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► **To cite this version:**

François Alouges, Giovanni Di Fratta. Homogenization of composite ferromagnetic materials. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 2015, 471 (2182), pp.20150365. 10.1098/rspa.2015.0365 . hal-02562413

**HAL Id: hal-02562413**

**<https://hal.science/hal-02562413>**

Submitted on 10 Feb 2024

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# Homogenization of Composite Ferromagnetic Materials

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October 22, 2021

## Abstract

Nowadays, nonhomogeneous and *periodic* ferromagnetic materials are the subject of a growing interest. Actually such periodic configurations often combine the attributes of the constituent materials, while sometimes, their properties can be strikingly different from the properties of the different constituents. These periodic configurations can be therefore used to achieve physical and chemical properties difficult to achieve with homogeneous materials. To predict the magnetic behavior of such composite materials is of prime importance for applications.

The main objective of this paper is to perform, by means of  $\Gamma$ -convergence and *two-scale convergence*, a rigorous derivation of the homogenized GIBBS-LANDAU free energy functional associated to a composite periodic ferromagnetic material, i.e. a ferromagnetic material in which the heterogeneities are periodically distributed inside the ferromagnetic media. We thus describe the  $\Gamma$ -limit of the GIBBS-LANDAU free energy functional, as the period over which the heterogeneities are distributed inside the ferromagnetic body shrinks to zero.

## 1 Introduction

Composite materials are an important class of natural or engineered heterogeneous media, composed of a mixture of two or more constituents with significantly different physical or chemical properties, firmly bonded together, which remain separate and distinct within the finished structure. Finding a model which considers the composite as a bulk and whose coefficients and terms are computed from suitable averages of those of its constituents and the geometry of the microstructure is the aim of homogenization theory. The study of composites and their homogenization is a subject with a long history, which has attracted the interest and the efforts of some of the most illustrious names in science [22]: In 1824, POISSON, in his first *Mémoire sur la théorie du magnétisme* [28], put the basis of the theory of induced magnetism assuming a model in which the body is composed of conducting spheres embedded in a nonconducting material. This paper is the origin of the basic models and ideas that prevailed in the theory of heterogeneous media in almost all domains of continuum mechanics, for almost a century after its appearance [22]. We refer the reader to the papers of MARKOV [22] and LANDAUER [19] for more historical details.

Nowadays, nonhomogeneous and *periodic* ferromagnetic materials are the subject of a growing interest. Actually such periodic configurations often combine the attributes of the constituent materials, while sometimes, their properties can be strikingly different from the properties of the different constituents [24]. These periodic configurations can be therefore used to achieve physical and chemical properties difficult to achieve with homogeneous materials. To predict the magnetic behavior of such composite materials is of prime importance for applications [24]. From a mathematical point of view, the study of composite materials, and more generally of media which involve microstructures, is the main source of inspiration for the *Mathematical Theory of Homogenization* which, roughly speaking, is a mathematical procedure which aims at understanding heterogeneous materials with highly oscillating heterogeneities (at the microscopic level) via an effective model [26].

The main objective of this paper is to perform, in the framework of DE GIORGI's notion of  $\Gamma$ -convergence [12] and ALLAIRE's notion of *two-scale convergence* [3] (see also the paper by NGUETSENG [27]), a mathematical homogenization study of the GIBBS-LANDAU free energy functional associated to a composite periodic ferromagnetic material, i.e. a ferromagnetic material in which the heterogeneities are periodically distributed inside the ferromagnetic media. Compared to earlier works related to the subject (see for instance [10, 11, 14, 30]) we consider here the full GIBBS-LANDAU functional for mixtures of different materials in the three-dimensional space.

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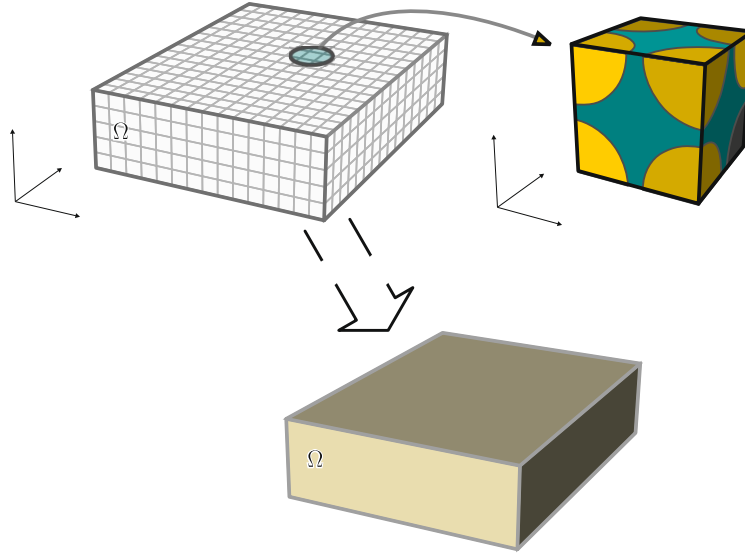


Figure 1: If we assume that the heterogeneities are evenly distributed inside the ferromagnetic media  $\Omega$ , we can model the material as periodic. As illustrated in the figure, this means that we can think of the material as being built up of small identical cubes  $Q_\varepsilon$ , the side length of which we call  $\varepsilon$ .

constant on the region  $\Omega$  occupied by the ferromagnet. Moreover one has to describe the local interactions of two grains with different magnetic properties at their touching interface [1].

From a mathematical point of view, this latter requirement is usually taken into account in two different ways. Either one adds to the model a surface energy term which penalizes jumps of the magnetization direction  $\mathbf{m}$  at the interface of both grains, or, and we stick on this later on, one simply considers a *strong coupling*, meaning that the direction of the magnetization does not jump through an interface. We insist on the fact that only the direction is continuous at an interface while the magnitude  $M_s$  is obviously discontinuous. Therefore, the natural mathematical setting for the problem turns out to be characterized by the assumption that the magnetization direction  $\mathbf{m}$  is in the “weak” SOBOLEV metric space  $(H^1(\Omega, \mathbb{S}^2), d_{L^2(\Omega, \mathbb{S}^2)})$ , i.e. on the metric subspace of  $H^1(\Omega, \mathbb{R}^3)$  constituted by the functions constrained to take values on the unit sphere of  $\mathbb{R}^3$  and endowed with the  $L^2(\Omega)$  metric. It is in this framework that we will conduct our work from now on. We start by recalling the basic idea of the mathematical theory of homogenization. Let  $\Omega \subset \mathbb{R}^3$  be the region occupied by the composite material. If we assume that the heterogeneities are regularly distributed, we can model the material as periodic. As illustrated in Fig.1, this means that we can think of the material as being built up of small identical cubes, the side length of which being called  $\varepsilon$ . Let  $Q = [0, 1]^3$  be the unit cube of  $\mathbb{R}^3$ . We let for  $y \in Q$ ,  $a_{\text{ex}}(y)$ ,  $M_s(y)$ ,  $\varphi_{\text{an}}(y, \mathbf{m})$  be the periodic repetitions of the functions that describe how the exchange constant  $a_{\text{ex}}$ , the saturation magnetization  $M_s$  and the anisotropy density energy  $\varphi_{\text{an}}(y, \mathbf{m})$  vary over the representative cell  $Q$  (see Fig. 1). Substituting  $x/\varepsilon$  for  $y$ , we obtain the «two-scale» functions  $a_\varepsilon(x) := a_{\text{ex}}(x/\varepsilon)$ ,  $M_\varepsilon(x) := M_s(x/\varepsilon)$  and  $\varphi_\varepsilon(x, \mathbf{m}) := \varphi_{\text{an}}(x/\varepsilon, \mathbf{m})$  that oscillate periodically with period  $\varepsilon$  as the variable  $x$  runs through  $\Omega$ , describing the oscillations of the material dependent parameters of the composite. At every scale  $\varepsilon$ , the energy associated to the  $\varepsilon$ -heterogeneous ferromagnet, will be given by the following generalized GIBBS-LANDAU energy functional

$$\begin{aligned} \mathcal{G}_\varepsilon^\varepsilon(\mathbf{m}) &:= \int_{\Omega} a_\varepsilon |\nabla \mathbf{m}|^2 d\tau + \int_{\Omega} \varphi_\varepsilon(\cdot, \mathbf{m}) d\tau \\ &\quad \underset{=: \mathcal{E}_\varepsilon(\mathbf{m})}{=} \quad \underset{=: \mathcal{A}_\varepsilon(\mathbf{m})}{=} \\ &\quad - \frac{\mu_0}{2} \int_{\Omega} \mathbf{h}_d[M_\varepsilon \mathbf{m}] \cdot M_\varepsilon \mathbf{m} d\tau - \mu_0 \int_{\Omega} \mathbf{h}_a \cdot M_\varepsilon \mathbf{m} d\tau. \end{aligned} \quad (5)$$

$\underset{=: \mathcal{W}_\varepsilon(\mathbf{m})}{=} \quad \underset{=: \mathcal{Z}_\varepsilon(\mathbf{m})}{=}$

The asymptotic  $\Gamma$ -convergence analysis of the family of functionals  $(\mathcal{G}_\varepsilon^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  as  $\varepsilon$  tends to 0, is the object of the present paper.

### 1.3 Statement of the main result

The main purpose of this paper is to analyze, by the means of both  $\Gamma$ -convergence and two-scale convergence techniques, the asymptotic behavior, as  $\varepsilon \rightarrow 0$ , of the family of Gibbs-Landau free energy functionals  $(\mathcal{G}_\varepsilon^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  expressed by (5). Let us make the statement more precise.

We consider the unit sphere  $\mathbb{S}^2$  of  $\mathbb{R}^3$  and, for every  $s \in \mathbb{S}^2$ , the tangent space of  $\mathbb{S}^2$  at a point  $s$  will be denoted by  $T_s(\mathbb{S}^2)$ . The class of admissible maps we are interested in is defined as

$$H^1(\Omega, \mathbb{S}^2) := \{ \mathbf{m} \in H^1(\Omega, \mathbb{R}^3) : \mathbf{m}(x) \in \mathbb{S}^2 \text{ for } \tau\text{-a.e. } x \in \Omega \},$$

where we have denoted by  $\tau$  the Lebesgue measure on  $\mathbb{R}^3$ . We consider  $H^1(\Omega, \mathbb{S}^2)$  as a metric space endowed with the metric structure induced by the classical  $L^2(\Omega, \mathbb{R}^3)$  metric. For every positive real number  $t > 0$ , we set  $Q_t := [0, t]^3$  and  $Q := Q_1 = [0, 1]^3$ . We recall that a function  $u : \mathbb{R}^3 \rightarrow \mathbb{R}$  is said to be  $Q$ -periodic if  $u(\cdot) = u(\cdot + e_i)$  for every  $e_i$  in the canonical basis  $(e_1, e_2, e_3)$  of  $\mathbb{R}^3$ .

For the energy densities appearing in the family  $(\mathcal{G}_\varepsilon^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  we assume the following hypotheses:

**[H<sub>1</sub>]** The *exchange parameter*  $a_{\text{ex}}$  is supposed to be a  $Q$ -periodic measurable function belonging to  $L^\infty(Q)$  which is bounded from below and above by two positive constants  $c_{\text{ex}} > 0, C_{\text{ex}} > 0$ , i.e.  $0 < c_{\text{ex}} \leq a_{\text{ex}}(y) \leq C_{\text{ex}}$  for  $\tau$ -a.e.  $y \in Q$ . In the setting of classical Calculus of Variations, this hypothesis guarantees that the exchange energy density, which has the form  $g(y, \xi) := a_{\text{ex}}(y)|\xi|^2$ ,  $\xi \in \mathbb{R}^{3 \times 3}$ , is a CARATHÉODORY integrand satisfying the following quadratic growth condition for  $\tau$ -a.e.  $y \in Q$

$$\forall \xi \in \mathbb{R}^{3 \times 3} \quad c_{\text{ex}}|\xi|^2 \leq g(y, \xi) \leq C_{\text{ex}}(1 + |\xi|^2). \quad (6)$$

Then we set  $a_\varepsilon(x) := a_{\text{ex}}(x/\varepsilon)$ .

**[H<sub>2</sub>]** The *anisotropy density energy*  $\varphi_{\text{an}} : \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^+$  is supposed to be a  $Q$ -periodic measurable function belonging to  $L^\infty(Q)$  with respect to the first variable, and globally lipschitz with respect to the second one (uniformly with respect to the first variable), i.e.  $\exists \kappa_L > 0$  such that

$$\text{ess sup}_{y \in Q} |\varphi_{\text{an}}(y, s_1) - \varphi_{\text{an}}(y, s_2)| \leq \kappa_L |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{S}^2. \quad (7)$$

We then set  $\varphi_\varepsilon(x, s) := \varphi_{\text{an}}(x/\varepsilon, s)$ . The hypotheses assumed on  $\varphi_{\text{an}}$  are sufficiently general to treat the most common classes of crystal anisotropy energy densities arising in applications. As a sake of example, for uniaxial anisotropy, the energy density reads as

$$\varphi_{\text{an}}(y, \mathbf{m}(x)) = \kappa(y)[1 - (\mathbf{u}(y) \cdot \mathbf{m}(x))^2], \quad (8)$$

the spatially dependent unit vector  $\mathbf{u}(\cdot)$  being the easy axis of the crystal. For cubic type anisotropy, the energy density reads as:

$$\varphi_{\text{an}}(y, \mathbf{m}(x)) = \kappa(y) \sum_{i=1}^3 [(\mathbf{u}_i(y) \cdot \mathbf{m}(x))^2 - (\mathbf{u}_i(y) \cdot \mathbf{m}(x))^4], \quad (9)$$

the mutually orthogonal unit vectors  $\mathbf{u}_i(\cdot)$  being the three *easy-axes* of the cubic crystal. Note that the anisotropy depends on the material both in strength ( $\kappa(y)$ ) and in direction ( $\mathbf{u}_i(y)$ ).

**[H<sub>3</sub>]** The *saturation magnetization*  $M_s$  is supposed to be a  $Q$ -periodic measurable function belonging to  $L^\infty(Q)$ , and we set  $M_\varepsilon(\cdot) = M_s(\cdot/\varepsilon)$ .

The main result of this paper is the following:

**THEOREM 1.1** Let  $(\mathcal{G}_\varepsilon^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  be a family of GIBBS-LANDAU free energy functionals satisfying the hypotheses **[H<sub>1</sub>]**, **[H<sub>2</sub>]** and **[H<sub>3</sub>]**. Then  $(\mathcal{G}_\varepsilon^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  is **equicoercive** in the metric space  $(H^1(\Omega, \mathbb{S}^2), d_{L^2(\Omega, \mathbb{S}^2)})$ . Moreover  $(\mathcal{G}_\varepsilon^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$   $\Gamma$ -converges in  $(H^1(\Omega, \mathbb{S}^2), d_{L^2(\Omega, \mathbb{S}^2)})$  to the functional  $\mathcal{G}_{\text{hom}} : H^1(\Omega, \mathbb{S}^2) \rightarrow \mathbb{R}^+$  defined by

$$\mathcal{G}_{\text{hom}}(\mathbf{m}) := \mathcal{E}_{\text{hom}}(\mathbf{m}) + \mathcal{A}_{\text{hom}}(\mathbf{m}) + \frac{\mu_0}{2} \mathcal{W}_{\text{hom}}(\mathbf{m}) + \mu_0 \mathcal{Z}_{\text{hom}}(\mathbf{m}). \quad (10)$$

The four terms that appear in (10) have the following expressions: The homogenized **exchange energy** is given by

$$\mathcal{E}_{\text{hom}}(\mathbf{m}) := \int_{\Omega} A_{\text{hom}} \nabla \mathbf{m}(x) : \nabla \mathbf{m}(x) \, dx, \quad (11)$$

where  $A_{\text{hom}}$  is the «classical» homogenized tensor  $A_{\text{hom}}$  given by the average

$$A_{\text{hom}} := \langle a_{\text{ex}}(y)(I + \nabla \varphi(y))^T (I + \nabla \varphi(y)) \rangle_Q, \quad \varphi := (\varphi_1, \varphi_2, \varphi_3), \quad (12)$$

where for every  $j \in \mathbb{N}_3$  the component  $\varphi_j$  is the unique (up to a constant) solution of the following scalar unit cell problem

$$\varphi_j := \operatorname{argmin}_{\varphi \in H_{\#}^1(Q)} \int_Q a_{\text{ex}}(y) [|\nabla \varphi(y) + e_j|^2] dy. \quad (13)$$

The homogenized **anisotropy energy** is given by

$$\mathcal{A}_{\text{hom}}(\mathbf{m}) := \int_{\Omega \times Q} \varphi_{\text{an}}(y, \mathbf{m}(x)) dy dx, \quad (14)$$

while the homogenized **magnetostatic self-energy** is given by

$$\mathcal{W}_{\text{hom}}(\mathbf{m}) := -\langle M_s \rangle_Q^2 \int_{\Omega} \mathbf{h}_d[\mathbf{m}] \cdot \mathbf{m} d\tau + \int_{\Omega \times Q} |\nabla_y v_{\mathbf{m}}(x, y)|^2 dx dy, \quad (15)$$

where, for every  $x \in \Omega$ , the scalar function  $v_{\mathbf{m}} : \Omega \times Q \rightarrow \mathbb{R}$ , is the unique solution of the cell problem:

$$\mathbf{m}(x) \cdot \int_Q M_s(y) \nabla_y \psi(y) dy = - \int_Q \nabla_y v_{\mathbf{m}}(x, y) \cdot \nabla_y \psi(y) dy, \quad \int_Q v_{\mathbf{m}}(x, y) dy = 0 \quad (16)$$

for all  $\psi \in H_{\#}^1(Q)$ .

Finally, the homogenized **interaction energy** is given by

$$\mathcal{Z}_{\text{hom}}(\mathbf{m}) = -\langle M_s \rangle_Q \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} d\tau. \quad (17)$$

The paper is organized as follows: In Section 2 we give a brief survey of the main mathematical concepts and results used throughout the paper. The equicoercivity of the family  $(\mathcal{G}_{\mathcal{L}}^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  is established in Section 3; the  $\Gamma$ -limit of the exchange energy family of functionals  $(\mathcal{E}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  is computed in Section 4; in Section 5 it is shown that the family of magnetostatic self-energies  $(\mathcal{W}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  continuously converges to  $\mathcal{W}_{\text{hom}}$ , while in Section 6 it is established the continuous convergence of the family of anisotropy energies  $(\mathcal{A}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  to  $\mathcal{A}_{\text{hom}}$  and the continuous convergence of the family of interaction energies  $(\mathcal{Z}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  to the functional  $\mathcal{Z}_{\text{hom}}$ . Eventually, the proof of MHT (Theorem 1.1) is completed in Section 7, and some well-known results in homogenization theory, though somewhat difficult to find in the literature are given in the appendix.

## 2 Mathematical Preliminaries

The purpose of this section is to fix some notations and to give a survey of the concepts and results that are used throughout this work. All results are stated without proof as they can be readily found in the references given below.

### 2.1 $\Gamma$ -convergence of a family of functionals

We start by recalling DE GIORGI's notion of  $\Gamma$ -convergence and some of its basic properties (see [12, 9]). Throughout this part we indicate with  $(X, d)$  a metric space and, for every  $m \in X$ , with  $\mathfrak{C}_d(m)$  the subset of all sequences of elements of  $X$  which converge to  $m$ .

**DEFINITION 2.1** ( *$\Gamma$ -convergence of a family of functionals*) Let  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be a sequence of functionals defined on  $X$  with values on  $\overline{\mathbb{R}}$ . The functional  $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$  is said to be the  $\Gamma$ -lim of  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  with respect to the metric  $d$ , if for every  $m \in X$  we have:

$$\forall (m_n) \in \mathfrak{C}_d(m) \quad \mathcal{F}(m) \leq \liminf_{n \rightarrow \infty} \mathcal{F}_n(m_n) \quad (18)$$

and

$$\exists (\bar{m}_n) \in \mathfrak{C}_d(m) \quad \mathcal{F}(m) = \lim_{n \rightarrow \infty} \mathcal{F}_n(\bar{m}_n). \quad (19)$$

In this case we write  $\mathcal{F} = \Gamma\text{-}\lim_{n \rightarrow \infty} \mathcal{F}_n$ .

If  $(\mathcal{F}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  is a family of functionals, we say that  $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$  is the  $\Gamma$ -lim of  $(\mathcal{F}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$  as  $\varepsilon \rightarrow 0$ , if for every  $\varepsilon_n \downarrow 0$  one has  $\mathcal{F} = \Gamma\text{-}\lim_{n \rightarrow \infty} \mathcal{F}_{\varepsilon_n}$ . In this case we write  $\mathcal{F} = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{F}_{\varepsilon}$ .

The condition (19) is sometimes referred to in literature as the existence of a recovery sequence.

One of the most important properties of  $\Gamma$ -convergence, and the reason why this kind of variational convergence is so important in the asymptotic analysis of variational problems, is that under appropriate compactness hypotheses it implies the convergence of (almost) minimizers of a family of equicoercive functionals to the minimum of the  $\Gamma$ -limit functional. More precisely, the following result holds:

THEOREM 2.2 (*Fundamental Theorem of  $\Gamma$ -convergence*) If  $(\mathcal{F}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  is a family of equicoercive functionals  $\Gamma$ -converging on  $X$  to the functional  $\mathcal{F}$ . Then  $\mathcal{F}$  is coercive and lower semicontinuous (therefore there exists a minimizer for  $\mathcal{F}$  on  $X$ ) and we have the convergence of minima values

$$\min_{m \in X} \mathcal{F}(m) = \lim_{\varepsilon \rightarrow 0} \inf_{m \in X} \mathcal{F}_\varepsilon(m). \quad (20)$$

Moreover, given  $\varepsilon_n \downarrow 0$  and  $(m_n)_{n \in \mathbb{N}}$  a converging sequence such that

$$\lim_{n \rightarrow \infty} \mathcal{F}_{\varepsilon_n}(m_n) = \lim_{n \rightarrow \infty} \left( \inf_{m \in X} \mathcal{F}_{\varepsilon_n}(m) \right), \quad (21)$$

its limit is a minimizer for  $\mathcal{F}$  on  $X$ . If (21) holds, the sequence  $(m_n)_{n \in \mathbb{N}}$  is said to be a sequence of almost-minimizers for  $\mathcal{F}$ .

Let us recall now that given two families of functional  $(\mathcal{F}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  and  $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$   $\Gamma$ -converging respectively to  $\mathcal{F}$  and  $\mathcal{G}$ , it is in general not the case (see [9]) that  $\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} (\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon) = \mathcal{F} + \mathcal{G}$ . A sufficient condition for that property to hold is that at least one of the two families of functionals satisfies a stronger type of convergence:

DEFINITION 2.3 We say that a family of functionals  $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  is *continuously convergent* in  $X$  to a functional  $\mathcal{G} : X \rightarrow \overline{\mathbb{R}}$ , and we will write  $\mathcal{G}_\varepsilon \xrightarrow{\Gamma\text{cont}} \mathcal{G}$ , if for every  $m_0 \in X$

$$\lim_{(m, \varepsilon) \rightarrow (m_0, 0)} \mathcal{G}_\varepsilon(m) = \mathcal{G}(m_0).$$

We then have (see [9] for a proof):

PROPOSITION 2.4 Let  $\mathcal{F} = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon$ . Suppose that the family of functionals  $(\mathcal{G}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  continuously converges to  $\mathcal{G}$ , and that  $\mathcal{G}_\varepsilon$  and  $\mathcal{G}$  are everywhere finite on  $X$ . Then  $\mathcal{G} = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{G}_\varepsilon$  and

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} (\mathcal{F}_\varepsilon + \mathcal{G}_\varepsilon) = \mathcal{F} + \mathcal{G}.$$

In particular if  $\mathcal{Z} : X \rightarrow \mathbb{R}$  is a continuous functional then  $\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} (\mathcal{F}_\varepsilon + \mathcal{Z}) = \mathcal{F} + \mathcal{Z}$  and  $\mathcal{Z}$  is called a continuous perturbation of the  $\Gamma$ -limit.

## 2.2 Two-scale convergence

The aim of this section is to present in a schematic way the main properties of two-scale convergence, a notion that is first due to NGUETSANG [27], developed as a methodology by ALLAIRE [3] and further investigated by many others (see [2] and references therein for instance).

We denote by  $C_{\#}^\infty(Q)$  the set of infinitely differentiable real functions over  $\mathbb{R}^3$  that are  $Q$ -periodic and define  $H_{\#}^1(Q)$  as the closure of  $C_{\#}^\infty(Q)$  in  $H_{\text{loc}}^1(\Omega)$ . Obviously any element of  $H_{\#}^1(Q)$  has the same trace on the opposite faces of  $Q$ .

A generalized version of the RIEMANN-LEBESGUE Lemma holds for the weak limit of rapidly oscillating functions. For a proof we refer the reader to [13].

PROPOSITION 2.5 Let  $\Omega \subset \mathbb{R}^3$  be any open set. Let  $1 \leq p < \infty$  and  $t > 0$  be a positive real number. Let  $u \in L^p(Q_t)$  be a  $Q_t$ -periodic function. Set  $u_\varepsilon(x) := u(x/\varepsilon)$   $\tau$ -a.e. on  $\Omega$ . Then, if  $p < \infty$ , as  $\varepsilon \rightarrow 0$

$$u_\varepsilon \rightharpoonup \langle u \rangle_{Q_t} := \frac{1}{|Q_t|} \int_{Q_t} u \, d\tau \quad \text{weakly in } L^p(\Omega).$$

If  $p = \infty$ , one has

$$u_\varepsilon \rightharpoonup \langle u \rangle_{Q_t} := \frac{1}{|Q_t|} \int_{Q_t} u \, d\tau \quad \text{weakly}^* \text{ in } L^\infty(\Omega).$$

DEFINITION 2.6 Let  $\Omega$  be an open set  $\Omega \subset \mathbb{R}^3$ , and let  $(\varepsilon_k)_{k \in \mathbb{N}}$  be a fixed sequence of positive real numbers (when it is clear from the context we will omit the subscript  $k$ ) converging to 0. The sequence of functions  $(u_\varepsilon) \in L^2(\Omega)$  is said to two-scale converge to a limit  $u \in L^2(\Omega \times Q)$ , if for any function  $\varphi \in \mathcal{D}[\Omega; C_{\#}^\infty(Q)]$  we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \varphi(x, x/\varepsilon) \, dx = \int_{\Omega \times Q} u(x, y) \varphi(x, y) \, dy \, dx \quad (22)$$

In this case we write  $u_\varepsilon \rightharpoonup u$ . We say that  $(u_\varepsilon)$  in  $L^2(\Omega)$  **strongly** two-scale converges to a limit  $u \in L^2(\Omega \times Q)$  if  $u_\varepsilon \rightharpoonup u$  and moreover

$$\|u\|_{\Omega \times Q} = \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{\Omega}.$$

The importance of this new notion of convergence relies on the following compactness results.

**PROPOSITION 2.7** For each bounded sequence  $(u_\varepsilon) \in L^2(\Omega)$ , there exists an  $u \in L^2(\Omega \times Q)$  such that, up to a subsequence,  $u_\varepsilon \rightharpoonup u$ .

Moreover, for bounded sequences in  $H^1(\Omega)$  we have the following result:

**PROPOSITION 2.8** Let  $(u_\varepsilon)$  be a sequence in  $H^1(\Omega)$  that converges weakly to a limit  $u \in H^1(\Omega)$ . Then  $u_\varepsilon \rightharpoonup u$  and there exists a function  $v \in L^2[\Omega; H^1_\#(Q)/\mathbb{R}]$  such that, up to a subsequence:

$$\nabla u_\varepsilon \rightharpoonup \nabla u + \nabla_y v.$$

Next we recall that if the sequence  $(u_\varepsilon)$  is bounded in  $L^2(\Omega)$ , it is possible to enlarge the class of test functions used in the definition of two-scale convergence.

**PROPOSITION 2.9** Let  $(u_\varepsilon)$  be a bounded sequence in  $L^2(\Omega)$  which two-scale converges to  $u \in L^2(\Omega \times Q)$ . Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \varphi(x, x/\varepsilon) dx = \int_{\Omega \times Q} u(x, y) \varphi(x, y) dy dx$$

for every  $\varphi \in L^2[\Omega; C_\#(Q)]$ .

Finally we recall a simple criteria that permits to «bypass» the problem concerning the convergence of the product of two  $L^2(\Omega)$ -weakly convergence sequences (cfr.[3, 20]).

**PROPOSITION 2.10** Let  $(u_\varepsilon)$  and  $(v_\varepsilon)$  be sequences in  $L^2(\Omega)$  that respectively two-scale converge to  $u$  and  $v$  in  $L^2(\Omega \times Q)$ . If at least one of them **strongly** two-scale converges, then

$$u_\varepsilon v_\varepsilon \rightarrow uv.$$

In particular, if  $(u_\varepsilon v_\varepsilon)$  is bounded in  $L^2(\Omega)$ , from the previous proposition, we have

$$\int_{\Omega} u_\varepsilon(x) v_\varepsilon(x) \varphi(x, x/\varepsilon) dx = \int_{\Omega \times Q} u(x, y) v(x, y) \varphi(x, y) dy dx$$

for every  $\varphi \in L^2[\Omega; C_\#(Q)]$ .

### 3 The equicoercivity of the composite Gibbs-Landau free energy functionals

This section is devoted to the proof of the equicoercivity of the family of GIBBS-LANDAU free energy functionals  $(\mathcal{G}_\mathcal{L}^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  expressed by (5). Equicoercivity has an important role in homogenization theory. In fact, the metric space in which to work, must be able to guarantee the equicoercivity of the family of functionals under consideration, i.e. the validity of the Fundamental Theorem of  $\Gamma$ -convergence.

**PROPOSITION 3.1** The family  $(\mathcal{G}_\mathcal{L}^\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  of GIBBS-LANDAU energy functionals is equicoercive on the metric space  $(H^1(\Omega, \mathbb{S}^2), d_{L^2(\Omega, \mathbb{S}^2)})$ .

**PROOF 3.2** According to the hypotheses  $[H_1]$ ,  $[H_2]$  and  $[H_3]$ , there exist positive constants  $c_{\text{ex}}, C_{\text{ex}}, C_s, C_{\text{an}} \in \mathbb{R}^+$  such that for every  $y \in Q$

$$0 < c_{\text{ex}} \leq a_{\text{ex}}(y) \leq C_{\text{ex}} \quad , \quad 0 \leq M_s(y) \leq C_s \quad , \quad 0 \leq \varphi(y, \mathbf{m}) \leq C_{\text{an}}.$$

Therefore, for every  $\varepsilon > 0$  on has

$$c_{\text{ex}} \|\nabla \mathbf{m}\|_{\Omega}^2 \leq \mathcal{G}_\mathcal{L}^\varepsilon(\mathbf{m}) \leq C^*(1 + \|\nabla \mathbf{m}\|_{\Omega}^2) \tag{23}$$

with  $C^* = \max \{C_{\text{ex}}, \frac{\mu_0}{2} C_s^2 |\Omega|, C_s |\Omega|^{1/2} \|\mathbf{h}_a\|_{\Omega}, C_{\text{an}} |\Omega|\}$ .

Now observe that for every constant in space magnetization  $\mathbf{u}$  and for every  $\varepsilon > 0$  on has  $\mathcal{G}_\mathcal{L}^\varepsilon(\mathbf{u}) \leq C^*$ ; thus

$$\inf_{\mathbf{m} \in H^1(\Omega, \mathbb{S}^2)} \mathcal{G}_\mathcal{L}^\varepsilon = \inf_{\mathbf{m} \in K} \mathcal{G}_\mathcal{L}^\varepsilon. \tag{24}$$

To finish, we simply observe that, due to RELICH-KONDRACHOV theorem,  $K$  is a compact subset of  $(H^1(\Omega, \mathbb{S}^2), d_{L^2(\Omega, \mathbb{S}^2)})$ .



## 4 The $\Gamma$ -limit of exchange energy functionals $\mathcal{E}_\varepsilon$

The fundamental constraint of micromagnetic theory, i.e. the fact that the domain of definition of the family  $\mathcal{E}_\varepsilon$  is a manifold value Sobolev space, plays a fundamental role in the homogenization process. In fact, although the unconstrained problem has been fully investigated (see [7, 21, 25]), it is not possible to get full information about the manifold constrained  $\Gamma$ -limit by just looking at the unconstrained one. This is due to the well-known fact (see [9]) that if  $(\mathcal{F}_\varepsilon)$  is a family of functionals defined in the metric space  $X$  and  $\Gamma$ -converging to  $\mathcal{F}$ , and  $\mathcal{G} = \Gamma\text{-lim}[\mathcal{G}_\varepsilon := \mathcal{F}_{\varepsilon|Y}]$  where  $(\mathcal{G}_\varepsilon) := (\mathcal{F}_{\varepsilon|Y})$  represents the family of functionals obtained as the restriction of  $(\mathcal{F}_\varepsilon)$  to some (metric) subspace  $Y$  of  $X$ , then  $\mathcal{F}|_Y \leq \mathcal{G}$ . Thus the identification of the manifold constrained  $\Gamma$ -limit requires more effort.

### 4.1 The tangential homogenization theorem

In what follows we make use of the following theorem due to BABADJIAN and MILLOT (see [5]) in which the dependence of the  $\Gamma$ -limit from the tangent bundle of the manifold is taken into account via the so-called *tangentially homogenized energy density*. We state the **tangential homogenization theorem** (THM) in a bit less general form which is adequate for our purposes.

**PROPOSITION 4.1** (THM) [5] Let  $\mathcal{M}$  be a connected smooth submanifold of  $\mathbb{R}^3$  without boundary and  $g : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^+$  be a Carathéodory function such that

1. For every  $\xi \in \mathbb{R}^{3 \times 3}$  the function  $g(\cdot, \xi)$  is  $Q$ -periodic, i.e. such that if  $(e_1, e_2, e_3)$  denotes the canonical basis of  $\mathbb{R}^3$ , one has

$$\forall i \in \mathbb{N}_3, \forall y \in \mathbb{R}^3, \forall \xi \in \mathbb{R}^{3 \times 3} \quad g(y + e_i, \xi) = g(y, \xi).$$

2. There exist  $0 < \alpha \leq \beta < \infty$  such that

$$\alpha|\xi|^2 \leq g(y, \xi) \leq \beta(1 + |\xi|^2) \quad \text{for a.e. } y \in \mathbb{R}^3 \text{ and all } \xi \in \mathbb{R}^{3 \times 3}.$$

Then the family

$$\mathcal{E}_\varepsilon(\mathbf{m}) := \int_{\Omega} g(x/\varepsilon, \nabla \mathbf{m}) d\tau \tag{25}$$

defined in the metric space  $(H^1(\Omega, \mathcal{M}), d_{L^2(\Omega, \mathcal{M})})$   $\Gamma$ -converges to the functional

$$\mathcal{E}_{\text{hom}}(\mathbf{m}) := \int_{\Omega} Tg_{\text{hom}}(\mathbf{m}, \nabla \mathbf{m}) d\tau, \tag{26}$$

where for every  $s \in \mathcal{M}$  and  $\xi \in [T_s(\mathcal{M})]^3$ ,  $Q_t := [0, t]^3$ ,

$$Tg_{\text{hom}}(s, \xi) = \lim_{t \rightarrow \infty} \left( \inf_{\varphi \in W_0^{1, \infty}(Q_t, T_s(\mathcal{M}))} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy \right), \tag{27}$$

is the **tangentially homogenized energy density**.

We refer the reader to [5] for a more general version and the proof.

#### 4.1.1 The role of tangent bundle.

Let us emphasize why the tangent bundle  $[\mathcal{T}(\mathcal{M})]^3 := \sqcup_{s \in \mathcal{M}} [T_s(\mathcal{M})]^3$  plays a role. In order to understand this, it is convenient to develop a minimizer  $\mathbf{m}_\varepsilon$  of  $\mathcal{E}_\varepsilon$  under the so-called multiscale expansion

$$\mathbf{m}_\varepsilon(x) = \mathbf{m}_0(x) + \varepsilon \mathbf{m}_1\left(x, \frac{x}{\varepsilon}\right) + o(\varepsilon), \tag{28}$$

where  $\mathbf{m}_0, \mathbf{m}_1$  are respectively a minimizer of the  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon$  and the null average first order corrector. Clearly, due to the constraint  $\mathbf{m}_\varepsilon(x) \in \mathcal{M}$  for a.e.  $x \in \Omega$ , we get

$$0 \equiv \mathbf{n}[\mathbf{m}_\varepsilon] \cdot \nabla \mathbf{m}_\varepsilon, \tag{29}$$

where we have denoted by  $\mathbf{n}$  the local normal field defined around  $\mathbf{m}_\varepsilon(x) \in \mathcal{M}$ . By passing to the two-scale limit in both terms of (29), we formally reach the equality  $0 \equiv \mathbf{n}[\mathbf{m}_0] \cdot (\nabla \mathbf{m}_0 + \nabla_y \mathbf{m}_1) \equiv \mathbf{n}[\mathbf{m}_0] \cdot \nabla_y \mathbf{m}_1$ , which shows that  $\mathbf{n}[\mathbf{m}_0(x)] \cdot \mathbf{m}_1(x, y)$  does not depend on  $y$ . Then, passing to the average over  $Q$  we get  $\mathbf{m}_1(x, y) \in T_{\mathbf{m}_0(x)}(\mathcal{M})$ . The rigorous formulation of the previous idea is the object of the next PROPOSITION:

PROPOSITION 4.2 Let  $\mathcal{M}$  be a connected smooth submanifold of  $\mathbb{R}^3$ , and let  $(\mathbf{m}_\varepsilon)$  be a sequence in  $H^1(\Omega, \mathcal{M})$  that converges weakly to a limit  $\mathbf{m} \in H^1(\Omega, \mathbb{R}^3)$ . Then

$$\mathbf{m} \in H^1(\Omega, \mathcal{M}) \quad \text{and} \quad \mathbf{m}_\varepsilon \rightharpoonup \mathbf{m}.$$

Moreover there exists a null average function  $\mathbf{v} \in L^2[\Omega; H^1_\#(Q)/\mathbb{R}]$  such that, up to the extraction of a a subsequence:

$$\nabla \mathbf{m}_\varepsilon \rightharpoonup \nabla \mathbf{m} + \nabla_y \mathbf{v} \quad \text{and} \quad \mathbf{v}(x, y) \in T_{\mathbf{m}_0(x)}(\mathcal{M}) \text{ for a.e. } (x, y) \in \Omega \times Q.$$

PROOF 4.3 In view of Proposition 2.8, we only need to prove that  $\mathbf{v}(x, y) \in T_{\mathbf{m}_0(x)}(\mathcal{M})$  for a.e.  $(x, y) \in \Omega \times Q$ . To this end, let us denote by  $\mathbf{n}(\mathbf{m})$  the normal vector at  $\mathbf{m} \in \mathcal{M}$  and observe that it is sufficient to prove that the scalar function  $\mathbf{n}(\mathbf{m}(x)) \cdot \mathbf{v}(x, y)$  does not depend on the  $y$  variable, i.e. that in the language of distributions on  $\Omega \times Y$  one has

$$\int_{\Omega \times Q} [\mathbf{n}(\mathbf{m}(x)) \cdot \mathbf{v}(x, y)] \operatorname{div}_y \boldsymbol{\varphi}(x, y) dy dx = 0 \quad \forall \boldsymbol{\varphi} \in \mathcal{D}[\Omega; C^\infty_\#(Q)]. \quad (30)$$

Indeed, as far as  $\mathbf{n}(\mathbf{m}(x)) \cdot \mathbf{v}(x, y)$  is independent from the  $y$  variable, since by assumption  $\langle \mathbf{v}(x, \cdot) \rangle_Q = 0$  for a.e.  $x \in \Omega$ , one has  $\mathbf{n}(\mathbf{m}(x)) \cdot \mathbf{v}(x, y) = \mathbf{n}(\mathbf{m}(x)) \cdot \langle \mathbf{v}(x, \cdot) \rangle_Q = 0$  and therefore  $\mathbf{v}(x, y) \in T_{\mathbf{m}(x)}(\mathcal{M})$  for a.e.  $(x, y) \in \Omega \times Q$ .

To prove (30) we note that since  $\mathbf{m}_\varepsilon \rightarrow \mathbf{m}$  in  $L^2(\Omega)$ , one also has  $\mathbf{n}(\mathbf{m}_\varepsilon) \rightarrow \mathbf{n}(\mathbf{m})$  in  $L^2(\Omega)$ . Therefore the family  $\mathbf{n}(\mathbf{m}_\varepsilon)$  strongly two-scale converges to  $\mathbf{n}(\mathbf{m})$  and moreover  $\mathbf{0} = (\mathbf{n}(\mathbf{m}_\varepsilon) \cdot \nabla, \mathbf{m}_\varepsilon)_\Omega \rightarrow (\mathbf{n}(\mathbf{m}) \cdot \nabla, \mathbf{m})_\Omega$ . Hence, due to Proposition 2.10 we get

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} [\mathbf{m}_\varepsilon(x) \cdot \nabla \mathbf{m}_\varepsilon(x)] \cdot \boldsymbol{\varphi}(x, x/\varepsilon) dx \\ &= \int_{\Omega \times Q} \mathbf{m}(x) \cdot [\nabla \mathbf{m}(x) + \nabla_y \mathbf{v}(x, y)] \cdot \boldsymbol{\varphi}(x, y) dy dx \\ &= \int_{\Omega \times Q} [\mathbf{m}(x) \cdot \nabla_y \mathbf{v}(x, y)] \cdot \boldsymbol{\varphi}(x, y) dy dx \\ &= - \int_{\Omega \times Q} [\mathbf{m}(x) \cdot \mathbf{v}(x, y)] \operatorname{div}_y \boldsymbol{\varphi}(x, y) dx, \end{aligned}$$

i.e. the desired relation (30).

#### 4.1.2 The role of tangentially homogenized energy density

Although the following considerations are not completely rigorous, they give a very good explanation of the idea behind the expression of the tangentially homogenized energy density (27); they are mainly based on the notion of two-scale convergence while [5] relies on  $\Gamma$ -convergence.

Let us denote by  $(\mathcal{E}_\mathcal{M}^\varepsilon)$  the family of functionals

$$\mathbf{m} \in (H^1(\Omega, \mathcal{M}), d_{L^2(\Omega, \mathcal{M})}) \mapsto \int_{\Omega} a_{\text{ex}} \left( \frac{x}{\varepsilon} \right) |\nabla \mathbf{m}(x)|^2 dx, \quad (31)$$

and let us denote by  $\mathcal{E}_\mathcal{M}$  its  $\Gamma$ -limit in the metric space  $(H^1(\Omega, \mathcal{M}), d_{L^2(\Omega, \mathcal{M})})$ . Since  $H^1(\Omega, \mathcal{M})$  is a metric subspace of  $H^1(\Omega, \mathbb{R}^3)$ , from the properties of  $\Gamma$ -convergence under restriction to subspaces (see [9]), we know that for every  $\mathbf{m}_0 \in H^1(\Omega, \mathcal{M})$  and for every  $(\mathbf{m}_\varepsilon) \in \mathcal{C}_{d_{L^2(\Omega, \mathcal{M})}}(\mathbf{m}_0)$

$$\mathcal{E}_{\mathbb{R}^3}(\mathbf{m}_0) \leq \mathcal{E}_\mathcal{M}(\mathbf{m}_0) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\mathcal{M}^\varepsilon(\mathbf{m}_\varepsilon). \quad (32)$$

Now, let  $\mathcal{M}$  be a convex smooth manifold, i.e. a smooth manifold lying on some subset of the boundary of a **convex bounded domain**  $\Theta_\mathcal{M}$ . Next, define the **tangent cone** of  $\mathcal{M}$  at  $s \in \mathcal{M}$  by the position

$$T_s(\mathcal{M}) := \{ \mathbf{v} \in \mathbb{R}^3 : \forall \lambda \geq 0, s + \lambda \mathbf{v} \notin \Theta_\mathcal{M} \},$$

and denote by  $\Pi_\mathcal{M}$  the nearest point projection on  $\Theta_\mathcal{M}$ .

Since  $\Pi_\mathcal{M}$  is a (Lipschitz) non-expansive map, one has  $\Pi_\mathcal{M}[\mathbf{u}] \in H^1(\Omega, \mathcal{M})$  for every  $\mathbf{u} \in H^1(\Omega, \Theta_\mathcal{M}^c)$ , moreover (see [4])

$$|\nabla \Pi_\mathcal{M}[\mathbf{u}]| \leq |\nabla \mathbf{u}| \quad \tau\text{-a.e. in } \Omega. \quad (33)$$

Let us now suppose that  $\mathbf{m}_0$  is sufficiently smooth so that for every test function  $\mathbf{m}_1 \in \mathcal{D}[\Omega; C_{\#}^{\infty}(Q, T_{\mathbf{m}_0}(\mathcal{M}))]$ , the family  $\mathbf{m}_{\varepsilon}(x) := \mathbf{m}_0(x) + \varepsilon \mathbf{m}_1(x, x/\varepsilon)$  belongs to  $H^1(\Omega, \Theta_{\mathcal{M}})$ . In this hypothesis one has

$$\Pi_{\mathcal{M}}[\mathbf{m}_{\varepsilon}] \rightarrow \mathbf{m}_0 \quad \text{in } (H^1(\Omega, \mathcal{M}), d_{L^2(\Omega, \mathcal{M})}).$$

Therefore, taking into account the estimates (33), (32) and the fact that  $\nabla_y \mathbf{m}_1$  is an admissible test function (see [3], Remark 1.11), we get (passing to the two-scale limit):

$$\begin{aligned} \mathcal{E}_{\mathbb{R}^3}(\mathbf{m}_0) &\leq \mathcal{E}_{\mathcal{M}}(\mathbf{m}_0) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_{\mathcal{M}}^{\varepsilon}(\Pi_{\mathcal{M}}[\mathbf{m}_{\varepsilon}]) \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_{\Omega} a_{\text{ex}}\left(\frac{x}{\varepsilon}\right) \left| \nabla \mathbf{m}_0(x) + \nabla_y \mathbf{m}_1\left(x, \frac{x}{\varepsilon}\right) \right|^2 dx \\ &= \int_{\Omega} \left[ \int_Q a_{\text{ex}}(y) |\nabla \mathbf{m}_0(x) + \nabla_y \mathbf{m}_1(x, y)|^2 dy \right] dx. \end{aligned}$$

Since  $\mathbf{m}_1 \in \mathcal{D}[\Omega; C_{\#}^{\infty}(Q, T_{\mathbf{m}_0}(\mathcal{M}))]$  is an arbitrary test function, passing to the infimum we finish with the following upper and lower bound for the manifold constrained homogenized functional:

$$\mathcal{E}_{\mathbb{R}^3}(\mathbf{m}_0) \leq \mathcal{E}_{\mathcal{M}}(\mathbf{m}_0) \leq \int_{\Omega} \left[ \inf_{\varphi \in C_{\#}^{\infty}(Q, T_s(\mathcal{M}))} \mathcal{I}[\xi, \varphi] \right]_{(s, \xi) := (\mathbf{m}_0(x), \nabla \mathbf{m}_0(x))} dx, \quad (34)$$

with

$$\mathcal{I}[\xi, \varphi] := \frac{1}{|Q|} \int_Q a_{\text{ex}}(y) |\xi + \nabla \varphi(y)|^2 dy. \quad (35)$$

Since the functional  $\mathcal{I}[\xi, \cdot] : C_{\#}^{\infty}(Q, T_s(\mathcal{M})) \rightarrow \mathbb{R}^+$  is continuous with respect to the  $H_{\#}^1(Q, T_s(\mathcal{M}))$  norm and  $C_{\#}^{\infty}(Q, T_s(\mathcal{M}))$  is dense in  $H_{\#}^1(Q, T_s(\mathcal{M}))$ , the infimum in (34) can be taken over  $H_{\#}^1(Q, T_s(\mathcal{M}))$ .

REMARK 4.4 Quite remarkably, as we prove below, when  $\mathcal{M} := \mathbb{S}^2$  the infimum appearing in the right-hand side of (34) does not depend on the  $s$ -variable and coincides with the infimum in the left-hand side of (34). Thus for  $\mathcal{M} := \mathbb{S}^2$  also the lower bound to  $\mathcal{E}_{\mathcal{M}}$  is sharp and  $\mathcal{E}_{\mathbb{R}^3} \equiv \mathcal{E}_{\mathbb{S}^2}$ .

## 4.2 The tangentially homogenized Exchange Energy $\mathcal{E}_{\text{hom}}$

Let us go back to the application of the tangentially homogenization result in our setting. We consider the family of exchange energy functionals, all defined in  $H^1(\Omega, \mathbb{S}^2)$ , given by  $(\mathcal{E}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ . Since  $[\mathbf{H}_1]$  holds, Proposition 4.1 ensures that the family  $(\mathcal{E}_{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$   $\Gamma$ -converges in the metric space  $(H^1(\Omega; \mathbb{S}^2), d_{L^2(\Omega, \mathbb{S}^2)})$ , i.e. with respect to the topology induces on  $H^1(\Omega, \mathbb{S}^2)$  by the strong  $L^2(\Omega, \mathbb{R}^3)$  topology, to the functional

$$\mathcal{E}_{\text{hom}} : H^1(\Omega, \mathbb{S}^2) \rightarrow \mathbb{R}^+ \quad , \quad \mathbf{m} \mapsto \mathcal{E}_{\text{hom}}(\mathbf{m}) = \int_{\Omega} Tg_{\text{hom}}(\mathbf{m}, \nabla \mathbf{m}) d\tau, \quad (36)$$

where for every  $s \in \mathbb{S}^2$  and every  $\xi \in [T_s(\mathbb{S}^2)]^3$ ,

$$Tg_{\text{hom}}(s, \xi) = \lim_{t \rightarrow \infty} \left[ \inf_{\varphi \in W_0^{1, \infty}(Q_t, T_s(\mathbb{S}^2))} \mathcal{I}_t[\xi, \varphi] \right] \quad (37)$$

with

$$\mathcal{I}_t[\xi, \varphi] := \frac{1}{|Q_t|} \int_{Q_t} a_{\text{ex}}(y) |\xi + \nabla \varphi(y)|^2 dy.$$

Equivalently, since  $\mathcal{I}_t[\xi, \cdot] : H_0^1(Q_t, T_s(\mathbb{S}^2)) \rightarrow \mathbb{R}^+$  is a continuous functional, and the subspace  $W_0^{1, \infty}(Q_t, T_s(\mathbb{S}^2))$  is dense in  $H_0^1(Q_t, T_s(\mathbb{S}^2))$ , we have for every  $s \in \mathbb{S}^2$  and every  $\xi \in [T_s(\mathbb{S}^2)]^3$

$$\inf_{\varphi \in W_0^{1, \infty}(Q_t, T_s(\mathbb{S}^2))} \mathcal{I}_t[\xi, \varphi] = \inf_{\varphi \in H_0^1(Q_t, T_s(\mathbb{S}^2))} \mathcal{I}_t[\xi, \varphi], \quad (38)$$

and hence

$$Tg_{\text{hom}}(s, \xi) = \lim_{t \rightarrow \infty} \left[ \inf_{\varphi \in H_0^1(Q_t, T_s(\mathbb{S}^2))} \mathcal{I}_t[\xi, \varphi] \right]. \quad (39)$$

This latter formulation is a more convenient one. Indeed, it can be easily verified, by LAX-MILGRAM theorem, that for every  $t \in \mathbb{N}$ , every  $s \in \mathbb{S}^2$  and every  $\xi \in [T_s(\mathbb{S}^2)]^3$ , there exists a unique solution  $\varphi_t(s, \xi) \in H_0^1(Q_t, T_s(\mathbb{S}^2))$  of the problem

$$\varphi_t(s, \xi) := \operatorname{argmin}_{\varphi \in H_0^1(Q_t, T_s(\mathbb{S}^2))} \mathcal{I}_t[\xi, \varphi]. \quad (40)$$

Moreover, for every  $t \in \mathbb{N}$  and every  $s \in \mathbb{S}^2$ , the function

$$\xi \in [T_s(\mathbb{S}^2)]^3 \mapsto \varphi_t(s, \xi) \in W_0^{1,\infty}(Q_t, T_s(\mathbb{S}^2)), \quad (41)$$

is a (continuous) linear map.

Let us now consider the problem that defines the **classical homogenization problem** (see [7, 21, 25]), namely

$$g_{\text{hom}}(\xi) := \lim_{t \rightarrow \infty} \left[ \inf_{\varphi \in H_0^1(Q_t, \mathbb{R}^3)} \mathcal{I}_t[\xi, \varphi] \right]. \quad (42)$$

Our aim is now to prove that the natural extension of  $g_{\text{hom}}$  to the tangent bundle  $[\mathcal{T}(\mathbb{S}^2)]^3 := \sqcup_{s \in \mathbb{S}^2} [T_s(\mathbb{S}^2)]^3$  coincides with the tangentially homogenized energy density  $Tg_{\text{hom}}(s, \xi)$ . To this end we observe that in the «classical» problem (42), the function space among which the minimization takes place is bigger than the one involved in the original problem (39), so that  $g_{\text{hom}}(\xi) \leq Tg_{\text{hom}}(s, \xi)$  for every  $(s, \xi) \in [\mathcal{T}(\mathbb{S}^2)]^3$ .

To prove that  $g_{\text{hom}}(\xi) \equiv Tg_{\text{hom}}(s, \xi)$  on  $[\mathcal{T}(\mathbb{S}^2)]^3$  it is thus sufficient to show that for every  $t \in \mathbb{N}$  and for every  $(s, \xi) \in [\mathcal{T}(\mathbb{S}^2)]^3$  there exists a function  $\psi_{t,\xi} \in H_0^1(Q_t, T_s(\mathbb{S}^2))$  such that  $\mathcal{I}_t[\xi, \psi_{t,\xi}] \leq g_{\text{hom}}(\xi)$ . Having this goal in mind, we observe that if  $\varphi_{t,\xi}$  is the unique solution of the minimization problem arising in (42), denoting by  $\psi_{t,\xi} := \varphi_{t,\xi} - (\varphi_{t,\xi} \cdot s)s$  the nearest point projection of  $\varphi_{t,\xi}$  on  $[T_s(\mathbb{S}^2)]^3$ , one has  $\psi_{t,\xi} \in H_0^1(Q_t, T_s(\mathbb{S}^2))$  and

$$\begin{aligned} |\xi + \nabla \varphi_{t,\xi}|^2 &= |\xi + \nabla \psi_{t,\xi}|^2 + |s \otimes \nabla(\varphi_{t,\xi} \cdot s)|^2 + 2(\xi + \nabla \psi_{t,\xi}) : s \otimes \nabla(\varphi_{t,\xi} \cdot s) \\ &= |\xi + \nabla \psi_{t,\xi}|^2 + |\nabla(\varphi_{t,\xi} \cdot s)|^2 + 2(\xi + \nabla \psi_{t,\xi}) \nabla(\varphi_{t,\xi} \cdot s) \cdot s \\ &= |\xi + \nabla \psi_{t,\xi}|^2 + |\nabla(\varphi_{t,\xi} \cdot s)|^2 \end{aligned}$$

where, in deriving the previous equalities, we have used the relations  $|a \otimes b|^2 = |a|^2|b|^2$ ,  $(A : B(a \otimes b)) = Ab \cdot Ba$ , and the orthogonality relations  $[\nabla \psi_{t,\xi}]^T s = 0$ ,  $\xi^T s = 0$ . Thus we have

$$\begin{aligned} \mathcal{I}_t[\xi, \psi_{t,\xi}] &= \frac{1}{|Q_t|} \int_{Q_t} a_{\text{ex}}(y) |\xi + \nabla \varphi_{t,\xi}(y)|^2 dy \\ &\quad - \frac{1}{|Q_t|} \int_{Q_t} a_{\text{ex}}(y) |\nabla(\varphi_{t,\xi} \cdot s)(y)|^2 dy \\ &= \mathcal{I}_t[\xi, \varphi_{t,\xi}] - \frac{1}{|Q_t|} \int_{Q_t} a_{\text{ex}}(y) |\nabla(\varphi_{t,\xi} \cdot s)(y)|^2 dy \\ &\leq \mathcal{I}_t[\xi, \varphi_{t,\xi}], \end{aligned}$$

and therefore, passing to the limit for  $t \rightarrow \infty$  we get  $g_{\text{hom}}(\xi) \equiv Tg_{\text{hom}}(s, \xi)$ . In particular  $Tg_{\text{hom}}(s, \xi)$  does not depend on  $s$ , and is given by (42).

**REMARK 4.5** The fact that tangential homogenization energy density  $Tg_{\text{hom}}(s, \xi)$  reduces to the «classical» one (i.e.  $g_{\text{hom}}$ ), which does not depend from the  $s$ -variable, is a quite remarkable fact. Indeed, it is possible to build elementary examples where the dependence on the  $s$ -variable in the tangential homogenization energy density is explicit (cfr. [5]). The independence from the  $s$ -variable in our framework, is mainly due to the very particular situation that for every  $y \in \mathbb{R}^3$ , the Carathéodory function  $g(y, \cdot) = a_{\text{ex}}(y) |\cdot|^2$  is invariant under the rotation group of the manifold under consideration (the 2-sphere  $\mathbb{S}^2$ ).

To complete the proof concerning the exchange energy part stated in Theorem 1.1, it is sufficient to recall that  $g_{\text{hom}}$  is a quadratic form in  $\xi$  (see [7, 21]), i.e. that there exists a symmetric and positive definite matrix  $A_{\text{hom}} \in \mathbb{R}^{3 \times 3}$  such that for every  $\xi \in \mathbb{R}^{3 \times 3}$

$$g_{\text{hom}}(\xi) = A_{\text{hom}} \xi : \xi. \quad (43)$$

Precisely, it is possible to show that the «classical» problem (42) may be equivalently reformulated by replacing homogeneous boundary conditions by periodic boundary conditions (see [25] and Appendix A), so that for every  $(s, \xi) \in [\mathcal{T}(\mathbb{S}^2)]^3$

$$Tg_{\text{hom}}(s, \xi) = g_{\text{hom}}(\xi) = \lim_{t \rightarrow \infty} \left[ \inf_{\varphi \in H_{\#}^1(Q_t, \mathbb{R}^3)} \mathcal{I}_t[\xi, \varphi] \right]. \quad (44)$$

Furthermore, from the convexity of the integrand, it is routinely seen that we can replace the limit for  $t \rightarrow \infty$  by the computation on the unit cell (though still with periodic boundary conditions). We are therefore left (see [7] and Appendix A) for every  $(s, \xi) \in [\mathcal{T}(\mathbb{S}^2)]^3$  with

$$Tg_{\text{hom}}(s, \xi) = g_{\text{hom}}(\xi) = \inf_{\varphi \in H_{\#}^1(Q, \mathbb{R}^3)} \int_Q a_{\text{ex}}(y) |\xi + \nabla \varphi(y)|^2 dy. \quad (45)$$

Now, LAX-MILGRAM theorem shows that for every  $\xi \in [T_s(\mathbb{S}^2)]^3$  there exists a unique solution  $\varphi_\xi$  of this latter problem, up to an additive constant that we may fix by requiring that  $\langle \varphi_\xi(y) \rangle_Q = \mathbf{0}$ . Moreover, the map  $\xi \in [T_s(\mathbb{S}^2)]^3 \mapsto \varphi_\xi \in H_{\#}^1(Q, \mathbb{R}^3)$  is a (continuous) *linear* map, and a direct computation, shows that  $A_{\text{hom}}$  can be expressed as:

$$A_{\text{hom}} := \langle a_{\text{ex}}(y)(I + \nabla\varphi(y))^T(I + \nabla\varphi(y)) \rangle_Q, \quad \varphi := (\varphi_1, \varphi_2, \varphi_3) \quad (46)$$

where for every  $j \in \mathbb{N}_3$  the component  $\varphi_j$  is the unique (up to a constant) solution of the *scalar unit cell problem*

$$\varphi_j := \underset{\varphi \in W_{\#}^{1,\infty}(Q, \mathbb{R}^3)}{\text{argmin}} \int_Q a_{\text{ex}}(y) [|\nabla\varphi(y) + e_j|^2] dx. \quad (47)$$

REMARK 4.6 By repeating the same argument followed above to deduce the equality  $Tg_{\text{hom}}(s, \xi) = g_{\text{hom}}(\xi)$ , it is now simple to show that the tangentially homogenized exchange energy density can be equivalently characterized by the position

$$Tg_{\text{hom}}(s, \xi) = \inf_{\varphi \in H_{\#}^1(Q, T_s(\mathbb{S}^2))} \int_Q a_{\text{ex}}(y) |\xi + \nabla\varphi(y)|^2 dy,$$

which is exactly the one deduced in equation (34).

## 5 The periodic homogenization of the demagnetizing field

This Section is devoted to show that the family of magnetostatic self-energies  $(\mathcal{W}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  continuously converges to  $\mathcal{W}_{\text{hom}}$ . To this end, let us first recall some essential facts concerning the demagnetizing field operator.

### 5.1 The Beppo-Levi space and the variational formulation for the demagnetizing field

From the mathematical point of view, assuming  $\Omega$  to be open, bounded and with a Lipschitz boundary, a given magnetization  $\mathbf{m} \in L^2(\Omega, \mathbb{R}^3)$  generates the stray field  $\mathbf{h}_d[\mathbf{m}] = \nabla u_{\mathbf{m}}$  where the potential  $u_{\mathbf{m}}$  solves:

$$\Delta u_{\mathbf{m}} = -\text{div}(\mathbf{m}\chi_\Omega) \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \quad (48)$$

In (48) we have denoted by  $\mathbf{m}\chi_\Omega$  the extension of  $\mathbf{m}$  to  $\mathbb{R}^3$  that vanishes outside  $\Omega$ .

Once introduced the weight  $\omega(x) = (1 + |x|^2)^{-1/2}$ , and the weighted Lebesgue space

$$L_\omega^2(\Omega) = \{u \in \mathcal{D}'(\Omega) : u\omega \in L^2(\Omega)\},$$

we define the BEPPO-LEVI space

$$BL^1(\Omega) = \{u \in \mathcal{D}'(\Omega) : u \in L_\omega^2(\Omega) \text{ and } \nabla u \in L^2(\Omega, \mathbb{R}^3)\}, \quad (49)$$

which is an Hilbert space when endowed with the scalar product  $(u, v)_{BL^1(\mathbb{R}^3)} := (\nabla u, \nabla v)_\Omega$ . It is straightforward to show, by the means of LAX-MILGRAM theorem, the existence and uniqueness of the solution of the variational formulation associated to equation (48): namely to find a potential  $u_{\mathbf{m}} \in BL^1(\mathbb{R}^3)$  such that for all  $\varphi \in BL^1(\mathbb{R}^3)$

$$(u_{\mathbf{m}}, \varphi)_{BL^1(\mathbb{R}^3)} := \int_{\mathbb{R}^3} \nabla u_{\mathbf{m}} \cdot \nabla \varphi d\tau = - \int_{\mathbb{R}^3} \mathbf{m} \cdot \nabla \varphi d\tau =: F_{\mathbf{m}}[\varphi]. \quad (50)$$

Thus, for every  $\mathbf{m} \in L^2(\mathbb{R}^3)$  there exists a unique  $u_{\mathbf{m}} \in BL^1(\mathbb{R}^3)$  such that (50) holds, and moreover, the following stability estimate holds:

$$\|u_{\mathbf{m}}\|_{BL^1(\mathbb{R}^3)} \equiv \|\nabla u_{\mathbf{m}}\|_\Omega^2 \leq \sup_{\substack{\varphi \in BL^1(\mathbb{R}^3) \\ \|\nabla \varphi\|_\Omega = 1}} |F_{\mathbf{m}}[\varphi]| \leq \|\mathbf{m}\|_{L^2(\mathbb{R}^3)}. \quad (51)$$

The quantity  $\mathbf{h}_d[\mathbf{m}] := \nabla u_{\mathbf{m}}$  is what is referred to as the demagnetizing field, and it is a linear and continuous operator from  $L^2(\mathbb{R}^3, \mathbb{R}^3)$  into  $L^2(\mathbb{R}^3, \mathbb{R}^3)$ . In particular  $\mathbf{m}\chi_\Omega \in L^2(\mathbb{R}^3)$  for every  $\mathbf{m} \in L^2(\Omega)$  and therefore  $\mathbf{h}_d$  is also a linear and continuous operator from  $L^2(\Omega, \mathbb{R}^3)$  into  $L^2(\mathbb{R}^3, \mathbb{R}^3)$ .

## 5.2 The two-scale limit of the demagnetizing field

In this subsection we make use of the notion of two-scale convergence, to characterize the behavior of the demagnetizing field operator under two-scale convergence. More precisely, we suppose to have a bounded sequence  $(\mathbf{m}_\varepsilon) \in L^2(\mathbb{R}^3)$  which two scale converges to some  $\mathbf{m}_\infty(x, y) \in L^2$  and we want to understand if the two-scale limit of the sequence  $\mathbf{h}_d[\mathbf{m}_\varepsilon]$  exists, and in the affirmative case to characterize in some analytic sense such a limit. This problem has already been treated in [30] although without justifying the use of two-scale compactness results in weighted space. That is why in this subsection we start by proving two compactness results concerning two-scale convergence in the weighted spaces  $L_\omega^2(\mathbb{R}^3)$  and  $BL^1(\mathbb{R}^3)$ .

The first one is a «weighted» variant of the compactness result stated in Proposition 2.7, and shows that a notion of two-scale convergence in  $L_\omega^2(\mathbb{R}^3)$  makes sense.

**PROPOSITION 5.1** Let  $(u_\varepsilon)$  be bounded sequence in  $L_\omega^2(\mathbb{R}^3)$ . There exists a function  $u \in L_{\text{loc}}^2(\mathbb{R}^3 \times Q)$  such that  $\langle u \rangle_Q \in L_\omega^2(\mathbb{R}^3)$  and, up to the extraction of a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} u_\varepsilon(x) \varphi(x, x/\varepsilon) dx = \int_{\mathbb{R}^3 \times Q} u(x, y) \varphi(x, y) dy dx \quad (52)$$

for all  $\varphi \in \mathcal{D}[\mathbb{R}^3; C_\#^\infty(Q)]$ . In this case we say that the sequence  $(u_\varepsilon)$   $L_\omega^2$ -two-scale converges to  $u$ .

**PROOF 5.2** Since  $(u_\varepsilon)$  is bounded in the Hilbert space  $L_\omega^2(\mathbb{R}^3)$ , there exists an element  $u_\infty \in L_\omega^2(\mathbb{R}^3)$  and a sequence  $(u_{\varepsilon(n)}) \subseteq (u_\varepsilon)$  such that

$$u_{\varepsilon(n)} \rightharpoonup u_\infty \quad \text{weakly in } L_\omega^2(\mathbb{R}^3). \quad (53)$$

This implies that for every bounded domain  $\Omega \subseteq \mathbb{R}^3$ , one has  $u_{\varepsilon(n)} \rightharpoonup u_\infty$  weakly in  $L^2(\Omega)$ . We now consider a sequence of bounded domain  $(\Omega_i)_{i \in \mathbb{N}}$  covering  $\mathbb{R}^3$ . Let us start with the index  $i = 1$ , i.e. with  $\Omega_1$ . According to the two-scale compactness result (see Proposition 2.7), there exists a subsequence  $u_{\varepsilon(n_{k_1})}$  and an element  $u_1 \in L^2(\Omega_1 \times Q)$  such that

$$u_{\varepsilon(n_{k_1})} \rightharpoonup u_1 \quad \text{in } L^2(\Omega_1 \times Q). \quad (54)$$

Now we consider  $i = 2$ , i.e.  $\Omega_2$ . Since  $u_{\varepsilon(n_{k_1})} \rightharpoonup u_\infty$  weakly in  $L_\omega^2(\mathbb{R}^3)$ , it is possible to extract a further subsequence  $(u_{\varepsilon(n_{k_2})})$  from  $u_{\varepsilon(n_{k_1})}$  such that  $u_{\varepsilon(n_{k_2})} \rightharpoonup u_2$  in  $L^2(\Omega_2 \times Q)$  for some suitable  $u_2 \in L^2(\Omega_2 \times Q)$ . Moreover, due to the unicity of the two-scale limit, one has

$$u_1|_{(\Omega_1 \cap \Omega_2) \times Q} \equiv u_2|_{(\Omega_1 \cap \Omega_2) \times Q} \quad (55)$$

whenever  $\Omega_1 \cap \Omega_2 \neq \emptyset$ . Proceeding in this way, we find for every  $i \in \mathbb{N}$  a subsequence  $u_{\varepsilon(n_{k_i})}$  such that

$$n_{k_i} \subseteq n_{k_{i-1}} \quad \text{and} \quad u_{\varepsilon(n_{k_i})} \rightharpoonup u_i \quad (56)$$

for some  $u_i \in L^2(\Omega_i \times Q)$ . We then define the **diagonal sequence** of indices defined by

$$n_{k_\infty(1)} := n_{k_1(1)}, n_{k_\infty(2)} := n_{k_2(2)}, \dots, n_{k_\infty(i)} = n_{k_i(i)}, \dots \quad (57)$$

From (56) we get that for every  $i \in \mathbb{N}$ , up to the first  $i - 1$  terms, the sequence of indices  $n_{k_\infty}$  is included in  $n_{k_i}$ , and this means that for every  $i \in \mathbb{N}$

$$u_{\varepsilon(n_{k_\infty})} \rightharpoonup u_i \quad \text{in } L^2(\Omega_i \times Q). \quad (58)$$

By observing again that  $u_i|_{(\Omega_i \cap \Omega_j) \times Q} \equiv u_j|_{(\Omega_i \cap \Omega_j) \times Q}$  if  $\Omega_i \cap \Omega_j \neq \emptyset$ , from the «*principe du recollement des morceaux*» (cfr. [31]) there exists a unique distribution  $u \in L_{\text{loc}}^2(\mathbb{R}^3 \times Q)$  such that  $u|_{\Omega_i \times Q} \equiv u_i$ , and therefore

$$\lim_{k_\infty \rightarrow \infty} \int_{\mathbb{R}^3} u_{\varepsilon(n_{k_\infty})}(x) \varphi(x, x/\varepsilon) dx = \int_{\mathbb{R}^3 \times Q} u(x, y) \varphi(x, y) dy dx \quad (59)$$

for every  $\varphi \in \mathcal{D}[\mathbb{R}^3; C_\#^\infty(Q)]$ . Moreover since

$$u_{\varepsilon(n_{k_\infty})} \rightharpoonup u_\infty \quad \text{in } L_\omega^2(\mathbb{R}^3) \quad (60)$$

and

$$\forall i \in \mathbb{N} \quad u_{\varepsilon(n_{k_\infty})} \rightharpoonup \langle u(x, y) \rangle_Q \quad \text{in } L^2(\Omega_i) \quad (61)$$

we get also  $\langle u(x, y) \rangle_Q \equiv u_\infty \in L_\omega^2(\mathbb{R}^3)$ . This completes the proof.

Exactly with the same diagonal argument, it is possible to prove the weighted variant of the compactness result stated in Proposition 2.8.

**PROPOSITION 5.3** Let  $(u_\varepsilon)$  be bounded sequence in  $BL^1(\mathbb{R}^3)$  weakly convergent to  $u_\infty$ . Then  $u_\varepsilon$   $L_\omega^2$ -two-scale converges to  $u_\infty \in L_\omega^2(\mathbb{R}^3)$  and there exists a function  $v \in L^2[\mathbb{R}^3; H_\#^1(Q)/\mathbb{R}]$  such that, up to the extraction of a subsequence:

$$\nabla u_\varepsilon \rightharpoonup \nabla u_\infty + \nabla_y v. \quad (62)$$

**PROOF 5.4** We start by observing that since  $u_{\varepsilon(n)} \rightharpoonup u_\infty$  weakly in  $BL^1(\mathbb{R}^3)$ ,  $u_{\varepsilon(n)} \rightharpoonup u_\infty$  in  $L_\omega^2(\mathbb{R}^3)$ , and therefore, according to the previous proposition, there exists a function  $u \in L_\omega^2(\mathbb{R}^3 \times Q)$  such that, up to a subsequence,

$$\begin{aligned} u_{\varepsilon(n)} &\rightharpoonup u(x, y) && \text{in } L_\omega^2(\mathbb{R}^3 \times Q) \\ u_{\varepsilon(n)} &\rightharpoonup u_\infty(x) \equiv \langle u(x, y) \rangle_Q && \text{in } L_\omega^2(\mathbb{R}^3). \end{aligned} \quad (63)$$

We now consider a sequence of bounded domain  $(\Omega_i)_{i \in \mathbb{N}}$ . Proceeding as in the proof of the previous Proposition 5.1, one proves that for every  $i \in \mathbb{N}$  there exists a subsequence  $u_{\varepsilon(n_{k_i})}$  such that

$$n_{k_i} \subseteq n_{k_{i-1}} \quad \text{and} \quad u_{\varepsilon(n_{k_i})} \rightharpoonup u_\infty(x) \equiv \langle u(x, y) \rangle_Q \equiv u(x, y) \quad \text{in } L^2(\Omega_i \times Q). \quad (64)$$

We then define the diagonal sequence of indices defined by the position  $n_{k_\infty(i)} = n_{k_i(i)}$ . From (64) we get that for every  $i \in \mathbb{N}$ , up to the first  $i - 1$  terms, the sequence of indices  $n_{k_\infty}$  is included in  $n_{k_i}$ , and this means that for every  $i \in \mathbb{N}$

$$u_{\varepsilon(n_{k_\infty})} \rightharpoonup u_\infty(x) \equiv \langle u(x, y) \rangle_Q \equiv u(x, y) \quad \text{in } L^2(\Omega_i \times Q). \quad (65)$$

Thus  $u \equiv u_\infty \in L_\omega^2(\mathbb{R}^3)$  in  $\mathbb{R}^3$ .

Next, we observe that since  $u_{\varepsilon(n)} \rightharpoonup u_\infty$  weakly in  $BL^1(\mathbb{R}^3)$  we have  $\nabla u_{\varepsilon(n)} \rightharpoonup \nabla u_\infty$  and  $(\nabla u_{\varepsilon(n)})$  bounded in  $L^2(\mathbb{R}^3)$ . Thus, according to the classical two-scale compactness result (see Proposition 2.8) there exists a function  $\kappa_\infty \in L^2(\mathbb{R}^3 \times Q)$  such that, up to a subsequence,

$$\nabla u_\varepsilon \rightharpoonup \kappa_\infty \quad \text{in } L^2(\mathbb{R}^3, \mathbb{R}^3). \quad (66)$$

Thus, for any test function  $[\varphi \otimes \psi_\#](x, y) := \varphi(x)\psi_\#(y) \in \mathcal{D}[\mathbb{R}^3; C_\#^\infty(Q)]$  with  $\text{div}_y \psi_\#(y) = 0$  one has

$$\int_{\mathbb{R}^3} u_\varepsilon(x) \text{div}_x [\varphi \otimes \psi_\#](x, x/\varepsilon) dx = - \int_{\mathbb{R}^3} \nabla u_\varepsilon(x) \cdot [\varphi \otimes \psi_\#](x, x/\varepsilon) dx. \quad (67)$$

Passing to the two-scale convergence on both sides we get that for a.e.  $x \in \mathbb{R}^3$  and for every  $\psi_\# \in C_\#^\infty(Q)$  such that  $\text{div}_y \psi_\#(y) = 0$  in  $Q$ .

$$\int_Q [\kappa_\infty(x, y) - \nabla u_\infty(x)] \cdot \psi_\#(y) dy = 0. \quad (68)$$

Since the orthogonal complement of the divergence-free functions is the space of gradients, for a.e.  $x \in \mathbb{R}^3$  there exists a unique function  $v(x, \cdot) \in H_\#^1(Q)/\mathbb{R}$  such that  $\nabla_y v(x, y) \equiv \kappa_\infty(x, y) - \nabla u_\infty(x)$ . Thus  $\nabla_y v \in L^2(\mathbb{R}^3 \times Q)$  and  $v(x, \cdot) \in H_\#^1(Q)/\mathbb{R}$ , i.e.  $v \in L^2[\mathbb{R}^3, H_\#^1(Q)/\mathbb{R}]$ . This completes the proof.

We are now ready to prove the two-scale convergence of the demagnetizing field operator (for this we follow the lines of [30]):

**PROPOSITION 5.5** Let  $(\mathbf{m}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  be a bounded family in  $L^2(\mathbb{R}^3, \mathbb{R}^3)$  that two-scale converges to  $\mathbf{m}(x, y)$ . Then the two-scale limit of  $(\mathbf{h}_d[\mathbf{m}_\varepsilon])_{\varepsilon \in \mathbb{R}^+} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$  exists and is given by

$$\mathbf{h}_d(x, y) = \mathbf{h}_d[\langle \mathbf{m}(x, \cdot) \rangle_Q] + \nabla_y v_{\mathbf{m}}(x, y) \quad (69)$$

where for every  $x \in \mathbb{R}^3$  the scalar function  $v_{\mathbf{m}}(x, \cdot)$  is the unique solution in  $H_\#^1(Q)$  of the cell problem

$$\Delta_y v_{\mathbf{m}}(x, y) = -\text{div}_y \mathbf{m}(x, y) \quad \text{in } H_\#^1(Q)/\mathbb{R} \quad (70)$$

and therefore of the variational cell problem

$$\int_Q \mathbf{m}(x, y) \cdot \nabla_y \psi(y) dy = - \int_Q \nabla_y v_{\mathbf{m}}(x, y) \cdot \nabla_y \psi(y) dy, \quad \int_Q v_{\mathbf{m}}(x, y) dy = 0 \quad (71)$$

for all  $\psi \in H_\#^1(Q)$ .

PROOF 5.6 Since  $(\mathbf{m}_\varepsilon)$  is bounded in  $L^2(\mathbb{R}^3)$ , due to the stability estimate (51), the sequence of magnetostatic potentials  $(u_{\mathbf{m}}^\varepsilon)$  solution of the problem  $\Delta u_{\mathbf{m}}^\varepsilon = -\operatorname{div}(\mathbf{m}_\varepsilon)$ , remains bounded in  $BL^1(\mathbb{R}^3)$ . This means that, up to a subsequence,  $(u_{\mathbf{m}}^\varepsilon) \rightharpoonup u_{\mathbf{m}}$  weakly in  $BL^1(\mathbb{R}^3)$  for some suitable  $u_{\mathbf{m}} \in BL^1(\mathbb{R}^3)$ . Thus, according to Proposition 5.3, there exist functions  $u_{\mathbf{m}} \in BL^1(\mathbb{R}^3)$  and  $v_{\mathbf{m}} \in L^2[\mathbb{R}^3; H_{\#}^1(Q)/\mathbb{R}]$  such that

$$(u_{\mathbf{m}}^\varepsilon) \rightharpoonup u_{\mathbf{m}} \text{ in } L_\omega^2 \quad \text{and} \quad \nabla u_{\mathbf{m}}^\varepsilon(x) \rightharpoonup \nabla u_{\mathbf{m}}(x) + \nabla_y v_{\mathbf{m}}(x, y). \quad (72)$$

In view of the previous limit relations,  $u_{\mathbf{m}}^\varepsilon$  is expected to behave as  $u_{\mathbf{m}}(x) + \varepsilon v_{\mathbf{m}}(x, x/\varepsilon)$ . This suggest to use, in the variational formulation of the magnetostatic problem expressed by equation (50), test functions having the form  $\varphi(x) + \varepsilon \psi(x, x/\varepsilon)$ , with  $\varphi \in \mathcal{D}(\mathbb{R}^3)$  and  $\psi \in \mathcal{D}[\mathbb{R}^3; C_{\#}^\infty(Q)]$ . This yields

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla u_{\mathbf{m}}^\varepsilon \cdot \left( \nabla \varphi + \varepsilon \nabla \psi \left( x, \frac{x}{\varepsilon} \right) + \nabla_y \psi \left( x, \frac{x}{\varepsilon} \right) \right) dx \\ = - \int_{\mathbb{R}^3} \mathbf{m}_\varepsilon \cdot \left( \nabla \varphi + \varepsilon \nabla \psi \left( x, \frac{x}{\varepsilon} \right) + \nabla_y \psi \left( x, \frac{x}{\varepsilon} \right) \right) dx. \end{aligned}$$

From the second of the two limit relations (72), we get

$$\begin{aligned} \int_{\mathbb{R}^3 \times Q} (\nabla u_{\mathbf{m}}(x) + \nabla_y v_{\mathbf{m}}(x, y)) \cdot (\nabla \varphi + \nabla_y \psi(x, y)) dy dx \\ = - \int_{\mathbb{R}^3 \times Q} \mathbf{m}(x, y) \cdot (\nabla \varphi + \nabla_y \psi(x, y)) dy dx. \end{aligned} \quad (73)$$

In particular, by choosing  $\psi \equiv 0$  we get

$$\begin{aligned} - \int_{\mathbb{R}^3} \langle \mathbf{m}(x, y) \rangle_Q \cdot \nabla \varphi(x) dx = \int_{\mathbb{R}^3 \times Q} (\nabla u_{\mathbf{m}}(x) + \nabla_y v_{\mathbf{m}}(x, y)) \cdot \nabla \varphi dy dx \\ = \int_{\mathbb{R}^3} \nabla u_{\mathbf{m}}(x) \cdot \nabla \varphi(x), \end{aligned} \quad (74)$$

where the last equality follows from the fact that  $\langle \nabla_y v_{\mathbf{m}}(x, y) \rangle_Q = 0$ . Thus, we reach the conclusion that the weak limit  $u_{\mathbf{m}}$  satisfies the variational formulation (74), i.e. is a solution of the «homogenized» equation

$$u_{\mathbf{m}}(x) = -\operatorname{div} \langle \mathbf{m}(x, y) \rangle_Q \quad \text{in } BL^1(\mathbb{R}^3). \quad (75)$$

On the other hand by choosing  $\varphi \equiv 0$  and  $\psi(x, y) = \psi_1(x)\psi_2(y)$  in (73) we get

$$\begin{aligned} \int_{\mathbb{R}^3} \langle (\nabla u_{\mathbf{m}}(x) + \nabla_y v_{\mathbf{m}}(x, y)) \cdot \nabla_y \psi_2(y) \rangle_Q \psi_1(x) dx \\ = - \int_{\mathbb{R}^3} \langle \mathbf{m}(x, y) \cdot \nabla_y \psi_2(y) \rangle_Q \psi_1(x) dx, \end{aligned}$$

and hence the so-called cell problem

$$\begin{aligned} - \int_Q \mathbf{m}(x, y) \cdot \nabla_y \psi_2(y) dy = \int_Q (\nabla u_{\mathbf{m}}(x) + \nabla_y v_{\mathbf{m}}(x, y)) \cdot \nabla_y \psi_2(y) dy \\ = \int_Q \nabla_y v_{\mathbf{m}}(x, y) \cdot \nabla_y \psi_2(y) dy, \end{aligned} \quad (76)$$

where, again, the last equality follows from the fact that  $\langle \nabla_y \psi_2(y) \rangle_Q = 0$ . Note that the variational formulation (76) can be more concisely expressed in the form

$$\Delta_y v_{\mathbf{m}}(x, y) = -\operatorname{div}_y \mathbf{m}(x, y) \quad \text{in } H_{\#}^1(Q)/\mathbb{R}, \quad (77)$$

and the well-posedness of the previous variational problem is again a direct consequence of LAX-MILGRAM theorem.

### 5.3 The continuous limit of magnetostatic self-energy functionals $\mathcal{W}_\varepsilon$

In what follows we will make use of Proposition 5.5, to prove the following



PROPOSITION 5.7 The family of magnetostatic self-energies

$$\mathcal{W}_\varepsilon : \mathbf{m} \in L^2(\Omega, \mathbb{S}^2) \mapsto -(\mathbf{h}_d[M_\varepsilon \mathbf{m}], M_\varepsilon \mathbf{m})_\Omega \quad (78)$$

continuously converges to the functional

$$\mathcal{W}_{\text{hom}} : \mathbf{m} \in L^2(\Omega, \mathbb{S}^2) \mapsto -\langle M_s \rangle_Q^2 (\mathbf{h}_d[\mathbf{m}], \mathbf{m})_\Omega + \|\nabla_y v_{\mathbf{m}}\|_{\Omega \times Q}^2 \quad (79)$$

where for every  $x \in \Omega$  the scalar function  $v_{\mathbf{m}} : \Omega \times Q \rightarrow \mathbb{R}$  is the unique solution of the following variational cell problem:

$$\mathbf{m}(x) \cdot \int_Q M_s(y) \nabla_y \psi(y) dy = - \int_Q \nabla_y v_{\mathbf{m}}(x, y) \cdot \nabla_y \psi(y) dy, \quad (80)$$

$$\int_Q v_{\mathbf{m}}(x, y) dy = 0, \quad (81)$$

for all  $\psi \in H_{\#}^1(Q)$ .

PROOF 5.8 We know (see Proposition 2.5) that  $|M_\varepsilon \mathbf{m}|^2 \equiv |M_\varepsilon|^2 \rightharpoonup \langle |M_\varepsilon|^2 \rangle_Q$  weakly\* in  $L^\infty(\Omega)$ . In particular, by choosing  $|\mathbf{m}|^2 \in L^1(\Omega)$  as a test function we get

$$\begin{aligned} \|M_\varepsilon\|_\Omega^2 &= (|M_\varepsilon \mathbf{m}|^2, |\mathbf{m}|^2)_\Omega \rightarrow (\langle |M_\varepsilon|^2 \rangle_Q, |\mathbf{m}|^2)_\Omega = |\Omega| \langle |M_\varepsilon|^2 \rangle_Q \\ &= \|M_s(y) \mathbf{m}(x)\|_{\Omega \times Q}^2 \end{aligned}$$

and therefore  $M_\varepsilon(x) \mathbf{m}(x) \rightharpoonup M_s(y) \mathbf{m}(x)$  strongly.

Next, since  $\mathbf{h}_d[M_\varepsilon \mathbf{m}] \cdot M_\varepsilon \mathbf{m}$  is bounded in  $L^2(\Omega)$ , from Proposition 2.10 and Proposition 5.5, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} -(\mathbf{h}_d[M_\varepsilon \mathbf{m}], M_\varepsilon \mathbf{m})_\Omega \\ = -\langle M_s \rangle_Q^2 (\mathbf{h}_d[\mathbf{m}], \mathbf{m})_\Omega - \int_{\Omega \times Q} \nabla_y v_{\mathbf{m}}(x, y) \cdot M_s(y) \mathbf{m}(x) dx dy. \end{aligned} \quad (82)$$

Now, we observe that for every  $x \in \Omega$  the scalar function  $v_{\mathbf{m}}(x, \cdot)$  is the unique solution of the variational cell problem (71), therefore setting  $\psi(\cdot) := v_{\mathbf{m}}(x, \cdot)$  in (71) we get

$$-\mathbf{m}(x) \cdot \int_Q M_s(y) \nabla_y v_{\mathbf{m}}(x, y) dy = \int_Q |\nabla_y v_{\mathbf{m}}(x, y)|^2 dy \quad (83)$$

and therefore

$$\lim_{\varepsilon \rightarrow 0} -(\mathbf{h}_d[M_\varepsilon \mathbf{m}], M_\varepsilon \mathbf{m})_\Omega = -\langle M_s \rangle_Q^2 (\mathbf{h}_d[\mathbf{m}], \mathbf{m})_\Omega + \|\nabla_y v_{\mathbf{m}}\|_{\Omega \times Q}^2 =: \mathcal{W}_{\text{hom}}(\mathbf{m}). \quad (84)$$

Now we show that the family  $\mathcal{W}_\varepsilon$  continuously converges to  $\mathcal{W}_{\text{hom}}$ . This amounts to prove that for every  $\mathbf{m} \in L^2(\Omega, \mathbb{S}^2)$  and every  $\varepsilon \in \mathbb{R}^+$ ,  $\varepsilon < \varepsilon_0$  and  $\|\mathbf{m} - \mathbf{m}_0\|_\Omega < \delta$  implies  $|\mathcal{W}_\varepsilon(\mathbf{m}) - \mathcal{W}_{\text{hom}}(\mathbf{m}_0)| < \eta$ . To this end, for every  $\mathbf{m}, \mathbf{m}_0 \in L^2(\Omega, \mathbb{S}^2)$  we split:

$$|\mathcal{W}_\varepsilon(\mathbf{m}) - \mathcal{W}_{\text{hom}}(\mathbf{m}_0)| \leq |\mathcal{W}_\varepsilon(\mathbf{m}) - \mathcal{W}_{\text{hom}}(\mathbf{m})| + |\mathcal{W}_{\text{hom}}(\mathbf{m}) - \mathcal{W}_{\text{hom}}(\mathbf{m}_0)|. \quad (85)$$

From (84) it follows the existence of a sufficiently small  $\varepsilon_0$  such that

$$\forall \varepsilon < \varepsilon_0 \quad |\mathcal{W}_\varepsilon(\mathbf{m}) - \mathcal{W}_{\text{hom}}(\mathbf{m})| < \frac{\eta}{2}. \quad (86)$$

On the other hand, since  $\mathbf{m} \mapsto \nabla_y v_{\mathbf{m}}$  is a linear and continuous map from  $L^2(\Omega, \mathbb{R}^3)$  into  $L^2(\Omega \times Q, \mathbb{R}^3)$  with  $\|\nabla_y v_{\mathbf{m}}\|_{\Omega \times Q} \leq c_M \|\mathbf{m}\|_\Omega$  (and  $c_M := \|M_s\|_Q$ ) one has

$$\begin{aligned} |\mathcal{W}_{\text{hom}}(\mathbf{m}) - \mathcal{W}_{\text{hom}}(\mathbf{m}_0)| &\leq \langle M_s \rangle_Q^2 |(\mathbf{h}_d[\mathbf{m}], \mathbf{m})_\Omega - (\mathbf{h}_d[\mathbf{m}_0], \mathbf{m}_0)_\Omega| \\ &\quad + \left| \|\nabla_y v_{\mathbf{m}}\|_{\Omega \times Q}^2 - \|\nabla_y v_{\mathbf{m}_0}\|_{\Omega \times Q}^2 \right| \\ &\leq \langle M_s \rangle_Q^2 |(\mathbf{h}_d[\mathbf{m} + \mathbf{m}_0], \mathbf{m} - \mathbf{m}_0)_\Omega| \\ &\quad + |(\nabla_y v_{\mathbf{m} + \mathbf{m}_0}, \nabla_y v_{\mathbf{m} - \mathbf{m}_0})_{\Omega \times Q}| \\ &\leq 2|\Omega|^{1/2} (\langle M_s \rangle_Q^2 + c_M^2) \|\mathbf{m} - \mathbf{m}_0\|_\Omega. \end{aligned} \quad (87)$$

and the previous estimate together with (86) clearly concludes the proof.

## 6 The homogenized anisotropy and interaction energies

This section is devoted to the proof of the continuous convergence of the family of anisotropy energy functionals  $\mathcal{A}_\varepsilon$  and of the family of interaction energy functionals  $\mathcal{Z}_\varepsilon$ , respectively to  $\mathcal{A}_{\text{hom}}$  and  $\mathcal{Z}_{\text{hom}}$ , whose expression is given by (14) and (17).

### 6.1 The continuous limit of the anisotropy energy functionals $\mathcal{A}_\varepsilon$

PROPOSITION 6.1 If the anisotropy energy density  $\varphi_{\text{an}} : \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}^+$  is  $Q$ -periodic with respect to the first variable and globally lipschitz with respect to the second one (uniformly with respect to the first variable) then the family  $\mathcal{A}_\varepsilon$  of anisotropy energies continuously converges to the homogenized anisotropy energy

$$\mathcal{A}_{\text{hom}} : \mathbf{m} \in L^2(\Omega, \mathbb{S}^2) \mapsto \int_{\Omega \times Q} \varphi_{\text{an}}(y, \mathbf{m}(x)) dy dx. \quad (88)$$

PROOF 6.2 Again, we have to prove that for every  $\mathbf{m}_0 \in L^2(\Omega, \mathbb{S}^2)$  one has

$$\lim_{(\varepsilon, \mathbf{m}) \rightarrow (0, \mathbf{m}_0)} \mathcal{A}_\varepsilon(\mathbf{m}) = \mathcal{A}_{\text{hom}}(\mathbf{m}_0).$$

For every  $\mathbf{m}, \mathbf{m}_0 \in L^2(\Omega, \mathbb{S}^2)$  we split

$$\begin{aligned} & \int_{\Omega \times Q} \varphi(x/\varepsilon, \mathbf{m}(x)) - \varphi(y, \mathbf{m}_0(x)) dy dx \\ &= \int_{\Omega \times Q} \varphi(x/\varepsilon, \mathbf{m}(x)) - \varphi(y, \mathbf{m}(x)) dy dx \\ & \quad + \int_{\Omega \times Q} \varphi(y, \mathbf{m}(x)) - \varphi(y, \mathbf{m}_0(x)) dy dx. \end{aligned} \quad (89)$$

Next, we observe that since  $\varphi(x/\varepsilon, \mathbf{m}(x)) \rightharpoonup \langle \varphi(y, \mathbf{m}(x)) \rangle_Q$  weakly\* in  $L^\infty(\Omega)$  (cfr. Lemma 2.5), there exists a sufficiently small  $\varepsilon_0$  such that for every  $\varepsilon < \varepsilon_0$

$$\left| \int_{\Omega \times Q} \varphi(x/\varepsilon, \mathbf{m}(x)) - \varphi(y, \mathbf{m}(x)) dy dx \right| < \frac{\eta}{2}. \quad (90)$$

On the other hand, by the global lipschitz continuity in the second variable of  $\varphi_{\text{an}}$ , we have

$$\int_{\Omega \times Q} |\varphi(y, \mathbf{m}(x)) - \varphi(y, \mathbf{m}_0(x))| dy dx \leq c_L \int_{\Omega} |\mathbf{m}(x) - \mathbf{m}_0(x)| dx \quad (91)$$

$$\leq c_L |\Omega|^{1/2} \|\mathbf{m} - \mathbf{m}_0\|_{\Omega}. \quad (92)$$

Substituting estimates (90) and (92) into (89) we get

$$|\mathcal{A}_\varepsilon(\mathbf{m}) - \mathcal{A}_{\text{hom}}(\mathbf{m}_0)| \leq \frac{\eta}{2} + c^* \|\mathbf{m} - \mathbf{m}_0\|_{\Omega} \quad \left( \text{with } c^* := c_L |\Omega|^{1/2} \right). \quad (93)$$

Therefore for every  $\mathbf{m} \in L^2(\Omega, \mathbb{S}^2)$  such that  $\|\mathbf{m} - \mathbf{m}_0\|_{\Omega} < \eta/(2c^*)$  we get  $|\mathcal{A}_\varepsilon(\mathbf{m}) - \mathcal{A}_{\text{hom}}(\mathbf{m}_0)| \leq \eta$ , and this concludes the proof.

COROLLARY 6.3 (*Uniaxial anisotropy energy density*). If  $\varphi(y, \mathbf{m}) = \kappa(y) |\mathbf{m}(x) \wedge \mathbf{u}(y)|^2$  then

$$\mathcal{A}_{\text{hom}}(\mathbf{m}) = \int_{\Omega} \langle \kappa \rangle_Q - \langle \kappa \mathbf{u} \otimes \mathbf{u} \rangle_Q : \mathbf{m} \otimes \mathbf{m} d\tau.$$

### 6.2 The continuous limit of interaction energy functionals $\mathcal{Z}_\varepsilon$

The convergence of  $(\mathcal{Z}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$  to  $\mathcal{Z}_\varepsilon$  is straightforward. Indeed this energy term is expressed by the product, with respect to the  $L^2(\Omega)$  scalar product, of the constant function  $\mathbf{h}_a$  and the weakly converging sequence  $(M_\varepsilon \mathbf{m})_{\varepsilon \in \mathbb{R}^+} \rightharpoonup \langle M_s \rangle_Q$  weakly\* in  $L^\infty(\Omega)$  (cfr. Lemma 2.5). Therefore repeating the same argument given in the previous subsection:

$$\mathcal{Z}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma\text{-cont}} \mathcal{Z}_{\text{hom}} \quad \text{with} \quad \mathcal{Z}_{\text{hom}}(\mathbf{m}) := -\langle M_s \rangle_Q \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} d\tau.$$

## 7 Proof of Theorem 1.1 completed

It is now easy to complete the proof of Theorem 1.1. Indeed the equicoercivity of the family of GIBBS-LANDAU free energy functionals  $(\mathcal{G}_\varepsilon^\xi)_{\varepsilon \in \mathbb{R}^+}$  expressed by (5) has been proved in Section 3. It is therefore sufficient to recall the stability properties of the  $\Gamma$ -limit under the sum of a continuously convergent family of functionals. In fact, what has been proved in the previous subsections, can be summarized by the following convergence scheme

$$\mathcal{E}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma} \mathcal{E}_{\text{hom}} \quad , \quad \mathcal{W}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma_{\text{cont}}} \mathcal{W}_{\text{hom}} \quad , \quad \mathcal{A}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma_{\text{cont}}} \mathcal{A}_{\text{hom}} \quad , \quad \mathcal{Z}_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\Gamma_{\text{cont}}} \mathcal{Z}_{\text{hom}}. \quad (94)$$

Thus, Proposition 2.4 completes the proof.

## 8 Conclusions and acknowledgment

We have given in this paper a complete theory for periodic microstructured magnetic materials. Obtained through a process of  $\Gamma$ -convergence the model derives rigorously the energy terms from the parameters of each constituent of the sample and the mixing geometry of the different materials in the unit periodic cell. We believe that the result applies to most of magnetic composites that are nowadays considered, e.g. those obtained from a mixing of hard and soft phases [15, 32] or the multilayer magnetic materials [16, 29]. In this latter case, the formula obtained furthermore simplifies since the exchange coefficient can be analytically computed. We leave the exploration of potential applications to a forthcoming work.

## A Restatement of some well-known result

In this appendix we state, in a bit general form, two results that we mentioned in the previous sections. We start with a result whose proof can be extracted from [25], which permits to pass, in the characterization of the homogenized energy density, from homogeneous boundary conditions to periodic ones. The proof, although well-known in the homogenization community, is difficult to find in the classical monographs on the subject:

**PROPOSITION A.1** Let  $\mathcal{V}$  be a vector subspace of  $\mathbb{R}^3$  and let  $g : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^+$  be a Carathéodory function satisfying the hypotheses of (THM) with  $\mathcal{M} := \mathcal{V}$  (see Proposition 4.1). Then, for every  $(s, \xi) \in \mathcal{V} \times \mathcal{V}^3$  one has  $Tg_{\text{hom}}(s, \xi) = T^\# g_{\text{hom}}(\xi)$  with

$$T^\# g_{\text{hom}}(\xi) := \lim_{t \rightarrow \infty} \left[ \inf_{\varphi \in H_\#^1(Q_t, \mathcal{V})} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy \right]. \quad (95)$$

**PROOF A.2** Since  $T_s(\mathcal{V}) \equiv \mathcal{V}$  it is obvious that  $Tg_{\text{hom}}$  does not depend on the  $s$ -variable. Moreover, since  $H_0^1(Q_t, \mathcal{V}) \subseteq H_\#^1(Q_t, \mathcal{V})$  one trivially has  $T^\# g_{\text{hom}}(\xi) \leq Tg_{\text{hom}}(s, \xi)$ . Let us therefore focus on the converse inequality. For any  $\varphi \in H_0^1(Q_t, \mathcal{V})$  and any  $\xi \in \mathcal{V}^3$  we consider the family  $(\mathbf{m}_\varepsilon(x) := \xi x + \varepsilon \varphi(x/\varepsilon)) \in H^1(\Omega, \mathcal{V})$  which converges to  $\xi x$  in the metric space  $(H^1(\Omega, \mathcal{V}), d_{L^2(\Omega, \mathcal{V})})$ . By the characterizing properties of the  $\Gamma$ -limit we have:

$$Tg_{\text{hom}}(s, \xi) = \frac{1}{|\Omega|} \mathcal{E}_{\text{hom}}(\xi x) \leq \frac{1}{|\Omega|} \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\mathbf{m}_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega|} \int_\Omega g\left(\frac{x}{\varepsilon}, \xi + \nabla \varphi\left(\frac{x}{\varepsilon}\right)\right) dx. \quad (96)$$

Since the integrand in the last member of the previous equation is  $Q_t$ -periodic and satisfies the hypotheses of the generalized RIEMANN-LEBESGUE Lemma (cfr. Proposition 2.5) we finish with the estimate

$$Tg_{\text{hom}}(s, \xi) \leq \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy \quad \forall \xi \in \mathcal{V}^3, \forall \varphi \in H_0^1(Q_t, \mathcal{V}). \quad (97)$$

Passing to the infimum we get the stated result.

Finally, in the hope to achieve our goal of making the material contained in this paper accessible to as large audience as possible, we prefer to state – again in a bit general form – and give a more «commented» proof of the following well-known result (cfr. [7]) which permits, in the convex case, to pass from variations that are periodic over an ensemble of  $t^3$  1-cells to variations that are periodic in the unit cell  $Q$ .

**PROPOSITION A.3** Let  $g : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^+$  be a Carathéodory integrand satisfying the hypotheses of (THM) (Proposition 4.1). If for any  $y \in \mathbb{R}^3$  the function  $g(y, \cdot)$  is convex, then for any convex subset  $\mathcal{C}$  of  $\mathbb{R}^3$  and every  $t \in \mathbb{N}$  the following equality holds:

$$\inf_{\varphi \in H_\#^1(Q_t, \mathcal{C})} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy = \inf_{\varphi \in H_\#^1(Q, \mathcal{C})} \int_Q g(y, \xi + \nabla \varphi(y)) dy. \quad (98)$$

PROOF A.4 One of the main ingredients of the argument, is in observing that for every  $t \in \mathbb{N}$  the following decomposition of the  $t$ -cube in 1-cubes holds:

$$Q_t = \bigsqcup_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} Q_1^{(i_1, i_2, i_3)} \quad \text{with} \quad Q_1^{(i_1, i_2, i_3)} := Q_1 + e_{(i_1, i_2, i_3)} \quad (99)$$

and  $e_{(i_1, i_2, i_3)} := (i_1 - 1)e_1 + (i_2 - 1)e_2 + (i_3 - 1)e_3$ . Thus, from the 1-periodicity of  $g$  in the first variable, we arrive to the intuitive and well-known fundamental equality:

$$\begin{aligned} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy &= \sum_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} \int_{Q_1^{(i_1, i_2, i_3)}} g(y, \xi + \nabla \varphi(y)) dy \\ &= \sum_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} \int_{Q_1} g(y, \xi + \nabla \varphi(y)) dy \\ &= |Q_t| \int_{Q_1} g(y, \xi + \nabla \varphi(y)) dy. \end{aligned} \quad (100)$$

Next, we observe that  $H_{\#}^1(Q, \mathcal{C}) \subseteq H_{\#}^1(Q_t, \mathcal{C})$  for every  $t \in \mathbb{N}$ , and therefore, from the previous equality we get:

$$\begin{aligned} \inf_{\varphi \in H_{\#}^1(Q_t, \mathcal{C})} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy &\leq \inf_{\varphi \in H_{\#}^1(Q, \mathcal{C})} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy \\ &= \inf_{\varphi \in H_{\#}^1(Q, \mathcal{C})} \int_Q g(y, \xi + \nabla \varphi(y)) dy. \end{aligned}$$

To complete the proof we now prove that the previous inequality is actually an equality, and therefore (in particular) that the left hand side of (98) does not depend on  $t \in \mathbb{N}$ . To this end it is sufficient to show that for every  $t \in \mathbb{N}$  and every  $\psi \in H_{\#}^1(Q_t, \mathcal{C})$  there exists a  $\varphi \in H_{\#}^1(Q, \mathcal{C})$  such that

$$\int_Q g(y, \xi + \nabla \varphi(y)) dy \leq \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \psi(y)) dy. \quad (101)$$

With this goal in mind, for every  $\psi \in H_{\#}^1(Q_t, \mathcal{C})$ , we define the convex combination

$$\varphi(y) := \frac{1}{|Q_t|} \sum_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} \psi(y + e_{(i_1, i_2, i_3)}). \quad (102)$$

Clearly  $\varphi$  belongs to  $H_{\#}^1(Q, \mathcal{C})$ . Moreover, by the convexity and 1-periodicity of  $g$  with respect to the first variable, from equality (100), we get:

$$\begin{aligned} \int_Q g(y, \xi + \nabla \varphi(y)) dy &= \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \varphi(y)) dy \\ &\leq \frac{1}{|Q_t|} \sum_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \psi(y + e_{(i_1, i_2, i_3)})) dy \\ &= \frac{1}{|Q_t|} \sum_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} \frac{1}{|Q_t|} \int_{Q_t} g(y + e_{(i_1, i_2, i_3)}, \xi + \nabla \psi(y + e_{(i_1, i_2, i_3)})) dy \\ &= \frac{1}{|Q_t|} \sum_{(i_1, i_2, i_3) \in \mathbb{N}_t^3} \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \psi(y)) dy \\ &= \frac{1}{|Q_t|} \int_{Q_t} g(y, \xi + \nabla \psi(y)) dy, \end{aligned}$$

and this completes the proof.

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