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Serge Troubetzkoy, Paulo Varandas

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THE ROLE OF CONTINUITY AND EXPANSIVENESS ON LEO AND PERIODIC SPECIFICATION PROPERTIES

SERGE TROUBETZKOY AND PAULO VARANDAS

ABSTRACT. In this short note we prove that a continuous map which is locally eventually onto and is expansive satisfies the periodic specification property. We also discuss the role of continuity as a key condition in the previous characterization. We include several examples to illustrate the relation between these concepts.

1. INTRODUCTION

There is a well known hierarchy of topological properties involving the topological indecomposability of a dynamical system, as transitivity, topological mixing, and the specification property, among many others. The relation between these and many others has been addressed by Akin, Auslander and Nagar [1]. The aim of this short note is to complement the above results, and to highlight the relation between the locally eventually onto (a dynamical property stronger than topological mixing) and the specification properties, and to make explicit the role of continuity on such characterization. For a survey of specification like properties we recommend the following article [9], while for a survey of mixing properties we recommend the article [1].

First let us recall some well known results. Blokh showed that for a continuous map of the interval $[0, 1]$ the periodic specification property is equivalent to topological mixing (see e.g., [4, 11]). So, while for continuous interval maps the picture is very well understood and most concepts of topological chaoticity coincide, this is no longer true for more general metric spaces or whenever continuity breaks down.

It is noticeable that while any locally eventually onto continuous map has dense periodic sets, it may not have periodic points (cf. [1, Theorem 2.30

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and Example 2.31]). In particular, a locally eventually onto continuous map need not satisfy the periodic specification property. Two results complement this discussion. First, expansiveness play a key role to bridge between the specification and periodic specification properties: a topological dynamical system satisfying the specification property and whose natural extension is expansive satisfies the periodic specification property (see e.g., [9, Lemma 6]). Second, Yan, Yin and Wang [14, Theorem 3.1] constructed an example of a topological mixing subshift, hence expansive, which does not even have the specification property.

While the locally eventually onto property need not ensure the periodic specification property, the following result shows that expansiveness can act as a sufficient condition for it. We refer the reader to Section 2 for definitions.

Theorem 1. *If the topological dynamical system (X, f) is locally eventually onto and expansive then it has the periodic specification property.*

Given the previous result it is natural to ask whether any locally eventually onto continuous map satisfies the specification property.

Remark 2. *It is worth mentioning that the situation is clear for continuous interval maps. Indeed, combining [5, Theorem B] and Blokh's theorem (cf. [11, Theorem 3.4]), it follows that the locally eventually onto property implies on the following conditions, which, for interval maps, are equivalent:*

- (i) f^2 is transitive,
- (ii) f^n is transitive for every $n \geq 1$,
- (iii) f is topologically mixing,
- (iv) f satisfies the specification property.

While the converse holds in the case of piecewise monotone continuous interval maps (cf. [5, Lemma 4.1]), it fails for general continuous interval maps. In particular there are continuous interval maps satisfying the specification property for which the locally eventually onto property fails (see e.g., [2, Example 3]).

On the positive direction, we notice that the same strategy used in Blokh's theorem ensures the following:

Theorem 3. *Every locally eventually onto, continuous and conformal map (i.e., mapping balls onto balls) on a compact metric space satisfies the periodic specification property.*

Remark 4. *In the definition of topological dynamical system, the assumption that the metric space is complete cannot be removed. Indeed, there exists a metric space $X \subset \{0, 1, 2\}^{\mathbb{N}}$ such that the shift map (X, σ) is locally eventually onto, it is clearly expansive, but fails even to present periodic points [1, Example 2.31].*

Our second goal concerns describing the consequences of discontinuities on locally eventually onto maps. This is a problem dual to the one considered by Buzzi [4], the study of the specification property for piecewise monotone interval maps. In the case of piecewise monotone continuous interval maps f , the transitivity for f^2 ensures the following “almost” locally eventually onto property: for any open interval A and any closed interval $J \subset (0, 1)$ there exists $N \geq 1$ so that $f^N(A) \supset J$ (see [2, Theorem 6]). However, while the key step in this argument explores the denseness of periodic points, the classical argument that ensures the denseness of periodic points for expanding maps does not apply for transitive piecewise expanding interval maps given that dynamical balls may fail to grow to a large scale.

We shall focus on important classes of dynamical systems known as β -expansions and β -shifts (see e.g., [3]). These can be realized by geometric models in the interval; for each $\beta > 1$, the β -map is the C^∞ -piecewise expanding interval map $T_\beta : [0, 1) \rightarrow [0, 1)$ given by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$

However, while the previous map is always expansive, and Markov for a countable set of parameters, T_β does not satisfy the specification property for Lebesgue almost every parameter $\beta > 1$ (cf. [4]). A characterization of the set of the values of β which lead to maps with specification can be found in [12]. The next result shows that continuity is essential in Theorem 1.

Theorem 5. *For Lebesgue almost every $\beta \in (1, +\infty)$ the map T_β :*

- (i) *is locally eventually onto;*
- (ii) *is expansive;*
- (iii) *does not satisfy the specification property [4].*

We complete this section with two final comments on the relation between the specification and the locally eventually onto properties for continuous maps in more general metric spaces. While any Anosov diffeomorphism satisfies the specification property (see e.g [8]), every volume preserving Anosov diffeomorphism is clearly not locally eventually onto. Nevertheless, on the converse direction, locally eventually onto maps displaying non-uniform expansion often satisfy some measure theoretical forms of specification (we refer the reader to [10, 13] for the precise formulations).

2. DEFINITIONS

Let (X, d) be a compact metric space, and $f : X \rightarrow X$ a continuous map. We refer to (X, f) as a *dynamical system*.

The map f is called *locally eventually onto* (*LEO*) if for every nonempty open set U there is an $n \in \mathbb{N}$ such that $f^n(U) = X$.

For integers $a \geq b \geq 0$ let $f^{[a,b]}(x) := \{f^j(x) : a \leq j \leq b\}$.

A family of orbit segments $\{f^{[a_j, b_j]}(x_j)\}_{j=1}^n$ is an N -spaced specification if $a_i - b_{i-1} \geq N$ for $2 \leq i \leq n$.

We say that a specification $\{f^{[a_j, b_j]}(x_j)\}_{j=1}^n$ is ε -shadowed by $y \in X$ if

$$d(f^k(y), f^k(x_i)) \leq \varepsilon \text{ for } a_i \leq k \leq b_i \text{ and } 1 \leq i \leq n.$$

We say that (X, f) has the *specification property* if for any $\varepsilon > 0$ there is a constant $N = N(\varepsilon)$ such that any N -spaced specification $\{f^{[a_j, b_j]}(x_j)\}_{j=1}^n$ is ε -shadowed by some $y \in X$. If additionally, y can be chosen in such a way that $f^{b_n - a_0 + N}(y) = y$ then (X, f) has the *periodic specification property*.

The dynamical system (X, f) is *positively expansive* if there exists $\alpha > 0$, called expansivity constant of f , such that if $x, y \in X$ and $x \neq y$, then for some $n \geq 0$, $d(f^n(x), f^n(y)) > \alpha$.

The dynamical system (X, f) is *expanding* if there are constants $\lambda > 1$ and $\delta_0 > 0$ such that, for all $x, y, z \in X$,

- (1) $d(f(x), f(y)) \geq \lambda d(x, y)$ whenever $d(x, y) < \delta_0$ and
- (2) $B(x, \delta_0) \cap f^{-1}(z)$ is a singleton whenever $d(f(x), z) < \delta_0$.

The set $B_n(x, \varepsilon) := \{y \in X : d(f^i x, f^i y) < \varepsilon \text{ for } 0 \leq i < n\}$ is called a *Bowen ball*.

A dynamical system (X, f) is called *conformal* if the image of every ball is a ball.

3. PROOFS

3.1. Proof of Theorem 1. Specification is a topological invariant, the proof just explores uniform continuity of the conjugacy, hence we can ask whether such a property holds for the continuous map f on (X, d) or on the metric space (X, d') , for a equivalent metric d' . For any expansive map Coven and Reddy constructed an adapted metric which made it expanding [6], thus we can assume that (X, f) is expanding (henceforth we will assume the metric d is the expanding).

From the locally eventually onto property, for each $y \in X$, and $\varepsilon > 0$ there is an $N(y, \varepsilon) \geq 1$ such that $f^{N(y, \varepsilon)}(B(y, \varepsilon/3)) = X$. Moreover, by compactness of X we can cover X by a finite collection of balls $\{B(y_i, \varepsilon/3)\}_i$. Let $N :=$

$\max_i \{N(y_i, \varepsilon)\}$. Then since any ball $B(y, \varepsilon)$ contains one of the $B(y_i, \varepsilon/3)$ we conclude that $f^N(B(y, \varepsilon)) = X$ for all $y \in X$.

Now since f is continuous and expanding, the image by f^m of a Bowen ball $B_m(x, \varepsilon)$ is $B(f^m(x), \varepsilon)$, for every $0 < \varepsilon < \delta_0$. Combining this with the previous paragraph yields $f^{m+N}B_m(x, \varepsilon) = X$ for each $x \in X$ and every $0 < \varepsilon < \delta_0$.

Fix $\varepsilon > 0$ and choose N as above. Consider an N -specification, i.e., a collection of orbit segments $\{f^{[a_j, b_j]}(x_j)\}_{j=1}^n$, with $a_i - b_{i-1} \geq N$ for $2 \leq i \leq n$. Setting $m_j := b_j - a_j$ and $N_j = m_j + N$ we have shown that

$$f^{N_j}(B_{m_j}(f^{a_j}(x_j), \varepsilon)) = X \supset B_{m_{j+1}}(f^{a_{j+1}}(x_{j+1}), \varepsilon),$$

and thus

$$B_{m_j}(f^{a_j}(x_j), \varepsilon) \cap f^{-N_j}(B_{m_{j+1}}(f^{a_{j+1}}(x_{j+1}), \varepsilon)) \neq \emptyset$$

hold for each $1 \leq j \leq n$. Iterating this, and noticing that f is expanding, yields that

$$(1) \quad \left\{ B_{m_1}(f^{a_1}(x_1), \varepsilon) \cap f^{-N_1 - N_2 - \dots - N_i}(B_{m_i}(f^{a_i}(x_i), \varepsilon)) \right\}_{2 \leq i \leq n}$$

is a nested sequence of compact sets. Any point in the intersection of these sets ε -shadows the specification, and thus we have shown the specification property holds.

Finally we must show that the periodic specification property holds. Fix $\varepsilon > 0$ and consider an arbitrary N -specification $\{f^{[a_j, b_j]}(x_j)\}_{j=1}^n$ with N chosen as above. We extend this to a longer N specification by choosing $a_{n+1} = b_n + N$, $b_{n+1} = a_{n+1} + m_1$ and $x_{m+1} = f^{a_1 - a_{n+1}}x_1$. Thus $m_{n+1} = m_1$ and $B_{m_{n+1}}(f^{a_{n+1}}x_{m+1}, \varepsilon) = B_{m_1}(f^{a_1}x_1, \varepsilon)$. Therefore the chain (1) of containments extends to

$$\left\{ B_{m_1}(f^{a_1}(x_1), \varepsilon) \cap f^{-N_1 - N_2 - \dots - N_i}(B_{m_i}(f^{a_i}(x_i), \varepsilon)) \right\}_{2 \leq i \leq n+1}.$$

The closure of the intersection of the extended chain of containments must contain a point fixed by $f^{N_1 + \dots + N_{m+1}}$, hence the periodic specification property holds. \square

3.2. Proof of Theorem 3. The strategy follows closely [4, Appendix A]. For that reason we just give a brief sketch of the proof. Let X be a compact metric space and $f : X \rightarrow X$ be a continuous, locally eventually onto conformal map. The key step is a uniform control on the images of Bowen balls. Indeed, while points in n -Bowen balls are within controlled distance to the original orbit during n iterates, it is the size of the image the Bowen ball by iteration of f^n which suggests how strong is the capability to obtain specification.

Claim: For any $\varepsilon > 0$ there exists $\zeta(\varepsilon) > 0$ so that

$$\text{diam}(f^n(B_n(x, \varepsilon))) \geq \zeta(\varepsilon) \quad \text{for every } n \geq 1 \text{ and } x \in X.$$

Proof of the claim. Fix $x \in X$. By conformality, for each $n \geq 1$ the set $f^n(B_n(x, \varepsilon))$ is a ball around $f^n(x)$. Recall also that

$$(2) \quad B_{n+1}(x, \varepsilon) = \bigcap_{j=0}^n f^{-j}(B(f^j(x), \varepsilon)) = B_n(x, \varepsilon) \cap f^{-n}(B(f^n(x), \varepsilon))$$

and clearly $B_n(x, \varepsilon) \cap f^{-n}(B(f^n(x), \varepsilon)) \subseteq B_n(x, \varepsilon)$.

In particular, by the conformality of f , for each $n \geq 1$ either: (i) the equality $B_{n+1}(x, \varepsilon) = B_n(x, \varepsilon)$ holds, or (ii) the set $B_n(x, \varepsilon) \cap f^{-n}(B(f^n(x), \varepsilon))$ is strictly contained in $B_n(x, \varepsilon)$. In the second case, there exists a point $y \in B_n(x, \varepsilon)$ so that $f^n(y) \notin B(f^n(x), \varepsilon)$. This shows that the ball $f^n(B_n(x, \varepsilon)) \supset B(f^n(x), \varepsilon)$, combining with (2) yields

$$f^n(B_{n+1}(x, \varepsilon)) = B(f^n(x), \varepsilon).$$

Altogether, this proves that for every $n \geq 1$ there exists $0 \leq j < n$ so that $f^n(B_n(x, \varepsilon)) = f^{n-j}(B(f^j(x), \varepsilon))$.

Thus, in order to prove the claim it is enough to show that the forward image of balls of a definite size do not degenerate: for any $\varepsilon > 0$ there exists $\zeta(\varepsilon) > 0$ such that $\text{diam}(f^n(B(z, \varepsilon))) \geq \zeta(\varepsilon)$ for every $n \geq 1$ and every $z \in X$.

Indeed, since f is locally eventually onto, for any given $z \in X$ there exists $N(z, \varepsilon) > 0$ such that $f^{N(z, \varepsilon)}(B(z, \varepsilon)) = X$; hence there exists $\zeta_z(\varepsilon) > 0$ such that $\text{diam}(f^n(B(z, \varepsilon))) \geq \zeta_z(\varepsilon)$ for every $n \geq 1$. The continuity of f and compactness of X ensures that $\min_{z \in X} \zeta_z(\varepsilon) > 0$, proving the claim. \square

We now claim that f satisfies the periodic specification property. Indeed, given $\varepsilon > 0$ let $N = N(\varepsilon) \geq 1$ be such that $f^N(B(x, \zeta(\varepsilon))) = X$ for every $x \in X$. Such $N \geq 1$ does exist as f is locally eventually onto and X is compact. The proof of the periodic specification property now follows as in Theorem 1. \square

3.3. Proof of Theorem 5. Since items (ii) and (iii) are known (see e.g., [4]) we need only prove that each T_β is locally eventually onto.

Fix $\beta > 1$ and take an arbitrary interval $J \subset [0, 1)$. We claim that there exists $N \geq 1$ so that $T_\beta^N(J) = [0, 1)$. We may assume without loss of generality that J is contained in some domain of smoothness for T_β . By the mean value theorem, $\text{Leb}(T_\beta(J)) \geq \beta \text{Leb}(J)$. If $T_\beta(J) \cap D_{T_\beta} = \emptyset$ then $\text{Leb}(T_\beta^2(J)) \geq \beta^2 \text{Leb}(J)$. Since the diameter is bounded, a recursive argument shows that $T_\beta^k(J) \cap D_{T_\beta} \neq \emptyset$ for some $k \geq 1$. In particular $T_\beta^k(J) \supset [0, a)$ for some $a \in (0, \frac{1}{\beta}]$. Since $T_\beta(0) = 0$, and T_β is monotone increasing in $[0, \frac{1}{\beta}]$ then there exists $N \geq 1$ so that $T_\beta^N(J) \supset [0, \frac{1}{\beta}]$. This assures that $T_\beta^{N+1}(J) = [0, 1)$. \square

4. EXAMPLES

We finish with some examples. The first example is a simple examples of piecewise expanding continuous maps which need not be neither expansive nor transitive.

Example 6. Consider the continuous and piecewise expanding interval map $f_0 : [0, \frac{1}{2}] \rightarrow [0, \frac{1}{2}]$ given by

$$f_0(x) = \begin{cases} 3x & \text{if } x \in [0, \frac{1}{6}] \\ -3x + 1 & \text{if } x \in (\frac{1}{6}, \frac{1}{3}] \\ 3x - 1 & \text{if } x \in (\frac{1}{3}, \frac{1}{2}]. \end{cases}$$

Let $f : [0, 1] \rightarrow [0, 1]$ be obtained by replication of the dynamics f_0 in intervals of exponential decreasing growth accumulating 1, defined by the relation

$$f(x) = 1 - 2^{-n} + 2^{-n} f_0(2^n(x - 1 + 2^{-n})), \quad x \in (1 - 2^{-n}, 1 - 2^{-(n+1)}].$$

and $f(0) = 0$, $f(1) = 1$. Clearly f is piecewise expanding, continuous, not expanding nor transitive.

The next example shows that transitivity is essential to avoid unattainable repelling points.

Example 7. Consider the continuous and C^1 -piecewise expanding interval map $f : [0, 1] \rightarrow [0, 1]$ given by

$$f(x) = \begin{cases} 3x & \text{if } x \in [0, \frac{1}{3}] \\ -2x + \frac{5}{3} & \text{if } x \in (\frac{1}{3}, \frac{2}{3}] \\ 2x - 1 & \text{if } x \in (\frac{2}{3}, 1]. \end{cases}$$

The map is not transitive as $f([\frac{1}{3}, 1]) = [\frac{1}{3}, 1]$, in other words, $[\frac{1}{3}, 1]$ is an f -invariant domain. Thus f is not locally eventually onto. Nevertheless, the attractor $\Lambda := \bigcap_{n \geq 0} f^n((0, 1]) = [\frac{1}{3}, 1]$ and $f|_{\Lambda}$ is locally eventually onto.

Finally we complete this note with an example showing that locally eventually onto is weaker than specification. We consider an example suggested by Lindenstrauss (cf. [1, Example 2.31]) of a locally eventually onto map having no periodic points.

Example 8. Consider the subshift $Y_0 \subset \{0, 1, 2\}^{\mathbb{N}}$ consisting of the set of sequences that admit no consecutive 0's, let and let $\pi : Y_0 \rightarrow \{1, 2\}^{\mathbb{N}}$ be given by supression of the 0's in the sequences belonging to Y_0 . Endowing the shift spaces with the usual distances, π is a continuous map on a compact metric space, hence it is uniformly continuous.

Consider a minimal subshift $X \subset (\{1, 2\}^{\mathbb{N}}, \sigma)$ and let $Y = \pi^{-1}(X)$. Akin et al proved that (Y, σ) is locally eventually onto (cf. Example 2.31 in [1]).

We claim that (Y, σ) does not satisfy the specification property. Recall that a factor of a map of a compact space with specification satisfies specification (cf. [7, Proposition 21.4]) This does not directly apply to our situation since we do not have compactness, however it is not hard to prove that the commuting diagram

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

together with the uniform continuity of π ensures that if (Y, σ) satisfies the specification property then so does (X, σ) . Second, (X, σ) does not satisfy the specification property. Indeed, if (X, T) has the specification property and its natural extension is expansive then (X, T) has the periodic specification property (see e.g., Lemma 6 in [9]). Altogether, this proves that (Y, σ) does not satisfy the specification property, as claimed.

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AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
POSTAL ADDRESS: I2M, LUMINY, CASE 907, F-13288 MARSEILLE CEDEX 9, FRANCE

E-mail address: `serge.troubetzkoy@univ-amu.fr`

URL: `www.i2m.univ-amu.fr/perso/serge.troubetzkoy/`

PAULO VARANDAS, CMUP AND DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA BAHIA, AV. ADEMAR DE BARROS S/N, 40170-110 SALVADOR, BRAZIL

E-mail address: `paulo.varandas@ufba.br`

URL: `https://sites.google.com/view/paulovarandas/`