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SAMPLING THE FLOW OF A BANDLIMITED FUNCTION

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ABSTRACT. We analyze the problem of reconstruction of a bandlimited function f from the space-time samples of its states $f_t = \phi_t * f$ resulting from the convolution with a kernel ϕ_t. It is well-known that, in natural phenomena, uniform space-time samples of f are not sufficient to reconstruct f in a stable way. To enable stable reconstruction, a space-time sampling with periodic nonuniformly spaced samples must be used as was shown by Lu and Vetterli. We show that the stability of reconstruction, as measured by a condition number, controls the maximal gap between the spacial samples. We provide a quantitative statement of this result. In addition, instead of irregular space-time samples, we show that uniform dynamical samples at sub-Nyquist spatial rate allow one to stably reconstruct the function $\widehat{f}$ away from certain, explicitly described blind spots. We also consider several classes of finite dimensional subsets of bandlimited functions in which the stable reconstruction is possible, even inside the blind spots. We obtain quantitative estimates for it using Remez-Turán type inequalities. En route, we obtain Remez-Turán inequality for prolate spheroidal wave functions. To illustrate our results, we present some numerics and explicit estimates for the heat flow problem.

1. INTRODUCTION

In this paper, we consider the sampling and reconstruction problem of signals $u = u(t, x)$ that arise as an evolution of an intial signal $f = f(x)$ under the action of convolution operators. The intial signal f is assumed to be in the Paley-Wiener space PW_c, $c > 0$ (fixed throughout this paper) given by

$$PW_c := \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq [-c, c] \right\}$$

with the Fourier transform normalized as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-it\xi} \, dt$.

The functions u are solutions of initial value problems stemming from a physical system. Thus, due to the semigroup properties of such solutions, there is a family of kernels $\{\phi_t : t > 0\}$ such that $u(t, x) = \phi_t * f(x)$, $\phi_{t+s} = \phi_t * \phi_s$ for all $t, s \in (0, \infty)$, and $f = \lim_{t \to 0+} \phi_t * f$, $f \in L^2(\mathbb{R})$.

As we are primarily interested in physical systems, we typically consider the following set of kernels:

$$(1.1) \quad \Phi_c = \{\phi \in L^1(\mathbb{R}) : \text{there exists } \kappa_\phi > 0 \text{ such that } \kappa_\phi \leqslant \widehat{\phi}(\xi) \leqslant 1 \text{ for } |\xi| \leqslant c, \widehat{\phi}(0) = 1\}.$$

Observe that $\phi \in L^1$ implies that $\widehat{\phi}$ is continuous and, therefore, the existence of $\kappa_\phi > 0$ such that $\widehat{\phi} \geqslant \kappa_\phi$ on $[-c, c]$ is equivalent to $\widehat{\phi} > 0$ on $[-c, c]$. We remark that some of our results hold for a less restrictive class of kernels.

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Example 1.1. A prototypical example is the diffusion process with $\widehat{\phi}_t(\xi) = e^{-t\sigma^2\xi^2}$, $t > 0$
It corresponds to the initial value problem (IVP) for the heat equation (with a diffusion
parameter $\sigma \neq 0$)

$$(1.2) \qquad \begin{cases} \partial_t u(x,t) = \sigma^2 \partial_x^2 u(x,t) & \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ u(x,0) = f(x) \end{cases},$$

for which the solution is given by $u(x,t) = (\phi_t * f)(x)$.

Other examples include the IVP for the fractional diffusion equation

$$\begin{cases} \partial_t u(x,t) = (\partial_x^2)^{\alpha/2} u(x,t) & \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ u(x,0) = f(x) \end{cases}, 0 < \alpha \leqslant 1,$$

for which the solution is given by $u(x,t) = (\phi_t * f)(x)$ with $\widehat{\phi}_t(\xi) = e^{-t|\xi|^\alpha}$, and the IVP for
the Laplace equation in the upper half plane

$$\begin{cases} \Delta u(x,y) = 0 & \text{for } x \in \mathbb{R} \text{ and } y > 0 \\ u(x,0) = f(x) \end{cases},$$

for which the solution is given by $u(x,y) = (\phi_y * f)(x)$ with $\widehat{\phi}_y(\xi) := e^{-y|\xi|}$.

The following problem serves as a motivation for this paper.

Problem 1. Let $\phi \in \Phi$, $L > 0$, and $\Lambda \subset \mathbb{R}$ be a discrete subset of $\mathbb{R}$. What are the conditions
that allow one to recover a function $f \in PW_c$ in a stable way from the data set

$$(1.3) \qquad \{(f * \phi_t)(\lambda) : \lambda \in \Lambda, 0 \leqslant t \leqslant L\}?$$

The set of measurements (1.3) is the image of an operator $\mathcal{T} : PW_c \to L^2(\Lambda \times [0,L])$. Thus,
the stable recovery of f from (1.3) amounts to finding conditions on Λ, ϕ and L such that $\mathcal{T}$
has a bounded inverse from $\mathcal{T}(PW_c)$ to PW_c or, equivalently, the existence of $A, B > 0$ such
that

$$(1.4) \qquad A\|f\|_2^2 \leqslant \int_0^L \sum_{\lambda \in \Lambda} |(f * \phi_t)(\lambda)|^2 \, dt \leqslant B\|f\|_2^2, \text{ for all } f \in PW_c.$$

If for a given ϕ and L the frame condition (1.4) is satisfied, we say that $\Lambda = \Lambda_{\phi,L}$ is a stable
sampling set.

Remark 1.2. It was shown in [5, Theorem 5.5] that $\Lambda_{\phi,L}$ is a stable sampling set for some
$L > 0$, if and only if $\Lambda_{\phi,1}$ is a stable sampling set. Thus, for qualitative results, we will only
consider the case of $L = 1$. For quantitative results, however, we may keep L in order to
estimate the optimal time length of measurements.

Remark 1.3. Whenever (1.4) holds, standard frame methods can be used for the stable re-
construction of f [11].

Let us discuss Problem 1 in more detail in the case of our prototypical example.

1.1. **Sampling the heat flow.** Consider the problem of sampling the temperature in a heat diffusion process initiated by a bandlimited function $f \in PW_c$:

$$f_t := f * \phi_t, \qquad 0 \leqslant t \leqslant 1,$$

where ϕ_t is the heat kernel at time t:

$$(1.5) \qquad \qquad \widehat{\phi_t}(\xi) = e^{-t\sigma^2 \xi^2},$$

with a parameter $\sigma \neq 0$. According to Shannon's sampling theorem, f can be stably reconstructed from equispaced samples $\{f(k/T) : k \in \mathbb{Z}\}$ if and only if the sampling rate T is bigger than or equal to the critical value $T = \dfrac{c}{\pi}$, known as the *Nyquist rate*. The Nyquist bound is universal in the sense that it also applies to irregular sampling patterns: if a bandlimited function can be stably reconstructed from its samples at $\Lambda \subseteq \mathbb{R}$, then the *lower Beurling density*

$$D^-(\Lambda) := \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#(\Lambda \bigcap [x - r, x + r])}{2r}$$

satisfies $D^-(\Lambda) \geqslant \dfrac{c}{\pi}$. Recall that the *upper Beurling density* is defined by

$$D^+(\Lambda) := \limsup_{r \to \infty} \sup_{x \in \mathbb{R}} \frac{\#(\Lambda \bigcap [x - r, x + r])}{2r}.$$

We are interested to know if the spatial sampling rate can be reduced by using the information provided by the following spatio-temporal samples:

$$(1.6) \qquad \qquad \{f_t(k/T) : k \in \mathbb{Z}, 0 \leqslant t \leqslant 1\}.$$

Observe that the amount of collected data in (1.6) is not smaller than that in the case of sampling at the Nyquist rate $T = \dfrac{c}{\pi}$. If $T < \dfrac{c}{\pi}$, however, the density of sensors is smaller, and thus such a sampling procedure may provide considerable cost savings.

Lu and Vetterli showed [16] that for all $T < \dfrac{c}{\pi}$ there exist bandlimited signals with norm 1 that almost vanish on the samples (1.6), i.e. stable reconstruction is impossible from (1.6). As a remedy, they introduced periodic, nonuniform sampling patterns $\Lambda \subseteq \mathbb{R}$ that do lead to a meaningful *spatio-temporal trade-off*: there exist sets $\Lambda \subseteq \mathbb{R}$ that have sub-Nyquist density and, yet, lead to the frame inequality:

$$(1.7) \qquad A\|f\|_2^2 \leqslant \int_0^1 \sum_{\lambda \in \Lambda} |(f_t)(\lambda)|^2 \, \mathrm{d}t \leqslant B\|f\|_2^2, \text{ for all } f \in PW_c,$$

with $A, B > 0$; see Example 4.1 for a concrete construction. The emerging field of dynamical sampling investigates such phenomena in great generality (see, e.g., [1, 2, 3, 4, 5]).

As follows from Example 4.1, the estimates (1.7) may hold with an arbitrary small sensor density. The meaningful trade-off between spatial and temporal resolution, however, is limited by the desired numerical accuracy. For example, in the following theorem we relate the *maximal gap* of a stable sampling set to the bounds from (1.7).

Theorem 1.4. *Let $\Lambda \subseteq \mathbb{R}$ be such that (1.7) holds. Then there exists an absolute constant $K > 0$ such that, for $R \geqslant K \max\left(\dfrac{B}{A}, \dfrac{1}{c}\right)$ and every $a \in \mathbb{R}$, we have $[a - R, a + R] \cap \Lambda \neq \varnothing$. In particular, we have $D^-(\Lambda) \geqslant K^{-1} \min\left(\frac{A}{B}, c\right)$ and $D^+(\Lambda) \leqslant KB$.*

Theorem 4.4, which is a more general version of the above result, provides a more explicit dependence of K on the parameters of the problem.

Besides the constraints implied by Theorem 1.4, the special sampling configurations of Lu and Vetterli that lead to (1.7) lack the simplicity of regular sampling patterns. In this article, we explore a different solution to the diffusion sampling problem. We consider sub-Nyquist equispaced spatial sampling patterns (1.6) with $T = \dfrac{c}{m\pi}$, $m \in \mathbb{N}$, and restrict the sampling/reconstruction problem to a subset $V \subseteq PW_c$, aiming for an inequality of the form:

$$(1.8) \qquad A\|f\|_2^2 \leqslant \int_0^1 \sum_{k \in \mathbb{Z}} \left| f_t\left(\frac{m\pi}{c}k\right) \right|^2 \, \mathrm{d}t \leqslant B\|f\|_2^2, \qquad f \in V.$$

Specifically, we consider the following signal models.

Away from blind spots. We will identify a set E with measure arbitrarily close to 1 such that (1.8) holds with $V = V_E = \{f \in PW_c : \operatorname{supp} \widehat{f} \subseteq E\}$. In effect, E is the set $[-c, c]\backslash\mathcal{O}$ where $\mathcal{O}$ is a small open neighborhood of a finite set, i.e., E avoids a certain number of "blind spots."

Theorem 1.5. *Let $\phi \in \Phi$ and $m \geqslant 2$ be an integer. Then for any $r > 0$ there exists a certain compact set $E \subseteq [-c, c]$ of measure at least $2c - r$ such that any $f \in V_E$ can be recovered from the samples*

$$\mathcal{M} = \left\{ f_t\left(\frac{m\pi}{c}k\right) : k \in \mathbb{Z}, 0 \leqslant t \leqslant 1 \right\}$$

in a stable way.

The set E in the above theorem depends only on ϕ and the choice of r. The stable recovery in this case means that (1.8) holds with $B = 1$ and some $A > 0$ which is estimated in a more explicit version of the above result, Theorem 2.8.

Prolate spheroidal wave functions. The Prolate Spheroidal Wave Functions (PSWFs) are eigenfunctions of an integral operator known as the time-band liminting operator or sinc-kernel operator

$$\mathcal{Q}_c f(x) = \int_{-1}^1 \frac{\sin \pi c(y - x)}{\pi(y - x)} f(y) \, \mathrm{d}y.$$

Using the min-max theorem, we get that $\psi_{n,c}$ is the norm-one solution of the following extremal problem

$$\max\left\{ \frac{\|f\|_{L^2(-1,1)}}{\|f\|_{L^2(\mathbb{R})}} : f \in PW_c, \ f \in \operatorname{span}\{\psi_{k,c} : \ k < n\}^{\perp} \right\}$$

where the condition $f \in \operatorname{span}\{\psi_{k,c} : \ k < n\}^{\perp}$ is void for $n = 0$. The family $(\psi_{n,c})_{n \geqslant 0}$ forms an orthogonal basis for PW_c and has the property to form an orthonormal sequence in $L^2(-1, 1)$.

We consider the N-dimensional space

(1.9) $$V_N = \operatorname{span}\{\psi_0^c, \ldots, \psi_N^c\} \subset PW_c.$$

The Landau-Pollak-Slepian theory shows that this subspace provides an optimal approximation of a bandlimited function that is concentrated on $[-1, 1]$. More precisely, $V = V_N$ minimizes the approximation error

$$\sup_{\substack{f \in PW_c \\ \|f\|_2 = 1}} \inf_{g \in V} \int_{-1}^{1} |f(x) - g(x)|^2 \, \mathrm{d}x,$$

among all N-dimensional subspaces of PW_c.

Sparse sinc translates with free nodes. In this model, we let

(1.10) $$V_N = \left\{ \sum_{n=1}^{N} c_n \operatorname{sinc} c(x - \lambda_n) : c_1, \ldots, c_N \in \mathbb{C}, \ \lambda_1, \ldots, \lambda_N \in \mathbb{R} \right\}$$

be the class of linear combinations of N arbitrary translates of the sinc kernel $\operatorname{sinc}(x) = \frac{\sin x}{x}$. Note that V_N is not a linear space. However, $V_N - V_N \subseteq V_{2N}$. Therefore, (1.8) with $V = V_{2N}$ implies

$$A\|f - g\|_2^2 \leq \int_0^1 \sum_{k \in \mathbb{Z}} \left| f_t\left(\frac{m\pi}{c} k\right) - g_t\left(\frac{m\pi}{c} k\right) \right|^2 \, \mathrm{d}t \leq B\|f - g\|_2^2, \qquad f, g \in V_N,$$

which ensures the numerical stability of the sampling problem $f \mapsto \{f_t(m\pi k/c) : k \in \mathbb{Z} : 0 \leq t \leq 1\}$ restricted non-linearly to the class V_N. In other words, if (1.8) holds with $V = V_{2N}$ then any $f \in V_N$ can be stably reconstructed from the samples (1.6).

Fourier polynomials. As our last model, we consider the Fourier image of the space of polynomials of degree at most N restricted to the unit interval. Explicitly,

(1.11) $$V_N = \left\{ \sum_{n=0}^{N} c_n D^n \operatorname{sinc} c \cdot : c_0, \ldots, c_N \in \mathbb{C} \right\},$$

where $D : PW_c \to PW_c$ is the differential operator $Df = f'$. Observe that the union of such V_N, $N \in \mathbb{N}$, is dense in PW_c.

In this article, we show that each of the above-mentioned signal models regularizes the diffusion sampling problem, albeit with possibly very large condition numbers.

Theorem 1.6. *Let $m \geq 2$ be an integer, Φ be given by $\widehat{\Phi}(\xi) = e^{-\sigma^2 \xi^2}$. Let $V = V_N$ be given by (1.9), (1.10), or (1.11). Then (1.8) holds with*

$$A = \frac{c\kappa_0(c)}{(\sigma c)^2 + m} \exp\left(-\kappa_1(c)N - m^2\left(-\kappa_2(c)\ln\sigma + \kappa_3(c)\sigma^2 + \ln m\right)\right), \qquad B = 1,$$

where the κ_j's are positive constants that depend on c only.

We provide a more precise expression for the lower frame constant in Theorem 3.5. Note that the lower bound deteriorates when $\sigma^2 \to 0$ (no diffusion) and $\sigma^2 \to +\infty$ (very rapid diffusion). This agrees with the intuition and numerical experiments for (non-bandlimited) sparse initial conditions presented in [20]: if σ^2 is very small, because of spatial undersampling, some components of f may be hidden from the sensors, while for large σ^2 the diffusion completely blurs out the signal and no information can be extracted.

Remark 1.7. To simplify the discussion we take $c = 1/2$ in this remark. There are instances when Theorem 1.6 applies for a signal $f \in V_N$ which cannot be recovered simply from its samples on, say, $2\mathbb{Z}$. As an example, we offer V_1 given by (1.10) with $\lambda_1 = 1$. The samples at time $t = 0$ are not sufficient to identify each signal since $\operatorname{sinc}(\cdot - 1) \in V_N$ vanishes on $m\mathbb{Z}$, $m \geqslant 2$. Similarly, for Theorem 1.5: the function $\sin(\omega \cdot) \operatorname{sinc}(\frac{\cdot}{a})$, with an appropriately chosen a and ω, belongs to V_E and vanishes on $m\mathbb{Z}$ for $m \geqslant 2$. In finite dimensional subspaces V_N, e.g., given by (1.9) and (1.11), sampling at time $t = 0$ with any $m \in \mathbb{N}$ may be sufficient for stable recovery. However, the expected error of reconstruction in the presence of noise will be reduced if temporal samples are used in addition to those at $t = 0$. Theorems 1.5 and 1.6 can be used together. For example, a function f can be reconstructed away from the blind spots using Theorem 1.5 and approximated around the blind spots using Theorem 1.6.

1.2. **Technical overview.** Lu and Vetterli explain the impossibility of subsampling the heat-flow of a bandlimited function on a grid (1.6) as follows [16]. The function with Fourier transform

$$\widehat{f} := \delta_{-T} - \delta_T$$

is formally bandlimited to $I = [-c, c]$ if $T < c$, and vanishes on the lattice $\frac{\pi}{T}\mathbb{Z}$. Moreover, f is an eigenfunction of the diffusion operator since

$$\widehat{f}_t = e^{-t\sigma^2(-T)^2}\delta_{-T} - e^{-t\sigma^2 T^2}\delta_T = e^{-t\sigma^2 T^2}\widehat{f},$$

see (1.2) and (1.5). Hence, all the diffusion samples (1.6) vanish, although $f \not\equiv 0$. While no Paley-Wiener function is infinitely concentrated at $\{-T, T\}$, a more formal argument can be given by regularization. If $\eta : \mathbb{R} \to \mathbb{R}$ is continuous and supported on $[-1, 1]$ and $\eta_\varepsilon(x) = \varepsilon^{-1}\eta(x/\varepsilon)$, then $f \cdot \widehat{\eta}_\varepsilon \in PW_c$ and provides a counterexample to (1.4), provided that ε is sufficiently small.

As we show below in Subsection 2.1, a similar phenomenon holds for more general diffusion kernels ϕ as in (1.1). Indeed, an analysis along the lines of the Papoulis sampling theorem [18] shows that the diffusion samples (1.6) of a function $f \in PW_c$ do not lead to a stable recovery of $\widehat{f}$. However, these samples do allow for the stable recovery *away from certain blind spots* determined by ϕ; that is, one can effectively recover $\widehat{f} \cdot \mathbf{1}_E$, for a certain subset $E \subseteq I$ of positive measure ($\mathbf{1}_E$ denotes the characteristic function of the set E). If we, furthermore, restrict the sampling problem to one of the finite dimensional spaces $V = V_N$ given by (1.9), (1.10), or (1.11), we may then infer all other values of $\widehat{f}$. The main tools, in this case, are *Remez-Turán-like* inequalities of the form:

$$\|\widehat{f}\mathbf{1}_I\| \leqslant C_E\|\widehat{f}\mathbf{1}_E\|, \qquad f \in V.$$

For Fourier polynomials (1.11) the classical Remez-Turán inequality provides an explicit constant C_E, while the case of sparse sinc translates (1.10) is due to Nazarov [17]. The corresponding inequality for prolate spheroidal wave functions (1.9) is new and a contribution of this article (our technique relies on [15]).

1.3. **Paper organization and contributions.** In Section 2, we show that uniform dynamical samples at sub-Nyquist rate allow one to stably reconstruct the function $\widehat{f}$ away from certain, explicitly described blind spots determined by the kernel ϕ. We also provide an upper and lower estimate for the lower frame bound in (1.8). The upper estimate relies on the standard formulas for Pick matrices (see, e.g. [7, 10]). The lower estimate is far more intricate and is based on the analysis of certain Vandermonde matrices. We also provide some numerics and explicit estimates in the case of the heat flow problem.

In Section 3, we restrict the problem to the sets $V = V_N$ given by (1.9), (1.10), or (1.11). We provide quantitative estimates for the frame bounds in (1.8). En route, we obtain an explicit Remez-Turán inequality for prolate spheroidal wave functions – a result which we find interesting in its own right.

In Section 4, we discuss the case of irregular spacial sampling. We recall that a stable reconstruction may be possible with sets Λ that have an arbitrarily small (but positive) lower density. Nevertheless, we show that the maximal gap between the spacial samples (and, hence, the lower Beurling density) is controlled by the condition number of the problem (i.e. the ratio $\frac{B}{A}$ of the frame bounds).

2. Recovering a bandlimited function away from the blind-spot

2.1. **Dynamical sampling in PW_c.** In this section, we recall some of the results on dynamical sampling from [4, 5] and adapt them for problems studied in this paper.

For $\phi \in L^1$, consider the function

$$\widehat{\phi}_p(x) = \sum_{k \in \mathbb{Z}} \widehat{\phi}(x - 2ck) \mathbf{1}_{[-c,c)}(x - 2ck),$$

that is, the $2c$-periodization of the piece of $\widehat{\phi}$ supported in $[-c, c]$. Recall that we consider kernels from the set Φ given by (1.1). Hence,

$$\kappa_\phi \leqslant \widehat{\phi}_p(\xi) \leqslant 1, \qquad \xi \in \mathbb{R}.$$

We also write

$$\widehat{f}_t(\xi) := \widehat{f}(\xi)\widehat{\phi}^t(\xi), \ f \in PW_c.$$

Next, we introduce the *sampled diffusion matrix*, which is the $m \times m$ matrix-valued function given by
(2.12)

$$\mathcal{B}_m(\xi) = \left(\int_0^1 \overline{(\widehat{\phi})_p^t \left(\frac{2c}{m}(\xi + j) \right)} (\widehat{\phi})_p^t \left(\frac{2c}{m}(\xi + k) \right) \, \mathrm{d}t \right)_{0 \leqslant j,k \leqslant m-1} = \int_0^1 \mathcal{A}_m^*(\xi, t)\mathcal{A}_m(\xi, t) \, \mathrm{d}t,$$

where

$$
\begin{aligned}
\mathcal{A}_m(\xi, t) &= \left((\widehat{\phi})_p^t \left(\frac{2c}{m}(\xi + k) \right) \right)_{k=0,\ldots,m-1} \\
&= \left((\widehat{\phi})_p^t \left(\frac{2c}{m}\xi \right) \quad \cdots \quad (\widehat{\phi})_p^t \left(\frac{2c}{m}(\xi + m - 1) \right) \right) \in \mathcal{M}_{1,m}(\mathbb{C}).
\end{aligned}
$$

Remark 2.1. Observe that the matrix function $\mathcal{B}_m$ is m-periodic. Its eigenvalues, however, are 1-periodic because the matrices $\mathcal{B}_m(\xi)$ and $\mathcal{B}_m(\xi + k)$, $k \in \mathbb{Z}$, are similar via a circular shift matrix.

The following lemma explains the role of the sampled diffusion matrix. In the lemma, we let

$$
(2.13) \qquad \mathbf{f}(\xi) = \left((\widehat{f})_p \left(\frac{2c}{m}(\xi + j) \right) \right)_{j=0,\ldots,m-1} = \begin{pmatrix} (\widehat{f})_p \left(\frac{2c}{m}\xi \right) \\ \vdots \\ (\widehat{f})_p \left(\frac{2c}{m}(\xi + m - 1) \right) \end{pmatrix} \in \mathcal{M}_{m,1}(\mathbb{C}).
$$

Note that if we recover $\mathbf{f}(\xi)$ for $\xi \in [0, 1]$ then we can recover f_p. Observe also that

$$
\begin{aligned}
(2.14) \qquad \int_0^1 \|\mathbf{f}(\xi)\|^2 \, d\xi &= \sum_{j=0}^{m-1} \int_0^1 \left| (\widehat{f})_p \left(\frac{2c}{m}(\xi + j) \right) \right|^2 \, d\xi = \frac{m}{2c} \sum_{j=0}^{m-1} \int_{2cj/m}^{2c(j+1)/m} |(\widehat{f})_p(u)|^2 \, du \\
&= \frac{m}{2c} \int_0^{2c} |(\widehat{f})_p(s)|^2 \, ds = \frac{m}{2c} \int_{-c}^{c} |\widehat{f}(s)|^2 \, ds
\end{aligned}
$$

In other words, $f \mapsto \sqrt{\frac{2c}{m}}\mathbf{f} : PW_c \to L^2([0,1], \mathcal{M}_{m,1}(\mathbb{C}))$ is an isometric isomorphism.

Lemma 2.2. *For $f \in PW_c$,*

$$
(2.15) \qquad \int_0^1 \sum_{k\in\mathbb{Z}} \left| f_t \left(\frac{m\pi}{c} k \right) \right|^2 \, dt = \left(\frac{c}{m\pi} \right)^2 \int_0^1 \mathbf{f}(\xi)^* \mathcal{B}_m(\xi)\mathbf{f}(\xi) \, d\xi.
$$

Proof. Observe that it suffices to prove the result in $PW_c \cap \mathcal{S}(\mathbb{R})$ (the Schwarz class). Consider the function

$$
b(\xi, t) = \sum_{k\in\mathbb{Z}} f_t \left(\frac{m\pi}{c} k \right) e^{-2i\pi k\xi}.
$$

Using the Poisson summation formula and the definition of f_t, we get

$$
\begin{aligned}
b(\xi, t) &= \frac{c}{m\pi} \sum_{j\in\mathbb{Z}} \widehat{f_t} \left(\frac{2c}{m}(\xi + j) \right) = \frac{c}{m\pi} \sum_{-\frac{m}{2}-\xi \leqslant j < \frac{m}{2}-\xi} \widehat{\phi}^t \left(\frac{2c}{m}(\xi + j) \right) \widehat{f} \left(\frac{2c}{m}(\xi + j) \right) \\
&= \frac{c}{m\pi} \sum_{j=0}^{m-1} (\widehat{\phi})_p^t \left(\frac{2c}{m}(\xi + j) \right) (\widehat{f})_p \left(\frac{2c}{m}(\xi + j) \right),
\end{aligned}
$$

Note that the functions $b(\cdot, t)$ are 1-periodic,

$$(2.16) \qquad b(\xi, t) = \frac{c}{m\pi} \mathcal{A}_m(\xi, t)\mathbf{f}(\xi),$$

and thus

$$\int_0^1 |b(\xi, t)|^2 \, \mathrm{d}t = \left(\frac{c}{m\pi}\right)^2 \mathbf{f}(\xi)^* \mathcal{B}_m(\xi)\mathbf{f}(\xi), \ \xi \in \mathbb{R}.$$

Combining the last equation with the Parseval's relation

$$(2.17) \qquad \int_0^1 |b(\xi, t)|^2 \mathrm{d}\xi = \sum_{k \in \mathbb{Z}} \left| f_t\left(\frac{m\pi}{c}k\right)\right|^2.$$

yields the desired conclusion. $\qquad\qquad\square$

Remark 2.3. Lemma 2.2 shows that the stability of reconstruction from spatio-temporal samples is controlled by the condition number of the self-adjoint matrices $\mathcal{B}_m(\xi)$ in (2.12). For symmetric $\phi \in \Phi$ and $m \geqslant 2$, however,

$$\inf_{\xi \in [0,1]} \lambda_{\min}\big(\mathcal{B}_m(\xi)\big) = \lambda_{\min}\big(\mathcal{B}_m(0)\big) = 0,$$

which precludes the stable reconstruction of all $f \in PW_c$, see, e.g., [4]. This adds to our explanation of the phenomenon of blind spots in Subsection 1.2. We can nonetheless hope to find a large set $\widetilde{E} \subseteq [0, 1]$ such that $\lambda_{\min}\big(\mathcal{B}_m(\xi)\big) \geqslant \kappa$ for $\xi \in \widetilde{E}$. Then, repeating the computation in (2.14), we get

$$(2.18) \qquad \begin{aligned} \int_0^1 \sum_{k \in \mathbb{Z}} \left| f_t\left(\frac{m\pi}{c}k\right)\right|^2 \, \mathrm{d}t &= \left(\frac{c}{m\pi}\right)^2 \int_0^1 \mathbf{f}(\xi)^* \mathcal{B}_m(\xi)\mathbf{f}(\xi) \, \mathrm{d}\xi. \geqslant \kappa \left(\frac{c}{m\pi}\right)^2 \int_{\widetilde{E}} \|\mathbf{f}(\xi)\|^2 \, \mathrm{d}\xi \\ &= \frac{c\kappa}{2m\pi^2} \int_E \|\widehat{f}(\xi)\|^2 \, \mathrm{d}\xi \end{aligned}$$

where $E = \left(\dfrac{2c}{m}(\widetilde{E} + \mathbb{Z})\right) \cap [-c, c]$.

In the following example, we offer some numerics. To simplify the computations, we represent $\mathcal{B}_m(\xi)$ in (2.12) as a Pick matrix (see, e.g., [7, 10]). For $\xi \in [-c, c)$, we write $\widehat{\phi}(\xi) = e^{-\psi(\xi)}$, so that $\psi \geqslant 0$ and $\psi(0) = 0$, and obtain for $j, k = 0, \ldots, m-1$,

$$(\mathcal{B}_m)_{jk}(\xi) = \int_0^1 \widehat{\phi}^t\left(\frac{2c}{m}(\xi + j')\right) \widehat{\phi}^t\left(\frac{2c}{m}(\xi + k')\right) \mathrm{d}t$$

where the indices j', k' are in the set

$$(2.19) \qquad I_\xi = \left\{n \in \mathbb{Z} : \frac{\xi + n}{m} \in [-1/2, 1/2)\right\},$$

m divides $|j - j'|$ and $|k - k'|$, and $j, k,$ and ξ are not 0 simultaneously. Thus

$$(2.20) \qquad \begin{aligned} (\mathcal{B}_m)_{jk}(\xi) &= \int_0^1 e^{-t\left(\psi\left(\frac{2c}{m}(\xi + j')\right) + \psi\left(\frac{2c}{m}(\xi + k')\right)\right)} \, \mathrm{d}t \\ &= \left(\psi\left(\frac{2c}{m}(\xi + j')\right) + \psi\left(\frac{2c}{m}(\xi + k')\right)\right)^{-1} \left(1 - e^{-\left(\psi\left(\frac{2c}{m}(\xi + j')\right) + \psi\left(\frac{2c}{m}(\xi + k')\right)\right)}\right) \end{aligned}$$

Observe that $(\mathcal{B}_m)_{00}(0) = 1$.

Example 2.4. Here, we choose ϕ to be the Gaussian function, i.e.,

$$\widehat{\phi}(\xi) = \widehat{\phi}_1(\xi) = e^{-\sigma^2 \xi^2}$$

for various values of $\sigma \neq 0$. Hence, $\psi(\xi) = \sigma^2 \xi^2$, and we get

$$(\mathcal{B}_m)_{jk}(\xi) = \frac{m^2}{4c^2\sigma^2} \cdot \frac{1 - e^{-\left(\frac{\sigma^2}{m^2}\left((\xi+j')^2+(\xi+k')^2\right)\right)}}{(\xi+j')^2 + (\xi+k')^2}$$

with j', k', and $(\mathcal{B}_m)_{00}(0)$ as above.

In Figure 1, we show the condition numbers of the matrices $\mathcal{B}_m(\xi)$ with $\xi = 0.45$, $c = 1/2$, $m \in \{2, 3, 5\}$, and σ varying from 1 to 200.
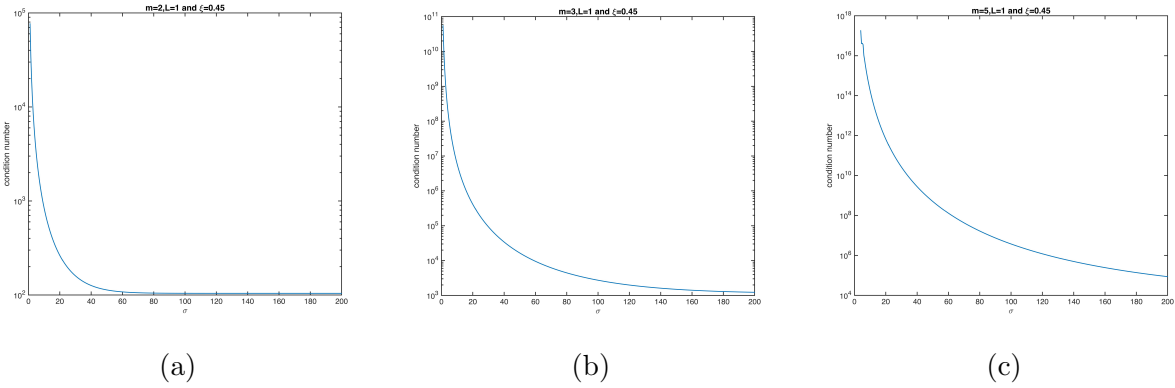

(a)   (b)   (c)

FIGURE 1. Condition numbers of $\mathcal{B}_m(\xi)$ for $m \in \{2, 3, 5\}$, $c = 1/2$, $\xi = 0.45$, and $\sigma \in [1, 200]$.

In Figure 2, we also show the condition numbers of the matrices $\mathcal{B}_m(\xi)$. This time, however, still $c = 1/2$, the parameter σ is fixed to be 200, whereas the point ξ is allowed to vary from 0.35 to 0.49. We still have $m \in \{2, 3, 5\}$.

2.2. **Estimating the minimal eigenvalue of the sampled diffusion matrix.** In this subsection, we use Vandermonde matrices to obtain a lower estimate for the eigenvalue $\lambda_{\min}^{(m)}(\xi)$ of the matrices $\mathcal{B}_m(\xi)$ in (2.12). We also present an upper estimate for $\lambda_{\min}^{(m)}(\xi)$, which follows from the general theory of Pick matrices [7, 10].

We begin with the following auxiliary result.

Lemma 2.5. *Let* $v_0, v_1, \ldots v_{m-1}$ *be* m *distinct non-zero real numbers and let* $\mathbf{v} = (v_0, \ldots, v_{m-1})$. *For* $k \in \mathbb{N}$, *define a function* $\Psi_k : \mathbb{R} \to \mathbb{R}$ *by* $\Psi_k(t) = \frac{1-t^2}{1-t^{2/k}}$ *if* $t \neq 1$ *and* $\psi_k(1) = k$. *For* $j = 0, \ldots, m-1$, *define*

$$\sigma_j^2 = \sum_{k=0}^{m-1} v_j^{2k} = \Psi_m(v_j^m) = \begin{cases} m & \text{if } v_j = 1 \\ \frac{1-v_j^{2m}}{1-v_j^2} & \text{otherwise} \end{cases}.$$
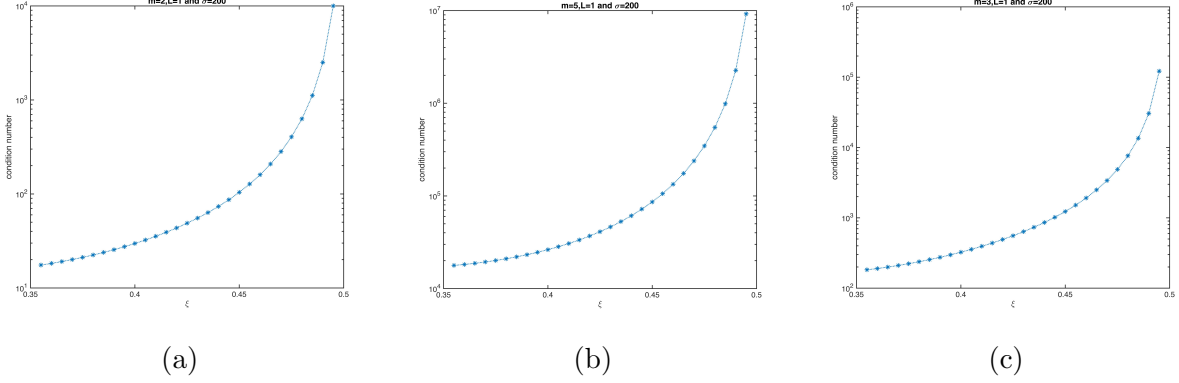
(a) (b) (c)

FIGURE 2. Condition numbers of $\mathcal{B}_m(\xi)$ for $m \in \{2, 3, 5\}$, $c = 1/2$, $\sigma = 200$, and $\xi \in [0.35, 0.49]$.

Let $\sigma = \left(\sum_{j=0}^{m-1} \sigma_j^2 \right)^{1/2}$, $\gamma_- = \min_j |v_j| > 0$, $\gamma_+ = \max_j |v_j|$ and let

$$\alpha = \left(\frac{m-1}{\sigma^2} \right)^{\frac{m-1}{2}} \prod_{0 \leqslant j < k \leqslant m-1} |v_j - v_k|.$$

For $N \in \mathbb{N}$, let W_N be the $(mN) \times m$ Vandermonde matrix associated to $\mathbf{v}_N = (v_0^{\frac{1}{N}}, v_1^{\frac{1}{N}}, \ldots v_{m-1}^{\frac{1}{N}})$, i.e.,

$$W_N = \left[v_j^{\frac{i-1}{N}} \right]_{1 \leqslant i \leqslant mN, 0 \leqslant j \leqslant m-1}.$$

Then for each $x \in \mathbb{C}^m$, we have

$$\alpha^2 \Psi_N(\gamma_-) \|x\|^2 \leqslant \|W_N x\|^2 \leqslant \sigma^2 \Psi_N(\gamma_+) \|x\|^2.$$

Proof. let V be the $m \times m$ Vandermonde matrix associated to $\mathbf{v}$:

$$V = [v_j^i]_{0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant m-1}.$$

Note that the Frobenius norm of V and its determinant are given by

$$\|V\|_F = \sigma \quad \text{and} \quad |\det V| = \prod_{0 \leqslant j < k \leqslant m-1} |v_j - v_k|.$$

Recall from [23] an estimate for the minimal singular value of an $m \times m$ matrix A:

(2.21) $$\sigma_{\min}(A) \geqslant \left(\frac{m-1}{\|A\|_F^2} \right)^{(m-1)/2} |\det A|.$$

Specifying this to V we get $\sigma_{\min}(V) \geqslant \alpha$. As $\|V\| \leqslant \|V\|_F$, it follows that, for all $x \in \mathbb{C}^m$,

(2.22) $$\alpha^2 \|x\|^2 \leqslant \|Vx\|^2 \leqslant \sigma^2 \|x\|^2.$$

Let D_N be the diagonal matrix with $\mathbf{v}_N$ on the main diagonal. Since

$$\|W_N x\|^2 = \langle W_N^* W_N x, x \rangle = \sum_{\ell=0}^{N-1} \langle (D_N^\ell)^* V^* V D_N^\ell x, x \rangle = \sum_{\ell=0}^{N-1} \|V D_N^\ell x\|^2,$$

we deduce from (2.22) that

$$\sum_{\ell=0}^{N-1} \alpha^2 \|D_N^\ell x\|^2 \leqslant \|W_N x\|^2 \leqslant \sum_{\ell=0}^{N-1} \sigma^2 \|D_N^\ell x\|^2.$$

Moreover, we have $\gamma_-^{\frac{2\ell}{N}}\|x\|^2 \leqslant \|D_N^\ell x\|^2 \leqslant \gamma_+^{\frac{2\ell}{N}}\|x\|^2$ by definition of D_N. The conclusion now follows by summing the two geometric sequences. $\square$

Note that the function Ψ_N is increasing on $(0, +\infty)$ and that, for $t \neq 1$, $t > 0$

$$(2.23) \qquad \lim_{N\to\infty} \frac{1}{N}\Psi_N(t) = \frac{1-t^2}{\lim_{N\to\infty} N(1 - e^{2\ln t/N})} = \left|\frac{1-t^2}{2\ln t}\right|.$$

Corollary 2.6. *With the notation of Lemma 2.5, assume further that $0 < \nu \leqslant v_j \leqslant 1$ and $m \geqslant 2$. Let*

$$(2.24) \qquad \widetilde{\alpha} = e^{-1/2}m^{-\frac{m-1}{2}} \prod_{0 \leqslant j < k \leqslant m-1} |v_j - v_k|.$$

Then for each $x \in \mathbb{C}^m$, we have

$$\widetilde{\alpha}^2 \Psi_N(\nu)\|x\|^2 \leqslant \|W_N x\|^2 \leqslant m^2 N\|x\|^2.$$

Proof. Indeed, $\nu \leqslant \gamma_- \leqslant \gamma_+ \leqslant 1$ so $\Psi_N(\nu) \leqslant \Psi_N(\gamma_-)$ and $\Psi_N(\gamma_+) \leqslant \Psi_N(1) = N$.

Further, since $v_j \leqslant 1$, $\sigma^2 \leqslant m^2$. Moreover, the derivative of $\left(\frac{t-1}{t}\right)^{(t-1)/2} = \left(1 - \frac{1}{t}\right)^{(t-1)/2}$ is

$$\frac{1}{2}\left(1 - \frac{1}{t}\right)^{(t-1)/2}\left(\frac{1}{t} + \ln\left(1 - \frac{1}{t}\right)\right) \leqslant 0$$

for $t \geqslant 1$. Thus,

$$\left(\frac{m-1}{m}\right)^{(m-1)/2} \geqslant \lim_{t\to+\infty} \exp\left[\frac{t-1}{2}\ln\left(1 - \frac{1}{t}\right)\right] = e^{-1/2}.$$

It follows that α in the statement of Lemma 2.5 satisfies

$$\alpha \geqslant \frac{\prod_{0 \leqslant j < k \leqslant m-1} |v_j - v_k|}{\sqrt{e}m^{(m-1)/2}},$$

and the result is established. $\square$

Proposition 2.7. *Let $\phi \in \Phi$. Define*

$$\Delta_m(\xi) = \prod_{0 \leqslant j < k \leqslant m-1} \left|\widehat{\phi}_p\left(\frac{2c}{m}(\xi + j)\right) - \widehat{\phi}_p\left(\frac{2c}{m}(\xi + k)\right)\right|.$$

Then, for each $x \in \mathbb{C}^m$, we have

$$\frac{1}{2em^{m^2}}\Delta_m(\xi)^2 \cdot \frac{1 - \kappa_\phi^{2/m}}{|\ln \kappa_\phi|}\|x\|^2 \leqslant \langle \mathcal{B}_m(\xi)x, x \rangle \leqslant m\|x\|^2.$$

Proof. We fix ξ and apply Corollary 2.6 to $v_j = (\hat{\phi})_p\left(\frac{2c}{m}(\xi + j)\right)^{\frac{1}{m}}$. With $\tilde{\alpha}$ given by (2.24),

$$\tilde{\alpha} = e^{-1/2}m^{-\frac{m-1}{2}} \prod_{0 \leqslant j < k \leqslant m-1} \left| \hat{\phi}_p\left(\frac{2c}{m}(\xi + j)\right)^{1/m} - \hat{\phi}_p\left(\frac{2c}{m}(\xi + k)\right)^{1/m} \right|,$$

we get

$$\tilde{\alpha}^2 \Psi_N(\kappa_\phi^{1/m})\|x\|^2 \leqslant \|W_N x\|^2 \leqslant m^2 N\|x\|^2.$$

On the other hand, $\frac{1}{mN}W_N^* W_N$ equals the left-end mN-term Riemann sum for the integral defining $\mathcal{B}_m(\xi)$. It follows that

$$\langle \mathcal{B}_m(\xi)x, x\rangle = \lim_{N \to \infty} \frac{1}{mN}\langle W_N^* W_N x, x\rangle = \lim_{N \to \infty} \frac{1}{mN}\|W_N x\|^2.$$

Using (2.23), we get

$$\tilde{\alpha}^2 \frac{1 - \kappa_\phi^{2/m}}{2m|\ln \kappa_\phi|}\|x\|^2 \leqslant \langle \mathcal{B}_m(\xi)x, x\rangle \leqslant m\|x\|^2.$$

Finally, note that if $0 < a, b \leqslant 1$, using the mean value theorem, there is an $\eta \in (a, b)$ such that

$$|a^{1/m} - b^{1/m}| = \frac{1}{m}|a - b|\eta^{-1+1/m} \geqslant \frac{1}{m}|a - b|.$$

Therefore

$$\begin{aligned} \tilde{\alpha} &= e^{-1/2}m^{-\frac{m-1}{2}} \prod_{0 \leqslant j < k \leqslant m-1} \left| \hat{\phi}_p\left(\frac{2c}{m}(\xi + j)\right)^{1/m} - \hat{\phi}_p\left(\frac{2c}{m}(\xi + k)\right)^{1/m} \right| \\ &\geqslant e^{-1/2}m^{-\frac{m-1}{2} - \frac{m(m-1)}{2}}\Delta(\xi) = e^{-1/2}m^{-\frac{m^2-1}{2}}\Delta(\xi) \end{aligned}$$

establishing the postulated estimates. $\qquad\square$

For an upper estimate of the minimal eigenvalue $\lambda_{\min}^{(m)}(\xi)$ we use the estimates of the singular values of Pick matrices by Beckerman-Townsend [7]. For $p_j \in \mathbb{C}$, $j = 1, \ldots, m$, and $0 < a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b$ let

(2.25)
$$(P_m)_{jk} = \frac{p_j + p_k}{x_j + x_k}, \qquad j, k = 1, \ldots, m,$$

be the corresponding Pick matrix. Then the smallest singular value $s_{\min}$ of P_m is bounded above by

(2.26)
$$s_{\min} \leqslant \min\left\{1, 4\left[\exp\left(\frac{\pi^2}{2\ln\left(\frac{4b}{a}\right)}\right)\right]^{-2\lfloor m/2 \rfloor}\right\} s_{\max},$$

where $s_{\max}$ is the largest singular value.

If $(\widetilde{P}_m)_{jk} = \frac{1 - c_j c_k}{x_j + x_k}$, then $\widetilde{P}_m$ is related to a Pick matrix of the form (2.25) via the diagonal matrix $D = \operatorname{diag}(1 + c_j)$:

$$\frac{1}{2}D^{-1}\widetilde{P}_m D^{-1} = P_m$$

with $p_j = \frac{1 - c_j}{1 + c_j}$, $c_j \neq -1$.

In our case, see (2.20), $x_j = \psi\left(\frac{2c}{m}(\xi + j)\right)$ and $c_j = e^{-x_j} \in (0, 1]$, so $\mathrm{Id} \leqslant D \leqslant 2\mathrm{Id}$ and the singular values of $\mathcal{B}_m(\xi)$ and the corresponding Pick matrix P_m differ at most by a factor 4. Therefore, (2.26) holds with $a(\xi) = \min\left\{\psi\left(\frac{2c}{m}(\xi + k)\right) : k \in I_\xi\right\}$ and $b(\xi) = \max\left\{\psi\left(\frac{2c}{m}(\xi + k)\right) : k \in I_\xi\right\}$, I_ξ defined in (2.19), and an additional factor 4 provided that $a(\xi) \neq 0$.

For our main examples, we have $\psi(\xi) = |\xi|^\alpha$, $\alpha > 0$. This yields

$$b(\xi) \leqslant c^\alpha \quad \text{and} \quad a(\xi) = \min\left\{\left|\frac{2c}{m}(\xi - k)\right|^\alpha : \frac{2c}{m}|\xi - k| \leqslant c, \ |\xi| \leqslant \frac{1}{2}\right\} = \left(\frac{2c}{m}|\xi|\right)^\alpha$$

So for the smallest singular value of $\mathcal{B}_m(\xi)$ we obtain the estimate

$$\lambda_{\min}^{(m)}(\xi) \leqslant 4^2 \left[\exp\left(\frac{\pi^2}{2\ln 4\left(\frac{m}{2|\xi|}\right)^\alpha}\right)\right]^{-2\lfloor m/2 \rfloor} m$$

(2.27)

$$\leqslant 16m \exp\left(-\frac{(m-1)\pi^2}{\ln 16 + 2\alpha \ln \frac{m}{2|\xi|}}\right).$$

Observe that the Beckerman-Townsend estimate (2.26) holds for *all* Pick matrices with the same values for $a = \min x_j$ and $b = \max x_j$ and is completely independent of the particular distribution of the x_j. Regardless, it shows that the condition number grows nearly exponentially with m, establishing limitations on how well the space-time trade-off can work numerically. Of course, the condition number may be much worse if two values x_j and x_{j+1} are close together (if $x_j = x_{j+1}$, then P_m is singular). Thus, (2.27) is an optimistic upper estimate for $\lambda_{\min}^{(m)}(\xi)$. By comparison, our lower estimate in Proposition 2.7 depends crucially on the distribution of the parameters x_j and is much harder to obtain. It does, however, establish an upper bound on the condition number and, thus, shows that the space-time trade-off may be useful. A precise result is formulated in the following subsection.

2.3. **Partial recoverability.**

Theorem 2.8. *Let $\phi \in \Phi$, $m \geqslant 2$ an integer and $\widetilde{E} \subseteq I = [0, 1]$ be a compact set. Assume that there exists $\delta > 0$ such that, for every $0 \leqslant j < k \leqslant m - 1$ and every $\xi \in \widetilde{E}$*

$$\left|\widehat{\phi}_p\left(\frac{2c}{m}(\xi + j)\right) - \widehat{\phi}_p\left(\frac{2c}{m}(\xi + k)\right)\right| \geqslant \delta.$$

Let $E = \left(\frac{2c}{m}(\widetilde{E} + \mathbb{Z})\right) \cap [-c, c]$. Then for any $f \in PW_c$, the function $\widehat{f}\mathbf{1}_E$ can be recovered from the samples

(2.28)
$$\mathcal{M} = \left\{f_t\left(\frac{m\pi}{c}k\right) : k \in \mathbb{Z}, 0 \leqslant t \leqslant 1\right\}$$

in a stable way. Moreover, we have

(2.29)
$$A\|\widehat{f}\mathbf{1}_E\|^2 \leqslant \int_0^1 \sum_{k \in \mathbb{Z}} \left|f_t\left(\frac{m\pi}{c}k\right)\right|^2 dt \leqslant \frac{c}{2\pi^2}\|\widehat{f}\|^2,$$

where

$$A = \frac{c}{4e\pi^2} \frac{\delta^{m(m-1)}}{m^{1+m^2}} \frac{\kappa_\phi^{2/m} - 1}{\ln \kappa_\phi}.$$

Proof. Recall from (2.15) that we need to estimate

$$\int_0^1 \sum_{k\in\mathbb{Z}} \left| f_t \left(\frac{m\pi}{c} k \right) \right|^2 \mathrm{d}t = \left(\frac{c}{m\pi} \right)^2 \int_0^1 \mathbf{f}(\xi)^* \mathcal{B}_m(\xi) \mathbf{f}(\xi) \, \mathrm{d}\xi.$$

The upper bound follows directly from Proposition 2.7, and (2.14):

$$\int_0^1 \mathbf{f}(\xi)^* \mathcal{B}_m(\xi) \mathbf{f}(\xi) \, \mathrm{d}\xi \leqslant m \int_0^1 \|\mathbf{f}(\xi)\|^2 \, \mathrm{d}\xi = \frac{m^2}{2c} \|\widehat{f}\|^2.$$

Let us now prove the lower bound using (2.18). First, $\Delta_m(\xi) \geqslant \delta^{\frac{m(m-1)}{2}}$. It follows from Proposition 2.7 that, if $\xi \in \widetilde{E}$ then

$$\mathbf{f}(\xi)^* \mathcal{B}_m(\xi) \mathbf{f}(\xi) \geqslant \frac{\kappa_\phi^{2/m} - 1}{2em^{m^2} \ln \kappa_\phi} \delta^{m(m-1)} \|\mathbf{f}(\xi)\|^2.$$

Taking $\kappa = \dfrac{\kappa_\phi^{2/m} - 1}{2em^{m^2} \ln \kappa_\phi} \delta^{m(m-1)}$ in (2.18) gives the result. $\qquad\square$

Remark 2.9. The condition number implied by the above theorem is not the best possible one can obtain through this method. For instance, a better estimate for the $\sigma_{\min}$ of a Vandermonde matrix may be used in place of (2.21).

However, the method will always lead to a deteriorating estimate of the condition number as m increases. This follows from the Beckerman-Townsend estimate (2.26) we discussed in the previous subsection.

Corollary 2.10. *Assume that $\phi \in \Phi$, $\widehat{\phi}$ is even and strictly decreasing on $\mathbb{R}_+$, and $m \geqslant 2$ is an integer. Given $\eta \in (0, \frac{1}{4})$, let $\widetilde{E} = [-\frac{1}{2} + \eta, -\eta] \cup [\eta, \frac{1}{2} - \eta]$ and $E = \left(\frac{2c}{m}(\widetilde{E} + \mathbb{Z}) \right) \cap [-c, c]$. Then there exists $A > 0$ such that, for any $f \in PW_c$,*

$$A\|\widehat{f}\mathbf{1}_E\|^2 \leqslant \int_0^1 \sum_{k\in\mathbb{Z}} \left| f_t \left(\frac{m\pi}{c} k \right) \right|^2 \mathrm{d}t \leqslant \|\widehat{f}\|^2.$$

Proof. We look into the main condition of Theorem 2.8: *there exists $\delta > 0$ such that, for every $0 \leqslant j < k \leqslant m - 1$ and every $\xi \in \widetilde{E}$*

(2.30)
$$\left| \widehat{\phi}_p \left(\frac{2c}{m}(\xi + j) \right) - \widehat{\phi}_p \left(\frac{2c}{m}(\xi + k) \right) \right| \geqslant \delta.$$

For a general $\phi \in \Phi$, the function $\widehat{\phi}$ is continuous and, therefore, $\widehat{\phi}_p$ is continuous, except possibly on $c + 2c\mathbb{Z}$ where a jump discontinuity occurs if $\widehat{\phi}(-c) \neq \widehat{\phi}(c)$. Under current assumptions, however, $\widehat{\phi}$ is even and, therefore $\widehat{\phi}_p$ is continuous everywhere.

For $0 \leqslant \ell \leqslant m-1$ and $\xi \in I$, we have $-\dfrac{1}{2m} \leqslant \dfrac{\xi + \ell}{m} \leqslant 1 - \dfrac{1}{2m}$ and

$$
\widehat{\phi_p}\left(\frac{2c}{m}(\xi + \ell)\right) = \begin{cases} \widehat{\phi}\left(\dfrac{2c}{m}(\xi + \ell)\right) & \text{if } \dfrac{\xi + \ell}{m} < 1/2 \\ \widehat{\phi}\left(\dfrac{2c}{m}(\xi + \ell - m)\right) & \text{if } \dfrac{\xi + \ell}{m} \geqslant 1/2 \end{cases}.
$$

Thus, the condition of Theorem 2.8 would be satisfied with $\widetilde{E} = I$ if $|\widehat{\phi}|$ were one-to-one on I, that is, either strictly decreasing or strictly increasing. However, $\widehat{\phi}$ is even and strictly decreasing on $\mathbb{R}_+$, so that $\widehat{\phi_p}$ is continuous, strictly decreasing on $[0, c]$ and strictly increasing on $[-c, 0]$. It follows that (2.30) may only fail in small intervals around the points $\xi \in I$ where $\widehat{\phi_p}\left(\dfrac{2c}{m}(\xi + j)\right) - \widehat{\phi_p}\left(\dfrac{2c}{m}(\xi + k)\right) = 0$ for some $j, k \in \mathbb{Z}$. Such points must satisfy

$$
\frac{\xi + j}{m} = 1 - \frac{\xi + \ell}{m}, \quad 0 \leqslant j \leqslant \frac{m-1}{2} < \ell \leqslant m - 1.
$$

Thus, we need $\xi = \frac{1}{2}(m - j - \ell)$, i.e. $\xi \in \{0, \pm\frac{1}{2}\}$. In view of the continuity of $\widehat{\phi_p}$, it follows that there exists $\eta > 0$ such that (2.30) holds for $\xi \in \widetilde{E} = [-\frac{1}{2} + \eta, -\eta] \cup [\eta, \frac{1}{2} - \eta]$. It remains to observe that with any given $\eta \in (0, \frac{1}{4})$ inequality (2.30) will hold for δ sufficiently small. $\qquad\square$

2.4. Explicit quantitative estimates for the Gaussian.

To obtain explicit estimates, we need to establish a precise relation between η and δ in the proof of Corollary 2.10. In other words, we need to estimate $\min_{\xi \in \widetilde{E}} \psi(\xi)$, where, as above, $\widetilde{E} = [-\frac{1}{2} + \eta, -\eta] \cup [\eta, \frac{1}{2} - \eta]$, $\eta \in (0, \frac{1}{4})$, and the function ψ is given by

$$
\omega(\xi) = \min_{j, k \in \mathbb{Z}} \left| \widehat{\phi_p}\left(\frac{2c}{m}(\xi + j)\right) - \widehat{\phi_p}\left(\frac{2c}{m}(\xi + k)\right) \right|.
$$

Lemma 2.11. *Let E and $\widetilde{E}$ be as in Corollary 2.10. Assume that the kernel $\phi \in \Phi$ is such that $\widehat{\phi}$ is differentiable on E and*

$$
\min_{\xi \in E} \left| \widehat{\phi}'(\xi) \right| \geqslant R.
$$

Then

$$
\min_{\xi \in \widetilde{E}} \omega(\xi) \geqslant \frac{4cR\eta}{m}.
$$

Proof. Observe that

$$
\min_{\xi \in \widetilde{E}} \min_{j, k \in \mathbb{Z}} \left| \left| \frac{2c}{m}(\xi + j) \right| - \left| \frac{2c}{m}(\xi + k) \right| \right| = \frac{2c}{m} 2\eta.
$$

With this, the assertion of the lemma follows immediately from the mean value theorem. $\qquad\square$

The above observation leads to the following explicit estimate for the Gaussian kernel.

Proposition 2.12. *Let $\widehat{\phi}(\xi) = e^{-\sigma^2\xi^2}$, $\sigma \neq 0$, and $m \geq 2$ be an integer. Given $\eta \in (0, \frac{1}{4})$, let $\widetilde{E} = [-\frac{1}{2} + \eta, -\eta] \cup [\eta, \frac{1}{2} - \eta]$ and $E = \left(\dfrac{2c}{m}(\widetilde{E} + \mathbb{Z})\right) \cap [-c, c]$. Then, for any $f \in PW_c$, we have*

$$(2.31) \qquad A\|\widehat{f}\mathbf{1}_E\|^2 \leq \int_0^1 \sum_{k \in \mathbb{Z}} \left| f_t\left(\frac{m\pi}{c}k\right) \right|^2 \, \mathrm{d}t \leq \|\widehat{f}\|^2,$$

where

$$(2.32) \qquad A = \frac{c}{2e\pi^2(2(\sigma c)^2 + m)} \frac{(4cR\eta)^{m(m-1)}}{m^{1-m+2m^2}} \quad with \quad R = 2\sigma^2 \min\left\{\eta e^{-(\sigma\eta)^2}, c e^{-(\sigma c)^2}\right\}.$$

Proof. Observe that Lemma 2.11 applies with R given by (2.32). It remains to apply Theorem 2.8 with $\kappa_\phi = e^{-(\sigma c)^2}$ and $\delta = 4cR\eta/m$. We deduce that (2.31) holds with

$$A = \frac{c}{4e\pi^2} \frac{\delta^{m(m-1)}}{m^{1+m^2}} \frac{\kappa_\phi^{2/m} - 1}{\ln\kappa_\phi} = \frac{c}{2e\pi^2} \frac{1 - e^{-\frac{2(\sigma c)^2}{m}}}{\frac{2(\sigma c)^2}{m}} \cdot \frac{(4cR\eta)^{m(m-1)}}{m^{2-m+2m^2}}.$$

Using $\dfrac{1 - e^{-t}}{t} \geq \dfrac{1}{t+1}$, we obtain the claimed bound. $\qquad\qquad\square$

We remark that the estimate in the above proposition is quite pessimistic. Our numerical experiments showed that the true bound may be much better.

3. Remez-Turán Property and Fixing the Blind Spots

In Theorem 2.8, the main issue is that the lower bound is only in terms of $\|\widehat{f}\mathbf{1}_E\|$ and not $\|\widehat{f}\|$ so that stability is not obtained. In this section, we consider a certain class of subsets of PW_c for which Theorem 2.8 does lead to stable reconstruction.

3.1. **Remez-Turán Property.**

Definition 3.1. Let $V \subset PW_c$ and write $\widehat{V} = \{\widehat{f} : f \in V\} \subset L^2([-c, c])$. We will say that $\widehat{V}$ has the Remez-Turán property if, for every $E \subset [-c, c]$ of positive Lebesgue measure, there exists $C = C(E, V)$ such that, for every $f \in V$,

$$(3.33) \qquad\qquad \|\widehat{f}\mathbf{1}_E\|_2 \geq C\|\widehat{f}\mathbf{1}_{[-c,c]}\|_2.$$

When V is a finite dimensional subspace of PW_c such that $\widehat{V}$ consists of analytic functions (restricted to I), then $\widehat{V}$ has the Remez-Turán property since $\|\widehat{f}\mathbf{1}_E\|_2$ is then a norm on V which, by finite dimensionality of V, is equivalent to $\|\widehat{f}\mathbf{1}_{[-c,c]}\|_2$. However, the previous argument does not provide any quantitative estimate on the constant $C(E, V)$. Let us start with two fundamental examples of spaces that have the Remez-Turán property, and for which quantitative estimates are known.

3.2. Fourier polynomials.

Let V_N be given by (1.11), so that $\widehat{V}_N = \{P\mathbf{1}_{[-c,c]}, P \in \mathbb{C}_N[x]\}$ is the space of polynomials of degree at most N, restricted to I. The quantitative form of the Remez-Turán property for $\widehat{V}_N$ is then known as the Remez Inequality [9]: for every polynomial of degree at most N,

$$(3.34) \qquad \|P\mathbf{1}_{[-c,c]}\|_2 \leqslant \left(\frac{8c}{|E|}\right)^{N+1/2} \|P\mathbf{1}_E\|_2.$$

3.3. Sparse sinc translates with free nodes.

Let V_N be given by (1.10), so that $\widehat{V}_N = \left\{P\mathbf{1}_{[-c,c]} : P(\xi) = \sum_{n=1}^N c_n e^{2i\pi\lambda_n\xi}\right\}$. Recall that $\widehat{V}_N$ is not a linear subspace. The fact that $\widehat{V}_N$ has the Remez-Turán property is a deep result of Nazarov [17]: for every exponential polynomial of order at most N, *i.e.* every P of the form $P(\xi) = \sum_{n=1}^N c_n e^{2i\pi\lambda_n\xi}$ one has

$$(3.35) \qquad \|P\mathbf{1}_{[-c,c]}\| \leqslant \left(\frac{\gamma c}{|E|}\right)^{N+1/2} \|P\mathbf{1}_E\|,$$

where γ is an absolute constant.

3.4. Prolate spheroidal wave functions (PSWF).

The Prolate spheroidal wave functions (PSWFs) denoted by $(\psi_{n,c}(\cdot))_{n\geqslant 0}$, are defined as the bounded eigenfunctions of the Sturm-Liouville differential operator $\mathcal{L}_c$, defined on $C^2([-1,1])$, by

$$(3.36) \qquad \mathcal{L}_c(\psi) = -(1-x^2)\frac{d^2\psi}{dx^2} + 2x\frac{d\psi}{dx} + c^2x^2\psi.$$

They are also the eigenfunctions of the finite Fourier transform $\mathcal{F}_c$, as well as the ones of the operator $\mathcal{Q}_c = \frac{c}{2\pi}\mathcal{F}_c^*\mathcal{F}_c$, which are defined on $L^2([-1,1])$ by

$$(3.37) \qquad \mathcal{F}_c(f)(x) = \int_{-1}^1 e^{icxy}f(y)\,\mathrm{d}y, \quad \text{and} \quad \mathcal{Q}_c(f)(x) = \int_{-1}^1 \frac{\sin(c(x-y))}{\pi(x-y)}f(y)\,\mathrm{d}y.$$

They are normalized so that $\|\psi_{n,c}\|_{L^2([-1,1])} = 1$ and $\psi_{n,c}(1) > 0$. We call $(\chi_n(c))_{n\geqslant 0}$ the corresponding eigenvalues of $\mathcal{L}_c$, $\mu_n(c)$ the eigenvalues of $\mathcal{F}_c$

$$(3.38) \qquad \mu_n(c)\psi_{n,c}(x) = \int_{-1}^1 \psi_{n,c}(y)e^{-icxy}\,\mathrm{d}y, \ x \in [-1,1].$$

and $\lambda_n(c)$ the ones of $\mathcal{Q}_c$ which are arranged in decreasing order. They are related by

$$\lambda_n(c) = \frac{c}{2\pi}|\mu_n(c)|^2.$$

A well known property is then that $\|\psi_{n,c}\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{\lambda_n(c)}}$. Further, their Fourier transform is given by

$$(3.39) \qquad \widehat{\psi_{n,c}}(\xi) = \int_{\mathbb{R}} \psi_{n,c}(x)e^{-ix\xi}\,\mathrm{d}x = (-1)^k\frac{2\pi}{c}\frac{\mu_n}{|\mu_n(c)|^2}\psi_{n,c}\left(\frac{\xi}{c}\right)\mathbf{1}_{|\xi|\leqslant c}$$

The crucial commuting property of $\mathcal{L}_c$ and $\mathcal{Q}_c$ has been first observed by Slepian and co-authors [21], whose name is closely associated with all properties of PSWFs, the spectrum

of the operators $\mathcal{L}_c$ and Q_c and almost time- and band-limited functions. Among the basic properties of PSWFs, we cite their analytic extension to the whole real line and their unique properties to form an orthonormal basis of $L^2([-1, 1])$ and an orthogonal basis of PW_c.

The prolate spheroidal wave functions admit a good representation in terms of the orthonormal basis of Legendre polynomials. In agreement with the standard practice, we will be denoting by P_k the classical Legendre polynomials, defined by the three-term recursion

$$P_{k+1}(x) = \frac{2k+1}{k+1} x P_k(x) - \frac{k}{k+1} P_{k-1}(x),$$

with the initial conditions

$$P_0(x) = 1, P_1(x) = x.$$

These polynomials are orthogonal in $L^2([-c, c])$ and are normalized so that

$$P_k(1) = 1 \quad \text{and} \quad \int_{-1}^{1} P_k(x)^2 \, \mathrm{d}x = \frac{1}{k+1/2}.$$

We will denote by $P_{k,c}$ the normalized Legendre polynomial $\widetilde{P}_{k,c}(x) = \sqrt{\frac{2k+1}{2c}} P_k\left(\frac{x}{c}\right)$ and the $P_{k,c}$'s then form an orthonormal basis of $L^2([-c, c])$.

We start from the following identity relating Bessel functions of the first kind to the finite Fourier transform of the Legendre polynomials, see [6]: for every $x \in \mathbb{R}$,

$$(3.40) \qquad \int_{-1}^{1} e^{ixy} P_k(y) \, \mathrm{d}y = 2i^k j_k(x), \ k \in \mathbb{N},$$

where j_k is the spherical Bessel function defined by $j_k(x) = (-x)^k \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^k \frac{\sin x}{x}$. Note that j_k has the same parity as k and recall that, for $x \geqslant 0$, $j_k(x) = \sqrt{\frac{\pi}{2x}} J_{k+1/2}(x)$ where J_α is the Bessel function of the first kind. In particular, from the well-known bound $|J_\alpha(x)| \leqslant \frac{|x|^\alpha}{2^\alpha \Gamma(\alpha+1)}$, valid for all $x \in \mathbb{R}$, we deduce that

$$|j_k(x)| \leqslant \sqrt{\pi} \frac{|x|^k}{2^{k+1} \Gamma(k+3/2)}, \ k \in \mathbb{N}.$$

Using the bound $\Gamma(x) \geqslant \sqrt{2\pi} x^{x-1/2} e^{-x}$ we get

$$(3.41) \qquad |j_k(x)| \leqslant \frac{e^{k+3/2}}{\sqrt{2}(2k+3)^{k+1}} |x|^k, \ k \in \mathbb{N}.$$

We have the following lemma.

Lemma 3.2. *Write* $\widehat{\psi_{n,c}} = \sum_{k \geqslant 0} \beta_k^n(c) P_{k,c}$. *Then for every* $k, \ell \geqslant 0$

$$|\beta_k^n| \leqslant \frac{10}{c^{3/2} |\lambda_n(c)|} \left(\frac{e}{2k+3}\right)^{k+1}$$

This bound is an adaptation of techniques from [15] to improve the proof of the exponential decay from [22].

Proof. Using (3.39), we have

$$
\begin{aligned}
\beta_k^n(c) &= \langle \psi_{n,c}, P_{k,c} \rangle_{L^2(I)} = \int_{-c}^{c} \widehat{\psi_{n,c}}(x) P_{k,c}(x) \, \mathrm{d}x \\
&= (-1)^k \frac{\mu_n(c)}{|\mu_n(c)|^2} \frac{2\pi}{c} \sqrt{\frac{2k+1}{2c}} \int_{-c}^{c} \psi_{n,c}(x/c) P_k\left(\frac{x}{c}\right) \mathrm{d}x \\
&= (-1)^k \pi \frac{\mu_n(c)}{c^{1/2}|\mu_n(c)|^2} \sqrt{4k+2} \int_{-1}^{1} \psi_{n,c}(x) P_k(x) \, \mathrm{d}x \\
&= \frac{(-1)^k \pi \sqrt{4k+2}}{c^{1/2}|\mu_n(c)|^2} \int_{-1}^{1} \int_{-1}^{1} \psi_{n,c}(y) e^{-icxy} \, \mathrm{d}y \, P_k(x) \, \mathrm{d}x
\end{aligned}
$$

with (3.38). Recalling that $\lambda_n(c) = \dfrac{c}{2\pi}|\mu_n(c)|^2$ and using Fubini, we get

$$
\begin{aligned}
\beta_k^n(c) &= \frac{(-1)^k 2\sqrt{4k+2}}{c^{3/2}\lambda_n(c)} \int_{-1}^{1} \int_{-1}^{1} P_k(x) e^{-icxy} \, \mathrm{d}x \, \psi_{n,c}(y) \, \mathrm{d}y \\
&= \frac{(-i)^k 4\sqrt{4k+2}}{c^{3/2}\lambda_n(c)} \int_{-1}^{1} \psi_\ell(y) j_k(y) \, \mathrm{d}y
\end{aligned}
$$

with (3.40). But then, from (3.41) and Cauchy-Schwarz, we deduce that

$$
\begin{aligned}
|\beta_k^n(c)| &\leqslant \frac{4\sqrt{4k+2}}{c^{3/2}\lambda_n(c)} \left(\int_{-1}^{1} j_k(y)^2 \, \mathrm{d}y \right)^{1/2} \\
&\leqslant \frac{4\sqrt{2k+1}e^{k+3/2}}{(2k+3)^{k+1}c^{3/2}\lambda_n(c)} \left(\int_{-1}^{1} |y|^{2k} \, \mathrm{d}y \right)^{1/2} \\
&= 4\sqrt{2e} \frac{1}{c^{3/2}\lambda_n(c)} \left(\frac{e}{2k+3} \right)^{k+1}.
\end{aligned}
$$

As $4\sqrt{2e} \leqslant 10$, the result follows. $\qquad\square$

We will also need the following estimate.

Lemma 3.3. *The eigenvalues* (3.38) *of* $\mathcal{Q}_c$ *satisfy*

$$
(3.42) \qquad \Lambda_N := \left(\sum_{n=0}^{N} \frac{1}{\lambda_n(c)} \right)^{1/2} \leqslant
\begin{cases}
\sqrt{3+ec} & \text{if } N \leqslant \max(ec, 2) \\
\left(\frac{2(N+1)}{ec} \right)^{\frac{2N+1}{2}} & \text{if } N \geqslant \max(ec, 2)
\end{cases}.
$$

Proof. Precise pointwise estimates of the $\lambda_n(c)$'s have been obtained in [15, Section 4 & Appendix C] and have been further improved in [8] to

$$
\lambda_n(c) \leqslant \left(\frac{ec}{2(n+1)} \right)^{2n+1} \qquad \text{for } n \geqslant \max\left(n, \frac{ec}{2} \right).
$$

while we always have $\lambda_n(c) < 1$.

It follows that

$$\sum_{k=0}^{N} \frac{1}{\lambda_n(c)} \geq \begin{cases} N+1 \leq 3 + ec & \text{if } N \leq \max(ec, 2) \\ \frac{1}{\lambda_N(c)} \geq \left(\frac{2(N+1)}{ec}\right)^{2N+1} & \text{if } N \geq \max(ec, 2) \end{cases}.$$

The result follows. □

We can now prove our Remez lemma for Prolate spheroidal wave functions.

Theorem 3.4 (Remez's Lemma for PSWF). *Let N be an integer and*

$$V_N = span\{\psi_{0,c}, \ldots, \psi_{N,c}\} \subset PW_c.$$

Then, for every $\psi \in \widehat{V}_N$ and every $E \subseteq [-c, c]$ of positive measure,

(3.43) $$\|\psi\| \leq 2 \left(\frac{8c}{|E|}\right)^{K(N)} \|\widehat{\psi} \mathbf{1}_E\|,$$

where

(3.44) $$K(N) = \begin{cases} \max\left(\left\lceil \frac{3200(3+ec)}{c^3} \right\rceil, \left\lceil \frac{4ec}{|E|} \right\rceil\right) & \text{if } N \leq \max(2, ec), \\ \max\left(20, N, \left\lceil \frac{8(N+1)}{|E|} \right\rceil\right) & \text{if } N \geq \max(2, ec). \end{cases}$$

Proof. Let $\psi = \sum_{n=0}^{N} c_\ell \psi_{n,c}$ so that, by orthogonality and the fact that $\|\psi_{n,c}\| = \lambda_n(c)^{-1/2}$,

$$\|\psi\| = \left(\sum_{n=0}^{N} \frac{|c_n|^2}{\lambda_n(c)}\right)^{1/2}.$$

On the other hand

$$\widehat{\psi} = \sum_{n=0}^{N} c_\ell \widehat{\psi_{n,c}} = \sum_{n=0}^{N} c_\ell \sum_{k \geq 0} \beta_k^n(c) P_{k,c}.$$

Let K be an integer that will be fixed later and write

$$\widehat{\psi} = \sum_{n=0}^{N} c_\ell \sum_{k=0}^{K} \beta_k^n(c) P_{k,c} + \sum_{n=0}^{N} c_\ell \sum_{k>K} \beta_k^n(c) P_{k,c} := F_K + R_K.$$

Note that F_K is a polynomial of degree K so that

(3.45) $$\|F_K \mathbf{1}_{[-c,c]}\| \leq \left(\frac{8c}{|E|}\right)^{K+\frac{1}{2}} \|F_K \mathbf{1}_E\|$$

by (3.34). On the other hand,

$$R_K = \sum_{k>K} \left(\sum_{n=0}^{N} c_n \beta_k^n(c)\right) P_{k,c}$$

so that

$$\|R_K \mathbf{1}_E\| \leq \|R_K \mathbf{1}_{[-c,c]}\| = \left(\sum_{k>K} \left|\sum_{n=0}^{N} c_n \beta_k^n(c)\right|^2\right)^{1/2} \leq \left(\sum_{k>K} \sum_{n=0}^{N} \lambda_n(c) |\beta_k^n(c)|^2\right)^{1/2} \|\psi\|$$

by Cauchy-Schwarz. We now apply Lemma 3.2 to get

$$
\begin{aligned}
\|R_K \mathbf{1}_E\| &\leq \frac{10}{c^{3/2}} \left(\sum_{k>K} \sum_{n=0}^{N} \frac{1}{\lambda_n(c)} \left(\frac{e}{2k+3} \right)^{2k+2} \right)^{1/2} \|\psi\| \\
&= \frac{10}{c^{3/2}} \left(\sum_{n=0}^{N} \frac{1}{\lambda_n(c)} \right)^{1/2} \left(\sum_{k>K} \left(\frac{e}{2k+3} \right)^{2k+2} \right)^{1/2} \|\psi\| \\
&\leq \frac{12}{c^{3/2}} \left(\sum_{n=0}^{N} \frac{1}{\lambda_n(c)} \right)^{1/2} \left(\frac{e}{2K+5} \right)^{K+1} \|\psi\|.
\end{aligned}
$$

Using Lemmas 3.3 and 3.2 we can rewrite this in the form $\|R_K \mathbf{1}_E\| \leq \Lambda_N \Phi_K \|\psi\|$ with

$$
\Lambda_N := \begin{cases} \sqrt{3+ec} & \text{if } N \leq \max(ec, 2) \\ \left(\frac{2(N+1)}{ec} \right)^{N+\frac{1}{2}} & \text{if } N \geq \max(ec, 2) \end{cases} \quad \text{and} \quad \Phi_K = \frac{12}{c^{3/2}} \left(\frac{e}{2K+5} \right)^{K+1}.
$$

Next

$$
\begin{aligned}
\|\widehat{\psi} \mathbf{1}_E\| &\geq \|F_K \mathbf{1}_E\| - \|R_K \mathbf{1}_E\| \geq \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}} \|F_K \mathbf{1}_{[-c,c]}\| - \|R_K \mathbf{1}_{[-c,c]}\| \\
&\geq \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}} \|\widehat{\psi}\| - \left(1 + \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}} \right) \|R_K \mathbf{1}_{[-c,c]}\| \\
&\geq \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}} \|\widehat{\psi}\| - 2\|R_K \mathbf{1}_{[-c,c]}\|
\end{aligned}
$$

since $E \subset [-c,c]$ implies $\left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}} \leq 1$. Therefore

$$
\|\widehat{\psi} \mathbf{1}_E\| \geq \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}} \left(1 - 2\Lambda_N \Phi_K \left(\frac{8c}{|E|} \right)^{K+\frac{1}{2}} \right) \|\widehat{\psi}\|.
$$

It remains to choose K so that $\Lambda_N \Phi_K \leq \frac{1}{4} \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}}$.

First, if $N \leq \max(ec, 2)$, then we want

$$
\left(\frac{e}{2K+5} \right)^{1/2} \left(\frac{e}{2K+5} \right)^{K+1/2} = \left(\frac{e}{2K+5} \right)^{K+1} \leq \frac{c^{3/2}}{48\sqrt{3+ec}} \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}}
$$

so that it is enough that $\frac{e}{2K+5} \leq \frac{c^3}{48^2(3+ec)}$ and $\frac{e}{2K+5} \leq \frac{|E|}{8c}$ so we take

$$
K = K(N) := \max \left(\left\lceil \frac{3200(3+ec)}{c^3} \right\rceil, \left\lceil \frac{4ec}{|E|} \right\rceil \right).
$$

On the other hand, if $N \geq \max(ec, 2)$, then we want

$$
\left(\frac{e}{2K+5} \right)^{1/2} \left(\frac{e}{2K+5} \right)^{K+1/2} = \left(\frac{e}{2K+5} \right)^{K+1} \leq \frac{1}{4} \left(\frac{ec}{2(N+1)} \right)^{N+\frac{1}{2}} \left(\frac{|E|}{8c} \right)^{K+\frac{1}{2}}
$$

Taking $K := K(N) = \max\left(20, N, \left\lceil \frac{8(N+1)}{|E|} \right\rceil\right)$, we get

$$\left(\frac{e}{2K+5}\right)^{K+1} \leqslant \frac{1}{4}\left(\frac{e}{2K}\right)^{K+1/2} \leqslant \frac{1}{4}\left(\frac{ec}{2(N+1)}\frac{|E|}{8c}\right)^{K+1/2}$$

which gives the desired estimate since $2(N+1) > ec$ and $K \geqslant N$. □

3.5. Sampling the heat flow.

Equipped with the Remez-Turán Property, we are ready to close the blind spots in Theorem 2.8. We do it only in the case of heat flow as it should be clear how to obtain similar estimates in the case of other kernels $\phi \in \Phi$.

Theorem 3.5. *Let* $\widehat{\phi}(\xi) = e^{-\sigma^2\xi^2}$, $\sigma \neq 0$, *and* $m \geqslant 2$ *be an integer. Let* $V = V_N$ *be given by* (1.9), (1.10), *or* (1.11). *Then, for every* $f \in V$,

$$(3.46) \qquad \kappa\|\widehat{f}\|^2 \leqslant \int_0^1 \sum_{k\in\mathbb{Z}}\left|f_t\left(\frac{m\pi}{c}k\right)\right|^2 \, dt \leqslant \|\widehat{f}\|^2,$$

where

$$(3.47) \qquad \kappa = \frac{c\kappa_0(c)}{(\sigma c)^2 + m}\exp\left(-\kappa_1(c)N - m^2\left(-\kappa_2(c)\ln\sigma + \kappa_3(c)\sigma^2 + \ln m\right)\right)$$

with κ_j *positive constants that depend on* c *only.*

Remark 3.6. For $V = V_N$ given by (1.10), (1.11) and for $V = V_N$ given by (1.9) when $N \geqslant \max(2, ec)$, κ_0, κ_1 do not depend on c.

Proof. To obtain this result, we take $\eta = \dfrac{1}{8}$ in Proposition 2.12. First note that if $\widetilde{E} = \left[-\dfrac{3}{8}, -\dfrac{1}{8}\right] \cup \left[\dfrac{1}{8}, \dfrac{3}{8}\right]$ and $E = \left(\dfrac{2c}{m}(\widetilde{E} + \mathbb{Z})\right) \cap [-c, c]$ then $\dfrac{c}{|E|} \geqslant \dfrac{1}{8}$ (say). Then (2.31) tells us that

$$(2.31) \qquad A\|\widehat{f}\mathbf{1}_E\|^2 \leqslant \int_0^1 \sum_{k\in\mathbb{Z}}\left|f_t\left(\frac{m\pi}{c}k\right)\right|^2 \, dt \leqslant \|\widehat{f}\|^2,$$

for any $f \in PW_c$, where

$$A = \frac{c}{2e\pi^2(2(\sigma c)^2 + m)}\frac{(cR/2)^{m(m-1)}}{m^{1-m+2m^2}} \quad \text{with} \quad R = 2\sigma^2\min\left\{\frac{1}{8}e^{-(\sigma/8)^2}, ce^{-(\sigma c)^2}\right\}.$$

Note that

$$\frac{cR}{2} = \min\left\{c\frac{\sigma^2}{8}e^{-(\sigma/8)^2}, c^2\sigma^2 e^{-(\sigma c)^2}\right\} < 1$$

so that

$$\frac{(cR/2)^{m(m-1)}}{m^{1-m+2m^2}} \geqslant \left(\frac{cR/2}{m}\right)^{m^2} = \exp\left(-m^2\left(-\gamma_1(c)\ln\sigma + \gamma_2(c)\sigma^2 + \ln m\right)\right).$$

Finally $A \geqslant \dfrac{c\gamma_0}{\left((\sigma c)^2 + m\right)}\exp\left(-m^2\left(-\gamma_1(c)\ln\sigma + \gamma_2(c)\sigma^2 + \ln m\right)\right)$ where $\gamma_0, \gamma_1(c), \gamma_2(c)$ are constants depending on c only.

It remains to fix the blind spots $\|\widehat{f}\mathbf{1}_E\|^2$ with the help of a Remez type inequality. For $V = V_N$ given by (1.10), (1.11) and $f \in V_N$, we simply have $\|\widehat{f}\mathbf{1}_E\|^2 \geqslant \gamma_3^{2N+1}\|f\|^2$ where $\gamma_3 < 1$ is a constant.

For $V = V_N$ given by (1.9), $\|\widehat{f}\mathbf{1}_E\|^2 \geqslant \gamma_3^{2K(N)}\|f\|^2$ where $K(N)$ is given by (3.44)

$$K(N) = \begin{cases} \max\left(\left\lceil \frac{3200(3+ec)}{c^3} \right\rceil, \left\lceil \frac{4ec}{|E|} \right\rceil \right) \leqslant \gamma_4(c) & \text{if } N \leqslant \max(2, ec), \\ \max\left(20, N, \left\lceil \frac{8(N+1)}{|E|} \right\rceil \right) \leqslant 64(N+1) & \text{if } N \geqslant \max(2, ec). \end{cases}$$

Adding the estimates for fixing the blind spot yields (3.47). $\qquad\qquad\square$

Remark 3.7. Theorem 3.5 immediately implies Theorem 1.6. We also note that if $V = V_N$ is given by (1.9) or (1.11), the reconstruction can be done from measurements at a finite number of spacial locations. Indeed, our results imply that in this case one can find the coefficients of f in its decomposition in a basis of V via simple least squares.

4. Sensor density, maximal spatial gaps and condition numbers

In this section, we discuss irregular spatio-temporal sampling. We establish that stable reconstruction from dynamical samples may occur when the set Λ has an arbitrarily small density. More importantly, however, we show that the density cannot be arbitrarily small for fixed frame bounds in (1.4). In fact, we provide an explicit estimate for the maximal spatial gap in terms of the condition number $\frac{B}{A}$.

Example 4.1. In this example, we take $c = 1/2$ to simplify discussion. Assume that $\phi \in \Phi$ is such that $\widehat{\Phi}$ is real, even, and decreasing on $[0, 1/2]$. Let $\Lambda_0 = m\mathbb{Z}$, with $m \in \mathbb{N}$ odd, $\Lambda_k = mn\mathbb{Z} + k$, where n is any fixed odd number and $k = 1, \ldots \frac{m-1}{2}$. Then $\Lambda = \bigcup_{k=0}^{\frac{m-1}{2}} \Lambda_k$ has density $D^-(\Lambda) \leqslant 1/n + 1/m$ and is a stable set of sampling, i.e., (1.7) is satisfied.

The claim in the last example follows by stringing together several theorems on dynamical sampling. Firstly, [4, Theorems 2.4 and 2.5] yield that any $f \in \ell^2(\mathbb{Z})$ can be recovered from the space-time samples $\{\phi^j * f(x_k) : j = 0, \ldots, m-1, x_k \in \Lambda\}$ and that the problem of sampling and reconstruction in PW_c on subsets of $\mathbb{Z}$ is equivalent to the sampling and reconstruction problem of sequences in $\ell^2(\mathbb{Z})$. Secondly, combining [5, Theorems 5.4 and 5.5] shows that for $\phi \in \Phi$, $f \in PW_c$ can be stably reconstructed from $\{\phi^j * f(x_k) : j = 0, \ldots, m-1, x_k \in \Lambda\}$ if and only if (1.7) is satisfied.

Example 4.1 thus shows that (1.7) can hold with sets having arbitrarily small densities. The goal of this section is to show that the maximal gap in such sets is controlled by the condition number B/A.

We first establish the following lemma, which parallels [13, Proposition 4.4].

Lemma 4.2. *Let $\phi \in \Phi$ be such that $\widehat{\phi}$ is $\mathcal{C}^1$-smooth on $I = [-c, c]$. Then there exists a finite constant $C_{\phi,L}$ such that*

(4.48) $$\int_0^L |(\operatorname{sinc}(c\cdot) * \phi_t)(x)|^2 \, \mathrm{d}t \leqslant \frac{C_{\phi,L}}{1 + x^2}, \text{ for all } x \in \mathbb{R}.$$

On the other hand, setting $c_{\phi,L} = \frac{2(\kappa_\phi^{2L}-1)}{\pi^2 \ln \kappa_\phi} > 0$, for $|x| \leqslant \pi/2c$, we have

$$(4.49) \qquad \int_0^L |(\mathrm{sinc} * \phi_t)(x)|^2 \, dt \geqslant c_{\phi,L}.$$

Proof. Firstly, writing the Fourier inversion formula shows that

$$(4.50) \qquad (\mathrm{sinc}(c\cdot) * \phi_t)(x) = \frac{1}{2c} \int_{-c}^c \left(\widehat{\phi}(\xi)\right)^t e^{ix\xi} \, d\xi$$

from which it follows that

$$(4.51) \qquad |(\mathrm{sinc}(c\cdot) * \phi_t)(x)| \leqslant \frac{1}{2c} \int_{-c}^c |\widehat{\phi}(\xi)|^t \, d\xi \leqslant 1,$$

due to $|\widehat{\phi}| \leqslant 1$.

Secondly, note that, due to its smoothness, $\widehat{\phi}'$ is bounded by $E_\Phi := \sup_{\xi \in [-c,c]} |\widehat{\phi}'(\xi)| < +\infty$ on $[-c, c]$ Then, integrating (4.50) by parts leads to

$$x(\mathrm{sinc}(c\cdot) * \phi_t)(x) = \frac{\widehat{\phi}^t(c)e^{icx} - \widehat{\phi}^t(-c)e^{-icx}}{2ic} - \frac{1}{2ic} \int_{-c}^c e^{ix\xi} t \widehat{\phi}^{t-1}(\xi) \widehat{\phi}'(\xi) \, d\xi,$$

and, as $\kappa_\phi \leqslant \widehat{\phi} \leqslant 1$ on I, we deduce that

$$|x(\mathrm{sinc}(c\cdot) * \phi_t)(x)| \leqslant \frac{1}{c} + \frac{E_\phi}{\kappa_\phi} t.$$

Consequently,

$$x^2 \int_0^L |(\mathrm{sinc}(c\cdot) * \phi_t)(x)|^2 \, dt \leqslant \int_0^L \left(\frac{1}{c} + \frac{E_\phi}{\kappa_\phi} t\right)^2 dt = \frac{\kappa_\phi}{3E_\phi} \left(\left(\frac{1}{c} + \frac{E_\phi}{\kappa_\phi} L\right)^3 - \frac{1}{c^3}\right),$$

and the estimate (4.48) follows in view of (4.51).

On the other hand (4.50) implies that

$$|(\mathrm{sinc}(c\cdot) * \phi_t)(x)| \geqslant |\Re(\mathrm{sinc}(c\cdot) * \phi_t)(x)| = \left|\frac{1}{2c} \int_{-c}^c \widehat{\phi}(\xi)^t \cos x\xi \, d\xi\right|.$$

But, for $|\xi| \leqslant c$, we have $\widehat{\phi}(\xi)^t \geqslant \kappa_\phi^t$. Further, if we also have $|x| \leqslant \pi/2c$, then $\cos 2x\xi \geqslant 0$. Therefore,

$$|(\mathrm{sinc}(c\cdot) * \phi_t)(x)| \geqslant \frac{1}{2c} \int_{-c}^c \widehat{\phi}(\xi)^t \cos x\xi \, d\xi \geqslant \kappa_\phi^t \frac{1}{2c} \int_{-c}^c \cos x\xi \, d\xi = \kappa_\phi^t \mathrm{sinc}(cx) \geqslant \frac{2}{\pi} \kappa_\phi^t$$

since $\mathrm{sinc}(cx)$ is decreasing on $[0, \pi/2c]$ and $\mathrm{sinc}\left(c\frac{\pi}{2c}\right) = \frac{2}{\pi}$. It follows that

$$\int_0^L |(\mathrm{sinc} * \phi_t)(x)|^2 \, dt \geqslant \frac{4}{\pi^2} \int_0^L \kappa_\phi^{2t} \, dt = \frac{2(\kappa_\phi^{2L} - 1)}{\pi^2 \ln \kappa_\phi} > 0,$$

and we get the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.3. If $\widehat{\phi}(\xi) = e^{-\sigma^2 \xi^2}$, $\sigma \neq 0$, then $\kappa_\phi = e^{-(\sigma c)^2}$ and we may take $E_\phi = \sqrt{\frac{2}{e}}|\sigma|$. Therefore, the constants $c_{\phi,L}$ and $C_{\phi,L}$ in the above lemma can be taken as

$$(4.52) \qquad C_{\phi,L} = \frac{1}{c^3} + \left(1 + \sigma^2 e^{(\sigma c)^2}\right)L^3 \quad \text{and} \quad c_{\phi,L} = \frac{2(1 - e^{-2L(\sigma c)^2})}{\pi^2(\sigma c)^2}.$$

For the estimate of $C_{\phi,L}$ we have used that

$$\frac{1}{3\alpha}(a + \alpha b) = a^2 b + \alpha a b^2 + \frac{\alpha^2}{3}b^3 \leqslant a^3 + \frac{b^3}{3}(1 + 2\alpha^{3/2} + \alpha^2) \leqslant a^3 + b^2(1 + \alpha^2)$$

with Hölder.

Theorem 4.4. *Let* $\phi \in \Phi$ *and assume that* $\widehat{\phi}$ *is* $\mathcal{C}^1$-*smooth on* $[-c, c]$. *Assume that* $\Lambda \subseteq \mathbb{R}$ *is a stable sampling set for Problem 1 with frame bounds A, B (i.e., (1.4) holds:*

$$(1.4) \qquad A\|f\|_2^2 \leqslant \int_0^L \sum_{\lambda \in \Lambda} |(f * \phi_t)(\lambda)|^2 \, dt \leqslant B\|f\|_2^2, \text{ for all } f \in PW_c.$$

Let $c_{\phi,L}$ and $C_{\phi,L}$ be the constants from Lemma 4.2. Then for $R \geqslant \max\left(\dfrac{\pi}{c}, \dfrac{8c}{\pi}\dfrac{B}{A}\dfrac{C_{\Phi,L}}{c_{\Phi,L}}\right)$ and every $a \in \mathbb{R}$, we have $[a-R, a+R] \cap \Lambda \neq \varnothing$. Further, we have $D^-(\Lambda) \geqslant \min\left(\dfrac{c}{2\pi}, \dfrac{\pi}{16c}\dfrac{A}{B}\dfrac{c_{\Phi,L}}{C_{\Phi,L}}\right)$ and $D^+(\Lambda) \leqslant 4\dfrac{B}{c_{\Phi,L}}$.

Proof. Denoting $I_a = [a - \pi/4c, a + \pi/4c]$, $a \in \mathbb{R}$, let us bound the covering number

$$n_\Lambda := \sup_{a \in \mathbb{R}} \#(\Lambda \cap I_a).$$

We use (4.49), i.e., the fact that $\int_0^L |(\mathrm{sinc}(c\cdot) * \phi_t)(x)|^2 \, dt \geqslant c_{\Phi,L}$ for $|x| \leqslant \pi/2c$, and our first observation to obtain

$$\begin{aligned} \#(\Lambda \cap I_a) &\leqslant \frac{1}{c_{\Phi,L}} \sum_{\lambda \in \Lambda \cap I_a} \int_0^L |(\mathrm{sinc}(c\cdot) * \phi_t)(\lambda - a)|^2 dt \\ &\leqslant \frac{1}{c_{\Phi,L}} \sum_{\lambda \in \mathbb{Z}} \int_0^L |(\mathrm{sinc}(c\cdot) * \phi_t)(\lambda - a)|^2 dt \leqslant \frac{B}{c_{\Phi,L}}\|\mathrm{sinc}\, c(t-a)\|^2 \end{aligned}$$

where we applied (1.4) to $f(t) = \mathrm{sinc}\, c(t - a)$ for all $a \in \mathbb{R}$. As $\widehat{f}(\xi) = \dfrac{\pi}{c}\mathbf{1}_{[-c,c]}$, Parseval's relation gives $\|f\|^2 = \frac{\pi}{c}$ hence

$$(4.53) \qquad n_\Lambda \leqslant \frac{\pi}{c}\frac{B}{c_{\Phi,L}}.$$

As a first consequence, this implies that $D^+(\Lambda) \leqslant 4\frac{B}{c_{\Phi,L}}$.

Now we assume that for some $a_0 \in \mathbb{R}$, and some $R \geqslant \dfrac{\pi}{c}$, $\Lambda \cap [a_0 - R, a_0 + R] = \varnothing$. As the Paley-Wiener space is invariant under translation, if (1.4) holds for Λ, it also holds for its translates, so that we may assume that $a_0 = 0$.

From Lemma 4.2, there exists $C_{\Phi,L}$ such that $\int_0^L |(\mathrm{sinc}(c\cdot) * \phi_t)(x)|^2 \mathrm{d}t \leqslant C_{\Phi,L}/(1 + x^2)$. Therefore, we have the following estimates

$$
\begin{aligned}
\frac{\pi}{c}A &\leqslant \sum_{\lambda \in \Lambda} \int_0^L |(\mathrm{sinc}(c\cdot) * \phi_t)(\lambda)|^2 \, \mathrm{d}t \leqslant \sum_{\lambda \in \Lambda} \frac{C_{\Phi,L}}{1 + \lambda^2} \\
&\leqslant \sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda \cap [R+k\pi/2c, R+(k+1)\pi/2c]} \frac{C_{\Phi,L}}{1+\lambda^2} + \sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda \cap [-R-(k+1)\pi/2c, -R-k\pi/2c]} \frac{C_{\Phi,L}}{1+\lambda^2} \\
&\leqslant 2n_\Lambda \sum_{k=0}^{\infty} \frac{C_{\Phi,L}}{1 + (R + k\pi/2c)^2} \leqslant 4\frac{C_{\Phi,L}B}{c_{\Phi,L}} \int_{R-\pi/2c}^{\infty} \frac{\mathrm{d}x}{1+x^2} \\
&\leqslant 4\frac{C_{\Phi,L}B}{c_{\Phi,L}} \int_{R/2}^{\infty} \frac{\mathrm{d}x}{x^2} = 8\frac{C_{\Phi,L}B}{c_{\Phi,L}R}
\end{aligned}
$$

since we assumed that $R \geqslant \pi/c$. It follows that $R \leqslant \dfrac{8c}{\pi} \dfrac{B}{A} \dfrac{C_{\Phi,L}}{c_{\Phi,L}}$. Finally, note that this implies that $D^-(\Lambda) \geqslant \frac{1}{2R}$. $\qquad\square$

Remark 4.5. Computing the explicit estimate for $\frac{C_{\Phi,L}}{c_{\Phi,L}}$, we observe that the maximal allowed gap in spacial measurements grows with L, which is to be expected. For the Gaussian, we may take the constant $\frac{C_{\Phi,L}}{c_{\Phi,L}}$ to be $O(L^2)$ (see (4.52)). The above results also shows that for $\mathcal{C}^1$-smooth functions ϕ, stable sampling sets must have positive lower density.

Remark 4.6. Theorem 4.4 immediately implies Theorem 1.4.

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References

[1] A. Aldroubi, C. Cabrelli, A.F. Çakmak, U. Molter, and A. Petrosyan. Iterative actions of normal operators. *J. Funct. Anal.*, 272(3):1121–1146, 2017.

[2] A. Aldroubi, C. Cabrelli, U. Molter, and S. Tang. Dynamical sampling. *Applied and Computational Harmonic Analysis*, 42(3):378–401, 2017. doi: 10.1016/j.acha.2015.08.014.

[3] A. Aldroubi, J. Davis, and I. Krishtal. Dynamical sampling: time-space trade-off. *Appl. Comput. Harmon. Anal.*, 34:495–503, 2013.

[4] A. Aldroubi, J. Davis, and I. Krishtal. Exact reconstruction of signals in evolutionary systems via spatiotemporal trade-off. *J. Fourier Anal. Appl.*, 21(1):11–31, 2015.

[5] A. Aldroubi, L.X. Huang, and A. Petrosyan. Frames induced by the action of continuous powers of an operator. *J. Math. Anal. Appl.*, 478:1059–1084, 2019.

[6] G. E. Andrews, R. Askey, and R. Roy. Special Functions, Cambridge University Press, 2000.

[7] B. Beckermann and A. Townsend. On the singular values of matrices with displacement structure. *SIAM J. Matrix Anal. Appl.*, 38:1227–1248, 2017.

[8] A. Bonami, Ph. Jaming and A. Karoui Non-asymptotic behaviour of the spectrum of the sinc-kernel operator and related applications. Available at *ArXiv:1804.01257.*

[9] P. Borwein and T. Erdélyi. Polynomials and Polynomial Inequalities, Springer, New York, 1995

[10] D. Fasino and V. Olshevsky. How bad are symmetric Pick matrices? In *Structured matrices in mathematics, computer science, and engineering, I (Boulder, CO, 1999)*, volume 280 of *Contemp. Math.*, pages 301–311. Amer. Math. Soc., Providence, RI, 2001.

[11] M. Fornasier and H. Rauhut. Continuous frames, function spaces, and the discretization problem. *J. Fourier Anal. Appl.*, 11(3):245–287, 2005.

[12] W. Gautschi. On inverses of Vandermonde and confluent Vandermonde matrices. *Numer. Math.*, 4:117–123, 1962.

[13] K. Gröchenig, J.L. Romero, J. Unnikrishnan, and M. Vetterli. On minimal trajectories for mobile sampling of bandlimited fields. *Appl. Comput. Harmon. Anal.*, 39(3):487–510, 2015.

[14] A. D. Güngör. Erratum to "An upper bound for the condition number of a matrix in spectral norm" [J. Comput. Appl. Math. 143 (2002) 141–144] *J. Comput. Appl. Math.*, 234:316, 2010.

[15] Ph. Jaming, A. Karoui, and S. Spektor. The approximation of almost time and band limited functions by their expansion in some orthogonal polynomials bases. *J. Approx. Theory*, 212:41–65, 2016.

[16] Y.M. Lu and M.Vetterli. Spatial super-resolution of a diffusion field by temporal oversampling in sensor networks. In *2009 IEEE International Conference on Acoustics, Speech and Signal Processing*. IEEE, apr 2009.

[17] F. Nazarov. Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. *Algebra i Analiz.*, 5:3–66, 1993.

[18] A. Papoulis. Generalized sampling expansion. *IEEE Trans. Circuits and Systems*, CAS-24(11):652–654, 1977.

[19] G. Piazza and T. Politi. An upper bound for the condition number of a matrix in spectral norm. *J. Comput. Appl. Math.*, 143(1):141–144, 2002.

[20] Juri Ranieri, Amina Chebira, Yue M. Lu, and Martin Vetterli. Sampling and reconstructing diffusion fields with localized sources. *2011 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 4016–4019, 2011.

[21] D. Slepian and H. O. Pollak Prolate spheroidal wave functions, Fourier analysis and uncertainty I. *Bell System Tech. J.* 43: 3009–3058, 1964.

[22] H. Xiao, V. Rokhlin, and N. Yarvin. Prolate spheroidal wave functions, quadrature and interpolation. *Inverse Problems*, 17:805–838, 2001.

[23] Y.-S. Yu and D.-H. Gu. A note on a lower bound for the smallest singular value. *Linear Algebra Appl.*, 253:25–38, 1997.

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