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Xavier Dussau

To cite this version:
Xavier Dussau. Elementary construction of the Nagel point. 2020. hal-02558108

HAL Id: hal-02558108
https://hal.science/hal-02558108
Preprint submitted on 29 Apr 2020

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Elementary construction of the Nagel point

Xavier Dussau*,1

1Laboratoire de mathématiques, Lycée Max Linder, Libourne, France

April 29, 2020

Abstract

In this short note, we give an easy construction of the Nagel point of a triangle.

1 Introduction

If you continue the sides of a triangle beyond every vertex at the distances equaling to the length of the opposite side, the resulting six points lie on a circle, which is called Conway’s circle (figure 1).

In this note, we investigate a similar construction which characterizes the Nagel point of a triangle ABC. Starting from point $A$, we transport the length of the opposite edge on each half-line starting from $A$. We obtain two points $A'$ and $A''$ which satisfy $\overrightarrow{AA'} = \frac{a}{c} \overrightarrow{AB}$ and $\overrightarrow{AA''} = \frac{a}{b} \overrightarrow{AC}$ (figure 2). Repeating this construction with vertices $B$ and $C$, we have four more points $B'$, $B''$, $C'$ and $C''$ which satisfy $\overrightarrow{BB'} = \frac{b}{a} \overrightarrow{BC}$, $\overrightarrow{BB''} = \frac{b}{c} \overrightarrow{BA}$, $\overrightarrow{CC'} = \frac{c}{b} \overrightarrow{CA}$ and $\overrightarrow{CC''} = \frac{c}{a} \overrightarrow{CB}$.

*Corresponding author: xavier.dussau@gmail.com
By construction,

\[ AA' = a, \quad BB' = b, \quad CC' = c, \]
\[ AA'' = a, \quad BB'' = b, \quad CC'' = c. \]

Figure 2. Easy construction of the Nagel point

Using barycentric coordinates, we prove that the three lines \( A''B', A'C'' \) and \( C'B'' \) are concurrent (figure 2) when \( ABC \) is a scalene triangle (a triangle with edges of different lengths).

Moreover, it turns out that the intersection point is the Nagel point of the triangle \( ABC \).

Figure 3 gives the original construction of the Nagel point which is defined by the following result "the lines joining the points of contact of an excircle with the sides of a triangle to the vertices opposite the respective sides, are concurrent".

Figure 3. Original construction of the Nagel Point

A construction of the Nagel point requiring only the incircle of the triangle can be found in Hoehn [Hoe07].

2 Main result

Our theorem will use basic facts of barycentric coordinates in the plane. In particular, a line, in barycentric coordinates, is given by a linear homogeneous equation \( \alpha X + \beta Y + \gamma W = 0 \) for some coefficients \( \alpha, \beta, \gamma \) not all of which are 0.

**Theorem 1.** Let \( ABC \) be a scalene triangle and the six points \( A', A'', B', B'', C' \) and \( C'' \) given by the relations:

\[ \overrightarrow{AA'} = \frac{a}{c} \overrightarrow{AB}, \quad \overrightarrow{AA''} = \frac{a}{b} \overrightarrow{AC}, \quad \overrightarrow{BB'} = \frac{b}{a} \overrightarrow{BC}, \]

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Then, the three lines $A'B'$, $A'C''$ and $C'B''$ are concurrent and the intersection point is the Nagel point.

Proof. The condition $\overrightarrow{AA'} = \frac{a}{c} \overrightarrow{AB}$ implies that $A'$ has barycentric coordinates $(a-c, -a, 0)$. We also deduce that $A''(a-b, 0, -a)$, $B'(0, b-a, b)$, $B''(-b, b-c, 0)$, $C'(-c, 0, c-b)$ and $C''(0, c, c-a)$.

Let $\alpha X + \beta Y + \gamma Z = 0$ be an equation of the line $A''B'$. Fix $\gamma = 1$. The barycentric coordinates of $A''$ and $B'$ satisfy this equation and we obtain $\beta = \frac{b}{b-a}$ and $\alpha = \frac{a}{a-b}$. Multiplying by $a-b$, we deduce that $aX - bY + (a-b)Z = 0$ is an equation of the line $A''B'$.

We can now proceed analogously for lines $A'C''$ and $C'B''$. It follows immediately that $aX + (a-c)Y - cZ = 0$ is an equation of $A'C''$ and $(b-c)X + bY - cZ = 0$ is an equation of $C'B'$.

Easy computations show that

$$\left| \begin{array}{ccc} a & -b & a-b \\ a & a-c & -c \\ b-c & b & -c \end{array} \right| = 0.$$

This determinant is 0 hence the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent or parallel. We leave it to the reader to verify that the coordinates $(-a+b+c, a-b+c, a+b-c)$ satisfy the three equations. Hence, the three lines are concurrent. The barycentric coordinates of the Nagel point are $(-a+b+c, a-b+c, a+b-c)$ (see [Kim]) so we can conclude that the intersection point of these three lines is the Nagel point.

References
