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A simplified method for random vibration analysis of structures with random parameters

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Abstract. Piezoelectric patches with adapted electrical circuits or viscoelastic dissipative materials are two solutions particularly adapted to reduce vibration of light structures. To accurately design these solutions, it is necessary to describe precisely the dynamical behaviour of the structure. It may quickly become computationally intensive to describe robustly this behaviour for a structure with nonlinear phenomena, such as contact or friction for bolted structures, and uncertain variations of its parameters. The aim of this work is to propose a non-intrusive reduced stochastic method to characterize robustly the vibrational response of a structure with random parameters. Our goal is to characterize the eigenspace of linear systems with dynamic properties considered as random variables. This method is based on a separation of random aspects from deterministic aspects and allows us to estimate the first central moments of each random eigenfrequency with a single deterministic finite elements computation. The method is applied to a frame with several Young's moduli modeled as random variables. This example could be expanded to a bolted structure including piezoelectric devices. The method needs to be enhanced when random eigenvalues are closely spaced. An indicator with no additional computational cost is proposed to characterize the "proximity" of two random eigenvalues.

1. Introduction

In order to accurately predict the vibrational response of a structure, uncertainty modeling and quantification in computational mechanics have received particular attention in recent years. This uncertainty, mainly due to the system parameters variability, is taken into account in a probabilistic approach, well adapted to numerical resolution [1]. We are looking for the eigenspace characterization of linear systems with dynamic properties considered as random variables.

The matrices of interest in the present paper are the result of a finite-dimensional approximation of an underlying continuous system and their randomness is tied to the uncertainty in the parameters of this system. For such systems, closed-form expressions are generally not available for the solution of the random eigenvalue problem. Current approaches to this problem include statistical sampling, perturbation techniques, and polynomial chaos representations coupled with Galerkin projections.

Methods based on statistical sampling provide a good framework to solve the random dynamic problem, nevertheless they need intensive computation to remain accurate [2, 3]. The total computational cost increases dramatically with the cost of each deterministic case. For these



reasons, different non-sampling methods have been developed over the last few decades. Two particular approaches are mainly used in the literature to approximate the statistical properties of the response of a system : the perturbation and the spectral method.

The perturbation method is based on an approximation of the random variables of interest through the truncation of its Taylor expansion. Its implementation is pretty easy but as high order perturbation terms are computationally intensive, the expansion is generally limited to second order. Moreover, variations of the system parameters should remain small to guarantee accurate estimation of statistical moments [4]. Many papers are based on the perturbation method to estimate statistics of random eigenvalues and eigenvectors [5, 6, 7].

The Spectral Stochastic Finite Element Method (SSFEM) was introduced by Ghanem and Spanos in [8], inspired by Wiener [9]. The method is based on a representation of the random variables and vectors of interest with respect to a basis set of orthogonal multidimensional polynomials of orthonormal random variables.

In order to characterize the eigenspace of a system with random dynamic properties, the Polynomial Chaos method gives a general and accurate framework but its implementation is rather complex. On the other hand, perturbation methods are easy to implement but their intrinsic assumptions limit their applications to small variations of the input parameters.

The aim of this paper is to propose an indicator to be able to discriminate which method is more adapted : perturbation method for most of the cases or SSFEM for some cases. The estimate of this indicator is based on a reduced number of deterministic finite element computations.

The first section describes the proposed method to optimize the computational cost. The appropriate indicator referred to as the "Proximity Factor" is developed in the next section. A three-degrees-of-freedom test case illustrates the accuracy of the method depending on the random eigenfrequencies configuration and the capability of the Proximity Factor to predict with sufficient accuracy and at low cost in with case a more accurate method like SSFEM must be used. In the last section, the method is applied to a more realistic system consisting of a frame with different random Young's moduli.

2. Eigenvalue problem

2.1. Problem presentation

The general eigenvalue problem of an undamped system can be expressed by

$$\lambda_k(\theta)\mathbf{M}(\theta)\Phi_k(\theta) = \mathbf{K}(\theta)\Phi_k(\theta) \quad (1)$$

where

$$\lambda_k(\theta) \in \mathbb{R}, \quad \Phi_k(\theta) \in \mathbb{R}^n, \quad \mathbf{M}(\theta) \in \mathbb{R}^{n \times n}, \quad \mathbf{K}(\theta) \in \mathbb{R}^{n \times n}, \quad \theta \in \Omega$$

λ_k and Φ_k are the k^{th} eigenvalue and the k^{th} associated eigenvector. The relationship between the eigenvalues and the natural frequencies of the system is $\lambda_k = \omega_k^2$. The eigenvector Φ_k is assumed to be mass-normalized such that $\Phi_k^T \mathbf{M} \Phi_k = \mathbf{1}$, the identity matrix.

(Ω, \mathcal{F}, p) is the probability space associated with the underlying physical experiments. The space of square integrable random variables is denoted by $L_2(\Omega)$ and forms a Hilbert space with the norm $\|\cdot\|_{L_2(\Omega)}$. Matrices $\mathbf{M}(\theta)$ and $\mathbf{K}(\theta)$ represent the mass and stiffness matrices of the structure. Their randomness is due to physical parameters of the structure such as mass density, Young's modulus or geometric properties. In this paper, $E[\cdot]$, $\text{Var}[\cdot]$ and $\sigma[\cdot]$ denote respectively the mathematical expectation, variance and standard deviation.

Let us first consider the case of a structure composed of n sub-structures with n different Young's moduli (Y_1, Y_2, \dots, Y_n) . The Young's moduli are assumed to be the only random

parameters : the mass matrix \mathbf{M} is supposed deterministic. The stiffness matrix can be written according to different stiffness sub-matrices relating to each sub-structure:

$$\mathbf{K}(\theta) = \sum_{i=1}^n \mathbf{K}_i(\theta) = \sum_{i=1}^n \gamma_i(\theta) \mathbf{K}_i \quad (2)$$

where $\gamma_i(\theta)$ is the random parameter corresponding to the i th Young's modulus and defined as:

$$\gamma_i(\theta) = \frac{Y_i(\theta)}{Y_{0i}} \quad ; \quad Y_{0i} = E[Y_i(\theta)] \quad \text{and} \quad E[\gamma_i(\theta)] = 1$$

To construct, as objectively as possible, the probability law of the input parameters, in this case the Young's moduli $Y_i(\theta)$, some available information has to be taken into account [1]. The pdf could be constructed through the maximum entropy principle [10]. For example, a real-valued random variable X such as $\text{Supp}(X) =]0, +\infty[$, $E[X] = m_x$ and $\text{Var}[X] = \sigma_x^2$ given follows a Gamma distribution. Young's moduli can then be modeled as random variables with Gamma distribution $Y_i \rightsquigarrow \text{Gamma}(\alpha_i, \beta_i) = \Gamma(\alpha_i, \beta_i)$ with $E[Y_i] = \alpha_i \beta_i$ and $\text{Var}[Y_i] = \alpha_i \beta_i^2$.

2.2. A simple resolution approach : Stochastic Model Reduction method

2.2.1. Initial Stochastic Model Reduction method. It is assumed that Young's moduli variations around an expected value do not change the eigenvector shapes of the structure. The matrix of random eigenvectors corresponds to the matrix of eigenvectors calculated with the mean parameters.

This assumption added to the mass normalization of eigenvectors Φ_k allows us to rewrite the k^{th} random frequency as :

$$\lambda_k(\theta) = \omega_k^2(\theta) = \sum_{i=1}^n \gamma_i(\theta) \Phi_k^T \mathbf{K}_i \Phi_k = \sum_{i=1}^n \lambda_{ki} \gamma_i(\theta) \quad (3)$$

where

$$\lambda_{ki} = \Phi_k^T \mathbf{K}_i \Phi_k \quad (4)$$

The coefficients of this expansion are deterministic. This simplified method is referred to hereafter as SMR1 method, for "Initial Stochastic Model Reduction method".

As the n Young's moduli $(Y_i)_{1 \leq i \leq n}$ are assumed to be independent random variables, each random eigenvalue $\lambda_k(\theta)$ obtained by (3) is completely determined. This complete probabilistic description allows us to have an exact estimate of the expected value and of the standard deviation of $\lambda_k(\theta)$. The expected value of the k^{th} eigenvalue is obtained directly :

$$E[\lambda_k] = \sum_{i=1}^n E[\gamma_i] \Phi_k^T \mathbf{K}_i \Phi_k = \sum_{i=1}^n \Phi_k^T \mathbf{K}_i \Phi_k = \sum_{i=1}^n \lambda_{ki} = \bar{\lambda}_k \quad (5)$$

where $\bar{\lambda}_k$ represents the deterministic eigenvalue associated with the deterministic eigenvalue problem. The variance of the k^{th} eigenvalue is :

$$\text{Var}[\lambda_k] = \sum_{i=1}^n \lambda_{ki}^2 \text{Var}[\gamma_i] = \sum_{i=1}^n \lambda_{ki}^2 \delta_i^2 \quad (6)$$

where $\delta_i = \sigma[Y_i]/E[Y_i]$ is the coefficient of variation of the random parameter Y_i .

2.2.2. Enhanced SMR method. The working assumption whereby eigenvectors can be assumed deterministic is valid if the eigenvalues are not too close. In this case, in the vicinity of the k^{th} natural frequency, the contributions of distant eigenshapes to the corresponding k^{th} eigenshape are negligible and can be neglected. The k^{th} eigenvalue is then approximated by equation (3). When eigenvalues are getting closer, the variability of the corresponding eigenshapes increases and has to be taken into account. In order to accurately estimate slightly close eigenvalues, the SMR1 method could be improved by taking into account the eigenshapes variability. For this purpose, it is proposed to expand the eigenvectors corresponding to close eigenvalues by their Taylor expansions around $E[Y(\theta)] = Y_0$. This improvement is referred hereafter as SMR2.

The k^{th} random eigenvalue $\lambda_k(\gamma(\theta))$ and the k^{th} eigenvector $\Phi_k(\gamma(\theta))$ given by its Taylor series expansions around the point $E[\gamma(\theta)] = \mathbf{1}$ can be written as:

$$\lambda_k(\theta) = \sum_{i=1}^n \gamma_i(\theta) \Phi_k^T(\gamma(\theta)) \mathbf{K}_i \Phi_k(\gamma(\theta)) \quad (7)$$

$$\Phi_k(Y(\theta)) = \Phi_k(Y_0) + \sum_{i=1}^n \left. \frac{\partial \Phi_k}{\partial \gamma_i} \right|_{\gamma_i=1} (\gamma_i(\theta) - 1) \quad (8)$$

The derivative of each eigenvector remains to be determined. Many papers on analysis and calculation of eigenderivatives of dynamic systems [11, 12, 13] are based on methods developed by Fox and Kapoor [14]. In order to minimize the computational cost of SMR2, Lin et al. [15] propose an efficient algorithm to compute eigenvector derivatives.

2.2.3. Application to complex structures. For complex structures, the set of random eigenvalues will contain well separated as well as close eigenvalues, the two refinement levels of the SMR method have to be jointly used to ensure the best estimate of the statistical moments of all the set of random eigenvalues. To this purpose, a simple resolution approach is summarized on figure 1. This approach consists in the adaptation of the invested computational resources to each considered random eigenvalue. First, all the random eigenvalue statistical moments are estimated through the SMR1 method, assuming that all the eigenvectors are deterministic. This first step provides an initial estimate of the statistical moments of all the eigenvalues which allows us to identify the subset of well separated eigenvalues. Since the SMR1 method is accurate for well separated random eigenvalues, these identifications will have a sufficient level of confidence. For random eigenvalues identified as close, it is then proposed to better estimate their statistical moments using the SMR2 method. The SMR2 method more demanding on computational cost is applied only on a reduced set of eigenvalues. When the quality of the SMR2 estimation is not sufficient enough, the random eigenvector could be modeled by their polynomial chaos (PC) expansion. The coefficients of the eigenvector PC expansion are rather complex to determine [16]. We propose to use the PC expansion only for the random eigenvalues for which the SMR2 accuracy is not sufficient.

The efficiency of the proposed approach now depends on the ability to qualify the proximity of two random eigenvalues. The notion of “close eigenvalue” appears in different papers about eigenvalue curve veering [17], statistical energy analysis [18], eigenvalues and eigenvectors derivatives [15] and, a fortiori, random eigenvalue problems [6]. In the next section, we introduce a proximity factor to discriminate the different cases and associated models.

3. Proximity Factor

3.1. Definition of a Proximity Factor

Du Bois *et al.* [17] propose a modal coupling factor analogous to the coupling factor of Perkins and Mote [19]. This factor is based on the stiffness and mass matrix sensitivities, which are

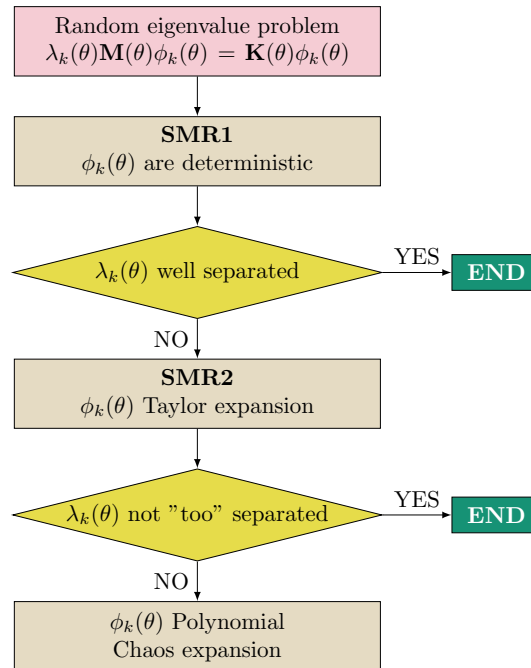


Figure 1. SMR approach for eigenvalue problem resolution

deterministic and available from commercial software. An indicator could be constructed on this basis in order to take into account the whole set of input random parameters but it would be the object of future work. In this paper, an indicator inspired by the statistical overlap factor of Manohar and Keane [18] seems to be more suitable because of its simplicity and its low computational cost. The statistical overlap is defined as the ratio between the standard deviation of the k^{th} natural frequency and the mean modal spacing. In order to take into account a potential difference between the standard deviation of two random eigenvalues, the Proximity Factor (PF) of two random eigenvalues λ_i, λ_{i+1} is defined as :

$$PF(\lambda_i) = \frac{2(\sigma[\lambda_i] + \sigma[\lambda_{i+1}])}{E[\lambda_{i+1}] - E[\lambda_i]} \quad (9)$$

As the expectation and the standard deviation of every random eigenvalue are estimated through the SMR methods, the proximity factor is obtained directly. The Proximity Factor allows us to predict the quality of the estimation computed with the SMR methods : when the proximity factor is greater than a limit value, initially assumed as $PF > 1$, the corresponding eigenvalues are assumed to be closely spaced and the SMR assumptions (random eigenvectors are deterministic for SMR1 and have small variations for SMR2) are not valid any more. To qualify the indicator, we need to ensure that a strong relation exist between the value of the indicator and the quality of the SMR results.

3.2. Characterization of the Proximity Factor

In order to qualify the Proximity Factor a simple three-degrees-of-freedom (Dof) undamped spring-mass system is considered. This example is taken from [20] and [6] and presented in figure 2. The advantage of this example is to easily drive the system eigenvalues with only one of the input random parameters. This allows us to characterize the quality of the SMR method when eigenvalues are well separated or close and the pertinence of the Proximity Factor as a good indicator.

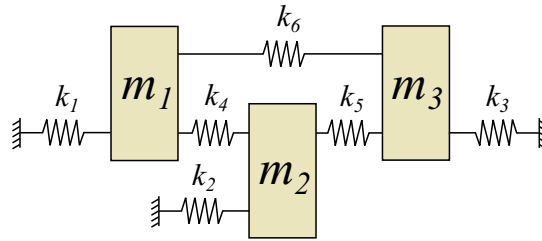


Figure 2. Three-degrees-of-freedom undamped spring-mass random system

The mass and stiffness matrices of this three Dof system are given by:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_2 + k_4 + k_5 & -k_5 \\ -k_6 & -k_5 & k_3 + k_5 + k_6 \end{bmatrix} \quad (10)$$

It is assumed that only spring stiffnesses k_i with $i = 1, \dots, 6$ are randomly varying and the vector of the random stiffnesses is noted $\mathbf{x} = [k_1, \dots, k_6]^t$. Each random stiffness is assumed to have a Gamma distribution with expectation $\bar{k}_i = 1 \text{ N}\cdot\text{m}^{-1}$ for $i = 1, \dots, 5$. The mean value of k_6 is varying between $1.275 \text{ N}\cdot\text{m}^{-1}$ and $3 \text{ N}\cdot\text{m}^{-1}$, this allows us to bring the second and third random eigenvalues closer. $\bar{k}_6 = 3 \text{ N}\cdot\text{m}^{-1}$ corresponds to the case of well separated eigenvalues, $\bar{k}_6 = 1.275 \text{ N}\cdot\text{m}^{-1}$ to the case of close eigenvalues. The standard deviation of each random stiffness is $\sigma_{k_i} = 0.15 \text{ N}\cdot\text{m}^{-1}$ for $i = 1, \dots, 6$.

A Monte-Carlo simulation allows us to compute the pdf and the two first statistical moments of the random eigenvalues. The samples of the six independent Gamma random variables k_i for $i = 1, \dots, 6$ are generated and the eigenvalues are computed from the eigenvalue problem (1). A simulation with 30000 samples guarantees the estimation of the two first statistical moments with an error range of $\pm 0.1\%$. Results from this Monte-Carlo simulation are considered as reference to evaluate the quality of the SMR method.

Figure 3 represents the relative error of the estimation of the expectation and the standard deviation of the 2nd and 3rd random eigenvalues of the three-Dof system. This figure illustrates the decrease of the quality of the SMR1 and SMR2 methods when the Proximity Factor is increasing : when the Proximity Factor is increasing, the quality of the first moments estimation decreases. For Proximity Factors less than 1 the relative error of the SMR1 method is less than 0.7% for the expectation estimation and less than 3% for the standard deviation. The same quality is obtained for higher Proximity Factor with the SMR2 method, then for a given quality, the limit value of the Proximity Factor should be adapted to the employed method. The Proximity Factor is then a good indicator for the quality of SMR1 and SMR2 methods.

3.3. Error on the Proximity Factor estimation

The Proximity Factor qualifies the quality of the SMR methods. Nevertheless, the error of the first moments estimation by SMR1 leads to a certain approximation of the Proximity Factor itself. Two eigenvalues are considered close depending on their expectation and their standard deviation. It is then proposed to study the estimation of the Proximity Factor with SMR1 and SMR2 when the expectation and the standard variation of the corresponding random eigenvalues vary.

Figure 4 presents the relative error between the reference Proximity Factor of the 2nd and 3rd random eigenvalues and its estimation obtained with SMR1 and SMR2 when the two random eigenvalues are getting closer. A set of 120 configurations have been simulated and for each simulation, the random input stiffness $k_6(\theta)$ is defined as a Gamma random variable with

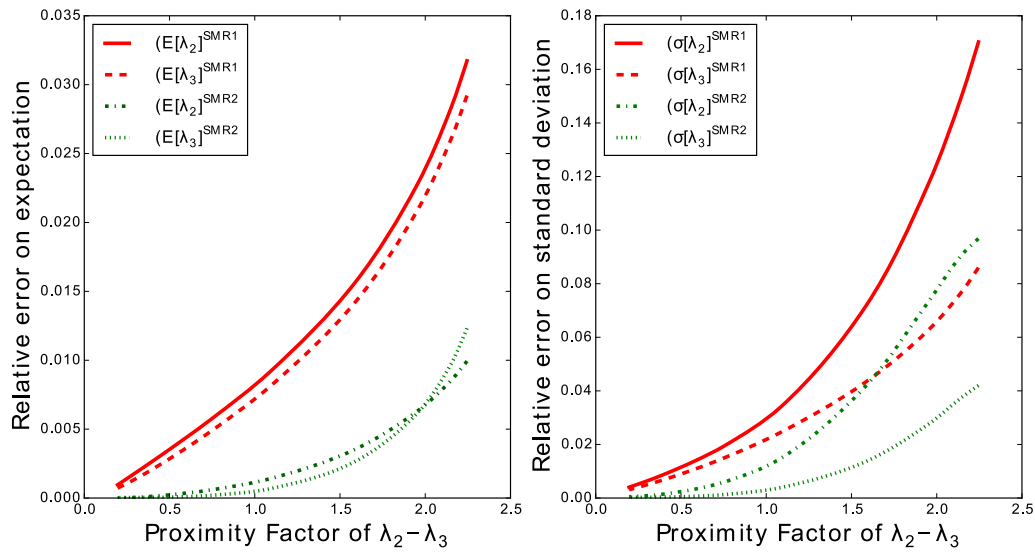


Figure 3. Error on the first moments estimation when PF increase

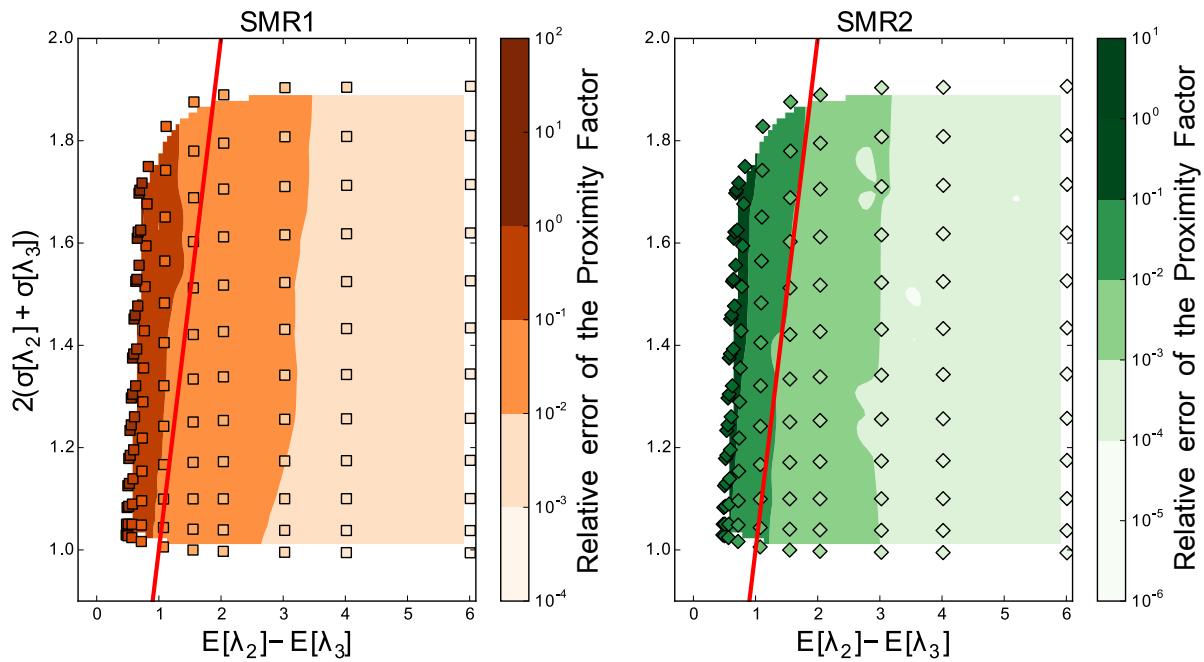


Figure 4. Relative error of the Proximity Factor respectively computed with SMR1 and SMR2

expectation $E[k_6]$ (10 values between 1.275, and 4) and coefficient of variation $\delta[k_6] = \sigma[k_6]/E[k_6]$ (12 values between 0.05 and 0.3). The reference values of PF are obtained through Monte-Carlo simulations with 30000 samples. It can be observed that for $PF < 1$, corresponding to the half plane behind the line with equation $y = x$, the estimation of PF with SMR1 and SMR2 are close to the reference values.

As expected, over the limit value $PF = 1$, the estimation of PF is less accurate. Nevertheless,

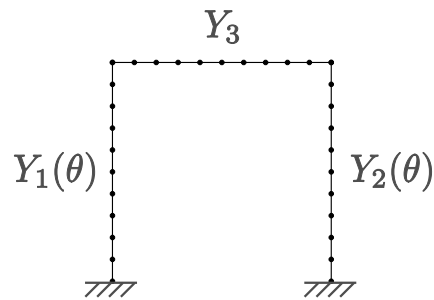


Figure 5. Definition of the three beams frame

the purpose is to identify the validity domain of the SMR1 and SMR2 methods, i.e. to identify with a certain precision the limit value of PF above which the method used to compute the random eigenvectors has to be refined. It can be noticed that the error of the estimation of the limit value $PF = 1$ with SMR1 is less than 10% while the error is around 1% with SMR2. It could therefore be considered that PF estimated with SMR1 allows us to identify when SMR2 needs to be used. The PF estimated with SMR2 is more reliable and could allow us to identify when the SMR2 method needs to be refined using the eigenvector Polynomial Chaos expansion.

3.4. Discussion

The PF estimation is more sensitive to the random eigenvalue proximity than the estimation of the two first moments themselves. Nevertheless, it is not critical because the main purpose is to choose the best suitable method to estimate the first moments. Considering the SMR1 results, the PF limit value $PF = 1$ is estimated with an error close to 8% but the corresponding expectation and standard deviation are estimated with relative errors respectively less than 0.8% and 3%. Even if the PF is not estimated with a high precision, the error of the first moments estimated with SMR1 is acceptable. The same approach can be done for the PF estimation with SMR2.

The PF limit value criterion is then a key point of the SMR method. Decreasing the limit value of PF allows us to increase the quality of the solution by refining the method used to compute the random eigenvector for a larger number of random eigenvalues but it implies a bigger investment in computational cost. As a partial conclusion, even if it is based on statistical moments estimations, the Proximity Factor is able to identify a certain validity domain of the two SMR methods. Its capability is based on the choice of the limit value which depends on the required quality or the available computational resources.

4. Application to a frame

In this section, an application of the SMR method is proposed on a structure with different Young's moduli, to test the method on a more realistic structure discretized by finite elements.

4.1. System modeling

The studied structure, presented in figure 5, is a frame composed of three substructures with two random Young's moduli denoted $Y_1(\theta)$ and $Y_2(\theta)$ and one deterministic Young's modulus, denoted Y_3 . The three substructures are beams with the same geometrical properties : length 250 mm, width 10 mm and thickness 1 mm. The three beams are assumed to have the same density $\rho = 2800 \text{ kg}\cdot\text{m}^{-3}$. The frame is modeled by finite elements with 30 beam elements and 3 Dof for each node. The base of the frame is assumed to be clamped. The complete system has 87 Dof.

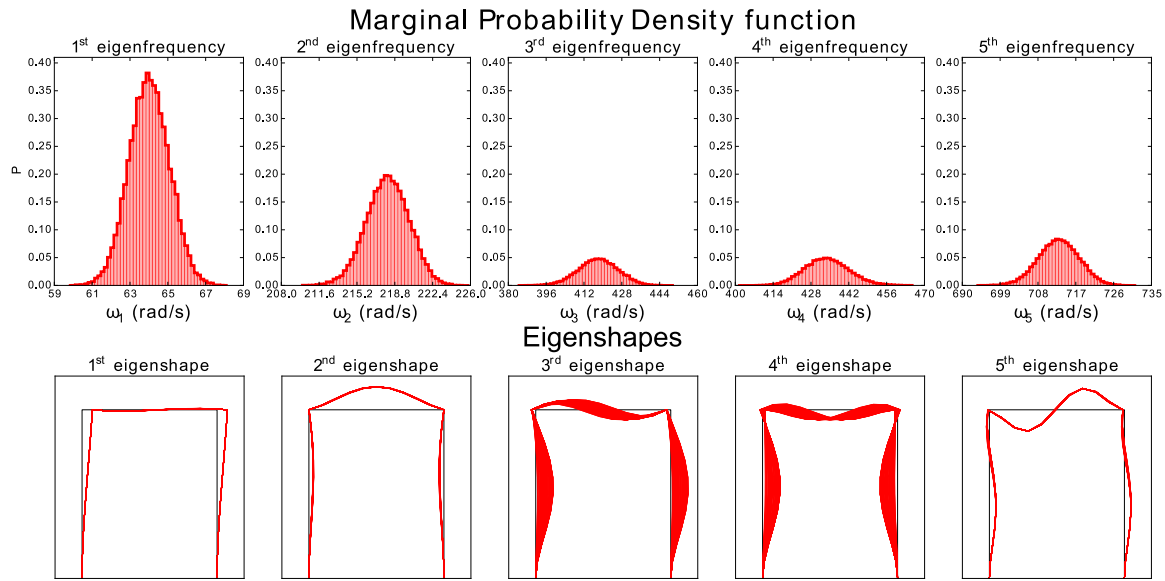


Figure 6. Marginal density of the 5 first random frequencies and the corresponding eigenshapes

The two input random parameters of the structure, $Y_1(\theta)$ and $Y_2(\theta)$, are modeled as independent random variables with Gamma distribution. Their expectations and standard deviations are respectively $E[Y_1] = E[Y_2] = 75$ GPa and $\sigma[Y_1] = \sigma[Y_2] = 5$ GPa (corresponding to a coefficient of variation $\delta = \sigma[Y_i]/E[Y_i] = 1/15 \simeq 0,066$) and correspond to small variations of the Young's moduli. The Young's deterministic modulus is $\bar{Y}_3 = 20$ GPa.

A Monte-Carlo simulation with 30 000 samples allows us to compute the pdf of each random variable of interest. It constitutes the reference framework to compare the results from the SMR method.

4.2. Numerical results

4.2.1. Illustration of the SMR founding assumption. The Monte-Carlo simulation allows us to estimate the first five random eigenfrequencies through 30000 draws of the two input Young's moduli. Figure 6 shows the marginal density functions of the first five random eigenvalues and the corresponding eigenvectors of the frame. It can be noticed that the 1st, 2nd and 5th eigenshapes do not vary significantly while the 3rd and 4th eigenshapes have high variations. This illustrates the SMR1 assumption that, under certain conditions, the random eigenvectors of the structure could be considered as deterministic. The behaviour of the 3rd and 4th random eigenmodes should be related to the relative closeness of the random eigenvalues. An overlap could be observed representing their marginal pdf on the same axis.

4.2.2. Five first random eigenvalues of the frame. Figure 7 shows the marginal pdf obtained with SMR1 and SMR2 in comparison with the marginal pdf from the Monte-Carlo simulation. For the three well separated eigenfrequencies, the SMR1 and SMR2 methods fit properly the reference marginal pdf. In the case of two close eigenfrequencies, as illustrated by the 3rd and 4th eigenfrequencies, the SMR1 method does not accurately estimate the marginal pdf whereas the SMR2 method is still fitting the reference results.

The relative errors of the two first statistical moments are presented on figures 8 and 9. It can be noticed that for the three random eigenfrequencies which are well separated, the relative error of the estimation of the two first moments with the SMR1 method is less than 0.5%. The

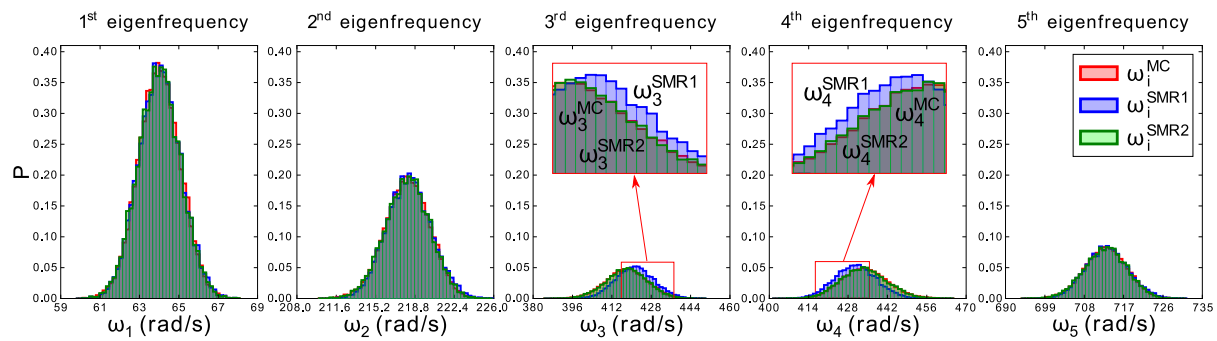


Figure 7. Marginal density functions of the first five eigenfrequencies estimated with SMR1 and SMR2

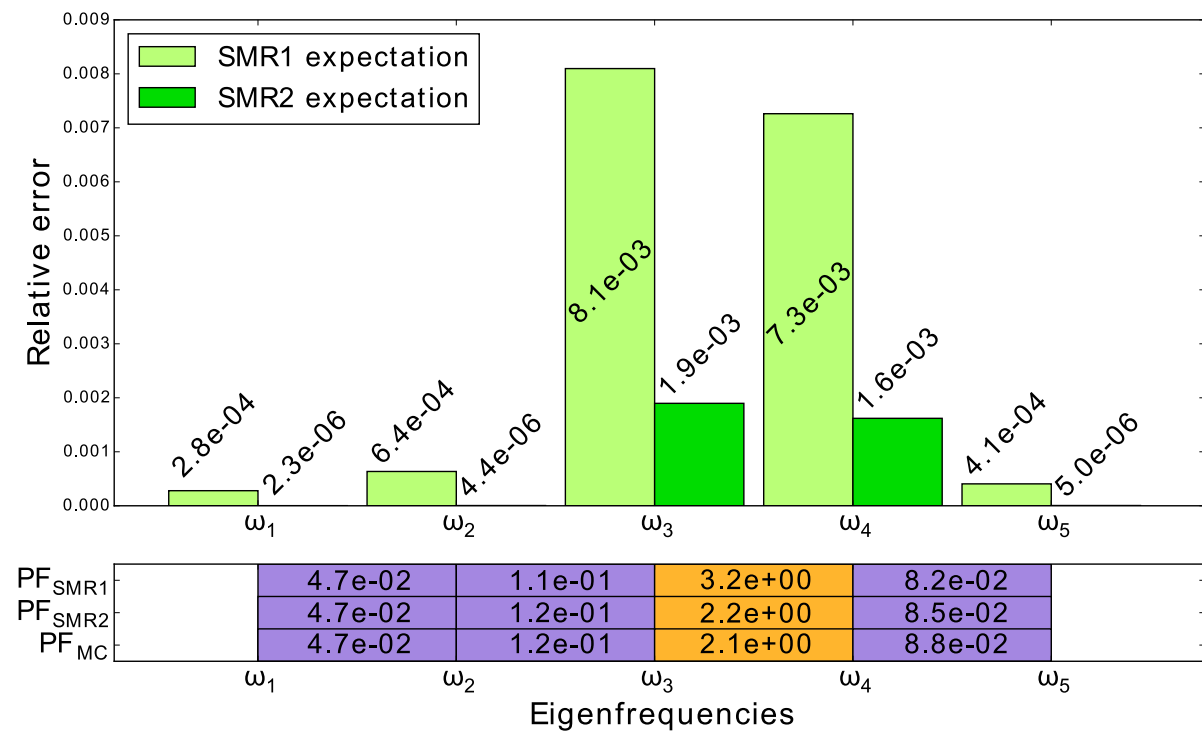


Figure 8. Relative error of the expectation estimated with SMR1 and SMR2

SMR2 method obviously gives better results but it is not necessary to invest in more complex computations while SMR1 results are sufficient. The table below the error diagrams 8 and 9 presents the approximation of the proximity factor computed from the first moments estimations of the SMR1 and SMR2 methods compared to the reference proximity factor from Monte-Carlo simulation. The proximity factor between the first and second random eigenfrequencies corresponds to the cell between these two eigenfrequencies and so on. The cell is purple if the proximity factor is under the limit value $PF = 1$ and is orange when it is over this limit. It can be noticed that PF_{SMR1} correctly identifies the two close random eigenfrequencies for which the method has to be refined.

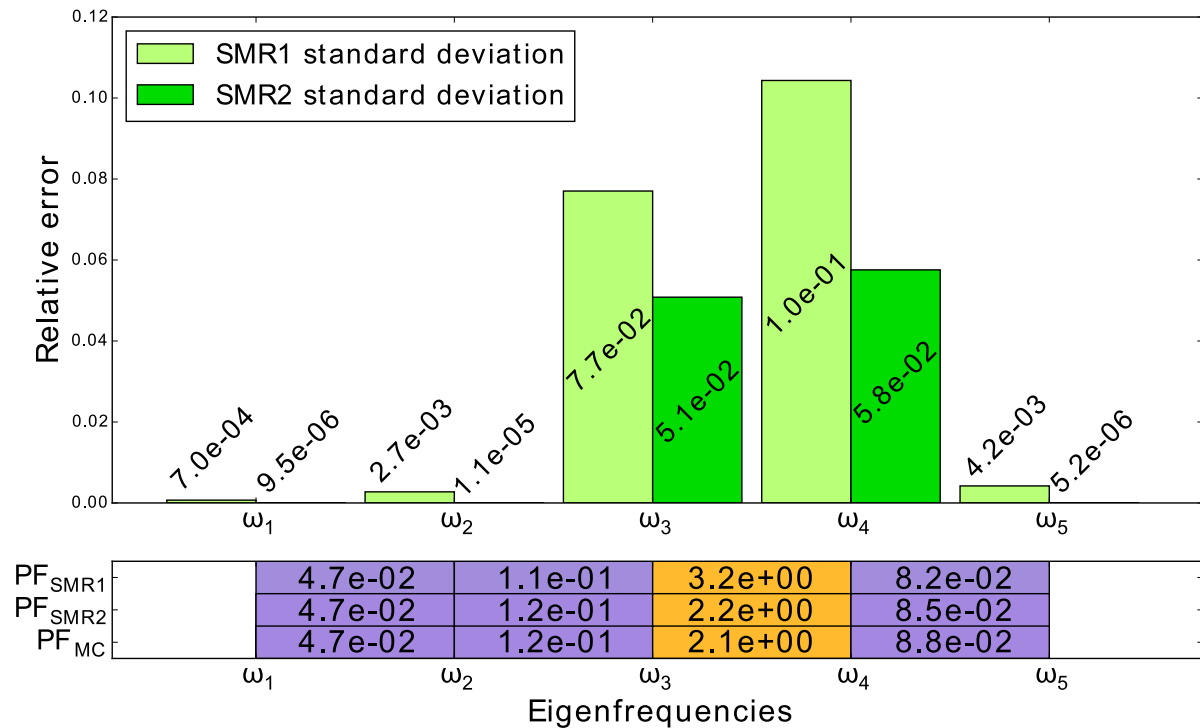


Figure 9. Relative error of the standard deviation estimated with SMR1 and SMR2

5. Conclusion

A non-intrusive method to estimate the two first statistical moments of the random eigenvalues of a structure is presented. Referred to as SMR for Stochastic Model Reduction method, it requires only a single deterministic finite element computation. The SMR method is based on the assumption that, in the vicinity of a given natural eigenfrequency, the dynamical behaviour of a system is mainly characterized by the modal propriety of the considered eigenfrequency if all other eigenfrequencies are well separated. The eigenvectors of the problem are assumed to be deterministic in first approximation. This assumption is no longer valid when eigenvalues become closely spaced. In this case, the system becomes more coupled. So it is proposed to refine the method by considering the eigenvectors' randomness through their first order Taylor expansion. To decide when the method should be refined, an indicator referred as Proximity Factor, based on the two first statistical moments, is proposed. Computationally free, the efficiency of this indicator depends on the choice of its limit value criterion.

A first case of a three-degrees-of-freedom system is used to validate the method and to characterize the behaviour of the Proximity Factor. This example highlights the criticality of the choice of the limit value of this indicator over which the method needs to be refined. In a second application case, a frame with random Young's moduli allows us to apply the SMR method to a wide range of random eigenfrequencies. We illustrate the basis assumption of the SMR method by plotting the random eigenshapes of the frame. The ability of the Proximity Factor to identify when the method needs to be refined is presented and the criticality of its limit value criterion is discussed.

We conclude that the SMR method is an efficient method due to its ratio accuracy/computational-time. The Proximity Factor is a key parameter of the method, allowing one to refine only when it is necessary. Particular attention should be paid to the choice of the Proximity Factor limit value depending on the tradeoff accuracy/computational-time imposed

by the designer.

The computational gain of the SMR method is even greater for large degree-of-freedom systems. The method could then be applied to more complex examples from industrial structures. Of particular interest is the case of a bolted assembly where the stiffness of each bolted joint would be considered as a random variable. The SMR method allows us to estimate the marginal pdf of each random eigenfrequency. Nevertheless, by the definition of the SMR method, the coupling between eigenvalues is not taken into account and the joint pdf cannot be accurately estimated. In order to consider the coupling between two random eigenvectors corresponding to two close random eigenvalues, another model of the random eigenvectors should be implemented. Further work will study the expansion of the random eigenvectors on the Polynomial Chaos.

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