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Gradient, scattering and other kinds of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory.

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Abstract

After Arthur Ashkin's pioneering work in optical levitation and manipulation, the study of optical forces exerted by laser beams on particles has become an active field of research. The present paper is a contribution to this issue. The interest of Bessel beams is that their intensity does not have any longitudinal gradient along the direction of propagation leading to a trivial separation between gradient and scattering forces. Beside the classical gradient and scattering forces, we shall however exhibit a new kind of optical forces associated with the existence of non zero axicon angles.

Keywords: optical forces; gradient and scattering optical forces; Bessel beams; generalized Lorenz-Mie theory; Rayleigh regime.

1 Introduction.

After Arthur Ashkin's pioneering work in optical levitation and manipulation, e.g. a collection of preprints in [1], a great deal of work has been devoted to the study of optical forces exerted by laser beams on particles, e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10] for reviews.

Many of them focused their attention on the classical gradient and scattering forces discussed by Ashkin according to the following definitions: the gradient force is proportional to the gradient of intensity and points in the direction of the intensity gradient, while the scattering force is proportional to the optical intensity and points in the direction of the incident light [11]. The study of gradient and scattering forces is particularly appealing in the Rayleigh regime where the formal computations are facilitated and are even amenable to analytic formulations, e.g. [12], [13], [14], [15], [16], [17], [18] [19], [20], [21], [22], [23], [24], [25], [26], [27] and some reviews above. I shall particularly rely on a paper by J.A. Lock who, in the framework of a study devoted to radiation trapping forces for laser tweezers, provided an analysis of gradient and scattering forces in the Rayleigh regime, in the framework of generalized Lorenz-Mie theory, in the case of an on-axis Gaussian beam described by using a localized approximation in the weak-beam confinement limit [28].

In the present paper, a similar study is carried out in the more difficult case of off-axis Bessel beams. The opportunity of studying off-axis beams instead of on-axis beams is partly due to the fact that some simplifications occur due to the fact that the gradient of intensity along the axis z of propagation is always zero for Bessel beams, in deep contrast with the case of Gaussian beams even in an on-axis configuration. Furthermore, this fact implies a trivial separation between longitudinal gradient and scattering forces since all longitudinal gradient forces are expected to be zero so that only scattering forces act along the longitudinal direction. Another interest of Bessel beams is that the beam shape coefficients encoding the structure of the beams in the framework of generalized Lorenz-Mie theory (GLMT) are available in closed forms, without the need of using a localized approximation. Furthermore, an unexpected result will be the discovery of a new kind of optical forces related to the existence of axicon angles in Bessel beams, which do not occur in Gaussian beams.

The paper is organized as follows. Section 2 provides a background to be used in the sequel, concerning (i) optical forces in the framework of GLMT, (ii) Poynting vector in the framework of GLMT, (iii) basic mathematical formulae for further use and (iv) the description of Bessel beams. Section 3 deals with optical forces for Rayleigh particles exerted by Bessel beams. In Section 4, the optical forces derived in the previous section are analyzed in terms of gradient and scattering forces, revealing the existence of an unexpected new kind of optical forces, namely scattering axicon forces which do not occur for more usual beams whose axicon angle is zero.

2 Physical and mathematical background.

The GLMT *stricto sensu* describes the interaction between an arbitrary (structured) shaped beam and a homogeneous spherical particle defined by its diameter and its complex refractive index, e.g. [29], [30], [31], and references

therein. Also, see reviews [32], [33], in particular for other GLMTs devoted to the interaction between laser beams and other kinds of scatterers. In the present paper, we consider a Rayleigh particle located at the origin O_P of a Cartesian coordinate system O_Pxyz illuminated by an off-axis Bessel beam propagating along the z -direction.

2.1 Optical forces.

In the GLMT framework, we express the optical force components F_i by using (unnormalized) pressure radiation cross-section components $C_{pr,i}$ ($i = x, y, z$). In the present paper, we deal only with the longitudinal cross-section (along z -direction) reading as:

$$\begin{aligned}
C_{pr,z} = & E_0 H_0^* \frac{\lambda^2}{4\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|m|)!}{(n-|m|)!} \right. \\
& \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*) g_{n,TM}^m g_{n+1,TM}^{m*} \\
& + (b_n + b_{n+1}^* - 2b_n b_{n+1}^*) g_{n,TE}^m g_{n+1,TE}^{m*}] \\
& + m \frac{2n+1}{n^2(n+1)^2} \frac{(n+|m|)!}{(n-|m|)!} \\
& \left. \text{Re}[i(2a_n b_n^* - a_n - b_n^*) g_{n,TM}^m g_{n,TE}^{m*}] \right\} \quad (1)
\end{aligned}$$

with notations being the ones of [29]. However, note that, strictly speaking, $C_{pr,z}$ of Eq.1 would be a pressure radiation cross-section, i.e. would be homogeneous to an area, if we used the normalization condition $E_0 H_0^*/2 = 1$. Here, we did not use this condition for the sake of consistency with later expressions concerning the Poynting vector. When using the normalization condition, the forces are related to the cross-sections according to $F_i = C_{pr,i}/c$ ([34], p.14). The time-dependence of the wave is $\exp(+i\omega t)$ which is the usual choice in GLMT.

Furthermore, specifically, a_n and b_n are the usual Mie coefficients of the usual Lorenz-Mie theory, $g_{n,TM}^m$ and $g_{n,TE}^m$, with TM and TE standing for "Transverse Magnetic" and "Transverse Electric" respectively, are the beam shape coefficients (BSCs) encoding the description of the beam, either in terms of scalar potentials, e.g. [31] or in terms of vector spherical wave functions [35], the star denotes a complex conjugation and δ_{nm} is the Kronecker symbol. Furthermore, we have reintroduced the normation prefactor $E_0 H_0^*/2$ which was set to 1 in original works, e.g. Eq.(3.106) in [31]. In the present paper, we shall consider only the longitudinal force of Eq.1 because it will lead to simpler computations than for the two other force components along x and y , and, more important, because we expect a natural separation between gradient and scattering forces along the z -direction.

2.2 Poynting vector.

For the same reasons, we shall only consider the z -component S_z of the Poynting vector defined as:

$$S_z = \frac{1}{2} \text{Re}(E_x H_y^* - E_y H_x^*) \quad (2)$$

which has been already evaluated to [36], [37]:

$$S_z = \frac{-E_0 H_0^*}{2r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} i c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (3)$$

$$(\sin \theta S_{np}^{mq} + \cos \theta C_{np}^{mq})$$

in which (r, θ, φ) are spherical coordinates attached to the Cartesian coordinates (x, y, z) , c_n^{pw} are prefactors appearing in the Bromwich formulation of the Lorenz-Mie theory [38], reading as:

$$c_n^{pw} = \frac{1}{ik} (-i)^n \frac{2n+1}{n(n+1)} \quad (4)$$

and :

$$S_{np}^{mq} = kr [-g_{n, TM}^m g_{p, TM}^{q*} \psi_p (\psi_n + \psi_n'') P_n^{|m|} \tau_p^{|q|} \quad (5)$$

$$+ g_{n, TE}^m g_{p, TE}^{q*} \psi_n (\psi_p + \psi_p'') P_p^{|q|} \tau_n^{|m|}$$

$$+ q g_{n, TM}^m g_{p, TE}^{q*} \psi_p' (\psi_n + \psi_n'') P_n^{|m|} \pi_p^{|q|}$$

$$+ m g_{n, TM}^m g_{p, TE}^{q*} \psi_n' (\psi_p + \psi_p'') P_p^{|q|} \pi_n^{|m|}]$$

$$C_{np}^{mq} = -g_{n, TM}^m g_{p, TM}^{q*} \psi_p \psi_n' (\tau_n^{|m|} \tau_p^{|q|} + m q \pi_n^{|m|} \pi_p^{|q|}) \quad (6)$$

$$+ g_{n, TM}^m g_{p, TE}^{q*} \psi_n \psi_p' (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|})$$

$$- g_{n, TE}^m g_{p, TM}^{q*} \psi_p \psi_n (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|})$$

$$+ g_{n, TE}^m g_{p, TE}^{q*} \psi_n \psi_p' (m q \pi_n^{|m|} \pi_p^{|q|} + \tau_n^{|m|} \tau_p^{|q|})$$

in which k is the wavenumber, $P_n^{|m|} = P_n^{|m|}(\cos \theta)$ are associated Legendre functions, $\tau_n^{|m|} = \tau_n^{|m|}(\cos \theta)$ and $\pi_n^{|m|} = \pi_n^{|m|}(\cos \theta)$ are generalized Legendre functions, and $\psi_n = \psi_n(kr)$ are Ricatti-Bessel functions, while a prime denotes a derivative with respect to the argument (and a double prime a second derivative). In Eqs.5-6, arguments are omitted for convenience

2.3 Mathematical background.

Associated Legendre functions are defined according to Hobson's notation [39]:

$$P_n^{|m|}(\cos \theta) = (-1)^{|m|} (\sin \theta)^{|m|} \frac{d^{|m|} P_n(\cos \theta)}{(d \cos \theta)^{|m|}} \quad (7)$$

in which $P_n(\cos \theta)$ are the usual Legendre polynomials. For the description of Bessel beams, we shall also need associated Legendre functions with a negative superscript. They can be obtained from [40]:

$$P_n^{-m}(\cos \theta) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) \quad (8)$$

Associated Legendre functions may then be evaluated according to:

$$\tau_n^m(\cos \theta) = \frac{dP_n^m(\cos \theta)}{d\theta} \quad (9)$$

$$\pi_n^m(\cos \theta) = \frac{P_n^m(\cos \theta)}{\sin \theta} \quad (10)$$

Ricatti-Bessel functions $\psi_n(kr)$ may be expressed in terms of spherical Bessel functions $j_n(kr)$ according to:

$$\psi_n(x) = x j_n(x) \quad (11)$$

Furthermore, we have:

$$\psi_n'(x) = (n+1)j_n(x) - x j_{n+1}(x) \quad (12)$$

and:

$$\lim_{x \rightarrow 0} j_n(x) = \frac{x^n}{(2n+1)!!} \quad (13)$$

2.4 Bessel beams.

Bessel beams have been introduced by Durnin and co-workers [41], [42]. They possess the appealing property of being self-healing and non diffracting, and, more important in the context of the present paper, they furthermore possess a propagation invariance property, namely the intensity of the beam is constant along the direction of propagation. There exist an infinity of kinds of Bessel beams, in particular depending on the value given to an arbitrary function $g(\alpha_0)$, in which α_0 is the axicon angle (or half-cone angle) of the beam, with different linear and circular polarizations [43], [44], [45], [46]. All of them generically exhibit the following structure:

$$K_i = k_i(x, y)e^{-ik_z z}, \quad i = x, y, z, \quad K = E \text{ or } H \quad (14)$$

in which $k_z = k \cos \alpha$ is the longitudinal wavenumber. Among all of them, we shall pay a particular attention to circularly symmetric Bessel beams of arbitrary order whose BSCs in an off-axis configuration read as [46]:

$$g_{n, TM}^m = -g(\alpha_0)e^{ik_z z_0} [i^{l-m+1} e^{i(l-m+1)\phi_0} A_n^m + i^{l-m-1} e^{i(l-m-1)\phi_0} B_n^m] \quad (15)$$

$$g_{n, TE}^m = ig(\alpha_0)e^{ik_z z_0} [i^{l-m+1} e^{i(l-m+1)\phi_0} A_n^m - i^{l-m-1} e^{i(l-m-1)\phi_0} B_n^m] \quad (16)$$

in which:

$$A_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m+1}(\sigma_0) [\tau_n^m(\cos \alpha_0) + m\pi_n^m(\cos \alpha_0)] \quad (17)$$

$$B_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m-1}(\sigma_0) [\tau_n^m(\cos \alpha_0) - m\pi_n^m(\cos \alpha_0)] \quad (18)$$

in which $\sigma_0 = k_t \rho_0$, $k_t = k \sin \alpha_0$ is the transverse wavenumber, $\rho_0 = (x_0^2 + y_0^2)^{1/2}$, $\phi_0 = \tan^{-1}(y_0/x_0)$, (x_0, y_0, z_0) denotes the location of the beam origin with respect to the origin of the particle coordinate system at which the scatterer

is located, and therefore specifies the off-axis location of the beam, and $J_k(\cdot)$ is the k -order Bessel function of the first kind. When $g(\alpha_0) = (1 + \cos \alpha_0)/4$, the beam reduces to a Davis circularly symmetric Bessel beam as discussed in [43], [47]. When $g(\alpha_0) = 1/2$, the beam reduces to another kind of Bessel beam discussed in [48], [49], [50]. In the sequel, it will be convenient to introduce the notation $C = \cos \alpha_0$.

3 Optical forces exerted by Bessel beams on Rayleigh particles.

3.1 General expressions of optical forces for Rayleigh particles.

Let us consider Eq.1 which contains an infinite number of Mie coefficients. For Rayleigh particles, the Mie coefficients reduce to [28], and [34], pp. 143-144:

$$a_1 = \frac{2i}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 + O(i\alpha^5) + \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (19)$$

$$b_1 = O(i\alpha^5) \quad (20)$$

in which m is the refractive index (here taken to be real) with respect to the surrounding medium and α the size parameter $\pi d/\lambda$. The other coefficients a_n and b_n ($n > 1$) involves still higher powers of α . Real parts are then proportional to α^6 and imaginary parts are proportional to α^3 while higher powers are discarded and we therefore retain only:

$$\text{Im}(a_1) = \frac{2}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \quad (21)$$

$$\text{Re}(a_1) = \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (22)$$

We then start from Eq.1, retain only the ($n = 1$)-partial wave and the Mie coefficient a_1 , leading to:

$$C_{pr,z} = E_0 H_0^* \frac{3\lambda^2}{8\pi} \text{Re}(a_1 G) \quad (23)$$

in which:

$$G = g_{1,TM}^{-1}(g_{2,TM}^{-1*} + ig_{1,TE}^{-1*}) + g_{1,TM}^1(g_{2,TM}^{1*} - ig_{1,TE}^{1*}) + \frac{1}{3}g_{1,TM}^0g_{2,TM}^{0*} \quad (24)$$

This becomes:

$$C_{pr,z} = E_0H_0^* \frac{3\lambda^2}{8\pi} [\text{Re}(a_1) \text{Re}(G) - \text{Im}(a_1) \text{Im}(G)] \quad (25)$$

in which we recall that $\text{Re}(a_1) \sim \alpha^6$ and $\text{Im}(a_1) \sim \alpha^3$. According to the literature, e.g. [12], [13], [20], [25], [28], and many other papers already quoted in the introduction, the α^6 -term and the α^3 -term correspond to the scattering and the gradient forces respectively so that $C_{pr,z}$ is the sum of two terms according to $C_{pr,z} = C_{pr,z}^s + C_{pr,z}^g$ in which, using Eqs.21-22, we obtain:

$$C_{pr,z}^s = E_0H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2 - 1}{m^2 + 2}\right)^2 \text{Re}(G)\alpha^6 \quad (26)$$

$$C_{pr,z}^g = -E_0H_0^* \frac{\lambda^2}{4\pi} \frac{m^2 - 1}{m^2 + 1} \text{Im}(G)\alpha^3 \quad (27)$$

in which the subscripts s and g recall that we expect that $C_{pr,z}^s$ and $C_{pr,z}^g$ correspond to scattering and gradient forces respectively (this expectation will not be perfectly fulfilled). We shall now specify these expressions to the case of circularly symmetric Bessel beams of arbitrary order whose BSCs in an off-axis configuration are given by Eqs.15-18. A remark is worthwhile to be done here, namely that G occur both in $C_{pr,z}^s$ and in $C_{pr,z}^g$, i.e. that both quantities would be expressed in terms of the same set of BSCs. This suggests that it is in general impossible to express the original GLMT series of fields as a decomposition of sub-fields leading to different kinds of optical forces related to different sets of BSCs, as confirmed in a more general context by [51], [52]. [53].

3.2 Term with imaginary part of G for circularly symmetric Bessel beams.

Let us consider Eq.24. It is convenient to rewrite it as:

$$G = G_1 + G_2 + G_3 \quad (28)$$

in which:

$$G_1 = g_{1, TM}^{-1} (g_{2, TM}^{-1*} + i g_{1, TE}^{-1*}) \quad (29)$$

$$G_2 = g_{1, TM}^1 (g_{2, TM}^{1*} - i g_{1, TE}^{1*}) \quad (30)$$

$$G_3 = \frac{1}{3} g_{1, TM}^0 g_{2, TM}^{0*} \quad (31)$$

We now evaluate these quantities using Eqs.15-18, and obtain, after a significant amount of computations:

$$G_1 = -|g(\alpha_0)|^2 \{-A_1^{-1} A_2^{-1} + A_1^{-1} A_1^{-1} - B_1^{-1} B_2^{-1} - B_1^{-1} B_1^{-1} + e^{2i\phi_0} [A_1^{-1} B_2^{-1} + A_1^{-1} B_1^{-1}] + e^{-2i\phi_0} [B_1^{-1} A_2^{-1} - B_1^{-1} A_1^{-1}]\} \quad (32)$$

$$G_2 = |g(\alpha_0)|^2 \{A_1^1 A_2^1 + A_1^1 A_1^1 + B_1^1 B_2^1 - B_1^1 B_1^1 + e^{2i\phi_0} [A_1^1 B_1^1 - A_1^1 B_2^1] - e^{-2i\phi_0} [B_1^1 A_2^1 + B_1^1 A_1^1]\} \quad (33)$$

$$G_3 = \frac{1}{3} |g(\alpha_0)|^2 [A_1^0 A_2^0 + B_1^0 B_2^0 - e^{2i\phi_0} A_1^0 B_2^0 - e^{-2i\phi_0} B_1^0 A_2^0] \quad (34)$$

Recalling that A_n^m and B_n^m are real quantities, we then have:

$$\text{Im}(G_3) = -\frac{1}{3} |g(\alpha_0)|^2 \sin 2\phi_0 [A_1^0 B_2^0 - B_1^0 A_2^0] \quad (35)$$

Expressing the coefficients A_n^m and B_n^m involved in Eq.35 by using Eqs.17-18, we obtain: $A_1^0 = J_{l+1} \tau_1^0$, $B_2^0 = J_{l-1} \tau_2^0$, $B_1^0 = J_{l-1} \tau_1^0$ and $A_2^0 = J_{l+1} \tau_2^0$, in which arguments σ_0 and $C = \cos \alpha_0$ are omitted, immediately implying, without the need to specify the expressions for the τ 's:

$$A_1^0 B_2^0 - B_1^0 A_2^0 = 0 \quad (36)$$

and therefore:

$$\text{Im}(G_3) = 0 \quad (37)$$

We now deal with $(G_1 + G_2)$, i.e. not with G_1 and G_2 separately. From Eqs.32-33, recalling again that A_n^m and B_n^m pertain to \mathbf{R} , we obtain:

$$\text{Im}(G_1 + G_2) = |g(\alpha_0)|^2 \sin 2\phi_0 G_{12} \quad (38)$$

in which:

$$\begin{aligned} G_{12} = & -A_1^{-1}B_2^{-1} - A_1^{-1}B_1^{-1} + B_1^{-1}A_2^{-1} - B_1^{-1}A_1^{-1} \\ & + A_1^1B_1^1 - A_1^1B_2^1 + B_1^1A_2^1 + B_1^1A_1^1 \end{aligned} \quad (39)$$

Evaluating the coefficients A_n^m and B_n^m involved in Eq.39, we obtain after a bit of algebra, omitting again the arguments:

$$\begin{aligned} G_{12} = & 2J_l J_{l+2} [-\tau_1^{-1}\pi_2^{-1} - \tau_1^{-1}\tau_1^{-1} + \pi_1^{-1}\tau_2^{-1} + \pi_1^{-1}\pi_1^{-1}] \\ & + \frac{1}{6} J_l J_{l-2} [3\tau_1^1\tau_1^1 - 3\pi_1^1\pi_1^1 + \tau_1^1\pi_2^1 - \pi_1^1\tau_2^1] \end{aligned} \quad (40)$$

The τ 's and π 's involved in Eq.40 are then evaluated using the expressions provided in subsection 2.3, leading to: $P_1^1 = -S$, $P_2^1 = -3SC$, $P_1^{-1} = S/2$, $P_2^{-1} = SC/2$, $\tau_1^1 = -C$, $\pi_1^1 = -1$, $\pi_2^1 = -3C$, $\tau_2^1 = 3(S^2 - C^2)$, $\tau_1^{-1} = C/2$, $\pi_1^{-1} = 1/2$, $\pi_2^{-1} = C/2$, $\tau_2^{-1} = (C^2 - S^2)/2$, in which we again use $C = \cos \alpha_0$ and furthermore introduced $S = \sin \alpha_0$, leading eventually to:

$$\text{Im}(G_1 + G_2) = 0 \quad (41)$$

Therefore, as a whole, we have $\text{Im}(G) = 0$ and:

$$C_{pr,z}^g = 0 \quad (42)$$

3.3 Term with real part of G for circularly symmetric Bessel beams.

For $C_{pr,z}^s$ of Eq.26, calculations run in a similar way. We however make a decomposition of G of Eq.24 which is different from the one of Eq.28, into three terms, according to:

$$G = G^{11} + G^{12} + G^0 \quad (43)$$

in which:

$$G^{11} = i[g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TE}^1g_{1,TM}^{1*}] \quad (44)$$

$$G^{12} = g_{1,TE}^{-1}g_{2,TE}^{-1*} + g_{1,TE}^1g_{2,TE}^{1*} \quad (45)$$

$$G^0 = \frac{1}{3}g_{1,TE}^0g_{2,TE}^{0*} \quad (46)$$

$C_{pr,z}^s$ may then be decomposed into three terms according to:

$$C_{pr,z}^s = C_{pr,z}^{s11} + C_{pr,z}^{s12} + C_{pr,z}^{s0} \quad (47)$$

in which:

$$C_{pr,z}^{s11} = E_0H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \text{Re}(G^{11})\alpha^6 \quad (48)$$

$$C_{pr,z}^{s12} = E_0H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \text{Re}(G^{12})\alpha^6 \quad (49)$$

$$C_{pr,z}^{s0} = E_0H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \text{Re}(G^0)\alpha^6 \quad (50)$$

$\text{Re}(G^{11})$ is then evaluated in the usual manner by (i) expressing the BSCs in terms of A_n^m 's and B_n^m 's leading to:

$$\text{Re}(G^{11}) = -|g(\alpha_0)|^2 [(A_1^{-1})^2 - (B_1^{-1})^2 - (A_1^1)^2 + (B_1^1)^2], \quad (51)$$

(ii) afterward expressing the A_n^m 's and B_n^m 's involved in Eq.51 according to: $A_1^{-1} = -J_{l+2}(\tau_1^{-1} - \pi_1^{-1})$, $B_1^{-1} = -J_l(\tau_1^{-1} + \pi_1^{-1})$, $A_1^1 = \frac{1}{2}J_l(\tau_1^1 + \pi_1^1)$, $B_1^1 = \frac{1}{2}J_{l-2}(\tau_1^1 - \pi_1^1)$ and (iii) evaluating the τ 's and π 's according to: $\tau_1^{-1} = C/2$, $\pi_1^{-1} = 1/2$, $\tau_1^1 = -C$, $\pi_1^1 = -1$, to obtain $\text{Re}(G^{11})$ and eventually:

$$C_{pr,z}^{s11} = -E_0 H_0^* \frac{\lambda^2}{24\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 |g(\alpha_0)|^2 \{(C-1)^2 [(J_{l+2})^2 + (J_{l-2})^2] - 2(C+1)^2 (J_l)^2\} \alpha^6 \quad (52)$$

We similarly obtain:

$$C_{pr,z}^{s0} = E_0 H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 |g(\alpha_0)|^2 C(1-C^2) [J_{l+1}^2 + J_{l-1}^2 - 2\cos(2\phi_0) J_{l+1} J_{l-1}] \alpha^6 \quad (53)$$

and:

$$\begin{aligned} C_{pr,z}^{s12} &= E_0 H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 |g(\alpha_0)|^2 \\ &\quad \left[\frac{C^3}{2} (J_{l+2}^2 + J_{l-2}^2 + 2J_l^2) - \frac{3C^2-1}{4} (J_{l+2}^2 + J_{l-2}^2 - 2J_l^2) \right. \\ &\quad \left. - C(C^2-1) (J_l J_{l+2} + J_l J_{l-2}) \cos(2\phi_0) \right] \alpha^6 \end{aligned} \quad (54)$$

4 Gradient, scattering and scattering axicon forces.

4.1 Gradient force.

On the one hand, we have found that $C_{pr,z}^g$ is 0, e.g. Eq.42. On the other hand, Eq.14 immediately implies that $\mathbf{E} \cdot \mathbf{E}^*$ does not depend on z , i.e. $\nabla(\mathbf{E} \cdot \mathbf{E}^*)$ is zero. This confirms our expectation that $C_{pr,z}^g$ must be interpreted as a gradient cross-section. Let us interestingly note that $[\varepsilon_0 \mathbf{E} \cdot \mathbf{E}^* + \mu_0 \mathbf{H} \cdot \mathbf{H}^*]$ and the Poynting vector $(\mathbf{E} \times \mathbf{H}^*)$ do not depend as well on z , meaning that their z -gradient components are both 0 as well, i.e. the time-averaged energy density and the Poynting vector as well have no gradient in the z -direction.

4.2 Scattering and scattering axicon forces.

We now intend to examine whether $C_{pr,z}^s$ indeed corresponds to scattering forces as expected, i.e. whether $C_{pr,z}^s$ is proportional to the intensity of the beam that we shall state as being given by the Poynting vector. More specifically, we have to consider the component S_z of the Poynting vector taken at the location of the particle. We then start from Eqs.3-6, in order to evaluate Eq.3. This equation is to be evaluated on the axis z (where the particle is located). Then all $\sin \theta$ -terms are cancelled and we are left with the coefficients C_{np}^{mq} only. Next, we remember that Eq.26 has been derived from Eq.24 which contains the following combinations of BSCs: $g_{1,TM}^{-1}g_{2,TM}^{-1*}$, $g_{1,TE}^{-1}g_{1,TE}^{-1*}$, $g_{1,TE}^{-1}g_{2,TE}^{-1*}$, $g_{1,TE}^{-1}g_{1,TE}^{-1*}$ and $g_{1,TE}^{-1}g_{2,TE}^{-1*}$. Therefore, among the quantities C_{np}^{mq} of Eq.6, we only have to consider the following quantities: C_{12}^{-1-1} , C_{21}^{-1-1} , C_{11}^{-1-1} , C_{12}^{11} , C_{21}^{11} , C_{11}^{11} , C_{12}^{00} and C_{21}^{00} . These quantities, restricted to the combinations of BSCs previously given, read as:

$$\widetilde{C}_{12}^{-1-1} = -g_{1,TM}^{-1}g_{2,TM}^{-1*}\psi_2\psi_1'(\tau_1^1\tau_2^1 + \pi_1^1\pi_2^1) \quad (55)$$

$$\widetilde{C}_{21}^{-1-1} = -g_{2,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_2'(\tau_2^1\tau_1^1 + \pi_2^1\pi_1^1) \quad (56)$$

$$\widetilde{C}_{11}^{-1-1} = -2(g_{1,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_1' - g_{1,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_1)\pi_1^1\tau_1^1 \quad (57)$$

$$\widetilde{C}_{12}^{11} = -g_{1,TE}^{-1}g_{2,TE}^{-1*}\psi_2\psi_1'(\tau_1^1\tau_2^1 + \pi_1^1\pi_2^1) \quad (58)$$

$$\widetilde{C}_{21}^{11} = -g_{2,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_2'(\tau_2^1\tau_1^1 + \pi_2^1\pi_1^1) \quad (59)$$

$$\widetilde{C}_{11}^{11} = -2(g_{1,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_1 - g_{1,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_1)\pi_1^1\tau_1^1 \quad (60)$$

$$\widetilde{C}_{12}^{00} = -g_{1,TE}^{-1}g_{2,TE}^{-1*}\psi_2\psi_1'\tau_1^0\tau_2^0 \quad (61)$$

$$\widetilde{C}_{21}^{00} = -g_{2,TE}^{-1}g_{1,TE}^{-1*}\psi_1\psi_2'\tau_2^0\tau_1^0 \quad (62)$$

in which the tilde is meant to insist on the fact that these quantities are given with the mentioned restrictions to the combinations of BSCs.

Next, we remember that the particle is located at the origin of the coordinate system, i.e. at $r = 0$. The quantities C_{np}^{mq} above do not occur alone in Eq.3 but as combinations of the form C_{np}^{mq}/r . We therefore have to consider the limit of these various quantities as $r \rightarrow 0$. We then recall Eqs.11-13 and establish:

$$\lim_{r \rightarrow 0} \frac{\psi_2(kr)\psi_1'(kr)}{r^2} = \lim_{r \rightarrow 0} \frac{\psi_1(kr)\psi_2'(kr)}{r^2} = \lim_{r \rightarrow 0} \frac{\psi_1(kr)\psi_1(kr)}{r^2} = 0 \quad (63)$$

so that we are left with the terms C_{11}^{-1-1}/r^2 and C_{11}^{11}/r^2 reading as:

$$C_{11}^{-1-1} = -\frac{8k^2}{9} g_{1, TM}^{-1} g_{1, TE}^{-1*} \quad (64)$$

$$C_{11}^{11} = \frac{8k^2}{9} g_{1, TM}^1 g_{1, TE}^{1*} \quad (65)$$

in which we have used $[\tau_1^1(\cos \theta)]_{\theta=0} = -1$, $[\pi_1^1(\cos \theta)]_{\theta=0} = -1$ and $\lim_{r \rightarrow 0} \psi_1'(kr)\psi_1'(kr)/r^2 = 4k^2/9$. Inserting these results into Eq.3 and noting that $c_1^{pw} c_1^{pw*} = 9/(4k^2)$, we obtain a reduced expression of S_z , now denoted as \widetilde{S}_z reading as:

$$\widetilde{S}_z = E_0 H_0^* \text{Re}[i(g_{1, TM}^{-1} g_{1, TE}^{-1*} - g_{1, TM}^1 g_{1, TE}^{1*})] \quad (66)$$

which is to be compared with Eqs.44, 48 leading to:

$$C_{pr,z}^{s11} = E_0 H_0^* \frac{\lambda^2}{6\pi} \left(\frac{m^2 - 1}{m^2 + 2}\right)^2 \text{Re}[i(g_{1, TM}^{-1} g_{1, TE}^{-1*} - g_{1, TM}^1 g_{1, TE}^{1*})] \alpha^6 \quad (67)$$

showing that $C_{pr,z}^{s11}$, otherwise given by Eq.52, is proportional to \widetilde{S}_z , therefore complying with the definition of a scattering force.

It then happens that $C_{pr,z}^{s0}$ of Eq.53 and $C_{pr,z}^{s12}$ of Eq.54 are associated with two unexpected extra-kinds of forces. The interpretation of Eq.53 is easy. It is zero when the axicon angle is zero, i.e. when $C = 1$. The term $C_{pr,z}^{s0}$ may then be called an axicon pressure radiation cross-section, associated with an axicon force. It is conjectured that such forces may occur whenever the beam description exhibits an axicon angle, i.e. for Bessel beams whatever the polarization, but also for other kinds of beams including other polarizations of Bessel beams, frozen waves, Mathieu and Lommel beams, which are formed by various superpositions of Bessel beams, e.g. [54], [55], [56], [57], [58], [59], [60],

[61], [62], [63], or Laguerre-Gauss beams focused by a lens, e.g. [64], [65], [66], [67], [68].

The term $C_{pr,z}^{s12}$ corresponds to another kind of axicon optical forces which, however, are not zero when the axicon angle is zero. However, in this case, we obtain from Eqs.52 and 54:

$$C_{pr,z}^{s11} = C_{pr,z}^{s12} = E_0 H_0^* \frac{\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 |g(0)|^2 J_l^2(\sigma_0) \alpha^6 \quad (68)$$

in which we reintroduced the argument of the Bessel function J_l , so that $C_{pr,z}^{s12}$, as well as $C_{pr,z}^{s11}$, is proportional to \widehat{S}_z , therefore complying as well to the usual definition of a scattering force, regardless of the on-axis ($\sigma_0 = 0$) or off-axis ($\sigma_0 \neq 0$) location of the beam. This makes sense because, when $\alpha_0 = 0$, the angular spectrum decomposition of the Bessel beam only contains incident plane waves which behaves identically irrespectively of their off-axis location. Furthermore, for an on-axis location, due to the relation $J_l(0) = \delta_{l0}$, we have $C_{pr,z}^{s11} = C_{pr,z}^{s12} = 0$ except for $l = 0$.

Next, instead of specifying first $\alpha_0 = 0$ (no axicon angle) and afterward $\sigma_0 = 0$ (on-axis location), we may consider an on-axis location straight away. Then, both $C_{pr,z}^{s11}$ and $C_{pr,z}^{s12}$ are zero (i.e. are proportional with a constant of proportionality equal to 0) for $l \neq 0, \pm 2$. If, furthermore $\alpha_0 = 0$, then, for $l = 0$, both $C_{pr,z}^{s11}$ and $C_{pr,z}^{s12}$ are equal and are given by Eq.68, i.e. they are proportional with a constant of proportionality equal to 1 and, for $l = \pm 2$, both $C_{pr,z}^{s11}$ and $C_{pr,z}^{s12}$ are found to be zero, i.e. they are proportional with a constant of proportionality equal to 0. Therefore, in all cases, $C_{pr,z}^{s12}$ may be viewed as a genuine scattering force, whether it is zero or not, in the on-axis location, while it provides a new kind of scattering axicon forces otherwise. Let us also remember that the dichotomy between the cases ($l \neq 0, \pm 2$) and ($l = 0, \pm 2$) correspond as well to a dichotomy between dark and non-dark beams, i.e. between beams possessing or not a hole of intensity on the axis [69].

5 Conclusion.

The study of the decomposition between gradient and scattering forces exerted by laser beams on Rayleigh particles is an issue propitious to the formal understanding of the role of such forces. The case of Bessel beams is of particular interest for the understanding of such issues because such beams have no gradient of intensity along the direction of propagation, therefore implying a natural separation between gradient forces (which should be zero) and scattering forces. Furthermore, although similar studies up to now focused on on-axis configurations, the study of off-axis configurations, although much more intensive from a formal computation point of view, might provide complementary insights. This

has been the case for the present study devoted to the study of longitudinal optical forces exerted by off-axis Bessel beams in the case of Rayleigh particles. The most important result of this paper is the occurrence of a new kind of optical forces which may be called scattering axicon forces associated with an axicon propagation term of the form $\exp(ik \cos \alpha_0 z)$ in which α_0 is an axicon angle, which are zero if the axicon angle is zero. Future works will be devoted to associated complementary studies, such as the systematic study of on-axis configurations for Bessel beams, or to the case of transverse optical forces in off-axis Bessel configurations.

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