Gel’fand-Calderon inverse problem, *Inverse Problems*, **27** (2011), 015001, 21pp.
- [19] R.G. Novikov, Multidimensional inverse scattering for the Schrödinger equation, e-preprint: <https://hal.archives-ouvertes.fr/hal-02465839v1>.

- [20] A. Papoulis, A new algorithm in spectral analysis and band-limited extrapolation. *IEEE Transactions on Circuits and Systems*, **22**(9) (1975), 735–742.
- [21] M. Santacesaria, A Holder-logarithmic stability estimate for an inverse problem in two dimensions, *Journal of Inverse and Ill-posed Problems*, **23**(1) (2015), 51–73.
- [22] V.K. Tuan, Stable analytic continuation using hypergeometric summation, *Inverse Problems*, **16**(1) (2000), 75–87.
- [23] S. Vessella, A continuous dependence result in the analytic continuation problem, *Forum Mathematicum*, **11**(6) (1999), 695–703.