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**ENVY-FREE CAKE CUTTING:
A POLYNOMIAL NUMBER OF QUERIES
WITH HIGH PROBABILITY**

GUILLAUME CHÈZE

ABSTRACT. In this article we propose a probabilistic framework in order to study the fair division of a divisible good, e.g. a cake, between n players. Our framework follows the same idea than the “Full independence model” used in the study of fair division of indivisible goods. We show that, in this framework, there exists an envy-free division algorithm satisfying the following probability estimate:

$$\mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{7+b}) = \mathcal{O}\left(n^{-\frac{b-1}{3}+1+o(1)}\right),$$

where μ_1, \dots, μ_n correspond to the preferences of the n players, $C(\mu_1, \dots, \mu_n)$ is the number of queries used by the algorithm and $b > 4$. In particular, this gives

$$\lim_{n \rightarrow +\infty} \mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{12}) = 0.$$

It must be noticed that nowadays few things are known about the complexity of envy-free division algorithms. Indeed, Procaccia has given a lower bound in $\Omega(n^2)$ and Aziz and Mackenzie have given an upper bound in $n^{n^{n^{n^n}}}$. As our estimate means that we have $C(\mu_1, \dots, \mu_n) < n^{12}$ with a high probability, this gives a new insight on the complexity of envy-free cake cutting algorithms. Our result follows from a study of Webb’s algorithm and a theorem of Tao and Vu about the smallest singular value of a random matrix.

INTRODUCTION

In this article we study the problem of fair resource allocation. It consists to share an heterogeneous good between different players or agents. This good can be for example: a cake, land, time or computer memory. This problem is old. For example, the Rhind mathematical papyrus contains problems about the division of loaves of bread and about partition of plots of land. In the Bible we find the famous “Cut and Choose” algorithm between Abraham and Lot, and in the greek mythology we find the trick at Mecone. More recently, the “Cut and Choose” protocol has been used in the United Nations Convention on the Law of the Sea (December 1982, Annex III, article 8).

The problem of fair division has been formulated in a scientific way by Steinhaus in 1948, see [33]. Nowadays, there exist several papers, see e.g. [1, 5, 6, 20, 21, 22, 25, 27, 30, 38], and books about this topic, see e.g. [3, 7, 28, 31]. These results

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appear in the mathematics, economics, political science, artificial intelligence and computer science literature. Recently, the cake cutting problem has been studied intensively by computer scientists for solving resource allocation problems in multi agents systems, see e.g. [8, 12, 13, 24].

Throughout this article, the cake will be an heterogeneous good represented by the interval $\mathcal{C} = [0, 1]$. This assumption is classical and not restrictive for our study. We also consider n players and we associate to each player a non-atomic probability measure μ_i on the interval $\mathcal{C} = [0, 1]$. More precisely, we suppose that the measures μ_i are absolutely continuous with respect to the Lebesgue measure. These measures represent the preferences, the utility functions of the players. We have $\mu_i(\mathcal{C}) = 1$ for all i .

The problem in this situation is to get a fair division of $\mathcal{C} = \mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$, where the i -th player get \mathcal{C}_i .

When we study fair divisions, we have to precise what is the meaning of “fair”. Indeed, several notions exist.

- We say that a division is *proportional* when for all i , we have

$$\mu_i(\mathcal{C}_i) \geq 1/n.$$

- We say that a division is *equitable* when for all $i \neq j$, we have

$$\mu_i(\mathcal{C}_i) = \mu_j(\mathcal{C}_j).$$

- We say that a division is *exact* in the ratios $(\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$, when for all i and j we have

$$\mu_i(\mathcal{C}_j) = \alpha_j.$$

- We say that a division is *envy-free* when for all $i \neq j$, we have

$$\mu_i(\mathcal{C}_i) \geq \mu_i(\mathcal{C}_j).$$

A practical problem is the computation of fair divisions. In order to describe algorithms we thus need a model of computation. There exist two main classes of cake cutting algorithms: discrete and continuous protocols (also called moving knife methods). Here, we study discrete algorithms. These kinds of algorithms can be described thanks to the classical model introduced by Robertson and Webb and formalized by Woeginger and Sgall in [40]. In this model we suppose that a mediator interacts with the agents. The mediator asks two type of queries: either cutting a piece with a given value, or evaluating a given piece. More precisely, the two type of queries allowed are:

- (1) *eval* _{i} (x, y): Ask agent i to evaluate the interval $[x, y]$. This means return $\mu_i([x, y])$.
- (2) *cut* _{i} (x, a): Ask agent i to cut a piece of cake $[x, y]$ such that $\mu_i([x, y]) = a$. This means: for given x and a , return y such that $\mu_i([x, y]) = a$.

In the Robertson-Webb model the mediator can adapt the queries from the previous answers given by the players. In this model, the complexity counts the number of queries necessary to get a fair division. For a rigorous description of this model we can consult: [9, 40].

We can remark that this model of computation does not take into account the nature and the number of operations performed by the mediator. The BSSRW model

of computation introduced in [16] allows to avoid these drawbacks.

The first studied notion of fair division has been proportional division, [33]. Proportional division is a simple and well understood notion. In [33] Steinhaus explains the Banach-Knaster algorithm which gives a proportional division. There also exists an optimal algorithm to compute a proportional division in the Robertson-Webb model, see [21, 22]. The complexity of this algorithm is in $\mathcal{O}(n \log(n))$. Furthermore, the portion \mathcal{C}_i given to the i -th player in this algorithm is an interval.

Exact divisions in the ratios $(\alpha_1, \dots, \alpha_n)$ exist for all ratios $(\alpha_1, \dots, \alpha_n)$. The existence of this kind of fair division follows from a convexity theorem given by Lyapounov, see e.g. [20]. When we have $\alpha_i = 1/n$, for all i , we just say that the division is exact. Unfortunately, there exist no algorithm to compute exact divisions, see [31].

Equitable fair division is of the same kind. Indeed, there exist equitable fair divisions where each \mathcal{C}_i is an interval, see [10, 14, 32]. However, there do not exist algorithms computing an equitable fair division, see [11, 15, 29].

Envy-free fair division is difficult to obtain in practice. Indeed, whereas envy-free fair divisions where each \mathcal{C}_i is an interval exist, there does not exist an algorithm in the Robertson-Webb model computing such divisions. These results have been proved by Stromquist in [34, 35].

The first envy-free algorithm for n players has been given by Brams and Taylor in [6]. This algorithm has been given approximatively 50 years after the first algorithm computing a proportional fair division. The Brams-Taylor algorithm has an unbounded complexity in the Robertson-Webb model. This means that we cannot bound the complexity of this algorithm in terms of the number of players only. It is only recently that a finite and bounded algorithm has been given to solve this problem, see [1]. The complexity of this algorithm is in $\mathcal{O}(n^{n^{n^{n^n}}})$. When $n = 2$, $n^{n^{n^{n^n}}}$ is bigger than the number of atoms in the universe... A lower bound for envy-free division algorithm has been given by Proccacia in [26]. This lower bound is in $\Omega(n^2)$.

We can remark that there is a huge difference between the complexity in the worst case $\mathcal{O}(n^{n^{n^{n^n}}})$ and the lower bound $\Omega(n^2)$. Therefore a natural question arises:

Can we design an envy-free algorithm such that in practice the number of queries is smaller than n^d , where d is a given degree, with a high probability ?

In order to answer to this question we have to define a probabilistic framework. When we consider indivisible goods there exist two probabilistic models, see e.g. [4].

The first model is the *Full correlation model*. In this model we suppose that all agents have the same preference and all preferences are equiprobable. When we want to share several indivisible goods, this case corresponds to the worst case.

However, in the cake cutting situation, if we suppose that $\mu_1 = \dots = \mu_n$ then we can easily obtain an envy-free division. Indeed, we ask to the first player to cut the cake in n equals portions. Thus the full correlation model is not an interesting model in the cake cutting situation.

The second model in the indivisible goods setting is the *Full independence model*. In this model we suppose that all preferences are equiprobable and that all agents have independent preferences. In the cake cutting setting, in order to obtain a similar situation we consider the following construction:

First, we divide the interval $[0, 1]$ into n witness intervals $W_j = [(j-1)/n, j/n]$, where $j = 1, \dots, n$.

Second, we remark that for all probabilistic measures μ_i on $[0, 1]$, the vector $(\mu_i(W_1), \dots, \mu_i(W_n))$ belongs to the standard $(n-1)$ -simplex. Indeed, as μ_i is a probabilistic measure we have for all $j = 1, \dots, n$, $\mu_i(W_j) \geq 0$, and

$$\mu_i(W_1) + \dots + \mu_i(W_n) = \mu_i(W_1 \sqcup \dots \sqcup W_n) = \mu_i([0, 1]) = 1.$$

When we consider a random measure μ_i , it is natural to suppose that all witness intervals play the same role. For example, there is no reason to suppose that the players usually prefer the first part W_1 of the cake.

Our probabilistic framework is thus the following:

We suppose that the distribution of $(\mu_i(W_1), \dots, \mu_i(W_n))$ follows a uniform distribution over the standard $(n-1)$ -simplex.

A classical way to obtain a uniform distribution on the standard $(n-1)$ -simplex is the following, see [19, Theorem 4.1]: Consider n independent random variables X_i with probability density function $f_i(x) = e^{-x}$. Set $S = \sum_{i=1}^n X_i$ and $Y_i = X_i/S$, then (Y_1, \dots, Y_n) follows the uniform distribution on the standard $(n-1)$ -simplex.

Furthermore, we are going to suppose that *agents have independent preferences*. This means for example that $\mu_1(W_j)$ is independent of $\mu_2(W_j)$. Thus, in the cake cutting situation the *Full independence model* means that we suppose that the following hypothesis holds:

(H): *Taking randomly a matrix of the following kind*

$$\mathcal{M} = \begin{pmatrix} \mu_1(W_1) & \dots & \mu_1(W_n) \\ \vdots & \dots & \vdots \\ \mu_n(W_1) & \dots & \mu_n(W_n) \end{pmatrix}$$

means that we consider a random matrix $\mathcal{M} = (m_{ij})$ where

$$m_{ij} = \frac{X_{ij}}{\sum_{j=1}^n X_{ij}}$$

and X_{ij} are independent exponential random variables, i.e. with a probability density function $f_{ij}(x) = e^{-x}$.

We remark that we have define what is a random matrix $\mathcal{M} = (\mu_i(W_j))$ and that we do not have define what is a random measure. Indeed, instead of taking random measures and then construct the matrix \mathcal{M} , we have directly define the

probability distribution for the matrix \mathcal{M} . This approach allows to obtain a simple and explicit probabilistic framework.

In the following, we are going to study an envy-free fair division algorithm. We will denote by $C(\mu_1, \dots, \mu_n)$ the number of queries used by this algorithm when the inputs are $\mu_1, \mu_2, \dots, \mu_n$.

Theorem 1. *If we suppose that the hypothesis (H) is satisfied, then we have the following result:*

There exists a protocol in the Robertson-Webb model of computation giving an envy-free fair division and such that for all $b > 4$ we have the following probability estimate

$$\mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{7+b}) = \mathcal{O}\left(n^{-\frac{b-1}{3}+1+o(1)}\right).$$

This theorem says: the bigger the number of queries, the smaller the probability.

We recall that $f(n) = \mathcal{O}(g(n))$ means that there exists a constant C and an integer n_0 such that for all $n \geq n_0$, we have $|f(n)| \leq Cg(n)$. The notation $o(1)$ refer to a function $f(n)$ such that $\lim_{n \rightarrow +\infty} f(n) = 0$.

Examples:

- If we choose $b = 5$ then

$$-\frac{b-1}{3} + 1 + o(1) = -\frac{1}{3} + o(1).$$

When n is big enough we can suppose $o(1) < 1/6$ and then in this case

$$\mathcal{O}\left(n^{-\frac{b-1}{3}+1+o(1)}\right) = \mathcal{O}(n^{-1/6}).$$

This gives

$$\lim_{n \rightarrow +\infty} \mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{12}) = 0.$$

- If we choose $b = 11$ then

$$-\frac{b-1}{3} + 1 + o(1) = -\frac{7}{3} + o(1).$$

When n is big enough we can suppose $o(1) < 1/3$ and then in this case

$$\mathcal{O}\left(n^{-\frac{b-1}{3}+1+o(1)}\right) = \mathcal{O}(n^{-2}).$$

Thus Theorem 1 gives

$$\mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{18}) = \mathcal{O}\left(\frac{1}{n^2}\right).$$

These bounds are not very sharp but they give a precise statement of the following idea: when n is big the probability that the algorithm uses more than n^{12} (or n^{18}) queries is very small.

Strategy of the algorithm and structure of the paper

The algorithm proposed in this article is just a slight modification of Webb's super envy-free division algorithm. Webb's algorithm constructs an envy-free division from the matrix $\mathcal{M} = (\mu_i(W_j))$ when $\det(\mathcal{M}) \neq 0$. The algorithm that we propose

works as follows: if $\det(\mathcal{M}) \neq 0$ then use Webb's algorithm else use another envy-free algorithm.

When the hypothesis (H) is satisfied the probability that $\det(\mathcal{M}) = 0$ is equal to zero. Thus, in practice our algorithm almost always corresponds to Webb's algorithm. As the number of queries needed in Webb's algorithm can be written in terms of the smallest singular value of \mathcal{M} , the strategy to prove Theorem 1 relies on a probabilistic study of the smallest singular value of \mathcal{M} .

The structure of this article is thus the following:

In the first section, we recall what is a super envy-free fair division and we also recall Webb's super envy-free algorithm. Then, we give our algorithm. In Section 2, we study the number of queries used by this algorithm. This leads us to recall some standard results on singular values of a matrix and to write the complexity of the algorithm in terms of the smallest singular value of the matrix \mathcal{M} . In Section 3, we use a theorem of Tao and Vu, see [36], about the probability that \mathcal{M} have a small singular value. This theorem will be the key point in the proof of Theorem 1.

1. THE ALGORITHM

1.1. Super envy-free algorithm. Super envy-free fair division is a strong kind of envy-free division. This notion has been introduced and studied by Barbanel, see [2, 3].

Definition 2. We say that a division is *super envy-free* when for all $i \neq j$, we have

$$\mu_i(\mathcal{C}_i) > \frac{1}{n} > \mu_i(\mathcal{C}_j).$$

This definition says that this division is proportional and all players think to have strictly more than other players. Of course, this kind of fair division is not always possible. For example, if $\mu_1 = \mu_2 = \dots = \mu_n$, then the previous inequality is not possible. Indeed, we cannot have $\mu_1(\mathcal{C}_1) > 1/n > \mu_2(\mathcal{C}_1) = \mu_1(\mathcal{C}_1)$. However, a super envy-free fair division exists when the measures μ_i are linearly independent.

Definition 3. Let μ_1, \dots, μ_n be n measures on a measurable set $(\mathcal{C}, \mathcal{B})$, where \mathcal{B} is the Borel σ -algebra. We say that these measures are linearly independent when they are linearly independent as functions from \mathcal{B} to $[0, 1]$.

Theorem 4 (Barbanel's theorem). *A super envy-free division exists if and only if the measures μ_1, \dots, μ_n are linearly independent.*

In the following we are going to use a witness matrix in order to know if the measures are linearly independent.

Definition 5. The witness matrix associated to the measures μ_1, \dots, μ_n is the matrix $\mathcal{M} = (\mu_i(W_j))$ where W_j is the interval $[(j-1)/n, j/n]$ and $j = 1, \dots, n$.

Remark 6. If $\det(\mathcal{M}) \neq 0$ then the measures μ_1, \dots, μ_n are linearly independent.

In [39], Webb gives a strategy to compute super envy-free fair division. In order to recall this strategy, we recall what is a ε *near-exact fair division*.

Definition 7. Let A be a measurable subset of \mathcal{C} .

We say that a division of $A = A_1 \sqcup \dots \sqcup A_n$ is ε near-exact in the ratios $(\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$, when for all i and j we have

$$\left| \frac{\mu_i(A_j)}{\mu_i(A)} - \alpha_j \right| < \varepsilon.$$

Now, we can describe Webb's algorithm.

Super Envy-free fair division algorithm

Inputs: A partition $\mathcal{C} = A_1 \sqcup \dots \sqcup A_n$, a matrix $\mathcal{M}_0 = (m_{ij})$ where $m_{ij} = \mu_i(A_j)$, \mathcal{M}_0 is non-singular.

Outputs: A super envy free division $\mathcal{C} = \mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$.

- (1) Compute $\mathcal{M}_0^{-1} = (\tilde{m}_{ij})$.
- (2) Set $\delta := \frac{n-1}{n(1-tn)}$ where $t = \min_{i,j}(\tilde{m}_{ij})$.
- (3) Set $\mathcal{N} := (n_{ij})$, where $n_{ii} := 1/n + \delta$ and $n_{ij} = 1/n - \delta/(n-1)$.
- (4) Compute $\mathcal{R} = (r_{ij}) := \mathcal{M}_0^{-1}\mathcal{N}$.
- (5) For $j = 1, \dots, n$ do
 Compute a $\varepsilon = \delta/n^2$ near-exact fair division of A_j in the ratios (r_{j1}, \dots, r_{jn}) ,
 this gives $A_j = A_{j1} \sqcup \dots \sqcup A_{jn}$.
- (6) For all $i = 1, \dots, n$ do
 $\mathcal{C}_i := A_{1i} \sqcup A_{2i} \sqcup \dots \sqcup A_{ni}$.

We remark that in Step 2, we have $t \leq 0$. Indeed, if $t > 0$ then the equality $\mathcal{M}_0\mathcal{M}_0^{-1} = I$ is impossible, because the coefficients m_{ij} are non-negative. Therefore in Step 2, we have $\delta > 0$. The formula used to define δ implies that the coefficients r_{ij} of \mathcal{R} are non-negative. It is a straightforward computation to check that the coefficients r_{ij} also satisfy $\sum_{j=1}^n r_{ij} = 1$.

In order to explain this algorithm, suppose that in Step 5 we compute an exact fair division in the ratios (r_{j1}, \dots, r_{jn}) instead of an ε near-exact fair division with these ratios. Then, by construction the partition $\mathcal{C} = \mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$ as the following property $\mu_i(\mathcal{C}_j) = n_{ij}$. This gives $\mu_i(\mathcal{C}_i) = 1/n + \delta$ and $\mu_i(\mathcal{C}_j) = 1/n - \delta/(n-1)$. Thus this partition gives a super envy-free division.

In practice, the computation of an exact fair division in the ratios (r_{j1}, \dots, r_{jn}) is impossible, since it has been proved that such algorithms cannot exist, see [31]. That is the reason why a ε near-exact algorithm is used. Indeed, a ε near-exact algorithm in the Robertson-Webb model exists, see [31, Theorem 10.2]. Therefore the idea is to choose a small enough ε in order to obtain a result very close to the theoretical result where $\varepsilon = 0$. Thus, we obtain in practice a partition where $\mu_i(\mathcal{C}_j)$ are very close to n_{ij} and then the division is super envy-free.

The number of queries used by the ε near-exact division algorithm is at most $n \times (2 + 2n^{3/2})/\varepsilon$, see [31, Theorem 10.2]. As already remarked in [31], this bound can be improved. However, we just want to get a bound on $C(\mu_1, \dots, \mu_n)$ in terms of a polynomial in n , thus this estimate is sufficient.

Therefore, the number of queries used by Webb's super envy-free algorithm is at

most $n^4 \times (2 + 2n^{3/2})/\delta$, since $\varepsilon = \delta/n^2$ and in Step 5 we compute n ε -near-exact fair divisions. Thanks to the definition of δ we get the following lemma.

Lemma 8. *The number of queries used by the super envy-free division algorithm in the Robertson-Wenn model is bounded by*

$$\frac{n^5 \times (2 + 2n^{3/2}) \times (1 - tn)}{n - 1} \in \mathcal{O}(|t|n^{6.5}).$$

1.2. An envy-free algorithm.

Envy-free fair division algorithm

Inputs: A cake $\mathcal{C} = [0, 1]$, n measures μ_1, \dots, μ_n .

Outputs: An envy free division $\mathcal{C} = \mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$.

- (1) % Construct the matrix $\mathcal{M} = (m_{ij})$ where $m_{ij} = \mu_i(W_j)$. %
 For all $i = 1, \dots, n$, do
 For all $j = 1, \dots, n$ do
 $m_{ij} := \text{eval}_i(W_j)$.
- (2) If $\det(\mathcal{M}) = 0$ then compute an envy-free fair division algorithm thanks to Aziz-Mackenzie's algorithm,
 Else compute a super envy-free fair division algorithm thanks to Webb's algorithm.

Remark 9. When $\det(\mathcal{M}) = 0$, we have to use an algorithm different from Webb's algorithm. Indeed, in this case Webb's algorithm is not defined (we cannot compute \mathcal{M}^{-1}). Furthermore, it is not necessary to use Aziz-Mackenzie's algorithm. The bound given in Theorem 1 will not change if we use another envy-free algorithm when $\det(\mathcal{M}) = 0$.

2. COMPLEXITY ANALYSIS

The number of queries used by our envy-free division algorithm is $\mathcal{O}(|t| \times n^{6.5})$ when $\det(\mathcal{M}) \neq 0$. In the next subsection, we are going to bound $|t|$ by σ_n^{-1} where σ_n is the smallest singular value of \mathcal{M} . Then, in the second subsection, we use an estimate on the probability $\mathbb{P}(\sigma_n \leq n^{-b})$ in order to prove our theorem.

2.1. The smallest singular value. We recall here the definition and a simple result about the singular values of a matrix.

Definition 10. The singular values of a matrix \mathcal{M} are the square roots of the eigenvalues of $\mathcal{M}^T \mathcal{M}$. They are denoted by $\sigma_1(\mathcal{M}) \geq \dots \geq \sigma_n(\mathcal{M})$.

Remark 11. We have $\det(\mathcal{M}) = 0 \iff \sigma_n(\mathcal{M}) = 0$.

The smallest singular value allows to bound the coefficients of the inverse of a matrix.

Proposition 12. *Let \mathcal{M} be a non-singular matrix, such that $\mathcal{M}^{-1} = (\tilde{m}_{ij})$.*

Let $\sigma_n(\mathcal{M})$ be the smallest singular value of \mathcal{M} . We have

$$\max_{ij} (|\tilde{m}_{ij}|) \leq \|\mathcal{M}^{-1}\|_2 = \sigma_n^{-1}(\mathcal{M}).$$

Proof. This is a classical result, see [23, Formula 2.3.8 page 56] and [18, Theorem 3.3]. \square

The previous proposition allows us to obtain an upper bound on the complexity of the super envy-free algorithm in terms of $\sigma_n(\mathcal{M})$.

Corollary 13. *When $n \geq 19$, the number of queries used by the super envy-free division algorithm in the Robertson-Webb model is bounded by*

$$n^7 \times \max(1, \sigma_n^{-1}(\mathcal{M})).$$

Proof. We have already remarked that in Step 2 of the super envy-free algorithm we have $t = \min_{i,j}(\tilde{m}_{ij}) \leq 0$. Then Proposition 12 gives

$$-t = |t| \leq \max_{i,j} |\tilde{m}_{ij}| \leq \sigma_n^{-1}(\mathcal{M}).$$

Then Lemma 8 implies that the number of queries used by the super envy-free division algorithm in the Robertson-Webb model is bounded by

$$\frac{n^5 \times (2 + 2n^{3/2}) \times (1 + n\sigma_n^{-1}(\mathcal{M}))}{n - 1}.$$

As we have supposed that $n \geq 19$, we get

$$\frac{2 + 2n^{3/2}}{n - 1} \leq \frac{n}{2}.$$

Then, we have

$$\begin{aligned} \frac{n^5 \times (2 + 2n^{3/2}) \times (1 + n\sigma_n^{-1}(\mathcal{M}))}{n - 1} &\leq \frac{n^6}{2} \times (1 + n\sigma_n^{-1}(\mathcal{M})) \\ &\leq \frac{n^7}{2} (1 + \sigma_n^{-1}(\mathcal{M})) \\ &\leq n^7 \times \max(1, \sigma_n^{-1}(\mathcal{M})). \end{aligned}$$

□

The singular value $\sigma_n(\mathcal{M})$ measures how far \mathcal{M} is from a singular matrix. Therefore, if $\sigma_n(\mathcal{M})$ is small then the measures μ_i are nearly linearly dependent and the previous corollary shows that the number of queries is big. This result satisfies the general result: if the agents have “very different” preferences it will be easier to get an envy-free fair division. A precise statement of this result with an explicit bound has been given in [17, Corollary 17].

2.2. Proof of Theorem 1.

We consider the event $C(\mu_1, \dots, \mu_n) \geq n^{7+b}$.

During the envy-free algorithm two situations appear:

First, $\det(\mathcal{M}) \neq 0$, then in this situation the algorithm used the super envy-free algorithm. Thanks to Corollary 13, the number of queries used in this situation satisfies

$$n^{7+b} \leq C(\mu_1, \dots, \mu_n) \leq n^7 \times \max(1, \sigma_n^{-1}(\mathcal{M})).$$

As $n > 1$, it follows

$$n^b \leq \sigma_n^{-1}(\mathcal{M}).$$

This means that we have the following inclusion

$$(\star) \quad \{\det(\mathcal{M}) \neq 0\} \cap \{C(\mu_1, \dots, \mu_n) \geq n^{7+b}\} \subset \{\sigma_n(\mathcal{M}) \leq n^{-b}\}.$$

The second situation corresponds to $\det(\mathcal{M}) = 0$. Thus the second situation corresponds to $\sigma_n(\mathcal{M}) = 0$ and obviously $\sigma_n(\mathcal{M}) \leq n^{-b}$. This gives the following inclusion

$$(\star\star) \quad \{\det(\mathcal{M}) = 0\} \subset \{\sigma_n(\mathcal{M}) \leq n^{-b}\}.$$

Thanks to (\star) and $(\star\star)$ we get

$$\{C(\mu_1, \dots, \mu_n) \geq n^{7+b}\} \subset \{\sigma_n(\mathcal{M}) \leq n^{-b}\}.$$

We deduce then the following inequality between probabilities

$$\mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{7+b}) \leq \mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b}).$$

In the next section we are going to prove the following proposition.

Proposition 14. *If we suppose that hypothesis (H) is satisfied then the following holds:*

Let $b > 4$ be a constant, then there exists a constant $c > 0$ depending on b such that

$$\mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b}) \leq c \left(n^{-\frac{b-1}{3}+1+o(1)} + ne^{-\sqrt{n}} + n^2 e^{-\sqrt{n}-1} \right).$$

In order to finish the proof of Theorem 1, we remark that for all $b > 4$ we have $ne^{-\sqrt{n}} + n^2 e^{-\sqrt{n}-1} = \mathcal{O}(n^{-\frac{b-1}{3}+1})$. Therefore, we get

$$\mathbb{P}(C(\mu_1, \dots, \mu_n) \geq n^{7+b}) \leq \mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b}) = \mathcal{O}\left(n^{-\frac{b-1}{3}+1+o(1)}\right),$$

which gives the desired estimate.

3. AN ESTIMATE FOR $\mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b})$

In this section we prove Proposition 14.

In order to bound $\mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b})$, we are going to introduce some notations.

As we suppose that hypothesis H is satisfied we have

$$\mathcal{M} = \mathcal{D}\mathcal{X},$$

where \mathcal{D} is the diagonal matrix with coefficients in the i -th row equal to $1/(\sum_{j=1}^n X_{ij})$, \mathcal{X} is the matrix with coefficients X_{ij} and X_{ij} are independent exponential random variables with probability density function $f_{ij} = e^{-x}$.

We also consider the two following events:

$$A = \{\sigma_n(\mathcal{M}) \leq n^{-b}\} \cap \{\sigma_n(\mathcal{D}) \leq n^{-3/2}\}.$$

$$B = \{\sigma_n(\mathcal{M}) \leq n^{-b}\} \cap \{\sigma_n(\mathcal{D}) \geq n^{-3/2}\}.$$

Obviously, we have

$$(\#) \quad \{\sigma_n(\mathcal{M}) \leq n^{-b}\} = A \cup B.$$

Now, we are going to bound $\mathbb{P}(A)$ and $\mathbb{P}(B)$.

Lemma 15. *We have*

$$\mathbb{P}(A) \leq ne^{-\sqrt{n}}.$$

Proof. We have $A \subset \{\sigma_n(\mathcal{D}) \leq n^{-3/2}\}$. Thus

$$\mathbb{P}(A) \leq \mathbb{P}(\sigma_n(\mathcal{D}) \leq n^{-3/2}).$$

As \mathcal{D} is a diagonal matrix with coefficients $1/(\sum_{j=1}^n X_{ij})$, we have

$$\begin{aligned} \sigma_n(\mathcal{D}) \leq n^{-3/2} &\Rightarrow \min_i \left(\frac{1}{X_{i1} + \dots + X_{in}} \right) \leq n^{-3/2} \\ &\Rightarrow X_{11} + \dots + X_{1n} \geq n^{3/2} \\ &\Rightarrow \exists i, X_{1i} \geq \sqrt{n}. \end{aligned}$$

We set $A_i = \{X_{1i} \geq \sqrt{n}\}$, we have

$$A \subset \cup_{i=1}^n A_i.$$

Furthermore,

$$\mathbb{P}(A_i) = \int_{\sqrt{n}}^{+\infty} e^{-x} dx = e^{-\sqrt{n}}.$$

Therefore, we get

$$\mathbb{P}(A) \leq \sum_{i=1}^n \mathbb{P}(A_i) = ne^{-\sqrt{n}}.$$

□

In order to give a bound on $\mathbb{P}(B)$ we are going to use the following theorem due to Tao and Vu, see [36], and see also the erratum in [37].

Theorem 16. *Let Y be a random variable with mean zero and bounded second moment, and let $\gamma \geq 1/2$, $a \geq 0$ be constants. Then there is a constant c depending on Y , γ , and a such that the following holds. Let \mathcal{Y} be the random matrix of size n whose entries are independent and identically distributed copies of Y , let M be a deterministic matrix satisfying $\|M\|_2 \leq n^\gamma$. Then*

$$\mathbb{P}(\sigma_n(M + \mathcal{Y}) \leq n^{-(2a+2)\gamma+1/2}) \leq c \left(n^{-a+o(1)} + \mathbb{P}(\|\mathcal{Y}\|_2 \geq n^\gamma) \right).$$

In the following we will need the following lemma.

Lemma 17. *Let \mathcal{A} and \mathcal{B} be two $n \times n$ matrices, we have*

$$\sigma_n(\mathcal{A}) \times \sigma_n(\mathcal{B}) \leq \sigma_n(\mathcal{AB}).$$

Proof. When \mathcal{M} is an $n \times n$ matrix we have $\|\mathcal{M}^{-1}\|_2^{-1} = \sigma_n(\mathcal{M})$, see [18, Theorem 3.3]. Thus

$$\frac{1}{\sigma_n(\mathcal{AB})} = \|(\mathcal{AB})^{-1}\|_2 = \|\mathcal{B}^{-1}\mathcal{A}^{-1}\|_2 \leq \|\mathcal{B}^{-1}\|_2 \times \|\mathcal{A}^{-1}\|_2.$$

Therefore $\sigma_n(\mathcal{AB}) \geq \|\mathcal{B}^{-1}\|_2^{-1} \times \|\mathcal{A}^{-1}\|_2^{-1}$, which gives the desired result. □

Proposition 18. *There exists a constant c such that the following holds*

$$\mathbb{P}(B) \leq c \left(n^{-\frac{b-1}{3}+1+o(1)} + n^2 e^{-\sqrt{n}-1} \right).$$

Proof. With our notations we have $\mathcal{M} = \mathcal{D}\mathcal{X}$ and by Lemma 17 we have

$$(\star) \quad \sigma_n(\mathcal{D}) \times \sigma_n(\mathcal{X}) \leq \sigma_n(\mathcal{M}).$$

If the event B is realized then by definition we have

$$\sigma_n(\mathcal{M}) \leq n^{-b} \text{ and } \sigma_n(\mathcal{D}) \geq n^{-3/2}.$$

The inequality (\star) implies

$$\sigma_n(\mathcal{X}) \leq n^{-b+3/2}.$$

We set $C = \{\sigma_n(\mathcal{X}) \leq n^{-b+3/2}\}$. We have thus shown $B \subset C$.

Now, we are going to applied Theorem 16 to $\sigma_n(\mathcal{X})$.

We set $Y = X - 1$, where X is the exponential distribution with the probability density function $f(x) = e^{-x}$.

Furthermore, we denote by M the $n \times n$ matrix with all its entries equal to 1. We denote by $Y = (Y_{ij})$ the $n \times n$ matrix where its coefficients Y_{ij} are independent and identically distributed copies of Y . Therefore, the matrix $M + \mathcal{Y}$ corresponds to our matrix \mathcal{X} .

Furthermore, we remark easily that we have $\|M\|_2 = n$. Then we can set $\gamma = 3/2$. Now, we are giving a bound on $\mathbb{P}(\|\mathcal{Y}\|_2 \geq n^{3/2})$. We recall the classical bound, see [23],

$$\|\mathcal{Y}\|_2 \leq n \max_{i,j} |Y_{ij}|,$$

where $\mathcal{Y} = (Y_{ij})$. Therefore, we have

$$\|\mathcal{Y}\|_2 \geq n^{3/2} \Rightarrow \max_{i,j} |Y_{ij}| \geq \sqrt{n}.$$

We denote by $C_{i,j}$ the following set

$$C_{i,j} = \{|Y_{ij}| \geq \sqrt{n}\}.$$

We deduce then the following inclusion

$$\{\|\mathcal{Y}\|_2 \geq n^{3/2}\} \subset \cup_{i,j=1}^n C_{i,j}.$$

By definition of \mathcal{Y} , we have $Y_{ij} = X_{ij} - 1$ where X_{ij} follow the exponential distribution. Therefore $X_{ij} \geq 0$ and

$$|Y_{ij}| \geq \sqrt{n} \iff |X_{ij} - 1| \geq \sqrt{n} \iff X_{ij} \geq \sqrt{n} + 1.$$

This gives

$$\mathbb{P}(C_{i,j}) = \mathbb{P}(|Y_{ij}| \geq \sqrt{n}) = \mathbb{P}(X_{ij} \geq \sqrt{n} + 1) = \int_{\sqrt{n}+1}^{+\infty} e^{-x} dx = e^{-\sqrt{n}-1},$$

it follows

$$\mathbb{P}(\|\mathcal{Y}\|_2 \geq n^{3/2}) \leq \sum_{i,j=1}^n \mathbb{P}(C_{i,j}) \leq n^2 e^{-\sqrt{n}-1}.$$

Then Theorem 16 gives

$$\mathbb{P}\left(\sigma_n(M + \mathcal{Y}) \leq n^{-3(2a+2)/2+1/2}\right) \leq c\left(n^{-a+o(1)} + n^2 e^{-\sqrt{n}-1}\right).$$

By construction, we have $M + \mathcal{Y} = \mathcal{X}$, then if we set

$$-b + \frac{3}{2} = -(2a + 2) \times \frac{3}{2} + \frac{1}{2}$$

then we have

$$a = \frac{b-1}{3} - 1$$

and

$$\mathbb{P}(\sigma_n(\mathcal{X}) \leq n^{-b+3/2}) \leq c \left(n^{-\frac{b-1}{3}+1+o(1)} + n^2 e^{-\sqrt{n}-1} \right).$$

□

Now, we can prove Proposition 14.

Thanks to (#), we have $\mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b}) \leq \mathbb{P}(A) + \mathbb{P}(B)$.

Furthermore, by Lemma 15 and Proposition 18, there exists a constant c such that

$$\mathbb{P}(\sigma_n(\mathcal{M}) \leq n^{-b}) \leq \mathbb{P}(A) + \mathbb{P}(B) \leq c \left(n^{-\frac{b-1}{3}+1+o(1)} + n e^{-\sqrt{n}} + n^2 e^{-\sqrt{n}-1} \right),$$

which gives the desired result.

CONCLUSION

We have shown that, with high probability, the use of an unbounded algorithm can be more efficient than a bounded algorithm. Indeed, in Webb's algorithm we cannot bound the number of queries in term of the number of players, but if we use this algorithm in our probabilistic framework, then $\mathbb{P}(C(\mu_1, \dots, \mu_n)) \geq n^{19}$ is small.

REFERENCES

- [1] H. Aziz and S. Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. In *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCSS 2016, 9-11 October 2016*, pages 416–427, 2016.
- [2] J. Barbanel. Super envy-free cake division and independence of measures. *Journal of Mathematical Analysis and Applications*, 197(1):54 – 60, 1996.
- [3] J. Barbanel. *The geometry of efficient fair division*. Cambridge University Press, 2005.
- [4] S. Bouveret and J. Lang. A general elicitation-free protocol for allocating indivisible goods. In *Proceedings of the 22st International Joint Conference on Artificial Intelligence (IJCAI'11)*, Barcelona, Spain, July 2011.
- [5] S. Brams, M. Jones, and C. Klamler. N -person cake-cutting: There may be no perfect division. *The American Mathematical Monthly*, 120(1):35–47, 2013.
- [6] S. Brams and A. Taylor. An envy-free cake division protocol. *The American Mathematical Monthly*, 102(1):9–18, 1995.
- [7] S. Brams and A. Taylor. *Fair division - from cake-cutting to dispute resolution*. Cambridge University Press, 1996.
- [8] S. Brânzei and P. Miltersen. A dictatorship theorem for cake cutting. In Qiang Yang and Michael Wooldridge, editors, *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015*, pages 482–488. AAAI Press, 2015.
- [9] S. Brânzei and N. Nisan. The query complexity of cake cutting. *ArXiv e-prints*, abs/1705.02946, 2017.
- [10] K. Cechlárová, J. Doboš, and E. Pillárová. On the existence of equitable cake divisions. *Information Sciences*, 228(Supplement C):239 – 245, 2013.
- [11] K. Cechlárová and E. Pillárová. On the computability of equitable divisions. *Discrete Optimization*, 9(4):249 – 257, 2012.
- [12] Y. Chen, J. Lai, D. Parkes, and A. Procaccia. Truth, justice, and cake cutting. *Games and Economic Behavior*, 77(1):284 – 297, 2013.
- [13] Y. Chevaleyre, P. Dunne, U. Endriss, J. Lang, M. Lematre, N. Maudet, J. Padget, S. Phelps, J. Rodriguez-Aguilar, and P. Sousa. Issues in multiagent resource allocation. *INFORMATICA*, 30:3–31, 2006.
- [14] G. Chèze. Existence of a simple and equitable fair division: A short proof. *Mathematical Social Sciences*, 87:92 – 93, 2017.

- [15] G. Chèze. Cake cutting: Explicit examples for impossibility results. *Mathematical Social Sciences*, 102:68 – 72, 2019.
- [16] G. Chèze. Don't cry to be the first! Symmetric fair division exist, <http://arxiv.org/abs/1804.03833>, 2018.
- [17] G. Chèze and L. Amodei. How to cut a cake with a gram matrix. *Linear Algebra and its Applications*, 560:114 – 132, 2019.
- [18] J. Demmel. *Applied Numerical Linear Algebra*. Society for Industrial and Applied Mathematics, USA, 1997.
- [19] L. Devroye. *Non-Uniform Random Variate Generation*. Springer-Verlag, 1986.
- [20] L. Dubins and E. Spanier. How to cut a cake fairly. *The American Mathematical Monthly*, 68(1):1–17, 1961.
- [21] J. Edmonds and K. Pruhs. Cake cutting really is not a piece of cake. *ACM Trans. Algorithms*, 7(4):51, 2011.
- [22] S. Even and A. Paz. A note on cake cutting. *Discrete Applied Mathematics*, 7(3):285 – 296, 1984.
- [23] G. Golub and C. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, third edition, 1996.
- [24] I. Kash, A. Procaccia, and N. Shah. No agent left behind: dynamic fair division of multiple resources. In *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS '13, Saint Paul, MN, USA, May 6-10, 2013*, pages 351–358, 2013.
- [25] O. Pikhurko. On envy-free cake division. *The American Mathematical Monthly*, 107(8):736–738, 2000.
- [26] A. Procaccia. Thou shalt covet thy neighbor's cake. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence, IJCAI'09*, pages 239–244, San Francisco, CA, USA, 2009. Morgan Kaufmann Publishers Inc.
- [27] A. Procaccia. Cake cutting: Not just child's play. *Commun. ACM*, 56(7):78–87, July 2013.
- [28] A. Procaccia. Cake cutting algorithms. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 13. Cambridge University Press, 2016.
- [29] A. Procaccia and J. Wang. A lower bound for equitable cake cutting. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, EC 17, page 479495, New York, NY, USA, 2017. Association for Computing Machinery.
- [30] J. Robertson and W. Webb. Near exact and envy-free cake division. *Ars Combinatoria*, 45:97–108, 1997.
- [31] J. Robertson and W. Webb. *Cake-cutting algorithms - be fair if you can*. A K Peters, 1998.
- [32] E. Segal-Halevi and B. Sziklai. Resource-monotonicity and population-monotonicity in connected cake-cutting. *Mathematical Social Sciences*, 95:19 – 30, 2018.
- [33] H. Steinhaus. The problem of fair division. *Econometrica*, 16(1):101–104, January 1948.
- [34] W. Stromquist. How to cut a cake fairly. *Amer. Math. Monthly*, 87(8):640–644, 1980.
- [35] W. Stromquist. Envy-free cake divisions cannot be found by finite protocols. *Electr. J. Comb.*, 15(1), 2008.
- [36] T. Tao and V. Vu. Smooth analysis of the condition number and the least singular value. *Math. Comput.*, 79(272):2333–2352, 2010.
- [37] T. Tao and V. Vu. Smooth analysis of the condition number and the least singular value, <https://arxiv.org/abs/0805.3167>, 2017.
- [38] W. Thomson. Children crying at birthday parties. Why? *Economic Theory*, 31(3):501–521, 2006.
- [39] W. Webb. An algorithm for super envy-free cake division. *Journal of Mathematical Analysis and Applications*, 239(1):175 – 179, 1999.
- [40] G. Woeginger and J. Sgall. On the complexity of cake cutting. *Discrete Optimization*, 4(2):213 – 220, 2007.