



HAL
open science

Elementary construction of the Nagel point

Xavier Dussau

► **To cite this version:**

| Xavier Dussau. Elementary construction of the Nagel point. 2020. hal-02558108

HAL Id: hal-02558108

<https://hal.science/hal-02558108>

Preprint submitted on 29 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Elementary construction of the Nagel point

Xavier Dussau^{*,1}

¹Laboratoire de mathématiques, Lycée Max Linder, Libourne, France

April 29, 2020

Abstract

In this short note, we give an easy construction of the Nagel point of a triangle.

1 Introduction

If you continue the sides of a triangle beyond every vertex at the distances equaling to the length of the opposite side, the resulting six points lie on a circle, which is called Conway's circle (figure 1).

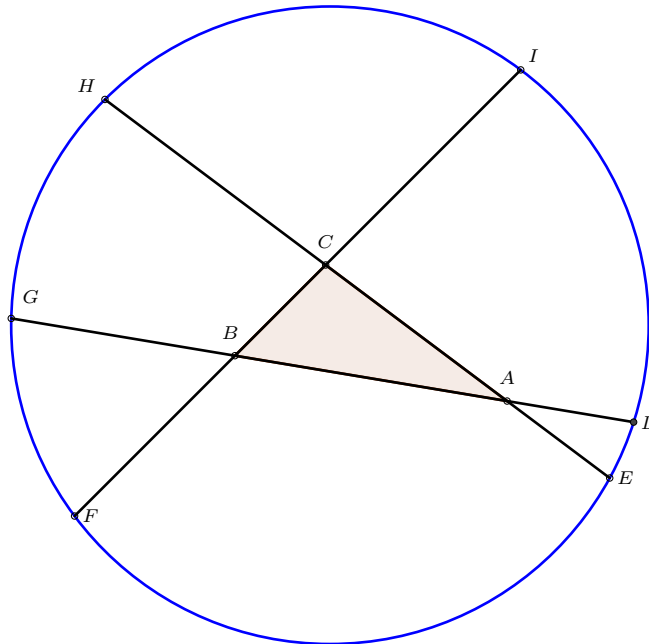


Figure 1. The Conway circle

In this note, we investigate a similar construction which characterizes the Nagel point of a triangle ABC. Starting from point A, we transport the length of the opposite edge on each half-line starting from A. We obtain two points A' and A'' which satisfy $\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}$ and $\overrightarrow{AA''} = \frac{a}{b}\overrightarrow{AC}$ (figure 2). Repeating this construction with vertices B and C, we have four more points B' , B'' , C' and C'' which satisfy $\overrightarrow{BB'} = \frac{b}{a}\overrightarrow{BC}$, $\overrightarrow{BB''} = \frac{b}{c}\overrightarrow{BA}$, $\overrightarrow{CC'} = \frac{c}{b}\overrightarrow{CA}$ and $\overrightarrow{CC''} = \frac{c}{a}\overrightarrow{CB}$.

^{*}Corresponding author: xavier.dussau@gmail.com

By construction,

$$AA' = a, \quad BB' = b, \quad CC' = c,$$

$$AA'' = a, \quad BB'' = b, \quad CC'' = c.$$

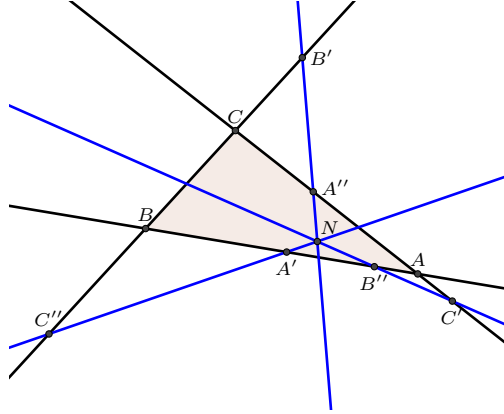


Figure 2. Easy construction of the Nagel point

Using barycentric coordinates, we prove that the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent (figure 2) when ABC is a scalene triangle (a triangle with edges of different lengths).

Moreover, it turns out that the intersection point is the Nagel point of the triangle ABC .

Figure 3 gives the original construction of the Nagel point which is defined by the following result "the lines joining the points of contact of an excircle with the sides of a triangle to the vertices opposite the respective sides, are concurrent".

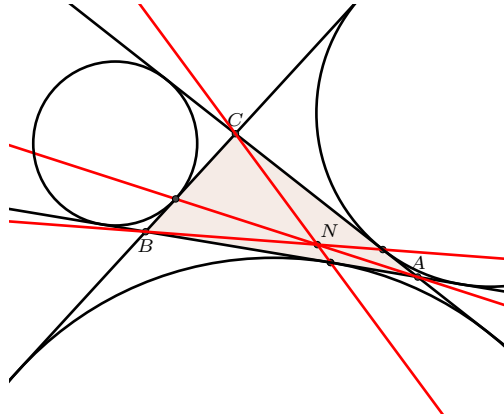


Figure 3. Original construction of the Nagel Point

A construction of the Nagel point requiring only the incircle of the triangle can be found in Hoehn [Hoe07].

2 Main result

Our theorem will use basic facts of barycentric coordinates in the plane. In particular, a line, in barycentric coordinates, is given by a linear homogeneous equation $\alpha X + \beta Y + \gamma W = 0$ for some coefficients α , β , and γ not all of which are 0.

Theorem 1. *Let ABC be a scalene triangle and the six points A' , A'' , B' , B'' , C' and C'' given by the relations:*

$$\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}, \quad \overrightarrow{AA''} = \frac{a}{b}\overrightarrow{AC}, \quad \overrightarrow{BB'} = \frac{b}{a}\overrightarrow{BC},$$

$$\overrightarrow{BB''} = \frac{b}{c}\overrightarrow{BA}, \quad \overrightarrow{CC'} = \frac{c}{b}\overrightarrow{CA}, \quad \overrightarrow{CC''} = \frac{c}{a}\overrightarrow{CB}.$$

Then, the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent and the intersection point is the Nagel point.

Proof. The condition $\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}$ implies that A' has barycentric coordinates $(a-c, -a, 0)$. We also deduce that $A''(a-b, 0, -a)$, $B'(0, b-a, b)$, $B''(-b, b-c, 0)$, $C'(-c, 0, c-b)$ and $C''(0, c, c-a)$.

Let $\alpha X + \beta Y + \gamma Z = 0$ be an equation of the line $A''B'$. Fix $\gamma = 1$. The barycentric coordinates of A'' and B' satisfy this equation and we obtain $\beta = \frac{b}{b-a}$ and $\alpha = \frac{a}{a-b}$. Multiplying by $a-b$, we deduce that $aX - bY + (a-b)Z = 0$ is an equation of the line $A''B'$.

We can now proceed analogously for lines $A'C''$ and $C'B''$. It follows immediately that $aX + (a-c)Y - cZ = 0$ is an equation of $A'C''$ and $(b-c)X + bY - cZ = 0$ is an equation of $C'B''$.

Easy computations show that

$$\begin{vmatrix} a & -b & a-b \\ a & a-c & -c \\ b-c & b & -c \end{vmatrix} = 0.$$

This determinant is 0 hence the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent or parallel. We leave it to the reader to verify that the coordinates $(-a+b+c, a-b+c, a+b-c)$ satisfy the three equations. Hence, the three lines are concurrent. The barycentric coordinates of the Nagel point are $(-a+b+c, a-b+c, a+b-c)$ (see [Kim]) so we can conclude that the intersection point of these three lines is the Nagel point. \square

References

[Hoe07] Larry Hoehn. A new characterization of the nagel point. *Missouri J. Math. Sci.*, 19:45–48, 2007.

[Kim] C. Kimberling. Encyclopaedia of triangle centers, <https://faculty.evansville.edu/ck6/encyclopedia/etc.html>.