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COEFFICIENTS OF DRINFELD MODULAR FORMS AND HECKE OPERATORS

by

Cécile Armana

Abstract. — Consider the space of Drinfeld modular forms of fixed weight and type for $\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(\mathbf{F}_q[T])$. It has a linear form b_n , given by the coefficient of $t^{m+n(q-1)}$ in the power series expansion of a type m modular form at the cusp infinity, with respect to the uniformizer t . It also has an action of a Hecke algebra. Our aim is to study the Hecke module spanned by b_1 . We give elements in the Hecke annihilator of b_1 . Some of them are expected to be nontrivial and such a phenomenon does not occur for classical modular forms. Moreover, we show that the Hecke module considered is spanned by coefficients b_n , where n runs through an infinite set of integers. As a consequence, for any Drinfeld Hecke eigenform, we can compute explicitly certain coefficients in terms of the eigenvalues. We give an application to coefficients of the Drinfeld Hecke eigenform h .

1. Introduction

Drinfeld modular forms are certain analogues over $\mathbf{F}_q[T]$ of classical modular forms, introduced by D. Goss [12, 13]. A Drinfeld modular form f has a power series expansion with respect to a canonical uniformizer t at the cusp infinity. If f has type m , this expansion is $\sum_{n \geq 0} b_n(f)t^{m+n(q-1)}$. On the space of Drinfeld modular forms of fixed weight and type, we have the linear form $b_n : f \mapsto b_n(f)$ and an action of a Hecke algebra. In the present work, we investigate the Hecke module spanned by b_1 .

Our interest in the problem comes from the torsion of rank-2 Drinfeld modules. In a previous work, we established a uniform bound on the torsion under an assumption on the latter Hecke module in weight 2 and type 1 (see [1, 2]). This condition was required for studying a Drinfeld modular curve at a neighborhood of the cusp infinity, namely for showing that the map from the curve (or rather a symmetric power) to a quotient of its Jacobian variety is a formal immersion at this cusp in a special fiber.

Before stating the main results, we fix some notations. Let $A = \mathbf{F}_q[T]$ be the ring of polynomials over a finite field \mathbf{F}_q in an indeterminate T , $K = \mathbf{F}_q(T)$ the field of rational functions, $K_\infty = \mathbf{F}_q((1/T))$ and \mathbf{C}_∞ the completion of an algebraic closure of K_∞ . For an ideal \mathfrak{n} of A , $k \in \mathbf{N}$ and $0 \leq m < q - 1$, we consider the \mathbf{C}_∞ -vector space $M_{k,m}(\Gamma_0(\mathfrak{n}))$ of Drinfeld modular forms of weight k and type m for the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $\mathrm{GL}_2(A)$ (see Section 4.1 for the definition). These are rigid analytic \mathbf{C}_∞ -valued functions

on $\mathbf{C}_\infty - K_\infty$ which have an interpretation as multi-differentials on the Drinfeld modular curve attached to $\Gamma_0(\mathfrak{n})$.

Let $\mathbf{T} = \mathbf{T}_{k,m}(\Gamma_0(\mathfrak{n}))$ be the Hecke algebra, that is the commutative subring of $\text{End}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})))$ spanned over \mathbf{C}_∞ by all Hecke operators T_P for P monic polynomial in A (see Section 4.2). Its restriction $\mathbf{T}' = \mathbf{T}'_{k,m}(\Gamma_0(\mathfrak{n}))$ to the subspace $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$ of doubly cuspidal forms (with expansion vanishing at order ≥ 2 at all cusps) stabilizes this subspace. As Goss first observed, doubly cuspidal Drinfeld modular forms play a role similar to classical cusp forms.

In this work, we are interested in the pairing between the space $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$ and the Hecke algebra \mathbf{T}' given by the coefficient b_1 of the expansion. More precisely, the dual space $\text{Hom}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ has a natural right action of \mathbf{T} (given by composition) and contains the linear form $b_n : f \mapsto b_n(f)$. Let $u = u_{k,m,\mathfrak{n}} : \mathbf{T}' \rightarrow \text{Hom}_{\mathbf{C}_\infty}(M_{k,m}^2(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ be \mathbf{C}_∞ -linear map defined by $s \mapsto b_1 s$. Our main results concern the kernel \mathbf{I} and the image $b_1 \mathbf{T}'$ of u .

Let A_{d+} be the set of monic polynomials of degree d in A . The first statement gives a family of elements of \mathbf{I} .

Theorem 1.1. — *The following elements of \mathbf{T}' belong to \mathbf{I} :*

1. $\sum_{P \in A_{1+}} P^{1-m} T_P + T_1$ if $m \in \{0, 1\}$.
2. $\sum_{P \in A_{d+}} C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}} T_P$ if $d \geq 1$ and $(i_0, \dots, i_{d-1}) \in \mathbf{N}^d$ is such that
 - (1) $0 \leq i_j \leq q - m$ for all $j \in \{0, \dots, d-1\}$
 - (2) $i_0 + \dots + i_{d-1} \leq (d-1)(q-1) - m$.

Here, $C_{P,j} \in A$ stands for the j th coefficient of the Carlitz module at P (see Section 3.1 for its definition).

3. $\sum_{P \in A_{d+}} P^l T_P$ if $0 \leq l \leq q - m$ and $d \geq 1 + (l + m)/(q - 1)$
 $\sum_{P \in A_{d+}} T_P$ if $d \geq 2$, or if $d = 1$ and $m = 0$.

These elements actually belong to the span over A of all Hecke operators. Moreover, they are universal in the sense that, for a given type m , they do not depend on the weight k nor on the ideal \mathfrak{n} .

In most cases, we believe that $\mathbf{I} \neq 0$, that is at least one element of Theorem 1.1 is a nontrivial endomorphism of $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$, hence the pairing is not perfect. Over the space $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ with \mathfrak{n} prime, the situation is as follows. If \mathfrak{n} has degree 3, we prove that $\mathbf{I} = 0$ (Theorem 7.7). If \mathfrak{n} has degree ≥ 5 , numerical experiments suggest that $\mathbf{I} \neq 0$ (Conjecture 6.9). Moreover, it may happen that some elements of Theorem 1.1 are zero in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$: examples of such a situation are explored in Section 6.3.

For the rest of the introduction, we restrict our attention to Drinfeld modular forms of type 0 or 1. Our second statement gives an infinite family of coefficients of Drinfeld modular forms in $b_1 \mathbf{T}'$.

Theorem 1.2. — *Assume q is a prime and $m \in \{0, 1\}$. Let \mathcal{S} be the set of natural integers of the form $c/(q-1)$, where $c \in \mathbf{N}$ is such that the sum of its base q digits is $q-1$. For every $n \in \mathcal{S}$, there exists $s_n \in \mathbf{T}'$, independent of k and \mathfrak{n} , satisfying*

$$b_n = b_1 s_n \in b_1 \mathbf{T}'.$$

Moreover, $b_1\mathbf{T}'$ is the \mathbf{C}_∞ -vector space spanned by b_n for all $n \in \mathcal{S}$.

The primality assumption on q is not essential (see Remark 7.3). As for the set \mathcal{S} , it is infinite of natural density zero and the first integer not belonging to \mathcal{S} is $q + 1$. For example, if $q = 3$, the first elements of \mathcal{S} are

$$1, 2, 3, 5, 6, 9, 14, 15, 18, 27, 41, 42, 45, 54, 81.$$

Theorem 1.2 relies on an explicit version, Theorem 7.2 (the elements s_n that we produce depend on whether the type is 0 or 1). The expression for s_n is rather natural: it is a A -linear combination of Hecke operators T_P , with P of fixed degree, involving Carlitz binomial coefficients in A .

Suppose now that $\mathbf{I} \neq 0$. Then the map u fails to be surjective (see Lemma 6.2). In particular, $b_1\mathbf{T}'$ does not contain all linear forms b_n for $n \geq 1$. It is then natural to ask what is the smallest integer n such that $b_n \notin b_1\mathbf{T}'$. Theorem 1.2 suggests that $n = q + 1$ might be a good candidate.

Both theorems bring new insight on Drinfeld Hecke eigenforms. Consider a Drinfeld modular form f which is an eigenform for the Hecke algebra \mathbf{T} . Theorem 1.1 translates into linear relations among eigenvalues of f , provided that $b_n(f) \neq 0$ for some $n \in \mathcal{S}$ (Proposition 6.5 and Corollary 7.5). Similarly, Theorem 7.2 gives explicit formulas for coefficients $b_n(f)$ ($n \in \mathcal{S}$) in terms of eigenvalues of f and $b_1(f)$. From Theorem 7.2, we also derive:

- multiplicity one statements in some spaces of Drinfeld modular forms of small dimension (Theorem 7.7); as far as we know, these are the only known results of this kind for Drinfeld modular forms.
- explicit expressions for some coefficients of the Drinfeld modular form h (Proposition 8.1). This extends previous work of Gekeler.

As a side remark, we give a brief account of the multiplicity one problem for Drinfeld modular forms. Since there exist two Hecke eigenforms for $\mathrm{GL}_2(A)$ with different weights and same system of eigenvalues (Goss [12]), the question of multiplicity one should be stated as: do eigenvalues and weight determine the Hecke eigenform, up to a multiplicative constant? (see Gekeler [7], Section 7). Böckle and Pink showed that this does not hold for doubly cuspidal forms of weight 5 for the group $\Gamma_1(T)$ when $q > 2$ by means of cohomological techniques (Example 15.4 of [3]). Except for Theorem 7.7 mentioned above, the question remains open for $\Gamma_0(\mathfrak{n})$.

We now compare our results with their analogues for classical modular forms. Consider the space $S_k(\Gamma_0(N))$ of cuspidal modular forms of weight k for the subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$ ($N \geq 1$). Let $(c_n(f))_{n \geq 1}$ be the Fourier coefficients of such a modular form f at the cusp infinity. Computing the action of the n th Hecke operator T_n on the Fourier expansion of f gives the well-known relation, for any $n \geq 1$

$$(3) \quad c_n(f) = c_1(T_n f).$$

In particular, the Hecke module spanned by the linear form c_1 , which now contains all coefficients c_n , is the whole dual space of $S_k(\Gamma_0(N))$ and the coefficient c_1 gives rise to a perfect pairing over \mathbf{C} between $S_k(\Gamma_0(N))$ and the Hecke algebra. Conjecture 6.9 and Theorem 1.2 thus suggest a phenomenon specific to the function field setting. For Drinfeld modular forms, the reason for not having straightforward statements about the

kernel and image of u is that the action of Hecke operators on the expansion is not well understood. Goss [12, 13, 11] and subsequently Gekeler [7] wrote down this action using Goss polynomials. But such polynomials are difficult to handle (see also Remark 5.3). In particular, a relation as general as (3) is lacking.

We now sketch the proofs of Theorems 1.1 and 1.2, which involve rather elementary techniques.

- We first compute the coefficient $b_1(T_P f)$, for any f and P , using Goss polynomials (Proposition 5.5). Note that the formula we get is more intricate than (3): it is a A -linear combination of several coefficients of f . For the next step, the crucial point is that the index of these coefficients depends only on the degree of P . This already proves that $b_1 \mathbf{T}'$ is contained in the \mathbf{C}_∞ -vector space spanned by b_n , for $n \in \mathcal{S}$ when $m \in \{0, 1\}$ (Corollary 5.8).
- We take advantage of characteristic p . For power sums of polynomials of a given degree in A , vanishing properties and closed formulas are well-known (see [21, III] for a survey). Here we use a variant consisting of power sums of coefficients of the Carlitz module. Such sums are studied in Section 3 and closed formulas are given in Proposition 3.5. In Section 3.4, we also explain their connection with Carlitz binomial coefficients and special values of Goss zeta function at negative integers.
- By taking adequate linear combinations of $b_1(T_P f)$, for P of fixed degree, and using results of Section 3, we obtain elements in the kernel \mathbf{I} (Theorem 1.1, Section 6) and in the image $b_1 \mathbf{T}'$ (Theorems 7.2 and 1.2).

For the study of the Hecke module $b_1 \mathbf{T}'$, our method has reached its limit and improving our results would require new ideas. Our approach might be used to tackle other Hecke modules $b_i \mathbf{T}'$: however, computing $b_i(T_P f)$ for any $i \geq 2$ is a harder combinatorial problem.

2. Notations

A tuple will always be a tuple of nonnegative integers. For such a tuple $\underline{i} = (i_0, \dots, i_s)$, let $\binom{i_0 + \dots + i_s}{\underline{i}}$ be the generalized multinomial coefficient $\frac{(i_0 + \dots + i_s)!}{i_0! \dots i_s!}$.

Let q be a power of a prime p and \mathbf{F}_q (resp. \mathbf{F}_p) be a finite field with q (resp. p) elements. We will use repeatedly the following theorem of Lucas: $\binom{i_0 + \dots + i_s}{\underline{i}}$ is nonzero in \mathbf{F}_p if and only if there is no carry over base p in the sum $i_0 + \dots + i_s$.

We keep the same notations as in the introduction. On $A = \mathbf{F}_q[T]$, we have the usual degree \deg with the convention $\deg 0 = -\infty$. By convention, any ideal of A that we will consider is nonzero. We will often identify an ideal \mathfrak{p} of A with the monic polynomial $P \in A$ generating \mathfrak{p} . Accordingly, $\deg \mathfrak{p}$ stands for $\deg P$.

Let $K_\infty = \mathbf{F}_q((1/T))$ be the completion of K at $1/T$ with the natural nonarchimedean absolute value $|\cdot|$ such that $|T| = q$. We write \mathbf{C}_∞ for the completion of an algebraic closure of K_∞ : it is an algebraically closed complete field for the canonical extension of $|\cdot|$ to \mathbf{C}_∞ .

For P, Q in A , (P) denotes the principal ideal generated by P , $P \mid Q$ means P divides Q and (P, Q) is the g.c.d. of P and Q . The integer part is denoted by $[\cdot]$.

3. Power sums of Carlitz coefficients

3.1. The Carlitz module. — Let $A\{\tau\}$ the noncommutative ring of polynomials in the indeterminate τ with coefficients in A for the multiplication given by $\tau a = a^q \tau$ ($a \in A$). By the map $\tau \mapsto X^q$, the ring $A\{\tau\}$ can be identified with the subring of $\text{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$ of additive polynomials of the form $\sum a_i X^{q^i}$ (where the multiplication law is given by composition). The Carlitz module is the rank-1 Drinfeld module $C : A \rightarrow A\{\tau\}$ defined by $C_T = T\tau^0 + \tau$. For $a \in A$, we put C_a for the image of a by C , as usual, and $C_a = \sum_{k=0}^{\deg a} C_{a,k} \tau^k$ with $C_{a,k} \in A$. In particular, $C_{a,0} = a$ and $C_{a,d} = 1$ if a is monic of degree d .

3.2. Deformation of the Carlitz module. — We study the dependence of $C_{a,k}$ in the coefficients of a , when a is viewed as a polynomial in T . For this purpose, we need a formal version of the Carlitz module. Let $\mathbf{F}_q[T, \mathbf{a}] = \mathbf{F}_q[T, \mathbf{a}_0, \mathbf{a}_1, \dots]$ be the polynomial ring in T and an infinite set of indeterminates $\{\mathbf{a}_i\}_{i \geq 0}$. Consider the ring homomorphism

$$\mathbf{C} : \mathbf{F}_q[T, \mathbf{a}] \longrightarrow \mathbf{F}_q[T, \mathbf{a}]\{\tau\}$$

defined by

$$\mathbf{C}_T = T\tau^0 + \tau, \quad \mathbf{C}_{\mathbf{a}_i} = \mathbf{a}_i \tau^0 \quad \text{for all } i \geq 0$$

where the noncommutative ring $\mathbf{F}_q[T, \mathbf{a}]\{\tau\}$ is defined in the obvious way. Let P be an element of $\mathbf{F}_q[T, \mathbf{a}]$ and d its degree as a polynomial in T . We define $\mathbf{C}_{P,0}, \dots, \mathbf{C}_{P,d}$ in $\mathbf{F}_q[T, \mathbf{a}]$ by $\mathbf{C}_P = \sum_{i=0}^d \mathbf{C}_{P,i} \tau^i$. These coefficients satisfy the following recursive formulas.

Lemma 3.1. — *Let $P \in \mathbf{F}_q[T, \mathbf{a}]$ be a monic of degree d in T . Write $P = Tb + c$, with $c \in \mathbf{F}_q[\mathbf{a}]$ and $b \in \mathbf{F}_q[T, \mathbf{a}]$ monic of degree $d-1$ in T . Then*

$$\begin{aligned} \mathbf{C}_{P,0} &= T\mathbf{C}_{b,0} + c = P \\ \mathbf{C}_{P,i} &= T\mathbf{C}_{b,i} + \mathbf{C}_{b,i-1}^q \quad (1 \leq i \leq d-1) \\ \mathbf{C}_{P,d} &= \mathbf{C}_{b,d-1}^q = 1. \end{aligned}$$

Proof. — Since \mathbf{C} is additive, we have $\mathbf{C}_{P,i} = \mathbf{C}_{Tb,i} + \mathbf{C}_{c,i}$. Moreover, $\mathbf{C}_{c,i}$ is c if $i=0$ and 0 otherwise. It remains to compute $\mathbf{C}_{Tb,i}$ in terms of $\mathbf{C}_{b,i}$. We have the following equalities in $\mathbf{F}_q[T, \mathbf{a}]\{\tau\}$:

$$\mathbf{C}_{Tb} = \mathbf{C}_T \mathbf{C}_b = (T\tau^0 + \tau) \left(\sum_{i=0}^{d-1} \mathbf{C}_{b,i} \tau^i \right) = T \left(\sum_{i=0}^{d-1} \mathbf{C}_{b,i} \tau^i \right) + \sum_{i=0}^{d-1} \mathbf{C}_{b,i}^q \tau^{i+1}.$$

By identification, we get our claim. \square

Lemma 3.2. — *Let $d \geq 1$ and $P \in \mathbf{F}_q[T, \mathbf{a}]$ monic of degree d in T . Write $P = T^d + n_{d-1}T^{d-1} + \dots + n_0$ with $n_0, \dots, n_{d-1} \in \mathbf{F}_q[\mathbf{a}]$. For all $0 \leq j \leq d-1$, one has*

$$\mathbf{C}_{P,j} = n_j^{q^j} + TQ_j \quad \text{with } Q_j \in \mathbf{F}_q[T, n_k \mid k > j].$$

In particular, if $P = T^d + \mathbf{a}_{d-1}T^{d-1} + \dots + \mathbf{a}_0$, the polynomial $\mathbf{C}_{P,j}$ is independent of \mathbf{a}_0 for $j \geq 1$.

Proof. — For $j = 0$, we have $\mathbf{C}_{P,0} = P = n_0 + T(n_1 + \dots + n_{d-1}T^{d-1})$ which has the expected form. For other coefficients, we proceed by induction on d . The statement is already proven for $d = 1$. Suppose the property satisfied for all monic polynomials of degree $< d$ in T . Let $P = T^d + n_{d-1}T^{d-1} + \dots + n_0$ and write $P = Tb + n_0$ with $b \in \mathbf{F}_q[T, n_1, \dots, n_{d-1}]$ monic of degree $< d$ in T . Let $1 \leq j \leq d-1$. By Lemma 3.1, we have

$$(4) \quad \mathbf{C}_{P,j} = T\mathbf{C}_{b,j} + \mathbf{C}_{b,j-1}^q.$$

By hypothesis, there exists $R_{j-1} \in \mathbf{F}_q[T, n_k \mid k > j]$ and $R_j \in \mathbf{F}_q[T, n_k \mid k > j+1]$ such that $\mathbf{C}_{b,j} = n_{j+1}^{q^j} + TR_j$ and $\mathbf{C}_{b,j-1} = n_j^{q^{j-1}} + TR_{j-1}$. Substituting in (4), we get $\mathbf{C}_{P,j} = n_j^{q^j} + T(n_{j+1}^{q^j} + TR_j + T^{q-1}R_{j-1}^q)$. Since $n_{j+1}^{q^j} + TR_j + T^{q-1}R_{j-1}^q$ belongs to $\mathbf{F}_q[T, n_k \mid k > j]$, the coefficient $\mathbf{C}_{P,j}$ has the expected form. The property is then established for any monic polynomial P of degree d . \square

3.3. Power sums of Carlitz coefficients. —

Notation 3.3. — Let $d \geq 1$. Recall that the set of monic polynomials of degree d in A is denoted by A_{d+} . For $P \in A_{d+}$ and $\underline{i} = (i_0, \dots, i_{d-1})$, let

$$C(P)^{\underline{i}} = C_{P,0}^{i_0} \cdots C_{P,d}^{i_d} = C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}}$$

(the last equality follows from $C_{P,d} = 1$). By convention, $0^0 = 1$. Let

$$S_d(i_0, \dots, i_{d-1}) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} \in A.$$

Note that for $d = 1$, the sum is just $S_1(i) = \sum_{P \in A_{1+}} P^i$. We will compute $S_d(i_0, \dots, i_{d-1})$ for small i_0, \dots, i_{d-1} .

Lemma 3.4. — Let $0 \leq i \leq 2(q-1)$ and $P \in A$. Then

$$\sum_{a \in \mathbf{F}_q} (P+a)^i = \begin{cases} -1 & \text{if } i = q-1 \text{ or } 2(q-1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The vanishing case is merely an application of Lemma 3.1 of Goss [10]. Since we need to compute the remaining cases, we give a full proof. Let $R_i(P) = \sum_{a \in \mathbf{F}_q} (P+a)^i$. Then by the binomial formula,

$$R_i(P) = \sum_{k=0}^i \binom{i}{k} P^{i-k} \left(\sum_{a \in \mathbf{F}_q} a^k \right).$$

Recall that $\sum_{a \in \mathbf{F}_q} a^k$ equals -1 if $k > 0$ and $k \equiv 0 \pmod{q-1}$, and 0 otherwise. Thus $R_{q-1}(P) = -1$ and $R_i(P) = 0$ if $0 \leq i < q-1$. Now let $i = q+j$ with $0 \leq j \leq q-2$. Then

$$R_i(P) = \sum_{a \in \mathbf{F}_q} (P^q + a)(P+a)^j = P^q R_j(P) + \sum_{a \in \mathbf{F}_q} a(P+a)^j.$$

Since $j \leq q - 2$, $R_j(P)$ is zero. Moreover, by the binomial formula,

$$\sum_{a \in \mathbf{F}_q} a(P+a)^j = \sum_{k=0}^j \binom{j}{k} P^{j-k} \left(\sum_{a \in \mathbf{F}_q} a^{k+1} \right)$$

which is 0 if $j < q - 2$ (resp. -1 if $j = q - 2$). \square

Proposition 3.5. — *Let $i_j \in \{0, \dots, 2(q-1)\}$ for all $j \in \{0, \dots, d-1\}$. Then*

$$S_d(i_0, \dots, i_{d-1}) = \begin{cases} (-1)^d & \text{if, for all } j, i_j \in \{q-1, 2(q-1)\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The sum $S_d(i_0, \dots, i_{d-1})$ is equal to

$$\sum_{a_0, \dots, a_{d-1} \in \mathbf{F}_q} C_{T^d + a_{d-1}T^{d-1} + \dots + a_0, 0}^{i_0} \cdots C_{T^d + a_{d-1}T^{d-1} + \dots + a_0, d-1}^{i_{d-1}}.$$

By Lemma 3.2, the polynomials $C_{T^d + \dots + a_0, 1}, \dots, C_{T^d + \dots + a_0, d-1}$ do not depend on a_0 , so we can rewrite the sum as

$$\sum_{a_1, \dots, a_{d-1} \in \mathbf{F}_q} C_{T^d + \dots + a_1 T, 1}^{i_1} \cdots C_{T^d + \dots + a_1 T, d-1}^{i_{d-1}} \left(\sum_{a_0 \in \mathbf{F}_q} (T^d + \dots + a_1 T + a_0)^{i_0} \right).$$

Let ϵ_j be -1 if $i_j \in \{q-1, 2(q-1)\}$ and 0 otherwise. Since $0 \leq i_0 \leq 2(q-1)$, Lemma 3.4 gives $\sum_{a_0 \in \mathbf{F}_q} (T^d + \dots + a_1 T + a_0)^{i_0} = \epsilon_0$. Then, again by Lemma 3.2, $S_d(i_0, \dots, i_{d-1})$ is equal to

$$\epsilon_0 \sum_{a_2, \dots, a_{d-1} \in \mathbf{F}_q} C_{T^d + \dots + a_2 T^2, 2}^{i_2} \cdots C_{T^d + \dots + a_2 T^2, d-1}^{i_{d-1}} \left(\sum_{a_1 \in \mathbf{F}_q} (TQ_1 + a_1^q)^{i_1} \right).$$

Since $0 \leq i_1 \leq 2(q-1)$, Lemma 3.4 yields $\sum_{a_1 \in \mathbf{F}_q} (TQ_1 + a_1^q)^{i_1} = \sum_{a_1 \in \mathbf{F}_q} (TQ_1 + a_1)^{i_1} = \epsilon_1$. Continuing in this fashion, we obtain $S_d(i_0, \dots, i_{d-1}) = \epsilon_0 \cdots \epsilon_{d-1}$. \square

3.4. Connection with Carlitz binomial coefficients and special zeta values.

— We recall Carlitz's analogue $\left\{ \begin{smallmatrix} a \\ k \end{smallmatrix} \right\}$ in $\mathbf{F}_q[T]$ of the binomial coefficient $\binom{n}{k}$ (the reader may consult Thakur's article [21] for examples of such analogies). Let $a \in A$ and $k \in \mathbf{N}$ with base q expansion $\sum_{i=0}^w k_i q^i$ ($0 \leq k_i < q$). We put $\left\{ \begin{smallmatrix} a \\ k \end{smallmatrix} \right\} = \prod_{i=0}^w C_{a, i}^{k_i}$ (if $i > \deg a$, $C_{a, i} = 0$ by convention). In particular, $\left\{ \begin{smallmatrix} a \\ q^i \end{smallmatrix} \right\} = C_{a, i}$. Note that if $0 \leq i_j < q$, then

$$C(P)^{\mathbf{i}} = C_{P, 0}^{i_0} \cdots C_{P, d-1}^{i_{d-1}} = \left\{ \begin{smallmatrix} P \\ i_0 + i_1 + \dots + i_{d-1} q^{d-1} \end{smallmatrix} \right\}.$$

In general ($i_j \geq q$), it is still possible to write $C_{P, 0}^{i_0} \cdots C_{P, d-1}^{i_{d-1}}$ in terms of several Carlitz binomials. We now explain how Proposition 3.5 might be proved using this formalism.

If x is an indeterminate, $\left\{ \begin{smallmatrix} x \\ k \end{smallmatrix} \right\}$ is a polynomial in $K[x]$ with degree k (because $\left\{ \begin{smallmatrix} x \\ k \end{smallmatrix} \right\}$ is also the exponential function of a finite lattice, see Equation 2.5 of [21] or [14]). Any polynomial f in $K[x]$ may therefore be written as a linear combination of $\left\{ \begin{smallmatrix} x \\ k \end{smallmatrix} \right\}$. Moreover, the coefficients of this combination can be recovered, in terms of $\left\{ \begin{smallmatrix} x \\ k \end{smallmatrix} \right\}$, by a Mahler inversion type formula due to Carlitz (Theorem 6 in [4], Lemma 3.2.14 in [14] or Theorem XIV in [21]). For $f = 1$, the coefficients in the binomial basis are easily

computable and, by the inversion, we obtain for $d \geq 0$ and $0 \leq i < q^d$ with base q expansion $\sum_{j=0}^{d-1} i_j q^j$,

$$S_d(i_0, \dots, i_{d-1}) = \sum_{P \in A_{d+}} \left\{ \begin{matrix} P \\ i \end{matrix} \right\} = \begin{cases} (-1)^d & \text{if } i = q^d - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is precisely a special case of Proposition 3.5 (see also [21] p. 14 and Theorem 3.2.16 in [14] for similar statements). It seems likely that Proposition 3.5 can be proved by Mahler inversion.

Finally, we explain how, by the previous observations, $S_d(i_0, \dots, i_{d-1})$ is related to special zeta values of Goss zeta function at negative integers. Consider the Carlitz zeta function $\zeta : \mathbf{N} \rightarrow K_\infty$ defined by $\zeta(k) = \sum_{P \in A, P \text{ monic}} P^{-k}$. In [10] Goss proved that ζ can be extended to \mathbf{Z} by summing over fixed degree: $\zeta(-k) = \sum_{i=0}^{\infty} (\sum_{P \in A_{i+}} P^k) \in A$ for $k \geq 0$. Now, let \mathfrak{p} be a prime ideal of A and $A_{\mathfrak{p}}$ the ring of integers of the completion of K at \mathfrak{p} . Following Thakur [21], one can attach to ζ an $A_{\mathfrak{p}}$ -valued zeta measure μ determined by its k th moment:

$$\int_{A_{\mathfrak{p}}} x^k d\mu = \begin{cases} \zeta(-k) & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases}$$

By Wagner's Mahler-inversion formula for continuous functions on $A_{\mathfrak{p}}$ ([14] or Theorem VI in [21]), the measure μ is uniquely determined by the coefficients of its divided power series i.e. the sequence $\mu_k = \int_{A_{\mathfrak{p}}} \left\{ \begin{matrix} x \\ k \end{matrix} \right\} d\mu$ ($k \geq 0$). Thakur has computed explicitly μ_k ([21], Theorem VII). It follows from his proof that, when $0 \leq i_j < q$ and $i = i_0 + \dots + i_{d-1}q^{d-1}$,

$$S_d(i_0, \dots, i_{d-1}) = \mu_{i+q^d}.$$

4. Drinfeld modular forms and Hecke operators

We collect some basic facts, and set up notation and terminology as well, for Drinfeld modular forms and Hecke operators.

4.1. Drinfeld modular forms. — The first occurrence of Drinfeld modular forms goes back to the seminal work of D. Goss [12, 13]. Subsequent developments in the 1980s are due to Gekeler [5, 7].

The so-called Drinfeld upper-half plane is $\Omega = \mathbf{C}_\infty - K_\infty$, which has a rigid analytic structure. For an ideal \mathfrak{n} of A , the Hecke congruence subgroup $\Gamma_0(\mathfrak{n})$ is the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A)$ such that $c \in \mathfrak{n}$. Fix an integer $k \geq 0$ and a class m in $\mathbf{Z}/(q-1)\mathbf{Z}$. From now on, m will denote the unique representative of such a class in $\{0, 1, \dots, q-2\}$. A *Drinfeld modular form* (for $\Gamma_0(\mathfrak{n})$) of weight k and type m is a rigid holomorphic function $f : \Omega \rightarrow \mathbf{C}_\infty$ such that

$$(5) \quad f\left(\frac{az+b}{cz+d}\right) = (ad-bc)^{-m}(cz+d)^k f(z) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{n})$$

and f is holomorphic at the cusps of $\Gamma_0(\mathfrak{n})$. We will not detail the second assumption and rather refer to [5] (V, Section 3) and [15] (Section 2). For our purpose, we need only the behaviour at the cusp infinity, which we now recall.

Let $\bar{\pi}$ be the period of the Carlitz module (well-defined up to multiplication by an element in \mathbf{F}_q^\times). The Carlitz exponential e is the holomorphic function $\mathbf{C}_\infty \rightarrow \mathbf{C}_\infty$ defined by

$$e(z) = z \prod_{\lambda \in \bar{\pi}A - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

It is surjective and \mathbf{F}_q -linear with kernel $\bar{\pi}A$. For $z \in \mathbf{C}_\infty - A$, let

$$t(z) = \frac{1}{e(\bar{\pi}z)} = \frac{1}{\bar{\pi}} \sum_{\lambda \in A} \frac{1}{z - \lambda}.$$

The function t , invariant by translations $z \mapsto z + a$ ($a \in A$), is then a uniformizer at the cusp infinity. Since any f satisfying (5) is invariant under such translations, it has a Laurent series expansion $f(z) = \sum_{i \geq i_0} a_i(f)t(z)^i$ with $i_0 \in \mathbf{Z}$ (the series does not converge on all Ω , but only for $|t(z)|$ small enough). Such a function is said to be *holomorphic at the cusp infinity* if the expansion has the form $\sum_{i \geq 0} a_i(f)t^i$. We call it the *t -expansion of f* (at infinity). Since Ω is a connected rigid analytic space, any Drinfeld modular form is uniquely determined by its t -expansion.

Let $M_{k,m}(\Gamma_0(\mathfrak{n}))$ be the space of Drinfeld modular forms of weight k and type m for $\Gamma_0(\mathfrak{n})$. It is a finite-dimensional vector space over \mathbf{C}_∞ whose dimension may be calculated explicitly thanks to Gekeler [5]. If $a_0(f) = 0$ (resp. $a_0(f) = a_1(f) = 0$) and similar conditions at other cusps, f is *cuspidal* (resp. *doubly cuspidal*) and the subspace of such functions is denoted by $M_{k,m}^1(\Gamma_0(\mathfrak{n}))$ (resp. $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$). Goss observed that doubly cuspidal forms play a role similar to classical cusp forms. For an interpretation of Drinfeld modular forms as differentials on a Drinfeld modular curve, one may refer to Section V.5 in [5].

Type and weight are not independent: namely, if $k - 2m \not\equiv 0 \pmod{q-1}$, the space $M_{k,m}(\Gamma_0(\mathfrak{n}))$ is zero. Therefore, from now on we assume $k \equiv 2m \pmod{q-1}$.

Since $\Gamma_0(\mathfrak{n})$ contains matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ for $\lambda \in \mathbf{F}_q^\times$, (5) implies $a_i(f) = 0$ when $i \not\equiv m \pmod{q-1}$. Thus any $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ has t -expansion of the form

$$\sum_{j \geq 0} a_{m+j(q-1)}(f)t^{m+j(q-1)}.$$

For $j \geq 0$, let

$$b_j(f) = a_{m+j(q-1)}(f).$$

Later on, we will use both notations for coefficients. A Drinfeld modular form of type > 0 (resp. > 1) is automatically cuspidal (resp. doubly cuspidal). If f is doubly cuspidal, the coefficient $b_0(f)$ may not vanish in general (it does when $m \in \{0, 1\}$).

4.2. Hecke algebra. — We define a formal Hecke algebra $\mathbf{R}_\mathfrak{n}$ which acts on the different spaces $M_{k,m}(\Gamma_0(\mathfrak{n}))$. In this section, we adopt the notation $\Gamma = \Gamma_0(\mathfrak{n})$.

Let $\Delta = \Delta_0(\mathfrak{n})$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with entries in A such that $ad - bc$ is monic, $c \in \mathfrak{n}$ and $(a) + \mathfrak{n} = A$. Let $\mathbf{R}_\mathfrak{n}$ be the \mathbf{C}_∞ -vector space spanned by formal sums of double cosets $\Gamma g \Gamma$ for $g \in \Delta$. One can make $\mathbf{R}_\mathfrak{n}$ a commutative algebra over \mathbf{C}_∞ (see Section 3.1 of [17] for a general treatment or Section 6.1 of [3] for Drinfeld modular forms).

For an ideal \mathfrak{p} of A , let $\Delta^{\mathfrak{p}}$ be the set of $g \in \Delta$ such that $\det g$ is the monic generator of \mathfrak{p} . The Hecke operator $T_{\mathfrak{p}}$ is then defined as the formal sum of all double cosets $\Gamma g \Gamma$ with $g \in \Delta^{\mathfrak{p}}$ in $\mathbf{R}_{\mathfrak{n}}$. For example, when \mathfrak{p} is prime, $T_{\mathfrak{p}} = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \Gamma$ where P is the monic generator of \mathfrak{p} .

As elements of $\mathbf{R}_{\mathfrak{n}}$ have coefficients in a field of characteristic p , the usual relation for the product gives

$$T_{\mathfrak{p}} T_{\mathfrak{p}'} = T_{\mathfrak{p}\mathfrak{p}'} \quad \text{for any ideals } \mathfrak{p}, \mathfrak{p}'$$

(see [11]). This is very different from Hecke operators for classical modular forms, where the above relation only holds for relatively prime ideals. One can check that $\mathbf{R}_{\mathfrak{n}}$ is the polynomial ring over \mathbf{C}_{∞} spanned by $T_{\mathfrak{p}}$ for \mathfrak{p} prime (such elements are algebraically independent over \mathbf{C}_{∞}).

As for the notation, $T_{\mathfrak{p}}$ depends on the subgroup $\Gamma_0(\mathfrak{n})$ but from the context, there will be no confusion on which Hecke algebra (or space of endomorphisms of Drinfeld modular forms) it belongs to.

For $\mathfrak{n} = A$, let us consider the formal Hecke algebra \mathbf{R}_A attached to the sets $\mathrm{GL}_2(A)$ and $\Delta_0(A)$. Let $\tilde{T}_{\mathfrak{p}}$ temporarily denotes the \mathfrak{p} th Hecke operator in \mathbf{R}_A . The map $\tilde{T}_{\mathfrak{p}} \mapsto T_{\mathfrak{p}}$, for \mathfrak{p} prime, defines an algebra homomorphism $\mathbf{R}_A \rightarrow \mathbf{R}_{\mathfrak{n}}$. We regard \mathbf{R}_A as a universal formal Hecke algebra, independent of \mathfrak{n} . Any algebraic relation among the Hecke operators in \mathbf{R}_A can be translated to the corresponding relation in $\mathbf{R}_{\mathfrak{n}}$ for any \mathfrak{n} .

4.3. Hecke operators on Drinfeld modular forms. — For $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with entries in A and $f : \Omega \rightarrow \mathbf{C}_{\infty}$, let

$$f_{|[v]_k} : z \mapsto (ad - bc)^{k-1} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Fix $g \in \Delta$. The group Γ acts on the left on the double coset $\Gamma g \Gamma$. Let $\{g_i\}_i$ be a finite system of representatives such that g_i has monic determinant. We define an action of $\Gamma g \Gamma$ on $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ by

$$f_{|[\Gamma g \Gamma]_k} = \sum_i f_{|[g_i]_k}$$

(independently of the choice of $\{g_i\}_i$). It extends, in a unique way, to a non-faithful action of $\mathbf{R}_{\mathfrak{n}}$ on $M_{k,m}(\Gamma_0(\mathfrak{n}))$. Let $\mathbf{T} = \mathbf{T}_{k,m}(\Gamma_0(\mathfrak{n}))$ be the commutative sub- \mathbf{C}_{∞} -algebra of $\mathrm{End}_{\mathbf{C}_{\infty}}(M_{k,m}(\Gamma_0(\mathfrak{n})))$ induced by $\mathbf{R}_{\mathfrak{n}}$.

For any $g \in \Delta^{\mathfrak{p}}$, a set of representatives of $\Gamma \backslash \Gamma g \Gamma$ with monic determinant is given by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \quad \alpha, \delta \text{ monic in } A, (\alpha\delta) = \mathfrak{p}, (\alpha) + A = \mathfrak{n}, \beta \in A/(\delta).$$

Therefore, the action of $T_{\mathfrak{p}}$ on the Drinfeld modular form f can be written more concretely as

$$(6) \quad T_{\mathfrak{p}}(f)(z) = P^{k-1} \sum_{\substack{\alpha, \delta \text{ monic} \in A \\ \beta \in A, \deg \beta < \deg \delta \\ \alpha\delta = P, (\alpha) + \mathfrak{n} = A}} \delta^{-k} f\left(\frac{\alpha z + \beta}{\delta}\right) = \frac{1}{P} \sum_{\alpha, \beta, \delta} \alpha^k f\left(\frac{\alpha z + \beta}{\delta}\right)$$

where P is the monic generator of \mathfrak{p} . This formula slightly differs from other references. Gekeler [7] (resp. Böckle [3], Section 6) considered $PT_{\mathfrak{p}}$ (resp. $P^{m+1-k}T_{\mathfrak{p}}$). In particular, our operator coincides with Böckle's when $k = m - 1$ (for instance, when $k = 2$ and

$m = 1$). In general, these variously normalized Hecke operators have the same eigenforms, however with different eigenvalues.

The Hecke algebra \mathbf{T} stabilizes the subspaces $M_{k,m}^1(\Gamma_0(\mathfrak{n}))$ et $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$ (see for example Proposition 6.9 of [3]). We denote by $\mathbf{T}' = \mathbf{T}'_{k,m}(\Gamma_0(\mathfrak{n}))$ the restriction of \mathbf{T} to $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$.

5. Hecke action on the first coefficient of Drinfeld modular forms

We recall some results on Goss polynomials for finite lattices and their role in the t -expansion of Drinfeld modular forms. Then we give an explicit formula for the action of Hecke operators on the first coefficient of this expansion.

5.1. Goss polynomials. — Let Λ be a \mathbf{F}_q -lattice in \mathbf{C}_∞ , i.e. a \mathbf{F}_q -submodule of \mathbf{C}_∞ having finite intersection with each ball of \mathbf{C}_∞ of finite radius. We assume Λ to be *finite* of dimension d over \mathbf{F}_q . The exponential corresponding to Λ

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right) \quad (z \in \mathbf{C}_\infty)$$

is an entire Λ -periodic \mathbf{F}_q -linear function. Since Λ is finite, it is a polynomial in z of the form

$$e_\Lambda(z) = \sum_{i=0}^d \lambda_i z^{q^i}$$

with coefficients $\lambda_i \in \mathbf{C}_\infty$ depending on Λ . Goss has considered Newton's sums associated to the reciprocal polynomial of $e_\Lambda(X - z) = e_\Lambda(X) - e_\Lambda(z) \in \mathbf{C}_\infty[z][X]$, namely

$$\begin{aligned} N_0 &= 0 \\ N_j(z) &= N_{j,\Lambda}(z) = \sum_{\lambda \in \Lambda} \frac{1}{(z+\lambda)^j} \quad (j \geq 1, z \in \mathbf{C}_\infty - \Lambda). \end{aligned}$$

Let

$$t_\Lambda(z) = e_\Lambda(z)^{-1} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} \quad (z \in \mathbf{C}_\infty - \Lambda).$$

Proposition 5.1 (Section 2 of [13], 3.4–3.9 in [7]). — *Let $j \geq 1$. There exists a unique polynomial $G_j = G_{j,\Lambda}(X) \in \mathbf{C}_\infty[X]$ such that the following equalities hold:*

1. if $j \leq q$ then $G_j(X) = X^j$
2. $G_j(X) = X \sum_{i \geq 0, j - q^i \geq 0} \lambda_i G_{j - q^i}(X)$.

The polynomial $G_j(X)$ is monic of degree j and satisfies $N_j = G_j(t_\Lambda)$. Moreover,

$$(7) \quad G_j(X) = \sum_{n=0}^{j-1} \sum_{\underline{i}} \binom{n}{\underline{i}} \lambda^{\underline{i}} X^{n+1}$$

for $\underline{i} = (i_0, \dots, i_d)$ running through $(d+1)$ -tuples such that

$$\begin{aligned} i_0 + \dots + i_d &= n \\ i_0 + i_1 q + \dots + i_d q^d &= j - 1 \end{aligned}$$

and λ^i denotes $\lambda_0^{i_0} \cdots \lambda_d^{i_d}$. The polynomial $G_j(X)$ is divisible by X^u where $u = \lfloor j/q^d \rfloor + 1$. We further put $G_{0,\Lambda}(X) = 0$.

Gekeler provided the explicit formula (7) using a generating function.

5.2. Hecke algebra and Goss polynomials. — Let \mathfrak{p} an ideal of A of degree $d \geq 0$ with monic generator P . Recall that C denotes the Carlitz module over \mathbf{C}_∞ (Section 3.1). As usual, for an indeterminate X , put $C_P(X) = \sum_{i=0}^d C_{P,i} X^{q^i}$. For our purpose, we consider the \mathbf{F}_q -lattice of dimension d

$$\Lambda_P = \text{Ker}(C_P) = \{x \in \mathbf{C}_\infty \mid C_P(x) = 0\}$$

whose j th Goss polynomial is denoted by $G_{j,P}$. Let

$$t_P(z) = t(Pz) = \frac{1}{e(\bar{\pi}Pz)} \quad (z \in \mathbf{C}_\infty - A).$$

Then, if $f_P(X)$ is the P th inverse cyclotomic polynomial $C_P(X^{-1})X^{q^d}$, one has

$$(8) \quad t_P = \frac{t^{q^d}}{f_P(t)}.$$

The following statement mildly extends Gekeler's formula 7.3 in [7] (which was established for $\text{GL}_2(A)$ and \mathfrak{p} prime) to $\Gamma_0(\mathfrak{n})$ and any \mathfrak{p} .

Proposition 5.2. — Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ with t -expansion $\sum_{i \geq 0} a_i t^i$. We have

$$(9) \quad T_{\mathfrak{p}} f = P^{k-1} \sum_{i \geq 0} \sum_{\substack{\delta \text{ monic in } A \\ \delta | P, (\frac{P}{\delta}) + \mathfrak{n} = A}} \delta^{-k} a_i G_{i,\delta}(\delta t \frac{P}{\delta})$$

Moreover, for fixed j , only a finite number of terms in the right-hand side contribute to t^j in the t -expansion of $T_{\mathfrak{p}} f$.

Proof. — Let δ be a monic divisor of P . Recall that e is the Carlitz exponential. We write $F(z)$ for $\sum_{\beta \in A, \deg \beta < \deg \delta} f((Pz/\delta + \beta)/\delta)$. For $|t(z)|$ small enough, $F(z)$ is

$$\begin{aligned} & \sum_{\beta \in A, \deg \beta < \deg \delta} \sum_{i \geq 0} a_i t \left(\frac{\frac{P}{\delta} z + \beta}{\delta} \right)^i = \sum_{i \geq 0} a_i \sum_{\beta \in A, \deg \beta < \deg \delta} e \left(\frac{\frac{P}{\delta} z + \beta}{\delta} \right)^{-i} \\ & = \sum_{i \geq 0} a_i \sum_{\beta \in A, \deg \beta < \deg \delta} \left(e \left(\frac{\bar{\pi} P z}{\delta^2} \right) + e \left(\frac{\bar{\pi} \beta}{\delta} \right) \right)^{-i} \end{aligned}$$

by additivity of e . According to the analytic theory of Drinfeld modules, the finite set $\{e(\bar{\pi}\beta/\delta) \mid \beta \in A, \deg \beta < \deg \delta\}$ is in bijection with the lattice $\Lambda_\delta = \text{Ker}(C_\delta)$. Let $w = Pz/\delta^2$. Then, by Proposition 5.1, $F(z)$ is

$$\sum_{i \geq 0} a_i \sum_{\lambda \in \Lambda_\delta} (e(\bar{\pi}w) + \lambda)^{-i} = \sum_{i \geq 0} a_i N_{i,\Lambda_\delta}(e(\bar{\pi}w)) = \sum_{i \geq 0} a_i G_{i,\Lambda_\delta}(e_{\Lambda_\delta}(e(\bar{\pi}w))^{-1}).$$

Observe that $e_{\Lambda_\delta}(z) = C_\delta(z)/\delta$ (both polynomials have the same set of zeros and the multiplicative constant is obtained by comparing the terms in z). By the properties of

the Carlitz exponential, we also have $C_\delta(e(\bar{\pi}w)) = C(\bar{\pi}zP/\delta) = t(zP/\delta)^{-1}$. Substituting, we get

$$F(z) = \sum_{i \geq 0} a_i G_{i, \Lambda_\delta} \left(\delta t \left(\frac{zP}{\delta} \right) \right) = \sum_{i \geq 0} a_i G_{i, \Lambda_\delta} (\delta t_{\frac{P}{\delta}}(z)).$$

Our claim follows from (6) and the last statement of Proposition 5.1. \square

Remark 5.3. — To obtain the t -expansion of $T_{\mathfrak{p}}f$ from Equation (9), it would suffice to compose the t -expansions of $t_{P/\delta}$ and Goss polynomials $G_{i, \delta}$. However, making this explicit seems to be a difficult problem. Indeed, a similar question arises when trying to make explicit the t -expansion of Drinfeld-Eisenstein series (see Section 6 of [7]) since it involves the t -expansion of $G_{i, \bar{\pi}A}(t_P)^{(1)}$. This is quite different from the classical situation where coefficients of Eisenstein series are well-known arithmetic functions.

5.3. Hecke module spanned by b_1 . —

Notation 5.4. — The dual space of $M_{k, m}(\Gamma_0(\mathfrak{n}))$ has the natural right action of \mathbf{T} , given by composition, and contains the following linear forms, for any $n \geq 1$:

$$a_{m+n(q-1)} = b_n : f \mapsto a_{m+n(q-1)}(f) = b_n(f).$$

Let $u = u_{k, m, n} : \mathbf{T}' \rightarrow \text{Hom}_{\mathbf{C}_\infty}(M_{k, m}^2(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ be the \mathbf{C}_∞ -linear map $s \mapsto b_1 s$. We write $b_1 \mathbf{T}'$ for the image of u .

We collect some remarks on the dimension of the \mathbf{C}_∞ -algebra \mathbf{T}' . The map u is not necessarily an isomorphism, therefore the dimension of \mathbf{T}' is unknown *a priori*. In the case $\mathbf{T}' = \mathbf{T}'_{2, 1}(\Gamma_0(\mathfrak{n}))$, one can prove that its dimension coincides with $\dim_{\mathbf{C}_\infty} M_{2, 1}^2(\Gamma_0(\mathfrak{n}))$, using results from automorphic forms and work of Gekeler and Reversat [15].

We keep Notation 3.3. The next statement gives a first description of $b_1 \mathbf{T}'$.

Proposition 5.5. — *Let $f \in M_{k, m}(\Gamma_0(\mathfrak{n}))$ with t -expansion $\sum_{i \geq 0} a_i(f)t^i$. Let \mathfrak{p} an ideal of A of degree d with monic generator P . Then $a_{m+(q-1)}(T_{\mathfrak{p}}f)$ is*

$$(10) \quad \sum_{\underline{n}} \binom{m+q-2}{\underline{n}} C(P)^{\underline{n}} a_{1+n_0+n_1q+\dots+n_dq^d}(f) + \varepsilon \sum_{\substack{Q|P, Q \in A_1+ \\ (Q)+\mathfrak{n}=A}} Q^{k-1} a_1(f)$$

where $\underline{n} = (n_0, \dots, n_d)$ is such that $n_0 + \dots + n_d = m + q - 2$ with $n_i \geq 0$ for all i and ε is defined by $\varepsilon = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases}$

Remark 5.6. — 1. In Example 7.4 of [7], Gekeler treated $a_i(T_{\mathfrak{p}}f)$ for \mathfrak{p} of degree 1, $i \geq 0$, and f modular for $\text{GL}_2(A)$. Proposition 5.5 supplements Gekeler's statement.

2. Actually, Propositions 5.2 and 5.5 work for holomorphic functions $f : \Omega \rightarrow \mathbf{C}_\infty$ having an expansion $\sum_{i \geq 0} a_i t^i$ for $|t(z)|$ small enough (Hecke operators are still defined on f via (6)). In particular, this applies to holomorphic functions $f : \Omega \rightarrow \mathbf{C}_\infty$ which are A -periodic ($f(z+a) = f(z)$, $a \in A$) and holomorphic at the cusp infinity.

⁽¹⁾The lattice $\bar{\pi}A$ is not finite but Goss polynomials can be defined in that more general setting (see [13, 7]).

Proof. — By Proposition 5.2, we have to find the coefficient of $t^{m+(q-1)}$ in the t -expansion of $G_{i,\delta}(\delta t_{P/\delta})$. First, if $i = 0$, then $G_{0,\delta}(X) = 0$ so the expansion of $G_{0,\delta}(\delta t_{P/\delta})$ has no $t^{m+(q-1)}$ -term.

Assume $i > 0$. By (8) the t -expansion of $t_{P/\delta}$ is divisible by $t^{q-d-\deg \delta}$. Moreover, it follows from the definition of Goss polynomials that $G_{i,\delta}(X)$ has X as a factor. Hence, the t -expansion of $G_{i,\delta}(\delta t_{P/\delta})$ is divisible by $t^{q-d-\deg \delta}$. Since $m < q - 1$, $G_{i,\delta}(\delta t_{P/\delta})$ has no $t^{m+(q-1)}$ -term when $d - \deg \delta \geq 2$. Now assume $0 \leq d - \deg \delta \leq 1$. Put $s = \deg \delta$. Recall that $e_{\Lambda_\delta}(z) = C_\delta(z)/\delta = \sum_{i=0}^s C_{\delta,i} z^{q^i}/\delta$. The explicit formula for Goss polynomials gives

$$G_{i,\delta}(X) = \sum_{j=0}^{i-1} \delta^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(\delta)^{\underline{n}} X^{j+1}$$

where $\underline{n} = (n_0, \dots, n_s)$ are such that $n_0 + \dots + n_s = j$ and $n_0 + n_1 q + \dots + n_s q^s = i - 1$.

Suppose that $s = d$, i.e. $\delta = P$. Then the corresponding partial sum in (9) is

$$\frac{1}{P} \sum_{i \geq 0} a_i G_{i,P}(Pt) = \frac{1}{P} \sum_{i \geq 0} a_i \sum_{j=0}^{i-1} P^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(P)^{\underline{n}} (Pt)^{j+1}.$$

The $t^{m+(q-1)}$ -term corresponds to $j = m + q - 2$; namely, it is

$$\sum_{\underline{n}} \binom{m+q-2}{\underline{n}} C(P)^{\underline{n}} a_{1+n_0+n_1 q+\dots+n_d q^d}(f)$$

with $\underline{n} = (n_0, \dots, n_d)$ such that $n_0 + \dots + n_d = m + q - 2$.

Next, suppose that $s = d - 1$. Using Equation (8), we get

$$(11) \quad G_{i,\delta}(\delta t_{P/\delta}) = \sum_{j=0}^{i-1} \delta^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(\delta)^{\underline{n}} \left(\delta \frac{t^q}{1 + \frac{P}{\delta} t^{q-1}} \right)^{j+1}$$

where (n_0, \dots, n_{d-1}) with $n_0 + \dots + n_{d-1} = j$ and $n_0 + n_1 q + \dots + n_{d-1} q^{d-1} = i - 1$. If $j \geq 1$, then $q(j+1) \geq 2q > m + q - 1$, thus there is no $t^{m+(q-1)}$ -term in the expansion of (11). Finally, we assume $j = 0$, in other words $n_0 = \dots = n_{d-1} = 0$ and $i = 1$. We have

$$G_{1,\delta}(\delta t_{P/\delta}) = \delta \frac{t^q}{1 + \frac{P}{\delta} t^{q-1}} = \delta t^q \sum_{n=0}^{+\infty} (-1)^n \frac{P^n}{\delta^n} t^{n(q-1)}.$$

This series has a $t^{m+(q-1)}$ -term if and only if $q - 1$ divides $m - 1$. This happens only if $m = 1$, and in that case the corresponding coefficient is δ . Thus we obtain (10). \square

Assume $m \in \{0, 1\}$. By (10), the linear form $b_1 T_{\mathfrak{p}} = a_{m+(q-1)} T_{\mathfrak{p}}$ is a A -linear combination of a_i , where i runs through the set of natural integers satisfying the condition: the expansion of i in base q has at most $d + 1$ digits, whose sum is equal to $m + q - 1$. In particular, the set of such i 's only depends on the degree d of \mathfrak{p} . This observation, also communicated to the author by D. Goss, will be used in Section 7. For the moment, we derive the following statement for $b_1 \mathbf{T}'$.

Notation 5.7. — Let \mathcal{S} be the set of natural integers of the form $c/(q-1)$ where $c \in \mathbf{N}$ is such that the sum of its base q digits is $q - 1$.

Corollary 5.8. — If $m \in \{0, 1\}$ then $b_1 \mathbf{T}'$ is contained in the \mathbf{C}_∞ -vector space spanned by b_n for $n \in \mathcal{S}$.

The reverse inclusion will be proved in Section 7. Finally, we state another straightforward application of Proposition 5.5.

Notation 5.9. — For $d \geq 1$ and $\underline{i} = (i_0, \dots, i_{d-1})$, let

$$\Theta_d(i_0, \dots, i_{d-1}) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} T_P = \sum_{P \in A_{d+}} C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}} T_P \in \mathbf{R}_A.$$

Corollary 5.10. — Let $f \in M_{k,m}(\Gamma_0(\mathbf{n}))$. With the notations of Proposition 5.5 and Section 3, the coefficient $a_{m+(q-1)}(\Theta_d(i_0, \dots, i_{d-1})f)$ is

$$\begin{aligned} \sum_{\substack{\underline{n}=(n_0, \dots, n_d) \\ n_0 + \dots + n_d = m+q-2}} \binom{m+q-2}{\underline{n}} S_d(n_0 + i_0, \dots, n_{d-1} + i_{d-1}) a_{1+n_0+n_1q+\dots+n_dq^d}(f) \\ + \varepsilon \sum_{P \in A_{d+}} C(P)^{\underline{i}} \sum_{\substack{Q|P, Q \in A_{1+} \\ (Q)+\mathbf{n}=A}} Q^{k-1} a_1(f) \end{aligned}$$

where ε is defined as in Proposition 5.5.

6. Annihilator of b_1 for the Hecke action

Notation 6.1. — Let $\mathbf{I} = \mathbf{I}_{k,m,\mathbf{n}}$ be the kernel of u i.e. the ideal of elements $s \in \mathbf{T}'$ such that $b_1 s = 0$ in the dual space of $M_{k,m}^2(\Gamma_0(\mathbf{n}))$.

In particular, \mathbf{I} is a sub- \mathbf{C}_∞ -algebra of \mathbf{T}' which maps doubly cuspidal forms to Drinfeld modular forms f satisfying $a_0(f) = b_0(f) = b_1(f) = 0$.

Lemma 6.2. — If the map $u : \mathbf{T}' \rightarrow \text{Hom}_{\mathbf{C}_\infty}(M_{k,m}^2(\Gamma_0(\mathbf{n})), \mathbf{C}_\infty)$ is surjective, then it is an isomorphism.

Proof. — Since u is surjective, we take an element t_n in the preimage of b_n for any $n \geq 1$. If s belongs to the ideal \mathbf{I} , so does $t_n s$. Hence, for any $f \in M_{k,m}^2(\Gamma_0(\mathbf{n}))$, the n th coefficient $b_n(sf)$ is zero for any $n \geq 1$. As the t -expansion uniquely determines a Drinfeld modular form, sf must be zero. Therefore s is zero as an endomorphism of $M_{k,m}^2(\Gamma_0(\mathbf{n}))$. \square

6.1. Proof of Theorem 1.1. —

Proof of Theorem 1.1. — Actually we prove a slightly more general statement: all the following equalities of linear forms will take place in the dual space of $M_{k,m}(\Gamma_0(\mathbf{n}))$ if $m \neq 1$ (resp. of $M_{k,m}^2(\Gamma_0(\mathbf{n}))$ if $m = 1$).

1. Without any assumption on m , we apply Corollary 5.10 to $d = 1$. For $i \geq 0$ we get

$$b_1 \left(\sum_{P \in A_{1+}} P^i T_P \right) = \sum_{n=0}^{m+q-2} \binom{m+q-2}{n} S_1(n+i) a_{1+n+q(m+q-2-n)}.$$

This follows also from Gekeler's example 7.4 in [7], although stated there for $\mathrm{GL}_2(A)$ and with a different normalization of Hecke operators.

Assume $m = 0$. The sum $S_1(n+1) = \sum_{Q \in A_{1+}} Q^{n+1}$ is nonzero if and only if $n = q - 2$, and $S_1(q-1) = -1$ (by Lemma 3.4 for instance). Taking $i = 1$, our expression simplifies as $b_1 \left(\sum_{P \in A_{1+}} PT_P \right) = -b_1$.

Assume $m = 1$. Since the sum $S_1(n)$ is nonzero if and only if $n = q - 1$, taking $i = 0$, we get $b_1 \left(\sum_{P \in A_{1+}} TP \right) = -b_1$.

2. Consider (i_0, \dots, i_{d-1}) as in the statement. By Corollary 5.10, we get that $b_1(\Theta_d(i_0, \dots, i_{d-1}))$ is

$$\sum_{\substack{\underline{n} \\ n_0 + \dots + n_d = m + q - 2}} \binom{m+q-2}{\underline{n}} S_d(n_0 + i_0, \dots, n_{d-1} + i_{d-1}) a_{1+n_0+n_1q+\dots+n_dq^d}.$$

We have $0 \leq n_j + i_j \leq 2(q-1)$, hence we can evaluate $S_d(n_0 + i_0, \dots, n_{d-1} + i_{d-1})$ thanks to Proposition 3.5. This sum is nonzero if and only if $n_j + i_j = q - 1$ or $2(q-1)$ for all $j \in \{0, \dots, d-1\}$. If this happens, we have

$$d(q-1) \leq \sum_{l=0}^{d-1} (n_l + i_l) \leq i_0 + \dots + i_{d-1} + m + q - 2$$

which contradicts $i_0 + \dots + i_{d-1} \leq (d-1)(q-1) - m$. Accordingly, the sum always vanishes and $b_1(\Theta_d(i_0, \dots, i_{d-1})) = 0$.

3. Apply the statement proved before to $i_0 = l$ and $i_1 = \dots = i_{d-1} = 0$.

□

It is worth pointing out that the elements of \mathbf{I} given in Theorem 1.1 are universal in the sense that, for a given type, they do not depend on the weight k nor the ideal \mathfrak{n} . Some of them, as $\sum_{P \in A_{d+}} TP$ for $d \geq 2$ for instance, are also independent of the type m . This means that, in the universal formal Hecke algebra \mathbf{R}_A , such an element is independent of k , m and \mathfrak{n} .

Remark 6.3. — This phenomenon does not occur for classical modular forms of weight 2 as we now explain. Let $S_2(\Gamma_0(N))$ be the complex space of weight-2 cusp forms for $\Gamma_0(N)$ ($N \geq 1$). We write $(c_n)_{n \geq 1}$ for the linear forms given by Fourier coefficients of such modular forms at the cusp infinity. The Hecke algebra \mathbf{T}_c of weight 2 for $\Gamma_0(N)$ is the subring of $\mathrm{End}(S_2(\Gamma_0(N)))$ spanned over \mathbf{C} by all Hecke operators T_n for $n \in \mathbf{N}$. Let u_c be the \mathbf{C} -linear map $\mathbf{T}_c \rightarrow \mathrm{Hom}_{\mathbf{C}}(S_2(\Gamma_0(N)), \mathbf{C})$ given by $s \mapsto c_1 s$. Relation (3) gives $c_n = u_c(T_n)$ for all $n \geq 1$, thus u_c is bijective. We claim that if there exists a \mathbf{C} -linear combination $s = \lambda_1 T_{i_1} + \dots + \lambda_j T_{i_j}$, with $j, \lambda_1, \dots, \lambda_j, i_1, \dots, i_j$ independent of N , such that $s = 0$ as an endomorphism of $S_2(\Gamma_0(N))$, then the coefficients $\lambda_1, \dots, \lambda_s$ must be zero. In fact, when N is prime, the Hecke operators $T_1, \dots, T_{g(N)}$ are \mathbf{C} -linearly independent in $\mathrm{End}(S_2(\Gamma_0(N)))$ for $g(N) = \dim S_2(\Gamma_0(N))$ (this follows from the cusp infinity not being a Weierstrass point on the modular curve $X_0(N)$). Choosing N prime such that $g(N)$ is large enough yields $\lambda_1 = \dots = \lambda_j = 0$ and proves our claim.

In Section 7.2, we will further our investigation of the ideal \mathbf{I} and prove that it vanishes in some cases (Theorem 7.7).

6.2. Linear relations for eigenvalues. —

Notation 6.4. — Let \mathfrak{p} an ideal of A with monic generator P . A *Hecke eigenform* f is a Drinfeld modular form which is an eigenform for all Hecke operators. We write $\lambda_P(f)$ for its eigenvalue for $T_P = T_{\mathfrak{p}}$.

For a Hecke eigenform f such that $b_1(f) \neq 0$, Theorem 1.1 yields linear relations among its eigenvalues. It seems rather remarkable that these relations are universal in the sense that, for a fixed type, they do not depend on the weight k nor on the level \mathfrak{n} .

Proposition 6.5. — *Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ be a Hecke eigenform with $b_1(f) \neq 0$. If $m = 1$, we assume further $f \in M_{k,m}^2(\Gamma_0(\mathfrak{n}))$.*

1. *If $m \in \{0, 1\}$, then*

$$\sum_{P \in A_{1+}} P^{1-m} \lambda_P(f) + 1 = 0.$$

2. *Let $d \geq 1$ and i_0, \dots, i_{d-1} satisfying (1) and (2). Then*

$$\sum_{P \in A_{d+}} C(P)^{i_0} \lambda_P(f) = 0.$$

3. *Let l and d be integers such that $0 \leq l \leq q - m$ and $d \geq (l + m)/(q - 1) + 1$. Then*

$$\sum_{P \in A_{d+}} P^l \lambda_P(f) = 0.$$

In particular, if $d \geq 2$, or f has type 0 and $d = 1$, then

$$\sum_{P \in A_{d+}} \lambda_P(f) = 0.$$

6.3. Linear relations for Hecke operators. — We explain how some relations of Proposition 6.5 may follow from linear relations among Hecke operators in characteristic zero or p . In other words, we prove or suggest that certain elements of \mathbf{I} given in Theorem 1.1 are zero in \mathbf{T}' .

Notation 6.6. — For an ideal \mathfrak{n} of A , let $\mathbf{H}_{\mathfrak{n}}$ be the abelian group of \mathbf{Z} -valued cuspidal harmonic cochains for $\Gamma_0(\mathfrak{n})$ on the Bruhat-Tits tree \mathcal{T} of $\mathrm{PGL}(2, K_{\infty})$ (we refer to Section 3 of [15] for the relevant definitions and properties). The group $\mathrm{GL}_2(K)$ acts on the left on the set of oriented edges $Y(\mathcal{T})$ of \mathcal{T} . We define an endomorphism $\theta_{\mathfrak{p}}$ of $\mathbf{H}_{\mathfrak{n}}$ by

$$(\theta_{\mathfrak{p}}F)(e) = \sum_{\substack{\alpha, \delta \text{ monic} \in A \\ \beta \in A, \deg \beta < \deg \delta \\ (\alpha\delta) = \mathfrak{p}, (\alpha) + \mathfrak{n} = A}} F \left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} e \right)$$

for $F \in \mathbf{H}_{\mathfrak{n}}$ and $e \in Y(\mathcal{T})$.

After scalar extension to the complex numbers \mathbf{C} , $\mathbf{H}_{\mathfrak{n}}$ is identified with a space of cuspidal automorphic forms on $\mathrm{GL}(2)$ over the adèles of K (by the strong approximation theorem). Moreover, using Teitelbaum's residue map [19], Gekeler and Reversat [15] gave an isomorphism between $\mathbf{H}_{\mathfrak{n}}/p\mathbf{H}_{\mathfrak{n}}$ and a subspace of Drinfeld modular forms, namely the subspace $M_{2,1}^2(\Gamma_0(\mathfrak{n}), \mathbf{F}_p)$ of $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ consisting of such forms with residues in \mathbf{F}_p .

It turns out that this isomorphism is Hecke-equivariant, with the normalizations we have adopted here for T_p and θ_p . Finally, $M_{2,1}^2(\Gamma_0(\mathbf{n}), \mathbf{F}_p)$ is an \mathbf{F}_p -vector space which, after scalar extension to \mathbf{C}_∞ , gives the whole space $M_{2,1}^2(\Gamma_0(\mathbf{n}))$. Put differently, the Hecke operator T_p acting on $M_{2,1}^2(\Gamma_0(\mathbf{n}))$ can be thought of as the mod p reduction of θ_p .

Lemma 6.7. — *Let \mathbf{n} be a prime. Assume $d \geq \deg(\mathbf{n}) - 1$. Then $\sum_{\deg p=d} \theta_p = 0$. In particular, $\sum_{\deg p=d} T_p = 0$ on $M_{2,1}^2(\Gamma_0(\mathbf{n}))$.*

Proof. — Let $F \in \mathbf{H}_n(\mathbf{C}) = \mathbf{H}_n \otimes_{\mathbf{Z}} \mathbf{C}$ be an eigenform for $(\theta_p)_p$ with eigenvalues $(\lambda_p)_p$. For $d > \deg(\mathbf{n}) - 3$, we have

$$(12) \quad \sum_{\deg p \leq d} \lambda_p = 0.$$

It is essentially a consequence of the cuspidality of F . Namely, by the structure of the quotient graph $\Gamma_0(\mathbf{n}) \backslash \mathcal{T}$, the edges of \mathcal{T} corresponding to matrices $\begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix}$ with $k \geq \deg \mathbf{n}$ are not in the support of F (see also [18]). Using the Fourier expansion of F and the relation between Fourier coefficients and Hecke eigenvalues $(\lambda_p)_p$ ((3.12') and (3.13) in [8]), we derive (12). Since \mathbf{n} is prime, there exists a basis of $\mathbf{H}_n(\mathbf{C})$ consisting of normalized eigenforms for $(\theta_p)_p$. Hence we have $\sum_{\deg p \leq d} \theta_p = 0$ if $d > \deg(\mathbf{n}) - 3$. An equivalent formulation is: $\sum_{\deg p \leq \deg(\mathbf{n})-2} \theta_p = 0$ and $\sum_{\deg p=d} \theta_p = 0$ if $d \geq \deg(\mathbf{n}) - 1$. This completes the proof. \square

Therefore, from the theory of automorphic forms, we know that certain elements of \mathbf{I} given in Theorem 1.1 are zero on $M_{2,1}^2(\Gamma_0(\mathbf{n}))$, because so they are on \mathbf{H}_n : this is the case for $\sum_{\deg p=d} T_p$ if \mathbf{n} is prime and $d \geq \deg(\mathbf{n}) - 1$.

It is now natural to ask whether some elements of \mathbf{I} in Theorem 1.1 can act nontrivially on \mathbf{H}_n and be zero in \mathbf{T}' (i.e. in characteristic p). We suggest that this happens.

Question 6.8. — *Assume \mathbf{n} is prime. Do the following relations among Hecke operators on $M_{2,1}^2(\Gamma_0(\mathbf{n}))$:*

1. $\sum_{\deg p \leq 1} T_p = 0$ if \mathbf{n} has degree 4
2. $\sum_{\deg p = \deg(\mathbf{n})-2} T_p = 0$ if \mathbf{n} has degree ≥ 4

hold?

We checked numerically such relations on several examples. We computed Hecke operators on $\mathbf{H}_n/p\mathbf{H}_n$, for \mathbf{n} prime, using Teitelbaum's modular symbols for $\mathbf{F}_q(T)$ [20]. The first relation has been checked for $q \in \{2, 3, 4, 5, 7\}$ and the second one for all primes \mathbf{n} of degree 5 and 6 in $\mathbf{F}_2[T]$. Note that, when $\deg \mathbf{n} = 4$, both relations are equivalent: indeed, we have $\sum_{\deg p \leq 2} \theta_p = 0$ (see proof of Lemma 6.7).

An affirmative answer to Question 6.8 would tell that some elements of \mathbf{I} would be zero in \mathbf{T}' but may be nonzero on the automorphic level, more precisely:

- $\sum_{\deg p \leq 1} T_p = 0$ in $\mathbf{T}'_{2,1}(\Gamma_0(\mathbf{n}))$ for \mathbf{n} prime of degree 4;
- $\sum_{\deg p = \deg(\mathbf{n})-2} T_p = 0$ in $\mathbf{T}'_{2,1}(\Gamma_0(\mathbf{n}))$ for \mathbf{n} prime of degree ≥ 4 .

In the next paragraph, we are interested in the reverse problem: finding nonzero elements in the ideal \mathbf{I} .

6.4. Nonzero elements in the annihilator. — The following conjecture suggests that, in general, the Hecke annihilator \mathbf{I} of b_1 is nonzero.

Conjecture 6.9. — Assume \mathfrak{n} is prime of degree ≥ 5 . Then $\sum_{\deg \mathfrak{p} \leq 1} T_{\mathfrak{p}} \in \mathbf{I}$ is nonzero as an endomorphism of $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$. In particular, the map

$$\begin{aligned} u : \mathbf{T}' &\longrightarrow \text{Hom}_{\mathbf{C}_\infty}(M_{2,1}^2(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty) \\ s &\longmapsto b_1 s \end{aligned}$$

is not surjective.

The last statement follows from Lemma 6.2. As in Section 6.3, we were able to compute the action of $\sum_{\deg \mathfrak{p} \leq 1} T_{\mathfrak{p}}$ on $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ on some examples. We checked Conjecture 6.9 for all primes \mathfrak{n} in $\mathbf{F}_2[T]$ of degree in $\{5, 6, 7, 8, 9\}$, in $\mathbf{F}_3[T]$ of degree in $\{5, 6, 7, 8\}$, in $\mathbf{F}_4[T]$ and $\mathbf{F}_5[T]$ of degree 5.

7. Proof and applications of Theorem 1.2

7.1. Explicit version of Theorem 1.2. —

Notation 7.1. — We call a *decomposition* of $c \in \mathbf{N}$ a tuple $\underline{c} = (c_0, \dots, c_d)$ such that $c = \sum_{j=0}^d c_j q^j$ and $0 \leq c_j < q$ for any $j \in \{0, \dots, d\}$, for some $d \geq 0$. The *length* of \underline{c} is $d + 1$. Note that we do not require $c_d \neq 0$. The base q expansion gives a decomposition of c . By putting zeros at the end of any decomposition of c , we obtain decompositions of larger length.

If $\underline{i} = (i_0, \dots, i_d)$ is a decomposition of $i \geq 0$, let

$$l(i) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} \sum_{\substack{Q|P, Q \in A_{1+} \\ (Q) + \mathfrak{n} = A}} Q^{k-1} \in A.$$

We prove Theorem 1.2 by establishing the following explicit version.

Theorem 7.2. — Assume q is a prime.

1. Suppose $m = 0$. Let $n = c/(q-1) \in \mathcal{S}$. We fix a decomposition (c_0, \dots, c_d) of c of length $d + 1$ for some $d \geq 0$ (therefore $c_0 + \dots + c_d = q - 1$). Let

$$t_{c_0, \dots, c_d} = (-1)^d \binom{q-2}{c_0-1, c_1, \dots, c_d}^{-1} \sum_{P \in A_{d+}} \{q^{d+1-c}\}_P^P \in \mathbf{R}_A.$$

Then, for any k and \mathfrak{n} , we have $b_n = b_1 t_{c_0, \dots, c_d}$ in the dual space of $M_{k,0}(\Gamma_0(\mathfrak{n}))$.

2. Suppose $m = 1$. Let $n = c/(q-1) \in \mathcal{S}$. We fix a decomposition (c_0, \dots, c_d) of c of length $d + 1$ for some $d \geq 0$ with $c_d \neq q - 1$ (therefore $c_0 + \dots + c_d = q - 1$). Let

$$t'_{c_0, \dots, c_d} = (-1)^d \binom{q-1}{c_0, \dots, c_d}^{-1} \sum_{P \in A_{d+}} \{q^{d+1-1-c}\}_P^P \in \mathbf{R}_A.$$

Then, for any k and \mathfrak{n} , we have

$$b_n = b_1 t'_{c_0, \dots, c_d} + (-1)^{d+1} \binom{q-1}{c_0, \dots, c_d}^{-1} l(q^{d+1} - 1 - c) a_1$$

in the dual space of $M_{k,1}(\Gamma_0(\mathfrak{n}))$.

3. Assume $m = 1$. Let $d \geq 1$ and

$$t_d = (-1)^d \sum_{P \in A_{d+}} \left(\{q^{d-1}\} - \sum_{i=0}^{d-1} \{q^{d-1-(q-1)q^i}\} \right) T_P \in \mathbf{R}_A.$$

Then, for any k and \mathbf{n} , we have

$$(13) \quad b_{q^d} = b_1 t_d + (-1)^d \left(-l(q^d - 1) + \sum_{i=0}^{d-1} l(q^d - 1 - (q-1)q^i) \right) a_1$$

in the dual space of $M_{k,1}(\Gamma_0(\mathbf{n}))$.

- Remark 7.3.** — 1. Since q is prime and $\sum_{j=0}^d c_j = q-1$, the multinomial coefficients $\binom{q-1}{c_0, \dots, c_d}$ and $\binom{q-2}{c_0-1, c_1, \dots, c_d}$ are nonzero in \mathbf{F}_p , by Lucas's theorem, hence invertible.
2. On doubly cuspidal forms, a_1 vanishes and the expressions of Theorem 7.2 simplify and provide Theorem 1.2. Moreover, since $b_1 \mathbf{T}'$ is contained in the \mathbf{C}_∞ -vector space spanned by b_n for $n \in \mathcal{S}$ (Corollary 5.8), we get the equality provided that q is prime and $m \in \{0, 1\}$.
3. For a given $n \in \mathcal{S}$, we get infinitely many expressions $s_n \in \mathbf{T}'$ such that $b_n = b_1 s_n$. The reason is that, in the first two items of Theorem 7.2, any decomposition of $c = (q-1)n$ gives rise to a formula for $s_n \in \mathbf{T}'$ satisfying the desired property. More generally, any element of $s_n + \mathbf{I}$ would satisfy the same property.
4. The primality assumption on q is not always essential: it is required to ensure that the multinomial coefficient $\binom{q-1}{c_0, \dots, c_d}$ for $m = 1$ (resp. $\binom{q-2}{c_0-1, c_1, \dots, c_d}$ for $m = 0$) is nonzero in \mathbf{F}_p . Hence, the assumption is unnecessary in (13). If q is not a prime, the first (resp. second) statement of Theorem 7.2 is true for $n = c/(q-1) \in \mathcal{S}$ such that there exists a decomposition (c_0, \dots, c_d) of c with $\binom{q-2}{c_0-1, c_1, \dots, c_d} \neq 0$ in \mathbf{F}_p (resp. $\binom{q-1}{c_0, \dots, c_d} \neq 0$ in \mathbf{F}_p) for some $d \geq 0$.

Before proving Theorem 7.2, we give an example.

Example 7.4 ($d = 1$). — We put

$$s_n = -\binom{q-1}{n-1}^{-1} \sum_{P \in A_{1+}} P^{n-1} T_P \quad \text{for } 1 \leq n \leq q-1$$

$$s_q = - \sum_{P \in A_{1+}} (P^{q-1} - 1) T_P.$$

Then $b_n(f) = b_1(s_n(f))$ for all $f \in M_{k,1}^2(\Gamma_0(\mathbf{n}))$ and $1 \leq n \leq q$. This is valid for q a power of a prime, by Remark 7.3 and Lucas's theorem. Using these formulas, we can recover the first q coefficients of any Hecke eigenform f in $M_{k,1}^2(\Gamma_0(\mathbf{n}))$ in terms of $b_1(f)$ and the eigenvalues.

Proof of Theorem 7.2. — 1. Assume that the type m is 0. We put $n_0 = c_0 - 1, n_1 = c_1, \dots, n_d = c_d$, so that $n_0 + \dots + n_d = q - 2$. By Corollary 5.10, $a_{q-1}(\Theta_d(q-1 - n_0, \dots, q-1 - n_{d-1}))$ is

$$\sum_{\underline{r}} \binom{q-2}{\underline{r}} S_d(r_0 + q - 1 - n_0, \dots, r_{d-1} + q - 1 - n_{d-1}) a_{1+r_0+r_1q+\dots+r_dq^d}$$

where $\underline{r} = (r_0, \dots, r_d)$ satisfies $r_0 + \dots + r_d = q - 2$. From $n_i \geq -1$, we get $0 \leq r_i + q - 1 - n_i \leq 2(q - 1)$ for all i . We can thus evaluate the sum $S_d(r_0 + q - 1 - n_0, \dots, r_{d-1} + q - 1 - n_{d-1})$ by Proposition 3.5: it is nonzero only if \underline{r} is such that $r_i \in \{n_i, q - 1 + n_i\}$, for all $i \in \{0, \dots, d - 1\}$. Since $r_i \leq q - 2$, we have $r_0 = n_0, \dots, r_{d-1} = n_{d-1}$ and by Proposition 3.5,

$$a_{q-1}(\Theta_d(q - 1 - n_0, \dots, q - 1 - n_{d-1})) = \binom{q-2}{\underline{n}} (-1)^d a_{1+n_0+n_1q+\dots+n_dq^d}.$$

Finally, $a_{1+n_0+\dots+n_dq^d} = a_{n(q-1)} = b_n$ and the conclusion follows.

2. Assume that the type m is 1. Since $q^{d+1} - 1 - c$ has base q expansion $\sum_{j=0}^d (q - 1 - c_j)q^j$, we have

$$\begin{aligned} \sum_{P \in A_{d+}} \{q^{d+1} - 1 - c\}^P T_P &= \sum_{P \in A_{d+}} C_{P,0}^{q-1-c_0} \dots C_{P,d-1}^{q-1-c_{d-1}} T_P \\ &= \Theta_d(q - 1 - c_0, \dots, q - 1 - c_{d-1}). \end{aligned}$$

By Corollary 5.10, $b_1(\Theta_d(q - 1 - c_0, \dots, q - 1 - c_{d-1}))$ is

$$\begin{aligned} \sum_{\underline{r}} \binom{q-1}{\underline{r}} S_d(r_0 + q - 1 - c_0, \dots, r_{d-1} + q - 1 - c_{d-1}) a_{1+r_0+r_1q+\dots+r_dq^d} \\ + l(q^{d+1} - 1 - c) a_1 \end{aligned}$$

with $\underline{r} = (r_0, \dots, r_d)$ such that $r_0 + \dots + r_d = q - 1$. From $c_i \geq 0$ and $0 \leq r_i \leq q - 1$, we get $0 \leq r_i + q - 1 - c_i \leq 2(q - 1)$. Thus the sum $S_d(r_0 + q - 1 - c_0, \dots, r_{d-1} + q - 1 - c_{d-1})$ can be evaluated thanks to Proposition 3.5: it is nonzero if and only if $r_i \in \{c_i, q - 1 + c_i\}$ for all $i \in \{0, \dots, d - 1\}$.

Suppose there exists $k \in \{0, \dots, d - 1\}$ with $r_k = q - 1 + c_k$. Then, according to the previous remarks, we have

$$q - 1 - r_d = \sum_{j=0}^{d-1} r_j = q - 1 + c_k + \sum_{j=0, j \neq k}^{d-1} r_j \geq q - 1 + \sum_{j=0}^{d-1} c_j = 2(q - 1) - c_d$$

hence $0 \leq q - 1 - c_d \leq -r_d$. This implies $r_d = 0$, thus $c_d = q - 1$, which is impossible. Therefore, we have $r_j = c_j$ for any $j \in \{0, \dots, d - 1\}$ and $r_d = c_d$ as a consequence. Proposition 3.5 then provides

$$\begin{aligned} b_1(\Theta_d(q - 1 - c_0, \dots, q - 1 - c_{d-1})) &= a_{1+c_0+c_1q+\dots+c_dq^d} \\ &\quad + (-1)^d \binom{q-1}{c_0, \dots, c_d}^{-1} l(q^{d+1} - 1 - c) a_1. \end{aligned}$$

Finally, $a_{1+c_0+c_1q+\dots+c_dq^d} = a_{1+n(q-1)} = b_n$, thus the statement is proved.

3. Assume that the type m is 1. We first compute $b_1(\Theta_d(q - 1, \dots, q - 1))$. According to Corollary 5.10, it is

$$\begin{aligned} \sum_{\substack{\underline{r}=(r_0, \dots, r_d) \\ r_0+\dots+r_d=q-1}} \binom{q-1}{\underline{r}} S_d(r_0 + q - 1, \dots, r_{d-1} + q - 1) a_{1+r_0+r_1q+\dots+r_dq^d} \\ + l(q^{d+1} - 1) a_1. \end{aligned}$$

By Proposition 3.5, the sum $S_d(r_0 + q - 1, \dots, r_{d-1} + q - 1)$ is nonzero if and only if $r_i \in \{0, q - 1\}$ for all $i \in \{0, \dots, d - 1\}$. This means that (r_0, \dots, r_{d-1}) is one of

the following:

$$(q-1, 0, \dots, 0), (0, q-1, 0, \dots, 0), \dots, (0, \dots, 0, q-1), (0, \dots, 0).$$

Thus $b_1(\Theta_d(q-1, \dots, q-1))$ equals

$$(14) \quad (-1)^d \left(a_{1+(q-1)} + \dots + a_{1+(q-1)q^{d-1}} + a_{1+(q-1)q^d} \right) + l(q^{d+1} - 1)a_1.$$

Next, we compute $b_1(\Theta(q-1, \dots, 0, \dots, q-1))$, the only zero term being at the $(j+1)$ th position ($0 \leq j \leq d-1$). From Corollary 5.10, it is

$$\sum_{\substack{\underline{r}=(r_0, \dots, r_d) \\ r_0 + \dots + r_d = q-1}} \binom{q-1}{\underline{r}} S_d(r_0 + q-1, \dots, r_j, \dots, r_{d-1} + q-1) a_{1+r_0+r_1q+\dots+r_dq^d} \\ + l(q^{d+1} - 1 - (q-1)q^j) a_1.$$

Again by Proposition 3.5, the sum is only over \underline{r} satisfying the following two properties:

$$r_i \in \{0, q-1\} \quad \text{for all } i \in \{0, \dots, d-1\}, i \neq j \\ r_j \in \{q-1, 2(q-1)\}.$$

Since $r_0 + \dots + r_d = q-1$, we have necessarily $r_j = q-1$, $r_i = 0$ for all $i \neq j$ and $r_d = 0$. Then

$$(15) \quad b_1(\Theta(q-1, \dots, 0, \dots, q-1)) = (-1)^d a_{1+(q-1)q^j} + l(q^{d+1} - 1 - (q-1)q^j) a_1$$

Combining (14) and (15), we get the claim. □

7.2. Applications. — Theorem 1.2 has the following straightforward consequence.

Corollary 7.5. — *Under the assumptions of Theorem 1.2, if f is a Hecke eigenform with $b_n(f) \neq 0$ for some $n \in \mathcal{S}$, then $b_1(f) \neq 0$.*

In particular, in Proposition 6.5, one can replace the assumption $b_1(f) \neq 0$ by: there exists $n \in \mathcal{S}$ such that $b_n(f) \neq 0$.

We now provide multiplicity one statements in certain spaces of Drinfeld modular forms.

Lemma 7.6. — 1. *Let $d = \dim M_{k,m}(\mathrm{GL}_2(A))$. The \mathbf{C}_∞ -linear map*

$$\begin{array}{ccc} M_{k,m}(\mathrm{GL}_2(A)) & \longrightarrow & \mathbf{C}_\infty^d \\ f & \longmapsto & (b_0(f), \dots, b_{d-1}(f)) \end{array}$$

is an isomorphism.

2. *Let $d = \dim M_{2,1}^2(\Gamma_0(\mathbf{n}))$. The \mathbf{C}_∞ -linear map*

$$\begin{array}{ccc} M_{2,1}^2(\Gamma_0(\mathbf{n})) & \longrightarrow & \mathbf{C}_\infty^d \\ f & \longmapsto & (b_1(f), \dots, b_d(f)) \end{array}$$

is an isomorphism.

Proof. — The first assertion follows readily from a formula relating, for a nonzero $f \in M_{k,m}(\mathrm{GL}_2(A))$, the orders of vanishing of f at elliptic, non-elliptic points and the cusp infinity of $\mathrm{GL}_2(A)$ (see (5.14) in Gekeler’s paper [7]).

For the second assertion, we consider the Drinfeld modular curve $X_0(\mathfrak{n})$ attached to $\Gamma_0(\mathfrak{n})$. This smooth projective algebraic curve over \mathbf{C}_∞ is the compactification of the affine curve $Y_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n}) \backslash \Omega$ over \mathbf{C}_∞ , the group $\Gamma_0(\mathfrak{n})$ acting on Ω via linear fractional transformations. Actually, the curve $Y_0(\mathfrak{n})$ is a coarse moduli scheme for rank-2 Drinfeld modules with a level structure determined by \mathfrak{n} . The cusps, i.e. the set $X_0(\mathfrak{n}) - Y_0(\mathfrak{n})$, is naturally in bijection with $\Gamma_0(\mathfrak{n}) \backslash \mathbf{P}^1(K)$. Since we assume \mathfrak{n} prime, the cusps are labeled $\{0, \infty\}$ as usual. Gekeler gave formulas for the genus $g = g(X_0(\mathfrak{n}))$ in terms of the degree of \mathfrak{n} ([5, 6]).

One can show that ∞ is not a Weierstrass point on $X_0(\mathfrak{n})$ with \mathfrak{n} prime (i.e. every holomorphic differential form on $X_0(\mathfrak{n})$ vanishes at ∞ at order $< g$). This is merely an adaptation of Ogg’s geometric argument [16] for the classical modular curve $X_0(pM)$ with p prime and $p \nmid M$. To adapt the proof, we need Gekeler’s description of the bad fiber at \mathfrak{n} of a model of $X_0(\mathfrak{n})$ over A ([6] p. 233): it consists of two copies of the projective line over $\mathbf{F}_\mathfrak{n} = A/\mathfrak{n}$ intersecting transversally at n supersingular points (i.e. points whose underlying rank-2 Drinfeld module is supersingular over $\mathbf{F}_\mathfrak{n}$). The full Atkin–Lehner involution $w_\mathfrak{n}$ interchanges the components. The reductions of the cusps are on distinct components hence are not supersingular points. The second ingredient is the analogue of Ogg’s formula (2) in [16]: the number n of supersingular points is $1 + g$, according to (5.4) in [6] and Gekeler’s formula for g . We leave the details to the reader.

The map $f \mapsto f(z)dz$ defines an isomorphism between $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ and the space of holomorphic differential forms on $X_0(\mathfrak{n})$ ([15] Proposition 2.10.2), hence both spaces have dimension $g = d$. Since ∞ is not a Weierstrass point, any Drinfeld modular form in $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ vanishes at ∞ at order $< d$. In other words, the linear map $M_{2,1}^2(\Gamma_0(\mathfrak{n})) \rightarrow \mathbf{C}_\infty^d$ given by $f \mapsto (b_1(f), \dots, b_d(f))$ is injective, hence bijective. \square

Theorem 7.7. — *Let M be one of the following spaces of Drinfeld modular forms:*

1. $M_{k,0}^1(\mathrm{GL}_2(A))$ with $k < (q+1)^2(q-1)$
2. $M_{k,1}^2(\mathrm{GL}_2(A))$ with $k < q^2(q+1)$
3. $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ with \mathfrak{n} prime of degree 3.

Then:

- Any eigenform in M for the operators $(T_\mathfrak{p})_{\deg \mathfrak{p}=1}$ is characterized in the space M by its eigenvalues, up to a multiplicative constant.
- The map $u : \mathbf{T}' \rightarrow \mathrm{Hom}_{\mathbf{C}_\infty}(M, \mathbf{C}_\infty)$ is an isomorphism.

Proof. — Consider the first two cases for M . By the cuspidality (resp. doubly cuspidality) condition and the assumption on the type, we have $b_0(f) = a_m(f) = 0$. Therefore, any function $f \in M$ is determined, in the space M , by its coefficients $b_1(f), \dots, b_{d-1}(f)$, according to Lemma 7.6. Now, if f is an eigenform for $(T_\mathfrak{p})_{\deg \mathfrak{p}=1}$, we know that $b_1(f), \dots, b_q(f)$ are determined by the eigenvalues (up to a multiplicative constant), thanks to Example 7.4. Recall that the dimension of $M_{k,m}(\mathrm{GL}_2(A))$ is $d = \lfloor (k - (q+1)m)/(q^2 - 1) \rfloor + 1$ (this follows from Gekeler’s formula (5.14) in [7]). Here, the assumptions on the weight k ensure that $d - 1 \leq q$. The conclusion follows.

The proof of the third case is similar, except that the dimension of M is q . Indeed, this dimension is equal to the genus of $X_0(\mathbf{n})$. By Gekeler's formula for the genus ([5, 6]), it is q when \mathbf{n} is prime of degree 3.

For the bijectivity of u , we need only to prove the surjectivity by Lemma 6.2. Consider the first two cases for M . As before, M has dimension $d - 1 \leq q$. Moreover, the image of u contains b_1, \dots, b_{d-1} (by Theorem 7.2) which are linearly independent (by Lemma 7.6), hence the conclusion. The proof of the third case is similar. \square

As a corollary, we get that the dimension of the \mathbf{C}_∞ -algebra \mathbf{T}' coincides with the dimension of the space of Drinfeld modular forms M , for M as in the statement.

7.3. Comment on A -structures. — Although we worked with \mathbf{C}_∞ -structures, most of the results of this paper could be transferred over the ring A . For instance, one could work with the subspace $M_{k,m}^2(\Gamma_0(\mathbf{n}); A) \subset M_{k,m}^2(\Gamma_0(\mathbf{n}))$ consisting of modular forms with expansion in $A[[t]]$ and the Hecke algebra \mathbf{T}'_A spanned over A by Hecke operators. Using Proposition 5.2, one may check that the map

$$\mathbf{T}'_A \rightarrow \text{Hom}_A(M_{k,m}^2(\Gamma_0(\mathbf{n}); A), A)$$

induced by $s \mapsto b_1 s$, is well-defined. We expect that $M_{k,m}^2(\Gamma_0(\mathbf{n}); A)$ is a A -structure of $M_{k,m}^2(\Gamma_0(\mathbf{n}))$ (i.e. there exists a basis of $M_{k,m}^2(\Gamma_0(\mathbf{n}))$ consisting of modular forms with coefficients in A). However, a general theory of such algebraic Drinfeld modular forms is still missing in the literature. Some instances of such a theory can be found in [12] (Section 2, for $M_{k,m}(\text{GL}_2(A))$) and [1] (Section 4.2, for $M_{2,1}^2(\Gamma_0(\mathbf{n}))$).

8. Coefficients of h

We use Theorem 7.2 to compute explicitly some coefficients of Gekeler's Drinfeld modular form h , defined in [7]. Recall that h has weight $q + 1$ and type 1 for $\text{GL}_2(A)$. It is defined as a certain Poincaré series and is also a $(q - 1)$ th root of the Drinfeld discriminant form Δ . Moreover, it is a cuspidal Hecke eigenform with $T_{\mathfrak{p}}h = h$ for any \mathfrak{p} (Corollary 7.6 in [7] with a different normalization of Hecke operators). The first

coefficients of h are $a_1(h) = -1$ and $b_1(h) = a_q(h) = \begin{cases} 0 & \text{if } q > 2 \\ 1 & \text{if } q = 2 \end{cases}$.

Proposition 8.1. — For P in A , let $\sigma_P = \sum_{Q|P, Q \in A_{1+}} Q^q$.

1. Assume q is a prime > 2 . Let $c \in \mathbf{N}$ such that $c = \sum_{j=0}^d c_j q^j$ with $0 \leq c_j < q$, $\sum_{j=0}^d c_j = q - 1$ and $c_d \neq q - 1$ (we do not necessarily assume $c_d \neq 0$). Then

$$(16) \quad b_{\frac{c}{q-1}}(h) = (-1)^d \binom{q-1}{c_0, \dots, c_d}^{-1} \sum_{P \in A_{d+}} \{q^{d+1-1-c}\}^P \sigma_P.$$

Moreover, for $d \geq 0$,

$$(17) \quad b_{q^d}(h) = (-1)^{d+1} \sum_{P \in A_{d+}} \left(-\{q^{d-1}\}^P + \sum_{i=0}^{d-1} \{q^{d-1-(q-1)q^i}\}^P \right) \sigma_P$$

2. Assume $q = 2$. Then for every $d \geq 0$, one has

$$(18) \quad b_{2^d}(h) = (-1)^d \sum_{P \in A_{d+}} \left(-\{2^d P\} + \sum_{i=0}^{d-1} \{2^{d-1-2^i} P\} \right) (1 + \sigma_P).$$

Remark 8.2. — We recover that the corresponding coefficients of h are polynomials in $T^q - T$ with coefficients in \mathbf{F}_q (indeed, they are elements of A which are invariant under $T \mapsto T + c$ for $c \in \mathbf{F}_q$). More generally, Gekeler proved that this property holds for any coefficient of h (Theorem 2.4 of [9]).

Taking $d = 1$ in Proposition 8.1, one can recover the first q coefficients of h . If q is a prime > 2 , then $b_i(h) = 0$ if $1 \leq i \leq q - 2$, $b_{q-1}(h) = -1$ and $b_q(h) = T^q - T$. They can also be obtained from the Taylor series $h = -tU_1^{-1} + o(t^{1+(q-1)(q^3-q^2)})$ with $U_1 = 1 - t^{(q-1)^2} + (T^q - T)t^{(q-1)q}$ (see Corollary 10.4 in [7]).

For $i \in \mathbf{N}$, let $[i] = T^{q^i} - T$. Using congruences and estimates on the degree of coefficients of h , Gekeler proved that for any $d \geq 1$,

$$(19) \quad b_{q^d}(h) = \begin{cases} [d] & \text{if } q > 2 \\ 1 + [d] & \text{if } q = 2 \end{cases}$$

(see Corollary 2.6 of [9]; note that his b_i denotes our $-b_i$). Equation (17) thus provides an alternative formula for $b_{q^d}(h)$. We have not been able to recover (19) from (17) and (18). Hence we derive some arithmetic identities in $\mathbf{F}_q[T]$ which may be nontrivial and of some interest.

Corollary 8.3. — Let q be a prime > 2 and $d \geq 1$.

1.

$$[d] = (-1)^{d+1} \sum_{P \in A_{d+}} \left(-\{q^d P\} + \sum_{i=0}^{d-1} \{q^{d-1-(q-1)q^i} P\} \right) \sigma_P.$$

2. For $0 \leq i \leq d - 1$,

$$(-1)^d [i] = \sum_{P \in A_{d+}} \{q^{d-1-(q-1)q^i} P\} \sigma_P.$$

3.

$$(-1)^d \sum_{i=1}^d [i] = \sum_{P \in A_{d+}} \{q^{d-1} P\} \sigma_P.$$

Proof. — The first one follows from (17) and (19). For the second one, we first apply (16) to $c = (q - 1)q^i$ with $0 \leq i \leq d - 1$ and get

$$(-1)^d b_{q^i}(h) = \sum_{P \in A_{d+}} \{q^{d+1-1-(q-1)q^i} P\} \sigma_P = \sum_{P \in A_{d+}} \{q^{d-1-(q-1)q^i} P\} \sigma_P$$

where the last equality follows from $q^{d+1} - 1 - (q - 1)q^i = (q - 1)q^d + \sum_{j=0}^{d-1} (q - 1)q^j - (q - 1)q^i$ and $\deg P = d$. With (19), we get the second claim. The third one is obtained by combining the first two identities. \square

TABLE 1. $q = 3, d \leq 4$

i	$b_i(h)$
1	0
2	-1
3	[1]
5	-[1]
6	-[1] ² - 1
9	[2] = [1] ³ + [1]
14	[1] ⁴ - 1
15	[1] ⁵ - [1] ³ + [1]
18	-[1] ⁶ + [1] ⁴ - [1] ² - 1
27	[3] = [1] ⁹ + [1] ³ + [1]
41	-[1] ¹³ + [1] ⁹ - [1] ⁷ - [1]
42	-[1] ¹⁴ + [1] ¹² - [1] ¹⁰ - [1] ⁸ - [1] ² - 1
45	[1] ¹⁵ - [1] ¹³ + [1] ¹¹ - [1] ⁹ + [1] ³ + [1]
54	-[1] ¹⁸ + [1] ¹² + [1] ¹⁰ - [1] ⁶ + [1] ⁴ - [1] ² - 1
81	[4] = [1] ²⁷ + [1] ⁹ + [1] ³ + [1]

In Table 1, we provide further examples of coefficients of h from Proposition 8.1. Observe that when i is even (resp. odd), $b_i(h)$ is an even (resp. odd) polynomial in $[1] = T^q - T$. This is more generally true for any coefficient when $q = 3$: it follows from the coefficients of h being balanced, a property established by Gekeler (Theorem 2.4 of [9]). Note that, in our table, the constant term is -1 when i is even: we wonder if such a statement holds more generally.

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