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Some works on the Generalized Andrica's Conjectures

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The Andrica's conjecture affirm that for every prime number $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ (I.1) and more precisely, that the upper bound of this function is reached for $p_n = 7$. Another conjecture, called generalized Andrica's conjecture, affirm that, for every prime number, there is an only real number x , such as : $(p_{n+1})^x - (p_n)^x = 1$ (I.2), and the minimum value of $x=x(p_n)$ is obtained for $p_n = 113$. Florentin Smarandache in 2000 [8], has listed a set of conjectures linked to Andrica's conjecture such as $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ (I.3) for all p_n and all $k > 2$. David Lowry-Duda in 2017 [4], has formulated the following open question : For any constant C , what is the smallest x such as $(p_{n+1})^x - (p_n)^x = C$ (I.4). In 2018 Matt Wisser [1],[2] from Victoria University of Wellington has proposed a variant of Andrica's conjecture and presents an original approach to prove that $\text{Ln}^k(p_{n+1}) - \text{Ln}^k(p_n)$ has an upper bound and gives some numerical values for $k \leq 5$.

In the present work, we are going to establish a set of results regarding :

Part 1 : the generalized Andrica's conjecture : $F_n(x) = (p_{n+1})^x - (p_n)^x = 1$.

We shall build some lower and upper bound functions $X^-(p_n)$ et $X^+(p_n)$ such as, for all p_n , $X^-(p_n) \leq x(p_n) \leq X^+(p_n)$. We shall study the behavior of these functions $X^-(p_n)$ et $X^+(p_n)$ and show under a weaker condition than Cramer's conjecture that x is minimal for $p_n = 113$.

We'll give the proof that $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ for all $k > 2$.

We'll show that all $x < 1$, $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound for all $x \in [0, 0.475[$ and Cramer's conjecture for all $x \in [0.475, 1[$.

Part 2 : We propose an alternative approach to prove that for all integer k , $L_k(p_n) = \text{Ln}^k(p_{n+1}) - \text{Ln}^k(p_n)$ has an upper bound and we'll build some lower and upper bound functions $L_k^-(p_n)$ and $L_k^+(p_n)$. We'll try to estimate the integer sequence $(p_{n,k})$ such as $\text{Max}_n (L_k(p_n)) = L_k(p_{n,k})$. We'll propose some upper bound values of $L_k(p_n)$ for all $k < 25$. We'll deduce that for all integers $k > 0$ and $r > 0$, $G_{k,r}(p_n) = \text{Ln}^k(p_{n+1})^r - \text{Ln}^k(p_n)^r$ has an upper bound value. We'll prove that $L_y(p_n) = \text{Ln}^y(p_{n+1}) - \text{Ln}^y(p_n) = 1$ has a solution $y(p_n) \in \mathbb{R}$ for all p_n and we will propose an approximation of $y(p_n)$.

Part 3 ; We'll generalize the different results obtained in part 1 and 2 to establish that the following functions on prime number accept upper values :

$H_{k,x}(p_n) = (p_{n+1})^x \cdot \text{Ln}^k(p_{n+1}) - (p_n)^x \cdot \text{Ln}^k(p_n)$ for all integer $k > 0$ and real $0 \leq x < 1$

$H_{x,y}(p_n) = (p_{n+1})^x \cdot \text{Ln}^y(p_{n+1}) - (p_n)^x \cdot \text{Ln}^y(p_n)$ for all real numbers (x,y) such as $0 \leq x < 1$ and $y > 0$

We'll study the relation between x and y which verify $H_{x,y}(p_n) = (p_{n+1})^x \cdot \text{Ln}^y(p_{n+1}) - (p_n)^x \cdot \text{Ln}^y(p_n) = 1$

Part 1 : The Generalised Andrica's conjecture

I.1 - Study of the equation $(p_{n+1})^x - (p_n)^x = 1$

We define the function $F_n(x) = p_{n+1}^x - p_n^x$ where $x \in \mathbb{R}$ and p_n is the n^{st} prime number and $g_n = p_{n+1} - p_n$ the gap between the two consecutive primes p_n and p_{n+1} .

We can express $F_n(x)$ as following

$$F_n(x) = (p_n + g_n)^x - p_n^x = p_n^x \cdot \left(1 + \frac{g_n}{p_n}\right)^x - p_n^x = p_n^x \cdot \left(\left(1 + \frac{g_n}{p_n}\right)^x - 1\right)$$

We have obviously, $F_n(0) = 0$ and $F_n(1) = p_{n+1} - p_n > 1$ for all p_n

According to the theorem of intermediate values, as $F_n(x)$ is a continuous function in $[0,1]$, there is at least one x such as $F_n(x) = 1$ and $0 < x < 1$. And as $F_n(x)$ is a growing function, x is unique. x is a function of p_n , $x = x(p_n)$, but to simplify the notation in the following, we'll keep the notation x .

We can make a taylor serie expansion of $F_n(x)$

$$F_n(x) = p_n^x \cdot \left(1 + \frac{g_n}{p_n}\right)^x - p_n^x = p_n^x \cdot \left(1 + \frac{x \cdot g_n}{p_n} + \frac{x \cdot (x-1)}{2} \cdot \left(\frac{g_n}{p_n}\right)^2 + O\left(\frac{g_n}{p_n}\right)^3\right) - p_n^x$$

$$F_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} + \frac{x \cdot (x-1)}{2} \cdot p_n^x \cdot \left(\frac{g_n}{p_n}\right)^2 + p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^3 \quad (I.5)$$

$$\text{As } 0 < x < 1, x-1 < 0 \quad \text{and} \quad \frac{x \cdot (x-1)}{2} \cdot p_n^x \cdot \left(\frac{g_n}{p_n}\right)^2 < 0 \quad \text{and} \quad p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^3 > 0$$

So we obtain a upper bound of $F_n(x)$ given by : $M_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n}$

$$\text{We are going to find } x \text{ such as} \quad M_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} = 1 \quad (I.6)$$

$$\text{which gives the equation} \quad x \cdot p_n^x = \frac{p_n}{g_n} \quad (I.7)$$

Before trying to estimate x , we are going first to study the function $F_{1/4}(x) = p_{n+1}^{1/4} - p_n^{1/4}$

$$F_{1/4}(x) = p_{n+1}^{1/4} - p_n^{1/4} = \frac{p_{n+1}^{1/2} - p_n^{1/2}}{p_{n+1}^{1/4} + p_n^{1/4}} = \frac{p_{n+1} - p_n}{(p_{n+1}^{1/4} + p_n^{1/4}) \cdot (p_{n+1}^{1/2} + p_n^{1/2})} < \frac{p_{n+1} - p_n}{4 \cdot p_n^{3/4}} = \frac{g_n}{4 \cdot p_n^{3/4}}$$

According to the best known result due to Baker, Harman and Pintz (BHP), stating that for all $p_n > 113$, $g_n < < (p_n)^a$ with $a=0,525$

$$F_{1/4}(x) = \frac{g_n}{4 \cdot p_n^{3/4}} < \frac{p_n^{0,525}}{4 \cdot p_n^{3/4}} = \frac{p_n^{21/40}}{4 \cdot p_n^{3/4}} = \frac{1}{4 \cdot p_n^{9/40}} < \frac{1}{4}$$

So we can limit our approach to the values of $x \in \left[\frac{1}{4}, 1\right]$ and we can write :

$$\frac{1}{4} p_n^x < x \cdot p_n^x < p_n^x$$

$$\text{We replace by (I.7) :} \quad \frac{1}{4} p_n^x < \frac{p_n}{g_n} < p_n^x \quad (I.8)$$

Which gives two inequalities to estimate x :

$$\frac{p_n}{g_n} < p_n^x \Leftrightarrow x > \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(p_n)} = 1 - \frac{\ln(g_n)}{\ln(p_n)} = x_0 \quad \text{and} \quad \frac{p_n}{g_n} > \frac{1}{4} \cdot p_n^x \Leftrightarrow x < \frac{\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} = 1 - \frac{\ln(g_n/4)}{\ln(p_n)} = x_1$$

So we obtain a lower x_0 and upper x_1 values for x and we are going to show that :

$$F_n(x_0) < 1 < F_n(x_1)$$

$$F_n(x_0) < x_0 \cdot p_n^{x_0} \cdot \frac{g_n}{p_n} = \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(p_n)} \cdot p_n^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(p_n)}} \cdot \frac{g_n}{p_n} = \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(p_n)} < 1 \quad \text{for all } p_n$$

$$F_n(x_1) > x_1 \cdot p_n^{x_1} \cdot \frac{g_n}{p_n} + \frac{x_1 \cdot (x_1 - 1)}{2} \cdot p_n^{x_1} \cdot \left(\frac{g_n}{p_n}\right)^2 = \frac{\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} \cdot p_n^{\frac{\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)}} \cdot \frac{g_n}{p_n} - \frac{\ln\left(\frac{4p_n}{g_n}\right)}{2 \ln(p_n)} \cdot \frac{\ln\left(\frac{g_n}{4}\right)}{\ln(p_n)} \cdot p_n^{\frac{\ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)}} \cdot \left(\frac{g_n}{p_n}\right)^2$$

$$F_n(x_1) > \frac{4 \ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} - \frac{2 \ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} \cdot \frac{\ln\left(\frac{g_n}{4}\right)}{\ln(p_n)} \cdot \frac{g_n}{p_n} = \frac{4 \ln\left(\frac{4p_n}{g_n}\right)}{\ln(p_n)} \cdot \left(1 - \frac{g_n \cdot \ln\left(\frac{g_n}{4}\right)}{2 p_n \cdot \ln(p_n)}\right) = 4 \left(1 - \frac{\ln\left(\frac{g_n}{4}\right)}{\ln(p_n)}\right) \cdot \left(1 - \frac{g_n \cdot \ln\left(\frac{g_n}{4}\right)}{2 p_n \cdot \ln(p_n)}\right)$$

Using the Baker, Harman and Pintz (BHP) result, stating that for all $p_n > 113$, $g_n \ll (p_n)^a$ with $a=0,525$

$$\frac{\ln\left(\frac{g_n}{4}\right)}{\ln(p_n)} = \frac{\ln(g_n) - \ln(4)}{\ln(p_n)} < \frac{0,525 \cdot \ln(p_n) - \ln(4)}{\ln(p_n)} < \frac{21}{40} \quad \text{and for } p_n \geq 2 \quad \frac{g_n \cdot \ln\left(\frac{g_n}{4}\right)}{2 p_n \cdot \ln(p_n)} < \frac{a \cdot p_n^a}{2 p_n} < \frac{0,2625}{p_n^{0,475}} < \frac{1}{5}$$

$$\text{In these conditions : } F_n(x_1) > 4 \cdot \left(1 - \frac{21}{40}\right) \cdot \left(1 - \frac{1}{5}\right) = 1,52 > 1 \quad \text{for all } p_n > 113$$

$$\text{So we have proven that : } F_n(x_0) < 1 < F_n(x_1) \text{ for all } p_n \quad \text{for all } p_n > 113 \quad (I.9)$$

We are going to build a sequence (x_i) such as $x_{i+1} < x_i$ and find an integer k such as $F(x_{k+1}) < 1 < F(x_k)$

$$\text{We define } x_2 = \frac{x_0 + x_1}{2} = \frac{\ln\left(\frac{p_n}{g_n}\right) + \ln\left(\frac{4p_n}{g_n}\right)}{2 \ln(p_n)} = \frac{\ln\left(4 \cdot \left(\frac{p_n}{g_n}\right)^2\right)}{2 \ln(p_n)} = \frac{\ln\left(2 \cdot \frac{p_n}{g_n}\right)}{\ln(p_n)}$$

$$\text{We define } x_3 = \frac{x_0 + x_2}{2} = \frac{\ln\left(\frac{p_n}{g_n}\right) + \ln\left(\frac{2p_n}{g_n}\right)}{2 \ln(p_n)} = \frac{\ln\left(2 \cdot \left(\frac{p_n}{g_n}\right)^2\right)}{2 \ln(p_n)} = \frac{\ln\left(\sqrt{2} \cdot \frac{p_n}{g_n}\right)}{\ln(p_n)}$$

And we can generalize to x_{k+2} defined by

$$x_{k+2} = \frac{x_0 + x_{k+1}}{2} = \frac{\ln\left(\frac{p_n}{g_n}\right) + \ln\left(\frac{2^{1/2^{k-1}} \cdot p_n}{g_n}\right)}{2 \ln(p_n)} = \frac{\ln\left(2^{1/2^{k-1}} \cdot \left(\frac{p_n}{g_n}\right)^2\right)}{2 \ln(p_n)} = \frac{\ln\left(2^{1/2^k} \cdot \frac{p_n}{g_n}\right)}{\ln(p_n)}$$

We must find k such as $F(x_{k+2}) = 1$

As $2^{1/2^k} \rightarrow 1^+$ when $k \rightarrow \infty$, we can replace $2^{1/2^k}$ by $1 + \varepsilon$ to simplify our analysis :

$$F_n(x_{k+2}) < x_{k+2} \cdot p_n^{x_{k+2}} \cdot \frac{g_n}{p_n} = \frac{\ln\left((1+\varepsilon) \cdot \frac{p_n}{g_n}\right)}{\ln(p_n)} \cdot p_n^{\frac{\ln\left((1+\varepsilon) \cdot \frac{p_n}{g_n}\right)}{\ln(p_n)}} \cdot \frac{g_n}{p_n} = (1 + \varepsilon) \cdot \frac{\ln\left((1+\varepsilon) \cdot \frac{p_n}{g_n}\right)}{\ln(p_n)}$$

To solve the equation $F_n(x_{k+2}) \approx 1$ we have to solve the following equation :

$$(1 + \varepsilon) \cdot \left(\ln(1 + \varepsilon) + \ln\left(\frac{p_n}{g_n}\right)\right) = \ln(p_n) \quad (I.10)$$

For k high, ε is small and we can make a taylor serie expansion of $\ln(1+\varepsilon) = \varepsilon + O(\varepsilon^2)$

$$(1 + \varepsilon) \cdot \left(\varepsilon + O(\varepsilon^2) + \ln\left(\frac{p_n}{g_n}\right)\right) = \ln(p_n)$$

We can develop to obtain an estimation of ε

$$\varepsilon + O(\varepsilon^2) + \ln\left(\frac{p_n}{g_n}\right) + \varepsilon \cdot \ln\left(\frac{p_n}{g_n}\right) = \ln(p_n) \quad \text{so} \quad \varepsilon \approx \frac{\ln\left(\frac{g_n}{e \cdot p_n}\right)}{\ln\left(\frac{e \cdot p_n}{g_n}\right)} \quad \text{and} \quad 1 + \varepsilon \approx \frac{\ln(e \cdot p_n)}{\ln\left(\frac{e \cdot p_n}{g_n}\right)}$$

Finally we obtain a first evaluation of x , noted X^+ :

$$X^+(p_n) \approx \frac{\ln\left(\frac{p_n}{g_n} \cdot \frac{\ln(e \cdot p_n)}{\ln\left(\frac{e \cdot p_n}{g_n}\right)}\right)}{\ln(p_n)} = \frac{\ln\left(\frac{p_n}{g_n} \cdot \left(1 - \frac{\ln(g_n)}{\ln\left(\frac{e \cdot p_n}{g_n}\right)}\right)\right)}{\ln(p_n)} = 1 - \frac{\ln(g_n)}{\ln(p_n)} - \frac{\ln\left(1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)}\right)}{\ln(p_n)} \quad (I.11)$$

Using a Taylor serie expansion, we can write (I.11) as follows :

$$X^+(p_n) = 1 - \frac{\ln(g_n)}{\ln(p_n)} + \frac{1}{\ln(p_n)} \cdot \left(\left(\frac{\ln(g_n)}{\ln(e \cdot p_n)} \right) - \frac{1}{2} \left(\frac{\ln(g_n)}{\ln(e \cdot p_n)} \right)^2 + O\left(\frac{\ln(g_n)}{\ln(e \cdot p_n)} \right)^3 \right) \quad \text{with } O\left(\frac{\ln(g_n)}{\ln(e \cdot p_n)} \right)^3 > 0 \quad (I.12)$$

We can extract from this formulation different functions given some lower bounds of x .

$$X^-(p_n) = 1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \cdot \left(1 - \frac{\ln(g_n)}{2 \cdot \ln(p_n) \cdot \ln(e \cdot p_n)} \right) \quad (I.13)$$

$$X^{2-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(p_n)} + \frac{\ln(g_n)}{\ln(p_n) \cdot \ln(e \cdot p_n)} = 1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \quad (I.14)$$

$$X^{3-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(p_n)} \quad \text{which correspond to } x_0 \text{ where we have already prove that } F(x_0) < 1$$

So we are going to demonstrate that $F(X^{2-}) < 1$ for all p_n .

$$F_n(X^{2-}) < X^{2-} \cdot p_n^{X^{2-}} \cdot \frac{g_n}{p_n} = \left(1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \right) \cdot p_n^{\left(1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \right)} \cdot \frac{g_n}{p_n} = \left(1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \right) \cdot p_n^{-\frac{\ln(g_n)}{\ln(e \cdot p_n)}} \cdot g_n$$

$$F_n(X^{2-}) < \left(1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \right) \cdot e^{-\frac{\ln(g_n) \cdot \ln(p_n)}{\ln(e \cdot p_n)}} \cdot e^{\ln(g_n)} = \left(1 - \frac{\ln(g_n)}{\ln(e \cdot p_n)} \right) \cdot e^{\frac{\ln(g_n)}{\ln(e \cdot p_n)}}$$

We have only to show that for every positive y such a $0 \leq y < 1$, the function $g(y) = (1 - y) \cdot e^y < 1$

This is obvious because $g(1)=0$, $g(0) = 1$ and $g'(y) = -y \cdot e^y < 0$ for all $y > 0$.

So if we take $y = \frac{\ln(g_n)}{\ln(e \cdot p_n)}$, then $F_n(X^{2-}) < 1$ and $X^{2-}(p_n)$ gives a good lower bound for x .

We present below in table I.1, the numerical values of X^+ , $F(X^+)$, X^{2-} , $F(X^{2-})$, X^- and $F(X^-)$ for a set of the 300 first prime numbers. We see that the local lower bounds are reach for the prime numbers corresponding to the first occurrence of the maximal gaps. It means that, to find the lower bound of x , we can limit our study to this subset of prime numbers, noted \mathbb{P}_{G_n} . Effectively, if we consider $\mathbb{P}_{G_n} = \{p_i \in \mathbb{P} / g_i = G_n\}$, for all $p_i \in \mathbb{P}_G$, if $P_n = \text{Min}\{p_i \in \mathbb{P}_{G_n}\}$ then $X^{2-}(p_i) > X^{2-}(P_n)$. In table I.2, we presents the numerical values of X^+ , $F(X^+)$, X^{2-} , $F(X^{2-})$, X^- and $F(X^-)$ for all prime numbers corresponding to the first occurrence of the maximal gaps $G_n = \text{Max}_{i \leq n}(g_i)$, up to $p_n = 218\ 209\ 405\ 436\ 543$ corresponding to the maximal gap $G_n = 906$.

Primes	Gaps	$X(P_n)$	$F(X(P_n))$	$X^2(P_n)$	$F(X^2(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1	1	1	1	1	1
3	2	0,669711655	0,851328799	0,719719798	0,979732227	0,733992458	1,018887569
5	2	0,734369162	0,913957339	0,756327983	0,978907397	0,761156522	0,993683791
7	4	0,529417297	0,757405268	0,58650709	0,950360565	0,614413829	1,058090378
11	2	0,796006903	0,959235065	0,804687007	0,990823929	0,806080038	0,995978635
13	4	0,611132103	0,854182017	0,640641635	0,96977175	0,65151272	1,01563719
17	2	0,819173336	0,97155371	0,824944888	0,994853341	0,825749936	0,99814351
19	4	0,648544605	0,890454187	0,669532525	0,979875239	0,67624737	1,01012675
23	6	0,566736314	0,829998067	0,59669363	0,963347411	0,609663343	1,02671729
29	2	0,841286872	0,980732053	0,845027558	0,997705074	0,845477012	0,999762777
31	6	0,595903329	0,860562023	0,61969188	0,973172523	0,628978628	1,020617914
37	4	0,699345252	0,929961567	0,711865052	0,991030592	0,715118524	1,00749506
41	2	0,852946519	0,98458737	0,855858261	0,998758836	0,856179242	1,000332774
43	4	0,708835099	0,936055362	0,720107404	0,99262813	0,722917695	1,007218142
47	6	0,630576301	0,892033379	0,648305109	0,982959573	0,654392898	1,016096755
53	6	0,63950619	0,899263623	0,65587674	0,985093601	0,661309415	1,015208888
59	2	0,863487529	0,987527879	0,865772774	0,999449078	0,866004576	1,000665917
61	6	0,649422092	0,90687384	0,66437439	0,98726838	0,669148918	1,014302542
67	4	0,733645296	0,950124554	0,742082767	0,99600428	0,74396113	1,006498093
71	2	0,868290073	0,988707403	0,870324947	0,999686399	0,870523295	1,000762774
73	6	0,66132252	0,915432903	0,674692296	0,989606412	0,678764977	1,013293769
79	4	0,741818079	0,954191216	0,749446632	0,996866919	0,751079964	1,006235718
83	6	0,669346342	0,920856432	0,681719573	0,991016748	0,685368643	1,012646942
89	8	0,621136945	0,892404769	0,63712936	0,984190579	0,642822962	1,018941877
97	4	0,751324442	0,958585794	0,758083885	0,997722624	0,759465317	1,005905712
101	2	0,876557025	0,990520698	0,878207956	0,999992617	0,878357746	1,0008563
103	4	0,753973197	0,959747982	0,760503701	0,997933869	0,761820831	1,005808709
107	2	0,877812781	0,990773432	0,879410316	1,000028695	0,879553637	1,000863072
109	4	0,75642073	0,960798364	0,762744664	0,998118895	0,764004535	1,005717102
113	14	0,539221472	0,831800413	0,561683725	0,964797943	0,57240055	1,034955938
127	4	0,7627909	0,963428556	0,768599099	0,998556077	0,769719252	1,005469944
131	6	0,695029908	0,936409558	0,704569735	0,994667638	0,707097175	1,010669337
137	2	0,88291395	0,991742404	0,884307184	1,000149681	0,884426453	1,000872585
139	10	0,611998449	0,889916672	0,627255531	0,983946511	0,632876861	1,020888288
149	2	0,884551402	0,992034558	0,885883209	1,000180362	0,885995474	1,000869959
151	6	0,702230988	0,940299	0,711067952	0,995469009	0,71333738	1,010126917
157	6	0,704146838	0,941300687	0,712803196	0,995666927	0,715007704	1,00998303
163	4	0,772505548	0,96716211	0,777585945	0,999106284	0,778517158	1,005070231
167	6	0,707132841	0,942834691	0,715512963	0,995962963	0,717619289	1,009759075
173	6	0,708812846	0,943683353	0,717040334	0,996122975	0,719092994	1,009633229
179	2	0,887974146	0,992617131	0,889183806	1,000232819	0,889282473	1,000856487
181	10	0,628525257	0,902158096	0,64179952	0,987379747	0,646396638	1,018630613
191	2	0,889136777	0,992806655	0,890306823	1,000247261	0,890401178	1,000849636
193	4	0,778642353	0,969356219	0,783297911	0,999386971	0,784123543	1,004805318
197	2	0,889682523	0,992894198	0,890834295	1,000253482	0,890926677	1,000846049
199	12	0,605150756	0,887737822	0,619879636	0,983504692	0,625445908	1,022176909
211	12	0,608790594	0,89050073	0,623090885	0,984302072	0,628423311	1,021616198
223	4	0,783633964	0,971051668	0,787963048	0,999580332	0,788710039	1,004584036
227	2	0,892116331	0,993273821	0,89318906	1,000277083	0,893273037	1,000827337
229	4	0,784526848	0,971346766	0,788799294	0,999611799	0,78953287	1,004543987
233	6	0,722252552	0,950110752	0,729329077	0,997241903	0,730990879	1,008630809
239	2	0,892974434	0,993403562	0,894020235	1,000283863	0,894101395	1,000819744
241	10	0,644925644	0,913196615	0,65642027	0,990190948	0,660157912	1,016532036
251	6	0,725419908	0,951535051	0,732242786	0,997466235	0,733821846	1,008395949
257	6	0,726410345	0,951973595	0,733155245	0,997533484	0,734709123	1,008322648
263	6	0,727371048	0,952395895	0,734040915	0,99759742	0,735570677	1,008251616
269	2	0,894893477	0,993686211	0,895880792	1,000296307	0,895955921	1,000801044
271	6	0,728608418	0,952935373	0,735182516	0,997677911	0,73668166	1,008160234
277	4	0,790716984	0,973326565	0,794611078	0,99980549	0,795256827	1,004263202
281	2	0,895584491	0,993785504	0,896551324	1,00029991	0,896624369	1,000793765
283	10	0,653509368	0,918555794	0,664143345	0,991444209	0,667486611	1,015479812
293	14	0,604941746	0,889725659	0,618681633	0,984111231	0,623878645	1,02223408

Table I.1

I.2 - Behaviour of the lower bound function $X^{2-}(p_n)$:

We see in tables 1 and 2 that the minimum value of X^{2-} , x , X^+ are reached for $p_n = 113$. To analyse the behavior of the function $X^{2-}(p_n)$, we compute the derivative. We suppose that $X^{2-}(p)$ is a derivable function of p on \mathbb{R} and $g=g(p)$ is a derivable function of p

$$\frac{dX^{2-}}{dp} = \frac{\ln(g)}{p \cdot \ln(e \cdot p)^2} - \frac{g'}{g \cdot \ln(e \cdot p)} = \frac{\ln(g)}{p \cdot \ln(e \cdot p)^2} \cdot \left(1 - \frac{g' \cdot p \cdot \ln(e \cdot p)}{g \cdot \ln(g)}\right)$$

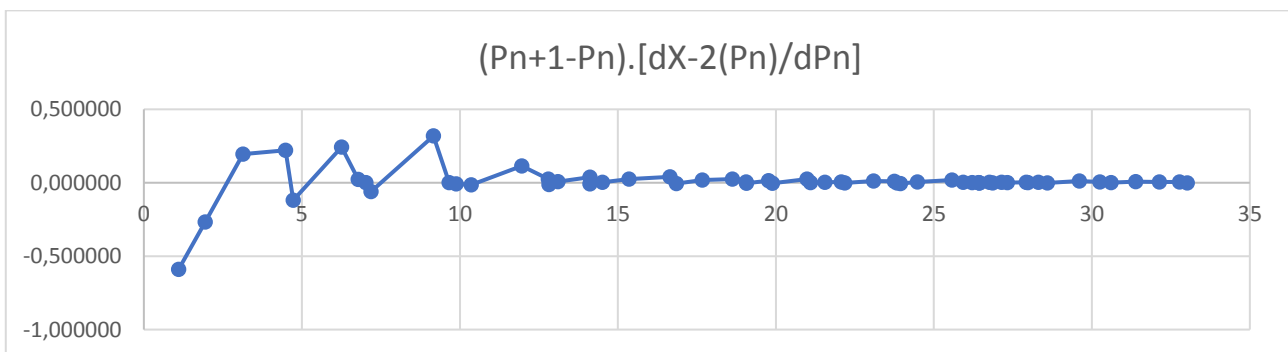
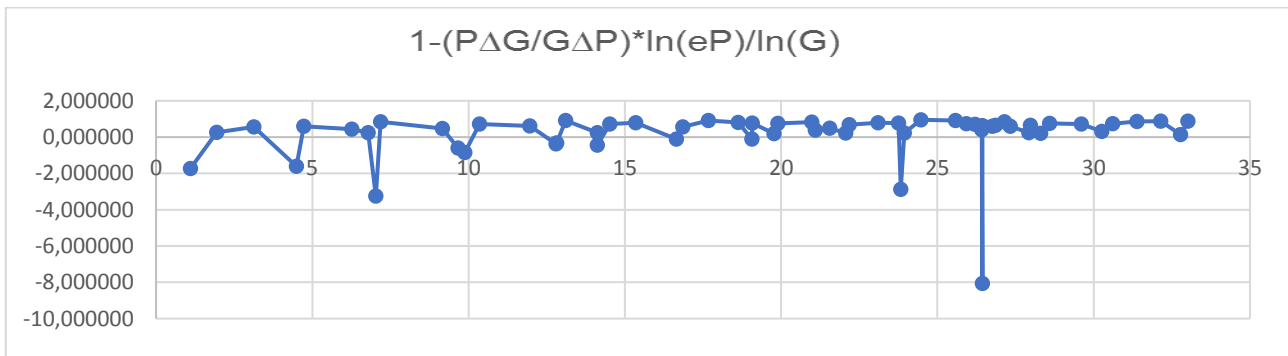
$\frac{\partial X^{2-}}{\partial p} = \frac{\ln(g)}{p \cdot \ln(e \cdot p)^2} > 0$ means, considering an even number $g=g_n$, for all $p_i \in \mathbb{P}_{g_n}$, $X^{2-}(p_i)$ is a growing function for all $p_i \in \mathbb{P}_{g_n}$ and the smallest value of $X^{2-}(p_i)$ is obtained for the smallest $p_i \in \mathbb{P}_{g_n}$, represented by p_{g_n} . If we consider the integer sequence (G_n) defined previously and P_n , the prime number corresponding to the first occurrence of G_n , we can insure that for all $p_i \in \mathbb{P}_{G_n}$, $\text{Min}_{p_i \in \mathbb{P}_{G_n}}(X^{2-}(p_i)) = X^{2-}(P_n)$. It means that we can limit the research of the lower value of x to the prime numbers belonging to the integer sequence (P_n) . In the following of this work P_n will systematically represent the prime number associated to the first occurrence of the maximal gap G_n .

$\frac{\partial X^{2-}}{\partial g} = -\frac{g'}{g \cdot \ln(e \cdot p)}$ can be positive or negative depending on the values of two consecutive prime gaps. If $g_{n+1} \leq g_n$ then $g' \approx g_{n+1} - g_n \leq 0$ and $\frac{\partial X^{2-}}{\partial g} > 0$ which means that $X^{2-}(p_n) \leq X^{2-}(p_{n+1})$. But in contrary, if $g_{n+1} > g_n$ then we'll not be able to conclude. It means that we have only to look at the situation where g_n is strictly growing. It is exactly what's happen in the integer sequence (G_n) .

In reality, we must find the first prime number P_n such as :

$$1 - \frac{G'_{n-1} \cdot P_{n-1} \cdot \ln(e \cdot P_{n-1})}{G_{n-1} \cdot \ln(G_{n-1})} \geq 0 \quad \text{and} \quad 1 - \frac{G'_{n-1} \cdot P_{n-1} \cdot \ln(e \cdot P_{n-1})}{G_{n-1} \cdot \ln(G_{n-1})} < 0 \quad (\text{I.15})$$

If we look at in table I.2 and the two graphics below, the first prime P_n which satisfy (I.15) is $P_n=113$ for which $X^{2-}(P_n) = 0,539221472$ is the minimal value of X^{2-} .



On these two graphics, the horizontal axis correspond to the values of $\ln(P_n)$. We can see that the first P_n for which $\frac{dX^{2-}}{dp} = 0$ and positive before and negative after is for $4 < \ln(P_n) < 5$ which correspond to $P_n = 113$ is table I.2.

We give below the behavior of the lower and upper bound functions of X^+ , $F(X^+)$, X^- , $F(X^-)$, X^* and $F(X^*)$ for all p_n corresponding to the first occurrence of G_n .

Primes	Gaps	$X^-(P_n)$	$F(X^-(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$	$X^*(P_n)$	$F(X^*(P_n))$
2	1	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000
3	2	0,669712	0,851329	0,719720	0,979732	0,733992	1,018888
7	4	0,529417	0,757405	0,586507	0,950361	0,614414	1,058090
23	6	0,566736	0,829998	0,596694	0,963347	0,609663	1,026717
89	8	0,621137	0,892405	0,637129	0,984191	0,642823	1,018942
113	14	0,539221	0,831800	0,561684	0,964798	0,572401	1,034956
523	18	0,601854	0,890154	0,614518	0,984072	0,619362	1,022454
887	20	0,615332	0,900121	0,626233	0,986536	0,630201	1,019937
1 129	22	0,615019	0,900464	0,625563	0,986457	0,629406	1,019729
1 327	34	0,569466	0,871111	0,582356	0,977503	0,587896	1,026978
9 551	36	0,647444	0,920507	0,654226	0,989804	0,656410	1,013193
15 683	44	0,645021	0,919440	0,651544	0,989145	0,653664	1,012908
19 609	52	0,636959	0,915312	0,643627	0,987907	0,645864	1,013504
31 397	72	0,623349	0,908074	0,630200	0,985549	0,632621	1,014446
155 921	86	0,656224	0,925360	0,661166	0,989084	0,662703	1,009780
360 653	96	0,669146	0,931515	0,673424	0,990210	0,674687	1,008237
370 261	112	0,658623	0,926558	0,663168	0,988930	0,664568	1,008975
492 113	114	0,664253	0,929244	0,668554	0,989489	0,669849	1,008386
1 349 533	118	0,684380	0,938347	0,687908	0,991353	0,688887	1,006572
1 357 201	132	0,677083	0,935139	0,680776	0,990561	0,681831	1,006997
2 010 733	148	0,677890	0,935502	0,681465	0,990510	0,682483	1,006763
4 652 353	154	0,691984	0,941594	0,695074	0,991745	0,695903	1,005659
17 051 707	180	0,705811	0,947223	0,708410	0,992755	0,709067	1,004612
20 831 323	210	0,700475	0,945092	0,703137	0,992208	0,703826	1,004782
47 326 693	220	0,711147	0,949306	0,713508	0,993033	0,714091	1,004139
122 164 747	222	0,724647	0,954356	0,726682	0,994015	0,727155	1,003462
189 695 659	234	0,728062	0,955585	0,730002	0,994222	0,730446	1,003274
191 912 783	248	0,725325	0,954602	0,727303	0,994004	0,727761	1,003358
387 096 133	250	0,734215	0,957750	0,736001	0,994597	0,736398	1,002973
436 273 009	282	0,729972	0,956264	0,731805	0,994260	0,732220	1,003079
1 294 268 491	288	0,742373	0,960524	0,743954	0,995050	0,744292	1,002586
1 453 168 141	292	0,743099	0,960765	0,744663	0,995089	0,744996	1,002553
2 300 942 549	320	0,744273	0,961154	0,745790	0,995126	0,746111	1,002469
3 842 610 773	336	0,747843	0,962323	0,749284	0,995322	0,749583	1,002327
4 302 407 359	354	0,746821	0,961991	0,748266	0,995243	0,748568	1,002339
10 726 904 659	382	0,753261	0,964057	0,754579	0,995594	0,754846	1,002104
20 678 048 297	384	0,759593	0,966025	0,760809	0,995941	0,761048	1,001916
22 367 084 959	394	0,759318	0,965941	0,760533	0,995919	0,760772	1,001915
25 056 082 087	456	0,754554	0,964464	0,755812	0,995615	0,756065	1,002003
42 652 618 343	464	0,758997	0,965843	0,760183	0,995857	0,760417	1,001871
127 976 334 671	468	0,768638	0,968726	0,769685	0,996357	0,769880	1,001616
182 226 896 239	474	0,771201	0,969468	0,772211	0,996481	0,772397	1,001551
241 160 624 143	486	0,772639	0,969882	0,773625	0,996546	0,773806	1,001512
297 501 075 799	490	0,774081	0,970290	0,775047	0,996613	0,775222	1,001477
303 371 455 241	500	0,773505	0,970130	0,774476	0,996584	0,774653	1,001485
304 599 508 537	514	0,772532	0,969850	0,773511	0,996529	0,773690	1,001499
416 608 695 821	516	0,774959	0,970536	0,775905	0,996645	0,776077	1,001441
461 690 510 011	532	0,774693	0,970460	0,775638	0,996628	0,775809	1,001441
614 487 453 523	534	0,776848	0,971073	0,777765	0,996728	0,777930	1,001394
738 832 927 927	540	0,777905	0,971364	0,778808	0,996770	0,778969	1,001364
1 346 294 310 749	582	0,779923	0,971921	0,780790	0,996847	0,780943	1,001308
1 408 695 493 609	588	0,779913	0,971918	0,780779	0,996853	0,780931	1,001312
1 968 188 556 461	602	0,781622	0,972376	0,782464	0,996921	0,782611	1,001257
2 614 941 710 599	652	0,781022	0,972208	0,781861	0,996888	0,782008	1,001265
7 177 162 611 713	674	0,787163	0,973907	0,787928	0,997128	0,788057	1,001148
13 829 048 559 701	716	0,789695	0,974480	0,790426	0,997223	0,790547	1,001087
19 581 334 192 423	766	0,789873	0,974651	0,790595	0,997211	0,790715	1,001003
42 842 283 925 351	778	0,794473	0,976006	0,795146	0,997574	0,795255	1,001007
90 874 329 411 493	804	0,798144	0,976349	0,798778	0,997467	0,798879	1,000671
171 231 342 420 521	806	0,801857	0,977203	0,802456	0,996582	0,802549	1,000580
218 209 405 436 543	906	0,799831	0,976318	0,800438	0,997131	0,800533	1,000244

Table I.2

We are going to try to estimate the minimum value of $X^{2^-}(p_n)$ using different upper bound of g_n and/or G_n .

I.2.1 - Baker, Harman and Pintz result

The best known result due to Baker, Harman and Pintz giving an upper bound value of g_n : $g_n < G_n \ll (p_n)^a$ with $a = 0,525$

$$X_{\text{BHP}}^{2^-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(e.p_n)} > 1 - \frac{\ln(p_n^a)}{\ln(e.p_n)} = \frac{(1-a)\ln(p_n)+1}{\ln(p_n)+1} > 1 - a = 0,475$$

This approach can only let us conclude that x is always higher than 0,475.

I.2.2 - Cramer's conjecture (under RH)

Cramer stated that $g_n \approx O(\ln^2(p_n))$.

It means that for p_n enough high, $g_n < G_n < O(\ln^2(p_n)) < \ln^{2+\varepsilon}(p_n)$ with $\varepsilon > 0$

$$X_{\text{Crm}}^{2^-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(e.p_n)} > 1 - \frac{\ln(\ln^{2+\varepsilon}(p_n))}{\ln(e.p_n)} = 1 - \frac{(2+\varepsilon).\ln(\ln(p_n))}{\ln(e.p_n)} = X_{m_1}(p_n)$$

We are going to calculate the derivative of the function $M_1(p_n)$ using (15)

$$\frac{dX_{m_1}(p_n)}{dp_n} = \frac{(2+\varepsilon).\ln(\ln(p_n))}{p_n.\ln(e.p_n)^2} - \frac{(2+\varepsilon).\ln^2(p_n)}{\ln^3(p_n).\ln(e.p_n)} = \frac{(2+\varepsilon).\ln(\ln(p_n)).\ln(p_n) - \ln(e.p_n)}{p_n.\ln(p_n).\ln(e.p_n)^2}$$

To find the lower bound of $X_{m_1}(p_n)$, we have to solve the equation $\frac{dX_{m_1}(p_n)}{dp_n} = 0$

$$\ln(\ln(p_n)).\ln(p_n) - \ln(e.p_n) = 0 \tag{I.16}$$

To simplify (I.16), we replace by $y = \ln(p_n)$. So we have to find Y which verify :

$$\ln(y).y - y - 1 = 0$$

An estimation of Y is given by $\ln(p_n) = y = (e^2 + 2.e)^{1/2} \approx 3.58$ and $g_n \approx \ln^2(p_n) \approx 12.82$

On table I.4 below, we gives the numerical values of $X_{m_1}(p_n)$ for all P_n corresponding to the first occurrence of a maximal gap and with $\varepsilon=0.5$. We see that the minimal value of $X_{m_1}(p_n)$ is reached for $p_n = 23$ which is the closest prime of the solution of (I.16).

For $p_n > 10\,726\,904\,659$, we have $x > X^{2^-}(p_n) > m_1(p_n) > 0.6$ and we know that for all p_n up to 10^{15} (as stated in table I.2) the lowest value of x is given for $p_n = 113$ (and $g_n = 14$), of course under the condition of Cramer's conjecture.

I.2.3 - Weaker hypothesis on maximal gap upper bound

Cramer has conjectured that $g_n < O(\ln^2(p_n))$ under Riemann hypothesis, and this statement hasn't never been contradicted by numerical computation up to very large numbers. If nobody up to now, has succeed on this point, we can think that the probability to find any prime number such as $g_n > \ln^r(p_n)$ for any real $r > 2$, is very low. In these conditions, we are going to show that if there is at least a real r , such that $g_n < \ln^r(p_n)$ for all prime numbers, then we'll prove that we can build a lower bound function $X_{m_r}(p_n)$ of $X^{2^-}(p_n)$, which is growing and higher than 0.6 for all p_n enough large.

If we suppose that there is a real r , such that $g_n < \ln^r(p_n)$, we can write :

$$X^{2^-}(p_n) = 1 - \frac{\ln(g_n)}{\ln(e.p_n)} > 1 - \frac{\ln(\ln^r(p_n))}{\ln(e.p_n)} = 1 - \frac{r.\ln(\ln(p_n))}{\ln(e.p_n)} = X_{m_r}(p_n)$$

We easily see that the functions $X_{m_r}(p_n)$ has a lower bound for $p_n = 23$ for all real $r > 0$ and that the limit of $m_r(p_n)$ when p_n tends toward ∞ , is 1. So, there is for each real r , a prime $p_{n,r}$ such as for any prime number p_n higher than $p_{n,r}$, $X^{2^-}(p_n) > 0.6$. As it has been calculated on all prime numbers below 218 209 405 436 543 that the lowest value of x has been reached for $p_n = 113$, we can conclude that if there is a real r , such that $g_n < \ln^r(p_n)$, which is a much more weakest hypothesis than Cramer's conjecture, then the generalized Andrica's conjecture will be verified, if and only if we can prove that for $p_n > 218 209 405 436 543$, $X^{2^-}(P_n) > 0,6$.

To insure this condition, we have to find $p_n < 218 209 405 436 543$ such as $X^{2^-}(P_n) > X_{m_r}(p_n) > 0,6$.

$$1 - \frac{r \ln(\ln(p_n))}{\ln(e \cdot p_n)} > 0,6 \quad \Leftrightarrow \quad \frac{r \ln(\ln(p_n))}{\ln(e \cdot p_n)} > 0,4$$

Which is satisfy for all $p_n > e^{16,533787} \approx 41 193 733$ which is coherent the numerical values of $X_{m_r}(p_n)$ in table 3.

This finish to establish that under the $g_n < \ln^r(p_n)$ with $r=2.5$, the minimum value of x is reached for $p_n = 113$.

I.2.4 - Upper bound of x thanks to Lower bound on maximal gaps

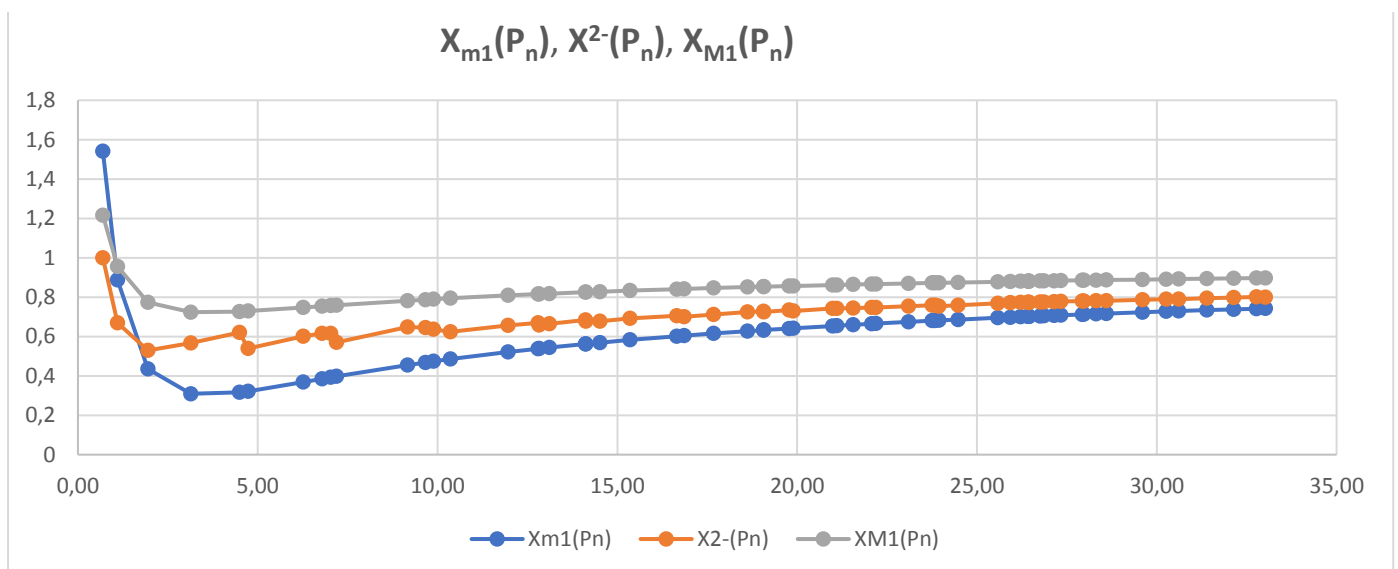
In 1931, Erik Westzynthius [13] proved that maximal gaps grows more than logarithmically.

$$\text{Limsup} \frac{G_n}{\ln(p_n)} = \infty \quad \text{which imply that for } n \text{ enough large } G_n > \ln(p_n)$$

In these conditions we can define for all P_n , an upper bound function of $X^{2^-}(P_n)$ independent of G_n .

$$X^{2^-}(P_n) = 1 - \frac{\ln(G_n)}{\ln(e \cdot P_n)} < 1 - \frac{\ln(\ln(P_n))}{\ln(e \cdot P_n)} = X_{M_1}(P_n)$$

The Graphic below shows the evolution of $X^{2^-}(P_n)$ surrounded by the lower and upper bound functions $X_{m_1}(P_n)$ and $X_{M_1}(P_n)$



We give below the behavior of the lower and upper bound functions X_{m1} , $F(X_{m1})$, X_{M1} , $F(X_{M1})$ of X^2 , $F(X^2)$, for all p_n corresponding to the first occurrence of G_n .

Primes	Gaps	Ln(P_n)	$X_{m1}(P_n)$	$F(X_{m1}(P_n))$	$X^2(P_n)$	$F(X^2(P_n))$	$X_{M1}(P_n)$	$F(X_{M1}(P_n))$
2	1	0,69	1,541171088	0,944224262	1	1	1,216468435	1,013547875
3	2	1,10	0,887964266	0,972790415	0,669711655	0,851328799	0,955185706	0,991653064
7	4	1,95	0,435038938	0,646085363	0,529417297	0,757405268	0,774015575	0,942802451
23	6	3,14	0,30915947	0,501750404	0,566736314	0,829998067	0,723663788	0,93699799
89	8	4,49	0,31606466	0,544656826	0,621136945	0,892404769	0,726425864	0,94854812
113	14	4,73	0,321954074	0,559151858	0,539221472	0,831800413	0,728781629	0,950513065
523	18	6,26	0,368381864	0,655514074	0,601854215	0,890153941	0,747352746	0,960716153
887	20	6,79	0,385217119	0,684714704	0,615332318	0,900120577	0,754086848	0,963413106
1129	22	7,03	0,392814909	0,696991051	0,615019462	0,900463798	0,757125964	0,964538042
1327	34	7,19	0,397856421	0,704838152	0,569466487	0,871110994	0,759142569	0,965258269
9551	36	9,16	0,455126154	0,779200356	0,647444164	0,92050724	0,782050462	0,972397117
15683	44	9,66	0,468115076	0,792913799	0,645021429	0,919439877	0,78724603	0,973822721
19609	52	9,88	0,473781404	0,798606188	0,636959143	0,915312439	0,789512562	0,974427525
31397	72	10,35	0,48535278	0,809735209	0,623349481	0,908074421	0,794141112	0,975632633
155921	86	11,96	0,521242266	0,840682918	0,656223564	0,925360302	0,808496907	0,979138011
360653	96	12,80	0,538060372	0,853662182	0,669146357	0,931515118	0,815224149	0,980669673
370261	112	12,82	0,538567801	0,854040818	0,658623112	0,926558073	0,81542712	0,98071484
492113	114	13,11	0,543984662	0,858038	0,664253312	0,92924402	0,817593865	0,981193239
1349533	118	14,12	0,562155146	0,870877496	0,684379777	0,938346626	0,824862058	0,982748459
1357201	132	14,12	0,562252856	0,870944278	0,677083335	0,935139132	0,824901142	0,982756618
2010733	148	14,51	0,568919576	0,875446017	0,677890351	0,935502058	0,827567831	0,98330822
4652353	154	15,35	0,582443168	0,884256345	0,69198383	0,941593507	0,832977267	0,984396545
17051707	180	16,65	0,601666374	0,896074454	0,705810832	0,94722331	0,840666549	0,985873904
20831323	210	16,85	0,604459938	0,897725542	0,700475184	0,945091969	0,841783975	0,986081864
47326693	220	17,67	0,615477106	0,904077702	0,711147249	0,949306378	0,846190843	0,986885518
122164747	222	18,62	0,627401596	0,910672158	0,724646546	0,954356428	0,850960638	0,987725887
189695659	234	19,06	0,632664021	0,913491391	0,728062423	0,955585458	0,853065608	0,988087026
191912783	248	19,07	0,632800768	0,913563916	0,725324968	0,954602135	0,853120307	0,988096394
387096133	250	19,77	0,640855053	0,917770579	0,734215264	0,957750126	0,856342021	0,988637526
436273009	282	19,89	0,642189298	0,918455169	0,729971915	0,956264032	0,856875719	0,988725927
1294268491	288	20,98	0,653837626	0,924285628	0,742372686	0,960523725	0,86153505	0,989481196
1453168141	292	21,10	0,655028989	0,924867293	0,743098577	0,960765261	0,862011596	0,989556879
2300942549	320	21,56	0,659669076	0,927107083	0,744273299	0,961154306	0,86386763	0,98984848
3842610773	336	22,07	0,664686723	0,929483461	0,747843175	0,962322969	0,865874689	0,990159333
4302407359	354	22,18	0,665770629	0,929990609	0,74682144	0,961990597	0,866308252	0,990225822
10726904659	382	23,10	0,674255309	0,933885169	0,753261311	0,964057438	0,869702124	0,990736961
20678048297	384	23,75	0,680062453	0,93647434	0,759592713	0,966024682	0,872024981	0,991078734
22367084959	394	23,83	0,680741857	0,936773278	0,759317567	0,965940595	0,872296743	0,991113901
25056082087	456	23,94	0,681718544	0,937201517	0,754554244	0,964464486	0,872687418	0,991178036
42652618343	464	24,48	0,686208291	0,93914777	0,758996737	0,965842515	0,874483316	0,991422653
127976334671	468	25,58	0,695051167	0,942875706	0,768638097	0,968725502	0,878020467	0,991942406
182226896239	474	25,93	0,69777919	0,943997636	0,7712014	0,969467998	0,879111676	0,992058754
241160624143	486	26,21	0,699903979	0,944862366	0,772638824	0,969881535	0,879961592	0,992191315
297501075799	490	26,42	0,701474411	0,945496231	0,774080862	0,970290184	0,880589765	0,992284775
303371455241	500	26,44	0,70161964	0,945554867	0,773505452	0,970129848	0,880647856	0,9923172
303 371 455 241	500	26,44	0,70161964	0,945554867	0,773505452	0,970129848	0,880647856	0,9923172
304 599 508 537	514	26,44	0,701649646	0,945566833	0,772532494	0,969849825	0,880659859	0,992296219
416 608 695 821	516	26,76	0,703955334	0,946489245	0,774958971	0,970535994	0,881582134	0,992393494
461 690 510 011	532	26,86	0,704703251	0,946786523	0,774692824	0,970460176	0,8818813	0,992500305
614 487 453 523	534	27,14	0,706762396	0,947600275	0,776848219	0,971072674	0,882704958	0,992622375
738 832 927 927	540	27,33	0,708072866	0,94811371	0,777905459	0,97136426	0,883229146	0,992671967
1 346 294 310 749	582	27,93	0,712251117	0,949732125	0,779922999	0,971920967	0,884900447	0,992843628
1 408 695 493 609	588	27,97	0,712561223	0,949851453	0,779913156	0,971918106	0,885024489	0,992889404
1 968 188 556 461	602	28,31	0,714827535	0,950713396	0,781621809	0,972376347	0,885931014	0,992797852
2 614 941 710 599	652	28,59	0,716721888	0,951429009	0,781022338	0,972208023	0,886688755	0,993087769
7 177 162 611 713	674	29,60	0,723233148	0,953845501	0,787162734	0,973907471	0,889293259	0,99395752
13 829 048 559 701	716	30,26	0,727287703	0,955310345	0,789694676	0,974479675	0,890915081	0,992370605
19 581 334 192 423	766	30,61	0,729384733	0,956071854	0,789873231	0,974651337	0,891753893	0,992919922
42 842 283 925 351	778	31,39	0,733976712	0,957698822	0,794472824	0,976005554	0,893590685	0,997802734
90 874 329 411 493	804	32,14	0,738226866	0,959186554	0,798144279	0,976348877	0,895290747	0,992675781
171 231 342 420 521	806	32,77	0,741692363	0,960258484	0,801857154	0,977203369	0,896676945	1,008789063
218 209 405 436 543	906	33,02	0,742991698	0,960548401	0,799831139	0,976318359	0,897196679	0,989257813

Table I.3

I.3 - Proof of the inequation $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ and $(p_{n+1})^{p/q} - (p_n)^{p/q} < p/q$

Florentin Smarandache, has listed some conjectures derived from the Andrica's conjecture, such as the fact that $(p_{n+1})^{1/k} - (p_n)^{1/k} < 2/k$ for $k \geq 2$. We are going to prove that for all integer $k > 2$, this assertion is true.

We'll start by using the Taylor serie expansion of (1.5) :

$$F_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} + \frac{x \cdot (x-1)}{2} \cdot p_n^x \cdot \left(\frac{g_n}{p_n}\right)^2 + p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^3$$

As $x-1 < 0$, $F_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} - p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^2$ and as according to Baker, Harman and Pintz result : $g_n \ll (p_n)^{0.525}$

$$F_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} < x \cdot p_n^{x-0.475}$$

If we replace x by $1/k$ with $k > 2$, then $p_n^{x-0.475} < 1$ and $F_n(1/k) < 1/k$.

So for any $k > 2$, we can conclude that $(p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$. But we cannot conclude for $k=2$ except if we use the Cramer's conjecture. The same approach applied to $F_n\left(\frac{p}{q}\right) = p_{n+1}^{\frac{p}{q}} - p_n^{\frac{p}{q}}$ let us to show that $p_{n+1}^{\frac{p}{q}} - p_n^{\frac{p}{q}} < \frac{p}{q}$ for all rational numbers such as $\frac{p}{q} < 0.475$.

I.4 - Study of the equation $(p_{n+1})^x - (p_n)^x = C$

In 2017, David Lowry-Duda [4], has formulated the following question : for any constant C , what is the smallest x such as $(p_{n+1})^x - (p_n)^x = C$ With the same approach developed before, we are going to build some lower and upper bound functions of x satisfying the condition $F_n(x) = (p_{n+1})^x - (p_n)^x = C$

We can start from the Taylor serie expansion of (5) :

$$F_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} + \frac{x \cdot (x-1)}{2} \cdot p_n^x \cdot \left(\frac{g_n}{p_n}\right)^2 + p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^3$$

Resolving $F_n(x) = C$ correspond to solve, at the first degree of the Taylor serie expansion the following equality:

$$F_n(x) = x \cdot p_n^x \cdot \frac{g_n}{p_n} = C$$

Which can written : $x \cdot p_n^x \cdot \frac{h_n}{p_n} = 1$ with $h_n = \frac{g_n}{C}$ (I.18)

By analogy to the previous development, the following function give a good upper bound of x :

$$X_c^+(p_n) = 1 - \frac{\ln(h_n)}{\ln(p_n)} - \frac{\ln\left(1 - \frac{\ln(h_n)}{\ln(e \cdot p_n)}\right)}{\ln(p_n)} \quad (I.19)$$

And with the same Taylor serie expansion, we build the following lower bound function :

$$X_c^-(p_n) = 1 - \frac{\ln(h_n)}{\ln(e \cdot p_n)} \left(1 - \frac{\ln(h_n)}{2 \cdot \ln(p_n) \ln(e \cdot p_n)}\right) \quad (I.20)$$

$$X_c^{2-}(p_n) = 1 - \frac{\ln(h_n)}{\ln(e \cdot p_n)} \quad (I.21)$$

$$X_c^{3-}(p_n) = 1 - \frac{\ln(h_n)}{\ln(p_n)} \quad (I.22)$$

Several situations can be studied :

If C is an even number, and p_n enough large :

$$\begin{aligned} \text{for all } p_n \text{ such as } g_n = C, & \quad x = X_C^+(p_n) = X_C^1(p_n) = X_C^2(p_n) = X_C^3(p_n) = 1 \\ \text{for all } p_n \text{ such as } g_n < C, & \quad x < X_C^3(p_n) < X_C^2(p_n) < X_C^1(p_n) < X_C^+(p_n) \\ \text{for all } p_n \text{ such as } g_n > C, & \quad X_C^3(p_n) < X_C^2(p_n) < X_C^1(p_n) < x < X_C^+(p_n) \end{aligned}$$

If C is real different from an even number, and p_n enough large :

$$\begin{aligned} \text{for all } p_n \text{ such as } g_n < C, & \quad x < X_C^3(p_n) < X_C^2(p_n) < X_C^1(p_n) < X_C^+(p_n) \\ \text{for all } p_n \text{ such as } g_n > C, & \quad X_C^3(p_n) < X_C^2(p_n) < X_C^1(p_n) < x < X_C^+(p_n) \end{aligned}$$

We give below some numerical results for $C=2$, $C=6$, $C=\Pi$ for the primes numbers up to 300 (tables I.4, I.6, I.8) and for all primes numbers corresponding to the first occurrence of maximal gaps (tables I.5, I.7, I.9). We have shown in paragraphe I.2 that the local minimal values of lower bound functions of $x(p_n)$ was necessary reached for the prime numbers corresponding to the first occurrence of a maximal gap. Thus, we can limit our research of the lower bound of $x(p_n)$ to the set of these prime numbers.

for C=2, we can see that the minimum value of x is obtain for $p_n = 113$

Primes	Gaps	$X^2(P_n)$	$F(X^2(P_n))$	$X(P_n)$	$F(X(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1,409383891	2,047543071	1,53457703	2,500299469	1,504935371	2,386283217
3	2	1	2	1	2	1	2
5	2	1	2	1	2	1	2
7	4	0,764708648	1,82846929	0,77894565	1,921360642	0,781651372	1,939467764
11	2	1	2	1	2	1	2
13	4	0,805566051	1,904538249	0,812937487	1,960658639	0,814055938	1,969302797
17	2	1	2	1	2	1	2
19	4	0,824272302	1,931595547	0,829516913	1,975152393	0,830224758	1,98109976
23	6	0,734345587	1,855595552	0,745602614	1,954348804	0,748098163	1,976881952
29	2	1	2	1	2	1	2
31	6	0,75222926	1,884605669	0,761169648	1,968050494	0,762987424	1,985429737
37	4	0,849672626	1,959584518	0,852801988	1,989476527	0,853155612	1,992881241
41	2	1	2	1	2	1	2
43	4	0,854417549	1,963700645	0,857235182	1,991419411	0,857542302	1,994462982
47	6	0,773488896	1,913560699	0,780152716	1,981011106	0,781367259	1,993542509
53	6	0,778964232	1,920051171	0,78511767	1,983743732	0,786206617	1,995220373
59	2	1	2	1	2	1	2
61	6	0,78504414	1,926805701	0,790664616	1,986490392	0,791626554	1,996878923
67	4	0,866822648	1,972905887	0,868931813	1,995446505	0,869139947	1,997684108
71	2	1	2	1	2	1	2
73	6	0,792340854	1,93429832	0,797366598	1,989395506	0,798192084	1,99858656
79	4	0,87090904	1,97547282	0,872816023	1,996458623	0,872997811	1,998470257
83	6	0,797260638	1,938983339	0,801911848	1,991120615	0,802654421	1,99956698
89	8	0,74742463	1,903309751	0,754531516	1,984302176	0,756012191	2,001583035
97	4	0,875662221	1,978189478	0,87735197	1,997455854	0,877506507	1,999226843
101	2	1	2	1	2	1	2
103	4	0,876986598	1,978897069	0,878619122	1,997700996	0,878766674	1,999408977
107	2	1	2	1	2	1	2
109	4	0,878210365	1,979532434	0,879791255	1,997915371	0,879932575	1,999566641
113	14	0,660244738	1,817742928	0,672455603	1,962793959	0,676192115	2,009317475
127	4	0,88139545	1,981105304	0,882847425	1,998420629	0,882973494	1,999930906
131	6	0,813008444	1,952107477	0,816594689	1,995461529	0,817115401	2,001832747
137	2	1	2	1	2	1	2
139	10	0,728798554	1,893592701	0,736251878	1,985141461	0,737949777	2,006579199
149	2	1	2	1	2	1	2
151	6	0,817423766	1,955307871	0,820745816	1,9963813	0,821214944	2,002248005
157	6	0,818598464	1,956126249	0,821852626	1,996605911	0,822308738	2,002343674
163	4	0,886252774	1,983290352	0,887522821	1,999053945	0,887628155	2,00036675
167	6	0,820429323	1,957374732	0,82357965	1,996939732	0,824016048	2,00248084
173	6	0,821459414	1,9580629	0,824552367	1,997119019	0,824977973	2,002551759
179	2	1	2	1	2	1	2
181	10	0,740350297	1,906233094	0,746835103	1,989208521	0,748233754	2,007553049
191	2	1	2	1	2	1	2
193	4	0,889321177	1,984546253	0,890485025	1,999375905	0,89057871	2,000574287
197	2	1	2	1	2	1	2
199	12	0,715291167	1,885862617	0,722948513	1,98534376	0,72480446	2,010201154
211	12	0,717915698	1,888844041	0,725350252	1,986279277	0,72713097	2,010321934
223	4	0,891816982	1,985501459	0,892899219	1,999597239	0,892984196	2,000708134
227	2	1	2	1	2	1	2
229	4	0,892263424	1,98566631	0,893331503	1,999633221	0,893414991	2,000728979
233	6	0,829699932	1,963213628	0,832360236	1,99834495	0,832706901	2,002967465
239	2	1	2	1	2	1	2
241	10	0,751813676	1,91744136	0,75742917	1,992415578	0,758574261	2,008046611
251	6	0,831641987	1,964339812	0,834206944	1,998583378	0,834536813	2,003029037
257	6	0,832249271	1,964685416	0,834784915	1,998654245	0,835109666	2,003045661
263	6	0,832838323	1,965017703	0,835345761	1,998721341	0,835665608	2,003060619
269	2	1	2	1	2	1	2
271	6	0,833597014	1,96544145	0,836068451	1,998805406	0,836382068	2,00307822
277	4	0,895358492	1,986761185	0,89633199	1,999854469	0,896405712	2,000849419
281	2	1	2	1	2	1	2
283	10	0,757813442	1,922813073	0,763008516	1,99379594	0,764036418	2,008137117
293	14	0,708703612	1,883783703	0,716173381	1,986495674	0,718038098	2,012963731

Table I.4

Primes	Gaps	$X^2(P_n)$	$F(X^2(P_n))$	$X'(P_n)$	$F(X'(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1,409383891	2,047543	1,530278251	2,483489	1,504935371	2,386283
3	2	1	2,000000	1	2,000000	1	2,000000
7	4	0,764708648	1,828469	0,778933874	1,921282	0,781651372	1,939468
23	6	1	2	1	2	1	2
89	8	0,74742463	1,903310	0,754530833	1,984294	0,756012191	2,001583
113	14	0,660244738	1,817743	0,672453768	1,962771	0,676192115	2,009317
523	18	0,697334538	1,877944	0,704651831	1,986836	0,706572305	2,016411
887	20	0,704336039	1,887031	0,710775283	1,989505	0,71241456	2,016443
1 129	22	0,70134897	1,885435	0,707693494	1,989378	0,709329902	2,017078
1 327	34	0,654092857	1,840721	0,662412781	1,979271	0,665023514	2,024794
9 551	36	0,715637772	1,901025	0,72004951	1,991679	0,721117553	2,014257
15 683	44	0,710042588	1,896989	0,714394162	1,990571	0,71547438	2,014497
19 609	52	0,700645609	1,889575	0,705178964	1,988964	0,706351815	2,015504
31 397	72	0,684395688	1,876054	0,6892055	1,985722	0,690539387	2,017238
155 921	86	0,709719093	1,897357	0,713242647	1,988831	0,71411858	2,012239
360 653	96	0,719390173	1,904783	0,722467068	1,989747	0,723199397	2,010516
370 261	112	0,708771355	1,896689	0,712078731	1,988086	0,712904392	2,011575
492 113	114	0,713390161	1,900272	0,716523928	1,988644	0,717290295	2,010868
1 349 533	118	0,730237193	1,912693	0,732814968	1,990572	0,733398526	2,008635
1 357 201	132	0,722923567	1,907433	0,72564192	1,989528	0,72627854	2,009255
2 010 733	148	0,722569141	1,907180	0,725220644	1,989280	0,72584262	2,009040
4 652 353	154	0,734370677	1,915600	0,736668582	1,990597	0,737178752	2,007639
17 051 707	180	0,745078709	1,922837	0,74703	1,991543	0,74744147	2,006340
20 831 323	210	0,739302677	1,918983	0,741319151	1,990729	0,741756429	2,006635
47 326 693	220	0,748268365	1,924920	0,750061221	1,991648	0,75043341	2,005786
122 164 747	222	0,759973562	1,932282	0,761520553	1,992802	0,761823377	2,004868
189 695 659	234	0,762614517	1,933883	0,76409272	1,993003	0,764378182	2,004625
191 912 783	248	0,759857059	1,932211	0,761368882	1,992693	0,761664998	2,004758
387 096 133	250	0,76758106	1,936834	0,768946946	1,993401	0,769203997	2,004229
436 273 009	282	0,763146727	1,934203	0,764556702	1,992900	0,764828242	2,004407
1 294 268 491	288	0,773906306	1,940482	0,7751245	1,993852	0,775346203	2,003721
1 453 168 141	292	0,774466944	1,940799	0,775672451	1,993891	0,775891186	2,003678
2 300 942 549	320	0,77500256	1,941102	0,776176768	1,993878	0,776389213	2,003578
3 842 610 773	336	0,777889325	1,942716	0,779007006	1,994093	0,779206095	2,003385
4 302 407 359	354	0,776721104	1,942066	0,777844819	1,993964	0,778046253	2,003412
10 726 904 659	382	0,782027354	1,944985	0,783055931	1,994363	0,783235047	2,003089
20 678 048 297	384	0,787596014	1,947956	0,788545719	1,994800	0,788706052	2,002818
22 367 084 959	394	0,787232318	1,947766	0,788182139	1,994760	0,78834282	2,002821
25 056 082 087	456	0,78234195	1,945155	0,783331223	1,994300	0,783503196	2,002969
42 652 618 343	464	0,786204209	1,947223	0,787137939	1,994603	0,787296811	2,002779
127 976 334 671	468	0,794720666	1,951623	0,795544507	1,995246	0,795678053	2,002410
182 226 896 239	474	0,796941664	1,952735	0,797736786	1,995400	0,797864025	2,002313
241 160 624 143	486	0,798114001	1,953317	0,798891566	1,995474	0,799015145	2,002261
297 501 075 799	490	0,799360966	1,953931	0,800122851	1,995557	0,800243055	2,002207
303 371 455 241	500	0,798767553	1,953638	0,799533385	1,995512	0,799654634	2,002222
304 599 508 537	514	0,797790876	1,953156	0,798564043	1,995429	0,798687156	2,002247
416 608 695 821	516	0,799932376	1,954211	0,800680394	1,995581	0,800798012	2,002165
461 690 510 011	532	0,799574122	1,954037	0,800321949	1,995549	0,800439786	2,002168
614 487 453 523	534	0,801476767	1,954968	0,802202736	1,995675	0,802315848	2,002097
738 832 927 927	540	0,802373792	1,955390	0,803088363	1,995727	0,803199107	2,002054
1 346 294 310 749	582	0,803883803	1,956132	0,804572378	1,995807	0,804678133	2,001975
1 408 695 493 609	588	0,803836491	1,956086	0,804524282	1,995816	0,804629947	2,001979
1 968 188 556 461	602	0,805272143	1,956792	0,805941896	1,995885	0,806043905	2,001916
2 614 941 710 599	652	0,804445595	1,956404	0,805114334	1,995796	0,805216696	2,001928
7 177 162 611 713	674	0,809813177	1,958916	0,810424135	1,996159	0,810514648	2,001701
13 829 048 559 701	716	0,811869856	1,959724	0,812454713	1,996231	0,812540265	2,001572
19 581 334 192 423	766	0,811804383	1,959641	0,812382996	1,996216	0,812467669	2,001617
42 842 283 925 351	778	0,815873821	1,961945	0,816413866	1,996460	0,816490906	2,001755
90 874 329 411 493	804	0,81905969	1,962341	0,819569007	1,996704	0,819640204	2,001221
171 231 342 420 521	806	0,822380232	1,963440	0,822861539	1,996399	0,822927392	1,998901
218 209 405 436 543	906	0,820207947	1,962402	0,820697478	1,996460	0,820765405	2,000854

Table I.5

for C=6, we can see that the minimum value of x is obtain for $p_n = 113$

Primes	Gaps	$X^2(P_n)$	$F(X^2(P_n))$	$X(P_n)$	$F(X(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	2,058242006	5,429906392	3,10659915	21,74110049	2,543549878	10,5221896
3	2	1,523494642	6,279179845	1,650509009	8,1143397	1,616783094	7,584732868
5	2	1,421014918	6,035992942	1,476323569	6,924027768	1,464286901	6,721071099
7	4	1,137636617	6,151373262	1,142505599	6,243963571	1,142099131	6,23618524
11	2	1,323321409	5,908912743	1,345138467	6,340889005	1,341327456	6,263337329
13	4	1,113736569	6,060056777	1,116258489	6,115366452	1,116082006	6,11148054
17	2	1,286603481	5,89107396	1,301106116	6,212492739	1,29881495	6,16062283
19	4	1,102794113	6,03128825	1,104588543	6,074183009	1,104474247	6,071442248
23	6	1	6	1	6	1	6
29	2	1,251554356	5,888990462	1,260952561	6,125926799	1,259622578	6,091847141
31	6	1	6	1	6	1	6
37	4	1,087935876	6,00564521	1,089006641	6,035182245	1,088947715	6,033553165
41	2	1,233074253	5,893177274	1,240389471	6,092458931	1,239419575	6,065666496
43	4	1,085160274	6,002480583	1,086124382	6,029901259	1,086072903	6,02843409
47	6	1	6	1	6	1	6
53	6	1	6	1	6	1	6
59	2	1,216367147	5,899630067	1,222108243	6,068456519	1,221394318	6,047209406
61	6	1	6	1	6	1	6
67	4	1,077903757	5,996376034	1,078625458	6,018754134	1,07859003	6,017653723
71	2	1,208755295	5,903289359	1,213867347	6,059288768	1,213251092	6,040272447
73	6	1	6	1	6	1	6
79	4	1,075513371	5,994995501	1,076165893	6,015857986	1,076134793	6,014862043
83	6	1	6	1	6	1	6
89	8	0,947585875	5,977776104	0,947891898	5,988006088	0,947903029	5,988378494
97	4	1,072732938	5,993741732	1,07331113	6,012926397	1,073284535	6,012042673
101	2	1,195652486	5,910462811	1,199799933	6,045829524	1,199327458	6,030257081
103	4	1,071958227	5,993455674	1,072516838	6,012188683	1,072491404	6,011334522
107	2	1,193662161	5,911636083	1,19767546	6,044023433	1,197222345	6,028932603
109	4	1,071242369	5,993214656	1,071783313	6,011536331	1,071758916	6,010708848
113	14	0,852062077	5,875588741	0,854376912	5,957195907	0,854633893	5,966322392
127	4	1,069379214	5,992687926	1,069876047	6,009965669	1,069854198	6,009204811
131	6	1	6	1	6	1	6
137	2	1,185576998	5,916590652	1,189077032	6,037280999	1,188696405	6,024040102
139	10	0,913922341	5,958413851	0,914673123	5,985599727	0,914719187	5,987271634
149	2	1,182981698	5,918237827	1,186327412	6,035309206	1,185968058	6,022626979
151	6	1	6	1	6	1	6
157	6	1	6	1	6	1	6
163	4	1,066537862	5,992144091	1,066972444	6,007902132	1,066954078	6,007235363
167	6	1	6	1	6	1	6
173	6	1	6	1	6	1	6
179	2	1,177556778	5,921755142	1,180595631	6,031470359	1,180277804	6,019902831
181	10	0,917588793	5,965123858	0,918242025	5,98978191	0,918280285	5,991229191
191	2	1,17571405	5,922969918	1,178653382	6,030249782	1,178348792	6,019044938
193	4	1,064742961	5,991946542	1,065141205	6,00678832	1,065124808	6,00617653
197	2	1,174849064	5,923543232	1,177742486	6,029690898	1,177443963	6,018653565
199	12	0,889859589	5,941454746	0,891005475	5,985502296	0,891097225	5,989042782
211	12	0,890874896	5,943445086	0,891987446	5,986591551	0,892075631	5,990024499
223	4	1,063283009	5,991860772	1,063653328	6,005981607	1,063638409	6,005412106
227	2	1,17099157	5,926121447	1,173686421	6,027304619	1,173413838	6,016993372
229	4	1,063021857	5,991852051	1,063387331	6,005846195	1,063372666	6,005284034
233	6	1	6	1	6	1	6
239	2	1,16963151	5,927037897	1,172258712	6,026503441	1,171994852	6,016440156
241	10	0,921227198	5,970746538	0,92179287	5,993047127	0,921824446	5,994294363
251	6	1	6	1	6	1	6
257	6	1	6	1	6	1	6
263	6	1	6	1	6	1	6
269	2	1,166589897	5,929099076	1,169070169	6,024784511	1,168825047	6,0152609
271	6	1	6	1	6	1	6
277	4	1,061211358	5,991842954	1,061544469	6,004977441	1,06153147	6,004464349
281	2	1,165494666	5,929844679	1,167923484	6,024189531	1,167684855	6,014855284
283	10	0,923131487	5,973316491	0,923654811	5,994441196	0,923683274	5,995592238
293	14	0,87316228	5,929747823	0,874578435	5,987533272	0,874710848	5,992964216

Table I.6

Primes	Gaps	$X^2(P_n)$	$F(X^2(P_n))$	$X'(P_n)$	$F(X'(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	2,058242006	5,429906	2,866061886	16,014810	2,543549878	10,522190
3	2	1,523494642	6,279180	1,648218643	8,077333	1,616783094	7,584733
7	4	1,137636617	6,151373	1,142504221	6,243937	1,142099131	6,236185
23	6	1	6	1	6	1	6
89	8	0,947585875	5,977776	0,947891897	5,988006	0,947903029	5,988378
113	14	0,852062077	5,875589	0,854376846	5,957194	0,854633893	5,966322
523	18	0,848667269	5,908727	0,850496592	5,989845	0,850705001	5,999155
887	20	0,845403599	5,910233	0,847164103	5,993858	0,847369574	6,003693
1 129	22	0,838178003	5,903176	0,840040717	5,994376	0,84026971	6,005682
1 327	34	0,78822248	5,829161	0,791341083	5,985179	0,791865722	6,011824
9 551	36	0,823722082	5,893108	0,825417441	5,997722	0,825647376	6,012051
15 683	44	0,813098687	5,879643	0,814906704	5,996559	0,815169162	6,013720
19 609	52	0,801586268	5,863506	0,803577822	5,994940	0,803887921	6,015665
31 397	72	0,781151637	5,832065	0,783464388	5,991114	0,783869076	6,019382
155 921	86	0,794507502	5,854218	0,796273279	5,992427	0,796559867	6,015162
360 653	96	0,799024737	5,861143	0,800603046	5,992530	0,800852551	6,013566
370 261	112	0,788254439	5,844720	0,790002852	5,990487	0,790296933	6,015356
492 113	114	0,791270225	5,849460	0,79293231	5,990842	0,79320713	6,014543
1 349 533	118	0,802919477	5,866924	0,804295318	5,992226	0,804507851	6,011818
1 357 201	132	0,795578617	5,856108	0,797058268	5,990875	0,797296933	6,012900
2 010 733	148	0,793383347	5,852798	0,79485401	5,990153	0,795094253	6,012893
4 652 353	154	0,80155224	5,864978	0,802834785	5,991180	0,803034526	6,011076
17 051 707	180	0,807316822	5,873217	0,808431623	5,991524	0,808599324	6,009526
20 831 323	210	0,800842798	5,863960	0,802019622	5,990203	0,802203672	6,010190
47 326 693	220	0,80710394	5,872922	0,808156669	5,991011	0,808315238	6,009002
122 164 747	222	0,815965557	5,885023	0,816874985	5,992201	0,817004643	6,007640
189 695 659	234	0,817378289	5,886890	0,818253133	5,992285	0,818376748	6,007329
191 912 783	248	0,814589129	5,883188	0,81549035	5,991808	0,815619959	6,007592
387 096 133	250	0,820464596	5,890909	0,821279622	5,992566	0,821392529	6,006786
436 273 009	282	0,81572756	5,884707	0,816581001	5,991734	0,81670286	6,007173
1 294 268 491	288	0,823885912	5,895270	0,824625054	5,992778	0,824725193	6,006112
1 453 168 141	292	0,82418463	5,895646	0,824917223	5,992796	0,82501628	6,006054
2 300 942 549	320	0,823707286	5,895045	0,824428159	5,992606	0,824525938	6,005963
3 842 610 773	336	0,825511345	5,897307	0,82620113	5,992773	0,826293587	6,005685
4 302 407 359	354	0,82411095	5,895553	0,82480828	5,992524	0,824902615	6,005764
10 726 904 659	382	0,827620454	5,899918	0,82826374	5,992884	0,828348764	6,005280
20 678 048 297	384	0,831980195	5,905194	0,832574466	5,993418	0,832650733	6,004834
22 367 084 959	394	0,831476152	5,904594	0,832072025	5,993330	0,83214876	6,004853
25 056 082 087	456	0,826384423	5,898393	0,827013848	5,992522	0,827097728	6,005179
42 652 618 343	464	0,82932703	5,902001	0,829922079	5,992887	0,829999832	6,004866
127 976 334 671	468	0,836060561	5,909994	0,836585996	5,993708	0,836651557	6,004240
182 226 896 239	474	0,837739018	5,911922	0,838246733	5,993896	0,838309343	6,004074
241 160 624 143	486	0,838491201	5,912788	0,838988842	5,993975	0,839049885	6,004001
297 501 075 799	490	0,839428983	5,913854	0,839916953	5,994065	0,839976413	6,003903
303 371 455 241	500	0,838807036	5,913157	0,83929843	5,993990	0,839358572	6,003960
304 599 508 537	514	0,837824465	5,912019	0,838321792	5,993830	0,838383083	6,004004
416 608 695 821	516	0,839514287	5,913938	0,839995603	5,994028	0,840054217	6,003845
461 690 510 011	532	0,839010045	5,913364	0,839492538	5,993950	0,839551506	6,003869
614 487 453 523	534	0,840512091	5,915078	0,840980636	5,994139	0,84103729	6,003773
738 832 927 927	540	0,841155181	5,915800	0,84161682	5,994158	0,841672384	6,003685
1 346 294 310 749	582	0,84186078	5,916552	0,842308497	5,994213	0,842362112	6,003605
1 408 695 493 609	588	0,84175408	5,916491	0,842201675	5,994184	0,842255317	6,003611
1 968 188 556 461	602	0,842757036	5,917507	0,843193754	5,994255	0,843245716	6,003494
2 614 941 710 599	652	0,841570579	5,916248	0,842009506	5,994110	0,842062179	6,003448
7 177 162 611 713	674	0,845713281	5,920593	0,846115356	5,994431	0,846162176	6,003296
13 829 048 559 701	716	0,847016685	5,921768	0,847403427	5,994385	0,847448031	6,002960
19 581 334 192 423	766	0,846564437	5,922028	0,846949048	5,994690	0,846993555	6,002899
42 842 283 925 351	778	0,849793598	5,925537	0,850152996	5,995544	0,850193597	6,003174
90 874 329 411 493	804	0,852209834	5,926880	0,852549622	5,994751	0,852587312	6,002075
171 231 342 420 521	806	0,85490854	5,926758	0,855229702	5,989502	0,855264594	5,996338
218 209 405 436 543	906	0,852504422	5,926270	0,852833878	5,990967	0,852870339	5,998291

Table I.7

for $C=\pi$, we can see that the minimum value of x is obtain for $p_n = 113$.
 We see on the table below that if $g_n = 2 < \pi$, $F(X^-) > \pi$ and if $g_n > \pi$, $F(X^-) < \pi$

Primes	Gaps	$x^2(P_n)$	$F(x^2(P_n))$	$X(P_n)$	$F(X(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1,676095911	3,109610634	2,039899064	5,291108782	1,906391423	4,371742995
3	2	1,215181579	3,269237792	1,236319435	3,424594516	1,233648572	3,40461796
5	2	1,173057463	3,196864583	1,182368483	3,276355629	1,181410632	3,268097053
7	4	0,918000053	3,06885969	0,919727951	3,086279455	0,919828444	3,087295278
11	2	1,132900713	3,152011303	1,136584206	3,191327206	1,136286752	3,18813544
13	4	0,932239016	3,104472295	0,933134102	3,114985026	0,933176679	3,115485905
17	2	1,117807872	3,142352513	1,120257341	3,171610684	1,120080331	3,169487759
19	4	0,938758217	3,116602868	0,939395117	3,124767172	0,939422364	3,125116892
23	6	0,84354238	3,041234565	0,847446301	3,094383925	0,847907725	3,100722641
29	2	1,103400988	3,136835205	1,104988635	3,158356801	1,104886979	3,156974576
31	6	0,854075	3,063801823	0,857175703	3,108715059	0,85751452	3,113660026
37	4	0,947610328	3,128563169	0,947990384	3,134178215	0,948004199	3,134382508
41	2	1,095804774	3,135267926	1,097040615	3,153335756	1,096966924	3,152255598
43	4	0,949263952	3,13023641	0,949606152	3,135446133	0,949618184	3,135629452
47	6	0,866595898	3,085728826	0,868907159	3,12205467	0,869135645	3,125667713
53	6	0,869820607	3,090533658	0,871954869	3,124843884	0,872160207	3,128163912
59	2	1,088937346	3,134550798	1,089907291	3,149826777	1,089853349	3,148975342
61	6	0,873401379	3,09547948	0,875350807	3,127640459	0,875532663	3,13065684
67	4	0,953587177	3,13384254	0,953843339	3,138086025	0,95385155	3,138222148
71	2	1,085808507	3,134419133	1,086672189	3,148520428	1,086625747	3,14776061
73	6	0,877698791	3,100890737	0,879441958	3,130592003	0,879598494	3,133272457
79	4	0,955011301	3,134803465	0,955242908	3,138756585	0,955250096	3,138879362
83	6	0,880596306	3,104228109	0,882209576	3,132342382	0,882350678	3,134812921
89	8	0,829700568	3,06765524	0,8329313	3,125024359	0,833352228	3,132573997
97	4	0,956667803	3,13579185	0,956873027	3,13942367	0,956879155	3,139532171
101	2	1,08042262	3,134440559	1,081123345	3,14665469	1,081087899	3,146035732
103	4	0,957129353	3,136043721	0,957327628	3,139589085	0,957333483	3,139693836
107	2	1,0796045	3,134468498	1,080282562	3,14641042	1,080248592	3,145811095
109	4	0,957555841	3,136267711	0,957747844	3,139734353	0,957753456	3,139835725
113	14	0,739090927	2,96776685	0,746291477	3,101765403	0,747853748	3,131589337
127	4	0,958665855	3,136812637	0,958842202	3,140079567	0,958847217	3,140172519
131	6	0,889871004	3,113341969	0,89111491	3,13674593	0,891214492	3,13862689
137	2	1,076281108	3,13464088	1,076872457	3,145514977	1,076844002	3,144990896
139	10	0,80489337	3,052445198	0,808750749	3,126434796	0,809339405	3,137876353
149	2	1,075214314	3,134714708	1,075779591	3,145258735	1,075752752	3,144757319
151	6	0,892471416	3,115500538	0,893623688	3,137680859	0,893713564	3,139417321
157	6	0,893163256	3,11604776	0,894291982	3,137909445	0,894379406	3,139608895
163	4	0,960358652	3,137544029	0,960512903	3,140520467	0,960517105	3,140601576
167	6	0,894241543	3,116878641	0,895334255	3,138249433	0,895417962	3,13989237
173	6	0,894848217	3,117334532	0,895921028	3,138432182	0,896002699	3,140043942
179	2	1,07298441	3,13489494	1,073497844	3,144768539	1,073474152	3,144312258
181	10	0,813203878	3,063901666	0,816560041	3,130978742	0,817046779	3,140823564
191	2	1,07222696	3,134963542	1,07272358	3,144615342	1,072700889	3,14417372
193	4	0,961428002	3,137948792	0,961569355	3,14075025	0,961573099	3,140824468
197	2	1,071871409	3,13499695	1,072360272	3,144545663	1,07233804	3,144110806
199	12	0,787047219	3,04021041	0,791331011	3,127258147	0,792056236	3,142230523
211	12	0,789010281	3,043049855	0,793169442	3,128309868	0,79386583	3,142808566
223	4	0,962297799	3,138247927	0,962429241	3,140911807	0,962432641	3,140980741
227	2	1,070285793	3,135154771	1,070741107	3,144251679	1,070720835	3,143846113
229	4	0,962453386	3,138298732	0,962583109	3,140938445	0,96258645	3,141006464
233	6	0,899701484	3,120695722	0,900624233	3,139685926	0,900690972	3,14106377
239	2	1,069726742	3,135213663	1,070170627	3,144154338	1,070151014	3,143758761
241	10	0,821450815	3,07389015	0,824357092	3,134517937	0,824757051	3,142951476
251	6	0,900845261	3,121417706	0,901734939	3,139930998	0,901798493	3,141257579
257	6	0,901202922	3,12163827	0,902082432	3,140003953	0,902145015	3,141314795
263	6	0,901549845	3,121849885	0,902419572	3,140073078	0,902481225	3,141368786
269	2	1,068476493	3,135351002	1,068895553	3,143948001	1,068877353	3,14357414
271	6	0,901996677	3,122119098	0,902853917	3,140159764	0,902914387	3,141436169
277	4	0,963532029	3,138629597	0,963650265	3,141105412	0,96365322	3,141167322
281	2	1,0680263	3,135402251	1,068436667	3,143877421	1,068418956	3,143511172
283	10	0,825767142	3,078612859	0,828455841	3,136023673	0,828815545	3,143782552
293	14	0,776304067	3,034231667	0,780709022	3,129085497	0,781500223	3,146423629

Table I.8

Primes	Gaps	$X^2(P_n)$	$F(X^2(P_n))$	$X(P_n)$	$F(X(P_n))$	$X^+(P_n)$	$F(X^+(P_n))$
2	1	1,676095911	3,109611	2,005827961	5,041618	1,906391423	4,371743
3	2	1,215181579	3,269238	1,236255033	3,424112	1,233648572	3,404618
7	4	0,918000053	3,068860	0,919727777	3,086278	0,919828444	3,087295
23	6	0,84354238	3,041235	0,84744591	3,094379	0,84790773	3,100723
89	8	0,829700568	3,067655	0,832931159	3,125022	0,833352228	3,132574
113	14	0,739090927	2,967767	0,746290839	3,101753	0,747853748	3,131589
523	18	0,759539599	3,022423	0,764158215	3,130242	0,765064316	3,151828
887	20	0,762321619	3,029395	0,766482808	3,133338	0,767287607	3,153838
1 129	22	0,75759231	3,025820	0,76177219	3,133358	0,762600385	3,155104
1 327	34	0,70922661	2,969019	0,715105693	3,123131	0,716570397	3,162722
9 551	36	0,760065643	3,033940	0,763206519	3,135461	0,763821057	3,155712
15 683	44	0,752403621	3,026773	0,755576595	3,134161	0,756221915	3,156457
19 609	52	0,742137094	3,016297	0,745500864	3,132412	0,746220404	3,157816
31 397	72	0,724167056	2,996706	0,727841017	3,128708	0,728696524	3,160260
155 921	86	0,744571224	3,019652	0,74729947	3,131217	0,747876203	3,155318
360 653	96	0,752123824	3,027444	0,754524738	3,131861	0,755013718	3,153561
370 261	112	0,741442741	3,016480	0,744049668	3,129981	0,744609189	3,154887
492 113	114	0,745402629	3,020635	0,747875448	3,130489	0,748396067	3,154117
1 349 533	118	0,760113121	3,035324	0,762151542	3,132306	0,762550279	3,151633
1 357 201	132	0,752788301	3,028170	0,754952238	3,131104	0,755391495	3,152419
2 010 733	148	0,751677199	3,027070	0,753801499	3,130678	0,754235107	3,152255
4 652 353	154	0,761985544	3,037129	0,763830503	3,131952	0,764187947	3,150661
17 051 707	180	0,770661579	3,045183	0,772240875	3,132735	0,772533304	3,149219
20 831 323	210	0,764598639	3,039600	0,766242773	3,131715	0,766557038	3,149636
47 326 693	220	0,772452628	3,046801	0,773917546	3,132639	0,774186229	3,148642
122 164 747	222	0,782988976	3,055991	0,784253518	3,133857	0,78447256	3,147545
189 695 659	234	0,785125072	3,057790	0,78633622	3,134029	0,78654354	3,147268
191 912 783	248	0,782354583	3,055453	0,783596407	3,133651	0,783812269	3,147446
387 096 133	250	0,789318747	3,061258	0,790441084	3,134408	0,790628728	3,146806
436 273 009	282	0,784759988	3,057484	0,785924379	3,133778	0,7861241	3,147053
1 294 268 491	288	0,794450338	3,065390	0,795457207	3,134804	0,795620677	3,146221
1 453 168 141	292	0,794903314	3,065749	0,795900248	3,134838	0,796061682	3,146171
2 300 942 549	320	0,795022554	3,065843	0,7959971	3,134769	0,7961548	3,146067
3 842 610 773	336	0,797464275	3,067759	0,798393632	3,134979	0,798541898	3,145838
4 302 407 359	354	0,796200619	3,066771	0,797136815	3,134806	0,797287277	3,145880
10 726 904 659	382	0,80076832	3,070308	0,801627629	3,135213	0,801762081	3,145491
20 678 048 297	384	0,805840056	3,074123	0,806633621	3,135702	0,806754073	3,145156
22 367 084 959	394	0,805418671	3,073811	0,806213059	3,135645	0,806333943	3,145162
25 056 082 087	456	0,800445534	3,070061	0,801277085	3,135056	0,801407443	3,145368
42 652 618 343	464	0,803929771	3,072699	0,804715091	3,135393	0,804835671	3,145132
127 976 334 671	468	0,811713359	3,078394	0,812406452	3,136125	0,812507935	3,144666
182 226 896 239	474	0,813711347	3,079811	0,814380561	3,136295	0,814477335	3,144549
241 160 624 143	486	0,81471098	3,080518	0,815365953	3,136371	0,815460075	3,144484
297 501 075 799	490	0,815830856	3,081301	0,816472793	3,136464	0,816564393	3,144420
303 371 455 241	500	0,815225714	3,080879	0,815871399	3,136410	0,815963887	3,144437
304 599 508 537	514	0,814246615	3,080186	0,814899061	3,136300	0,814993094	3,144473
416 608 695 821	516	0,816202453	3,081563	0,816833755	3,136467	0,816923626	3,144372
461 690 510 011	532	0,815784191	3,081260	0,816415944	3,136428	0,816506117	3,144366
614 487 453 523	534	0,81752217	3,082464	0,81813553	3,136564	0,818222118	3,144291
738 832 927 927	540	0,818314815	3,082995	0,818918758	3,136628	0,819003587	3,144244
1 346 294 310 749	582	0,819494175	3,083820	0,820077495	3,136702	0,820158809	3,144117
1 408 695 493 609	588	0,819422451	3,083776	0,820005289	3,136711	0,820086574	3,144146
1 968 188 556 461	602	0,820680244	3,084614	0,821248201	3,136782	0,821326771	3,144033
2 614 941 710 599	652	0,819705756	3,083931	0,820274197	3,136642	0,820353329	3,144075
7 177 162 611 713	674	0,824569854	3,087227	0,82508968	3,137115	0,825159791	3,143913
13 829 048 559 701	716	0,8263169	3,088287	0,826815381	3,137039	0,82688184	3,143890
19 581 334 192 423	766	0,826092444	3,088165	0,826586534	3,136948	0,826652506	3,143631
42 842 283 925 351	778	0,829816487	3,090912	0,830277841	3,138184	0,830337926	3,144104
90 874 329 411 493	804	0,832686	3,092896	0,833121494	3,138000	0,833177115	3,143677
171 231 342 420 521	806	0,835750937	3,093506	0,836162509	3,135498	0,836213974	3,143555
218 209 405 436 543	906	0,833483358	3,092773	0,833903266	3,135864	0,833956603	3,141357

Table I.9

We can conclude that we have reached to build a function $X^{2^k}(p_n)$ which give a good approximation and a good lower bound of $x(p_n)$ which verify $F_n(x) = (p_{n+1})^x - (p_n)^x = 1$, for any prime number p_n . We have proved that $x(p_n)$ couldn't be lower than 0.475 and under Cramer's conjecture or the weaker hypothesis $g_n < \ln^k(p_n)$, we have proposed, $x(p_n)$ reached a lower bound for $p_n = 113$. We have also proved that $F_n(1/k) = (p_{n+1})^{1/k} - (p_n)^{1/k} < 1/k$ for all integer $k > 2$ and for $k=2$ if we take the weaker hypothesis $g_n < \ln^k(p_n)$. And at last, we have also build a lower bound function $X_{C^2}(p_n)$ of $x(p_n)$ verifying $F_n(x) = (p_{n+1})^x - (p_n)^x = C$. We are going, in the following part, to study the behavior of the functions $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ and show that $F_x(p_n)$ are positive upper bounded functions which tends toward 0 for all $0 \leq x < 1$.

I.5 - Study of $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ for $x < 1$

In this paragraph, our goal is to study the function $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ for $0 \leq x < 1$. We'll show that for $x < 0,475$, $F_x(p_n)$ is an upper bounded function which tends towards 0 when $p_n \rightarrow \infty$ and for $0,475 \leq x < 1$, under Cramer's conjecture or the weaker condition proposed above, $F_x(p_n)$ is a bounded function which tends toward 0 when $p_n \rightarrow \infty$. We'll build an upper bound function $M_x(p_n)$ of $F_x(p_n)$ and we'll analyse the behavior of $M_x(p_n)$. We'll also build a lower bound function $m_x(p_n)$ of $F_x(p_n)$. We'll analyse the behavior of $m_x(p_n)$ and gives an estimation of $p'_{n,x}$ giving the maximal value of $m_x(p_n)$. Finally, we'll try to estimate the value of $p'_{n,x}$ such as for all $p_n > p'_{n,x}$, $F_x(p_n) < M_x(p_n) < m_x(p'_{n,x})$ and will gives the maximal values of $F_x(p_n)$ for different $0 \leq x < 1$.

I.5.1 -Upper bound function of $F_x(p_n)$

We are going to show that for any real x such as $0 < x < 1$, the function $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound and tends toward 0 when $n \rightarrow \infty$, considering two different situations.:

- 1- Using the Baker, Harman and Pintz result, $g_n < p_n^{0,525}$, we'll prove that for any real x such as $0 < x < 0.475$, $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound and tends toward 0 when $p_n \rightarrow \infty$.
- 2- Based on Cramer's conjecture or the weaker condition on prime gaps proposed in I.2.3, we'll state that for any real x such as $0,475 \leq x < 1$, $F_x(p_n) = (p_{n+1})^x - (p_n)^x$ has an upper bound and tends toward 0 when $p_n \rightarrow \infty$.

We start from the Taylor serie expansion of (I.5)

$$F_x(p_n) = x \cdot p_n^x \cdot \frac{g_n}{p_n} - p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^2 < x \cdot p_n^x \cdot \frac{g_n}{p_n} \quad (I.22)$$

For the same reasons than the one exposed in paragraph I.2, the research of an upper bound of (I.22) can be reduced to all primes corresponding to the first occurrence of maximal gaps.

Replacing in (I.22), g_n by G_n , we can establish a new upper bound function of $F_x(p_n)$:

$$F_x(p_n) < x \cdot p_n^x \cdot \frac{g_n}{p_n} < x \cdot p_n^x \cdot \frac{G_n}{p_n} = M_x(p_n)$$

Due to BHP result : $G_n < p_n^a$ with $a=0,525$ then :

$$|F_x(p_n) - M_x(p_n)| < p_n^x \cdot O\left(\frac{G_n}{p_n}\right)^2 < O\left(\frac{G_n}{p_n^{1-x/2}}\right)^2 < O\left(\frac{p_n^a}{p_n^{1-x/2}}\right)^2 = O\left(\frac{1}{p_n^{0,95-x}}\right) \quad (I.23)$$

For all $x < 0,95$, $|F_x(p_n) - M_x(p_n)| \rightarrow 0$ when $n \rightarrow \infty$

Thanks to that, we can focus our study to the behavior of $M_x(p_n)$ which appears as a good estimation of $F_x(p_n)$.

If $x < 0,475$, we can write :

$$F_x(p_n) < M_x(p_n) < x \cdot p_n^x \cdot \frac{p_n^a}{p_n} = \frac{x}{p_n^{1-a-x}}$$

$$\lim_{n \rightarrow \infty} F_x(p_n) < \lim_{n \rightarrow \infty} M_x(p_n) < \lim_{n \rightarrow \infty} \left(\frac{x}{p_n^{1-a-x}}\right) = 0$$

If $0,475 \leq x < 1$, based on Cramer's conjecture on the weaker condition proposed in I.2.3, where there is a real r such as $G_n < \ln^r(p_n)$ then :

$$F_x(p_n) < M_x(p_n) < x \cdot p_n^x \cdot \frac{\ln^r(p_n)}{p_n} = x \cdot \frac{\ln^r(p_n)}{p_n^{1-x}}$$

$$\lim_{n \rightarrow \infty} F_x(p_n) < \lim_{n \rightarrow \infty} M_x(p_n) < \lim_{n \rightarrow \infty} \left(x \cdot \frac{\ln^r(p_n)}{p_n^{1-x}}\right) = 0$$

The table I.10 below presents the digital values of $F_x(p_n)$ for different values of $x < 1$ and for all prime numbers corresponding to the first occurrences of the maximal prime gaps G_n up to $P_n \leq 218\,209\,405\,436\,543$. Then for all $0 \leq x < 1$, $F_x(p_n)$ is a bounded positive function which tends forward to 0 when $n \rightarrow \infty$. Thus, there is one $p_{n,x}$ such as $F_x(p_{n,x}) = \text{Max}_{n \in \mathbb{N}}(F_x(p_n))$ and $F_x(p_{n,k}) < M_x(p_{n,k}) < \text{Max}_n M_x(p_n)$

We are now going to study more in detail the first and second derivatives functions of $F_x(p_n)$ and $M_x(p_n)$ to try to better estimate the values of $p_{n,k}$ where $F'_x(p_n) \approx M'_x(p_n) = 0$ and $F''_x(p_n) \approx M''_x(p_n) < 0$.

$$M'_x(p_n) = x^2 \cdot p_n^{x-1} \cdot \frac{G_n}{p_n} + x \cdot p_n^x \cdot \frac{G'_n}{p_n} - x \cdot p_n^x \cdot \frac{G_n}{p_n^2}$$

$$M'_x(p_n) = x \cdot p_n^x \cdot \frac{G_n}{p_n} \cdot \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right) = M_x(p_n) \cdot \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right)$$

As specified above, G'_n can be estimate by $\Delta G_n / \Delta p_n$:

$$M'_x(p_n) = \frac{M_x(p_n)}{p_n} \cdot \left(\frac{p_n \cdot G'_n}{G_n} - (1-x)\right) = \frac{M_x(p_n)}{p_n} \cdot \left(\frac{\Delta G_n}{\Delta p_n} \cdot \frac{p_n}{G_n} - (1-x)\right) \quad (I.24)$$

Regarding the second derivative $M''_x(p_n)$, we have :

$$M''_x(p_n) = M'_x(p_n) \cdot \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right) + M_x(p_n) \cdot \left(\frac{G''_n}{G_n} - \frac{(G'_n)^2}{G_n^2} + \frac{(1-x)}{p_n^2}\right)$$

$$M''_x(p_n) = M_x(p_n) \cdot \left(\frac{G'_n}{G_n} - \frac{(1-x)}{p_n}\right)^2 + M_x(p_n) \cdot \left(\frac{G''_n}{G_n} - \frac{(G'_n)^2}{G_n^2} + \frac{(1-x)}{p_n^2}\right)$$

$$M''_x(p_n) = M_x(p_n) \cdot \left(\frac{G''_n}{G_n} - \frac{2(1-x)G'_n}{p_n \cdot G_n} + \frac{(1-x)(2-x)}{p_n^2}\right) \quad (I.25)$$

We are looking p_n such as $M'_x(p_n) = 0$ and $M''_x(p_n) < 0$

If $M'_x(p_n) = 0$ is equivalent to $\frac{G'_n}{G_n} - \frac{(1-x)}{p_n} = 0$. Then we can replace in (25), $\frac{G'_n}{G_n} = \frac{(1-x)}{p_n}$ (I.26)

$$M''_x(p_n) = M_x(p_n) \cdot \left(\frac{G''_n}{G_n} - \frac{x(1-x)}{p_n^2} \right) \quad (I.27)$$

We can write (I.26) as follow $\frac{G'_n}{(1-x)} = \frac{G_n}{p_n}$ and using the approximation of (24), $G'_n = \frac{\Delta G_n}{\Delta p_n}$, we have to find all p_n

such as :

$$\frac{\Delta G_n}{\Delta p_n} \cdot \frac{p_n}{G_n} \approx (1-x) \quad (I.28)$$

To simplify the notation, we'll write G_n and G_{n+} two consecutive gaps of this integer sequence and p_n and p_{n+} the two primes numbers associated to these gaps.

In table I.7, we can see that the highest value of $F_x(p_n)$ is reached for the first p_n satisfying to the following conditions :

$$\frac{\Delta G_n}{\Delta p_n} \cdot \frac{p_n}{G_n} > 1-x \quad \text{and} \quad \frac{\Delta G_{n+}}{\Delta p_{n+}} \cdot \frac{p_{n+}}{G_{n+}} < 1-x$$

For $x = 0.1$, $1-x = 0.9$ then the first p_n for which $\frac{\Delta G_n}{\Delta p_n} \cdot \frac{p_n}{G_n} < 0.9$ is $p_n = 7$

For $x = 0.2$ up to $x = 0.5$, the first p_n for which $\frac{\Delta G_n}{\Delta p_n} \cdot \frac{p_n}{G_n} < 1-x$ is $p_n = 23$

Primes	Gaps	$\Delta G_n \cdot P_n / G_n \cdot \Delta P_n$	$\ln(P_n)$	0,950	0,900	0,895	0,800	0,700	0,600	0,500	0,400	0,300	0,200	0,100
2	1		0,69											
3	2	1,5000	1,10	1,774	1,569	1,5494	1,2157	0,9275	0,6933	0,5040	0,3518	0,2303	0,1340	0,0585
7	4	0,8750	1,95	3,406	2,893	2,8452	2,0662	1,4531	1,0013	0,6709	0,4316	0,2603	0,1396	0,0562
23	6	0,4792	3,14	4,844	3,899	3,8152	2,5031	1,5817	0,9792	0,5893	0,3405	0,1845	0,0888	0,0321
89	8	0,3371	4,49	6,059	4,576	4,4487	2,5853	1,4378	0,7833	0,4149	0,2110	0,1006	0,0426	0,0135
113	14	2,0179	4,73	10,469	7,807	7,5799	4,2997	2,3311	1,2381	0,6393	0,3169	0,1473	0,0608	0,0188
523	18	0,2835	6,26	12,494	8,648	8,3345	4,1038	1,9169	0,8771	0,3902	0,1667	0,0667	0,0237	0,0063
887	20	0,2437	6,79	13,524	9,120	8,7662	4,1074	1,8209	0,7908	0,3339	0,1353	0,0514	0,0174	0,0044
1 129	22	0,4241	7,03	14,699	9,794	9,4031	4,3066	1,8640	0,7903	0,3258	0,1289	0,0478	0,0158	0,0039
1 327	34	2,3654	7,19	22,531	14,890	14,2831	6,4401	2,7420	1,1436	0,4637	0,1805	0,0659	0,0214	0,0052
9 551	36	0,0645	9,16	21,626	12,956	12,3065	4,6049	1,6112	0,5522	0,1840	0,0589	0,0177	0,0047	0,0009
15 683	44	0,4650	9,66	25,785	15,069	14,2788	5,0972	1,6972	0,5536	0,1756	0,0534	0,0153	0,0039	0,0007
19 609	52	0,7684	9,88	30,135	17,416	16,4838	5,7609	1,8758	0,5983	0,1855	0,0552	0,0154	0,0038	0,0007
31 397	72	0,7399	10,35	40,756	23,006	21,7235	7,2601	2,2554	0,6863	0,2031	0,0577	0,0154	0,0036	0,0006
155 921	86	0,2038	11,96	44,934	23,412	21,9308	6,2948	1,6661	0,4320	0,1089	0,0263	0,0060	0,0012	0,0002
360 653	96	0,1835	12,80	48,099	24,033	22,4178	5,9420	1,4462	0,3448	0,0799	0,0178	0,0037	0,0007	0,0001
370 261	112	5,5052	12,82	56,042	27,964	26,0820	6,8959	1,6740	0,3981	0,0920	0,0204	0,0042	0,0008	0,0001
492 113	114	0,0709	13,11	56,237	27,665	25,7665	6,6309	1,5645	0,3616	0,0812	0,0175	0,0035	0,0006	0,0001
1 349 533	118	0,0534	14,12	55,347	25,888	23,9902	5,6096	1,1965	0,2500	0,0508	0,0099	0,0018	0,0003	0,0000
1 357 201	132	18,7722	14,12	61,896	28,943	26,8205	6,2680	1,3362	0,2790	0,0567	0,0110	0,0020	0,0003	0,0000
2 010 733	148	0,3326	14,51	68,048	31,201	28,8556	6,4965	1,3315	0,2673	0,0522	0,0098	0,0017	0,0003	0,0000
4 652 353	154	0,0686	15,35	67,899	29,853	27,4940	5,7158	1,0772	0,1989	0,0357	0,0062	0,0010	0,0001	0,0000
17 051 707	180	0,1986	16,65	74,372	30,643	28,0388	5,1524	0,8528	0,1383	0,0218	0,0033	0,0005	0,0001	0,0000
20 831 323	210	0,7874	16,85	85,903	35,042	32,0315	5,7752	0,9369	0,1489	0,0230	0,0034	0,0005	0,0001	0,0000
47 326 693	220	0,0812	17,67	86,376	33,819	30,7864	5,1344	0,7673	0,1123	0,0160	0,0022	0,0003	0,0000	0,0000
122 164 747	222	0,0147	18,62	83,125	31,039	28,1220	4,2860	0,5826	0,0776	0,0100	0,0012	0,0001	0,0000	0,0000
189 695 659	234	0,1441	19,06	85,711	31,308	28,3037	4,1371	0,5381	0,0686	0,0085	0,0010	0,0001	0,0000	0,0000
191 912 783	248	4,8864	19,07	90,786	33,142	29,9605	4,3744	0,5684	0,0723	0,0090	0,0011	0,0001	0,0000	0,0000
387 096 133	250	0,0159	19,77	88,363	31,146	28,0571	3,8324	0,4642	0,0551	0,0064	0,0007	0,0001	0,0000	0,0000
436 273 009	282	1,0067	19,89	99,080	34,715	31,2534	4,2207	0,5051	0,0592	0,0068	0,0007	0,0001	0,0000	0,0000
1 294 268 491	288	0,0314	20,98	95,833	31,800	28,4743	3,4680	0,3723	0,0392	0,0040	0,0004	0,0000	0,0000	0,0000
1 453 168 141	292	0,1253	21,10	96,603	31,871	28,5209	3,4357	0,3646	0,0379	0,0038	0,0004	0,0000	0,0000	0,0000
2 300 942 549	320	0,2375	21,56	103,461	33,358	29,7833	3,4345	0,3481	0,0346	0,0033	0,0003	0,0000	0,0000	0,0000
3 842 610 773	336	0,1187	22,07	105,884	33,275	29,6331	3,2546	0,3134	0,0296	0,0027	0,0002	0,0000	0,0000	0,0000
4 302 407 359	354	0,4758	22,18	110,928	34,664	30,8522	3,3524	0,3191	0,0298	0,0027	0,0002	0,0000	0,0000	0,0000
10 726 904 659	382	0,1224	23,10	114,357	34,140	30,2473	3,0134	0,2618	0,0223	0,0018	0,0001	0,0000	0,0000	0,0000
20 678 048 297	384	0,0108	23,75	111,245	32,138	28,3809	2,6566	0,2162	0,0172	0,0013	0,0001	0,0000	0,0000	0,0000
22 367 084 959	394	0,3361	23,83	113,694	32,717	28,8809	2,6833	0,2166	0,0171	0,0013	0,0001	0,0000	0,0000	0,0000
25 056 082 087	456	1,2669	23,94	130,841	37,438	33,0295	3,0358	0,2423	0,0189	0,0014	0,0001	0,0000	0,0000	0,0000
42 652 618 343	464	0,0418	24,48	129,642	36,122	31,7831	2,7773	0,2102	0,0156	0,0011	0,0001	0,0000	0,0000	0,0000
127 976 334 671	468	0,0128	25,58	123,769	32,642	28,5641	2,2486	0,1525	0,0101	0,0007	0,0000	0,0000	0,0000	0,0000
182 226 896 239	474	0,0425	25,93	123,160	31,912	27,8764	2,1220	0,1389	0,0089	0,0006	0,0000	0,0000	0,0000	0,0000
241 160 624 143	486	0,1010	26,21	124,522	31,816	27,7535	2,0571	0,1309	0,0082	0,0005	0,0000	0,0000	0,0000	0,0000
297 501 075 799	490	0,0431	26,42	124,235	31,412	27,3717	1,9888	0,1240	0,0076	0,0004	0,0000	0,0000	0,0000	0,0000
303 371 455 241	500	1,0336	26,44	126,647	31,990	27,8731	2,0215	0,1257	0,0077	0,0005	0,0000	0,0000	0,0000	0,0000
304 599 508 537	514	6,7558	26,44	130,167	32,872	28,6414	2,0764	0,1291	0,0079	0,0005	0,0000	0,0000	0,0000	0,0000
416 608 695 821	516	0,0144	26,76	128,643	31,983	27,8228	1,9579	0,1180	0,0070	0,0004	0,0000	0,0000	0,0000	0,0000
461 690 510 011	532	0,3080	26,86	131,953	32,638	28,3778	1,9776	0,1180	0,0069	0,0004	0,0000	0,0000	0,0000	0,0000
614 487 453 523	534	0,0151	27,14	130,569	31,837	27,6421	1,8747	0,1087	0,0062	0,0003	0,0000	0,0000	0,0000	0,0000
738 832 927 927	540	0,0660	27,33	130,825	31,607	27,4170	1,8272	0,1040	0,0058	0,0003	0,0000	0,0000	0,0000	0,0000
1 346 294 310 749	582	0,1599	27,93	136,833	32,081	27,7452	1,7466	0,0936	0,0049	0,0003	0,0000	0,0000	0,0000	0,0000
1 408 695 493 609	588	0,2304	27,97	137,931	32,266	27,8983	1,7487	0,0933	0,0049	0,0002	0,0000	0,0000	0,0000	0,0000
1 968 188 556 461	602	0,0818	28,31	138,872	31,947	27,5765	1,6744	0,0864	0,0044	0,0002	0,0000	0,0000	0,0000	0,0000
2 614 941 710 599	652	0,3101	28,59	148,286	33,631	28,9893	1,7134	0,0859	0,0042	0,0002	0,0000	0,0000	0,0000	0,0000
7 177 162 611 713	674	0,0513	29,60	145,746	31,426	26,9536	1,4473	0,0656	0,0029	0,0001	0,0000	0,0000	0,0000	0,0000
13 829 048 559 701	716	0,1220	30,26	149,821	31,266	26,7268	1,3485	0,0572	0,0024	0,0001	0,0000	0,0000	0,0000	0,0000
19 581 334 192 423	766	0,2222	30,61	157,522	32,307	27,5659	1,3457	0,0552	0,0022	0,0001	0,0000	0,0000	0,0000	0,0000
42 842 283 925 351	778	0,0284	31,39	153,857	30,339	25,7971	1,1689	0,0443	0,0016	0,0001	0,0000	0,0000	0,0000	0,0000
90 874 329 411 493	804	0,0612	32,14	153,141	29,080	24,6304	1,0390	0,0365	0,0013	0,0000	0,0000	0,0000	0,0000	0,0000
171 231 342 420 521	806	0,0053	32,77	148,645	27,356	23,0850	0,9169	0,0303	0,0010	0,0000	0,0000	0,0000	0,0000	0,0000
218 209 405 436 543	906	0,5127	33,02	165,125	30,002	25,2910	0,9818	0,0317	0,0010	0,0000	0,0000	0,0000	0,0000	0,0000

Table I.10

I.5.2 -Lower bound function of $F_x(p_n)$:

Our purpose is to build a lower bound function of $F_x(p_n)$ and to estimate the lowest prime $P_{n,\max}$ where the upper bound function becomes lower than the maximal value of the lower bound function. It means that the maximal value of $F_x(p_n)$ will necessarily occur for a $p_n < P_{n,\max}$. And if $P_{n,\max} < 218\,209\,405\,436\,543$ we'll be able to compute and find the highest value of $F_x(p_n)$ in this interval and to conclude that this value is a global maximal value for $F_x(p_n)$.

Starting from the Taylor serie expansion of (5)

$$F_x(p_n) = x \cdot p_n^x \cdot \frac{g_n}{p_n} \left(1 - \frac{(1-x)}{2} \cdot \frac{g_n}{p_n}\right) + p_n^x \cdot O\left(\frac{g_n}{p_n}\right)^2 \text{ with } O\left(\frac{g_n}{p_n}\right)^2 > 0$$

$$F_x(p_n) > x \cdot p_n^x \cdot \frac{g_n}{p_n} \left(1 - \frac{(1-x)}{2} \cdot \frac{g_n}{p_n}\right)$$

As specified in the previous paragraph, we can limit our study to the prime numbers corresponding to the first occurrence of maximal gaps. As previously, P_n is the prime number associated to the first occurrence of the maximal gap G_n .

$$F_x(P_n) > x \cdot P_n^x \cdot \frac{G_n}{P_n} \left(1 - \frac{(1-x)}{2} \cdot \frac{G_n}{P_n}\right) \quad (I.29)$$

In 1931, Erik Westzynthius proved that maximal gaps grows more than logarithmically.

$$\text{Limsup } \frac{G_n}{\ln(P_n)} = \infty \text{ which imply that for } n \text{ enough large } G_n > \ln(P_n) \quad (I.30)$$

Thanks to BHP result $G_n < P_n^a$ with $a=0,525$ and (I.30), we can define a lower bound function of $F_x(p_n)$:

$$F_x(P_n) > x \cdot P_n^x \cdot \frac{G_n}{P_n} \left(1 - \frac{(1-x)}{2} \cdot \frac{G_n}{P_n}\right) > x \cdot P_n^x \cdot \frac{\ln(P_n)}{P_n} \left(1 - \frac{(1-x)}{2} \cdot \frac{P_n^a}{P_n}\right) = m_x(P_n)$$

$$m_x(P_n) = x \cdot \frac{\ln(P_n)}{P_n^{1-x}} \cdot \left(1 - \frac{(1-x)}{2P_n^{1-a}}\right)$$

We are going to estimate the first derivative of $m_x(P_n)$:

$$m'_x(P_n) = x \left(\frac{1}{P_n^{2-x}} - (1-x) \frac{\ln(P_n)}{P_n^{2-x}} \right) \cdot \left(1 - \frac{(1-x)}{2P_n^{1-a}}\right) + x \cdot \frac{\ln(P_n)}{P_n^{1-x}} \cdot \frac{(1-x) \cdot (1-a)}{2P_n^{2-a}}$$

$$m'_x(P_n) = \frac{x}{2P_n^{3-a-x}} \left((1 - (1-x)\ln(P_n)) \cdot \left(2P_n^{1-a} - (1-x)\right) + (1-x) \cdot (1-a) \cdot \ln(P_n) \right)$$

$$m'_x(P_n) = \frac{x}{2P_n^{3-a-x}} \left((1 - (1-x)\ln(P_n)) \cdot \left(2P_n^{1-a} - (1-x)\right) + (1-x) \cdot (1-a) \cdot \ln(P_n) \right)$$

$$m'_x(P_n) = \frac{x}{2P_n^{3-a-x}} \cdot \left(\left(2P_n^{1-a} - (1-x)\right) - (1-x) \left(2P_n^{1-a} - (2-a)\right) \cdot \ln(P_n) \right) \quad (I.31)$$

The maximal value of $m_x(P_n)$ is reached for $m'_x(P_{n,x}) = 0$, and $P_{n,x}$ satisfying to :

$$\ln(P_{n,x}) = \frac{2P_n^{1-a} - (1-x)}{(1-x)(2P_n^{1-a} - (2-a))} = \frac{1}{(1-x)} \cdot \frac{1 - \frac{(1-x)}{2P_n^{1-a}}}{1 - \frac{(2-a)}{2P_n^{1-a}}} \approx \frac{1}{(1-x)} \cdot \frac{1 - \frac{(1-x)}{2e^{(1-a)/(1-x)}}}{1 - \frac{(2-a)}{2e^{(1-a)/(1-x)}}} > \frac{1}{(1-x)} \quad (I.32)$$

Thus $P_{n,x} = e^{\frac{1}{(1-x)} \cdot \frac{1 - \frac{(1-x)}{2e^{(1-a)/(1-x)}}}{1 - \frac{(2-a)}{2e^{(1-a)/(1-x)}}}} \approx e^{\frac{1}{(1-x)}} = P'_{n,x}$ In the table 8 below, we give for different x, the numerical values of $P_{n,x}$, $P'_{n,x}$, $m_x(P_{n,x})$, $m_x(P'_{n,x})$. We can see that $\frac{|m_x(P_n) - m_x(P'_n)|}{m_x(P_n)} < 1.5\%$ for $0.1 \leq x \leq 0.9$

x	$P'_{n,x} \approx$	$m_x(P'_{n,x})$	$P_{n,x} \approx$	$m_x(P_{n,x})$	$ m_x(P_{n,x}) - m_x(P'_{n,x}) / m_x(P_{n,x})$
0,1	3	0,030024624	4	0,030445391	1,38%
0,2	3	0,071653628	5	0,072093963	0,61%
0,3	4	0,129666555	7	0,129622113	0,03%
0,4	5	0,211916534	9	0,210941533	0,46%
0,5	7	0,332310923	13	0,330249272	0,62%
0,6	12	0,518160095	21	0,515412608	0,53%
0,7	28	0,831952717	45	0,829582868	0,29%
0,8	148	1,457830517	204	1,456915542	0,06%
0,9	22026	3,309482719	23385	3,309463809	0,00%

We can see in table I.11 above that the upper bound of the lower bound function $m_x(p_n)$ of $F_x(P_n)$ is correctly estimated if we compare with the upper bounds of $m_x(p_n)$ appearing in the table I.13 and I.15.

I.5.3 -Estimation of P_{no} such as for $P_n > P_{no}$ $M_x(P_n) < \text{Max}_{pn} m_x(P_n)$:

As specified above, the local maximal values of $F_x(P_n)$ are reached only for the prime numbers (P_n), where P_n is the first occurrence of a maximal gap G_n . If we consider a maximal gap G_n , we define $\mathbb{P}_{G_n} = \{p_i \in \mathbb{P} / g_i = G_n\}$ and $P_n = \text{Min}\{p_i \in \mathbb{P}_{G_n}\}$ and $\mathbb{P}_{(G_n)} = \{P_n \in \mathbb{P} \text{ such as } P_n = \text{Min}\{p_i \in \mathbb{P}_{G_n}\} \}$. We consider $P_{n,k}$ such as $F_x(P_{n,k}) = \text{Max}_{P_n \in \mathbb{P}_{(G_n)}} F_x(P_n)$. Our approach is to show that there is one P_{no} enough large such as for all $P_n > P_{no}$, $F_x(P_n) < M_x(P_n) < \text{Max}_{pn} m_x(P_n) = m_x(P'_{n,x})$.

In the previous paragraph, we have shown that $F_x(P_{n,k})$ was correctly estimated by $F_x(P'_{n,k}) = F_x\left(e^{\frac{1}{(1-x)}}\right)$.

We are going to search the best value of $\alpha = \alpha(x) > 1$, such as :

$$M_x\left(e^{\frac{\alpha}{(1-x)}}\right) < m_x\left(e^{\frac{1}{(1-x)}}\right) \quad \text{and} \quad P_{no} = e^{\frac{\alpha}{(1-x)}} \quad (I.33)$$

We can consider the two different cases define at the beginning of this study :

For $0 < x < 0,475$, then $G_n < P_n^a$ with $a = 0,525$

$$x \cdot P_n^x \cdot \frac{G_n}{P_n} < x \cdot P_n^x \cdot \frac{P_n^a}{P_n} < x \cdot \frac{\ln(P'_{n,x})}{P'_{n,x}^{1-x}} \cdot \left(1 - \frac{(1-x)}{2P'_{n,x}^{1-a}}\right)$$

$$\frac{1}{P_n^{1-x-a}} < \frac{\ln(P'_{n,x})}{P'_{n,x}^{1-x}} \cdot \left(1 - \frac{(1-x)}{2P'_{n,x}^{1-a}}\right)$$

And we can replace P_n and $P'_{n,x}$ by their expression defined in (33) :

$$\frac{1}{e^{\frac{k(1-x-a)}{1-x}}} < \frac{\ln\left(e^{\frac{1}{(1-x)}}\right)}{e} \cdot \left(1 - \frac{(1-x)}{2e^{\frac{(1-a)}{(1-x)}}}\right)$$

And finally :

$$\alpha > \frac{1-x}{1-a-x} \cdot \ln\left(\frac{e^{(1-x)}}{1 - \frac{(1-x)}{2} \cdot e^{\frac{(a-1)}{(1-x)}}}\right) \quad (I.34)$$

Table I.12 below, gives the for value of $\alpha(x)$ for different values of $x < 0,475$

x	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45
α	2,88	2,89	2,92	2,99	3,11	3,36	3,84	5,08	11,56

Table I.12

Thanks to that, we can now estimate the values of $P_{no} = e^{\frac{\alpha}{(1-x)}}$ and we 'll compare with the numerical values of table I.13

For $x=0.1$, $\alpha = 2.89$ and $p_{no,x} = e^{\frac{2,89}{(1-x)}} \approx 25$. So we can see in the table below that $F_x(p_n) \leq 0,059$ for all $P_n < 25$ and for $P_n > 25$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{pn} m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The nearest and lowest prime satisfying to this condition is $P_n = 89$

For $x=0.2$, $\alpha = 2,99$ and $p_{no,x} = e^{\frac{2,99}{0,8}} \approx 42$. So we can see in the table below that $F_x(P_n) < 0,14$ for all $P_n \leq 42$ and for $P_n > 42$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{pn} m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The nearest and lowest prime satisfying to this condition is $P_n = 89$

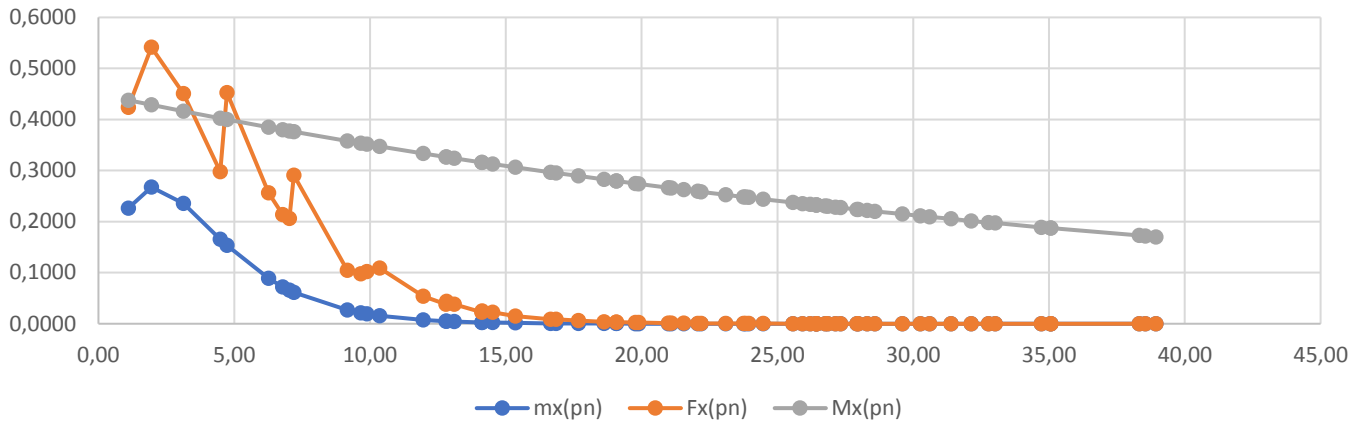
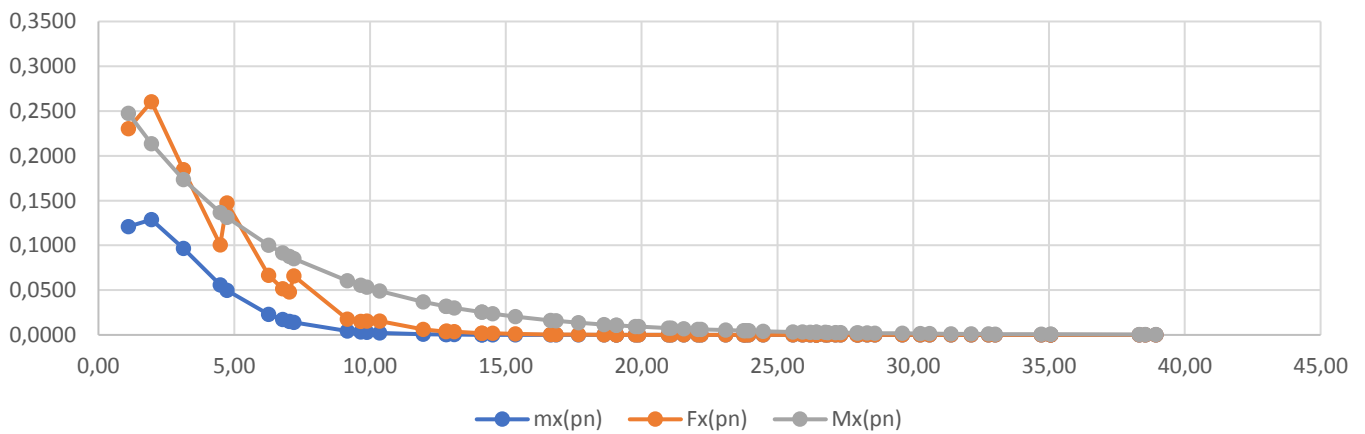
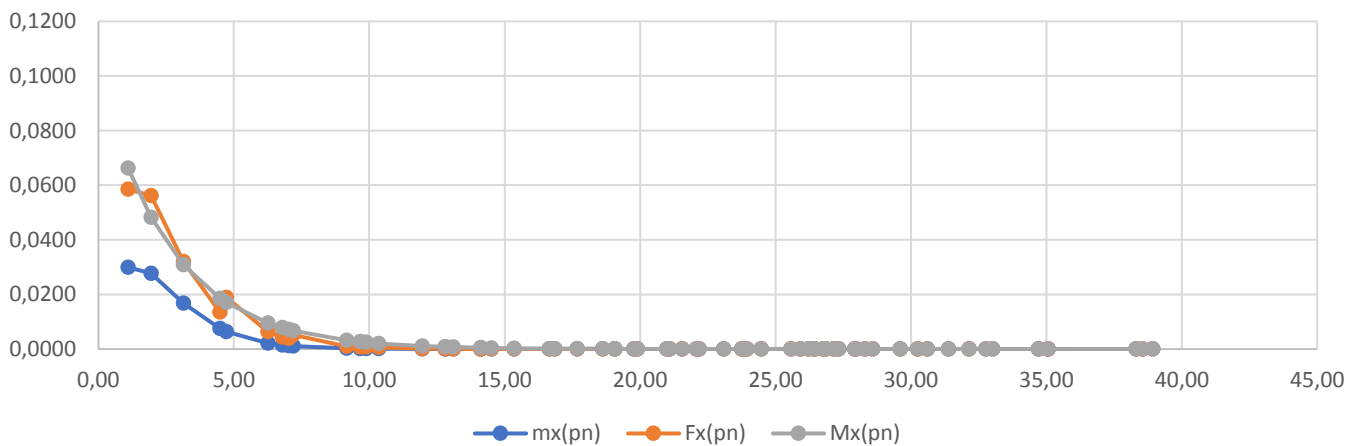
For $x=0.3$, $\alpha = 3,36$ and $p_{no,x} = e^{\frac{3,36}{0,7}} \approx 121$. So we can see in the table below that $F_x(p_n) < 0,27$ for all $P_n \leq 121$ and for $P_n > 121$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{pn} m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime satisfying to this condition is $P_n = 523$.

For $x=0.4$, $\alpha = 5,08$ and $p_{no,x} = e^{\frac{5,08}{0,6}} \approx 4753$. So we can see in the table below that $F_x(p_n) < 0,44$ for all $P_n \leq 4753$ and for $P_n > 4753$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{pn} m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime satisfying to this condition is $P_n = 9551$.

For $x=0.45$, $\alpha = 11,56$ and $p_{no,x} = e^{\frac{11,56}{0,55}} \approx 143\ 397\ 866\ 765$. So we can see in the table below that $F_x(p_n) < 0,55$ for all $P_n \leq 143\ 397\ 866\ 765$ and for $P_n > 143\ 397\ 866\ 765$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{pn} m_x(p_n) = m_x(e^{\frac{1}{(1-x)}})$. The closest and lowest prime satisfying to this condition is $P_n = 1\ 294\ 268\ 491$.

Remark: The value of $P_{no,x}$ obtained for $x=0,45$ seems to be over-estimated but it's due to the fact that the formulation of α contains an undefined for $x = 1-a$. This is only due to the fact we use the BHP result as upper bound for (G_n) . For all x values not to close from $1-a$, the estimation of $P_{no,x}$ is quite good and totally coherent with the numerical results of table I.13 below.

We can conclude that for all $x < 0.475$, $F_x(p_n)$ is a bounded function who tends toward 0 when $p_n \rightarrow \infty$ and the upper bound of $F_x(p_n)$ for every x is reached for $p_n < 218\ 209\ 405\ 436\ 543$ and thus can be estimated numerically for all $x < 0.45$.

$m_{0,45}(p_n) \quad F_{0,45}(p_n) \quad M_{0,45}(p_n)$  $m_{0,3}(p_n) \quad F_{0,3}(p_n) \quad M_{0,3}(p_n)$  $m_{0,1}(p_n) \quad F_{0,1}(p_n) \quad M_{0,1}(p_n)$ 

We can see on the graphics and table above that $F_x(p_n)$ is under $M_x(p_n)$ for all $p_n > 113$ and independently of the x . We can also see in the three graphics above, that the upper bound functions $M_x(p_n)$ (in grey) becomes always lower than the maximal value of $m_x(p_n)$ (in blue).

For $0,475 \leq x < 1$, then from Cramer's conjecture, $G_n < \ln^2(P_n)$
 (we could also use the weaker condition where we suppose there is $r \in \mathbb{R}$ such as $G_n < \ln^r(P_n)$)

$$x \cdot P_n^x \cdot \frac{G_n}{P_n} < x \cdot P_n^x \cdot \frac{\ln^2(P_n)}{P_n} < x \cdot \frac{\ln(P_{n,x})}{P_{n,x}^{1-x}} \cdot \left(1 - \frac{(1-x)}{2P_{n,x}^{1-a}}\right)$$

$$\frac{\ln^2(P_n)}{P_n^{1-x}} < \frac{\ln(P_{n,x})}{P_{n,x}^{1-x}} \cdot \left(1 - \frac{(1-x)}{2P_{n,x}^{1-a}}\right)$$

And we can replace p_n and $p'_{n,x}$ by their literal expression :

$$\frac{\ln^2\left(e^{\frac{\alpha}{(1-x)}}\right)}{e^\alpha} < \frac{\ln\left(e^{\frac{1}{(1-x)}}\right)}{e} \cdot \left(1 - \frac{(1-x)}{2 \cdot e^{\frac{(1-a)}{(1-x)}}}\right)$$

$$1 < (1-x) \frac{e^{\alpha-1}}{\alpha^2} \cdot \left(1 - \frac{(1-x)}{2 \cdot e^{\frac{(1-a)}{(1-x)}}}\right) \tag{I.35}$$

As (I.35) is an implicate function of α , we are going to estimate α , in the table below

$\alpha \backslash x$	0,5	0,55	0,6	0,65	0,7	0,75	0,8	0,85	0,9	0,95
5,2000	0,9170	0,8406	0,7629	0,6832	0,6009	0,5151	0,4250	0,3296	0,2278	0,1184
5,3396	1,0000	0,9167	0,8319	0,7450	0,6552	0,5617	0,4634	0,3594	0,2484	0,1291
5,4776	1,0909	1,0000	0,9075	0,8127	0,7148	0,6127	0,5055	0,3920	0,2709	0,1408
5,6293	1,2020	1,1019	1,0000	0,8956	0,7876	0,6752	0,5571	0,4320	0,2986	0,1551
5,7990	1,3422	1,2304	1,1167	1,0000	0,8795	0,7539	0,6220	0,4824	0,3334	0,1732
5,9934	1,5262	1,3991	1,2697	1,1371	1,0000	0,8573	0,7073	0,5485	0,3791	0,1970
6,2224	1,7803	1,6320	1,4811	1,3264	1,1665	1,0000	0,8250	0,6398	0,4422	0,2298
6,5029	2,1578	1,9781	1,7952	1,6077	1,4139	1,2120	1,0000	0,7755	0,5359	0,2785
6,8657	2,7824	2,5507	2,3148	2,0730	1,8232	1,5629	1,2895	1,0000	0,6911	0,3591
7,3796	4,0263	3,6910	3,3497	2,9998	2,6383	2,2616	1,8659	1,4470	1,0000	0,5196
8,2595	7,7483	7,1029	6,4462	5,7728	5,0771	4,3522	3,5908	2,7846	1,9244	1,0000

Table I.14

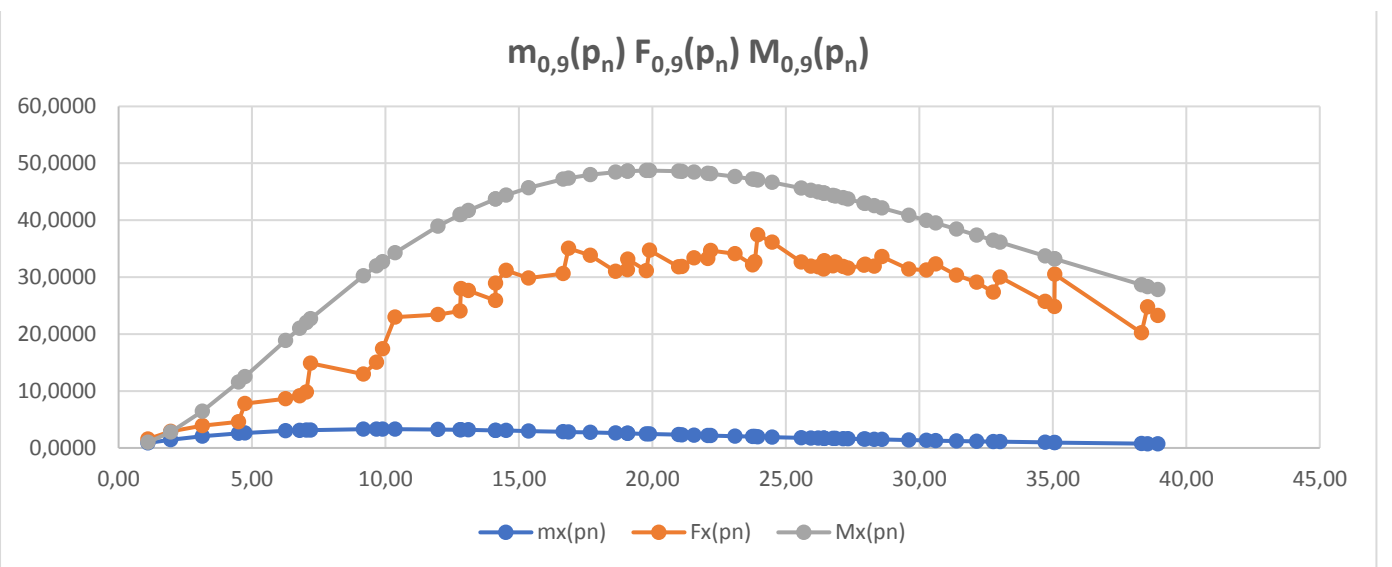
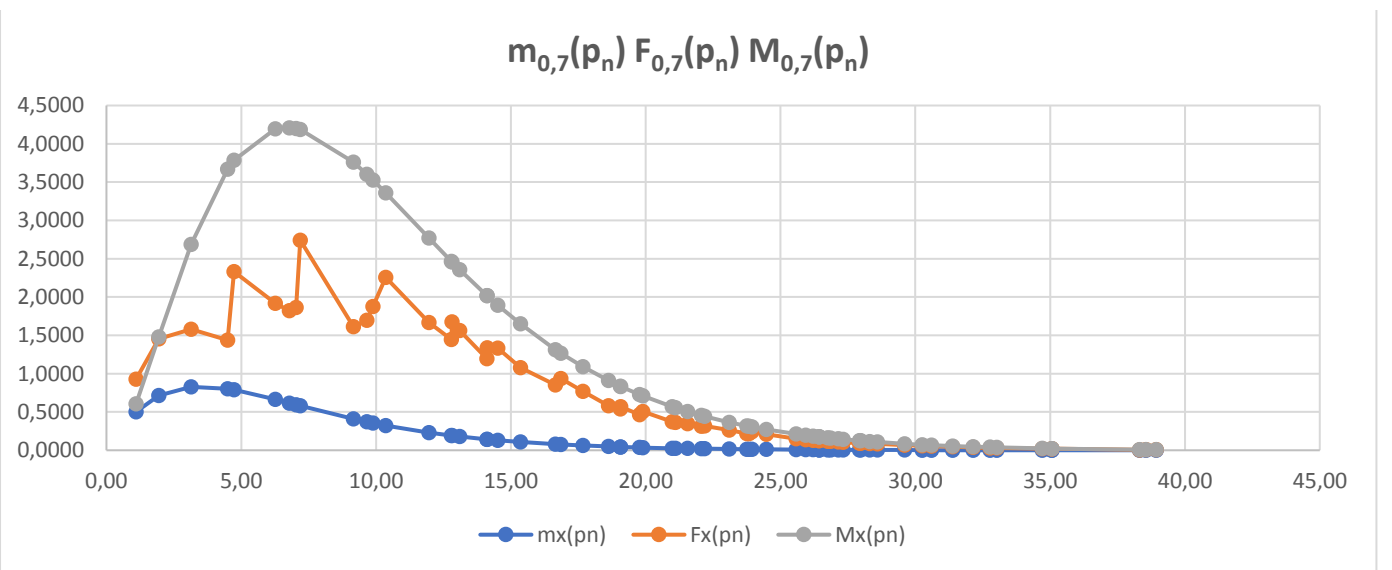
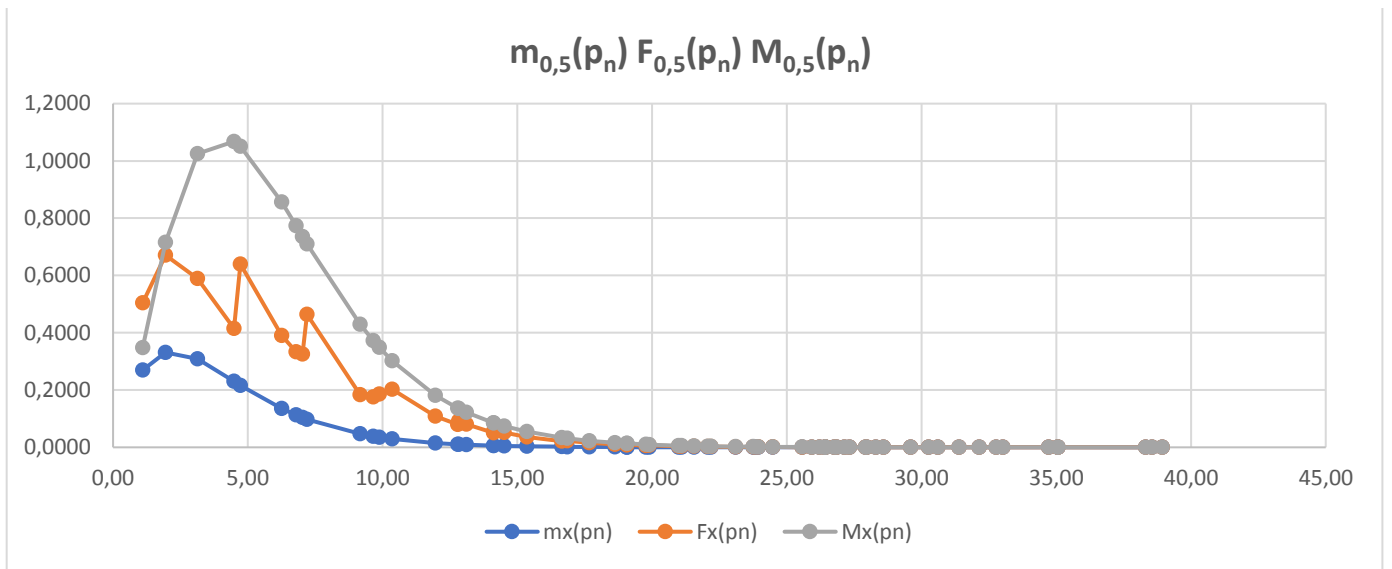
For $x=0.5$, $\alpha \approx 5.3396$ and $p_{no,x} = e^{\frac{5.3396}{0.5}} \approx 43442$. So we can see in the table below that $F_x(p_n) < 0,671$ for all $p_n \leq 43442$ and for $p_n > 43442$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{p_n} m_x(p_n) = m_x\left(e^{\frac{1}{(1-x)}}\right)$. The closest prime value satisfying this condition is $P_n = 31397$

For $x=0.6$, $\alpha \approx 5.6293$ and $p_{no,x} = e^{\frac{5.6293}{0.4}} \approx 1\,294\,001$. So we can see in the table below that $F_x(p_n) < 1,24$ for all $p_n \leq 1\,294\,001$ and for $p_n > 1\,294\,001$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{p_n} m_x(p_n) = m_x\left(e^{\frac{1}{(1-x)}}\right)$. The closest and lowest prime value satisfying this condition is $P_n = 1349\,533$

For $x=0.7$, $\alpha \approx 5.9934$ and $p_{no,x} = e^{\frac{5.9934}{0.3}} \approx 474\,608\,114$. So we can see in the table below that $F_x(p_n) < 2,75$ for all $p_n \leq 474\,608\,114$ and for $p_n > 474\,608\,114$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{p_n} m_x(p_n) = m_x\left(e^{\frac{1}{(1-x)}}\right)$. The closest and lowest prime value satisfying this condition is $P_n = 387\,096\,133$

For $x=0.8$, $\alpha \approx 6.5029$ and $p_{no,x} = e^{\frac{6.5029}{0.2}} \approx 132\,089\,389\,169\,128$. So we can see in the table below that $F_x(p_n) < 2,75$ for all $p_n \leq 132\,089\,389\,169\,128$. and for $p_n > 132\,089\,389\,169\,128$, we have proved here that $F_x(p_n) < M_x(p_n) < \text{Max}_{p_n} m_x(p_n) = m_x\left(e^{\frac{1}{(1-x)}}\right)$. The closest and lowest prime value satisfying this condition is $P_n = 90\,874\,329\,411\,493$

We can conclude that for all $0,475 < x < 1$, under Cramer's conjecture, $F_x(p_n)$ is a bounded function who tends toward 0 when $p_n \rightarrow \infty$ and the upper bound of $F_x(p_n)$ for every x is reached for $p_n < 218\,209\,405\,436\,543$ and thus can be estimated numerically for all $0,475 < x < 0,8$.



Under Cramer's conjecture (in grey), we see that for all prime up to the 66th maximal gap, $F_x(p_n)$ remains under $M_x(p_n)$. For $x < 0,85$, the value of P_{no} will be lower than the prime number $P_{n,max}$ corresponding to the highest known maximal gap $G_{n,max} = 1550$. In these conditions we can see the grey curve be under the maximal value of the blue one, which insure that the upper bound of $F_x(p_n)$ is reached for a prime number lower than $P_{n,max}$.

Part 2 : Study of $L_k(p_n) = \ln^k(p_{n+1}) - \ln^k(p_n)$ as a variant of Andrica's conjecture

In 2018 Matt Visser [1] [2] from Victoria University of Wellington has proposed a variant of Andrica's conjecture. He developed an original approach to estimate some upper bounds of the function $L_k(p_n) = \ln^k(p_{n+1}) - \ln^k(p_n)$. We propose a new approach to prove that for all k , $L_k(p_n)$ is a bounded function which tends toward 0 when $n \rightarrow \infty$. As we have processed in Part 1, we intend to build a lower and an upper bound functions $L_k^-(p_n)$ and $L_k^+(p_n)$, to state that these functions are bounded and tends toward 0 when $n \rightarrow \infty$. We'll try to estimate the prime number $P_{n_0, k}$ such as for all prime $P_n > P_{n_0, k}$, $L_k^+(p_n) < \text{Max}_{p_n} L_k^-(p_n)$. And we'll conclude, if $P_{n_0} < P_{n, \text{max}}$ which is the prime number associated to the highest maximal gap known $G_{n, \text{max}} = 1550$, that $L_k(p_n)$ has an upper bound and that this upper bound is reached for $P_n < P_{n, \text{max}}$. We'll also state a recurrence relation on k to the upper bound of $L_k(p_n)$ with the upper bound of $L_{k-1}(p_n)$.

In a second time, we'll prove that for all p_n , there is a real y such as $L_y(p_n) = \ln^y(p_{n+1}) - \ln^y(p_n) = 1$ and try to build an explicite formulation of $y(p_n)$.

I – Upper and Lower bound functions of $L_k(p_n)$:

We are first going to factorize $L_k(p_n)$:

$$L_k(p_n) = \ln^k(p_{n+1}) - \ln^k(p_n) = (\ln(p_{n+1}) - \ln(p_n)) \cdot \sum_{i=1}^k \ln^{k-i}(p_{n+1}) \cdot \ln^{i-1}(p_n)$$

As $p_{n+1} > p_n$ then $\ln(p_{n+1}) > \ln(p_n)$, Thus we can give a first lower and upper bounds estimation of $F_k(p_n)$:

$$k \cdot \ln^{k-1}(p_n) \cdot (\ln(p_{n+1}) - \ln(p_n)) < L_k(p_n) < k \cdot \ln^{k-1}(p_{n+1}) \cdot (\ln(p_{n+1}) - \ln(p_n)) \quad (\text{II.1})$$

We are going to use a Talor serie expansion of $\ln(p_{n+1}) - \ln(p_n)$:

$$\ln(p_{n+1}) - \ln(p_n) = \ln\left(p_n \cdot \left(1 + \frac{g_n}{p_n}\right)\right) - \ln(p_n) = \ln\left(1 + \frac{g_n}{p_n}\right) = \frac{g_n}{p_n} \cdot \left(1 - \frac{g_n}{2p_n}\right) + O\left(\frac{g_n}{p_n}\right)^3$$

$$\frac{g_n}{p_n} \cdot \left(1 - \frac{g_n}{2p_n}\right) < \ln(p_{n+1}) - \ln(p_n) < \frac{g_n}{p_n}$$

We can replace into (II.1) :

$$k \cdot \ln^{k-1}(p_n) \cdot \frac{g_n}{p_n} \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < k \cdot \ln^{k-1}(p_{n+1}) \cdot \frac{g_n}{p_n}$$

$$k \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < k \cdot \frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} \cdot \frac{p_{n+1}}{p_n} \cdot g_n$$

$$k \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < k \cdot \frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} \cdot g_n \cdot \left(1 + \frac{g_n}{p_n}\right) \quad (\text{II.2})$$

If we consider the function $M_k(p_n) = k \cdot \frac{\ln^{k-1}(p_n)}{p_n}$, (II.2) can be written :

$$M_k(p_n) \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_{n+1}) \cdot g_n \cdot \left(1 + \frac{g_n}{p_n}\right) \quad (\text{II.3})$$

we can easily see that the function $\frac{\ln^{k-1}(p_n)}{p_n}$ is decreasing for all $p_n > e^{k-1}$, thus $M_k(p_{n+1}) < M_k(p_n)$

$$M_k(p_n) \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_n) \cdot g_n \cdot \left(1 + \frac{g_n}{p_n}\right) \quad \text{for } p_n > e^{k-1} \quad (\text{II.4})$$

$$\left(1 - \frac{g_n}{2p_n}\right) < \frac{L_k(p_n)}{M_k(p_n) \cdot g_n} < \left(1 + \frac{g_n}{p_n}\right)$$

thanks to BHP result, we know that $g_n < p_n^a$ with $a = 0.525$

$$\left(1 - \frac{1}{2p_n^{1-a}}\right) < \frac{L_k(p_n)}{M_k(p_n) \cdot g_n} < \left(1 + \frac{1}{p_n^{1-a}}\right)$$

$$-\frac{1}{2p_n^{1-a}} < \frac{L_k(p_n)}{M_k(p_n) \cdot g_n} - 1 < \frac{1}{p_n^{1-a}}$$

Thus, $\limlim_{p_n \rightarrow \infty} \left(\frac{L_k(p_n)}{M_k(p_n) \cdot g_n}\right) = 1$ which imply that for n enough large, $M_k(p_n) \cdot g_n$ and $L_k(p_n)$ have the same behavior.

An interesting corollary of this result is the fact that the value of $p_{n,k}$ such as $L_k(p_{n,k}) = \text{Max}_{p_i \in \mathbb{P}} L_k(p_i)$ must be higher than e^{k-1} . the proof of this point will be presented in the next paragraph.

Our purpose is now to build two function $L_k^-(p_n)$ and $L_k^+(p_n)$ independent from g_n :

As for all integer, $g_n \geq 2$ and $t \ g_n < p_n^a$ with $a = 0.525$:

$$2k \cdot \frac{\ln^{k-1}(p_n)}{p_n} \left(1 - \frac{1}{2p_n^{1-a}}\right) < L_k(p_n) < k \cdot \frac{\ln^{k-1}(p_n)}{p_n^{1-a}} \cdot \left(1 + \frac{1}{p_n^{1-a}}\right) \quad (\text{II.5})$$

then we can define : $L_k^-(p_n) = 2k \cdot \frac{\ln^{k-1}(p_n)}{p_n} \left(1 - \frac{1}{2p_n^{1-a}}\right)$ and $L_k^+(p_n) = k \cdot \frac{\ln^{k-1}(p_n)}{p_n^{1-a}} \cdot \left(1 + \frac{1}{p_n^{1-a}}\right)$ which satisfy

$$L_k^-(p_n) < L_k(p_n) < L_k^+(p_n) \quad \text{for } p_n > e^{k-1} \quad (\text{II.6})$$

Thus we have built two functions $L_k^-(p_n)$ and $L_k^+(p_n)$, independent from g_n , satisfying to (II.6), which are two positive bounded functions which tends toward 0 when $n \rightarrow \infty$. Thus we can state that for all integer k , $L_k(p_n)$ has an upper bound and tends toward 0 when $n \rightarrow \infty$.

II – An approach based on the maximal gaps integer sequence

For the same reason exposed in Part 1, we can limit our study to prime numbers corresponding to the first occurrence of maximal G_N . gaps show that we can concentrate our analysis on maximal gaps. We'll write (G_N) the integer sequence corresponding to all maximal gaps and (P_N) , the integer sequence of prime numbers corresponding to the first occurrence of all element of (G_N) .

For each $G_N \in (G_N)$, we define $\mathbb{P}_{G_N} = \{p_i \in \mathbb{P} / g_i = G_N\}$ and $P_N = \text{Min}_{\mathbb{P}_{G_N}}(p_i)$

We can rewrite (II.4) as following :

$$M_k(p_n) \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < M_k(p_n) \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_n) \cdot g_n \cdot \left(1 + \frac{g_n}{p_n}\right) < M_k(p_n) \cdot G_n \cdot \left(1 + \frac{G_n}{p_n}\right) \quad (\text{II.7})$$

As our purpose is to estimate as well as possible $p_{n,k}$ such as $L_k(p_{n,k}) = \text{Max}_{p_i \in \mathbb{P}} L_k(p_i)$, we can conjecture that $p_{n,k} \in (P_N)$. To try to state this hypothesis, we suppose that $p_{n,k} \notin (P_N)$, there is G_N such as $p_{n,k} \in \mathbb{P}_{G_N}$ and by definition $p_{n,k} > P_N$. It's easy to prove that $L_k(p_{n,k}) < L_k(P_N)$.

$$L_k(P_N) - L_k(p_{n,k}) = \ln^k(P_N + G_N) - \ln^k(P_N) - (\ln^k(p_{n,k} + G_N) - \ln^k(p_{n,k}))$$

$$L_k(P_N) - L_k(p_{n,k}) = (\ln^k(p_{n,k}) - \ln^k(P_N)) - (\ln^k(p_{n,k} + G_N) - \ln^k(P_N + G_N))$$

We can remark that $\ln^k(x + d) - \ln^k(y + d)$ is decreasing when d grows, for $x > y > e^{k-1}$.

We can also insure that $p_{n,k}$ and P_N cannot be lower than e^{k-1} . Effectively $M_k(P_N)$ is a strictly growing function up to $p_n = e^{k-1}$ and G_N is also a growing function. So for all $p_n < e^{k-1}$, the product $M_k(P_N) \cdot G_N$ of two growing function is also a growing function. Which imply that the maximum value cannot be smaller than e^{k-1} .

Thus we can conclude that $p_{n,k} > P_N > e^{k-1}$ and $L_k(P_N) - L_k(p_{n,k}) > 0$ which contradict our hypothesis $L_k(p_{n,k}) = \text{Max}_{p_i \in \mathbb{P}} L_k(p_i)$ and $p_{n,k} \notin (P_N)$. We can conclude that $p_{n,k} \in (P_N)$ and limit our study to prime numbers belonging to the integer sequence (P_N) .

We can verify in tables II.1, II.2 and II.3 that for $k \leq 25$, $p_{n,k} > e^{k-1}$ and then belong to (P_N) .

In these conditions, (II.7) can be written :

$$M_k(P_N) \cdot G_N \cdot \left(1 - \frac{G_N}{2P_N}\right) < L_k(P_N) < M_k(P_N) \cdot G_N \cdot \left(1 + \frac{G_N}{P_N}\right) \quad (\text{II.8})$$

We have also:
$$1 - \frac{1}{2P_N^{1-a}} < 1 - \frac{G_N}{2P_N} < \frac{L_k(P_N)}{M_k(P_N) \cdot G_N} < 1 + \frac{G_N}{P_N} < 1 + \frac{1}{P_N^{1-a}} \quad (\text{II.9})$$

From (II.8) we can state that the behavior of $L_k(P_N)$ will be correctly represented by the behavior of $M_k(P_N) \cdot G_N$. Effectively, we have :

$$|L_k(P_N) - M_k(P_N) \cdot G_N| < M_k(P_N) \cdot \frac{G_N^2}{P_N} = k \cdot \ln^{k-1}(P_N) \cdot \frac{G_N^2}{P_N^2} < k \cdot \frac{\ln^{k-1}(P_N)}{P_N^{2-2a}} \quad \text{with } a=0,525$$

Which tends towards to 0 when $N \rightarrow \infty$

Thanks to Erik Westzynthius (I.30) result regarding Maximal gaps and BHP result, we can write, $\ln(P_N) < G_N < P_N^a$ and replace into (II.8)

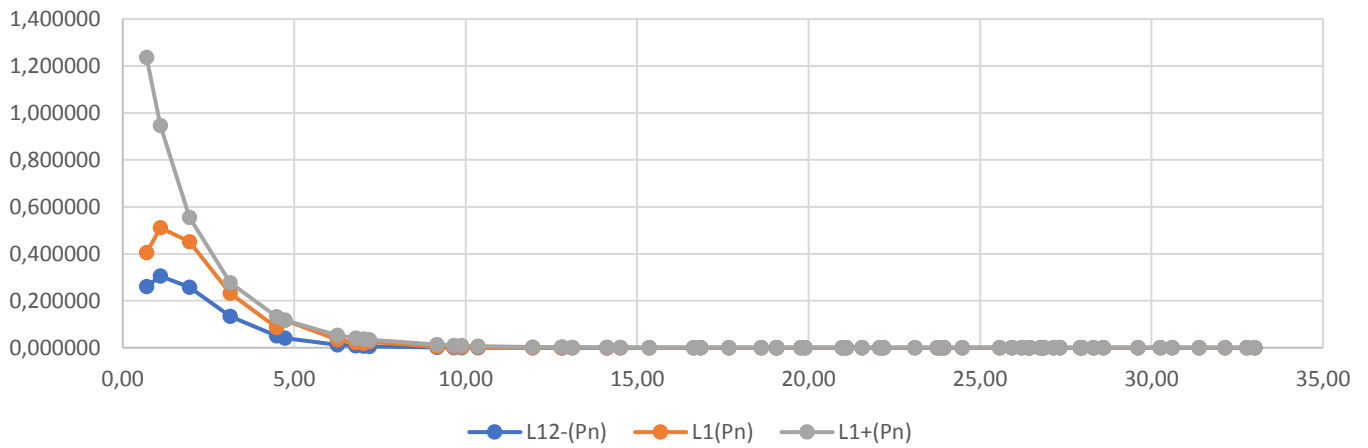
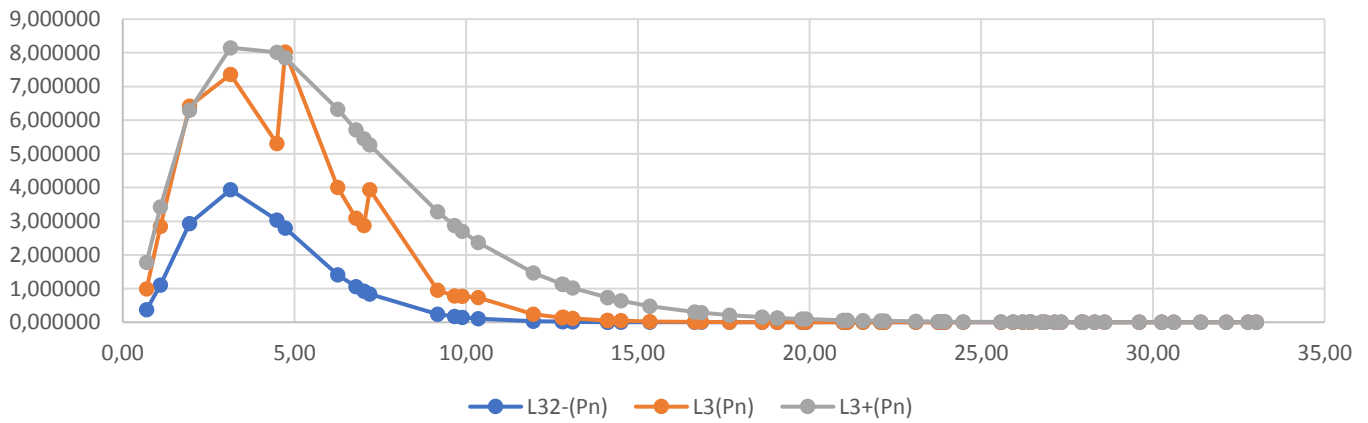
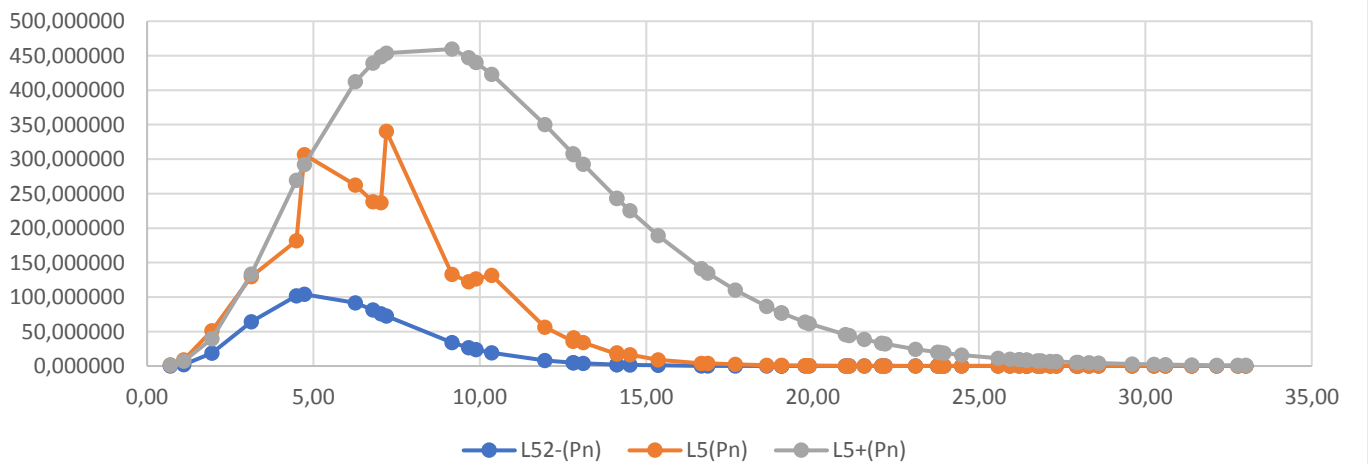
$$\begin{aligned} M_k(P_N) \cdot \ln(P_N) \cdot \left(1 - \frac{1}{2P_N^{1-a}}\right) < L_k(P_N) < M_k(P_N) \cdot P_N^a \cdot \left(1 + \frac{1}{P_N^{1-a}}\right) \\ k \cdot \frac{\ln^{k-1}(P_N)}{P_N} \cdot \ln(P_N) \cdot \left(1 - \frac{1}{2P_N^{1-a}}\right) < L_k(P_N) < k \cdot \frac{\ln^{k-1}(P_N)}{P_N} \cdot P_N^a \cdot \left(1 + \frac{1}{P_N^{1-a}}\right) \\ k \cdot \frac{\ln^k(P_N)}{P_N} \cdot \left(1 - \frac{1}{2P_N^{1-a}}\right) < L_k(P_N) < k \cdot \frac{\ln^k(P_N)}{P_N^{1-a}} \cdot \left(1 + \frac{1}{P_N^{1-a}}\right) = L_k^+(P_N) \end{aligned} \quad (\text{II.10})$$

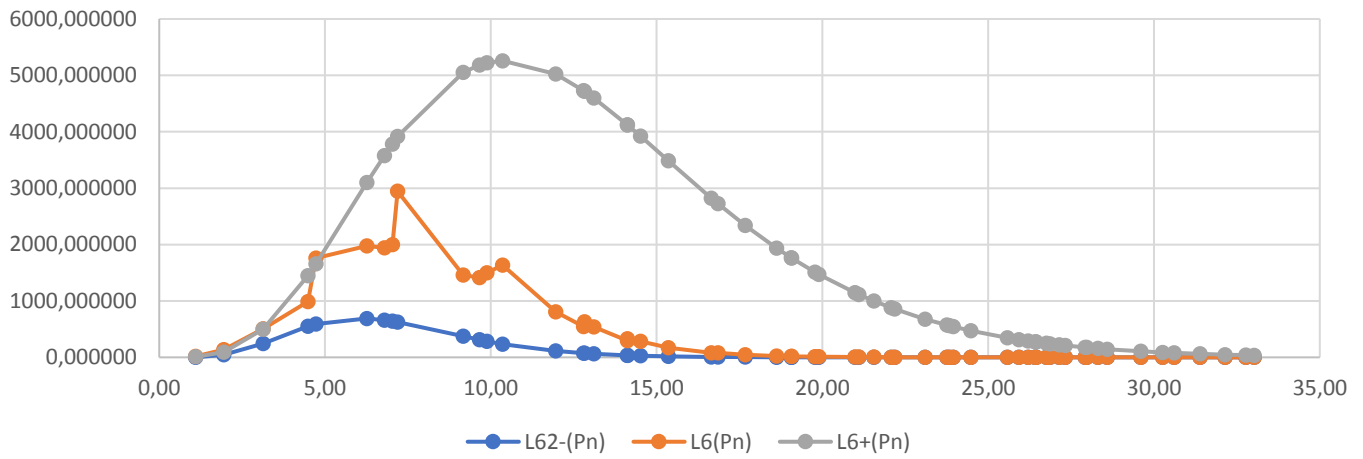
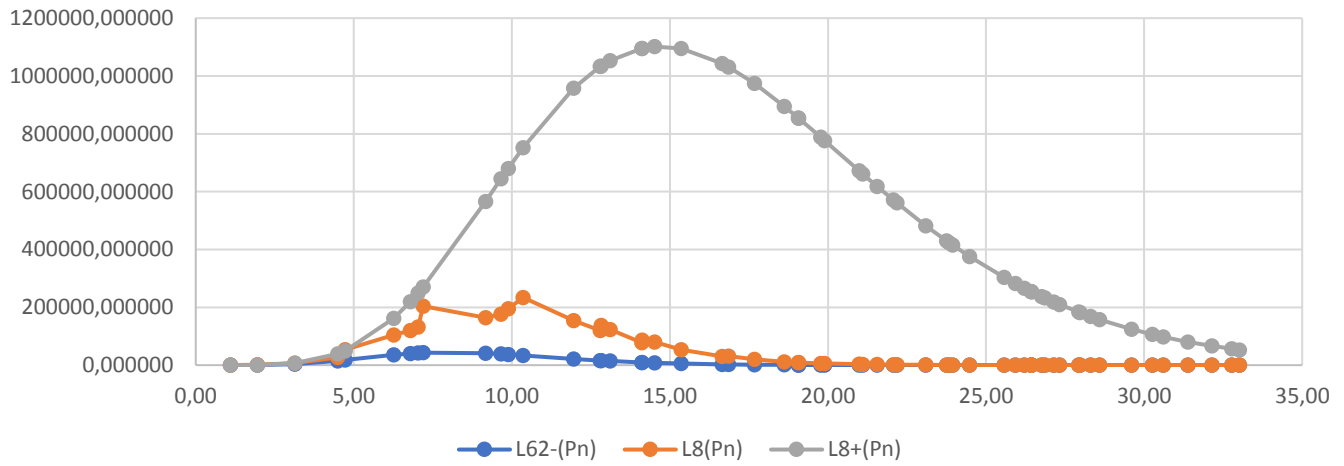
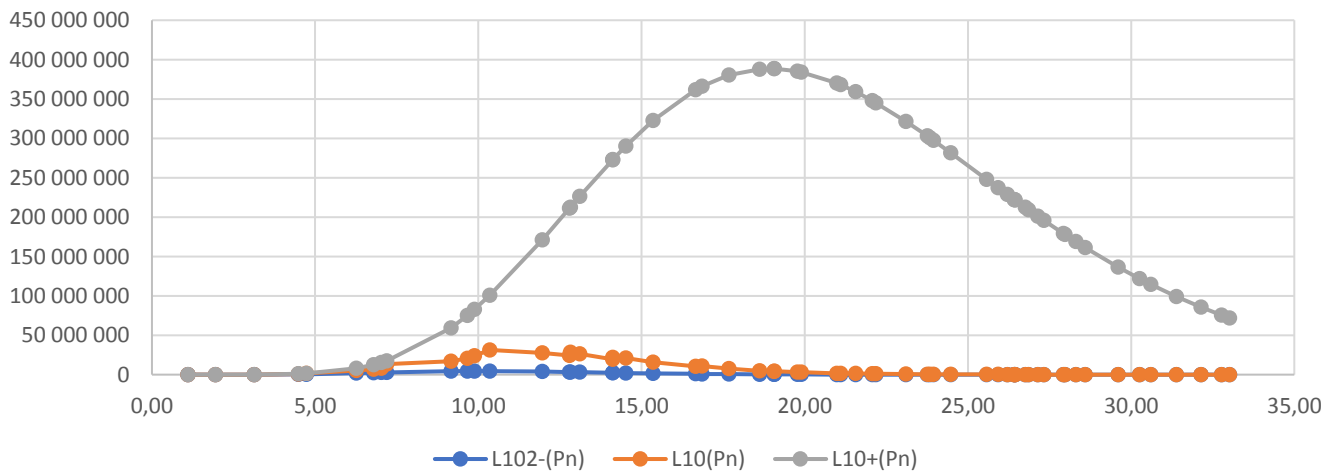
We can define a new lower bound function which satisfy (II.10) only for prime numbers belonging to (P_N) :

$$L_k^{2-}(P_N) = k \cdot \frac{\ln^k(P_N)}{P_N} \cdot \left(1 - \frac{1}{2P_N^{1-a}}\right) \quad (\text{II.11})$$

We presents in table II.1 and II.2, the numerical values of $L_k^{2-}(P_N)$, $L_k(P_N)$ and $L_k^+(P_N)$ for $k < 10$ and for all $P_{N(n)} \leq 218\ 209\ 405\ 436\ 543$. We have computed for all $P_N \in (P_N)$, and all $k < 30$, the numerical values of $L_k^{2-}(P_N)$, $L_k(P_N)$ and $L_k^+(P_N)$ and the results are summarized in table II.3 which present the upper bound of these three functions.

Primes	Gaps	ln(Pn)	k= 1,000		k= 2,000			k= 3,000			k= 4,000			k= 5,000			
			L ₁ ² (P _n)	L ₁ (P _n)	L ₁ ⁺ (P _n)	L ₂ ² (P _n)	L ₂ (P _n)	L ₂ ⁺ (P _n)	L ₃ ² (P _n)	L ₃ (P _n)	L ₃ ⁺ (P _n)	L ₄ ² (P _n)	L ₄ (P _n)	L ₄ ⁺ (P _n)	L ₅ ² (P _n)	L ₅ (P _n)	L ₅ ⁺ (P _n)
2	1	0,69	0,260	0,405	1,237	0,360	0,726	1,715	0,375	0,993	1,783	0,346	1,226	1,648	0,300	1,440	1,428
3	2	1,10	0,305	0,511	0,946	0,671	1,383	2,078	1,105	2,843	3,424	1,619	5,253	5,015	2,223	9,198	6,887
7	4	1,95	0,258	0,452	0,554	1,005	1,963	2,157	2,932	6,419	6,296	7,608	18,723	16,336	18,506	51,377	39,735
23	6	3,14	0,133	0,232	0,276	0,836	1,507	1,733	3,933	7,355	8,151	16,444	31,911	34,078	64,451	129,858	133,565
89	8	4,49	0,050	0,086	0,133	0,450	0,780	1,191	3,031	5,303	8,018	18,142	32,044	47,986	101,790	181,534	269,239
113	14	4,73	0,042	0,117	0,117	0,394	1,118	1,107	2,792	8,026	7,850	17,601	51,218	49,479	104,009	306,445	292,380
523	18	6,26	0,012	0,034	0,054	0,150	0,425	0,673	1,406	3,999	6,318	11,731	33,467	52,731	91,787	262,572	412,592
887	20	6,79	0,008	0,022	0,041	0,104	0,303	0,562	1,057	3,092	5,718	9,568	28,032	51,753	81,182	238,235	439,115
1 129	22	7,03	0,006	0,019	0,037	0,087	0,272	0,516	0,922	2,868	5,445	8,645	26,920	51,035	75,959	236,855	448,412
1 327	34	7,19	0,005	0,025	0,034	0,078	0,364	0,488	0,840	3,938	5,264	8,056	37,824	50,471	72,408	340,571	453,655
9 551	36	9,16	0,001	0,004	0,013	0,018	0,069	0,239	0,242	0,948	3,284	2,954	11,590	40,124	33,839	132,794	459,638
15 683	44	9,66	0,001	0,003	0,010	0,012	0,054	0,198	0,172	0,785	2,875	2,221	10,107	37,034	26,822	122,069	447,201
19 609	52	9,88	0,001	0,003	0,009	0,010	0,052	0,182	0,148	0,776	2,704	1,947	10,232	35,634	24,050	126,433	440,243
31 397	72	10,35	0,000	0,002	0,007	0,007	0,047	0,153	0,106	0,737	2,369	1,464	10,175	32,703	18,955	131,710	423,275
155 921	86	11,96	0,000	0,001	0,003	0,002	0,013	0,082	0,033	0,237	1,470	0,524	3,771	23,431	7,838	56,362	350,207
360 653	96	12,80	0,000	0,000	0,002	0,001	0,007	0,059	0,017	0,131	1,129	0,297	2,230	19,259	4,755	35,675	308,035
370 261	112	12,82	0,000	0,000	0,002	0,001	0,008	0,058	0,017	0,149	1,119	0,292	2,550	19,137	4,680	40,875	306,711
492 113	114	13,11	0,000	0,000	0,002	0,001	0,006	0,052	0,014	0,119	1,021	0,240	2,086	17,850	3,929	34,176	292,442
1 349 533	118	14,12	0,000	0,000	0,001	0,000	0,002	0,035	0,006	0,052	0,733	0,118	0,984	13,798	2,076	17,354	243,456
1 357 201	132	14,12	0,000	0,000	0,001	0,000	0,003	0,035	0,006	0,058	0,732	0,117	1,095	13,778	2,068	19,335	243,191
2 010 733	148	14,51	0,000	0,000	0,001	0,000	0,002	0,029	0,005	0,047	0,641	0,088	0,900	12,410	1,602	16,331	225,147
4 652 353	154	15,35	0,000	0,000	0,001	0,000	0,001	0,021	0,002	0,023	0,482	0,048	0,479	9,858	0,917	9,195	189,182
17 051 707	180	16,65	0,000	0,000	0,000	0,000	0,000	0,012	0,001	0,009	0,306	0,018	0,195	6,784	0,375	4,058	141,215
20 831 323	210	16,85	0,000	0,000	0,000	0,000	0,000	0,011	0,001	0,009	0,285	0,015	0,193	6,394	0,326	4,065	134,687
47 326 693	220	17,67	0,000	0,000	0,000	0,000	0,000	0,008	0,000	0,004	0,212	0,008	0,103	4,993	0,182	2,267	110,304
122 164 747	222	18,62	0,000	0,000	0,000	0,000	0,000	0,005	0,000	0,002	0,150	0,004	0,047	3,722	0,092	1,092	86,643
189 695 659	234	19,06	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,001	0,127	0,003	0,034	3,239	0,066	0,814	77,183
191 912 783	248	19,07	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,001	0,127	0,003	0,036	3,227	0,066	0,855	76,945
387 096 133	250	19,77	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,001	0,098	0,002	0,020	2,577	0,039	0,494	63,707
436 273 009	282	19,89	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,001	0,093	0,001	0,020	2,479	0,036	0,506	61,658
1 294 268 491	288	20,98	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,062	0,001	0,008	1,735	0,016	0,216	45,509
1 453 168 141	292	21,10	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,059	0,001	0,008	1,670	0,014	0,199	44,032
2 300 942 549	320	21,56	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,050	0,000	0,006	1,432	0,010	0,150	38,583
3 842 610 773	336	22,07	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,041	0,000	0,004	1,204	0,007	0,104	33,224
4 302 407 359	354	22,18	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,039	0,000	0,004	1,159	0,006	0,100	32,137
10 726 904 659	382	23,10	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,028	0,000	0,002	0,848	0,003	0,051	24,471
20 678 048 297	384	23,75	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,021	0,000	0,001	0,675	0,002	0,030	20,042
22 367 084 959	394	23,83	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,021	0,000	0,001	0,657	0,002	0,028	19,565
25 056 082 087	456	23,94	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,020	0,000	0,001	0,631	0,002	0,030	18,893
42 652 618 343	464	24,48	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,016	0,000	0,001	0,524	0,001	0,020	16,023
127 976 334 671	468	25,58	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,010	0,000	0,000	0,355	0,000	0,008	11,333
182 226 896 239	474	25,93	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,009	0,000	0,000	0,312	0,000	0,006	10,123
241 160 624 143	486	26,21	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,008	0,000	0,000	0,282	0,000	0,005	9,250
297 501 075 799	490	26,42	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,007	0,000	0,000	0,262	0,000	0,004	8,644
303 371 455 241	500	26,44	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,007	0,000	0,000	0,260	0,000	0,004	8,589
304 599 508 537	514	26,44	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,007	0,000	0,000	0,260	0,000	0,004	8,578
416 608 695 821	516	26,76	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,006	0,000	0,000	0,232	0,000	0,003	7,749
461 690 510 011	532	26,86	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,006	0,000	0,000	0,223	0,000	0,003	7,494
614 487 453 523	534	27,14	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,006	0,000	0,000	0,201	0,000	0,002	6,825
738 832 927 927	540	27,33	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,005	0,000	0,000	0,188	0,000	0,002	6,425
1 346 294 310 749	582	27,93	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,000	0,151	0,000	0,001	5,270
1 408 695 493 609	588	27,97	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,004	0,000	0,000	0,148	0,000	0,001	5,191
1 968 188 556 461	602	28,31	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,000	0,131	0,000	0,001	4,644
2 614 941 710 599	652	28,59	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,003	0,000	0,000	0,118	0,000	0,001	4,223
7 177 162 611 713	674	29,60	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,081	0,000	0,000	3,004
13 829 048 559 701	716	30,26	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,002	0,000	0,000	0,063	0,000	0,000	2,401
19 581 334 192 423	766	30,61	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,056	0,000	0,000	2,131
42 842 283 925 351	778	31,39	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,041	0,000	0,000	1,625
90 874 329 411 493	804	32,14	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,031	0,000	0,000	1,250
171 231 342 420 521	806	32,77	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,024	0,000	0,000	1,000
218 209 405 436 543	906	33,02	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,000	0,000	0,022	0,000	0,000	0,918

$L_1^{2-}(P_n) L_1(P_n) L_1^+(P_n)$  $L_3^{2-}(P_n) L_3(P_n) L_3^+(P_n)$  $L_5^{2-}(P_n) L_5(P_n) L_5^+(P_n)$ 

$L_6^{-}(P_n) L_6(P_n) L_6^{+}(P_n)$  $L_8^{-}(P_n) L_8(P_n) L_8^{+}(P_n)$  $L_{10}^{-}(P_n) L_{10}(P_n) L_{10}^{+}(P_n)$ 

II.3 – Behaviour of the lower and upper bound functions $L_k^-(p_n)$ and $L_k^+(p_n)$

As our purpose is to estimate the value of P_{n_0} when $L_k^+(P_N)$ becomes lower than $\text{Max}_{P_N \in \mathbb{P}} L_k^{2-}(P_N^{2-}) = L_k^{2-}(P_{N,k}^{2-})$, we need to $P_{N,k}^{2-}$ and thus compute the first derivative of $L_k^{2-}(P_N)$

First derivative of $L_k^{2-}(p_n)$

$$\frac{dL_k^{2-}(P_N)}{dP_N} = 2k^2 \cdot \frac{\ln^{k-1}(P_N)}{2P_N^{3-a}} \cdot \left(P_N^{1-a} - \frac{1}{k} \left(P_N^{1-a} - 1 + \frac{a}{2} \right) \cdot \ln(P_N) - \frac{1}{2} \right)$$

If we define $P_{N,k}^{2-}$ as the prime number for which $L_k^{2-}(P_{N,k}^{2-}) = \text{Max}_{P_N \in \mathbb{P}_{\text{Max}}} L_k^{2-}(P_N)$, $P_{N,k}$ must satisfy :

$$\frac{dL_k^{2-}(P_{N,k}^{2-})}{dP_N} = 0 \quad \text{which imply} \quad P_N^{1-a} - \frac{1}{k} \left(P_N^{1-a} - 1 + \frac{a}{2} \right) \cdot \ln(P_N) - \frac{1}{2} = 0$$

If P_N is large, then $P_{N,k} = e^k$ gives a first estimation. But a better estimation can be given by :

$$P_{N,k}^{2-} \approx e^k \left(\frac{1 - \frac{1}{2} e^{k(a-1)}}{1 - \frac{(2-a)}{2} \cdot e^{k(a-1)}} \right) \quad (\text{II.12})$$

First derivative of $L_k^+(p_n)$

$$\frac{dL_k^+(P_N)}{dP_N} = (k-1) \cdot \frac{\ln^{k-2}(P_N)}{2P_N^{3-2a}} \cdot \left(P_N^{1-a} - \frac{(1-a)}{k-1} (P_N^{1-a} + 2) \cdot \ln(P_N) + 1 \right)$$

If we define $P_{N,k}^+$ as the prime number for which $L_k^+(P_{N,k}^+) = \text{Max}_{P_N \in \mathbb{P}_{\text{Max}}} L_k^+(P_N)$, $P_{N,k}$ must satisfy :

$$\frac{dL_k^+(P_{N,k}^+)}{dP_N} = 0 \quad \text{which imply} \quad P_N^{1-a} - \frac{(1-a)}{k-1} (P_N^{1-a} + 2) \cdot \ln(P_N) + 1 = 0$$

If P_N is large, then $P_{N,k} = e^{\frac{k-1}{1-a}}$ gives a first estimation. But a better estimation can be given by :

$$P_{N,k}^+ \approx e^{\frac{k-1}{1-a}} \left(\frac{1 - e^{1-k}}{1 - 2 \cdot e^{1-k}} \right) \quad (\text{II.13})$$

k	Gn	$P_{N,k}^{2-}$	$P_{N,k}$	$P_{N,k}^+$	$L_{N,k}^{2-}(P_{N,k}^{2-})$	$L_{N,k}(P_{N,k})$	$L_{N,k}^+(P_{N,k}^+)$
1	2	4	3	1		0,51	
2	4	10	7	4	0,88	1,96	2,18
3	6	25	113	39	3,57	3,51	8,31
4	14	64	113	406	17,40	51,22	52,87
5	34	167	1327	3897	100,51	340,57	469,29
6	34	440	1327	34737	674,21	2943,90	5260,50
7	34	1166	1327	296647	5163,60	24740,35	70657,82
8	72	3113	31397	2479105	44526,46	234027,85	1102190,69
9	72	8350	31397	20510675	427331,21	2726439,61	19552555,81
10	72	22487	31397	168935955	4520449,10	31371060,15	388457386,43
11	112	60725	370261	1388758205	52276549,54	399607495,22	8541447795,12
12	112	164318	370261	11407170239	656291658,80	5589614319,57	205891790335,23
13	148	445269	2010733	93666147312	8890522390,57	83617440511,50	5398385259199,99
14	148	1207795	2010733	769001833747	129275727735,77	1306983392428,00	152949269629900,00
15	210	3278408	20831323	6313174159164	2008483436487,93	22526940330496,00	4656405979805580,00
16	210	8903032	20831323	51827276934671	33206213701666,20	404931625041920,00	151588979197246000,00
17	210	24185331	20831323	425466247153719	582112367132113,00	7250390520168450,00	5254836747375160000,00
18	282	65714327	436273009	3492772409868440	10785366340923100,00	13930191191203000,00	193244272347804000000,00
19	282	178579276	436273009	28673113449022000	210596296266572000,00	2925199322721030000,00	751403220173360000000,00
20	354	485337734	4302407359	235385214084396000	4322390679783290000,00	61730375961021500000,00	3080169932824650000000,00
21	456	1319121962	25056082087	1932339407684920000	93031549951990100000,00	1466708506822120000000,00	132756984271301000000000,00
22	456	3585455134	25056082087	15863083359450200000	209525444748286000000,00	367917830755920000000,00	600175065414524000000000,00
23	456	9745761797	25056082087	130224226887788000000	4928231173564370000000,00	9210000389131800000000,00	2839806838089180000000000,00
24	456	26490812582	25056082087	1069044943513680000000	12084009167384000000000,00	230115926560123000000000,00	14035418445338100000000000,00
25	464	72007876002	42652618343	877607118872905000000	30837322738430800000000,00	581344786051355000000000,00	723261941905702000000000000,00

Table II.3

II.4 – Estimation of p_{no} such as for $P_N > p_{no}$, $L_k^2-(P_N) < L_k^2-(P_{N,k}^2-)$

We are going to try to estimate for which value of p_n , the upper bound function becomes lower than the maximum of the lower bound function.

$$L_k(p_n) < L_k^+(p_n) < L_k^2-(p_{n,k}^2-) = \text{Max}_{p_n \in \mathbb{P}} L_k^2-(p_n) \quad (\text{II.14})$$

As $P_{N,k}^2- \approx e^{k \cdot \left(\frac{1-\frac{1}{2} \cdot e^{k(a-1)}}{1-\frac{(2-a)}{2} \cdot e^{k(a-1)}} \right)}$ is a relatively complex formulation, we are going to use the following approximation $P_N^2- \approx e^k$. Effectively for instance, for $k=4$, $\frac{1-\frac{1}{2} \cdot e^{k(a-1)}}{1-\frac{(2-a)}{2} \cdot e^{k(a-1)}} \approx 1,04$ and $P_N^2- \approx e^k \approx 55$ and $P_{N,k}^2- \approx 64$. As in the integer sequence (P_N) , $P_4 = 23$ and $P_5 = 89$ the approximation of $P_{N,k}^2-$ proposed above is coherent.

$$\text{So we have to estimate} \quad L_k^2-(e^k) = k \cdot \frac{\ln^k(e^k)}{e^k} \cdot \left(1 - \frac{1}{2e^{k(1-a)}} \right) < L_k^2-(P_{n,k}^2-)$$

We are going to look for a real $\alpha > 1$ such as $p_{no} \approx e^{\alpha \frac{(k-1)}{(1-a)}}$ which satisfy the following inequality

$$L_k^+ \left(e^{\alpha \frac{(k-1)}{(1-a)}} \right) = k \cdot \frac{\ln^{k-1} \left(e^{\alpha \frac{(k-1)}{(1-a)}} \right)}{e^{\alpha \cdot (k-1)}} \cdot \left(1 + \frac{1}{e^{\alpha \cdot (k-1)}} \right) < k \cdot \frac{\ln^k(e^k)}{e^k} \cdot \left(1 - \frac{1}{2e^{k(1-a)}} \right) = L_k^2-(e^k)$$

$$\frac{\left(\alpha \frac{(k-1)}{(1-a)} \right)^{k-1}}{e^{\alpha \cdot (k-1)}} < \frac{k^k}{e^k} \cdot \frac{\left(1 - \frac{1}{2e^{k(1-a)}} \right)}{\left(1 + \frac{1}{e^{\alpha \cdot (k-1)}} \right)} \Leftrightarrow \frac{\alpha}{e^\alpha} < \left(\frac{1-a}{k-1} \right) \cdot \left(\frac{k}{e} \right)^{k-1} \cdot \left(\frac{1 - \frac{1}{2e^{k(1-a)}}}{1 + \frac{1}{e^{\alpha \cdot (k-1)}}} \right)^{k-1}$$

$$\text{For } k > 1 \text{ we can state that} \quad \left(\frac{1-a}{k-1} \right) \cdot \left(\frac{k}{e} \right)^{k-1} \cdot \left(\frac{1 - \frac{1}{2e^{k(1-a)}}}{1 + \frac{1}{e^{\alpha \cdot (k-1)}}} \right)^{k-1} > \left(\frac{1-a}{k-1} \right) \cdot \frac{k}{e} \cdot \left(\frac{1 - \frac{1}{2e^{k(1-a)}}}{1 + \frac{1}{e^{k-1}}} \right)^{k-1} > \frac{1-a}{e}$$

As for all integer k , $\left(\frac{k}{k-1} \right) \cdot \left(\frac{1 - \frac{1}{2e^{k(1-a)}}}{1 + \frac{1}{e^{k-1}}} \right)^{k-1} > 1$. So we have to solve $\frac{\alpha}{e^\alpha} < \frac{1-a}{e}$ which gives $\alpha \approx \frac{11}{4}$

The table beside gives for $k < 7$, the numerical value of $P_{no,k} \approx e^{\frac{11}{4} \frac{(k-1)}{(1-a)}}$ satisfying (II.14) and compare with $P_{No,k}$ in table II.1 and table II.2. We see that our estimation over-estimate the value of $P_{No,k}$

k	$P_{no,k}$	$P_{No,k}$ in table II.1-II.2
2	327	523
3	106 825	9 551
4	34 914 789	492 113
5	11 411 583 088	122 164 747
6	3 729 772 743 500	10 726 904 659

We can conclude that for all $p_n > e^{\frac{11}{4} \frac{(k-1)}{(1-a)}}$ and all integer $k > 1$, $L_k(p_n) < L_k^+(p_n) < L_k^2-(p_{n,k}^2-)$ which means that $P_{N,k} < e^{\frac{11}{4} \frac{(k-1)}{(1-a)}}$ for all integer k .

We have tested numerically all prime of the integer sequence (P_N) corresponding to the prime numbers associated with the first occurrence of the maximal gaps, up to $P_n = 218\,209\,405\,436\,543$ and $G_n = 906$. We were limited by the accuracy of Excel. We intend in a future work to push the investigation up to $P_n =$

18 361 375 334 787 046 697 corresponding to the highest known maximal gap $G_n = 1550$ and $n = 423\,731\,791\,997\,205\,041$, to be able to gives upper bound of $L_k(p_n)$ for some values of $k > 7$.

Thanks to the previous results we can validate the following upper bounds of $L_k(p_n) = \ln^k(p_{n+1}) - \ln^k(p_n)$ for $k \leq 7$ (see table II.3)

$$\begin{aligned} L_1(p_n) &= \ln(p_{n+1}) - \ln(p_n) < 0,52 & L_2(p_n) &= \ln^2(p_{n+1}) - \ln^2(p_n) < 2 \\ L_3(p_n) &= \ln^3(p_{n+1}) - \ln^3(p_n) < 4 & L_4(p_n) &= \ln^4(p_{n+1}) - \ln^4(p_n) < 52 \\ L_5(p_n) &= \ln^5(p_{n+1}) - \ln^5(p_n) < 341 & L_6(p_n) &= \ln^6(p_{n+1}) - \ln^6(p_n) < 2944 \\ & & L_7(p_n) &= \ln^7(p_{n+1}) - \ln^7(p_n) < 24741 \end{aligned}$$

II.5 – Better bounds estimation from recurrence on k

We start from relation (II.4):

$$M_k(p_n) \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_k(p_n) < M_k(p_n) \cdot g_n \cdot \left(1 + \frac{g_n}{p_n}\right) \quad \text{for } p_n > e^{k-1}$$

And according to BHP result, $g_n < (p_n)^a$ with $a=0,525$

$$M_k(p_n) \cdot g_n \cdot \left(1 - \frac{1}{2p_n^{1-a}}\right) < L_k(p_n) < M_k(p_n) \cdot g_n \cdot \left(1 + \frac{1}{p_n^{1-a}}\right) \quad (\text{II.15})$$

The advantage of this formulation is the fact that we can establish a relation between $L_{k-1}(p_n)$ and $L_k(p_n)$ for all p_n and without g_n . For $L_{k-1}(p_n)$, we can apply (II.15):

$$M_{k-1}(p_n) \cdot g_n \cdot \left(1 - \frac{1}{2p_n^{1-a}}\right) < L_{k-1}(p_n) < M_{k-1}(p_n) \cdot g_n \cdot \left(1 + \frac{1}{p_n^{1-a}}\right) \quad (\text{II.16})$$

From (II.15) and (II.16) we can get :

$$\frac{k}{k-1} \cdot \ln(p_n) C(p_n) < \frac{L_k(p_n)}{L_{k-1}(p_n)} < \frac{k}{k-1} \cdot \frac{\ln(p_n)}{C(p_n)} \quad \text{with} \quad C(p_n) = \frac{\left(1 - \frac{1}{2p_n^{1-a}}\right)}{\left(1 + \frac{1}{p_n^{1-a}}\right)} < 1 \quad \text{for all } p \quad (\text{II.17})$$

Consider for all integer k , $P_{n,k}$ the prime number for which $F_k(P_{n,k})$ is maximum. In this conditions, we have :

$$\frac{L_k(P_{n,k-1})}{L_{k-1}(P_{n,k-1})} \leq \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k-1})} \leq \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k})}$$

Thanks to (II.16) :

$$\begin{aligned} \frac{k}{k-1} \cdot \ln(P_{n,k-1}) C(P_{n,k-1}) &< \frac{L_k(P_{n,k-1})}{L_{k-1}(P_{n,k-1})} \leq \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k-1})} \leq \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k})} < \frac{k}{k-1} \cdot \frac{\ln(P_{n,k})}{C(P_{n,k})} \\ \frac{k}{k-1} \cdot \ln(P_{n,k-1}) C(P_{n,k-1}) &< \frac{L_k(P_{n,k})}{L_{k-1}(P_{n,k-1})} < \frac{k}{k-1} \cdot \frac{\ln(P_{n,k})}{C(P_{n,k})} \\ \frac{k}{k-1} \cdot \ln(P_{n,k-1}) C(P_{n,k-1}) \cdot L_{k-1}(P_{n,k-1}) &< L_k(P_{n,k}) < \frac{k}{k-1} \cdot \frac{\ln(P_{n,k})}{C(P_{n,k})} \cdot L_{k-1}(P_{n,k-1}) \end{aligned} \quad (\text{II.18})$$

We define two new lower and upper bound functions :

$$L_k^{3-}(P_{n,k}) = \frac{k}{k-1} \cdot \ln(P_{n,k-1}) C(P_{n,k-1}) \cdot L_{k-1}(P_{n,k-1}) \quad \text{and} \quad L_k^{2+}(P_{n,k}) = \frac{k}{k-1} \cdot \frac{\ln(P_{n,k})}{C(P_{n,k})} \cdot L_{k-1}(P_{n,k-1})$$

The table (II.4) below, will give for each integer k up to 30, the numerical values of the lower and upper bounds of $L_k(p_{n,k})$ presented in (II.18).

Thanks to (II.18), we can see that the lower bound of $L_k(p_{n,k})$ is only dependent of the level k-1 and gives some numerical values very close from $L_k(p_{n,k})$. It means that we can estimate $L_k(p_{n,k})$ from $L_{k-1}(P_{n,k-1})$ and $P_{n,k-1}$ without any information about prime gaps.

K	Pn,k	Gaps	$L_k^3(P_n)$	$L_k(P_n)/L_k^3(P_n)$	$L_k(P_n)$	$L_k^{2+}(P_n)$	$L_k(P_n)/L_k^{2+}(P_n)$
1	3	2			0,67		
2	7	4	0,65	3,03676571	1,96	4,52	0,43426304
3	113	14	3,29	2,44040167	8,03	16,26	0,49369214
4	113	14	43,32	1,18222958	51,22	59,07	0,86705919
5	1327	34	259,20	1,31394732	340,57	483,44	0,70447933
6	1327	34	2 798,49	1,05196132	2 943,90	3 085,98	0,95396059
7	1327	34	23 518,28	1,05196240	24 740,35	25 934,32	0,95396157
8	31 397	72	193 612,23	1,20874515	234 027,85	295 991,73	0,79065671
9	31 397	72	2 696 458,70	1,01111862	2 726 439,61	2 756 143,97	0,98922249
10	31 397	72	31 026 092,56	1,01111863	31 371 060,15	31 712 845,42	0,98922250
11	370 261	112	353 423 636,96	1,13067564	399 607 495,22	443 967 035,74	0,90008371
12	370 261	112	5 570 605 585,19	1,00341233	5 589 614 319,57	5 608 555 617,79	0,99662278
13	2 010 733	148	77 379 188 888,64	1,08061924	83 617 440 511,50	88 022 066 199,56	0,94995998
14	2 010 733	148	1 304 994 775 159,13	1,00152385	1 306 983 392 428,00	1 308 968 402 035,86	0,99848353
15	20 831 323	210	20 293 666 177 617,60	1,11004784	22 526 940 330 496,00	23 610 295 435 578,90	0,95411514
16	20 831 323	210	404 728 764 895 014,00	1,00050122	404 931 625 041 920,00	405 134 344 485 731,00	0,99949962
17	20 831 323	210	7 246 758 264 556 170,00	1,00050122	7 250 390 520 168 450,00	7 254 020 257 045 710,00	0,99949962
18	436 273 009	282	129 305 834 748 391 000,00	1,07730569	139 301 911 911 203 000,00	152 740 268 939 947 000,00	0,91201824
19	436 273 009	282	2 924 853 870 792 310 000,00	1,00011811	2 925 199 322 721 030 000,00	2 925 544 719 503 790 000,00	0,99988194
20	4 302 407 359	354	61 248 838 739 379 200 000,00	1,00786198	61 730 375 961 021 500 000,00	68 305 940 868 206 000 000,00	0,90373363
21	25 056 082 087	456	1 437 739 666 357 580 000 000,00	1,02014888	1 466 708 506 822 120 000 000,00	1 552 027 277 194 470 000 000,00	0,94502753
22	25 056 082 087	456	36 791 148 800 458 500 000 000,00	1,00001724	36 791 783 075 559 200 000 000,00	36 792 417 582 700 100 000 000,00	0,99998275
23	25 056 082 087	456	920 984 122 302 756 000 000 000,00	1,00001724	921 000 003 891 318 000 000 000,00	921 015 883 428 386 000 000 000,00	0,99998276
24	25 056 082 087	456	23 011 195 895 870 800 000 000 000,00	1,00001724	23 011 592 656 012 300 000 000 000,00	23 011 989 461 651 200 000 000 000,00	0,99998276
25	42 652 618 343	464	573 946 744 068 867 000 000 000 000,00	1,01288977	581 344 786 051 355 000 000 000 000,00	586 716 086 234 403 000 000 000 000,00	0,99084515
26	2 614 941 710 599	652	14 798 170 925 310 900 000 000 000 000,00	1,11625268	16 518 497 946 691 300 000 000 000 000,00	17 286 874 263 177 600 000 000 000 000,00	0,95555146
27	2 614 941 710 599	652	490 465 738 715 257 000 000 000 000 000,00	1,00000123	490 466 342 952 388 000 000 000 000 000,00	490 467 598 386 435 000 000 000 000 000,00	0,99999744
28	2 614 941 710 599	652	14 542 905 529 616 300 000 000 000 000 000,00	1,00000187	14 542 932 725 262 400 000 000 000 000 000,00	14 542 960 671 127 900 000 000 000 000 000,00	0,99999808
29	19 581 334 192 423	766	430 665 088 119 679 000 000 000 000 000,00	1,05459080	454 175 441 613 020 000 000 000 000 000,00	460 991 742 408 187 000 000 000 000 000 000,00	0,98521383
30	19 581 334 192 423	766	14 379 621 528 494 800 000 000 000 000 000,00	0,99999813	14 379 594 583 688 900 000 000 000 000 000,00	14 379 642 481 382 800 000 000 000 000 000 000,00	0,99999667

II.6 – Analysis of the integer sequence of k, (Pn,k):

When we look at the numerical results of the table (II.3) and (II.4), we can see that the values of the prime numbers corresponding to the maximal values of $L_k(p_n)$, is a growing integer sequence. We are going to prove this assertion below .

If we consider that the relation (II.17):

$$\frac{k}{k-1} \cdot \ln(p_n) \cdot C(p_n) < \frac{L_k(p_n)}{L_{k-1}(p_n)} < \frac{k}{k-1} \cdot \frac{\ln(p_n)}{C(p_n)}$$

$$\frac{k}{k-1} \cdot \ln(p_n) \cdot C(p_n) \cdot L_{k-1}(p_n) < L_k(p_n) < \frac{k}{k-1} \cdot \frac{\ln(p_n)}{C(p_n)} \cdot L_{k-1}(p_n)$$

$$\frac{k}{k-1} \cdot \ln(p_n) \cdot (1 - C(p_n)) \cdot L_{k-1}(p_n) < \frac{k}{k-1} \cdot \ln(p_n) \cdot L_{k-1}(p_n) - L_k(p_n) < \frac{k}{k-1} \cdot \ln(p_n) \cdot \left(\frac{1-C(p_n)}{C(p_n)}\right) \cdot L_{k-1}(p_n) \quad (II.19)$$

As $C(p_n) = \frac{\left(1 - \frac{1}{2p_n^{1-a}}\right)}{\left(1 + \frac{1}{p_n^{1-a}}\right)} < 1$ we have $1 - C(p_n) = 1 - \left(\frac{1 - \frac{1}{2p_n^{1-a}}}{1 + \frac{1}{p_n^{1-a}}}\right) = \left(\frac{\frac{3}{2p_n^{1-a}}}{1 + \frac{1}{p_n^{1-a}}}\right)$

then for $p_n > 3$ we have: $\frac{1}{p_n^{1-a}} < 1 - C(p_n) < \frac{3}{2p_n^{1-a}}$ and $\frac{1}{2} < C(p_n) < 1$

We can replace into (II.19):

$$0 < \frac{k}{k-1} \cdot \frac{\ln(p_n)}{p_n^{1-a}} \cdot L_{k-1}(p_n) < \frac{k}{k-1} \cdot \ln(p_n) \cdot L_{k-1}(p_n) - L_k(p_n) < \frac{3k}{k-1} \cdot \frac{\ln(p_n)}{p_n^{1-a}} \cdot L_{k-1}(p_n)$$

$$0 < \frac{1}{p_n^{1-a}} < 1 - \frac{L_k(p_n)}{\frac{k}{k-1} \ln(p_n) \cdot L_{k-1}(p_n)} < \frac{3}{p_n^{1-a}}$$

$$\frac{k}{k-1} \cdot \ln(p_n) \cdot L_{k-1}(p_n) \cdot \left(1 - \frac{3}{p_n^{1-a}}\right) < L_k(p_n) < \frac{k}{k-1} \cdot \ln(p_n) \cdot L_{k-1}(p_n) \cdot \left(1 - \frac{1}{p_n^{1-a}}\right) \quad (II.20)$$

(II.20) shows that $L_k(p_n) = \frac{k}{k-1} \cdot \ln(p_n) \cdot L_{k-1}(p_n) \cdot \left(1 - O\left(\frac{1}{p_n^{1-a}}\right)\right)$ with $\frac{1}{p_n^{1-a}} < O\left(\frac{1}{p_n^{1-a}}\right) < \frac{3}{p_n^{1-a}}$ (II.21)

Which means that we can link the behavior of $L_k(p_n)$ to the behavior of $L_{k-1}(p_n)$. We are going now to compute the first derivative of the two parts of (II.21).

$$\frac{dL_k(p_n)}{dp_n} = \frac{k}{k-1} \cdot \left(L_{k-1}(p_n) \cdot \left(\frac{1}{p_n} + \ln(p_n) \cdot O\left(\frac{1}{p_n^{2-a}}\right) \right) + \ln(p_n) \cdot \frac{dL_{k-1}(p_n)}{dp_n} \right) \quad (II.22)$$

Considering the prime number $p_{n,k-1}$ defined by $L_{k-1}(p_n) = \text{Max}_{p_n \in \mathbb{P}} L_{k-1}(p_n)$ and which imply $\frac{dL_{k-1}(p_n)}{dp_n} = 0$

then : $\frac{dL_k(p_{n,k-1})}{dp_n} = \frac{k}{k-1} \cdot \left(\frac{L_{k-1}(p_{n,k-1})}{p_{n,k-1}} + \ln(p_{n,k-1}) \cdot \frac{dL_{k-1}(p_{n,k-1})}{dp_n} \right) = \frac{k}{k-1} \cdot \frac{L_{k-1}(p_{n,k-1})}{p_{n,k-1}} > 0$ (II.23)

Which means that for all integer k , $p_{n,k-1} \leq p_{n,k}$

Then we can conclude that for $p_{n,k}$ enough large, $p_{n,k-1} \leq p_{n,k}$ and we can verify in tables 1 and 2 that it is true for all $p_{n,k}$.

II.6 - $G_{k,r}(p_n) = \ln^k(p_{n+1})^r - \ln^k(p_n)^r$ bounds estimation from recurrence on k

The function $G_{k,r}(p_n)$ is linked to $L_k(p_n)$ by : $G_{k,r}(p_n) = \ln^k(p_{n+1})^r - \ln^k(p_n)^r = r^k L_k(p_n)$

All the proofs, in the previous paragraphs, regarding $L_k(p_n)$ are systematically applicable to $G_{k,r}(p_n)$

$$G_1(p_n) = \ln(p_{n+1})^r - \ln(p_n)^r < 0,52 * r \quad G_2(p_n) = \ln^2(p_{n+1})^r - \ln^2(p_n)^r < 2 * r^2$$

$$G_3(p_n) = \ln^3(p_{n+1})^r - \ln^3(p_n)^r < 4 * r^3 \quad G_4(p_n) = \ln^4(p_{n+1})^r - \ln^4(p_n)^r < 52 * r^4$$

$$G_5(p_n) = \ln^5(p_{n+1})^r - \ln^5(p_n)^r < 341 * r^5 \quad G_6(p_n) = \ln^6(p_{n+1})^r - \ln^6(p_n)^r < 2944 * r^6$$

$$G_7(p_n) = \ln^7(p_{n+1})^r - \ln^7(p_n)^r < 24741 * r^7$$

II.7 - $L_y(p_n) = \ln^y(p_{n+1}) - \ln^y(p_n) = 1$ has a solution for all p_n

As proved in (II.4), we can write for all real $y \in \mathbb{R}^+$

$$M_y(p_n) \cdot g_n \cdot \left(1 - \frac{g_n}{2p_n}\right) < L_y(p_n) < M_y(p_n) \cdot g_n \cdot \left(1 + \frac{g_n}{p_n}\right) \quad \text{for } p_n > e^{k-1}$$

With $M_y(p_n) = y \cdot \frac{\ln^{y-1}(p_n)}{p_n}$. So we can focus our study on $M_y(p_n) \cdot g_n$ which represent a very good approximation of $L_y(p_n)$. We are going to show that $M_y(p_n) \cdot g_n = 1$ has a solution $y=y(p_n)$ for all p_n .

$$M_y(p_n) \cdot g_n = y \cdot \frac{\ln^{y-1}(p_n)}{p_n} \cdot g_n = 1 \quad \Leftrightarrow \quad y \cdot \ln^{y-1}(p_n) = \frac{p_n}{g_n} \quad (\text{II.24})$$

We have shown previously that $0 < L_y(p_n) < 1$, so we can insure that if y satisfy (II.24), then $y(p_n) > 1$ for all p_n

We can take the logarithm of booth parts of (II.24):

$$\ln(y) + (y - 1) \cdot \ln(\ln(p_n)) = \ln\left(\frac{p_n}{g_n}\right) \quad (\text{II.25})$$

We may have a first solution of (II.25), supposing that $\ln(y)$ is very small :

$$y_0 - 1 \approx \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n))} \quad (\text{II.26})$$

We can see in table II.5 below that the values of y_0 correspond to the value of $L_{y_0}(p_n)$ which is normal because we have neglected the term $\ln(y)$ into (II.25). For all p_n , $L_{y_0}(p_n) > 1$

We are going to refine this first solution by searching ε such as $y_1 = y_0 - \varepsilon_0$

$$y_1 - 1 = \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon_0 \quad \text{and} \quad y_1 = \frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon_0 \quad (\text{II.27})$$

We replace (II.27) into (II.24) and we obtain :

$$\left(\frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon_0\right) \cdot \ln^{\frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right) - \varepsilon_0}{\ln(\ln(p_n))}}(p_n) = \frac{p_n}{g_n}$$

$$\left(\frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon\right) \cdot e^{\ln\left(\frac{p_n}{g_n}\right) - \varepsilon_0 \cdot \ln(\ln(p_n))} = \frac{p_n}{g_n}$$

$$\left(\frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon\right) \cdot e^{-\varepsilon_0 \cdot \ln(\ln(p_n))} = 1$$

Considering $\varepsilon \cdot \ln(\ln(p_n))$ enough small, we can replace $e^{-\varepsilon_0 \cdot \ln(\ln(p_n))}$ by its taylor serie expansion

$$\left(\frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right)}{\ln(\ln(p_n))} - \varepsilon_0\right) \cdot (1 - \varepsilon_0 \cdot \ln(\ln(p_n))) \approx 1$$

$$\left(\frac{\ln\left(\frac{p_n \cdot \ln(p_n)}{g_n}\right)}{\ln(\ln(p_n))} - 1\right) \approx \varepsilon_0 \cdot \left(\ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right) + 1\right)$$

Which gives an estimation of ε : $\varepsilon_0 \approx \frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}$ then $y_1 - 1 \approx \frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}$ (II.28)

In table II.5, we can see that the values of $L_{y_0}(p_n)$ are closer to 1, with $L_{y_0}(p_n) > L_{y_1}(p_n) > 1$. We are going to iterate a second time this process to have a better estimation of y .

We define $y_2 = y_1 - \varepsilon_1$ $y_2 - 1 = \frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} - \varepsilon_1$ (II.29)

If we replace y_2 into (II.24), we have :

$$\begin{aligned} & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - \varepsilon_1 \right) \cdot e^{\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} - \varepsilon_1 \cdot \ln(\ln(p_n))} = \frac{p_n}{g_n} \\ & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - \varepsilon_1 \right) \cdot e^{\ln\left(\frac{p_n}{g_n}\right) - 1 + \frac{\ln(e \cdot \ln(p_n))}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} - \varepsilon_1 \cdot \ln(\ln(p_n))} = \frac{p_n}{g_n} \\ & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - \varepsilon_1 \right) \cdot e^{-1 + \frac{\ln(e \cdot \ln(p_n))}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} - \varepsilon_1 \cdot \ln(\ln(p_n))} = 1 \\ & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - \varepsilon_1 \right) \cdot e^{-\varepsilon_1 \cdot \ln(\ln(p_n))} = e^{1 - \frac{\ln(e \cdot \ln(p_n))}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}} \end{aligned} \quad (II.30)$$

As before, we can replace $e^{-\varepsilon_0 \cdot \ln(\ln(p_n))}$ and the second term of (II.28) by its taylor serie expansion:

$$\begin{aligned} & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - \varepsilon_1 \right) \cdot (1 - \varepsilon_1 \cdot \ln(\ln(p_n))) \approx e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}} \\ & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}} \right) \approx \varepsilon_1 \cdot \left(1 + \frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} \right) \\ & \left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} + 1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}} \right) \approx \varepsilon_1 \cdot \left(1 - \frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} \right) \\ & \varepsilon_1 \approx \frac{\left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n))} + \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right) \cdot \left(1 - e^{\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}} \right) \right)}{\left(\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right) + \ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right) \right)} \end{aligned}$$

$$y_2 \approx 1 + \frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n)) \cdot \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)} - \frac{\left(\frac{\ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right)}{\ln(\ln(p_n))} + \ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right) \cdot \left(1 - e^{-\frac{\ln\left(\frac{p_n}{g_n}\right)}{\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right)}} \right) \right)}{\left(\ln\left(e \cdot \frac{p_n}{g_n} \cdot \ln(p_n)\right) + \ln\left(\frac{p_n}{g_n}\right) \cdot \ln\left(\frac{p_n}{g_n} \cdot \ln(p_n)\right) \right)}$$

In table II.5, y_2 seems to give a good estimation of y (represented by y_{ref}), with an accuracy better than 1,5% .

Primes	Gaps	ln(Pn)	Y ₀	M _{Y0}	ε ₀	Y ₁	M _{Y1}	ε ₁	Y ₂	M _{Y2}	Y _{ref}	M _{Yref}	Y _{ref} -Y ₂
2	1	0,69	-0,89	-0,47	-1,43	0,53	0,23	-0,98	1,52	0,58	3,01	1,00	1,49
3	2	1,10	5,31	10,87	2,88	2,44	1,93	0,99	1,44	0,84	1,62	1,00	0,18
7	4	1,95	1,84	1,60	0,38	1,46	0,95	0,14	1,33	0,77	1,50	1,00	0,17
23	6	3,14	2,18	2,02	0,34	1,84	1,15	0,19	1,65	0,82	1,76	1,00	0,11
89	8	4,49	2,60	2,53	0,33	2,28	1,35	0,22	2,06	0,87	2,13	1,01	0,07
113	14	4,73	2,34	2,25	0,29	2,05	1,25	0,18	1,87	0,85	1,95	1,01	0,08
523	18	6,26	2,84	2,80	0,30	2,54	1,46	0,21	2,33	0,90	2,37	0,99	0,04
887	20	6,79	2,98	2,96	0,30	2,68	1,51	0,22	2,47	0,91	2,50	0,99	0,03
1 129	22	7,03	3,02	3,00	0,29	2,73	1,53	0,22	2,51	0,92	2,55	1,01	0,04
1 327	34	7,19	2,86	2,83	0,28	2,58	1,47	0,20	2,37	0,90	2,41	0,99	0,04
9 551	36	9,16	3,52	3,51	0,29	3,23	1,71	0,23	3,01	0,96	3,02	1,00	0,01
15 683	44	9,66	3,59	3,59	0,28	3,31	1,74	0,23	3,08	0,97	3,09	0,99	0,01
19 609	52	9,88	3,59	3,59	0,28	3,31	1,74	0,22	3,09	0,97	3,10	1,01	0,01
31 397	72	10,35	3,60	3,60	0,28	3,32	1,74	0,22	3,10	0,97	3,11	0,99	0,01
155 921	86	11,96	4,02	4,02	0,28	3,75	1,89	0,23	3,52	1,01	3,52	1,01	0,00
360 653	96	12,80	4,23	4,23	0,27	3,95	1,97	0,23	3,73	1,04	3,71	0,99	-0,02
370 261	112	12,82	4,18	4,18	0,27	3,90	1,95	0,23	3,68	1,03	3,67	1,00	-0,01
492 113	114	13,11	4,25	4,25	0,27	3,98	1,97	0,23	3,75	1,04	3,74	1,00	-0,01
1 349 533	118	14,12	4,53	4,53	0,27	4,26	2,07	0,23	4,03	1,07	4,01	1,01	-0,02
1 357 201	132	14,12	4,49	4,49	0,27	4,22	2,06	0,23	3,99	1,07	3,97	1,00	-0,02
2 010 733	148	14,51	4,56	4,56	0,27	4,29	2,08	0,23	4,06	1,07	4,04	1,01	-0,02
4 652 353	154	15,35	4,78	4,78	0,27	4,51	2,16	0,23	4,28	1,10	4,25	1,01	-0,03
17 051 707	180	16,65	5,07	5,07	0,27	4,81	2,27	0,23	4,58	1,13	4,54	1,01	-0,04
20 831 323	210	16,85	5,07	5,07	0,27	4,81	2,27	0,23	4,58	1,13	4,54	1,01	-0,04
47 326 693	220	17,67	5,28	5,28	0,26	5,01	2,34	0,23	4,78	1,16	4,73	0,99	-0,05
122 164 747	222	18,62	5,52	5,52	0,26	5,26	2,43	0,23	5,03	1,19	4,97	0,99	-0,06
189 695 659	234	19,06	5,62	5,62	0,26	5,35	2,47	0,23	5,12	1,20	5,07	1,01	-0,05
191 912 783	248	19,07	5,60	5,60	0,26	5,34	2,46	0,23	5,11	1,20	5,05	1,00	-0,06
387 096 133	250	19,77	5,78	5,78	0,26	5,51	2,52	0,23	5,28	1,22	5,22	0,99	-0,06
436 273 009	282	19,89	5,77	5,77	0,26	5,50	2,52	0,23	5,28	1,22	5,21	0,99	-0,07
1 294 268 491	288	20,98	6,03	6,03	0,26	5,77	2,62	0,23	5,54	1,25	5,47	0,99	-0,07
1 453 168 141	292	21,10	6,06	6,06	0,26	5,80	2,63	0,23	5,57	1,25	5,50	1,01	-0,07
2 300 942 549	320	21,56	6,14	6,14	0,26	5,88	2,66	0,23	5,65	1,26	5,58	0,99	-0,07
3 842 610 773	336	22,07	6,25	6,25	0,26	5,99	2,70	0,23	5,77	1,28	5,69	1,00	-0,08
4 302 407 359	354	22,18	6,26	6,26	0,26	6,01	2,70	0,23	5,78	1,28	5,70	0,99	-0,08
10 726 904 659	382	23,10	6,46	6,46	0,26	6,21	2,77	0,23	5,98	1,30	5,90	1,01	-0,08
20 678 048 297	384	23,75	6,62	6,62	0,26	6,36	2,83	0,23	6,14	1,32	6,05	1,00	-0,09
22 367 084 959	394	23,83	6,63	6,63	0,26	6,37	2,83	0,23	6,15	1,32	6,06	0,99	-0,09
25 056 082 087	456	23,94	6,61	6,61	0,26	6,36	2,83	0,23	6,13	1,32	6,05	1,00	-0,08
42 652 618 343	464	24,48	6,73	6,73	0,25	6,48	2,87	0,23	6,25	1,34	6,17	1,00	-0,09
127 976 334 671	468	25,58	6,99	6,99	0,25	6,74	2,97	0,23	6,51	1,37	6,42	1,00	-0,09
182 226 896 239	474	25,93	7,07	7,07	0,25	6,82	2,99	0,23	6,59	1,38	6,50	0,99	-0,10
241 160 624 143	486	26,21	7,13	7,13	0,25	6,88	3,02	0,23	6,65	1,39	6,56	1,00	-0,10
297 501 075 799	490	26,42	7,18	7,18	0,25	6,92	3,03	0,23	6,70	1,39	6,60	1,00	-0,10
303 371 455 241	500	26,44	7,18	7,18	0,25	6,92	3,03	0,23	6,70	1,39	6,60	1,00	-0,10
304 599 508 537	514	26,44	7,17	7,17	0,25	6,92	3,03	0,23	6,69	1,39	6,59	0,99	-0,10
416 608 695 821	516	26,76	7,24	7,24	0,25	6,99	3,06	0,23	6,76	1,40	6,66	0,99	-0,10
461 690 510 011	532	26,86	7,25	7,25	0,25	7,00	3,06	0,23	6,78	1,40	6,68	1,01	-0,10
614 487 453 523	534	27,14	7,32	7,32	0,25	7,07	3,09	0,23	6,84	1,41	6,74	0,99	-0,10
738 832 927 927	540	27,33	7,36	7,36	0,25	7,11	3,10	0,23	6,88	1,42	6,78	1,00	-0,10
1 346 294 310 749	582	27,93	7,48	7,48	0,25	7,23	3,14	0,23	7,00	1,43	6,90	1,01	-0,10
1 408 695 493 609	588	27,97	7,48	7,48	0,25	7,23	3,14	0,23	7,01	1,43	6,90	0,99	-0,11
1 968 188 556 461	602	28,31	7,55	7,55	0,25	7,30	3,17	0,23	7,08	1,44	6,97	0,99	-0,11
2 614 941 710 599	652	28,59	7,59	7,59	0,25	7,35	3,19	0,23	7,12	1,45	7,02	1,00	-0,10
7 177 162 611 713	674	29,60	7,82	7,82	0,25	7,57	3,27	0,23	7,34	1,47	7,23	1,00	-0,11
13 829 048 559 701	716	30,26	7,95	7,95	0,25	7,70	3,31	0,23	7,47	1,49	7,36	1,00	-0,11
19 581 334 192 423	766	30,61	8,00	8,01	0,25	7,76	3,34	0,23	7,53	1,50	7,42	1,00	-0,11
42 842 283 925 351	778	31,39	8,18	8,18	0,25	7,93	3,40	0,22	7,71	1,52	7,59	1,01	-0,12
90 874 329 411 493	804	32,14	8,33	8,34	0,25	8,09	3,46	0,22	7,86	1,54	7,75	1,00	-0,12
171 231 342 420 521	806	32,77	8,47	8,46	0,24	8,23	3,50	0,22	8,01	1,56	7,89	1,02	-0,12
218 209 405 436 543	906	33,02	8,49	8,50	0,24	8,25	3,51	0,22	8,03	1,56	7,90	0,99	-0,13

Part 3 : A third variant of Andrica's conjecture

We propose in this part to study the behavior of a third variant of the Generalized Andrica's conjecture represented by the set of function $H_{k,x}(p_n) = (p_{n+1})^x \cdot \ln^k(p_{n+1}) - (p_n)^x \cdot \ln^k(p_n)$ and the values of $x \in \mathbb{R}$ and $k \in \mathbb{N}$ where $H_{k,x}(p_n)$ is bounded.

III.1 – Upper bound functions of $H_{k,x}(p_n)$:

$$H_{k,x}(p_n) = (p_{n+1})^x \cdot \ln^k(p_{n+1}) - (p_n)^x \cdot \ln^k(p_{n+1}) + (p_n)^x \cdot \ln^k(p_{n+1}) - (p_n)^x \cdot \ln^k(p_n)$$

$$H_{k,x}(p_n) = ((p_{n+1})^x - (p_n)^x) \cdot \ln^k(p_{n+1}) + (p_n)^x \cdot (\ln^k(p_{n+1}) - \ln^k(p_n)) \quad (III.1)$$

$$H_{k,x}(p_n) < \left(x \cdot (p_n)^x \cdot \frac{g_n}{p_n} \right) \ln^k(p_{n+1}) + (p_n)^x \cdot \left(k \cdot \frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} \cdot g_n \cdot \left(1 + \frac{g_n}{p_n} \right) \right)$$

$$H_{k,x}(p_n) < x \cdot (p_n)^x \cdot \frac{\ln^k(p_{n+1})}{p_{n+1}} \cdot g_n \cdot \left(1 + \frac{g_n}{p_n} \right) + k \cdot (p_n)^x \cdot \frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} \cdot g_n \cdot \left(1 + \frac{g_n}{p_n} \right)$$

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} \cdot g_n (x \cdot \ln(p_{n+1}) + k) \cdot \left(1 + \frac{g_n}{p_n} \right)$$

As we have state in (II.2), for all $p_n > e^{k-1}$, $\frac{\ln^{k-1}(p_{n+1})}{p_{n+1}} < \frac{\ln^{k-1}(p_n)}{p_n}$

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \left(x \cdot \left(\ln(p_n) + \ln \left(1 + \frac{g_n}{p_n} \right) \right) + k \right) \cdot \left(1 + \frac{g_n}{p_n} \right)$$

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \left(x \cdot \left(\ln(p_n) + \frac{g_n}{p_n} - \frac{1}{2} \left(\frac{g_n}{p_n} \right)^2 \right) + k \right) \cdot \left(1 + \frac{g_n}{p_n} \right)$$

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \left(x \cdot \ln(p_n) + k + (x \cdot \ln(p_n) + k + x) \cdot \frac{g_n}{p_n} + \frac{x}{2} \left(\frac{g_n}{p_n} \right)^2 \right)$$

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \cdot (x \cdot \ln(p_n) + k) \cdot \left(1 + \frac{(x \cdot \ln(p_n) + k + 2x) \cdot g_n}{x \cdot \ln(p_n) + k} \cdot \frac{g_n}{p_n} \right)$$

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^{k-1}(p_n)}{p_n} \cdot g_n \cdot (x \cdot \ln(p_n) + k) \cdot \left(1 + \frac{\left(x + \frac{k+2x}{\ln(p_n)} \right) \cdot g_n}{x + \frac{k}{\ln(p_n)}} \cdot \frac{g_n}{p_n} \right)$$

Thanks to BHP result about prime gaps, $g_n < p_n^a$ with $a = 0,525$.

$$H_{k,x}(p_n) < (p_n)^x \cdot \frac{\ln^k(p_n)}{p_n} \cdot g_n \cdot \left(x + \frac{k}{\ln(p_n)} \right) \cdot \left(1 + \frac{1}{p_n^{1-a}} \cdot \frac{\left(x + \frac{k+2x}{\ln(p_n)} \right)}{x + \frac{k}{\ln(p_n)}} \right) \quad (III.2)$$

We can reformulate $\frac{\left(x + \frac{k+2x}{\ln(p_n)} \right)}{x + \frac{k}{\ln(p_n)}} = 1 + \frac{2x}{x \cdot \ln(p_n) + k} < 2$ for all integer k and all $x < 1$ and all prime number

We define $M_{k,x}(p_n) = \frac{\ln^k(p_n)}{p_n^{1-x}} \cdot g_n \cdot \left(x + \frac{k}{\ln(p_n)} \right)$ and $\varepsilon = \frac{2}{p_n^{1-a}}$.

$$H_{k,x}(p_n) < M_{k,x}(p_n) \cdot (1 + \varepsilon) \quad (III.3)$$

III.2 – Lower bound functions of $H_{k,x}(p_n)$

We start from (III.1) : $H_{k,x}(p_n) = ((p_{n+1})^x - (p_n)^x) \cdot \ln^k(p_{n+1}) + (p_n)^x \cdot (\ln^k(p_{n+1}) - \ln^k(p_n))$ (III.4)

As $\ln^k(p_{n+1}) > \ln^k(p_n)$ and $\ln^k(p_{n+1}) - \ln^k(p_n) > k \cdot \ln^{k-1}(p_n) \cdot (\ln(p_{n+1}) - \ln(p_n))$

$$H_{k,x}(p_n) > (p_n)^x \cdot \left(\left(1 + \frac{g_n}{p_n}\right)^x - 1 \right) \cdot \ln^k(p_{n+1}) + k \cdot (p_n)^x \cdot \ln^{k-1}(p_n) \cdot (\ln(p_{n+1}) - \ln(p_n))$$

$$H_{k,x}(p_n) > (p_n)^x \cdot \left(\left(1 + \frac{g_n}{p_n}\right)^x - 1 \right) \cdot \ln^k(p_{n+1}) + k \cdot (p_n)^x \cdot \ln^{k-1}(p_n) \cdot \ln \left(1 + \frac{g_n}{p_n}\right)$$

Thanks to $\left(1 + \frac{g_n}{p_n}\right)^x - 1 > x \cdot \frac{g_n}{p_n} \cdot \left(1 - \frac{1-x}{2} \cdot \frac{g_n}{p_n}\right)$ and $\ln \left(1 + \frac{g_n}{p_n}\right) > \frac{g_n}{p_n} \cdot \left(1 - \frac{1}{2} \cdot \frac{g_n}{p_n}\right)$

$$H_{k,x}(p_n) > (p_n)^x \cdot x \cdot \frac{g_n}{p_n} \cdot \left(1 - \frac{1-x}{2} \cdot \frac{g_n}{p_n}\right) \cdot \ln^k(p_{n+1}) + k \cdot (p_n)^x \cdot \ln^{k-1}(p_n) \cdot \frac{g_n}{p_n} \cdot \left(1 - \frac{1}{2} \cdot \frac{g_n}{p_n}\right)$$

$$H_{k,x}(p_n) > (p_n)^x \cdot \ln^k(p_{n+1}) \cdot \frac{g_n}{p_n} \cdot \left(x \cdot \left(1 - \frac{1-x}{2} \cdot \frac{g_n}{p_n}\right) + \frac{k}{\ln(p_n)} \cdot \left(1 - \frac{1}{2} \cdot \frac{g_n}{p_n}\right) \right)$$

$$H_{k,x}(p_n) > (p_n)^x \cdot \ln^k(p_{n+1}) \cdot \frac{g_n}{p_n} \cdot \left(x \cdot \left(1 - \frac{1-x}{2} \cdot \frac{g_n}{p_n}\right) + \frac{k}{\ln(p_n)} \cdot \left(1 - \frac{1}{2} \cdot \frac{g_n}{p_n}\right) \right)$$

$$H_{k,x}(p_n) > (p_n)^x \cdot \ln^k(p_{n+1}) \cdot \frac{g_n}{p_n} \cdot \left(x + \frac{k}{\ln(p_n)} - \frac{g_n}{2p_n} \cdot \left(x(1-x) + \frac{k}{\ln(p_n)} \right) \right)$$

$$H_{k,x}(p_n) > (p_n)^x \cdot \ln^k(p_{n+1}) \cdot \frac{g_n}{p_n} \cdot \left(x + \frac{k}{\ln(p_n)} \right) \cdot \left(1 - \frac{g_n}{2p_n} \cdot \frac{x(1-x) + \frac{k}{\ln(p_n)}}{x + \frac{k}{\ln(p_n)}} \right)$$

Thanks to BHP result about prime gaps, $g_n < p_n^a$ with $a = 0,525$.

$$H_{k,x}(p_n) > (p_n)^x \cdot \ln^k(p_{n+1}) \cdot \frac{g_n}{p_n} \cdot \left(x + \frac{k}{\ln(p_n)} \right) \cdot \left(1 - \frac{1}{2p_n^{1-a}} \cdot \frac{x(1-x) + \frac{k}{\ln(p_n)}}{x + \frac{k}{\ln(p_n)}} \right) > M_{k,x}(p_n) \cdot \left(1 - \frac{1}{2p_n^{1-a}} \right) \quad (III.5)$$

As we can easily verify that $\frac{x(1-x) + \frac{k}{\ln(p_n)}}{x + \frac{k}{\ln(p_n)}} < 1$ thus $1 - \frac{1}{2p_n^{1-a}} \cdot \frac{x(1-x) + \frac{k}{\ln(p_n)}}{x + \frac{k}{\ln(p_n)}} > 1 - \frac{1}{2p_n^{1-a}}$

We define $\varepsilon' = \frac{1}{2p_n^{1-a}}$, we can state:

$$M_{k,x}(p_n) \cdot (1 - \varepsilon') < H_{k,x}(p_n) \quad (III.6)$$

We can finally give the lower and upper bound functions of $H_{k,x}(p_n)$:

$$M_{k,x}(p_n) \cdot (1 - \varepsilon') < H_{k,x}(p_n) < M_{k,x}(p_n) \cdot (1 + \varepsilon) \quad (III.7)$$

As ε and ε' tends toward to 0 when $n \rightarrow \infty$, we can confirm that $M_{k,x}(p_n)$ represents a good estimator $H_{k,x}(p_n)$ and we can limit our study in the next paragraphs to $M_{k,x}(p_n)$ behavior.

III.3 – Behavior of $M_{k,x}(p_n)$

At first, we can easily state that $M_{k,x}(p_n) = \frac{\ln^k(p_n)}{p_n^{1-x}} \cdot g_n \cdot \left(x + \frac{k}{\ln(p_n)}\right)$ is a bounded function, for all integer k and $x < 1-a$, which tends toward 0 when $n \rightarrow \infty$.

$$0 < M_{k,x}(p_n) < \frac{\ln^k(p_n)}{p_n^{1-a-x}} \cdot \left(x + \frac{k}{\ln(p_n)}\right) \quad \text{and} \quad \lim_{p_n \in \mathbb{P}} M_{k,x}(p_n) < \lim_{p_n \in \mathbb{P}} \left(\frac{\ln^k(p_n)}{p_n^{1-a-x}} \cdot \left(x + \frac{k}{\ln(p_n)}\right)\right) = 0$$

We are going to compute the derivative function of $M_{k,x}(p_n)$

To derivate $M_{k,x}(p_n)$ we shall consider that g is a derivable function on \mathbb{R} which satisfy $g_n = g(p_n)$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \left(\frac{k \ln^{k-1}(p_n)}{p_n^{2-x}} - (1-x) \frac{\ln^k(p_n)}{p_n^{2-x}}\right) \cdot g_n \cdot \left(x + \frac{k}{\ln(p_n)}\right) - \frac{\ln^k(p_n)}{p_n^{1-x}} \cdot g_n \cdot \frac{k}{p_n \cdot \ln^2(p_n)} + \frac{\ln^k(p_n)}{p_n^{1-x}} \cdot g'_n \cdot \left(x + \frac{k}{\ln(p_n)}\right)$$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \frac{\ln^{k-2}(p_n)}{p_n^{2-x}} \cdot \left((k \ln(p_n) - (1-x) \cdot \ln^2(p_n)) \cdot \left(x + \frac{k}{\ln(p_n)}\right) \cdot g_n - k \cdot g_n + \ln^2(p_n) \cdot p_n \cdot g'_n \cdot \left(x + \frac{k}{\ln(p_n)}\right) \right)$$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \frac{\ln^{k-2}(p_n)}{p_n^{2-x}} \cdot \left((k \ln(p_n) - (1-x) \cdot \ln^2(p_n)) \cdot \left(x + \frac{k}{\ln(p_n)}\right) \cdot g_n - k \cdot g_n + \ln^2(p_n) \cdot p_n \cdot g'_n \cdot \left(x + \frac{k}{\ln(p_n)}\right) \right)$$

$$\frac{dM_{k,x}(p_n)}{dp_n} = \frac{\ln^{k-2}(p_n)}{p_n^{2-x}} \cdot g_n \cdot \left(x \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + x \right) \cdot \ln^2(p_n) + k \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + 2x \right) \ln(p_n) + k(k-1) \right) \quad (\text{III.8})$$

If we consider that $p_{n,(k,x)}$ is the prime number such as $M_{k,x}(p_n)$ is maximal then $p_{n,(k,x)}$ is a solution of the equality

$$x \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + x \right) \cdot \ln^2(p_n) + k \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + 2x \right) \ln(p_n) + k(k-1) = 0 \quad (\text{III.9})$$

The positive solution of this equation is given by ;

$$\ln(p_n) = \frac{-k \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + 2x \right) + \sqrt{k^2 \left(p_n \cdot \frac{g'_n}{g_n} - 1 + 2x \right)^2 - 4k(k-1)x \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + x \right)}}{2x \cdot \left(p_n \cdot \frac{g'_n}{g_n} - 1 + x \right)}$$

Or

$$\ln(p_n) = \frac{k \cdot \left(1 - 2x - p_n \cdot \frac{g'_n}{g_n} \right) - \sqrt{k^2 \left(1 - 2x - p_n \cdot \frac{g'_n}{g_n} \right)^2 + 4k(k-1)x \cdot \left(1 - x - p_n \cdot \frac{g'_n}{g_n} \right)}}{2x \cdot \left(1 - x - p_n \cdot \frac{g'_n}{g_n} \right)}$$

This previous formulation has the inconvenient to be undetermined when $1 - x - p_n \cdot \frac{g'_n}{g_n} = 0$

For this reason we propose the following formulation which avoid this question:

$$\ln(p_n) = \frac{2(k-1)}{\sqrt{\left(1 - 2x - p_n \cdot \frac{g'_n}{g_n} \right)^2 + \frac{4(k-1)x}{k} \cdot \left(1 - x - p_n \cdot \frac{g'_n}{g_n} \right) + \left(1 - 2x - p_n \cdot \frac{g'_n}{g_n} \right)}} \quad (\text{III.10})$$

(III.10) gives an implicit formulation of $\ln(p_n)$ but we cannot obtain an explicit formulation. But it remains very interesting to estimate the upper bounds of lower or upper bound functions for which we can replace the term $p_n \cdot \frac{g'_n}{g_n}$ by a derivative function. We are going to develop this point below.

III.4 – Lower and upper bound functions independent from prime gaps :

As developed in Part 2, paragraph II.3, we'll limit our study to the set of prime numbers P_n corresponding to the first occurrence of each maximal gap G_n .

Let's define new lower and upper bound functions $H_{k,x}^-(P_n)$ and $H_{k,x}^+(P_n)$ totally independent from g_n .

Thanks to BHP result $G_n < P_n^a$ with $a = 0,525$: $H_{k,x}^+(P_n) = \frac{\ln^k(P_n)}{P_n^{1-x-a}} \cdot \left(x + \frac{k}{\ln(P_n)}\right) > M_{k,x}(P_n)$

Thanks to Erik Westzynthius result $G_n > \ln(P_n)$: $H_{k,x}^-(P_n) = \frac{\ln^{k+1}(P_n)}{P_n^{1-x}} \cdot \left(x + \frac{k}{\ln(P_n)}\right) < M_{k,x}(P_n)$

Thus we can compete (III.6):

$$H_{k,x}^-(P_n) \cdot \left(1 - \frac{1}{2P_n^{1-a}}\right) > M_{k,x}(P_n) \cdot \left(1 - \frac{1}{2P_n^{1-a}}\right) < H_{k,x}(P_n) < M_{k,x}(P_n) \cdot \left(1 + \frac{2}{P_n^{1-a}}\right) < H_{k,x}^+(P_n) \cdot \left(1 + \frac{2}{P_n^{1-a}}\right)$$

Thanks (III.9) we are not obliged to compute again the first derivative of $H_{k,x}^-(P_n)$ and $H_{k,x}^+(P_n)$

To get the prime P_n corresponding to the maximal value of $H_{k,x}^+(P_n)$ we have only to replace G_n by P_n^a into (III.9):

$$\ln(P_{n,(k,x)}^+) = \frac{2(k-1)}{\sqrt[2]{(1-2x-a)^2 + \frac{4(k-1)x}{k} \cdot (1-x-a) + (1-2x-a)}}$$

The table III.1 below gives the values $P_{n,(k,x)}^+$ for k and x . We can compare the values below with the prime numbers corresponding to the upper bound of $H_{k,x}^+(P_n)$ in tables III.3 up to III.6.

k\x	0,10	0,20	0,30	0,40
2	20	124	8 074	39 478 306 814
3	318	5 419	2 793 112	25 876 456 234 780 500
4	4 750	218 245	895 312 015	16 423 661 488 402 600 000 000
5	69 774	8 558 992	279 942 845 553	10 302 943 121 513 000 000 000 000 000
6	1 017 588	331 665 944	86 553 563 248 745	6 427 494 466 447 410 000 000 000 000 000 000
7	14 781 580	12 770 066 641	26 599 709 464 644 400	3 997 491 307 593 910 000 000 000 000 000 000 000 000
8	214 198 646	489 799 265 264	8 145 073 594 203 300 000	2 481 516 795 284 010 000 000 000 000 000 000 000 000 000 000

To get the prime P_n corresponding to the maximal value of $H_{k,x}^-(P_n)$ we have only to compute the derivative :

$$\frac{dH_{k,x}^-(P_n)}{dP_n} = \frac{\ln^{k-1}(P_n)}{P_n^{2-x}} \left(-x(1-x) \cdot \ln^2(p_n) - k \cdot \left(1 - 2x - \frac{x}{k}\right) \ln(p_n) + k^2\right) \quad (III.10)$$

Who has the following solution :

$$\ln(P_{n,(k,x)}^+) = \frac{2k}{\sqrt[2]{\left(1-2x-\frac{x}{k}\right)^2 + 4x \cdot (1-x) + \left(1-2x-\frac{x}{k}\right)}} \quad (III.11)$$

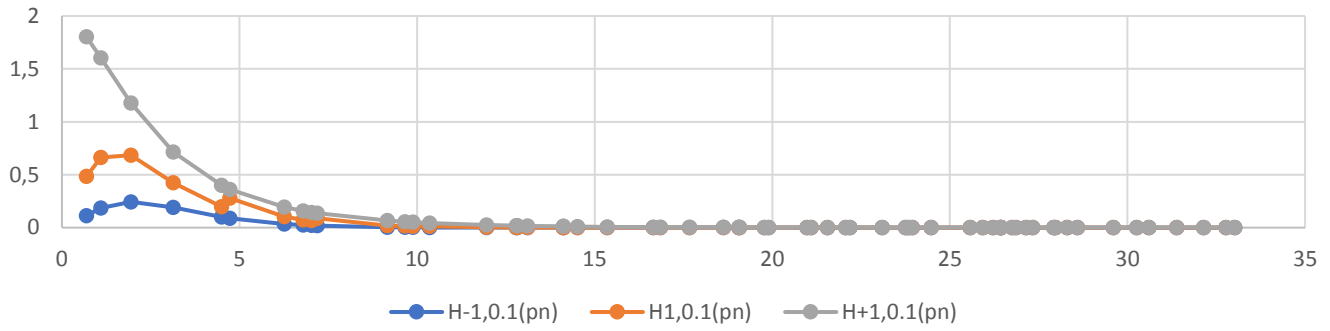
The table III.2 below gives the numerical values of $P_{n,(k,x)}^+$ estimated from (III.11) and must be compared with the true values of $P_{n,(k,x)}^+$ as it is appearing in tables III.3 up to III.6. The values below correspond to the integer which gives the maximal value of $H_{k,x}^-(P_n)$ while tables III.3 up to III.6 gives only the nearest prime corresponding to a first occurrence of the maximal gaps.

k\x	0,10	0,20	0,30	0,40
1	3	5	7	12
2	10	16	28	59
3	31	55	115	305
4	95	193	476	1594
5	290	671	1978	8371
6	880	2337	8228	44076
7	2671	8150	34257	232451
8	8113	28426	142709	1227131

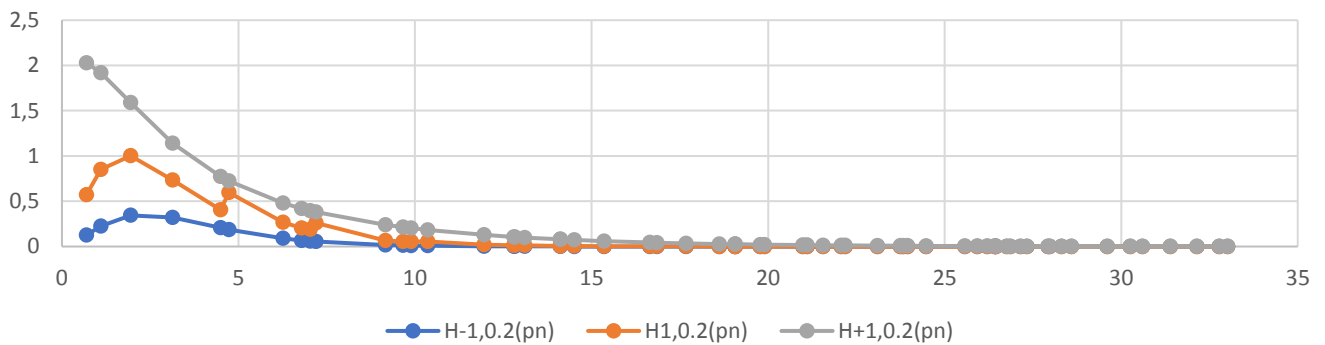
The table III.3 below give the numeral values of $H_{k,x}^-(p_n)$, $H_{k,x}(p_n)$ and $H_{k,x}^+(p_n)$ for $k=1$ and $x \in \{0.1, 0.2, 0.3, 0.4\}$ for all prime numbers corresponding to the first occurrence of the maximal gaps G_n .

Primes	Gaps	$\ln(p_n)$	$H_{1,0.1}^-(p_n)$	$H_{1,0.1}(p_n)$	$H_{1,0.1}^+(p_n)$	$H_{1,0.2}^-(p_n)$	$H_{1,0.2}(p_n)$	$H_{1,0.2}^+(p_n)$	$H_{1,0.3}^-(p_n)$	$H_{1,0.3}(p_n)$	$H_{1,0.3}^+(p_n)$	$H_{1,0.4}^-(p_n)$	$H_{1,0.4}(p_n)$	$H_{1,0.4}^+(p_n)$
k	x		1	1	1	1	1	1	1	1	1	1	1	1
			0,1	0,1	0,1	0,2	0,2	0,2	0,3	0,3	0,3	0,4	0,4	0,4
2	1	0,69	0,1114	0,4833	1,8023	0,1272	0,5724	2,0302	0,1446	0,6741	2,2815	0,1639	0,7903	2,5584
3	2	1,10	0,1844	0,6643	1,6031	0,2262	0,8520	1,9187	0,2752	1,0808	2,2860	0,3326	1,3589	2,7126
7	4	1,95	0,2433	0,6838	1,1766	0,3438	1,0018	1,5885	0,4761	1,4346	2,1231	0,6494	2,0193	2,8140
23	6	3,14	0,1898	0,4252	0,7149	0,3216	0,7331	1,1404	0,5249	1,2150	1,7824	0,8342	1,9591	2,7426
89	8	4,49	0,1009	0,1968	0,4000	0,2070	0,4062	0,7727	0,4010	0,7909	1,4393	0,7484	1,4834	2,6133
113	14	4,73	0,0884	0,2787	0,3617	0,1873	0,5954	0,7227	0,3736	1,1958	1,3881	0,7165	2,3090	2,5938
523	18	6,26	0,0345	0,1032	0,1940	0,0894	0,2678	0,4805	0,2137	0,6412	1,1186	0,4866	1,4626	2,5033
887	20	6,79	0,0243	0,0739	0,1581	0,0673	0,2050	0,4208	0,1710	0,5211	1,0449	0,4124	1,2585	2,4841
1129	22	7,03	0,0207	0,0665	0,1442	0,0589	0,1899	0,3966	0,1538	0,4961	1,0137	0,3808	1,2298	2,4769
1327	34	7,19	0,0185	0,0894	0,1357	0,0538	0,2608	0,3813	0,1431	0,6941	0,9936	0,3605	1,7516	2,4725
9551	36	9,16	0,0045	0,0180	0,0662	0,0168	0,0667	0,2403	0,0555	0,2207	0,7875	0,1727	0,6867	2,4360
15683	44	9,66	0,0031	0,0145	0,0557	0,0123	0,0567	0,2148	0,0431	0,1982	0,7443	0,1414	0,6500	2,4285
19609	52	9,88	0,0027	0,0142	0,0515	0,0107	0,0569	0,2042	0,0384	0,2038	0,7257	0,1289	0,6841	2,4250
31397	72	10,35	0,0019	0,0131	0,0438	0,0080	0,0558	0,1838	0,0300	0,2102	0,6879	0,1059	0,7414	2,4172
155921	86	11,96	0,0006	0,0040	0,0253	0,0028	0,0204	0,1285	0,0127	0,0914	0,5729	0,0528	0,3809	2,3834
360653	96	12,80	0,0003	0,0022	0,0191	0,0016	0,0122	0,1066	0,0080	0,0598	0,5198	0,0362	0,2720	2,3600
370261	112	12,82	0,0003	0,0025	0,0189	0,0016	0,0140	0,1060	0,0078	0,0687	0,5182	0,0357	0,3129	2,3592
492113	114	13,11	0,0002	0,0020	0,0172	0,0013	0,0115	0,0994	0,0067	0,0583	0,5012	0,0314	0,2735	2,3503
1349533	118	14,12	0,0001	0,0009	0,0122	0,0007	0,0056	0,0793	0,0038	0,0316	0,4447	0,0197	0,1646	2,3145
1357201	132	14,12	0,0001	0,0010	0,0122	0,0007	0,0063	0,0792	0,0038	0,0352	0,4444	0,0196	0,1835	2,3143
2010733	148	14,51	0,0001	0,0008	0,0107	0,0005	0,0052	0,0724	0,0030	0,0307	0,4239	0,0163	0,1664	2,2988
4652353	154	15,35	0,0000	0,0004	0,0080	0,0003	0,0029	0,0599	0,0018	0,0186	0,3827	0,0109	0,1098	2,2627
17051707	180	16,65	0,0000	0,0001	0,0052	0,0001	0,0013	0,0445	0,0009	0,0094	0,3257	0,0058	0,0632	2,1998
20831323	210	16,85	0,0000	0,0001	0,0048	0,0001	0,0013	0,0425	0,0008	0,0096	0,3176	0,0053	0,0660	2,1895
47326693	220	17,67	0,0000	0,0001	0,0037	0,0001	0,0007	0,0352	0,0005	0,0059	0,2862	0,0035	0,0441	2,1454
122164747	222	18,62	0,0000	0,0000	0,0027	0,0000	0,0004	0,0282	0,0003	0,0032	0,2533	0,0022	0,0264	2,0915
189695659	234	19,06	0,0000	0,0000	0,0023	0,0000	0,0003	0,0255	0,0002	0,0025	0,2392	0,0018	0,0218	2,0656
191912783	248	19,07	0,0000	0,0000	0,0023	0,0000	0,0003	0,0254	0,0002	0,0027	0,2389	0,0018	0,0229	2,0649
387096133	250	19,77	0,0000	0,0000	0,0018	0,0000	0,0002	0,0216	0,0001	0,0017	0,2178	0,0012	0,0157	2,0225
436273009	282	19,89	0,0000	0,0000	0,0017	0,0000	0,0002	0,0210	0,0001	0,0018	0,2144	0,0012	0,0165	2,0152
1294268491	288	20,98	0,0000	0,0000	0,0012	0,0000	0,0001	0,0162	0,0001	0,0009	0,1856	0,0007	0,0092	1,9474
1453168141	292	21,10	0,0000	0,0000	0,0011	0,0000	0,0001	0,0158	0,0001	0,0008	0,1827	0,0006	0,0088	1,9400
2300942549	320	21,56	0,0000	0,0000	0,0010	0,0000	0,0001	0,0142	0,0000	0,0007	0,1717	0,0005	0,0074	1,9107
3842610773	336	22,07	0,0000	0,0000	0,0008	0,0000	0,0000	0,0125	0,0000	0,0005	0,1602	0,0004	0,0059	1,8778
4302407359	354	22,18	0,0000	0,0000	0,0008	0,0000	0,0000	0,0122	0,0000	0,0005	0,1578	0,0004	0,0058	1,8705
10726904659	382	23,10	0,0000	0,0000	0,0006	0,0000	0,0000	0,0098	0,0000	0,0003	0,1393	0,0002	0,0037	1,8112
20678048297	384	23,75	0,0000	0,0000	0,0005	0,0000	0,0000	0,0084	0,0000	0,0002	0,1273	0,0002	0,0026	1,7684
22367084959	394	23,83	0,0000	0,0000	0,0004	0,0000	0,0000	0,0082	0,0000	0,0002	0,1259	0,0002	0,0026	1,7633
25056082087	456	23,94	0,0000	0,0000	0,0004	0,0000	0,0000	0,0080	0,0000	0,0002	0,1239	0,0001	0,0028	1,7559
42652618343	464	24,48	0,0000	0,0000	0,0004	0,0000	0,0000	0,0070	0,0000	0,0001	0,1151	0,0001	0,0021	1,7211
127976334671	468	25,58	0,0000	0,0000	0,0002	0,0000	0,0000	0,0054	0,0000	0,0001	0,0987	0,0001	0,0011	1,6495
182226896239	474	25,93	0,0000	0,0000	0,0002	0,0000	0,0000	0,0050	0,0000	0,0001	0,0939	0,0001	0,0009	1,6266
241160624143	486	26,21	0,0000	0,0000	0,0002	0,0000	0,0000	0,0046	0,0000	0,0000	0,0903	0,0000	0,0008	1,6084
297501075799	490	26,42	0,0000	0,0000	0,0002	0,0000	0,0000	0,0044	0,0000	0,0000	0,0877	0,0000	0,0007	1,5949
303371455241	500	26,44	0,0000	0,0000	0,0002	0,0000	0,0000	0,0044	0,0000	0,0000	0,0874	0,0000	0,0007	1,5936
304599508537	514	26,44	0,0000	0,0000	0,0002	0,0000	0,0000	0,0044	0,0000	0,0000	0,0874	0,0000	0,0008	1,5934
416608695821	516	26,76	0,0000	0,0000	0,0002	0,0000	0,0000	0,0040	0,0000	0,0000	0,0836	0,0000	0,0006	1,5732
461690510011	532	26,86	0,0000	0,0000	0,0002	0,0000	0,0000	0,0039	0,0000	0,0000	0,0824	0,0000	0,0006	1,5666
614487453523	534	27,14	0,0000	0,0000	0,0001	0,0000	0,0000	0,0037	0,0000	0,0000	0,0791	0,0000	0,0005	1,5483
738832927927	540	27,33	0,0000	0,0000	0,0001	0,0000	0,0000	0,0035	0,0000	0,0000	0,0770	0,0000	0,0005	1,5366
1346294310749	582	27,93	0,0000	0,0000	0,0001	0,0000	0,0000	0,0030	0,0000	0,0000	0,0707	0,0000	0,0004	1,4985
1408695493609	588	27,97	0,0000	0,0000	0,0001	0,0000	0,0000	0,0030	0,0000	0,0000	0,0703	0,0000	0,0004	1,4956
1968188556461	602	28,31	0,0000	0,0000	0,0001	0,0000	0,0000	0,0028	0,0000	0,0000	0,0670	0,0000	0,0003	1,4746
2614941710599	652	28,59	0,0000	0,0000	0,0001	0,0000	0,0000	0,0026	0,0000	0,0000	0,0643	0,0000	0,0003	1,4568
7177162611713	674	29,60	0,0000	0,0000	0,0001	0,0000	0,0000	0,0020	0,0000	0,0000	0,0556	0,0000	0,0002	1,3944
13829048559701	716	30,26	0,0000	0,0000	0,0000	0,0000	0,0000	0,0017	0,0000	0,0000	0,0505	0,0000	0,0001	1,3546
19581334192423	766	30,61	0,0000	0,0000	0,0000	0,0000	0,0000	0,0016	0,0000	0,0000	0,0481	0,0000	0,0001	1,3337
42842283925351	778	31,39	0,0000	0,0000	0,0000	0,0000	0,0000	0,0013	0,0000	0,0000	0,0429	0,0000	0,0001	1,2874
90874329411493	804	32,14	0,0000	0,0000	0,0000	0,0000	0,0000	0,0011	0,0000	0,0000	0,0384	0,0000	0,0000	1,2438
171231342420521	806	32,77	0,0000	0,0000	0,0000	0,0000	0,0000	0,0009	0,0000	0,0000	0,0350	0,0000	0,0000	1,2078
218209405436543	906	33,02	0,0000	0,0000	0,0000	0,0000	0,0000	0,0009	0,0000	0,0000	0,0338	0,0000	0,0000	1,1942

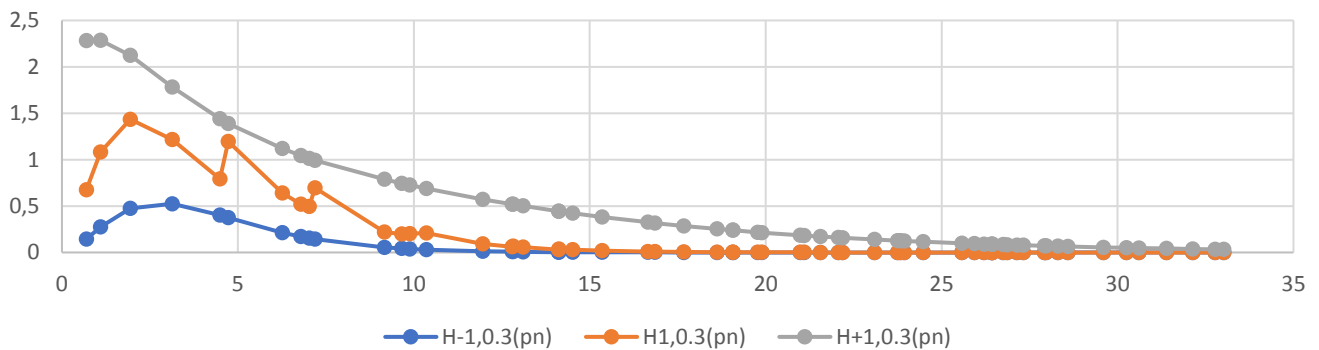
$$H_{1,0.1}^-(p_n) \quad H_{1,0.1}(p_n) \quad H_{1,0.1}^+(p_n)$$



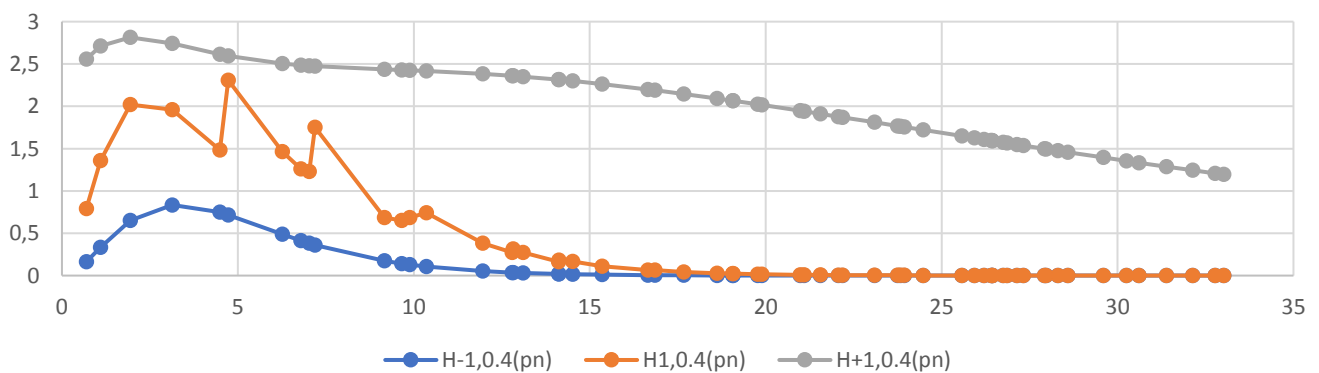
$$H_{1,0.2}^-(p_n) \quad H_{1,0.2}(p_n) \quad H_{1,0.2}^+(p_n)$$



$$H_{1,0.3}^-(p_n) \quad H_{1,0.3}(p_n) \quad H_{1,0.3}^+(p_n)$$

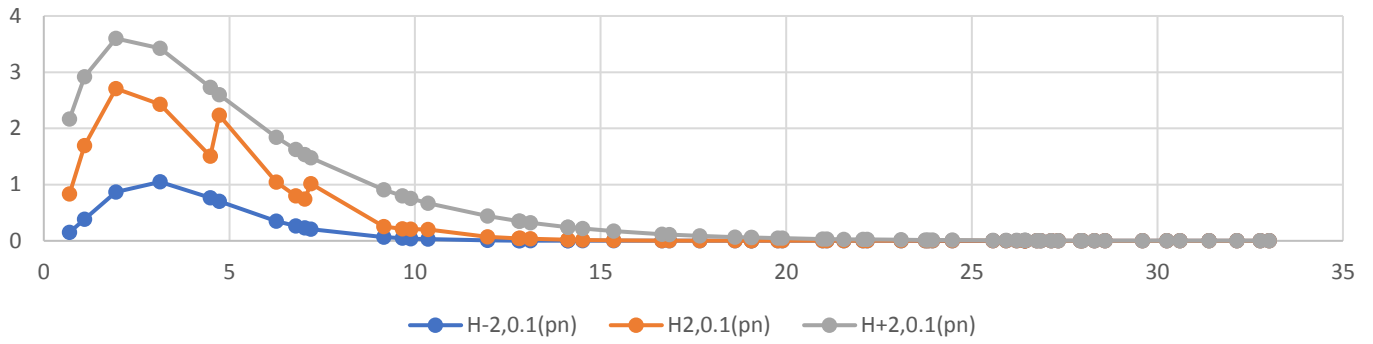
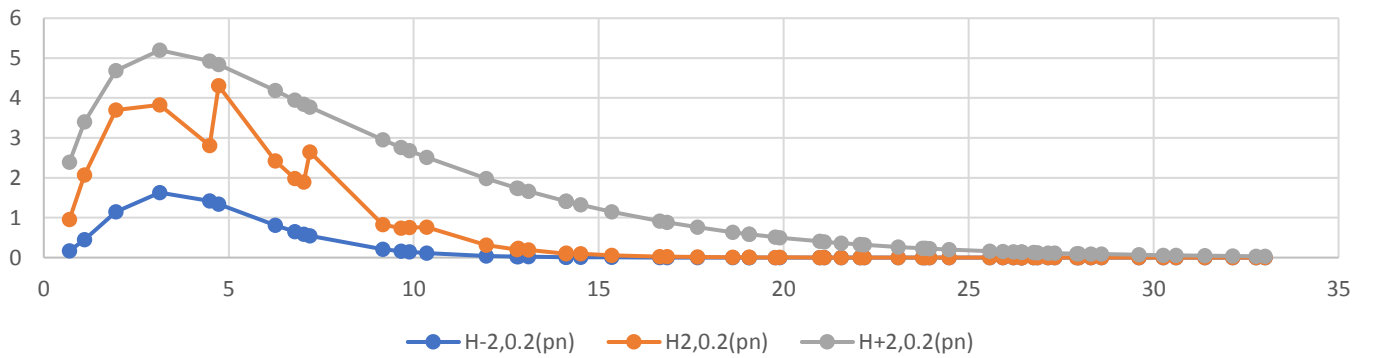
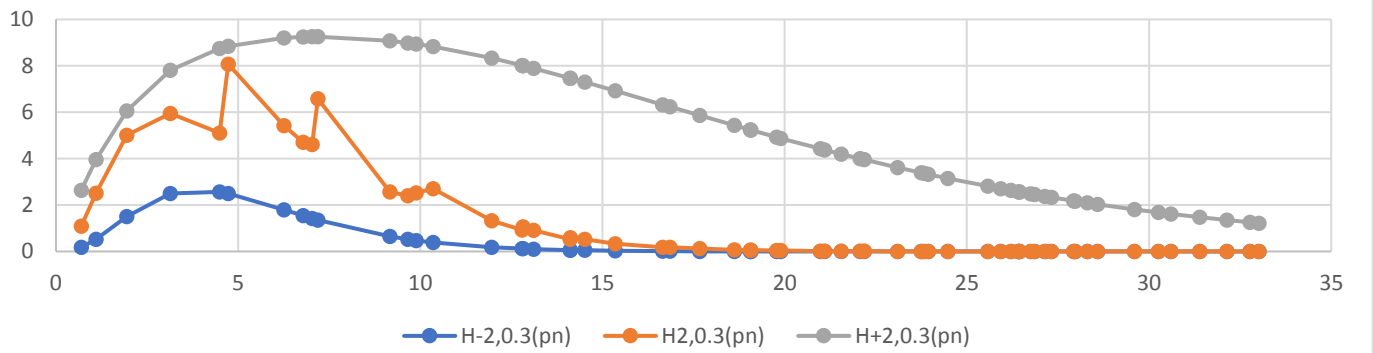
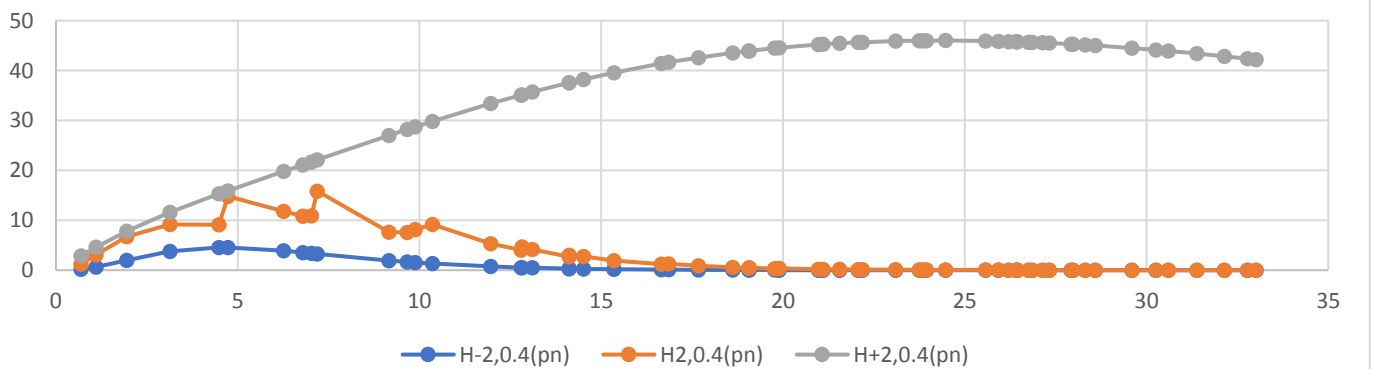


$$H_{1,0.4}^-(p_n) \quad H_{1,0.4}(p_n) \quad H_{1,0.4}^+(p_n)$$

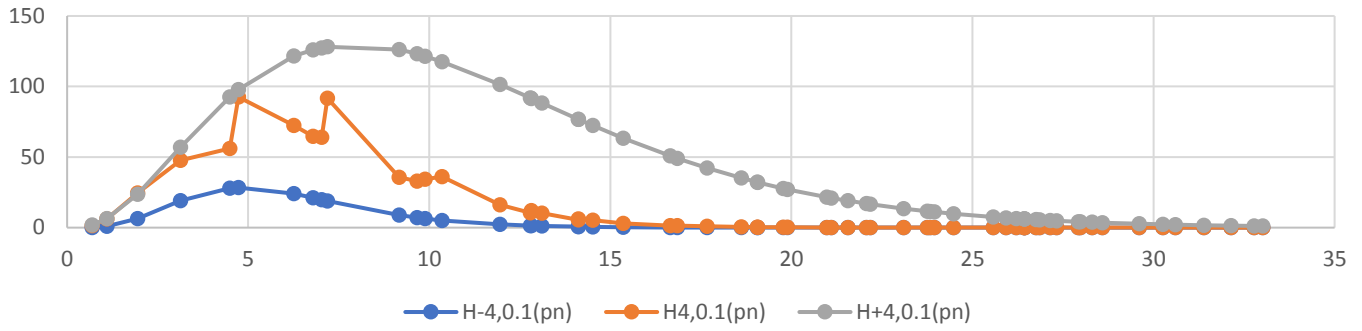


The table III.4 below give the numeral values of $H_{k,x}^-(p_n)$, $H_{k,x}(p_n)$ and $H_{k,x}^+(p_n)$ for $k=2$ and $x \in \{0.1, 0.2, 0.3, 0.4\}$ for all prime numbers corresponding to the first occurrence of the maximal gaps G_n .

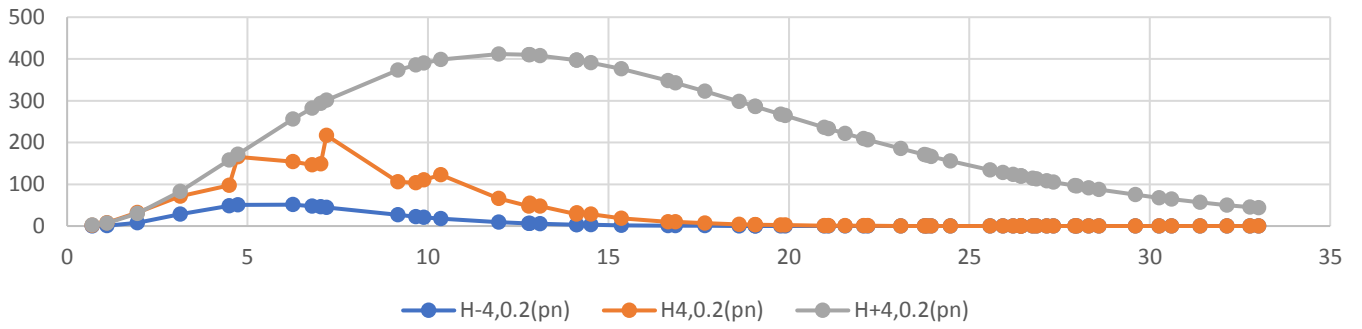
Primes	Gaps	$\ln(p_n)$	$H_{2,0.1}^-(p_n)$	$H_{2,0.1}(p_n)$	$H_{2,0.1}^+(p_n)$	$H_{2,0.2}^-(p_n)$	$H_{2,0.2}(p_n)$	$H_{2,0.2}^+(p_n)$	$H_{2,0.3}^-(p_n)$	$H_{2,0.3}(p_n)$	$H_{2,0.3}^+(p_n)$	$H_{2,0.4}^-(p_n)$	$H_{2,0.4}(p_n)$	$H_{2,0.4}^+(p_n)$
k	x		2	2	2	2	2	2	2	2	2	2	2	2
			0,1	0,1	0,1	0,2	0,2	0,2	0,3	0,3	0,3	0,4	0,4	0,4
2	1	0,69	0,1495	0,8322	2,1683	0,1656	0,9516	2,3922	0,1832	1,0866	2,6371	0,2025	1,2390	2,9048
3	2	1,10	0,3852	1,6955	2,9207	0,4523	2,0704	3,4020	0,5298	2,5198	3,9558	0,6192	3,0580	4,5922
7	4	1,95	0,8699	2,7080	3,5997	1,1505	3,7002	4,6827	1,5114	5,0168	6,0649	1,9744	6,7576	7,8248
23	6	3,14	1,0480	2,4263	3,4273	1,6282	3,8293	5,1982	2,4938	5,9530	7,8086	3,7760	9,1445	11,6367
89	8	4,49	0,7655	1,5055	2,7280	1,4189	2,8062	4,9294	2,5671	5,1036	8,7495	4,5608	9,1129	15,3159
113	14	4,73	0,7015	2,2359	2,5977	1,3407	4,3053	4,8413	2,4962	8,0727	8,8479	4,5588	14,8435	15,9291
523	18	6,26	0,3491	1,0454	1,8438	0,8084	2,4260	4,1844	1,8028	5,4205	9,2022	3,9154	11,7951	19,7839
887	20	6,79	0,2634	0,8019	1,6268	0,6509	1,9841	3,9480	1,5427	4,7085	9,2441	3,5528	10,8571	21,1039
1129	22	7,03	0,2304	0,7425	1,5353	0,5864	1,8918	3,8410	1,4287	4,6143	9,2526	3,3790	10,9250	21,7066
1327	34	7,19	0,2103	1,0188	1,4765	0,5458	2,6482	3,7698	1,3546	6,5818	9,2549	3,2613	15,8681	22,1102
9551	36	9,16	0,0633	0,2515	0,9057	0,2080	0,8267	2,9489	0,6444	2,5620	9,0844	1,9223	7,6435	26,9926
15683	44	9,66	0,0459	0,2110	0,7985	0,1599	0,7351	2,7598	0,5233	2,4063	8,9898	1,6462	7,5706	28,1886
19609	52	9,88	0,0396	0,2102	0,7542	0,1417	0,7518	2,6770	0,4754	2,5225	8,9410	1,5316	8,1278	28,7208
31397	72	10,35	0,0290	0,2028	0,6680	0,1095	0,7662	2,5077	0,3867	2,7069	8,8265	1,3099	9,1707	29,8264
155921	86	11,96	0,0097	0,0697	0,4384	0,0439	0,3165	1,9846	0,1845	1,3311	8,3308	0,7403	5,3424	33,3924
360653	96	12,80	0,0053	0,0402	0,3498	0,0267	0,2007	1,7439	0,1229	0,9240	8,0178	0,5386	4,0497	35,1101
370261	112	12,82	0,0052	0,0459	0,3473	0,0263	0,2300	1,7367	0,1213	1,0619	8,0075	0,5331	4,6672	35,1622
492113	114	13,11	0,0043	0,0373	0,3215	0,0221	0,1930	1,6604	0,1054	0,9186	7,8944	0,4773	4,1593	35,7175
1349533	118	14,12	0,0021	0,0173	0,2435	0,0120	0,1002	1,4102	0,0635	0,5312	7,4722	0,3193	2,6723	37,5731
1357201	132	14,12	0,0021	0,0192	0,2431	0,0119	0,1116	1,4089	0,0633	0,5922	7,4697	0,3186	2,9813	37,5830
2010733	148	14,51	0,0015	0,0157	0,2179	0,0094	0,0955	1,3199	0,0517	0,5281	7,2980	0,2713	2,7697	38,2560
4652353	154	15,35	0,0008	0,0083	0,1721	0,0055	0,0555	1,1451	0,0335	0,3359	6,9224	0,1915	1,9222	39,5961
17051707	180	16,65	0,0003	0,0034	0,1187	0,0024	0,0262	0,9123	0,0168	0,1817	6,3279	0,1100	1,1891	41,4092
20831323	210	16,85	0,0003	0,0034	0,1120	0,0021	0,0265	0,8803	0,0151	0,1881	6,2359	0,1008	1,2566	41,6602
47326693	220	17,67	0,0001	0,0018	0,0883	0,0013	0,0156	0,7588	0,0097	0,1204	5,8603	0,0703	0,8754	42,6106
122164747	222	18,62	0,0001	0,0008	0,0668	0,0007	0,0080	0,6369	0,0057	0,0685	5,4329	0,0460	0,5490	43,5533
189695659	234	19,06	0,0001	0,0006	0,0586	0,0005	0,0062	0,5865	0,0045	0,0552	5,2382	0,0377	0,4633	43,9353
191912783	248	19,07	0,0000	0,0006	0,0584	0,0005	0,0065	0,5852	0,0045	0,0581	5,2331	0,0375	0,4882	43,9449
387096133	250	19,77	0,0000	0,0004	0,0474	0,0003	0,0040	0,5122	0,0030	0,0382	4,9289	0,0273	0,3447	44,4815
436273009	282	19,89	0,0000	0,0004	0,0457	0,0003	0,0041	0,5007	0,0028	0,0400	4,8779	0,0258	0,3658	44,5645
1294268491	288	20,98	0,0000	0,0002	0,0329	0,0001	0,0019	0,4057	0,0015	0,0210	4,4267	0,0156	0,2142	45,2074
1453168141	292	21,10	0,0000	0,0001	0,0318	0,0001	0,0018	0,3967	0,0014	0,0198	4,3801	0,0148	0,2046	45,2644
2300942549	320	21,56	0,0000	0,0001	0,0276	0,0001	0,0014	0,3624	0,0011	0,0163	4,1979	0,0119	0,1769	45,4691
3842610773	336	22,07	0,0000	0,0001	0,0236	0,0001	0,0010	0,3275	0,0008	0,0125	4,0001	0,0094	0,1425	45,6583
4302407359	354	22,18	0,0000	0,0001	0,0228	0,0001	0,0010	0,3202	0,0008	0,0123	3,9574	0,0089	0,1416	45,6946
10726904659	382	23,10	0,0000	0,0000	0,0172	0,0000	0,0006	0,2667	0,0005	0,0075	3,6226	0,0057	0,0951	45,9176
20678048297	384	23,75	0,0000	0,0000	0,0141	0,0000	0,0003	0,2335	0,0003	0,0050	3,3945	0,0042	0,0678	46,0037
22367084959	394	23,83	0,0000	0,0000	0,0137	0,0000	0,0003	0,2298	0,0003	0,0049	3,3679	0,0040	0,0668	46,0100
25056082087	456	23,94	0,0000	0,0000	0,0133	0,0000	0,0004	0,2246	0,0003	0,0053	3,3297	0,0038	0,0729	46,0177
42652618343	464	24,48	0,0000	0,0000	0,0112	0,0000	0,0002	0,2014	0,0002	0,0038	3,1550	0,0030	0,0561	46,0306
127976334671	468	25,58	0,0000	0,0000	0,0080	0,0000	0,0001	0,1605	0,0001	0,0019	2,8159	0,0017	0,0317	45,9429
182226896239	474	25,93	0,0000	0,0000	0,0071	0,0000	0,0001	0,1491	0,0001	0,0016	2,7130	0,0015	0,0266	45,8835
241160624143	486	26,21	0,0000	0,0000	0,0065	0,0000	0,0001	0,1407	0,0001	0,0014	2,6335	0,0013	0,0236	45,8263
297501075799	490	26,42	0,0000	0,0000	0,0061	0,0000	0,0001	0,1346	0,0001	0,0012	2,5752	0,0011	0,0212	45,7775
303371455241	500	26,44	0,0000	0,0000	0,0061	0,0000	0,0001	0,1341	0,0001	0,0012	2,5698	0,0011	0,0215	45,7727
304599508537	514	26,44	0,0000	0,0000	0,0061	0,0000	0,0001	0,1339	0,0001	0,0012	2,5687	0,0011	0,0220	45,7717
416608695821	516	26,76	0,0000	0,0000	0,0055	0,0000	0,0001	0,1254	0,0001	0,0010	2,4838	0,0010	0,0187	45,6892
461690510011	532	26,86	0,0000	0,0000	0,0053	0,0000	0,0000	0,1227	0,0000	0,0010	2,4564	0,0009	0,0183	45,6599
614487453523	534	27,14	0,0000	0,0000	0,0049	0,0000	0,0000	0,1155	0,0000	0,0008	2,3816	0,0008	0,0157	45,5722
738832927927	540	27,33	0,0000	0,0000	0,0046	0,0000	0,0000	0,1111	0,0000	0,0007	2,3343	0,0007	0,0144	45,5112
1346294310749	582	27,93	0,0000	0,0000	0,0038	0,0000	0,0000	0,0978	0,0000	0,0005	2,1857	0,0005	0,0113	45,2890
1408695493609	588	27,97	0,0000	0,0000	0,0037	0,0000	0,0000	0,0969	0,0000	0,0005	2,1748	0,0005	0,0111	45,2708
1968188556461	602	28,31	0,0000	0,0000	0,0034	0,0000	0,0000	0,0902	0,0000	0,0004	2,0957	0,0004	0,0095	45,1303
2614941710599	652	28,59	0,0000	0,0000	0,0031	0,0000	0,0000	0,0849	0,0000	0,0004	2,0304	0,0004	0,0089	45,0028
7177162611713	674	29,60	0,0000	0,0000	0,0022	0,0000	0,0000	0,0683	0,0000	0,0002	1,8121	0,0002	0,0053	44,4923
13829048559701	716	30,26	0,0000	0,0000	0,0018	0,0000	0,0000	0,0593	0,0000	0,0002	1,6813	0,0002	0,0040	44,1158
19581334192423	766	30,61	0,0000	0,0000	0,0016	0,0000	0,0000	0,0550	0,0000	0,0001	1,6152	0,0001	0,0035	43,9027
42842283925351	778	31,39	0,0000	0,0000	0,0012	0,0000	0,0000	0,0463	0,0000	0,0001	1,4748	0,0001	0,0024	43,3916
90874329411493	804	32,14	0,0000	0,0000	0,0010	0,0000	0,0000	0,0393	0,0000	0,0001	1,3501	0,0001	0,0016	42,8625
171231342420521	806	32,77	0,0000	0,0000	0,0008	0,0000	0,0000	0,0342	0,0000	0,0000	1,2523	0,0000	0,0011	42,3901
218209405436543	906	33,02	0,0000	0,0000	0,0007	0,0000	0,0000	0,0324	0,0000	0,0000	1,2166	0,0000	0,0011	42,2034

$H^-_{2,0.1}(p_n) \quad H_{2,0.1}(p_n) \quad H^+_{2,0.1}(p_n)$  $H^-_{2,0.2}(p_n) \quad H_{2,0.2}(p_n) \quad H^+_{2,0.2}(p_n)$  $H^-_{2,0.3}(p_n) \quad H_{2,0.3}(p_n) \quad H^+_{2,0.3}(p_n)$  $H^-_{2,0.4}(p_n) \quad H_{2,0.4}(p_n) \quad H^+_{2,0.4}(p_n)$ 

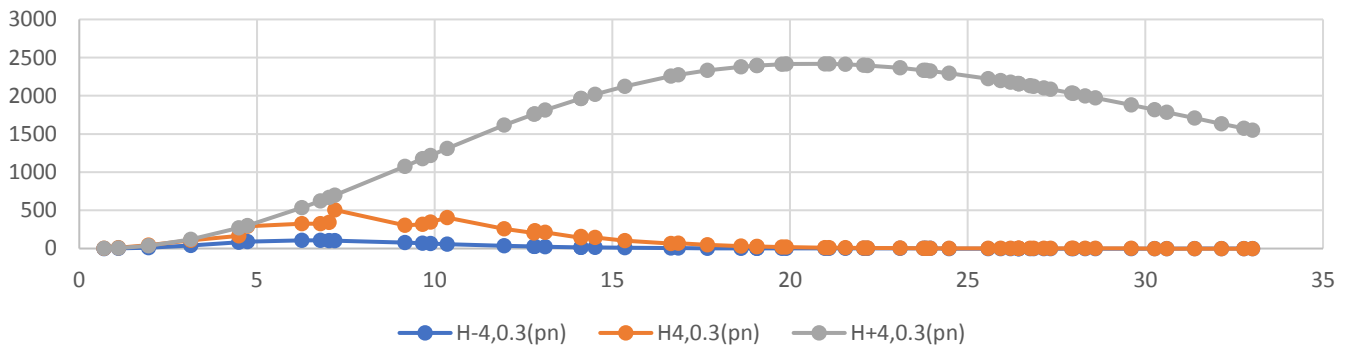
$$H_{4,0.1}^-(p_n) \quad H_{4,0.1}(p_n) \quad H_{4,0.1}^+(p_n)$$



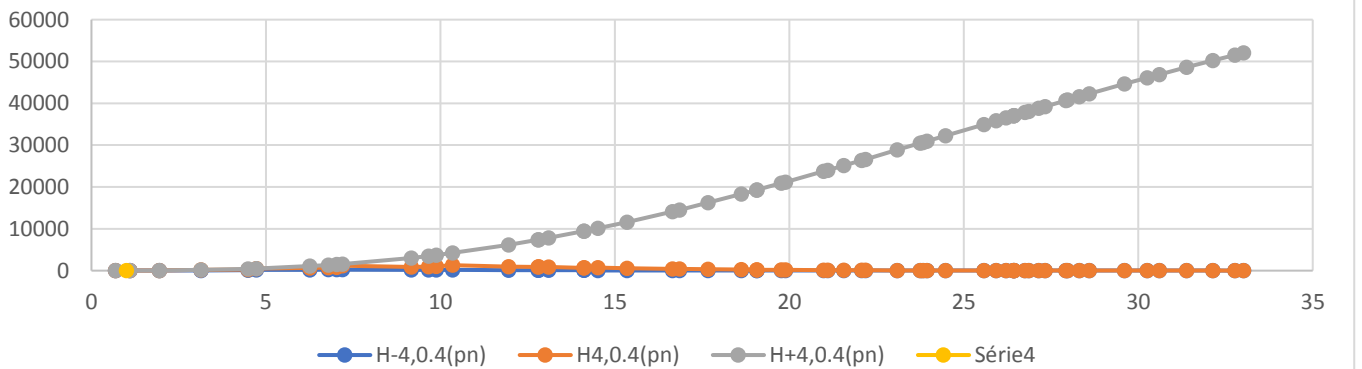
$$H_{4,0.2}^-(p_n) \quad H_{4,0.2}(p_n) \quad H_{4,0.2}^+(p_n)$$



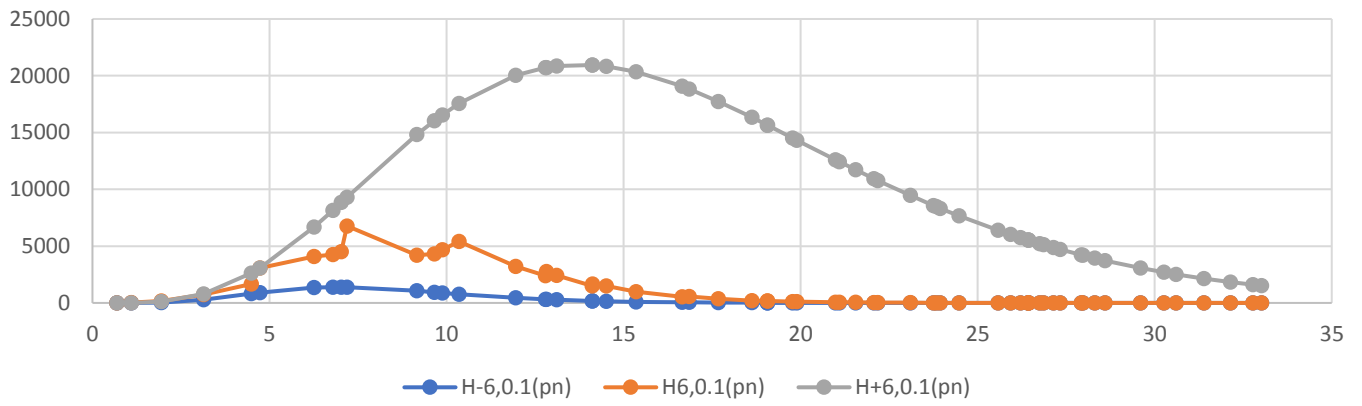
$$H_{4,0.3}^-(p_n) \quad H_{4,0.3}(p_n) \quad H_{4,0.3}^+(p_n)$$



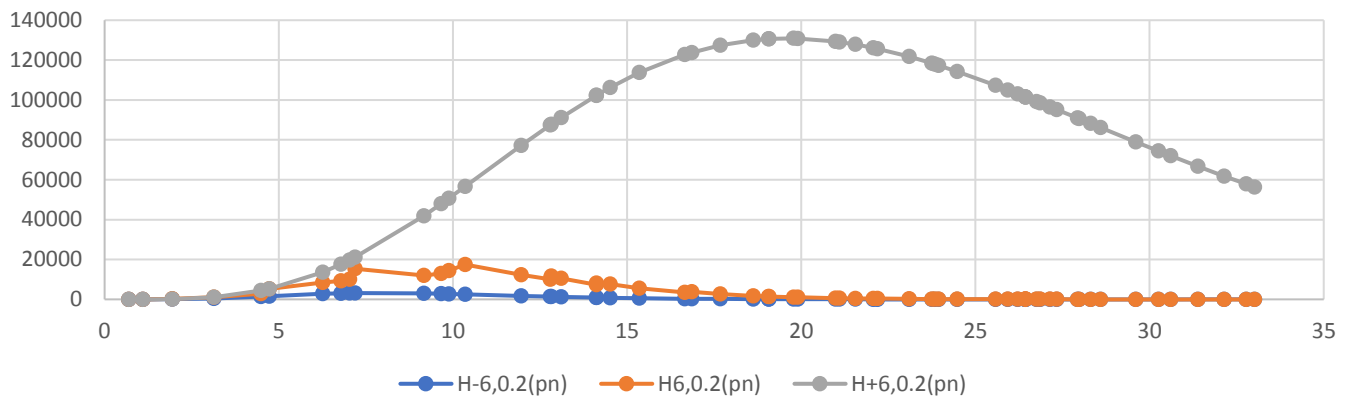
$$H_{4,0.4}^-(p_n) \quad H_{4,0.4}(p_n) \quad H_{4,0.4}^+(p_n) \quad \text{Série4}$$



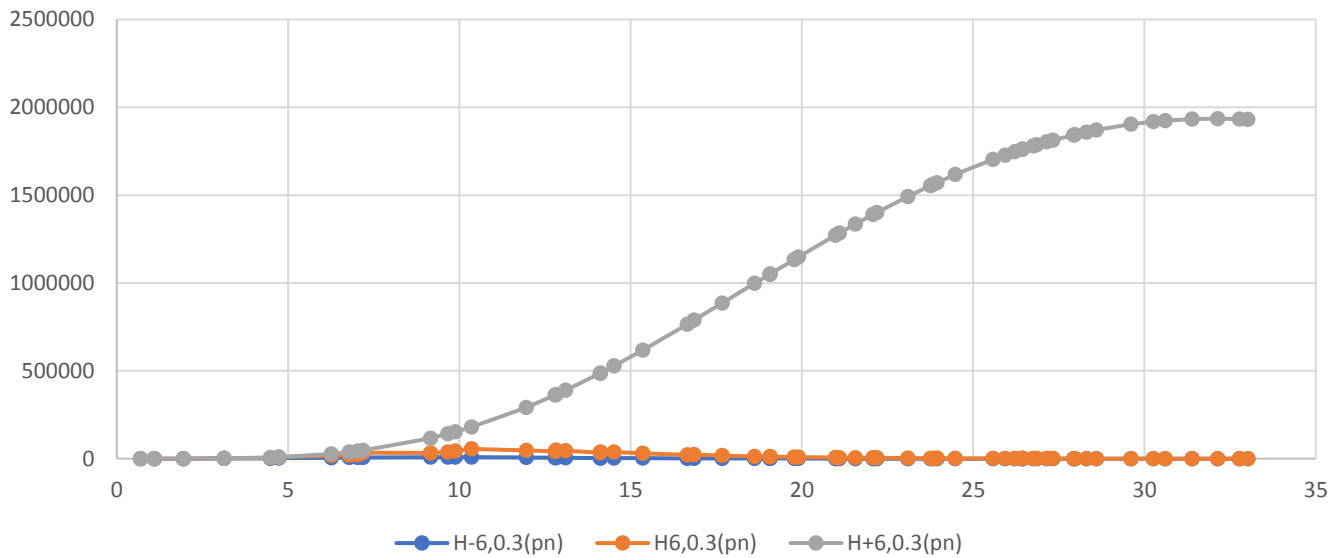
$$H^-_{6,0.1}(p_n) \quad H_{6,0.1}(p_n) \quad H^+_{6,0.1}(p_n)$$



$$H^-_{6,0.2}(p_n) \quad H_{6,0.2}(p_n) \quad H^+_{6,0.2}(p_n)$$



$$H^-_{6,0.3}(p_n) \quad H_{6,0.3}(p_n) \quad H^+_{6,0.3}(p_n)$$



III.5 - $H_{y,x}(p_n) = (p_{n+1})^x \cdot \ln^y(p_{n+1}) - (p_n)^x \cdot \ln^y(p_n) = 1$ has a set of solutions (x,y) for all p_n

This question represent a generalization of the cases studied in paragraphs I.1 and II.7.

We are going to study the field of $(x,y) \in \mathbb{R}^2$ which satisfy to $H_{y,x}(p_n) = (p_{n+1})^x \cdot \ln^y(p_{n+1}) - (p_n)^x \cdot \ln^y(p_n) = 1$

We have state in III.3 that that $M_{y,x}(p_n) = \frac{\ln^y(p_n)}{p_n^{1-x}} \cdot g_n \cdot \left(x + \frac{y}{\ln(p_n)}\right)$ was representing a good estimation of $H_{k,x}(p_n)$. So we are going to study the relation :

$$\frac{\ln^y(p_n)}{p_n^{1-x}} \cdot g_n \cdot \left(x + \frac{y}{\ln(p_n)}\right) = 1 \quad \Leftrightarrow \quad p_n^x \cdot \ln^y(p_n) \cdot \left(x + \frac{y}{\ln(p_n)}\right) = \frac{p_n}{g_n} \quad (III.12)$$

As before we start by taking the logarithm of the two parts of (III.12) :

$$x \cdot \ln(p_n) + y \cdot \ln(\ln(p_n)) + \ln\left(x + \frac{y}{\ln(p_n)}\right) = \ln\left(\frac{p_n}{g_n}\right) \quad (III.13)$$

(III.12) gives an implicit relation between y , x and p_n and we have to build an as good as possible estimation of $y = y(x, p_n)$. We can obtain a first approximation y_0 of y , neglecting the term $\ln\left(x + \frac{y}{\ln(p_n)}\right)$.

$$y_0 = \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)\right)}{\ln(\ln(p_n))}$$

We define

$$y_1 = y_0 - \varepsilon = \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)\right)}{\ln(\ln(p_n))} - \varepsilon \quad (III.14)$$

We now replace (III.14) into (III.12):

$$p_n^x \cdot e^{\left(\ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)\right) - \varepsilon \cdot \ln(\ln(p_n))} \cdot \left(x + \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)\right) - \varepsilon \cdot \ln(\ln(p_n))}{\ln(p_n) \cdot \ln(\ln(p_n))}\right) = \frac{p_n}{g_n}$$

$$e^{-\varepsilon \cdot \ln(\ln(p_n))} \cdot \left(x + \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)\right) - \varepsilon \cdot \ln(\ln(p_n))}{\ln(p_n) \cdot \ln(\ln(p_n))}\right) = 1$$

Considering ε small, we apply a Taylor serie expansion of $e^{-\varepsilon \cdot \ln(\ln(p_n))}$

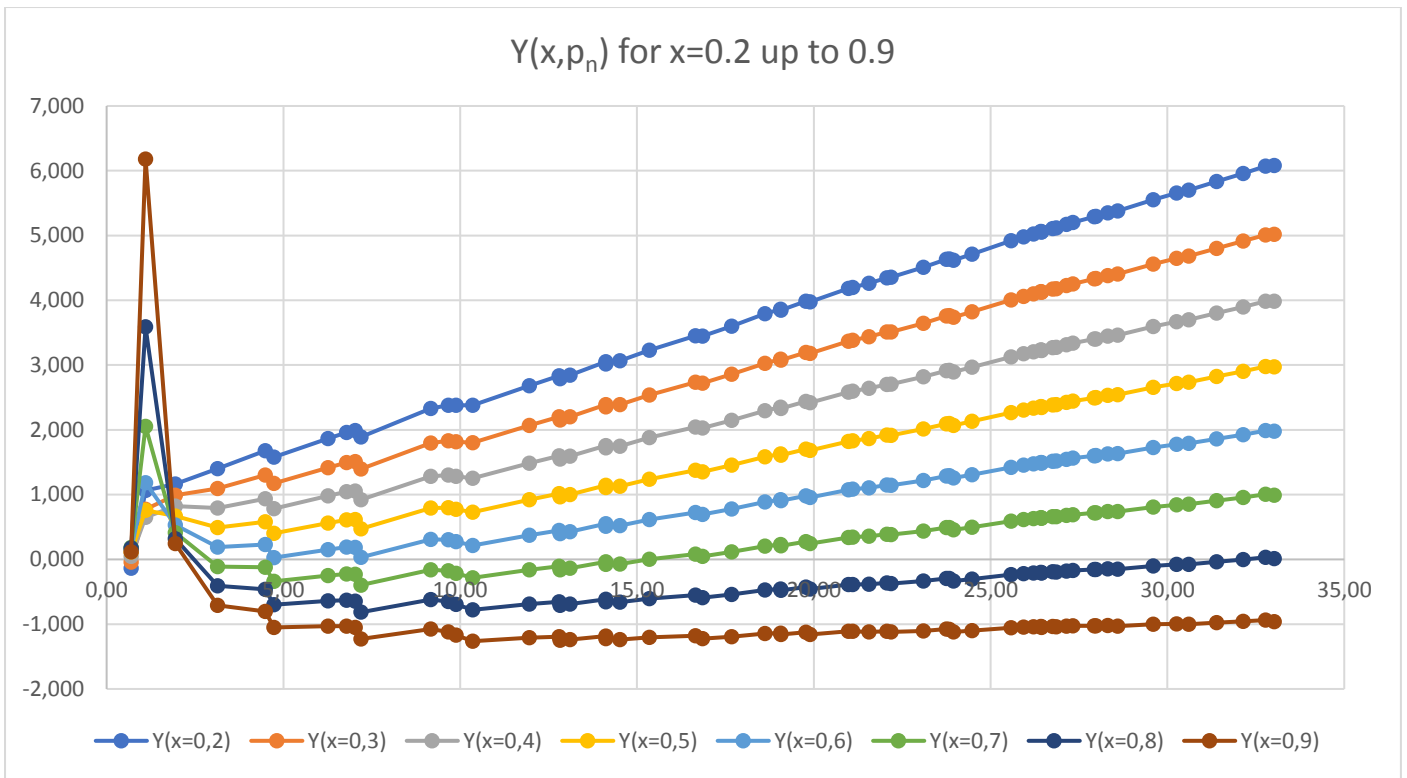
$$\left(1 - \varepsilon \cdot \ln(\ln(p_n))\right) \cdot \left(x \cdot \ln(p_n) \cdot \ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n) - \varepsilon \cdot \ln(\ln(p_n))\right) = \ln(p_n) \cdot \ln(\ln(p_n))$$

$$\varepsilon = \frac{(x-1) \cdot \ln(p_n) \cdot \ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)}{\ln(\ln(p_n)) \cdot (x \cdot \ln(p_n) \cdot \ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n) + 1)}$$

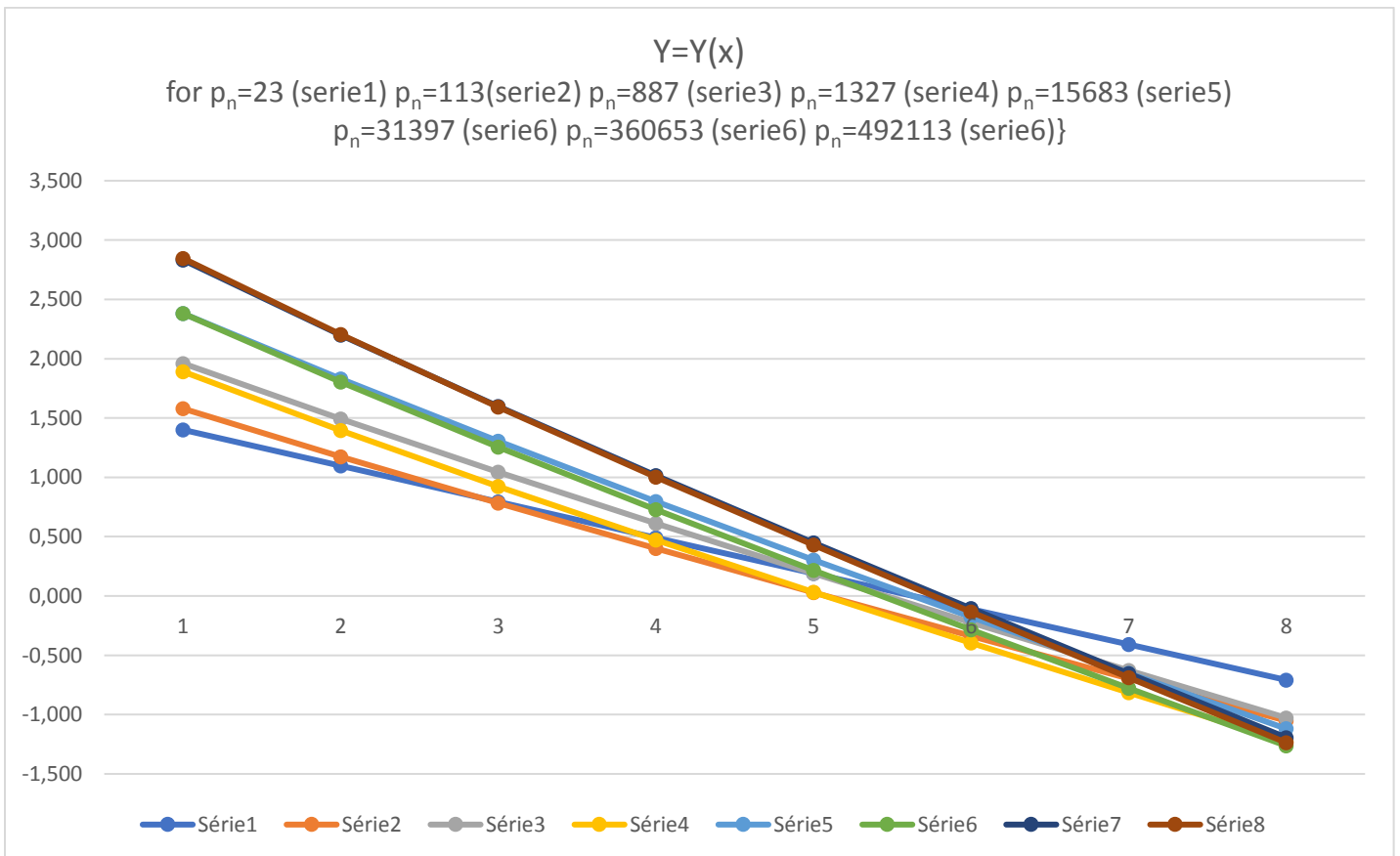
And

$$y_1 = \frac{\left(\ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)\right)}{\ln(\ln(p_n))} - \frac{(x-1) \cdot \ln(p_n) \cdot \ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n)}{\ln(\ln(p_n)) \cdot (x \cdot \ln(p_n) \cdot \ln(\ln(p_n)) + \ln\left(\frac{p_n}{g_n}\right) - x \cdot \ln(p_n) + 1)}$$

In table III.7 below, we have the estimation of y given by y_1 for different values of x and in red the corresponding value of $H_{y,x}(p_n)$. We see that y_1 gives a better estimation of y for the highest values of x .



The horizontal axis correspond to $\ln(P_n)$ for all P_n corresponding to the first occurrence of the maximal gaps.



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