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Twisted differential operators and q -crystals

Michel Gros, Bernard Le Stum & Adolfo Quirós*

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Abstract

We describe explicitly the q -PD-envelopes considered by Bhatt and Scholze in their recent theory of q -crystalline cohomology and explain the relation with our notion of a divided polynomial twisted algebra. Together with an interpretation of crystals on the q -crystalline site, that we call q -crystals, as modules endowed with some kind of stratification, it allows us to associate a module on the ring of twisted differential operators to any q -crystal.

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Introduction

In their recent article [3], B. Bhatt and P. Scholze have introduced two new cohomological theories with a strong crystalline flavor, the prismatic and the q -crystalline cohomologies, allowing them to generalize some of their former results obtained with M. Morrow on p -adic integral cohomology. As explained in ([5], section 6) these tools could also be a way for us, once a theory of coefficients for these new cohomologies is developed, to get rid of some non-canonical choices in our construction of the twisted Simpson correspondence ([6], corollary 8.9). At the end, this correspondance should hopefully appear just as an *avatar* of a deeper canonical equivalence of crystals on the prismatic and the q -crystalline sites, whose general pattern should look like a “ q -deformation” of Oyama’s reinterpretation ([10], theorem 1.4.3) of Ogus-Vologodsky’s correspondance as an equivalence between categories of crystals (see [5], section 6, for a general overview).

In this note, we start elaborating on one aspect of this hope by showing (theorem 7.3) how to construct a functor from the category of crystals on Bhatt-Scholze’s q -crystalline site (that we call for short *q -crystals*) to the category of modules over the ring D_q of twisted differential operators of [6], section 5. The construction of this functor has also the independent interest of describing explicitly, at least locally, the kind of structure hiding behind a q -crystal. Indeed, the ring $D_q = D_{A/R,q}$ is defined only for a very special class of algebras A over a base ring R and some element $q \in R$ (let’s call them in this introduction, forgetting additional data, simply *twisted R -algebras*; see [8] or [6] for a precise definition) and it is obviously only for them that the functor will be constructed. The construction consists then of two main steps. The first one, certainly the less standard, is to describe concretely the *q -PD-envelope* introduced a bit abstractly in [3], lemma 16.10, for these twisted R -algebras and to relate it to the algebra appearing in the construction of D_q . The second step is to develop the q -analogs of the usual calculus underlying the theory of classical crystals: hyperstratification, connection... for the q -crystalline site of twisted R -algebras.

Let us now describe briefly the organization of this note. In section 1, we recall for the reader’s convenience some basic vocabulary and properties of δ -rings used in [3]. In section 2, we essentially rephrase some of our considerations about twisted powers [6], using this time δ -structures instead of relative Frobenius. Following [3], we introduce in section 3 the notion of q -PD-envelope and we give its explicit description for a polynomial algebra (proposition 3.5), which will be useful in section 7. We also prove (theorem 3.6) that, in this way, we recover exactly our divided power algebra. Section 4 addresses the question of completion of these constructions, while in section 5 we combine the results from the former two sections in order to identify (theorem 5.2) the complete q -PD-envelope of a diagonal embedding arising from a twisted R -algebra. Finally, after some preparation (proposition 6.3), we deal in section 6 with the q -analogs of the notions of hyper-stratification, differential operators, connection and the corresponding equivalence of categories, and we end up in section 7 with the definition of a q -crystal and the construction of the promised functor. The appendix contains a discussion, in a more informal style, of the relation between usual crystals and the specialization to $q = 1$ of q -crystals.

Everything below depends on a prime p fixed once for all. We usually stick to the one dimensional case, leaving the reader to imagine how to extend the results to higher dimensions (which is most of the time quite straightforward). Also, since we are mainly interested in local questions, we concentrate on the affine case.

The first author (M.G.) heartily thanks the organizers, Bhargav Bhatt and Martin Olsson, for their invitation to the *Simons Symposium on p -adic Hodge Theory* (April 28-May 4, 2019) allowing him to follow the progress on topics related to those treated in this note.

1 δ -structures

In this section, we briefly review the notion of a δ -ring. We start from the Witt vectors point of view since it automatically provides all the standard properties.

As a set, the *ring of (p -typical) Witt vectors* of length two¹ on a commutative ring A is $W_1(A) = A \times A$. It is endowed with the unique *natural* ring structure such that both the *projection map*

$$W_1(A) \rightarrow A, \quad (f, g) \mapsto f$$

and the *ghost map*

$$W_1(A) \rightarrow A, \quad (f, g) \mapsto f^p + pg$$

are ring homomorphisms. A δ -*structure* on A is a section

$$A \rightarrow W_1(A), \quad f \mapsto (f, \delta(f))$$

of the projection map in the category of rings. This is equivalent to giving a p -derivation: a map $\delta : A \rightarrow A$ such that $\delta(0) = \delta(1) = 0$,

$$\forall f, g \in A, \quad \delta(f + g) = \delta(f) + \delta(g) - \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} f^{p-k} g^k \quad (1)$$

and

$$\forall f, g \in A, \quad \delta(fg) = f^p \delta(g) + \delta(f) g^p + p \delta(f) \delta(g). \quad (2)$$

Note that a δ -structure is uniquely determined by the action of δ on the generators of the ring. A δ -*ring* is a commutative ring endowed with a δ -structure and δ -rings make a category in the obvious way. We refer the reader to Joyal's note [7] for a short but beautiful introduction to the theory. When R is a fixed δ -ring, a δ - R -algebra is an R -algebra endowed with a compatible δ -structure.

Examples. 1. There exists only one δ -structure on the ring \mathbb{Z} of rational integers which is given by

$$\delta(n) = \frac{n - n^p}{p} \in \mathbb{Z}.$$

2. If $f \in \mathbb{Z}[x]$, then there exists a unique δ -structure on $\mathbb{Z}[x]$ such that $\delta(x) = f$.
3. There exists no δ -structure at all when $p^k A = 0$, unless $A = 0$ (show that $v_p(\delta(n)) = v_p(n) - 1$ when $v_p(n) > 0$ and deduce that $v_p(\delta^k(p^k)) = 0$).
4. It may also happen that there exists no δ -structure at all even when A is p -torsion-free: take $p = 2$ and $A = \mathbb{Z}[\sqrt{-1}]$.

¹We use the more recent index convention for truncated Witt vectors.

A *Frobenius* on a commutative ring A is a (ring) morphism $\phi : A \rightarrow A$ that satisfies

$$\forall f \in A, \quad \phi(f) \equiv f^p \pmod{p}.$$

If A is a δ -ring, then the map

$$\phi : A \rightarrow A, \quad f \mapsto f^p + p\delta(f),$$

obtained by composition of the section with the ghost map, is a Frobenius on A and this construction is functorial. Moreover, we have $\phi \circ \delta = \delta \circ \phi$. Note that the multiplicative condition (2) may be rewritten in the asymmetric but sometimes more convenient way

$$\forall f, g \in A, \quad \delta(fg) = \delta(f)g^p + \phi(f)\delta(g) = f^p\delta(g) + \delta(f)\phi(g). \quad (3)$$

Conversely, when A is p -torsion-free, any Frobenius of A comes from a unique δ -structure through the formula

$$\delta(f) = \frac{\phi(f) - f^p}{p} \in A.$$

We will systematically use the fact that δ -rings have all limits and colimits and that they both preserve the underlying rings: actually the forgetful functor is conservative and has both an adjoint (δ -envelope) and a coadjoint (Witt vectors). We refer the reader to the article of Borger and Wieland [4] for a plethystic interpretation of these phenomena.

Definition 1.1. *If R is a δ -ring and A is an R -algebra, then its δ -envelope A^δ is a δ -ring which is universal for morphisms of R -algebras into δ - R -algebras.*

Actually, it follows from the above discussion that the forgetful functor from δ - R -algebras to δ -algebras has an adjoint $A \mapsto A^\delta$ (so that δ -envelopes always exist and preserve colimits).

Example. We have² $R[x]^\delta = R[\{x_i\}_{i \in \mathbb{N}}]$ (polynomial ring with infinitely many variables) with the unique δ -structure such that $\delta(x_i) = x_{i+1}$ for $i \in \mathbb{N}$. More generally, since δ -envelope preserves colimits, we have $R[\{x_k\}_{k \in F}]^\delta = R[\{x_{k,i}\}_{k \in F, i \in \mathbb{N}}]$ with the unique δ -structure such that $\delta(x_{k,i}) = x_{k,i+1}$ for $k \in F, i \in \mathbb{N}$.

Definition 1.2. *A δ -ideal in a δ -ring A is an ideal I which is stable under δ . In general, the δ -closure I_δ of an ideal I is the smallest δ -ideal containing I .*

Equivalently, a δ -ideal is the kernel of a morphism of δ -rings. It immediately follows from formulas (1) and (2) above that the condition for an ideal to be a δ -ideal may be checked on generators. As a consequence, if $I = (\{f_i\}_{i \in E})$, then

$$I_\delta = \left(\left\{ \delta^j(f_i) \right\}_{i \in E, j \in \mathbb{N}} \right).$$

Also, if R is a δ -ring, A an R -algebra and $I \subset A$ an ideal, we have

$$(A/I)^\delta = A^\delta / (IA^\delta)_\delta.$$

This provides a convenient tool to describe a δ -envelope by choosing a presentation:

$$A = R[\{x_k\}_{k \in F}] / (\{f_i\}_{i \in E}) \Rightarrow A^\delta = R[\{x_{k,i}\}_{k \in F, i \in \mathbb{N}}] / \left(\left\{ \delta^j(f_i) \right\}_{i \in E, j \in \mathbb{N}} \right)$$

(with $x_k \mapsto x_{k,0}$).

²This is in fact the same as the free δ -ring $\mathbb{Z}\{e\}$ of [3] or the ring of p -typical symmetric functions Λ_p of [4].

Example. We usually endow the polynomial ring $\mathbb{Z}[q]$ with the unique δ -structure such that $\delta(q) = 0$. Then the principal ideal generated by $q-1$ is a δ -ideal: since $q^p - 1 \equiv (q-1)^p$ both modulo p and modulo $q-1$, we can write $q^p - 1 - (q-1)^p = p(q-1)c$ in the unique factorization domain $\mathbb{Z}[q]$, and then define $\delta(q-1) = c(q-1)$.

We will need the following:

Lemma 1.3. *Assume that p lies in the Jacobson radical of both A and B . Then any δ -structure on $A \times B$ induces a δ -structure on both A and B .*

Proof. It is enough to do it for A , and we only need to check that $\delta(1, 0) = (0, 0)$. If we write $\delta(1, 0) = (f, g)$, then we have

$$\delta(1, 0) = \delta((1, 0)^2) = 2(f, g)(1, 0)^p + p(f, g)^2 = (2f + pf^2, pg^2)$$

It follows that $f = 2f + pf^2$ and $g = pg^2$ or, in other words, that $(1 + pf)f = 0$ and $(1 - pg)g = 0$. Since p lies in the Jacobson radical of A (resp. B) then $1 + pf$ (resp. $1 - pg$) is invertible in A (resp. B) and it follows that $f = 0$ (resp. $g = 0$). \square

Note that the condition will be satisfied when A and B are both (p) -adically complete.

Note also that *some* condition is necessary because the result does not hold for example when $A = B = \mathbb{Q}$.

It is important for us to also recall that a p -derivation δ is systematically I -adically continuous with respect to any finitely generated ideal I containing p . In particular, δ will then automatically extend in a unique way to the I -adic completion of A .

Finally, we will use the convenient terminology of *rank one element* to mean $\delta(f) = 0$ (and consequently $\phi(f) = f^p$) and *distinguished element* to mean $\delta(f) \in A^\times$ (and therefore $p \in (f^p, \phi(f)) \subset (f, \phi(f))$).

2 δ -rings and twisted divided powers

We fix a δ -ring R as well as a rank one element q in R . Alternatively, we may consider R as a $\mathbb{Z}[q]$ -algebra, where q is then seen as a parameter, in which case we would still write q instead of $q1_R \in R$. As a δ -ring, R is endowed with a Frobenius endomorphism ϕ . Note that $\phi(q) = q^p$ and the action on q -analogs of natural numbers³ is given by $\phi((n)_q) = (n)_{q^p}$.

We let A be a δ - R -algebra and we fix some element x in A such that $\delta(x) \in R$. Note that in [6], we only considered the rank one case (which means that $\delta(x) = 0$). Let us stress the fact that, even if the general settings only requires $\delta(x) \in R$, our main results from [6] do not generalize (see the remark at the end of this section).

The R -algebra A is automatically endowed with a Frobenius ϕ which is semi-linear (with respect to the Frobenius ϕ of R) and satisfies $\phi(x) = x^p$ in the rank one case. Although we could consider the *relative* Frobenius⁴, which is the R -linear map

$$F : A' := R_{\phi} \frown_R A \rightarrow A, \quad a_{\phi} \frown f \mapsto af^p + pa\delta(f),$$

³We write $(n)_q := \frac{q^n - 1}{q - 1}$.

⁴We used to put a star and write $F_{A/R}^*$ in [6].

so that $\phi(f) = F(1 \otimes f)$, we will stick here to the *absolute* Frobenius ϕ and modify our formulas from [6] accordingly. Let us remark however that A' has a natural δ -structure and that F is then a morphism of δ -rings.

There exists a unique structure of δ - A -algebra on the polynomial ring $A[\xi]$ that satisfies $\delta(x + \xi) = \delta(x)$ that we call the *symmetric* δ -structure. It is given by

$$\delta(\xi) = \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^{p-i} \xi^i \quad (4)$$

(which depends on x but not on q or δ). Recall that we introduced in section 4 of [8] the *twisted powers*

$$\xi^{(n)_q} := \xi(\xi + x - qx) \cdots (\xi + x - q^{n-1}x) \in A[\xi]$$

(that clearly depend on both the choice of q and x). We will usually drop the index q and simply write $\xi^{(n)}$. They form an alternative basis for $A[\xi]$ (as an A -module) which is better adapted to working with q -analogs: if ϕ denotes the Frobenius of $A[\xi]$ attached to the symmetric δ -structure, then we have the fundamental congruence

$$\phi(\xi) \equiv \xi^{(p)} \pmod{(p)_q}.$$

This follows from corollary 7.6 of [6] and lemma 2.12 of [8] in the rank one case, but may also be checked directly. There exist more general explicit formulas for higher twisted powers but we will only consider the rank one case. In this situation, we showed in proposition 7.5 of [6] that⁵

$$\phi(\xi^{(n)}) = \sum_{i=n}^{pn} a_{n,i} x^{pn-i} \xi^{(i)}$$

with

$$a_{n,i} := \sum_{j=0}^n (-1)^{n-j} q^{\frac{p(n-j)(n-j-1)}{2}} \binom{n}{j}_{q^p} \binom{pj}{i}_q \in R \quad (5)$$

(in which we use the q -analogs of the binomial coefficients as in section 2 of [8] for example).

In section 2 of [6], we also introduced the ring of *twisted divided polynomials* $A\langle \xi \rangle_q$ which depends on the choice⁶ of both q and x (but does not require any δ -structure on A). This is a commutative A -algebra. As an A -module, it is free on some generators $\xi^{[n]_q}$ (called the *twisted divided powers*) indexed by $n \in \mathbb{N}$. The multiplication rule is quite involved:

$$\xi^{[n]_q} \xi^{[m]_q} = \sum_{0 \leq i \leq n, m} q^{\frac{i(i-1)}{2}} \binom{n+m-i}{n}_q \binom{n}{i}_q (q-1)^i x^i \xi^{[n+m-i]_q}.$$

Again, we will usually drop the index q and simply write $\xi^{[n]}$. Heuristically⁷, we have

$$\forall n \in \mathbb{N}, \quad \xi^{[n]} = \frac{\xi^{(n)}}{(n)_q!}.$$

In section 7 of [6], we showed that, when A is a δ -ring and x has rank one, the Frobenius of A extends naturally to $A\langle \xi \rangle_q$. Heuristically, we have

$$\forall n \in \mathbb{N}, \quad \phi\left(\xi^{[n]}\right) = \frac{\phi(\xi^{(n)})}{(n)_{q^p}!}.$$

⁵In [6] we actually used $A_{n,i}$ and $B_{n,i}$ instead of $a_{n,i}$ and $b_{n,i}$.

⁶Actually, it depends on q and $y := (1-q)x$.

⁷More precisely, there exists a unique natural homomorphism $A[\xi] \rightarrow A\langle \xi \rangle_q, \xi^{(n)} \mapsto (n)_q! \xi^{[n]}$.

Actually, we showed a little more: in this situation, there exists a semi-linear *divided Frobenius* $[\phi]$ on $A\langle\xi\rangle_q$ such that, heuristically again,

$$\forall n \in \mathbb{N}, \quad [\phi]\left(\xi^{[n]}\right) = \frac{\phi(\xi^{[n]})}{(p)_q^n}. \quad (6)$$

We also proved that

$$[\phi]\left(\xi^{[n]}\right) = \sum_{i=n}^{pn} b_{n,i} x^{pn-i} \xi^{[i]} \quad (7)$$

with

$$b_{n,i} = \frac{(i)_q!}{(n)_{q^p}!(p)_q^n} a_{n,i} \in R$$

and $a_{n,i}$ as in (5).

When A is p -torsion-free, the Frobenius of $A\langle\xi\rangle_q$ corresponds to a unique δ -structure on $A\langle\xi\rangle_q$ and this provides a natural δ -structure in general using the isomorphism

$$A\langle\xi\rangle_q \simeq A \otimes_{\mathbb{Z}[q,x]} \mathbb{Z}[q,x]\langle\xi\rangle_q$$

(where q and x are seen as parameters).

Remark. The rank one condition is crucial for the last results to hold. Otherwise, the Frobenius of $A[\xi]$ will *not* extend to $A\langle\xi\rangle_q$. This is easily checked when $p = 2$. Let us denote by δ_c the δ structure given by $\delta_c(x) = c \in R$ and by ϕ_c the corresponding Frobenius, so that

$$\phi_c(x) = x^2 + 2c = \phi_0(x) + 2c \quad \text{and} \quad \phi_c(\xi) = \xi^2 + 2x\xi = \phi_0(\xi).$$

Recall now from the first remark after definition 7.10 in [6] that

$$\phi_0(\xi) = (1+q)(\xi^{[2]} + x\xi).$$

Then, the following computation shows that $\phi_c(\xi^{(2)})$ is *not* divisible by $(2)_{q^2} = 1+q^2$ in general (so that $\phi_c(\xi^{[2]})$ does *not* exist in $A\langle\xi\rangle_q$):

$$\begin{aligned} \phi_c(\xi^{(2)}) &= \phi_c(\xi(\xi + (1-q)x)) \\ &= \phi_0(\xi)(\phi_0(\xi) + (1-q^2)(\phi_0(x) + 2c)) \\ &= \phi_0(\xi^{(2)}) + 2c(1-q^2)\phi_0(\xi) \\ &= (1+q^2)\phi_0(\xi^{[2]}) + 2c(1-q)(1+q)^2(\xi^{[2]} + x\xi). \end{aligned}$$

3 q -divided powers and twisted divided powers

As before, we let R be a δ -ring with fixed rank one element q . We consider the maximal ideal $(p, q-1) \subset \mathbb{Z}[q]$ and we assume from now on that R is actually a $\mathbb{Z}[q]_{(p,q-1)}$ -algebra.

Note that, under this new hypothesis, we have $(k)_q \in R^\times$ as long as $p \nmid k$ so that $R/(p)_q$ is a q -divisible ring of q -characteristic p in the sense of [8], which was a necessary condition for the main results of [6] to hold.

We have the following congruences when $k \in \mathbb{N}$:

$$(p)_{q^k} \equiv p \pmod{q-1} \quad \text{and} \quad (p)_{q^k} \equiv (k)_q^{p-1} (q-1)^{p-1} \pmod{p}, \quad (8)$$

which both imply that $(p)_{q^k} \in (p, q-1)$. It also follows that $(p)_{q^k}$ is a *distinguished* element in R since

$$\delta((p)_{q^k}) \equiv \delta(p) = 1 - p^{p-1} \equiv 1 \pmod{(p, q-1)}$$

(the first congruence follows from the fact that $(q-1)$ is a δ -ideal).

We recall now the following notion from [3]:

Definition 3.1. 1. If B is a δ - R -algebra and $J \subset B$ is an ideal, then (B, J) is a δ -pair.

We may also say that the surjective map $B \twoheadrightarrow \overline{B} := B/J$ is a δ -thickening.

2. If, moreover, B is $(p)_q$ -torsion-free and

$$\forall f \in J, \quad \phi(f) - (p)_q \delta(f) \in (p)_q J, \quad (9)$$

then (B, J) is a q -PD-pair⁸. We may also say that J is a q -PD-ideal or has q -divided powers or that the map $B \twoheadrightarrow \overline{B}$ is a q -PD-thickening.

Remarks. 1. Condition (9) may be split in two, as Bhatt and Scholze do: one may first require that $\phi(J) \subset (p)_q B$, then introduce the map $\gamma : J \rightarrow B, f \mapsto \phi(f)/(p)_q - \delta(f)$ and also require that $\gamma(J) \subset J$.

2. In the special case where J is a δ -ideal, condition (9) simply reads $\phi(J) \subset (p)_q J$, and this implies that $\phi^k(J) \subset (p^k)_q J$ for all $k \in \mathbb{N}$.

3. In general, the elements $f \in J$ that satisfy the property in condition (9) form an ideal and the condition can therefore be checked on generators.

Examples. 1. In the case $q = 1$, condition (9) simply reads

$$\forall f \in J, \quad f^p \in pJ$$

and (B, J) is a q -PD-pair if and only if B is p -torsion-free and J has *usual* divided powers (use lemma 2.36 of [3]). In other words, a 1-PD-pair is a p -torsion-free PD-pair endowed with a lifting of Frobenius (the δ -structure).

2. If B is a $(p)_q$ -torsion-free δ - R -algebra, then the *Nygaard ideal* $\mathcal{N} := \phi^{-1}((p)_q B)$ has q -divided powers (this is shown in [3], lemma 16.7). Note that \mathcal{N} is the first piece of the *Nygaard filtration*.

3. If B is a $(p)_q$ -torsion-free δ - R -algebra, then $J := (q-1)B$ has q -divided powers. This is also shown in [3], but actually simply follows from the equalities

$$\phi(q-1) = q^p - 1 = (p)_q (q-1),$$

since $(q-1)$ is a δ -ideal.

4. With the notations of the previous section, we may endow $A\langle \xi \rangle_q$ with the augmentation ideal $I^{[1]}$ generated by all the $\xi^{[n]}$ for $n \geq 1$. When A is $(p)_q$ -torsion-free, this is a q -PD pair as the fundamental equality (6) shows (since $I^{[1]}$ is clearly a δ -ideal).

⁸There are a lot more requirements, that we may ignore at this moment, in definition 16.2 of [3].

For later use, let us also mention the following:

Lemma 3.2. 1. If (B, J) is a q -PD-pair, B' is a $(p)_q$ -torsion-free δ -ring and $B \rightarrow B'$ is a morphism of δ -rings, then (B', JB') is a q -PD-pair.

2. The category of δ -pairs has all colimits and they are given by

$$\varinjlim (B_e, J_e) = (B, J)$$

with $B_e = \varinjlim B_e$ and $J = \sum J_e B$. Colimits preserve q -PD-pairs as long as they are $(p)_q$ -torsion free.

Proof. Both assertions concerning q -PD-pairs follow from the fact that the property in condition (9) may be checked on generators. The first part of the second assertion is a consequence of the fact that the category of δ -rings has all colimits and that they preserve the underlying rings. \square

Remarks. 1. As a consequence of the first assertion, we see that if (B, J) is a q -PD-pair and $\mathfrak{b} \subset B$ is a δ -ideal such that B/\mathfrak{b} is $(p)_q$ -torsion-free, then $(B/\mathfrak{b}, J + \mathfrak{b}/\mathfrak{b})$ is also a q -PD-pair.

2. As a particular case of the second assertion, we see that fibered coproducts in the category of δ -pairs are given by

$$(B_1, J_1) \otimes_{(B, J)} (B_2, J_2) = (B_1 \otimes_B B_2, B_1 \otimes_B J_2 + J_1 \otimes_B B_2).$$

Note that if B_1 is B -flat and B_2 is $(p)_q$ torsion-free, then $B_1 \otimes_B B_2$ is automatically $(p)_q$ -torsion-free.

Definition 3.3. Let (B, J) be a δ -pair. Then (if it exists) its q -PD envelope $(B^{[1]_q}, J^{[1]_q})$ is a q -PD-pair that is universal for morphisms to q -PD-pairs: there exists a morphism of δ -pairs $(B, J) \rightarrow (B^{[1]_q}, J^{[1]_q})$ such that any morphism $(B, J) \rightarrow (B', J')$ to a q -PD-pair extends uniquely to $(B^{[1]_q}, J^{[1]_q})$.

Remarks. 1. It might be necessary/useful to add the condition $\overline{B} \simeq \overline{B^{[1]_q}}$, but this is not clear yet.

2. In the case J is a δ -ideal, it is sufficient to check the universal property when J' also is a δ -ideal: we may always replace J' with JB' .

Examples. 1. When $q = 1$, B is p -torsion-free and J is generated by a regular sequence modulo p , then the q -PD-envelope of B is its usual PD-envelope (corollary 2.38 of [3]).

2. If B is $(p)_q$ -torsion-free and J already has q -divided powers, then $B^{[1]_q} = B$.

3. If $J = B$, then $B^{[1]_q} = B[\{1/(n)_q\}_{n \in \mathbb{N}}]$ ($\neq 0$ in general).

The rest of the section will be devoted to giving non trivial examples. But before, as a consequence of lemma 3.2, let us mention the following:

Lemma 3.4. 1. Let (B, J) be a δ -pair and $\mathfrak{b} \subset B$ a δ -ideal. Assume that (B, J) has a q -PD-envelope and $B^{[]_q} / \mathfrak{b}B^{[]_q}$ is $(p)_q$ -torsion-free. Then,

$$\left(B^{[]_q} / \mathfrak{b}B^{[]_q}, (J^{[]_q} + \mathfrak{b}B^{[]_q}) / \mathfrak{b}B^{[]_q} \right)$$

is the q -PD-envelope of $(B/\mathfrak{b}, J + \mathfrak{b}/\mathfrak{b})$.

2. Let $\{(B_e, J_e)\}_{e \in E}$ be a commutative diagram of δ -pairs all having a q -PD-envelope, $B := \varinjlim B_e$ and $J = \sum J_e B$. If $B' := \varinjlim B_e^{[]_q}$ is $(p)_q$ -torsion free, then $(B', \sum J_e^{[]_q} B')$ is the q -PD-envelope of (B, J) .

Proof. Concerning both assertions, we already know from lemma 3.2 (and the remarks thereafter) that the given pair is a q -PD-pair and the universal property follows from the universal properties of quotient and colimit respectively. \square

For the next example, the case $q = 1$ follows from lemma 2.36 of [3] (but see also their lemma 16.10 for the general case). We use the notations from definitions 1.1 and 1.2 and write

$$B[\{f_i/g_i\}_{i \in E}] := B[\{x_i\}_{i \in E}] / (g_i x_i - f_i)_{i \in E} \quad \text{for } f_i, g_i \in B.$$

Proposition 3.5. Assume R is $(p)_q$ -torsion-free. If $B := R[x]^\delta$ and $J := (x)_\delta$, then

$$B^{[]_q} = B \left[\left\{ \frac{\phi(\delta^i(x))}{(p)_q} \right\}_{i \in \mathbb{N}} \right]^\delta$$

and

$$J^{[]_q} = (x)_\delta + \left(\left\{ \frac{\phi(\delta^i(x))}{(p)_q} \right\}_{i \in \mathbb{N}} \right)_\delta.$$

Moreover, the ring $B^{[]_q}$ is faithfully flat (over R).

Proof. First of all, using the first assertion in lemma 3.4, we may assume that $R = \mathbb{Z}[q]_{(p, q-1)}[\{y_e\}_{e \in E}]$ so that, in particular, R is a $(p)_q$ -torsion-free domain and $R/(p)_q$ is p -torsion-free. This will be intensively used. Concretely, if we let

$$B' := R[\{x\} \cup \{z_i\}_{i \in \mathbb{N}}]^\delta / \left(\phi(\delta^i(x)) - (p)_q z_i \right)_\delta$$

and

$$J' := (\{\bar{x}\} \cup \{\bar{z}_i\}_{i \in \mathbb{N}})_\delta \subset B'$$

(where the bars indicate the residue classes), then we have to show that B' is faithfully flat (and in particular $(p)_q$ -torsion free) and that $\phi(J') \subset (p)_q B'$ (and therefore is a q -PD ideal because this is already a δ -ideal). The universal property will then follow automatically.

We start by showing that B' is faithfully flat. We can write

$$B' = \bigotimes_{i \in \mathbb{N}, R[x]^\delta} B_i \quad \text{with } B_i := R[x, z_i]^\delta / \left(\phi(\delta^i(x)) - (p)_q z_i \right)_\delta.$$

Since a (possibly infinite) tensor product of faithfully flat modules is still faithfully flat, it is sufficient to prove that each B_i is faithfully flat. We have

$$B_i \simeq R[x]^\delta \underset{\phi_i}{\frown} \otimes_{R[x]^\delta} \underset{p_i}{\nearrow} R[z]^\delta$$

where ϕ_i (resp. p_i) is the unique morphism of δ -rings such that $\phi_i(x) = \phi(\delta^i(x))$ (resp. $p_i(x) = (p)_q z$). Since $R[z]^\delta$ is a polynomial ring (in infinitely many variables), it is faithfully flat and we are therefore reduced to showing that ϕ_i is a faithfully flat morphism. We know from lemma 2.11 of [3] that ϕ is faithfully flat and it is therefore sufficient to verify that the endomorphism of $R[x]^\delta$ sending x to $\delta^i(x)$ is faithfully flat. But this is simply a shift in the variables that turns $R[x]^\delta$ into a polynomial ring (in i variables) over itself.

As a consequence, B' is $(p)_q$ -torsion-free and $B'/(p)_q$ is p -torsion-free (because the same properties hold in the case $B' = R$). Now, we let

$$J'' := J' \cap \phi^{-1}((p)_q J') = J' \cap \phi^{-1}((p)_q B'),$$

where the last equality is shown as follows: if $f \in J'$ and $\phi(f) = (p)_q g$ with $g \in B'$, then $\phi(f) \in J'$ because J' is a δ -ideal and therefore $(p)_q g \in J'$, but $B'/J' = R$ is $(p)_q$ -torsion-free and therefore $g \in J'$.

We can also show that, if $c \in R$ satisfies $\phi(c) \equiv p \pmod{(p)_q}$, then B'/J'' is c -torsion-free: if $cf \in J''$, then $cf \in J'$ and $\phi(c)\phi(f) = \phi(cf) \in (p)_q B'$. Since $B'/J' = R$ is a domain and $c \neq 0$, the first condition implies that $f \in J'$. On the other hand, since $B'/(p)_q$ is p -torsion-free and $\phi(c) \equiv p \pmod{(p)_q}$, the second condition implies that $\phi(f) \in (p)_q B'$. Note that this applies in particular to the case $c = p$ or $c = (p)_{q^k}$ with $k \in \mathbb{N}$.

We will be done if we show that $J' \subset J''$. We may first notice that, since $\phi(\delta^i(\bar{x})) = (p)_q \bar{z}_i$, we already have $\delta^i(\bar{x}) \in J''$ and it only remains to show that $\delta^j(\bar{z}_i) \in J''$. We have in B' :

$$\begin{aligned} \phi^{k+1}((p)_q \bar{z}_i) &= \phi^{k+2}(\delta^i(\bar{x})) \\ &= \phi^{k+1}(\delta^i(\bar{x})^p) + p\phi^{k+1}(\delta^{i+1}(\bar{x})) \\ &= (p)_{q^{p^k}}^p \phi^k(\bar{z}_i)^p + p(p)_{q^{p^k}} \phi^k(\bar{z}_{i+1}). \end{aligned}$$

The case $k = 0$ reads

$$\phi((p)_q \bar{z}_i) = (p)_q^p \bar{z}_i^p + p(p)_q \bar{z}_{i+1}.$$

It implies that $(p)_q \bar{z}_i \in J''$, and therefore also $\bar{z}_i \in J''$ since B'/J'' is $(p)_q$ -torsion-free, as we saw above (case $c = (p)_q$). In general, we have

$$(p)_{q^{p^{k+1}}} \phi^{k+1}(\bar{z}_i) = \phi^{k+1}((p)_q \bar{z}_i) = (p)_{q^{p^k}}^p \phi^k(\bar{z}_i)^p + p(p)_{q^{p^k}} \phi^k(\bar{z}_{i+1}).$$

Since B'/J'' is also $(p)_{q^{p^{k+1}}}$ -torsion-free, we obtain by induction on $k \in \mathbb{N}$ that $\phi^k(\bar{z}_i) \in J''$. We can now prove by induction on $j \in \mathbb{N}$ that $\phi^k(\delta^j(\bar{z}_i)) \in J''$: we have

$$p\phi^k(\delta^{j+1}(\bar{z}_i)) = p\delta(\phi^k(\delta^j(\bar{z}_i))) = \phi^{k+1}(\delta^j(\bar{z}_i)) - \phi^k(\delta^j(\bar{z}_i))^p \in J''$$

and we are done because B'/J'' is p -torsion-free (case $c = p$ above). Specializing to the case $k = 0$ provide us with $\delta^j(\bar{z}_i) \in J''$. \square

Remarks. 1. It is very likely that one may write down an alternative proof of this proposition by reducing modulo $q - 1$ in the spirit of lemma 16.10 of [3]. Note that their lemma goes somehow beyond our scope because they consider an ideal which is not necessarily a δ -ideal (but they assume that it contains $q - 1$).

2. Using the second assertion of lemma 3.4, the proposition extends immediately to several (possibly infinitely many) variables.

3. More generally, assume that B is a δ -ring, that $J \subset B$ is a δ -ideal and that the quotient map $B \rightarrow B/J$ admits a section as morphism of δ -rings. We may then write $R := B/J$ and consider B as a δ - R -algebra via the section map. There exists a surjective morphism of δ - R -algebras $C := R[\{x_e\}_{e \in E}]^\delta \twoheadrightarrow B$ sending $K := (\{x_e\}_{e \in E})$ onto J . If we denote by \mathfrak{c} the kernel of this map, then it follows from lemma 3.4 that

$$\left(C^{[]_q} / \mathfrak{c} C^{[]_q}, (K + \mathfrak{c}) C^{[]_q} / \mathfrak{c} C^{[]_q} \right)$$

is the q -PD-envelope of (B, J) , as long as one can show that the quotient ring $C^{[]_q} / \mathfrak{c} C^{[]_q}$ is $(p)_q$ -torsion free.

Example. Assume A is a $(p)_q$ -torsion-free δ - R -algebra with $\delta(x) \in R$ as in section 2 and endow $A[\xi]$ with the symmetric δ -structure. We consider the δ -pair $(A[\xi], (\xi))$. Then

$$A[\xi]^{[]_q} = A \left[\xi, \frac{\phi(\xi)}{(p)_q} \right]^\delta,$$

$$(\xi)^{[]_q} = \left(\xi, \frac{\phi(\xi)}{(p)_q} \right)_\delta,$$

and $A[\xi]^{[]_q}$ is faithfully flat over A (this follows from the above discussion). We will show shortly that we can do a lot better when x has rank one.

The next example is fundamental for us (we use the notations of the previous section). In the case $q = 1$, this is a consequence of lemma 2.35 of [3] (recall that we denote by $I^{[1]}$ the augmentation ideal of $A\langle \xi \rangle_q$):

Theorem 3.6. *If A is a $(p)_q$ -torsion-free δ - R -algebra with fixed rank one element x and $A[\xi]$ is endowed with the symmetric δ -structure, then $(A\langle \xi \rangle_q, I^{[1]})$ is the q -PD-envelope of $(A[\xi], (\xi))$.*

Proof. We may assume that $R = \mathbb{Z}[q]_{(p, q-1)}$ and $A = R[x]$. We endow $A\langle \xi \rangle_q$ with the degree filtration by the A -submodules F_n generated by $\xi^{[k]}$ for $k \leq n$. Formula (7) shows that Frobenius sends F_n into F_{pn} . Since this is clearly also true for the p th power map and A is p -torsion-free, we see that the same holds for δ as well. At this point, we may also notice that, although δ is not an additive map, formula (1) shows that it satisfies

$$\forall u, v \in F_n, \quad (u \equiv v \pmod{F_{n-1}}) \Rightarrow (\delta(u) \equiv \delta(v) \pmod{F_{pn-1}}). \quad (10)$$

Now, it is not difficult to compute the following leading terms

$$\phi(\xi^{[n]}) \equiv \frac{(np)_q!}{(n)_{q^p}!} \xi^{[np]} \pmod{F_{np-1}}$$

and

$$\left(\xi^{[n]} \right)^p \equiv \frac{(np)_q!}{((n)_q!)^p} \xi^{[np]} \pmod{F_{np-1}}.$$

It follows that

$$\delta(\xi^{[n]}) \equiv d_n \xi^{[np]} \pmod{F_{np-1}} \quad \text{with} \quad d_n = \frac{(((n)_q!)^p - (n)_{q^p}!)(np)_q!}{p(n)_{q^p}!((n)_q!)^p} \in R.$$

We claim that $d_{p^r} \in R^\times$ when $r \in \mathbb{N}$. Since R is a local ring and $q - 1$ belongs to the maximal ideal, we may assume that $q = 1$ and in this case,

$$d_{p^r} = \frac{((p^r!)^p - p^r!)p^{r+1}!}{pp^r!(p^r!)^p} = \frac{((p^r!)^{p-1} - 1)p^{r+1}!}{p(p^r!)^p}.$$

We have

$$v_p(d_{p^r}) = v_p(p^{r+1}!) - pv_p(p^r!) - 1 = \frac{p^{r+1} - 1}{p - 1} - p \frac{p^r - 1}{p - 1} - 1 = 0.$$

We can now show by induction that

$$\delta^r([\phi](\xi)) \equiv c_r \xi^{[p^r]} \pmod{F_{p^{r+1}-1}} \quad (11)$$

with $c_r \in R^\times$. Assuming that the formula holds for $r - 1$, we will have, thanks to property (10) and the asymmetric condition (3),

$$\begin{aligned} \delta^r([\phi](\xi)) &\equiv \delta(c_{r-1} \xi^{[p^r]}) \pmod{F_{p^{r+1}-1}} \\ &\equiv c_{r-1}^p \delta(\xi^{[p^r]}) + \delta(c_{r-1}) \phi(\xi^{[p^r]}) \pmod{F_{p^{r+1}-1}} \\ &\equiv (c_{r-1}^p d_{p^r} + (p)_q^{p^r} \delta(c_{r-1}) b_{p^r, p^{r+1}}) \xi^{[p^r+1]} \pmod{F_{p^{r+1}-1}} \end{aligned}$$

with $b_{n,i} \in R$ as in (7). Since $c_{r-1}^p d_{p^r} \in R^\times$, it is then sufficient to recall that $(p)_q$ belongs to the maximal ideal $(p, q - 1)$ and the coefficient is therefore necessarily invertible, as asserted.

Now, if $n \in \mathbb{N}$ has p -adic expansion $n = \sum_{r \geq 0} k_r p^r$, we set

$$v_n := \xi^{k_0} \prod_{r \geq 0} (\delta^r([\phi](\xi)))^{k_{r+1}}.$$

Formula (11) shows that $\{v_n\}_{n \in \mathbb{N}}$ is a basis for the free A -module $A\langle \xi \rangle_q$.

Assume now that (B, J) is a q -PD-pair and that $u : (A[\xi], (\xi)) \rightarrow (B, J)$ is a morphism of δ -pairs. Since (ξ) is a δ -ideal, we may assume that J also is a δ -ideal so that $\phi(J) \subset (p)_q B$. Since $f := u(\xi) \in J$ and B is $(p)_q$ -torsion-free, there exists a unique $g \in B$ such that $\phi(f) = (p)_q g$ and we may then extend u uniquely to $A\langle \xi \rangle_q$ by sending v_n to $f^{k_0} \prod_{r \geq 0} \delta^r(g)^{k_{r+1}}$. So far, this is only a morphism of A -modules, but in order to show that it is in fact a morphism of rings, we may actually assume that $q - 1 \in R^\times$, in which case $A[\xi] = A\langle \xi \rangle_q$ and the question becomes trivial. \square

Remarks. 1. The computations in the case $q = 1$ have been also carried out by Bhatt and Scholze in the proof of their lemma 2.34.

2. In our proof, it is actually not obvious at all from the original formulas that $d_{p^r} \in R^\times$. For example, in the simplest non trivial case $p = 2$ and $r = 1$, we have $d_2 = q + q^2 + q^3$, which is a non-trivial unit.
3. Our proof also shows that $A\langle \xi \rangle_q$ is generated as a δ - $A[\xi]$ -algebra by $[\phi](\xi)$ but this was already known from the example after proposition 3.5.

4. The result in our theorem is closely related to Pridham's work in [11]. For example, his lemma 1.3 shows that (when A is not merely a δ -ring but actually a λ -ring)

$$\lambda^n \left(\frac{\xi}{q-1} \right) = \frac{1}{(q-1)^n} \xi^{[n]} \quad \left(= \left(\frac{\xi}{q-1} \right)^{[n]} \right)$$

when $q-1 \in R^\times$ (in which case $A[\xi] = A\langle \xi \rangle_q$).

5. Note also that, as a corollary of our theorem, one recovers the existence of the q -logarithm (after completion, but see below) as in lemma 2.2.2 of [1].

4 Complete q -PD-envelopes

As before, R denotes a δ -ring with fixed rank one element q and we assume that R is actually a $\mathbb{Z}[q]_{(p,q-1)}$ -algebra. Any R -algebra B will be implicitly endowed with its $(p, q-1)$ -adic topology and we will denote by \widehat{B} or B^\wedge its completion for this topology, which this is automatically a $\mathbb{Z}_p[[q-1]]$ -algebra (as in [3]). We recall that, if B is a δ - R -algebra, then δ is necessarily continuous and extends therefore uniquely to \widehat{B} . We also want to mention that a complete ring is automatically $(p)_q$ -complete. Actually, the $(p, q-1)$ -adic topology and the $(p, (p)_q)$ -adic topology coincide thanks to congruences (8), and we will often simply call this the adic topology.

Remark. Besides the usual completion \widehat{B} of B , one may also consider its *derived completion* ([12, Tag 091N]) $R \varprojlim B_n^\bullet$, where B_n^\bullet denotes the Koszul complex

$$\begin{array}{ccccc} B & \longrightarrow & B \oplus B & \longrightarrow & B \\ f & \longmapsto & (p^n f, -(q-1)^n f) & & \\ & & (f, g) & \longmapsto & (q-1)^n f + p^n g. \end{array}$$

More generally, one can consider the derived completion functor

$$K^\bullet \mapsto \widehat{K^\bullet} := R \varprojlim (K^\bullet \otimes_B^L B_n^\bullet)$$

on complexes of B -modules. If M is a B -module and $M[0]$ denotes the corresponding complex concentrated in degree zero, then there exists a canonical map $\widehat{M[0]} \rightarrow \widehat{M}[0]$ which is *not* an isomorphism in general (but see below).

We recall that an abelian group M has *bounded p^∞ -torsion* if

$$\exists l \in \mathbb{N}, \forall s \in M, \forall m \in \mathbb{N}, \quad p^m s = 0 \Rightarrow p^l s = 0.$$

We will need below the following result:

Lemma 4.1. *If M has bounded p^∞ -torsion, so does its (p) -adic completion \widehat{M} .*

Proof. We use the notations of the definition and we assume that $p^m s_n \rightarrow 0$ when $n \rightarrow \infty$ for some $m \geq l$. Thus, given $k \in \mathbb{N}$, if we write $k' = k + m - l$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we can write $p^m s_n = p^{k'} t$ with $t \in M$. But then $p^m (s_n - p^{k'-m} t) = 0$ and therefore already $p^l (s_n - p^{k'-m} t) = 0$ or $p^l s_n = p^k t$. This shows that $p^l s_n \rightarrow 0$. \square

Definition 4.2. An R -algebra B is bounded⁹ if B is $(p)_q$ -torsion free and $B/(p)_q$ has bounded p^∞ -torsion.

Remarks. 1. If B is bounded, then it is not difficult to see that $\widehat{B[0]} = \widehat{B}[0]$ (the computations are carried out in the proof of lemma 3.7 of [3]). In other words, for our purpose, there is no reason in this situation to introduce the notion of derived completion.

2. It is equivalent to say that B is a complete bounded δ -ring or that $(B, (p)_q)$ is a bounded prism in the sense of Bhatt and Scholze.

Proposition 4.3. If B is a bounded R -algebra, then \widehat{B} also is bounded and the ideal $(p)_q\widehat{B}$ is closed in \widehat{B} (or equivalently $\widehat{B}/(p)_q\widehat{B}$ is complete).

Proof. The point consists in showing that the adic topology on $(p)_qB$ is identical to the topology induced by the adic topology of B . Since the topology is defined by the ideal $(p, (p)_q)$, it is actually sufficient to show that the (p) -adic topology on $(p)_qB$ is identical to the topology induced by the (p) -adic topology of B :

$$\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \quad (p)_qB \cap p^m B \subset p^n (p)_qB.$$

In other words, we have to show that

$$\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \forall f \in B, \quad (\exists g \in B, p^m f = (p)_q g) \Rightarrow (\exists h \in B, p^m f = (p)_q p^n h).$$

But since $B/(p)_q$ has bounded p^∞ -torsion, we know that

$$\exists l \in \mathbb{N}, \forall f \in B, \forall m \in \mathbb{N}, \quad (\exists g \in B, p^m f = (p)_q g) \Rightarrow (\exists h \in B, p^l f = (p)_q h).$$

It is therefore sufficient to set $m = n + l$.

It follows that the sequence

$$0 \longrightarrow B \xrightarrow{(p)_q} B \longrightarrow B/(p)_q \longrightarrow 0$$

is strict exact, and then the sequence

$$0 \longrightarrow \widehat{B} \xrightarrow{(p)_q} \widehat{B} \longrightarrow \widehat{B}/(p)_q \longrightarrow 0$$

is also strict exact. In particular, \widehat{B} is $(p)_q$ -torsion free and the ideal $(p)_q\widehat{B}$ is closed in \widehat{B} . Moreover, $\widehat{B}/(p)_q = \widehat{B}/(p)_q$ has bounded p^∞ -torsion thanks to lemma 4.1. \square

Remark. We have in fact shown that the adic topology on $(p)_qB$ is identical to the topology induced by the adic topology of B . It formally follows that the same holds for the ideal $(q-1)B$. In particular, $(q-1)\widehat{B}$ also is a closed ideal.

The next result is basic but very useful in practice:

Lemma 4.4. If R is bounded and B is flat (over R) then B is also bounded.

⁹We should say q -bounded or even q - p -bounded.

Proof. Upon tensoring with B over R , the exact sequence

$$0 \longrightarrow R \xrightarrow{(p)_q} R \longrightarrow R/(p)_q \longrightarrow 0$$

provides an exact sequence

$$0 \longrightarrow B \xrightarrow{(p)_q} B \longrightarrow B/(p)_q \longrightarrow 0.$$

This shows that B is $(p)_q$ -torsion free. Also, upon tensoring with $B/(p)_q$, which is flat over $R/(p)_q$, the exact sequence

$$0 \longrightarrow R/(p)_q[p^n] \longrightarrow R/(p)_q \xrightarrow{p^n} R/(p)_q \longrightarrow R/((p)_q, p^n) \longrightarrow 0$$

provides another exact sequence

$$0 \longrightarrow B \otimes_R (R/(p)_q)[p^n] \longrightarrow B/(p)_q \xrightarrow{p^n} B/(p)_q \longrightarrow B/((p)_q, p^n) \longrightarrow 0.$$

This shows that $B \otimes_R (R/(p)_q)[p^n] = (B/(p)_q)[p^n]$. It follows that $B/(p)_q$ has bounded p^∞ -torsion (with the same bound as $R/(p)_q$). \square

Definition 4.5. A δ -pair (B, J) is complete if B is complete and J is closed (or equivalently $\overline{B} := B/J$ is also complete). More generally, the completion of a δ -pair (B, J) is the δ -pair $(\widehat{B}, \overline{J})$ where \overline{J} denotes the closure of $J\widehat{B}$ in \widehat{B} .

Remarks. 1. We usually have $J\widehat{B} \neq \overline{J}$ unless B is noetherian.

2. Completion is clearly universal for morphisms to complete δ -pairs.

Example. If B is a bounded δ -ring, then $(\widehat{B}, 0)$, $(\widehat{B}, (q-1)\widehat{B})$ and $(\widehat{B}, \phi^{-1}((p)_q\widehat{B}))$ are complete bounded q -PD-pairs.

We may now make the following definition:

Definition 4.6. If (B, J) is a δ -pair, then (if it exists) its complete q -PD envelope $(\widehat{B}^{\lceil \lceil q}, \widehat{J}^{\lceil \lceil q})$ is a complete q -PD-pair which is universal for morphisms to complete q -PD-pairs.

Proposition 4.7. Let (B, J) be a δ -pair. If the (non complete) q -PD-envelope exists and is bounded, then its completion is the complete q -PD-envelope of (B, J) .

Recall that when R itself is bounded, it will be sufficient to show that $B^{\lceil \lceil q}$ is flat.

Proof. It follows from proposition 4.3 that $\widehat{B}^{\lceil \lceil q}$ is $(p)_q$ -torsion free. But this proposition also tells us that $(p)_q\widehat{B}^{\lceil \lceil q}$ is a closed ideal. In particular, condition (9) is closed and therefore satisfied by $\overline{J}^{\lceil \lceil q}$. Thus, we see that $(\widehat{B}^{\lceil \lceil q}, \overline{J}^{\lceil \lceil q})$ is a q -PD-pair. The universal property is then automatic. \square

Examples. 1. If B is bounded and J is a q -PD-ideal, then $\widehat{B}^{\lceil \lceil q} = \widehat{B}$.

2. If $J = B$, then $B^{\widehat{\lceil} \rceil}_q = 0$. This fundamental property does not rely on the proposition. It follows from example 3 after definition 3.3, which also shows that it is *not* true before completion.
3. If A is a bounded complete unramified δ - R -algebra and if we look at the δ -pair (P, I) , where $P := A \otimes_R A$ and I is the kernel of multiplication $P \rightarrow A$, then $P^{\widehat{\lceil} \rceil}_q = A$. This reduces to the previous cases since $(P, I) \simeq (N, N) \times (A, 0)$.
4. If R is bounded, then $(R[x]^\delta, (x)_\delta)$ has an explicit complete q -PD-envelope, namely the completion of the ring described in proposition 3.5.
5. If A is a complete bounded δ - R -algebra with rank one element x , then $(\widehat{A\langle \xi \rangle}_q, \overline{I^{(1)}})$ is the complete q -PD-envelope of $(A[\xi], \xi)$.

We will also need later the following elementary result:

Lemma 4.8. *If (B, J) is a q -PD-pair and $f \in J$, then f is topologically nilpotent.*

Proof. We have $f^p = \phi(f) + p\delta(f) \in (p)_q B + pB \subset (p, q-1)B$. □

5 Complete q -PD-envelope of a diagonal embedding

As usual, R denotes a δ -ring with fixed rank one element q and we assume that R is actually a $\mathbb{Z}[q]_{(p, q-1)}$ -algebra. Also, any R -algebra will be implicitly endowed with its $(p, q-1)$ -adic topology

We let A be a *complete* R -algebra with a fixed *topologically étale* (that is, formally étale and topologically finitely presented) *coordinate* x . Then, there exists a unique δ -structure on A once $\delta(x)$ is fixed. We will assume that $\delta(x) \in R$ as we did in section 2.

We let $P := A \otimes_R A$ (which is not complete in general), we denote by $p_1, p_2 : A \rightarrow P$ the canonical maps and set $\xi := p_2(x) - p_1(x) \in P$. Unless otherwise specified, we will use the “left” action $f \mapsto p_1(f)$ of A on P in order to turn P into an A -algebra. The ring P comes with its tensor product δ -structure and formula (4) holds since we assumed that $\delta(x) \in R$. It follows that P has a Frobenius ϕ , but, as in section 2, if we set

$$A' := R_{\phi, \rceil} \widehat{\otimes}_R A \quad \text{and} \quad P' := A' \otimes_R A' = R_{\phi, \rceil} \otimes_R P,$$

we could as well consider the relative Frobenius $F : P' \rightarrow P$ (which is a morphism of δ -algebras) as we did in section 7 of [6].

Lemma 5.1. *If two maps $u_1, u_2 : (P, I) \rightarrow (B, J)$ to a complete q -PD-pair coincide when restricted to $A[\xi]$, then they must be equal.*

Proof. The statement means that any map $(P, I) \otimes_{(A[\xi], \xi)} (P, I) \rightarrow (B, J)$ will factor (necessarily uniquely) through the multiplication map

$$(P, I) \otimes_{(A[\xi], \xi)} (P, I) \rightarrow (P, I).$$

Since x is a topologically étale coordinate, if we denote by N the kernel of multiplication from $Q := A \otimes_{R[x]} A$ to A , then the exact sequence

$$0 \rightarrow N \rightarrow Q \rightarrow A \rightarrow 0$$

splits as a sequence of *rings* (see [12, Tag 02FL] in the algebraic setting), and therefore also as a sequence of δ -rings thanks to lemma 1.3. Therefore, letting $M := A \otimes_R N$, there exists a sequence of isomorphisms

$$P \otimes_{A[\xi]} P \simeq A \otimes_R Q \simeq A \otimes_R (N \times A) \simeq M \times P.$$

Actually, there even exists an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & P \otimes_{A[\xi]} I + I \otimes_{A[\xi]} P & \longrightarrow & P \otimes_{A[\xi]} P & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & M \times I & \longrightarrow & M \times P & \longrightarrow & A \longrightarrow 0, \end{array}$$

where the upper right map is total multiplication and the lower right map is composition of projection and multiplication. In particular, there exists an isomorphism of δ -pairs

$$(P, I) \otimes_{(A[\xi], \xi)} (P, I) = (P \otimes_{A[\xi]} P, P \otimes I + I \otimes P) \simeq (M \times P, M \times I) = (M, M) \times (P, I).$$

Let us also note that there exists a commutative diagram

$$\begin{array}{ccc} (P, I) \otimes_{(A[\xi], \xi)} (P, I) & \longrightarrow & (P, I) \\ \downarrow \simeq & & \parallel \\ (M, M) \times (P, I) & \longrightarrow & (P, I), \end{array}$$

where the upper map is multiplication and the lower map is the projection. To finish the proof, it is therefore sufficient to show that the only map $(M, M) \rightarrow (B, J)$ is the zero map. But any such map factors through the *complete* q -PD-envelope of (M, M) which is the zero ring (because we use completions). \square

Since $q - 1$ is topologically nilpotent on A , there exists a unique endomorphism σ of the R -algebra A such that $\sigma(x) = qx$. In particular, A is canonically a *twisted algebra* in the sense of [9]. We extend σ to P in an asymmetric way by setting $\sigma_P := \sigma \otimes \text{Id}_A$ (we will simply write σ when no confusion can arise). If we let

$$I^{(n+1)} := I\sigma(I) \cdots \sigma^n(I)$$

be the *twisted powers* of the ideal I , then the induced maps

$$A[\xi]/(\xi^{(n+1)}) \rightarrow \widehat{P}/\widehat{I}^{(n+1)}$$

are bijective for all $n \in \mathbb{N}$. With the vocabulary of [9], this means that x is a *topological* (because we take into account the topology of A) *q -coordinate* on the adic ring A .

Let us consider now the canonical map $A[\xi] \rightarrow \widehat{A\langle \xi \rangle}_q$. Since the image $(k)_q! \xi^{[k]}$ of $\xi^{(k)}$ goes to zero when k goes to infinity, we see that this canonical map factors uniquely through the twisted power series ring

$$A[[\xi]]_q := \varprojlim A[\xi]/(\xi^{(n+1)}).$$

We define the *Taylor map (of level zero)* as the composite

$$\begin{array}{ccccccc} \theta : A & \xrightarrow{p_2} & P & \longrightarrow & \varprojlim \widehat{P}/\widehat{I}^{(n+1)} & \xrightarrow{\simeq} & A[[\xi]]_q \longrightarrow \widehat{A\langle \xi \rangle}_q \\ f \mapsto & & p_2(f) \mapsto & & \sum \partial_q^{[k]}(f) \xi^k \mapsto & & \sum \partial_q^k(f) \xi^{[k]} \end{array}$$

(the lower line may be understood as a definition for the action of q -derivatives, but we may also simply consider $\partial_q^{[k]}(f)$ and $\partial_q^k(f)$ as a convenient notation for the coefficients of the series). The Taylor map of level zero extends uniquely by A -linearity to a morphism of δ -pairs

$$\tilde{\theta} : (P, I) \rightarrow \left(\widehat{A\langle \xi \rangle}_q, \widehat{I}^{[1]} \right)$$

that the next result shows to be universal.

Theorem 5.2. *If A is bounded, then the q -PD-pair $\left(\widehat{A\langle \xi \rangle}_q, \widehat{I}^{[1]} \right)$ is the complete q -PD-envelope of (P, I) .*

Proof. We give ourselves a map $u : (P, I) \rightarrow (B, J)$ to a complete q -PD-pair. Thanks to theorem 3.6 and proposition 4.7, the restriction of u to $A[\xi]$ extends uniquely to a morphism $v : \left(\widehat{A\langle \xi \rangle}_q, \widehat{I}^{[1]} \right) \rightarrow (B, J)$. Moreover, by functoriality of the Taylor map, the diagram

$$\begin{array}{ccc} (A[\xi], \xi) & \longrightarrow & \left(\widehat{A\langle \xi \rangle}_q, \widehat{I}^{[1]} \right) \\ \downarrow & & \parallel \\ (P, I) & \xrightarrow{\tilde{\theta}} & \left(\widehat{A\langle \xi \rangle}_q, \widehat{I}^{[1]} \right) \end{array}$$

is commutative. We may then apply lemma 5.1 to the maps $u_1 = u$ and

$$u_2 : (P, I) \xrightarrow{\tilde{\theta}} \left(\widehat{A\langle \xi \rangle}_q, \widehat{I}^{[1]} \right) \xrightarrow{v} (B, J). \quad \square$$

6 Hyper q -stratifications

Although we will keep our previous setting (R is a δ -ring with fixed rank one element q which is actually a $\mathbb{Z}[q]_{(p, q-1)}$ -algebra and A is a complete R -algebra with fixed topologically étale coordinate x), what follows is way more general.

We denote by $e : \widehat{A\langle \xi \rangle}_q \rightarrow A$ the augmentation map and by

$$p_1, p_2 : \widehat{A\langle \xi \rangle}_q \rightarrow \widehat{A\langle \xi \rangle}_q \hat{\otimes}'_A \widehat{A\langle \xi \rangle}_q$$

the obvious maps (we use the notation $\hat{\otimes}'$ to indicate that we use the Taylor map θ for the A -structure on the left hand side). We will also consider the diagonal map

$$\begin{aligned} \widehat{A\langle \xi \rangle}_q &\xrightarrow{\Delta} \widehat{A\langle \xi \rangle}_q \hat{\otimes}'_A \widehat{A\langle \xi \rangle}_q \\ \xi^{[n]} &\longmapsto \sum_{i+j=n} \xi^{[i]} \otimes \xi^{[j]}. \end{aligned} \quad (12)$$

Definition 6.1. *A hyper- q -stratification (of level zero)¹⁰ on an A -module M is an $\widehat{A\langle \xi \rangle}_q$ -linear isomorphism*

$$\epsilon : \widehat{A\langle \xi \rangle}_q \hat{\otimes}'_A M \simeq M \otimes_A \widehat{A\langle \xi \rangle}_q$$

such that

$$e^*(\epsilon) = \text{Id}_M \quad \text{and} \quad \Delta^*(\epsilon) = p_1^*(\epsilon) \circ p_2^*(\epsilon). \quad (13)$$

¹⁰We could also say *hyper- q -PD-stratification*.

At this point, we need to recall some notions from our previous articles (taking into account the topology of A). A q -derivation of A with value in an A -module M is a map $D : A \rightarrow M$ that satisfies the twisted Leibniz rule

$$\forall f, g \in A, \quad D(fg) = fD(g) + \sigma(g)D(f).$$

The continuous q -derivations with values in A form a free A -module $\mathbb{T}_{A/R,q}$ on one generator $\partial_{A,q}$ determined by the condition $\partial_{A,q}(x) = 1$. A q -derivation on an A -module M with respect to some q -derivation D_A of A is a map $D_M : M \rightarrow M$ that satisfies the twisted Leibniz rule

$$\forall f \in A, \forall s \in M, \quad D_M(fs) = D_A(f)s + \sigma(f)D_M(s).$$

We call an action of the A -module $\mathbb{T}_{A/R,q}$ on M *topologically quasi-nilpotent* if $D^k(s) \rightarrow 0$ for all $D \in \mathbb{T}_{A/R,q}$ and $s \in M$. Actually, giving an R -linear action of $\mathbb{T}_{A/R,q}$ by q -derivations of M amounts to specifying a q -derivation $\partial_{M,q}$ on M with respect to $\partial_{A,q}$ and the action is topologically quasi-nilpotent if and only if $\partial_{M,q}^k(s) \rightarrow 0$ for all $s \in M$. In order to introduce the (*complete*) *module of q -differential forms*, we recall that we extend in an asymmetric way the endomorphism σ to $P := A \otimes_R A$ and that we denote by I the kernel of multiplication and by $I^{(n)}$ its twisted powers. We may then set

$$\Omega_{A/R,q} = \widehat{I}/\widehat{I}^{(2)}.$$

This is an A -module of rank one with basis $d_q x$, where $d_q : A \rightarrow \Omega_{A/R,q}$ is the universal continuous q -derivation into a complete module¹¹. A q -connection on an A -module M is an R -linear map

$$\nabla : M \rightarrow M \otimes_A \Omega_{A/R,q}$$

that satisfies the twisted Leibniz rule

$$\forall f \in A, \forall s \in M, \quad \nabla(fs) = s \otimes d_q(f) + \sigma(f)\nabla(s).$$

A q -connection on M is clearly equivalent to an action of $\mathbb{T}_{A/R,q}$ by continuous q -derivations via the formula

$$\nabla(s) = \partial_{M,q}(s) \otimes d_q(x).$$

We may also notice that a q -connection or a q -derivation is automatically continuous when M is finitely generated. We need now to introduce the *ring of q -differential operators (of level zero)* which is the non-commutative polynomial ring $\mathbb{D}_{A/R,q}$ in one generator ∂_q over A with the commutation rule

$$\partial_q \circ f = f\partial_q + \partial_{A,q}(f).$$

Clearly, a q -connection is equivalent to a structure of a $\mathbb{D}_{A/R,q}$ -module (through $\partial_q s = \partial_{M,q}(s)$). Note that we may define in general a q -differential operator (of order at most n and level 0) as an A -linear map

$$u : A\langle \xi \rangle_q / I^{[n+1]} \otimes'_A M \rightarrow N,$$

and that we may compose such operators using the maps

$$A\langle \xi \rangle_q / I^{[m+n+1]} \xrightarrow{\Delta_{n,m}} A\langle \xi \rangle_q / I^{[n+1]} \otimes'_A A\langle \xi \rangle_q / I^{[m+1]}$$

¹¹There is however no natural isomorphism between $\Omega_{A/R,q}$ and the usual module of continuous derivations $\Omega_{A/R}$.

induced by (12). In the case $M = N = A$, q -differential operators form a ring that we can identify with $D_{A/R,q}$. More precisely, the basis $\{\xi^{[n]}\}_{n \in \mathbb{N}}$ of $A\langle\xi\rangle_q$ is “dual” to the basis $\{\partial_q^k\}_{k \in \mathbb{N}}$ of $D_{A/R,q}$. Let us finally recall the notion of q -Taylor structure (of level zero) on an A -module M : this is a compatible family of A -linear maps $\theta_n : M \rightarrow M \otimes_A A\langle\xi\rangle_q/I^{[n+1]}$ such that

$$\forall m, n \in \mathbb{Z}_{\geq 0}, \quad (\theta_n \otimes \text{Id}_{A\langle\xi\rangle_q/I^{[m+1]}}) \circ \theta_m = (\text{Id}_M \otimes \Delta_{n,m}) \circ \theta_{m+n}.$$

Alternatively, we could consider the corresponding q -stratification (of level zero), which is a compatible family of $A\langle\xi\rangle_q/I^{[n+1]}$ -linear isomorphisms

$$\epsilon_n : A\langle\xi\rangle_q/I^{[n+1]} \otimes'_A M \simeq M \otimes_A A\langle\xi\rangle_q/I^{[n+1]}$$

satisfying a cocycle condition of the same type as that in (13). Such a structure is actually equivalent to a $D_{A/R,q}$ -module structure via the formula

$$\theta_n(s) = \epsilon_n(1 \otimes s) = \sum_{k=0}^n \partial_q^k s \otimes \xi^{[k]}.$$

Topological quasi-nilpotency travels through all these equivalences.

In order to give an alternative description in the case of q -Taylor structures, we set

$$A\langle\langle\xi\rangle\rangle_q := \varprojlim A\langle\xi\rangle_q/I^{[n+1]}.$$

Then, when M is a finitely presented (in which case tensor product commutes with infinite product) A -module with an action by q -derivations, there exists a *formal q -Taylor map of level zero*

$$\theta : M \rightarrow \varprojlim (M \otimes_A A\langle\xi\rangle_q/I^{[n+1]}) \simeq M \otimes_A A\langle\langle\xi\rangle\rangle_q, \quad s \mapsto \sum \partial_q^k s \otimes \xi^{[k]}.$$

Lemma 6.2. *An R -linear action of $T_{A/R,q}$ by continuous q -derivations on a finitely presented A -module M is topologically quasi-nilpotent if and only if the formal q -Taylor map of level zero factors (uniquely) through $M \otimes_A \widehat{A\langle\xi\rangle}_q$.*

Proof. Since

$$\widehat{A\langle\xi\rangle}_q = \left\{ \sum_{n=0}^{\infty} f_n \xi^{[n]} \in A\langle\langle\xi\rangle\rangle_q, \quad f_n \rightarrow 0 \right\},$$

this is clear from the definition. □

Proposition 6.3. *The following categories are all isomorphic:*

1. A -modules endowed with a topologically quasi-nilpotent q -Taylor structure (or q -stratification) of level zero over R ,
2. topologically quasi-nilpotent $D_{A/R,q}$ -modules,
3. A -modules with a topologically quasi-nilpotent q -connection relative to R ,
4. A -modules endowed with a topologically quasi-nilpotent R -linear action by q -derivations of the tangent module $T_{A/R,q}$.

If we restrict to finitely presented modules, then these categories are also isomorphic to the category of hyper- q -stratified modules.

Proof. It follows from the previous discussion that the first four categories are all isomorphic. Therefore, it only remains to show that on a finitely presented A -module M it is equivalent to give a hyper- q -stratification or a topologically quasi-nilpotent q -Taylor structure of level zero. First of all, if we start with an hyper- q -stratification ϵ , one easily checks that we obtain a q -Taylor structure on M by considering the composite

$$\theta_n : M \rightarrow A\langle\xi\rangle_q/I^{[n+1]} \otimes'_A M \simeq M \otimes_A A\langle\xi\rangle_q/I^{[n+1]}, \quad s \mapsto \epsilon(1 \otimes' s) \pmod{I^{[n+1]}}$$

for each n , and it follows from lemma 6.2 that this structure is topologically quasi-nilpotent. Conversely, if we are given a topologically quasi-nilpotent q -Taylor structure of level zero on M , then the same lemma provides us with an A -linear map

$$M \rightarrow M_A \otimes_A \widehat{A\langle\xi\rangle}_q \subset M \otimes_A A\langle\langle\xi\rangle\rangle_q, \quad s \mapsto \sum \partial^k s \otimes \xi^{[k]}.$$

Extending scalars provides us with an $\widehat{A\langle\xi\rangle}_q$ -linear map

$$\epsilon : \widehat{A\langle\xi\rangle}_q \otimes'_A M \rightarrow M \otimes_A \widehat{A\langle\xi\rangle}_q.$$

In order to show that this map is bijective, it is sufficient to check that the inverse is induced in the same way by the map

$$M \mapsto A\langle\langle\xi\rangle\rangle_q \otimes'_A M, \quad s \mapsto \sum (-1)^s \xi^{[k]} \otimes \partial^k s,$$

but this is straightforward. It only remains the easy task to check that we do obtain an hyper- q -stratification (and that our constructions are quasi-inverse one to the other): this is left to the reader. \square

7 q -crystals

All δ -rings live over the local ring $\mathbb{Z}[q]_{(p,q-1)}$ (with q of rank one) and they are endowed with their $(p, q-1)$ -adic topology. We fix a morphism of bounded q -PD-pairs $(R, \mathfrak{r}) \rightarrow (A, \mathfrak{a})$.

The absolute (big) q -crystalline site q -CRIS is the category opposite¹² to the category of complete bounded¹³ q -PD-pairs (B, J) (or equivalently complete q -PD-thickenings $B \twoheadrightarrow \overline{B}$ with B bounded). We may actually consider the slice category q -CRIS/ R over $(\widehat{R}, \widehat{\mathfrak{r}})$: an object is a morphism $(R, \mathfrak{r}) \rightarrow (B, J)$ to a complete bounded q -PD-pair (and a morphism is the opposite of a morphism of q -PD-pairs which is compatible with the structural maps). Now, if we denote by FSch the category of $(p, q-1)$ -adic formal schemes (over $\mathrm{Spf}(\mathbb{Z}[q]_{(p,q-1)}) = \mathrm{Spf}(\mathbb{Z}_p[[q-1]])$), then we may consider the functor

$$q\text{-CRIS} \rightarrow \mathrm{FSch}, \quad (B, J) \mapsto \mathrm{Spf}(\overline{B}) \quad (\text{with } \overline{B} = B/J).$$

If \mathcal{X} is a $(p, q-1)$ -adic formal scheme, then the absolute q -crystalline site q -CRIS(\mathcal{X}) of \mathcal{X} will be the corresponding fibered site over \mathcal{X} : an object is a pair made of a complete bounded q -PD-pair (B, J) and a morphism $\mathcal{X} \rightarrow \mathrm{Spf}(\overline{B})$ of formal schemes.

¹²We will always write morphisms of pairs in the usual way and call it an opposite morphism if we consider it as a morphism in the q -crystalline site.

¹³As in [3], we could add some other technical assumptions: see their definition 16.2.

We will actually mix both constructions and consider, if \mathcal{X} is a $(p, q - 1)$ -adic formal \overline{R} -scheme, the q -crystalline site $q\text{-CRIS}(\mathcal{X}/R)$ of \mathcal{X}/R : an object is a pair made of a morphism $(R, \mathfrak{r}) \rightarrow (B, J)$ to a complete bounded q -PD-pair together with a morphism $\mathcal{X} \rightarrow \text{Spf}(\overline{B})$ of formal \overline{R} -schemes. We will write

$$q\text{-CRIS}(\overline{A}/R) := q\text{-CRIS}(\text{Spf}(\overline{A})/R).$$

For the moment, we endow the category $q\text{-CRIS}(\mathcal{X}/R)$ with the *coarse* topology so that a sheaf is simply a presheaf and we denote by $(\mathcal{X}/R)_{q\text{-CRIS}}$ the corresponding topos¹⁴. A sheaf (of sets) E on $q\text{-CRIS}(\mathcal{X}/R)$ is thus a family of sets E_B together with a compatible family of maps $E_B \rightarrow E_{B'}$ for any morphism opposite to $(B, J) \rightarrow (B', J')$ in $q\text{-CRIS}(\mathcal{X}/R)$. In particular, we may consider the sheaf of rings $\mathcal{O}_{\mathcal{X}/R, q}$ that sends B to itself (we will write $\mathcal{O}_{\overline{A}/R, q}$ when $\mathcal{X} = \text{Spf}(\overline{A})$ as above). A sheaf of $\mathcal{O}_{\mathcal{X}/R, q}$ -modules is a family of B -modules E_B endowed with a compatible family of semi-linear maps $E_B \rightarrow E_{B'}$, or better, of linear maps $B' \otimes_B E_B \rightarrow E_{B'}$, called the *transition maps*.

Definition 7.1. *If \mathcal{X} is a $(p, q - 1)$ -adic formal \overline{R} -scheme, then a q -crystal on \mathcal{X}/R is a sheaf of $\mathcal{O}_{\mathcal{X}/R, q}$ -modules whose transition maps are all bijective.*

Remark. An $\mathcal{O}_{\mathcal{X}/R, q}$ -module E is finitely presented if and only if it is a q -crystal and all E_B are finitely presented B -modules.

At this point, it is convenient to generalize the notion of hyper- q -stratification:

Definition 7.2. *If $(R, \mathfrak{r}) \rightarrow (B, J)$ is a morphism to a complete q -PD-pair, then an hyper- q -stratification (of level zero) on a B -module N is a $(B \otimes_R B)^{\widehat{\cap}}_q$ -linear isomorphism*

$$\epsilon : (B \otimes_R B)^{\widehat{\cap}}_q \otimes_B N \simeq N \otimes_B (B \otimes_R B)^{\widehat{\cap}}_q$$

satisfying the usual normalization and cocycle conditions.

- Remarks.**
1. This definition assumes that the completed q -PD-envelopes exist.
 2. It follows from theorem 5.2 that this is consistent with definition 6.1 when $B = A$ is a complete R -algebra and x is a topologically étale coordinate of rank one on A .
 3. If \mathcal{X} is a $(p, q - 1)$ -adic formal \overline{R} -scheme and $(B, J) \in q\text{-CRIS}(\mathcal{X}/R)$, then there is an obvious functor $E \mapsto E_B$ from q -crystals on \mathcal{X}/R to hyper- q -stratified modules on B/R obtained by composing the transition maps

$$(B \otimes_R B)^{\widehat{\cap}}_q \otimes_B E_B \simeq E_{(B \otimes_R B)^{\widehat{\cap}}_q} \simeq E_B \otimes_B (B \otimes_R B)^{\widehat{\cap}}_q.$$

Theorem 7.3. *If A is complete and x is a topologically étale coordinate of rank one on A , then there exists a functor $E \mapsto E_A$ from finitely presented q -crystals on \overline{A}/R to finitely presented topologically quasi-nilpotent $D_{A/R, q}$ -modules.*

Proof. Using the previous remarks, this follows from proposition 6.3. □

We do not know yet how far this functor is from being an equivalence.

¹⁴Unlike in [3], we follow the standard notations for sites and topos.

Proposition 7.4. *Assume that A is complete with rank one topologically étale coordinate x . We denote by S the q -PD-envelope of $R[x_1]^\delta$ and we endow $U := S \widehat{\otimes}_R A$ with the unique δ -structure such that $\delta(x) = x_1$. Then, U is a covering of (the final object of the topos associated to) $q\text{-CRIS}(\overline{A}/R)$.*

Proof. The strategy is standard (see theorem 6.6 of [2] for example). If $(B, J) \in q\text{-CRIS}(\overline{A}/R)$, then there always exists a lifting $p_B : A \rightarrow B$ of the structural map $\overline{A} \rightarrow \overline{B}$ because B is complete, A is formally smooth and topologically of finite type over R , and, as we saw in lemma 4.8, J is made of topologically nilpotent elements. We may then lift p_B (uniquely) to a morphism of q -PD-pairs $\tilde{p}_B : U \rightarrow B$: we use successively the universal properties of the polynomial ring $R[x_1]$ (sending x_1 to $\delta(p_B(x))$), of the δ -envelope $R[x_1]^\delta$, of the q -PD-envelope and finally of completion. \square

Remarks. 1. If we denote by \mathfrak{s} the augmentation ideal of S , then we are actually considering the q -PD-pair $(U, \mathfrak{u}) = (S, \mathfrak{s}) \widehat{\otimes}_R (A, \mathfrak{a})$, which is a q -PD-thickening of \overline{A} .

2. It is important to notice that the morphism p_B in the proof is *not* a morphism of δ -rings (in which case we would not need to introduce the q -PD-ring U).

Corollary 7.5. *The following categories are equivalent:*

1. q -crystals on \overline{A}/R (with $\overline{A} = A/\mathfrak{a}$),
2. hyper- q -stratified modules on U/R .

This equivalence preserves finitely presented modules. \square

Remarks. 1. We have stayed into the framework of [3] and have only considered the coarse topology on the q -crystalline site. In order to get more properties of the functor defined in theorem 7.3 than its sole existence, it could be necessary to use finer topologies like in [1] for the prismatic site.

2. We are mostly interested in the coefficient problem but as long as cohomology is concerned, one can say a lot more. The canonical morphism $U \rightarrow A$ induces an isomorphism on the Čech-Alexander complexes (assuming $q - 1 \in \mathfrak{r}$): using derived Nakayama lemma, one may assume that $q = 1$ in which case we are computing usual crystalline cohomology (see [3], remark 16.15). As a consequence, one recovers the comparison theorem 16.21 of [3] between q -de Rham cohomology and q -crystalline cohomology: our q -PD-envelope $A\langle\xi\rangle_q$ is explicit enough to see that q -de Rham cohomology is isomorphic to Čech-Alexander cohomology.

8 Appendix: 1-crystals vs usual crystals

There already exists a category of crystals (the *usual* ones) that deserves to be compared with this new category of q -crystals when $q = 1$.

Let us recall what we know so far. First of all, a 1-PD-pair is the same thing as a p -torsion-free *usual* PD-pair endowed with a δ -structure. We also know that when B is a p -torsion-free δ -ring and J is generated by a regular sequence modulo p , then the 1-PD-envelope of (B, J) exists and is identical to the *usual* PD-envelope. Note that the result

actually also holds when J is generated by an infinite regular sequence modulo p (because both envelopes commute with colimits).

We defined the *1-crystalline site* 1-CRIS as the category opposite to the category of p -adically complete bounded *1-PD*-pairs (B, J) . We may also consider the *crystalline site* CRIS that we define¹⁵ as in [12, Tag 07KH] to be the category opposite to the category of p -adically complete *usual PD*-pairs (B, J) . All the results from section 7 have analogs on the site CRIS . Moreover, since a *1-PD*-pair has a natural *PD*-structure, there exists a functor $1\text{-CRIS} \rightarrow \text{CRIS}$ that forgets the δ -structure. If R is a bounded q -*PD*-pair and \mathcal{X} is a p -adic formal \overline{R} -scheme, then this functor provides us with a morphism of topos

$$(\mathcal{X}/R)_{\text{CRIS}} \rightarrow (\mathcal{X}/R)_{1\text{-CRIS}}$$

and pulling back preserves crystals.

Assume now that we are in the situation of proposition 7.4 so that we are given a morphism $(R, \mathfrak{r}) \rightarrow (A, \mathfrak{a})$ of *1-PD*-pairs where A is p -adically complete bounded with rank one topologically étale coordinate x . Recall that $R[x_1]^\delta = R[x_1, x_2, \dots]$ and that $(x_1)_\delta = (x_1, x_2, \dots)$. In particular, this is a (infinite) regular sequence and the *1-PD*-envelope S of $(R[x_1]^\delta, (x_1)_\delta)$ is therefore also the *usual PD*-envelope. It formally follows that there exists an equivalence of categories between *1*-crystals and *usual* crystals on \overline{A}/R .

Classically, it is not exactly our site CRIS that is considered. For all $n \in \mathbb{N}$, there exists the category CRIS_n of *usual PD*-pairs (B, J) over \mathbb{Z}/p^n and one then looks at the *formal crystalline site* $\bigcup_{n \in \mathbb{N}} \text{CRIS}_n$. This is actually a subcategory of our site CRIS and there also exists an obvious family of functors $\text{CRIS} \rightarrow \text{CRIS}_n$ which are given by reduction modulo p^n that allows us to go back and forth. If we assume that $p^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$, then we may consider the category of crystals on the *formal crystalline site* of \overline{A}/R . This is actually the subcategory of our previous category of crystals (or *1*-crystals since, as we have explained, they are equivalent in this situation) E on \overline{A}/R which is defined by the extra condition that E_A must be complete.

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¹⁵We will explain below how this relates to the classical definition.

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