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Abstract. We introduce a moment map picture for *holomorphic string algebroids* where the Hamiltonian gauge action is described by means of Morita equivalences, as suggested by higher gauge theory. The zero locus of our moment map is given by the solutions of the *Calabi system*, a coupled system of equations which provides a unifying framework for the classical Calabi problem and the Hull-Strominger system. Our main results are concerned with the geometry of the moduli space of solutions, and assume a technical condition which is fulfilled in examples. We prove that the moduli space carries a pseudo-Kähler metric with Kähler potential given by the *dilaton functional*, a topological formula for the metric, and an infinitesimal Donaldson-Uhlenbeck-Yau type theorem. Finally, we relate our topological formula to a physical prediction for the *gravitino mass* in order to obtain a new conjectural obstruction for the Hull-Strominger system.

Contents

1. Introduction

Back to the work of Atiyah and Bott [7], the interaction of Yang-Mills theory with symplectic geometry and, in particular, the idea of *moment map*, has had an important impact in our understanding of the moduli theory for holomorphic

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 750885 GENERALIZED. MGF was partially supported by the Spanish MINECO under ICMAT Severo Ochoa project No. SEV-2015-0554, and under grant No. MTM2016-81048-P. RR was supported by a Marie Skłodowska-Curie Individual Fellowship. CT was supported by the French government "Investissements d'Avenir" program ANR–11–LABX–0020–01 and ANR project EMARKS No ANR–14–CE25–0010.

vector bundles in algebraic geometry. The seed relation between stable bundles on a Riemann surface and flat unitary connections observed in [7, 36], was largely expanded with the Donaldson-Uhlenbeck-Yau Theorem [16, 41]. This important result, initially conjectured by Hitchin and Kobayashi, establishes a correspondence between the moduli space of solutions of the Hermite-Yang-Mills equations and the moduli space of slope-stable bundles on a compact Kähler manifold. A key upshot is that certain moduli spaces in algebraic geometry, constructed via Mumford's theory of stability, are endowed with natural symplectic structures.

Our main goal in the present work is to explore a new scenario where the 'moment map picture' arises tightly bound up with recent developments in higher gauge theory. Inspired by the Atiyah and Bott construction, our starting point is a class of holomorphic bundle-like objects on a compact complex manifold X, known as *string algebroids* [24]. A string algebroid Q is a special class of holomorphic Courant algebroid, which can be thought of as the 'higher Atiyah algebroid' of a holomorphic principal bundle for the (complexified) *string group* [43]. In the case of our interest, the geometric content of Q comprises, in particular, a holomorphic principal G-bundle P over X with vanishing first Pontryagin class $p_1(P) = 0$ and a holomorphic extension

$$0 \longrightarrow T^*X \longrightarrow Q \longrightarrow A_P \longrightarrow 0$$

of the holomorphic Atiyah algebroid A_P of P by the holomorphic cotangent bundle. We assume G to be a complex reductive Lie group with a fixed symmetric bilinear form $\langle\,,\rangle$ on its Lie algebra.

In this work we shall study gauge theoretical aspects of holomorphic string algebroids. For this, we start by developing basic aspects of the theory, such as gauge symmetries and a *Chern correspondence* in our setting. Gauge symmetries are described in Theorem 3.14, where we construct a category whose objects are string algebroids and whose morphisms are (isomorphism classes of) *Morita equivalences*. Two string algebroids Q, Q' are Morita equivalent if they can be obtained by reduction from the same complex string algebroid E (the analogous concept in the smooth category), which we represent by

$$\begin{array}{ccccc} & & E & & \\ & \swarrow & & \searrow & \\ Q & \longleftarrow & Q_L & & Q_{L'} \longrightarrow Q'. \end{array}$$

Morita equivalences should be thought of as 'higher gauge symmetries' (see Remark 3.15) and the Picard group of self-equivalences plays the role of the 'complex gauge group'.

The Chern correspondence (Lemmas 5.11 and 7.5) requires the study of a notion of *compact form* for Q by means of *real string algebroids*

$$E_{\mathbb{R}} \subset E.$$

A compact form $E_{\mathbb{R}}$ determines a reduction $P_h \subset P$ to a maximal compact subgroup $K \subset G$ (see Definition 5.4). The Chern correspondence associates to

each compact form on Q a horizontal subspace

$$W \subset E_{\mathbb{R}},$$

which provides the analogue of the Chern connection in our context. In agreement with structural properties of connections in higher gauge theory [25, 39], any such $W \subset E_{\mathbb{R}}$ determines the classical Chern connection θ^h of $P_h \subset P$ and a real $(1,1)$-form ω satisfying a structure equation (see Proposition 5.13).

We move on to study the geometry of the infinite-dimensional space of horizontal subspaces W on a fixed compact form $E_{\mathbb{R}}$, whose associated $(1,1)$-form ω is hermitian. Via the Chern correspondence, this space has a (possibly degenerate) pseudo-Kähler structure for each choice of volume form μ on X and *level* $\ell \in \mathbb{R}$. There is a global Kähler potential given by $-\log$ of the *dilaton functional* M_ℓ, that is,

$$-\log M_\ell := -\log \int_X e^{-\ell f_\omega} \frac{\omega^n}{n!}, \tag{1.1}$$

where $f_\omega := \frac{1}{2}\log(\omega^n/n!\mu)$. In Proposition 7.14 we prove that there is a natural Hamiltonian action for a subgroup of Morita Picard preserving the compact form, with zero locus for the moment map given by solutions of the coupled equations

$$\begin{aligned} F \wedge \omega^{n-1} &= 0, & F^{0,2} &= 0, \\ d(e^{-\ell f_\omega}\omega^{n-1}) &= 0, & dd^c\omega + \langle F \wedge F\rangle &= 0. \end{aligned} \tag{1.2}$$

Here F is the curvature of a connection in the principal K-bundle underlying $E_{\mathbb{R}}$, which is determined by $W \subset E_{\mathbb{R}}$.

The equations (1.2) were first found in [22] for $\ell = 1$ in a holomorphic setting, in relation to the critical locus of the dilaton functional M_1. By Proposition 7.14, they can be regarded as a natural analogue of the Hermite-Yang-Mills equations for string algebroids. Following [22], we will refer to (1.2) as the *Calabi system*. These moment map equations provide a unifying framework for the classical Calabi problem, which is recovered when K is trivial (see Section 7.2), and the Hull-Strominger system [31, 40]. For the latter, we assume that X is a (non-necessarily Kähler) Calabi-Yau threefold with holomorphic volume form Ω and we take $\ell = 1$ and

$$\mu = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}. \tag{1.3}$$

To our knowledge, Corollary 7.15 provides the first moment map interpretation of the Hull-Strominger system in the mathematics literature (see [17, 21, 37] for recent reviews covering this topic). As a matter of fact, this was our original motivation when we initiated the present work.

Our main results, discussed briefly over the next section, are devoted to the geometry of the moduli space of solutions of (1.2). Assuming a technical Condition A which is fulfilled in examples (see Section 8.4), we shall prove that the moduli space carries a (possibly degenerate) pseudo-Kähler metric with Kähler potential (1.1) (see Theorem 8.8), a topological formula for the metric (see Theorem 8.13), and an infinitesimal Donaldson-Uhlenbeck-Yau type

Theorem (see Theorem 8.19). Interestingly, the non-degeneracy of the metric is very sensitive to the level $\ell \in \mathbb{R}$.

Main results. Throughout this section we fix a solution W of the Calabi system (1.2) on a compact form $E_{\mathbb{R}}$. Via the Chern correspondence, W determines a string algebroid Q with underlying holomorphic principal G-bundle P. In addition, W determines two cohomological quantities which play an important role in the present paper, namely, a balanced class and an 'Aeppli class of Q'

$$\mathfrak{b} := \frac{1}{(n-1)!}[e^{-\ell f_\omega}\omega^{n-1}] \in H^{n-1,n-1}_{BC}(X,\mathbb{R}), \qquad \mathfrak{a} = [E_{\mathbb{R}}] \in \Sigma_A(Q,\mathbb{R}). \quad (1.4)$$

The space $\Sigma_A(Q,\mathbb{R})$ is constructed via Bott-Chern secondary characteristic classes and is affine for a subspace of $H^{1,1}_A(X,\mathbb{R})$ (see Proposition 6.8). For the sake of clarity, we will assume throughout this introduction that G is semisimple and the following cohomological conditions are satisfied

$$h^{0,1}_A(X) = 0, \qquad h^{0,2}_{\bar\partial}(X) = 0, \qquad h^0(\operatorname{ad} P) = 0. \quad (1.5)$$

The results in the paper are hence stronger and more precise than the presented below. On the other hand, our main results assume Condition A. In a nutshell, this technical condition states that any element in the kernel of the linearization of (1.2) along the Aeppli class $\mathfrak{a}$ determines an infinitesimal automorphism of Q (see Remark 8.7). This is very natural, as it typically follows for geometric PDE with a moment map interpretation. In Proposition 8.20 we discuss a class of non-Kähler examples of solutions of (1.2) where Condition A applies, obtained via deformation of a Kähler metric.

Our main theorem relies on a gauge fixing mechanism for infinitesimal variations $(\dot\omega, \dot b, \dot a) \in \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(P_h)$ of the Calabi system (1.2), which requires Condition A (see Proposition 8.6). To state the result, we use the decomposition $\dot\omega = \dot\omega_0 + (\Lambda_\omega\dot\omega)\omega/n$ into primitive and non-primitive parts with respect to the hermitian form ω. Denote by $\mathcal{M}_\ell$ the moduli space of solutions of the Calabi system (see Section 8.1). A precise statement is given in Theorem 8.8.

Theorem 1.1. *Assume Condition A and* (1.5). *Then, the tangent space to $\mathcal{M}_\ell$ at $[W]$ inherits a pseudo-Kähler structure with (possibly degenerate) metric*

$$\begin{aligned} g_\ell(\dot\omega,\dot b,\dot a) = {} & \frac{\ell-2}{M_\ell}\int_X \langle \dot a \wedge J\dot a\rangle \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\ & + \frac{2-\ell}{2M_\ell}\int_X (|\dot\omega_0|^2 + |\dot b^{1,1}_0|^2)e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ & + \frac{2-\ell}{2M_\ell}\left(\frac{\ell}{2} - \frac{n-1}{n}\right)\int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega\dot\omega|^2)e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ & + \left(\frac{2-\ell}{2M_\ell}\right)^2\left(\left(\int_X \Lambda_\omega\dot\omega e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2 + \left(\int_X \Lambda_\omega \dot b e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2\right). \end{aligned} \quad (1.6)$$

Ignoring topological issues, the significance of our main theorem is that the 'smooth locus' of the moduli space $\mathcal{M}_\ell$ inherits a (possibly degenerate) pseudo-Kähler metric g_ℓ with Kähler potential (1.1). An interesting upshot of our formula for the moduli space metric is that along the 'bundle directions', given formally by the first line in formula (1.6), the metric is conformal to the Atiyah-Bott-Donaldson pseudo-Kähler metric on the moduli space of Hermite-Yang-Mills connections with fixed hermitian metric ω (see [7, 16, 33]). Observe that the signature depends on $\langle\,,\rangle$. The conformal factor is given up to multiplicative constants by the inverse of the ℓ-dilaton functional M_ℓ in (1.1). This statement must be handled very carefully, since the hermitian metric ω in our picture varies in a complicated way from point to point in the moduli space.

Motivated by this observation, in Theorem 8.13 we study the structure of the metric (1.6) along the fibres of a natural map from $\mathcal{M}_\ell$ to the moduli space of holomorphic principal G-bundles, proving the following formula:

$$g_\ell = \frac{2-\ell}{2M_\ell}\left(\frac{2-\ell}{2M_\ell}(\mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathfrak{b})^2 - \mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathrm{Re}\,\dot{\mathfrak{b}} + \frac{2-\ell}{2M_\ell}(\mathrm{Im}\,\dot{\mathfrak{a}}\cdot\mathfrak{b})^2 - \mathrm{Im}\,\dot{\mathfrak{a}}\cdot\mathrm{Im}\,\dot{\mathfrak{b}}\right). \tag{1.7}$$

Here, $\dot{\mathfrak{b}} \in H^{n-1,n-1}_{BC}(X)$ and $\dot{\mathfrak{a}} \in H^{1,1}_A(X)$ are 'complexified variations' of the Bott-Chern class and the Aeppli class of the solution in (1.4), obtained via gauge fixing (see Lemma 8.10). Formula (1.7) shows that the moduli space metric (1.6) is 'semi-topological', in the sense that fibre-wise it can be expressed in terms of classical cohomological quantities.

When the structure group K is trivial, X is a Kähler Calabi-Yau threefold and we take the volume form as in (1.3) and $\ell = 1$, equation (1.7) matches Strominger's formula for the special Kähler metric on the 'complexified Kähler moduli' for X [13, Eq. (4.1)]. As a consequence of our framework, this classical moduli space is recovered, along with its special Kähler metric, by pseudo-Kähler reduction in Theorem 1.1. As an application of (1.7), in Section 8.4 we show that any stable vector bundle V over X satisfying

$$c_1(V) = 0, \qquad c_2(V) = c_2(X)$$

determines a deformation of the moduli special Kähler geometry to an explicit family of pseudo-Kähler metrics (see also Example 8.21).

On a (non necessarily Kähler) Calabi-Yau threefold (X, Ω), (7.25) is equivalent to the Hull-Strominger system [31, 40] provided that $\ell = 1$ and we take μ as in (1.3). For this interesting case, the physics of string theory predicts that the fibre-wise moduli metric (1.7) should be positive definite (see Conjecture 8.14 and Appendix A.3). This way, we obtain a physical prediction relating the variations of the Aeppli classes and balanced classes of solutions.

Conjecture 1.2. *If (X, P) admits a solution of the Hull-Strominger system, then* (1.7) *is positive definite. In particular, the variations of the Aeppli and balanced classes of nearby solutions must satisfy*

$$\mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathrm{Re}\,\dot{\mathfrak{b}} < \frac{1}{2\int_X \|\Omega\|_\omega \frac{\omega^3}{6}}(\mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathfrak{b})^2. \tag{1.8}$$

Formula (1.8) provides a potential obstruction to the existence of solutions of the Hull-Strominger system around a given a solution. We expect this phenomenon to be related to some obstruction to the global existence, which goes beyond the slope stability of the bundle and the balanced property of the manifold (cf. [45]). It would be interesting to obtain a physical explanation for the inequality (1.8).

Our last result can be regarded as an infinitesimal Donaldson-Uhlenbeck-Yau type theorem, relating the moduli space of solutions of the Calabi system with a Teichmüller space for string algebroids (see Section 8.3). A precise formulation can be found in Theorem 8.19.

Theorem 1.3. *Assume Condition A and* (1.5)*. Then, the tangent to the moduli space $\mathcal{M}_\ell$ at $[W]$ is canonically isomorphic to the tangent to the Teichmüller space for string algebroids at $[Q]$.*

This strongly suggests that—if we shift our perspective and consider the Calabi system as equations for a compact form on fixed Bott-Chern algebroid Q along a fixed Aeppli class—the existence of solutions should be related to a stability condition in the sense of Geometric Invariant Theory. This was essentially the point of view taken in [22]. The precise relation with stability in our context is still unclear, as the balanced class $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$ of the solution varies in the moduli space $\mathcal{M}_\ell$. The conjectural stability condition which characterizes the existence of solutions should be related to the *integral of the moment map*, given by the dilaton functional M_ℓ. We speculate that there is a relation between this new form of stability and the conjectural inequality (1.8). The global structure of the moduli space $\mathcal{M}_\ell$ is also a mistery to us. An important insight for future studies of this structure might be provided by the moduli space metric in our Theorem 1.1.

The moduli space of solutions of the Hull-Strominger system has been an active topic of research over the last years, including a remarkable physical construction of the moduli metric in [14, 35] and a very recent symplectic intepretation of the system in the physics literature [6] (see also [3, 5, 8, 15, 23] and references therein). Our formula for the Kähler potential (8.21), with $\ell = 1$, shall be compared with [14, Eq. (1.3)], which puts forward the case $\ell = 0$. The Morita setting in Section 3 establishes an interesting parallelism with recent developments in generalized Kähler geometry [9], which were inspirational for our work. It would be interesting to pursue further links between these two frameworks in the future.

Acknowledgments: The authors would like to thank Luis Álvarez-Cónsul, Vestislav Apostolov, Jean-Michel Bismut, Marco Gualtieri, Nigel Hitchin, Fernando Marchesano, Jock McOrist, Carlos Shahbazi and Martin Ziegler for helpful conversations.

2. String algebroids and liftings

2.1. **Holomorphic string algebroids.** Let X be a complex manifold of dimension n. A holomorphic Courant algebroid $(Q, \langle\,,\rangle, [\,,], \pi)$ over X consists of a holomorphic vector bundle $Q \to X$, with sheaf of sections denoted also by

Q, together with a holomorphic non-degenerate symmetric bilinear form $\langle\,,\rangle$, a holomorphic vector bundle morphism $\pi : Q \to TX$, and a homomorphism of sheaves of $\underline{\mathbb{C}}$-vector spaces

$$[\,,]\colon Q \otimes_{\underline{\mathbb{C}}} Q \to Q,$$

satisfying natural axioms, for sections $u, v, w \in Q$ and $\phi \in \mathcal{O}_X$,

(D1): $[u,[v,w]] = [[u,v],w] + [v,[u,w]]$,
(D2): $\pi([u,v]) = [\pi(u), \pi(v)]$,
(D3): $[u, \phi v] = \pi(u)(\phi)v + \phi[u,v]$,
(D4): $\pi(u)\langle v, w\rangle = \langle [u,v], w\rangle + \langle v, [u,w]\rangle$,
(D5): $[u,v] + [v,u] = 2\pi^* d\langle u, v\rangle$.

Given a holomorphic Courant algebroid Q over X with surjective anchor map π, there is an associated holomorphic Lie algebroid

$$A_Q := Q/(\operatorname{Ker}\pi)^\perp.$$

Furthermore, the holomorphic subbundle

$$\mathrm{ad}_Q := \operatorname{Ker}\pi/(\operatorname{Ker}\pi)^\perp \subset A_Q$$

inherits the structure of a holomorphic bundle of quadratic Lie algebras.

Let G be a complex Lie group with Lie algebra $\mathfrak{g}$, and consider a bi-invariant symmetric bilinear form $\langle\,,\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. Let $p\colon P \to X$ be a holomorphic principal G-bundle over X. Consider the holomorphic Atiyah Lie algebroid $A_P := TP/G$ of P, with anchor map $dp\colon A_P \to TX$ and bracket induced by the Lie bracket on TP. The holomorphic bundle of Lie algebras $\operatorname{Ker} dp = \operatorname{ad} P \subset A_P$ fits into the short exact sequence of holomorphic Lie algebroids

$$0 \to \operatorname{ad} P \to A_P \to TX \to 0.$$

Definition 2.1. A *string algebroid* is a tuple (Q, P, ρ), where P is a holomorphic principal G-bundle over X, Q is a holomorphic Courant algebroid over X, and ρ is a bracket-preserving morphism inducing a short exact sequence

$$0 \longrightarrow T^*X \longrightarrow Q \xrightarrow{\rho} A_P \longrightarrow 0, \tag{2.1}$$

such that the induced map of holomorphic Lie algebroids $\rho\colon A_Q \to A_P$ is an isomorphism restricting to an isomorphism $\mathrm{ad}_Q \cong (\operatorname{ad} P, \langle\,,\rangle)$.

We are interested in the classification of these objects up to isomorphism, as given in the following definition.

Definition 2.2 ([24])**.** A morphism from (Q, P, ρ) to (Q', P', ρ') is a pair (φ, g), where $\varphi\colon Q \to Q'$ is a morphism of holomorphic Courant algebroids and $g\colon P \to P'$ is a homomorphism of holomorphic principal bundles covering the identity on X, such that the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^*X & \longrightarrow & Q & \xrightarrow{\rho} & A_P & \longrightarrow & 0, \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle dg} & & \\
0 & \longrightarrow & T^*X & \longrightarrow & Q' & \xrightarrow{\rho'} & A_{P'} & \longrightarrow & 0.
\end{array}$$

We say that (Q, P, ρ) is isomorphic to (Q', P', ρ') if there exists a morphism (φ, g) such that φ and g are isomorphisms.

To recall the basic classification result that we need, we introduce now some notation which will be used in the rest of the paper. Given a holomorphic principal G-bundle P over X, denote by $\mathcal{A}_P$ the space of connections θ on the underlying smooth bundle $\underline{P}$ whose curvature F_θ satisfies $F_\theta^{0,2} = 0$ and whose $(0,1)$-part induces P. Given $\theta \in \mathcal{A}_P$, the Chern-Simons three-form $CS(\theta)$ is a G-invariant complex differential form of degree three on the total space of $\underline{P}$ defined by

$$CS(\theta) = -\frac{1}{6}\langle \theta \wedge [\theta, \theta]\rangle + \langle F_\theta \wedge \theta\rangle \in \Omega^3_{\mathbb{C}}(\underline{P}),$$

which satisfies

$$dCS(\theta) = \langle F_\theta \wedge F_\theta\rangle.$$

Proposition 2.3 ([24], Prop. 2.8)**.** *The isomorphism classes of string algebroids are in one-to-one correspondence with the set*

$$H^1(\mathcal{S}) = \{(P, H, \theta) : (H, \theta) \in \Omega^{3,0} \oplus \Omega^{2,1} \times \mathcal{A}_P \mid dH + \langle F_\theta \wedge F_\theta\rangle = 0\}/\sim,$$

where $(P, H, \theta) \sim (P', H', \theta')$ *if there exists an isomorphism* $g\colon P \to P'$ *of holomorphic principal* G*-bundles and, for some* $B \in \Omega^{2,0}$*,*

$$H' = H + CS(g\theta) - CS(\theta') - d\langle g\theta \wedge \theta'\rangle + dB. \tag{2.2}$$

Remark 2.4. The notation $H^1(\mathcal{S})$ comes from the fact that the isomorphism classes can be understood as the first cohomology of a certain sheaf $\mathcal{S}$ (see [24, Sec. 3.1] for more details). Implicitly, we shall use a smooth version of this sheaf (and its first cohomology) in Proposition 2.11.

Remark 2.5. Recall that given a pair of connections θ, θ' on a smooth principal G-bundle $\underline{P}$ over X, there is an equality (see e.g. [24])

$$CS(\theta') - CS(\theta) - d\langle \theta' \wedge \theta\rangle = 2\langle a, F_\theta\rangle + \langle a, d^\theta a\rangle + \frac{1}{3}\langle a, [a, a]\rangle \in \Omega^3_{\mathbb{C}}$$

where $a = \theta' - \theta$ is a smooth 1-form with values in the adjoint bundle of $\underline{P}$. By an abuse of notation, we omit the pullback of the right-hand side to the total space of $\underline{P}$.

2.2. **Liftings.** Our next goal is to understand string algebroids in terms of smooth data. For this, we will extend the *lifting plus reduction* method introduced in [27]. Our construction can be regarded as a higher analogue of the well-known construction of holomorphic vector bundles in terms of Dolbeault operators.

Let X be a complex manifold. We denote by $\underline{X}$ the underlying smooth manifold. A smooth complex Courant algebroid $(E, \langle\,,\rangle, [\,,], \pi)$ over X consists of a smooth complex vector bundle $E \to X$ together with a smooth non-degenerate symmetric bilinear form $\langle\,,\rangle$, a smooth vector bundle morphism $\pi : E \to T\underline{X} \otimes \mathbb{C}$ and a bracket $[\,,]$ on smooth sections satisfying the same axioms (D1)-(D5) as a holomorphic Courant algebroid (see Section 2.1).

We fix the data G, $\mathfrak{g}$, $\langle\,,\rangle$ as in the previous section. Let $\underline{P}$ be a smooth principal G-bundle over X with vanishing first Pontryagin class

$$p_1(\underline{P}) = 0 \in H^4(X, \mathbb{C}).$$

We consider the Atiyah Lie algebroid $A_{\underline{P}}$, fitting into the short exact sequence of smooth complex Lie algebroids

$$0 \to \operatorname{ad} \underline{P} \to A_{\underline{P}} \to T\underline{X} \otimes \mathbb{C} \to 0, \tag{2.3}$$

where $T\underline{X} \otimes \mathbb{C}$ denotes the complexified smooth tangent bundle of X. Recall that $A_{\underline{P}}$ is defined as the quotient of the complexification of the real Atiyah algebroid of $\underline{P}$, regarded as a principal bundle with real structure group, by the ideal $\operatorname{ad} P^{0,1}$, whereby $\operatorname{ad} \underline{P} \cong \operatorname{ad} P^{1,0}$ in (2.3).

Definition 2.6. A *complex string algebroid* is a tuple $(E, \underline{P}, \rho_c)$, where $\underline{P}$ is a smooth principal G-bundle over X, the bundle E is a smooth complex Courant algebroid over X, and ρ_c is a bracket-preserving morphism inducing a short exact sequence

$$0 \longrightarrow T^*\underline{X} \otimes \mathbb{C} \longrightarrow E \xrightarrow{\rho_c} A_{\underline{P}} \longrightarrow 0,$$

such that the induced map of complex Lie algebroids $\rho_c \colon A_E \to A_{\underline{P}}$ is an isomorphism restricting to an isomorphism $\operatorname{ad}_E \cong (\operatorname{ad} \underline{P}, \langle\,,\rangle)$.

Here, the notion of morphism is analogous to Definition 2.2, and it is therefore omitted. The basic device to produce a string algebroid out of a complex string algebroid is provided by the following definition.

Definition 2.7. Given $(E, \underline{P}, \rho_c)$ a complex string algebroid, a lifting of $T^{0,1}X$ to E is an isotropic, involutive subbundle $L \subset E$ mapping isomorphically to $T^{0,1}X$ under $\pi \colon E \to T\underline{X} \otimes \mathbb{C}$.

Our next result shows how to obtain a string algebroid for any lifting $L \subset E$.

Proposition 2.8. *Let $(E, \underline{P}, \rho_c)$ be a complex string algebroid. Then, a lifting $L \subset E$ of $T^{0,1}X$ determines a string algebroid (Q_L, P_L, ρ_L), with*

$$Q_L = L^\perp / L$$

where $L^\perp$ denotes the orthogonal complement of $L \subset E$.

Proof. We will follow closely [27, App. A]. Consider the reduction of E by L, given by the orthogonal bundle $L^\perp/L$. Arguing as in [12, Thm. 3.3], we obtain that Q_L inherits the structure of a smooth Courant algebroid over X, with surjective $\mathbb{C}$-linear anchor map

$$\begin{aligned} \pi_{Q_L} \colon Q_L &\to T^{1,0}X \cong T\underline{X} \\ [e] &\mapsto \pi(e), \end{aligned}$$

for π the anchor map of E. Furthermore, Q_L has a natural structure of holomorphic vector bundle given by the Dolbeault operator

$$\bar\partial^L_V e = [s(V), \tilde e] \qquad \mod L,$$

where $V \in \Gamma(T^{0,1}X)$, $\tilde e \in \Gamma(L^\perp)$ is any lift of $[e] \in \Gamma(Q_L)$ to $L^\perp$, and $s = \pi^{-1}_{|L} \colon T^{0,1}X \to L$. By construction, this endows Q_L with a canonical

structure of holomorphic transitive Courant algebroid over X. To endow Q_L with the structure of a string algebroid, we note that the image of $\rho_L := \rho_{E|L^\perp}$ is an involutive subbundle of $A_{\underline{P}}$. This determines uniquely a G-invariant (integrable) almost complex structure on $\underline{P}$, such that $T^{1,0}P/G = \operatorname{Im}\rho_L$ and the induced map $A_{Q_L} \to T^{1,0}P/G$ is an isomorphism of holomorphic Lie algebroids. □

In the following lemma we observe that every string algebroid comes from reduction.

Lemma 2.9. *Let (Q, P, ρ) be a string algebroid.*

i) There is a structure of complex string algebroid with lifting on

$$E_Q = Q \oplus T^{0,1}X \oplus (T^{0,1}X)^*, \qquad L = T^{0,1}X,$$

such that, for any $e, q \in \Gamma(Q)$, $V, W \in \Gamma(T^{0,1}X)$, $\xi, \eta \in \Omega^{0,1}$, the anchor map, the pairing, the bracket, and the bracket-preserving map are given respectively by

$$\begin{aligned}
\pi(e + V + \xi) &:= \pi(e) + V,\\
\langle e + V + \xi, e + V + \xi\rangle &:= \langle e, e\rangle + \xi(V),\\
[e + V + \xi, q + W + \eta] &:= [e, q] + \bar\partial^Q_V q - \bar\partial^Q_W e + [V, W] + L_V\eta - i_W d\xi,\\
\rho(e + V + \xi) &:= \rho(e) + \theta^{0,1}V,
\end{aligned}$$

where $\theta^{0,1}$ denotes the partial connection on $\underline{P}$ determined by the holomorphic principal bundle P.

ii) The reduced string algebroid Q_L is canonically isomorphic to Q via the map induced by the natural projection $L^\perp = Q \oplus T^{0,1}X \to Q$.

Proof. A direct proof of *i*) follows by a laborious but straightforward check using the axioms in Definition 2.1 and it is omitted (see Remark 2.10 and Remark 2.14 below for an alternative, shorter proof). Part *ii*) follows easily from Proposition 2.8. □

Remark 2.10. The construction of E_Q in Lemma 2.9 boils down to the fact that Q forms a *matched pair* with the standard Courant structure on $T^{0,1}X \oplus (T^{0,1}X)^*$ (cf. [26, 27]).

To finish this section, we recall the classification of complex string algebroids. This will be useful for some of the calculations in Section 4. Given a smooth principal G-bundle, we denote by $\mathcal{A}_{\underline{P}}$ the space of connections on $\underline{P}$.

Proposition 2.11 ([24], App. A)**.** *The isomorphism classes of complex string algebroids are in one-to-one correspondence with the set*

$$H^1(\underline{\mathcal{S}}) = \{(\underline{P}, H_c, \theta_c) : (H_c, \theta_c) \in \Omega^3_{\mathbb{C}} \times \mathcal{A}_{\underline{P}} \mid dH_c + \langle F_{\theta_c} \wedge F_{\theta_c}\rangle = 0\}/\sim, \quad (2.4)$$

where $(\underline{P}, H_c, \theta_c) \sim (\underline{P}', H_c', \theta_c')$ if there exists an isomorphism $g\colon \underline{P} \to \underline{P}'$ of smooth principal G-bundles and (2.2) is satisfied for some $B \in \Omega^2_{\mathbb{C}}$.

2.3. **Explicit models.** We describe now concrete models for string algebroids, either in the holomorphic or smooth categories, which will be used throughout the paper. We refer to [24, Prop. 2.4] for the fact that the model in the next definition satisfies the axioms in Definition 2.1 .

Definition 2.12. For any triple (P, H, θ) as in Proposition 2.3, we denote by

$$Q_0 = T^{1,0}X \oplus \operatorname{ad} P \oplus (T^{1,0}X)^*$$

the string algebroid with Dolbeault operator

$$\bar\partial_0(V + r + \xi) = \bar\partial V + i_V F_\theta^{1,1} + \bar\partial^\theta r + \bar\partial\xi + i_V H^{2,1} + 2\rangle F_\theta^{1,1}, r\rangle,$$

non-degenerate symmetric bilinear form, or pairing,

$$\langle V + r + \xi, V + r + \xi\rangle_0 = \xi(V) + \langle r, r\rangle,$$

bracket on $\mathcal{O}_{Q_0}$,

$$\begin{aligned}[V + r + \xi, W + t + \eta]_0 = [V, W] - F_\theta^{2,0}(V, W) + \partial_V^\theta t - \partial_W^\theta r \\ + i_V\partial\eta + \partial(\eta(V)) - i_W\partial\xi + i_V i_W H^{3,0}, \\ + 2\langle\partial^\theta r, t\rangle + 2\langle i_V F_\theta^{2,0}, t\rangle - 2\langle i_W F_\theta^{2,0}, r\rangle,\end{aligned}$$

anchor map $\pi_0(V+r+\xi) = V$, and bracket-preserving map $\rho_0(V+r+\xi) = V+r$, where we use the connection θ to identify $A_P \cong T^{1,0}X \oplus \operatorname{ad} P$.

We turn next to the the case of complex string algebroids. Since this case has not been considered previously in the literature, we give a few more details of the construction. Given a triple $(\underline{P}, H_c, \theta_c)$ as in Proposition 2.11, we can associate a complex string algebroid as follows: consider the smooth complex vector bundle

$$E_0 = (T\underline{X} \otimes \mathbb{C}) \oplus \operatorname{ad}\underline{P} \oplus (T^*\underline{X} \otimes \mathbb{C})$$

with the $\mathbb{C}$-valued pairing

$$\langle V + r + \xi, V + r + \xi\rangle = \xi(V) + \langle r, r\rangle \tag{2.5}$$

and anchor map $\pi(V + r + \xi) = V$. Endowed with the bracket

$$\begin{aligned}[V + r + \xi, W + t + \eta] = [V, W] - F_{\theta_c}(V, W) + d_V^{\theta_c} t - d_W^{\theta_c} r - [r, t] \\ + L_V\eta - i_W d\xi + i_V i_W H_c \\ + 2\langle d^{\theta_c} r, t\rangle + 2\langle i_V F_{\theta_c}, t\rangle - 2\langle i_W F_{\theta_c}, r\rangle,\end{aligned} \tag{2.6}$$

the bundle E_0 becomes a smooth complex Courant algebroid (the Jacobi identity for the bracket is equivalent to the four-form equation in (2.4)). The connection θ_c gives a splitting of the Atiyah sequence (2.3), so that $A_{\underline{P}} \cong (T\underline{X} \otimes \mathbb{C}) \oplus \operatorname{ad}\underline{P}$, and in this splitting the Lie bracket on sections of $A_{\underline{P}}$ is

$$[V + r, W + t] = [V, W] - F_{\theta_c}(V, W) + d_V^{\theta_c} t - d_W^{\theta_c} r - [r, t].$$

Then, one can readily check that

$$\rho_0(V + r + \xi) = V + r \tag{2.7}$$

defines a structure of complex string algebroid $(\underline{P}, E_0, \rho)$, in the sense of Definition 2.6, where we again use θ_c to identify $A_{\underline{P}} \cong (T\underline{X} \otimes \mathbb{C}) \oplus \operatorname{ad}\underline{P}$.

Definition 2.13. For any triple $(\underline{P}, H_c, \theta_c)$ as in (2.4), we denote by

$$E_0 = (T\underline{X} \otimes \mathbb{C}) \oplus \operatorname{ad}\underline{P} \oplus (T^*\underline{X} \otimes \mathbb{C})$$

the complex string algebroid described by the pairing (2.5), the bracket (2.6), and the bracket-preserving map (2.7).

Remark 2.14. By using the explicit models Q_0 and E_0 in Definition 2.12 and Definition 2.13, combined with Propositions 2.3 and 2.11, one can obtain a short proof of Lemma 2.9.

We next obtain explicit characterizations of liftings of $T^{0,1}X$ in terms of differential forms. Given $(\gamma, \beta) \in \Omega^2_{\mathbb{C}} \oplus \Omega^1(\operatorname{ad}\underline{P})$ we can define an orthogonal automorphism (γ, β) of E_0 by (see [24])

$$(\gamma, \beta)(V + r + \xi) = V + i_V\beta + r + i_V\gamma - \langle i_V\beta, \beta\rangle - 2\langle\beta, r\rangle + \xi. \tag{2.8}$$

Lemma 2.15. *Let E_0 be the complex string algebroid determined by a triple $(\underline{P}, H_c, \theta_c)$, as in Definition 2.13. There is a one-to-one correspondence between liftings of $T^{0,1}X$ to E_0 and elements*

$$(\gamma, \beta) \in \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\operatorname{ad}\underline{P})$$

satisfying

$$\begin{aligned} \Big(H_c + d\gamma - 2\langle\beta, F_{\theta_c}\rangle - \langle\beta, d^{\theta_c}\beta\rangle - \frac{1}{3}\langle\beta, [\beta,\beta]\rangle\Big)^{1,2+0,3} &= 0, \\ F^{0,2}_{\theta_c} + \bar\partial^{\theta_c}\beta + \frac{1}{2}[\beta,\beta] &= 0. \end{aligned} \tag{2.9}$$

More precisely, given (γ, β) satisfying (2.9)*, the lifting is*

$$L = \{(-\gamma, -\beta)(V^{0,1}),\ V^{0,1} \in T^{0,1}X\}, \tag{2.10}$$

and, conversely, any lifting is uniquely expressed in this way.

Proof. An isotropic subbundle $L \subset E_0$ mapping isomorphically to $T^{0,1}X$ under π is necessarily of the form (2.10) for a suitable $(\tilde\gamma, \tilde\beta) \in \Omega^{1,1+0,2} \oplus \Omega^1(\operatorname{ad}\underline{P})$ (see [20, Sec. 3.1]). Observe that, for any $V^{0,1} \in T^{0,1}X$,

$$(-\tilde\gamma, -\tilde\beta)(V^{0,1}) = (-\gamma, -\beta)(V^{0,1}),$$

where $\beta = \tilde\beta^{0,1}$ and $\gamma = \tilde\gamma + \langle\tilde\beta^{0,1} \wedge \tilde\beta^{1,0}\rangle$, and the pair

$$(\gamma, \beta) \in \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\operatorname{ad}\underline{P})$$

is uniquely determined by L. By the proof of [23, Prop. 4.3] we have

$$(\gamma, \beta)[(-\gamma, -\beta)\cdot, (-\gamma, -\beta)\cdot]_{\theta_c, H_c} = [\cdot, \cdot]_{\theta_c+\beta, H_c'} \tag{2.11}$$

where $[\cdot, \cdot]_{H_c, \theta_c}$ denotes the Dorfman bracket (2.6) and

$$H_c' = H_c + d\gamma - 2\langle\beta, F_{\theta_c}\rangle - \langle\beta, d^{\theta_c}\beta\rangle - \frac{1}{3}\langle\beta, [\beta,\beta]\rangle. \tag{2.12}$$

Then, by formula (2.6) for the bracket, L is involutive if and only if

$$F^{0,2}_{\theta_c+\beta} = F^{0,2}_{\theta_c} + \bar\partial^{\theta_c}\beta + \frac{1}{2}[\beta,\beta] = 0, \qquad (H_c')^{1,2+0,3} = 0, \tag{2.13}$$

and the proof follows. □

We describe now the isomorphism class of the reduced string algebroid in Proposition 2.8, in terms of the explicit model in the previous lemma.

Proposition 2.16. *Let E_0 be the complex string algebroid determined by a triple $(\underline{P}, H_c, \theta_c)$, as in Definition 2.13. If $L = (-\gamma, -\beta)T^{0,1}X$, as in* (2.10)*, then the isomorphism class of (Q_L, P_L, ρ_L) is (see Proposition 2.3)*

$$[(P_L, H_c^{3,0+2,1} + \partial\gamma^{1,1} - 2\langle\beta, F_{\theta_c}^{2,0}\rangle, \theta_c + \beta)] \in H^1(\mathcal{S}), \tag{2.14}$$

where P_L denotes $\underline{P}$ endowed with the holomophic structure $\theta_c^{0,1} + \beta$.

Proof. By the second equation in (2.9) it follows that $\theta_c^{0,1}+\beta$ induces a structure of holomorphic principal bundle on $\underline{P}$, called P_L. Now, we have

$$L^\perp = \{(-\gamma, -\beta)(W + t + \eta^{1,0}) \mid \ W \in T\underline{X} \otimes \mathbb{C},\ t \in \operatorname{ad}\underline{P},\ \eta^{1,0} \in (T^{1,0}X)^*\}$$

and therefore there is a smooth bundle isomorphism

$$\begin{aligned} Q_L &\to T^{1,0}X \oplus \operatorname{ad}\underline{P} \oplus (T^{1,0}X)^* \\ [(-\gamma, -\beta)(W + t + \eta^{1,0})] &\mapsto W^{1,0} + t + \eta^{1,0}. \end{aligned} \tag{2.15}$$

Let us now express the holomorphic Courant structure in terms of (2.15). Firstly, note that (see (2.11))

$$\begin{aligned} (\gamma, \beta)[(-\gamma, -\beta)(V^{0,1}), (-\gamma, -\beta)(W^{1,0} + t + \eta^{1,0})] \\ = \bar\partial_{V^{0,1}} W^{1,0} - F_{\theta_c'}(V^{0,1}, W^{1,0}) + \bar\partial^{\theta_c'}_{V^{0,1}} t \\ + \bar\partial_{V^{0,1}} \eta^{1,0} + i_{V^{0,1}} i_{W^{1,0}} H_c' + 2\langle i_{V^{0,1}} F_{\theta_c'}, t\rangle, \end{aligned}$$

where H_c' is as in (2.12) and $\theta_c' = \theta_c + \beta$. Since L is involutive, we have $(H_c')^{1,2+0,3} = 0$ (see (2.13)), and

$$\begin{aligned} \bar\partial^L(W^{1,0} + t + \eta^{1,0}) = \bar\partial W^{1,0} + i_{W^{1,0}} F_{\theta_c'}^{1,1} + \bar\partial^{\theta_c'} t \\ + \bar\partial\eta^{1,0} + i_{W^{1,0}}(H_c'^{2,1}) + 2\langle F_{\theta_c'}^{1,1}, t\rangle. \end{aligned}$$

Therefore, using the connection θ_c' to identify

$$A_{P_L} = T^{1,0}X \oplus \operatorname{ad}\underline{P}$$

it follows that

$$\begin{aligned} \rho_L \colon Q_L &\to A_{P_L} \\ [(-\gamma, -\beta)(W + t + \eta^{1,0})] &\mapsto W^{1,0} + t \end{aligned}$$

is holomorphic, and hence Q_L is a string algebroid. To finish, arguing as for the Dolbeault operator, we notice that, in terms of (2.15), the bracket of Q_L is given by

$$\begin{aligned} [V + r + \xi, W + t + \eta] = [V, W] - F_{\theta_c'}(V, W) + \partial_V^{\theta_c'} t - \partial_W^{\theta_c'} r - [r, t] \\ + \partial(i_V\eta) + i_V\partial\eta - i_W\partial\xi + i_V i_W H_c'^{3,0} \\ + 2\langle\partial^{\theta_c'} r, t\rangle + 2\langle i_V F_{\theta_c'}^{2,0}, t\rangle - 2\langle i_W F_{\theta_c'}^{2,0}, r\rangle, \end{aligned}$$

for $V + r + \xi, W + t + \eta$ holomorphic sections of $T^{1,0}X \oplus \operatorname{ad}\underline{P} \oplus (T^{1,0}X)^*$. Then, by [24, Prop. 2.4] it follows that the isomorphism class of (Q_L, P_L, ρ_L) is (2.14), as claimed. □

3. Morita equivalence

3.1. **Definition and basic properties.** We introduce now our notion of Morita equivalence for string algebroids over a fixed complex manifold X. We follow the notation in Proposition 2.8. For simplicity, when it is clear from the context, we will denote a complex string algebroid $(E, \underline{P}, \rho_c)$ (resp. string algebroid (Q, P, ρ)) over X simply by E (resp. Q). We fix the structure group of all our principal bundles (either smooth or holomorphic) to be a complex Lie group G.

Definition 3.1. A Morita equivalence between a pair of string algebroids Q and Q' is given by a complex string algebroid E, a pair of liftings $L \subset E \supset L'$ of $T^{0,1}X$ and string algebroid isomorphisms ψ, ψ', fitting in a diagram

$$\begin{array}{ccccc} & & E & & \\ & \swarrow & & \searrow & \\ Q \xleftarrow{\psi} & Q_L & & Q_{L'} & \xrightarrow{\psi'} Q', \end{array}$$

where the discontinuous arrows refer to the partial maps

$$L^\perp \to Q_L = L^\perp/L, \qquad L'^\perp \to Q_{L'} = L'^\perp/L'.$$

The set of Morita equivalences between Q and Q' will be denoted by $\mathrm{Hom}(Q, Q')$.

Our goal in this section is to characterize when $\mathrm{Hom}(Q, Q') \neq \emptyset$ for any given pair of string algebroids. It will be helpful to study the triples (E, L, ψ), consisting of a reduction and an isomorphism, thought of as 'half of a Morita equivalence'.

Definition 3.2. Let Q be a string algebroid. A *Morita brick* for Q is a tuple (E, L, ψ), given by a complex string algebroid E, a lifting $L \subset E$ of $T^{0,1}X$ and a string algebroid isomorphism $\psi\colon Q_L \to Q$.

Remark 3.3. Actually, a Morita brick (E, L, ψ) can be seen as a Morita equivalence of Q with itself of the form (ψ, L, E, L, ψ). However, it is helpful to single out these objects in order to develop the theory.

By Lemma 2.9, there always exists a Morita brick for a given string algebroid Q, given by

$$E = E_Q, \qquad L = T^{0,1}X, \qquad \psi = \mathrm{Id}_Q. \tag{3.1}$$

Here Id_Q denotes the isomorphism $Q_L := L^\perp/L \to Q$ induced by the natural projection $L^\perp = Q \oplus T^{0,1}X \to Q$. Furthermore, as we will see shortly, this is essentially the unique Morita brick, up to the right notion of isomorphism.

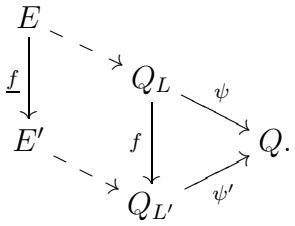
To introduce the following definition, observe that given pairs (E, L) and (E', L'), an isomorphism

$$\underline{f}\colon E \to E' \qquad \text{such that} \qquad \underline{f}(L) = L'$$

induces, upon restriction to $L^\perp$, an isomorphism $f\colon Q_L \to Q_{L'}$ and a commutative diagram

$$\begin{array}{ccc} E & \dashrightarrow & Q_L \\ {\scriptstyle \underline{f}}\downarrow & & \downarrow{\scriptstyle f} \\ E' & \dashrightarrow & Q_{L'}. \end{array} \tag{3.2}$$

Definition 3.4. We say that two Morita bricks (E, L, ψ), (E', L', ψ') for the same string algebroid Q are isomorphic if there exists an isomorphism $\underline{f} : E \to E'$ such that $\underline{f}(L) = L'$, thus inducing an isomorphism $f : Q_L \to Q_{L'}$, and $\psi = \psi' \circ f$. That is, the following diagram commutes



In the following result, we observe that there is a natural forgetful map from the set of isomorphism classes of string algebroids $H^1(\mathcal{S})$ to the set of isomorphism classes of complex string algebroids $H^1(\underline{\mathcal{S}})$ (see Proposition 2.3 and Proposition 2.11). This provides a lift of the map which sends a holomorphic principal G-bundle P to the underlying smooth principal bundle $\underline{P}$ (see Remark 3.6).

Lemma 3.5. *Using the notation in Lemma 2.9, there is a well-defined map*

$$\begin{aligned} s\colon H^1(\mathcal{S}) &\to H^1(\underline{\mathcal{S}}) \\ [Q] &\mapsto [E_Q]. \end{aligned} \tag{3.3}$$

Proof. Given an isomorphism $f\colon Q \to Q'$, we can define an induced isomorphism of complex string algebroids

$$\underline{f} := f \oplus \mathrm{Id}_{T^{0,1}X} \oplus \mathrm{Id}_{(T^{0,1}X)^*}\colon E_Q \to E_{Q'}.$$

□

Remark 3.6. Alternatively, relying on the classification in Proposition 2.3 and Proposition 2.11, we can also write (3.3) as

$$s([(P, H, \theta)]) = [(\underline{P}, H, \theta)],$$

where $\underline{P}$ denotes the smooth complex principal G-bundle underlying P.

We are now ready to prove the uniqueness of Morita bricks up to (unique) isomorphism.

Lemma 3.7. *Let Q be a string algebroid. Given a Morita brick (E, L, ψ) for Q, there exists a unique isomorphism*

$$\begin{array}{ccccc} E & & \dashrightarrow & & \\ \underline{f}\downarrow & & & Q_L & \xrightarrow{\psi} \\ E_Q & & \dashrightarrow & \downarrow f & & Q. \\ & & & Q & \xrightarrow{\mathrm{Id}_Q} \end{array} \tag{3.4}$$

to the Morita brick $(E_Q, T^{0,1}X, \mathrm{Id}_Q)$ in (3.1)*. Consequently, any Morita brick (E, L, ψ) for Q satisfies, for the map s in* (3.3)*,*

$$[E] = s([Q]) \in H^1(\underline{\mathcal{S}}).$$

Proof. The isotropic splitting $L \subset E$ gives a decomposition of E into $L^\perp$, which contains L, and $(T^{0,1}X)^* \subset E$. Combining this with $L \cong T^{0,1}X$, the definition of Q_L and ψ, we get

$$E = L^\perp + (T^{0,1}X)^* \cong Q_L + T^{0,1}X + (T^{0,1}X)^* \cong_\psi Q + T^{0,1}X + (T^{0,1}X)^* = E_Q.$$

This is an isomorphism of Courant algebroids with the Courant algebroid structure given in Lemma 2.9. This isomorphism tautologically sends L to $T^{0,1}X$, and induces a map from Q_L to Q which makes the diagram 3.4 commutative.

The uniqueness follows from the fact that the first isomorphism above is the only one that sends L to $T^{0,1}X$ via projection, and the second one is the only one satisfying $\psi = \mathrm{Id}_Q \circ f$. Finally, the last statement follows from the fact that $[E] = [E_Q] = s([Q])$, as defined in Lemma 3.5. □

With the previous result at hand, we can now decide whether there exists a Morita equivalence between any two string algebroids. The condition $\mathrm{Hom}(Q, Q') \neq \emptyset$ holds, precisely, when Q and Q' can be reduced from the same complex string algebroid (up to isomorphism).

Proposition 3.8. *Let Q and Q' be string algebroids. Then,* $\mathrm{Hom}(Q, Q') \neq \emptyset$ *if and only if*

$$s([Q]) = s([Q']) \in H^1(\underline{\mathcal{S}}). \tag{3.5}$$

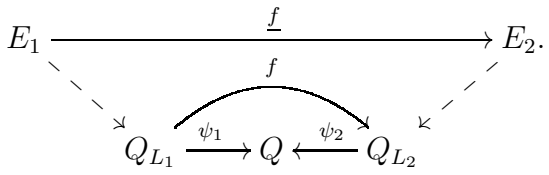
Proof. A Morita equivalence contains Morita bricks for Q and Q', so by Lemma 3.7, the existence of $\Lambda \in \mathrm{Hom}(Q, Q')$ implies (3.5). Conversely, if (3.5) holds, take Morita bricks (E_1, L_1, ψ_1) and (E_1', L', ψ_1'), which exist by Lemma 2.9, and any isomorphism $\underline{f} : E_1 \to E_1'$, the tuple

$$(\psi_1 \circ f^{-1}, \underline{f}(L_1), E_1', L', \psi_1')$$

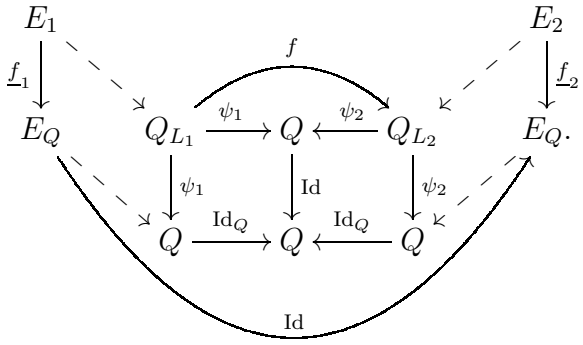
is a Morita equivalence between Q and Q'. □

3.2. **The Morita category.** Building on Lemma 3.7, we show next that a pair of Morita bricks for Q admit a unique isomorphism, which we will use for the definition of the composition of Morita equivalences.

Lemma 3.9. *Let Q be a string algebroid and a pair of Morita bricks (E_1, L_1, ψ_1) and (E_2, L_2, ψ_2) for Q. Then, there exists a unique isomorphism $\underline{f}\colon E_1 \to E_2$ such that $\underline{f}(L_1) = L_2$ making the following diagram commutative*



Proof. The statement follows as a direct consequence of Lemma 3.7, with the isomorphism given by $\underline{f}_2^{-1} \circ \underline{f}_1$ in the following diagram:



(3.6)

□

We are now ready to define the composition law on Morita equivalences.

Definition 3.10. The composition law for Morita equivalences

$$\mathrm{Hom}(Q, Q') \times \mathrm{Hom}(Q', Q'') \to \mathrm{Hom}(Q, Q'')$$

is defined by

$$(\psi_1, L_1, E_1, L_1', \psi_1') \circ (\psi_2', L_2', E_2, L_2'', \psi_2'') = (\psi_1 \circ f^{-1}, \underline{f}(L_1), E_2, L_2'', \psi_2'')$$

where $\underline{f}\colon E_1 \to E_2$ is the unique isomorphism such that $\underline{f}(L_1') = L_2'$ given in Lemma 3.9, which we call composing isomorphism, whereas $f\colon Q_{L_1} \to Q_{\underline{f}(L_1)}$ is the induced isomorphism.

In the following result we check that the composition law is associative.

Lemma 3.11. *For $\Lambda_1 \in \mathrm{Hom}(Q, Q'), \Lambda_2 \in \mathrm{Hom}(Q', Q''), \Lambda_3 \in \mathrm{Hom}(Q'', Q''')$,*

$$(\Lambda_1 \circ \Lambda_2) \circ \Lambda_3 = \Lambda_1 \circ (\Lambda_2 \circ \Lambda_3).$$

Proof. We set the notation

$$\begin{aligned} \Lambda_1 &= (\psi_1, L_1, E_1, L_1', \psi_1'), \\ \Lambda_2 &= (\psi_2', L_2', E_2, L_2'', \psi_2''), \\ \Lambda_3 &= (\psi_3'', L_3'', E_3, L_3''', \psi_3'''). \end{aligned} \tag{3.7}$$

On the one hand,

$$(\Lambda_1 \circ \Lambda_2) \circ \Lambda_3 = (\psi, \underline{f}_{23} \circ \underline{f}_{12}(L_1), E_3, L_3''', \psi_3''')$$

where $\psi = \psi_1 \circ (f_{23} \circ f_{12})^{-1}$ and

$$\underline{f}_{12} \colon E_1 \to E_2, \qquad \underline{f}_{23} \colon E_2 \to E_3,$$

are uniquely determined by

$$\underline{f}_{12}(L_1') = L_2', \qquad \underline{f}_{23}(L_2'') = L_3''$$

and the corresponding diagram (3.6). On the other hand

$$\Lambda_1 \circ (\Lambda_2 \circ \Lambda_3) = (\tilde{\psi}, \tilde{\underline{f}}_{13}(L_1), E_3, L_3''', \psi_3''')$$

where $\tilde{\psi} = \psi_1 \circ \tilde{f}_{13}^{-1}$ and

$$\underline{f}_{13} \colon E_1 \to E_3,$$

is uniquely determined by

$$\underline{f}_{13}(L_1') = \underline{f}_{23}(L_2')$$

and the diagram (3.6). Since

$$\underline{f}_{23} \circ \underline{f}_{12}(L_1') = \underline{f}_{23}(L_2') = \underline{f}_{13}(L_1'),$$

we have $\underline{f}_{23} \circ \underline{f}_{12} = \underline{f}_{13}$ and hence the composition is associative. □

Remark 3.12. A Morita equivalence of the form $(\mathrm{Id}_Q, T^{0,1}X, E_Q, T^{0,1}X, \mathrm{Id}_Q)$ provides a left identity for the product, as

$$(\mathrm{Id}_Q, T^{0,1}X, E_Q, T^{0,1}X, \mathrm{Id}_Q) \circ \Lambda_2 = (\psi_1 \circ f^{-1}, \underline{f}(L_1), E_2, L_2'', \psi_2'') = \Lambda_2 \quad (3.8)$$

where $\underline{f} \colon E_Q \to E_2$ is uniquely determined by $\underline{f}(T^{0,1}X) = L_2'$ and the condition $\mathrm{Id}_Q = \psi_2' \circ f$. However, it is not unique (we could replace E_Q by E_0, for instance) and does not provide a right inverse, which does not exist.

In order to overcome the shortcomings of Definition 3.10 mentioned in Remark 3.12, we pass to the quotient by introducing the following notion of isomorphism. Our construction shall be compared with the algebraic definition of Morita equivalence for bimodules.

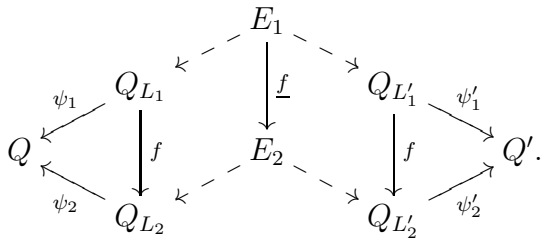
Definition 3.13. Let Q and Q' be string algebroids. Given Morita equivalences, for $j = 1, 2$,

$$\Lambda_j = (\psi_j, L_j, E_j, L_j', \psi_j') \in \mathrm{Hom}(Q, Q'),$$

an isomorphism between Λ_1 and Λ_2 is an isomorphism of complex string algebroids

$$\underline{f} \colon E_1 \to E_2,$$

such that $\underline{f}(L_1) = L_2$ and $\underline{f}(L_1') = L_2'$, inducing a commutative diagram



Whenever there exists such an isomorphism we will say that Λ_1 and Λ_2 are isomorphic, and denote it by $\Lambda_1 \cong \Lambda_2$. The set of isomorphism classes of Morita equivalences between Q and Q' will be denoted by $\mathbf{Hom}(Q, Q')$.

We are now ready to prove the main result of this section, which gives the structural properties for the composition law (3.10) after passing to the quotient. We express this in standard categorical language.

Theorem 3.14. *There exists a groupoid whose objects are string algebroids Q, and whose morphisms are isomorphism classes of Morita equivalences. In particular, the composition law*

$$\boldsymbol{Hom}(Q, Q') \times \boldsymbol{Hom}(Q', Q'') \to \boldsymbol{Hom}(Q, Q'') \tag{3.9}$$

is defined by

$$[\Lambda_1] \circ [\Lambda_2] = [\Lambda_1 \circ \Lambda_2]$$

and we have

i) for $[\Lambda_1] \in \boldsymbol{Hom}(Q, Q'), [\Lambda_2] \in \boldsymbol{Hom}(Q', Q''), [\Lambda_3] \in \boldsymbol{Hom}(Q'', Q''')$,

$$([\Lambda_1] \circ [\Lambda_2]) \circ [\Lambda_3] = [\Lambda_1] \circ ([\Lambda_2] \circ [\Lambda_3]),$$

ii) the identity morphism $\mathrm{Id}^Q \in \boldsymbol{Hom}(Q, Q)$ *is*

$$\mathrm{Id}^Q = [(\mathrm{Id}_Q, T^{0,1}X, E_Q, T^{0,1}X, \mathrm{Id}_Q)],$$

where E_Q is defined as in Lemma 2.9.

Proof. We first check that (3.9) is well defined. With the notation of (3.7), let $\Lambda_1 \circ \Lambda_2$ be defined via the composing isomorphism $\underline{f}_{12}$ from Lemma 3.9. Consider different representatives $\underline{g}\Lambda_1$ and $\underline{h}\Lambda_2$, as in Definition 3.13. The composing isomorphism for the composition $\underline{g}\Lambda_1 \circ \underline{h}\Lambda_2$ is then $\underline{h} \circ \underline{f}_{12} \circ \underline{g}^{-1}$, and $\Lambda_1 \circ \Lambda_2$ is isomorphic to $\underline{g}\Lambda_1 \circ \underline{h}\Lambda_2$ via $\underline{h}$.

Associativity, that is, *i*), follows directly from Lemma 3.11. As for *ii*), we have from (3.8) that Id^Q is a left identity. We see that Id^Q is a right identity: let $\underline{f}$ be the composing isomorphism for $\mathrm{Id}^Q \circ \Lambda_2$. We have that $\underline{f}^{-1}$ is the composing isomorphism for $\Lambda_2 \circ \mathrm{Id}^Q$, so $\Lambda_2 \circ \mathrm{Id}^Q$ is isomorphic to Λ_2 via $\underline{f}$, that is

$$[\mathrm{Id}^Q] \circ [\Lambda_2] = [\Lambda_2] \circ [\mathrm{Id}^Q] = [\Lambda_2].$$

Finally, the fact that our category is a grupoid follows from

$$[(\psi, L, E, L', \psi')]^{-1} = [(\psi', L', E, L, \psi)].$$

□

Remark 3.15. We expect that the previous result can be strengthened by proving that there exists a 2-category, whose objects are string algebroids Q, whose 1-morphisms are the Morita equivalences Λ, and whose 2-morphisms are isomorphisms of Morita equivalences, in the sense of Definition 3.13. Checking the details goes beyond the scope of the present work.

4. Hamiltonian Morita Picard

4.1. **The Morita Picard.** Let (Q, P, ρ) be a string algebroid with complex structure group G over a complex manifold X. As usual, (Q, P, ρ) will be denoted simply by Q. Let $\underline{P}$ be the smooth G-bundle underlying P, and let $\mathcal{G}_{\underline{P}}$ be the corresponding gauge group. In this section we study the *Morita Picard* of Q, that is, the group of automorphisms of Q in the groupoid constructed in Theorem 3.14,

$$\mathbf{Pic}(Q) := \mathbf{Hom}(Q, Q).$$

We start by identifying it with the automorphism group of E_Q.

Proposition 4.1. *Let E_Q be the complex string algebroid in Lemma 2.9. There is a canonical group isomorphism*

$$\begin{aligned} \varphi\colon \operatorname{Aut}(E_Q) &\to \boldsymbol{Pic}(Q) \\ \underline{f} &\mapsto [(\mathrm{Id}_Q, T^{0,1}X, E_Q, \underline{f}(T^{0,1}X), f^{-1})]. \end{aligned}$$

Proof. We use the notation $L_0 := T^{0,1}X$ for simplicity. By Theorem 3.14,

$$\begin{aligned} \varphi(\underline{f}_1) \circ \varphi(\underline{f}_2) &= [(\mathrm{Id}_Q, L_0, E_Q, \underline{f}_1(L_0), f_1^{-1})] \circ [(\mathrm{Id}_Q, L_0, E_Q, \underline{f}_2(L_0), f_2^{-1})] \\ &= [(f_1, \underline{f}_1^{-1}(L_0), E_Q, \underline{f}_2(L_0), f_2^{-1})] \\ &= [(\mathrm{Id}_Q, L_0, E_Q, (\underline{f}_1 \circ \underline{f}_2)(L_0), (f_1 f_2)^{-1})] = \varphi(\underline{f}_1 \circ \underline{f}_2), \end{aligned}$$

and hence φ is a homomorphism. If $\varphi(\underline{f}) = \mathrm{Id}^Q$ then

$$[(\mathrm{Id}_Q, L_0, E_Q, \underline{f}(L_0), f^{-1})] = [(\mathrm{Id}_Q, L_0, E_Q, L_0, \mathrm{Id}_Q)],$$

and hence $\underline{f} = \mathrm{Id}_{E_Q}$ by Lemma 3.9. Finally, given $[\Lambda] \in \mathbf{Pic}(Q)$, by Lemma 3.7 we can choose a representative of the form $(\mathrm{Id}_Q, L_0, E_Q, L, \psi)$ for suitable (L, ψ). By Lemma 3.9 there exists a unique $\underline{f} \in \operatorname{Aut}(E_Q)$ such that

$$L = \underline{f}(L_0), \qquad \psi = f^{-1},$$

and hence $\varphi(\underline{f}) = [\Lambda]$. □

We use now the previous result to obtain a structural property of the group $\mathbf{Pic}(Q)$, based on the characterization of the automorphism group of a string algebroid in [24, Prop. 2.11]. To state the result, recall from [24, App. A] that there is a group homomorphism

$$\sigma_{\underline{P}}\colon \mathcal{G}_{\underline{P}} \to H^3(X, \mathbb{C}),$$

defined by

$$\sigma_{\underline{P}}(g) = [CS(g\theta_c) - CS(\theta_c) - d\langle g\theta_c \wedge \theta_c\rangle] \in H^3(X, \mathbb{C})$$

for any choice of connection θ_c on $\underline{P}$. This defines a short exact sequence of groups (cf. [24, Prop. 2.11])

$$0 \longrightarrow \Omega^2_{\mathbb{C},cl} \longrightarrow \operatorname{Aut}(E_Q) \longrightarrow \operatorname{Ker}\sigma_{\underline{P}} \longrightarrow 1,$$

where $\Omega^2_{\mathbb{C},cl}$ the additive group of closed complex 2-forms on X. The proof of the next result is immediate from Proposition 4.1.

Corollary 4.2. *There is a canonical exact sequence*

$$0 \longrightarrow \Omega^2_{\mathbb{C},cl} \longrightarrow \boldsymbol{Pic}(Q) \longrightarrow \operatorname{Ker} \sigma_{\underline{P}} \longrightarrow \mathcal{G}_{\underline{P}} \xrightarrow{\sigma_{\underline{P}}} H^3(X,\mathbb{C}). \tag{4.1}$$

To obtain a more explicit description of $\mathbf{Pic}(Q)$, we choose a representative $[(P,H,\theta)] = [Q] \in H^1(\mathcal{S})$ and consider the model $Q_0 \cong Q$ in Definition 2.12. Then, we have an identification (see Lemma 2.9)

$$E_{Q_0} = E_0,$$

for the complex string algebroid E_0 determined by $(\underline{P}, H, \theta)$ (see Definition 2.13). Relying on [23, Cor. 4.2]–which characterizes $\operatorname{Aut}(E_0)$ in terms of differential forms (cf. [24, Lem. 2.9])–, we obtain the following result by direct application of Proposition 4.1.

Lemma 4.3. *Let Q_0 be given by (P,H,θ). There is a canonical bijection between $\boldsymbol{Pic}(Q_0)$ and the set of pairs $(g,\tau) \in \mathcal{G}_{\underline{P}} \times \Omega^2_{\mathbb{C}}$ satisfying*

$$d\tau = CS(g^{-1}\theta) - CS(\theta) - d\langle g^{-1}\theta \wedge \theta\rangle, \tag{4.2}$$

where (g,τ) acts on $V + r + \xi \in E_0$ by

$$(g,\tau)\cdot(V + r + \xi) = V + g(r + i_V a^g) + \xi + i_V B - \langle i_V a^g, a^g\rangle - 2\langle a^g, r\rangle \tag{4.3}$$

for $a^g := g^{-1}\theta - \theta$. Via this bijection, the group structure on $\boldsymbol{Pic}(Q_0)$ reads

$$(g,\tau)(g',\tau') = (gg', \tau + \tau' + \langle g'^{-1}a^g \wedge a^{g'}\rangle).$$

The following result–characterizing the Lie algebra of $\mathbf{Pic}(Q_0)$–has been stated in [23, 24] without a proof. As it is key for our development in Section 4.2, we include a detailed proof here. We follow the notation in Lemma 4.3.

Lemma 4.4. *Let Q_0 be given by (P,H,θ). There is a canonical bijection*

$$\operatorname{Lie} \boldsymbol{Pic}(Q_0) = \{(s,B) \mid d(B - 2\langle s, F_\theta\rangle) = 0\} \subset \Omega^0(\operatorname{ad}\underline{P}) \times \Omega^2_{\mathbb{C}}.$$

Via this bijection, the adjoint action of $\boldsymbol{Pic}(Q_0)$ reads

$$(g,\tau)(s,B) = (gs, B - \langle a^g \wedge [s,a^g]\rangle - 2\langle d^\theta s \wedge a^g\rangle), \tag{4.4}$$

for any $(g,\tau) \in \boldsymbol{Pic}(Q_0)$, and the Lie bracket structure is

$$[(s_0,B_0),(s_1,B_1)] = ([s_0,s_1], 2\langle d^\theta s_0 \wedge d^\theta s_1\rangle). \tag{4.5}$$

Proof. Let (g_t,τ_t) be a one-parameter family in $\mathbf{Pic}(Q_0)$ with $(g_0,\tau_0) = (\mathrm{Id}_{\underline{P}}, 0)$. Set $a_t = a^{g_t}$, and note that $(\dot{a}_t)_{|t=0} = d^\theta s$. Taking derivatives in (4.2) at $t = 0$, it follows that

$$(s,B) := (\dot{g}_t, \dot{\tau}_t)_{|t=0} \in \Omega^0(\operatorname{ad}\underline{P}) \times \Omega^2_{\mathbb{C}} \tag{4.6}$$

satisfies

$$d(B - 2\langle s, F_\theta\rangle) = 0 \tag{4.7}$$

(see Remark 2.5). Conversely, given $(s,B) \in \Omega^0(\operatorname{ad}\underline{P}) \times \Omega^2_{\mathbb{C}}$ satisfying (4.7), we define

$$(g_t,\tau_t) \in \mathcal{G}_{\underline{P}} \times \Omega^2_{\mathbb{C}}$$

by $g_t = e^{ts}$ and $\tau_t = t(B - 2\langle s, F_\theta\rangle) + \mu_t$, where

$$\mu_t = \int_0^t (2\langle s, F_{\theta_u}\rangle + \langle a_u \wedge d^{\theta_u} s\rangle) du$$

and $\theta_t = g_t^{-1}\theta$. Notice that $(\dot{\tau}_t)_{|t=0} = B$, as required. Setting

$$C_t := CS(\theta_t) - CS(\theta) - d\langle \theta_t \wedge \theta \rangle,$$

we have (see [22, Lem. 3.24])

$$\dot{C}_t = 2\langle d^{\theta_t} s, F_{\theta_t} \rangle + d\langle a_t \wedge d^{\theta_t} s \rangle = d(2\langle s, F_{\theta_t} \rangle + \langle a_t \wedge d^{\theta_t} s \rangle),$$

and therefore

$$d\dot{\tau}_t - \dot{C}_t = d\dot{\mu}_t - \dot{C}_t = 0.$$

From $\tau_0 = 0 = C_0$ it follows that $(g_t, \tau_t) \in \mathbf{Pic}(Q_0)$ for all t.

We prove next formula (4.4) for the adjoint action. For $(g_j, \tau_j) \in \mathbf{Pic}(Q_0)$, with $j = 0, 1$, denote $a_j := g_j^{-1}\theta - \theta$. Using that

$$a^{g_0 g_1} = g_1^{-1} g_0^{-1} \theta - \theta = g_1^{-1} a_0 + a_1,$$

we obtain

$$\begin{aligned}(g_0, \tau_0)(g_1, \tau_1)(g_0, \tau_0)^{-1} &= (g_0, \tau_0)(g_1, \tau_1)(g_0^{-1}, -\tau_0) \\ &= (g_0 g_1 g_0^{-1}, \tau_1 + \langle g_1^{-1} a_0 \wedge a_1 \rangle + \langle g_0 a^{g_0 g_1} \wedge a^{g_0^{-1}} \rangle) \\ &= (g_0 g_1 g_0^{-1}, \tau_1 + \langle a_0 \wedge g_1^{-1} a_0 \rangle + \langle a^{g_1^{-1}} \wedge a_0 \rangle + \langle a_0 \wedge a_1 \rangle).\end{aligned}$$

Assume now that $(g_1, \tau_1) = (g_t, \tau_t)$ is a one-parameter family of elements in $\mathbf{Pic}(Q_0)$, and define (s_1, B_1) as in (4.6). Taking derivatives in the previous expression it follows that

$$(g_0, \tau_0)(s_1, B_1) = (g_0 s_1, B_1 - \langle a_0 \wedge [s_1, a_0] \rangle - \langle d^\theta s_1 \wedge a_0 \rangle + \langle a_0 \wedge d^\theta s_1 \rangle),$$

as claimed in (4.4).

Finally, assume that $(g_0, \tau_0) = (g_t, \tau_t)$ is a one-parameter family of elements in $\mathbf{Pic}(Q_0)$, and define (s_0, B_0) as in (4.6). By taking derivatives in the last formula we have

$$[(s_0, B_0), (s_1, B_1)] = ([s_0, s_1], -2\langle d^\theta s_1 \wedge d^\theta s_0 \rangle),$$

which proves (4.5). For completeness, we check that $\operatorname{Lie} \mathbf{Pic}(Q_0)$ is closed for this bracket structure

$$\begin{aligned}d(2\langle d^\theta s_0 \wedge d^\theta s_1 \rangle) - 2\langle [s_0, s_1], F_\theta \rangle) = 2\langle [F_\theta, s_1] \wedge d^\theta s_0 \rangle - 2\langle d^\theta s_1 \wedge [F_\theta, s_0] \rangle \\ - 2\langle [d^\theta s_0, s_1], F_\theta \rangle - 2\langle [s_0, d^\theta s_1], F_\theta \rangle = 0.\end{aligned}$$

□

To finish, we observe from the first part of the proof of Lemma 4.4 that the differential of $\sigma_{\underline{P}}$ in (4.1) applied to $s \in \operatorname{Lie} \mathcal{G}_{\underline{P}} = \Omega^0(\operatorname{ad} \underline{P})$ vanishes identically,$d\sigma_{\underline{P}}(s) = -[d\langle s, F_{\theta_c} \rangle] = 0$. Therefore, at the infinitesimal level (4.1) induces a short exact sequence

$$0 \longrightarrow \Omega^2_{\mathbb{C},cl} \longrightarrow \operatorname{Lie} \mathbf{Pic}(Q) \longrightarrow \Omega^0(\operatorname{ad} \underline{P}) \longrightarrow 0. \tag{4.8}$$

4.2. **Hamiltonian automorphisms.** In this section we define a normal subgroup

$$\mathbf{Pic}_A(Q) \subset \mathbf{Pic}(Q)$$

by means of the Aeppli cohomology of the complex manifold X, which is the key to our moment map picture in Section 7. To fix ideas, we shall think of $\mathbf{Pic}(Q)$ as an analogue of the group of symplectomorphisms of a complex symplectic manifold, while the elements in $\mathbf{Pic}_A(Q)$ will play the role of complex Hamiltonian symplectomorphisms.

Consider the Aeppli cohomology groups of the complex manifold X,

$$H_A^{p,q}(X) = \frac{\operatorname{Ker}(dd^c\colon \Omega^{p,q} \to \Omega^{p+1,q+1})}{\operatorname{Im}(\partial \oplus \bar\partial\colon \Omega^{p-1,q} \oplus \Omega^{p,q-1} \to \Omega^{p,q})}. \tag{4.9}$$

Our first goal is to define a Lie algebra homomorphism

$$\mathbf{a}\colon \operatorname{Lie}\mathbf{Pic}(Q) \to H_A^{1,1}(X),$$

where the vector space $H_A^{1,1}(X)$ is regarded as an abelian Lie algebra. For this, notice that for any choice of representative $[(P,H,\theta)] = [Q] \in H^1(\mathcal{S})$ and isomorphism $Q \cong Q_0$, Lemma 4.4 implies that there is a natural map

$$\begin{aligned} \mathbf{a}_0\colon \operatorname{Lie}\mathbf{Pic}(Q_0) &\to H_A^{1,1}(X) \\ (s,B) &\mapsto [B^{1,1} - 2\langle s, F_\theta^{1,1}\rangle]. \end{aligned} \tag{4.10}$$

Lemma 4.5. *There is a canonical linear map*

$$\mathbf{a}\colon \operatorname{Lie}\boldsymbol{Pic}(Q) \to H_A^{1,1}(X), \tag{4.11}$$

which is invariant under the adjoint action of $\boldsymbol{Pic}(Q)$*. In particular,* (4.11) *is a Lie algebra homomorphism and there is a normal Lie subalgebra*

$$\operatorname{Ker}\mathbf{a} \subset \operatorname{Lie}\boldsymbol{Pic}(Q).$$

Moreover, for any choice of representative $[(P,H,\theta)] = [Q] \in H^1(\mathcal{S})$ *and isomorphism* $Q \cong Q_0$*, the induced homomorphism* $\mathbf{a}_0$ *coincides with* (4.10)*.*

Proof. By Proposition 4.1 an element $[\Lambda] \in \mathbf{Pic}(Q)$ corresponds uniquely to $\underline{f} \in \operatorname{Aut}(E_Q)$. Given an isomorphism $\psi\colon Q \to Q_0$ (for a choice of representative (P,H,θ) of $[Q] \in H^1(\mathcal{S})$), arguing as in the proof of Lemma 3.5 we obtain an isomorphism

$$\underline{\psi} := \psi \oplus \operatorname{Id}_{T^{0,1}X} \oplus \operatorname{Id}_{(T^{0,1}X)^*}\colon E_Q \to E_0$$

inducing an identification $\operatorname{Aut}(E_Q) \cong \operatorname{Aut}(E_0)$. Thus, by Lemma 4.4, an element $\zeta \in \operatorname{Lie}\mathbf{Pic}(Q)$ determines uniquely a pair $(s,B) \in \operatorname{Lie}\mathbf{Pic}(Q_0)$. Then, we define

$$\mathbf{a}(\zeta) = \mathbf{a}_0(s,B) = [B^{1,1} - 2\langle s, F_\theta^{1,1}\rangle] \in H_A^{1,1}(X).$$

To check that $\mathbf{a}$ is invariant under the adjoint $\mathbf{Pic}(Q)$-action, it is enough to check that $\mathbf{a}_0$ is invariant under the adjoint $\mathbf{Pic}(Q_0)$-action. Following Lemma 4.4, we define a closed complex two-form

$$D := B - \langle a^g \wedge [s,a^g]\rangle - 2\langle d^\theta s \wedge a^g\rangle - 2\langle gs, F_\theta\rangle,$$

so that $[D^{1,1}] = \mathbf{a}_0((g,\tau)(s,B)) \in H^{1,1}_A(X)$, and calculate

$$\begin{aligned} D &= B + \langle [a^g, a^g], s\rangle - 2d\langle s, a^g\rangle + 2\langle s, d^\theta a^g\rangle - 2\langle s, F_{g^{-1}\theta}\rangle \\ &= B - 2\langle s, F_\theta\rangle - 2d\langle s, a^g\rangle, \end{aligned} \tag{4.12}$$

which proves the invariance of $\mathbf{a}_0$. Here we have used the invariance of the pairing $\langle\,,\rangle$ combined with

$$g^{-1}F_\theta = F_{g^{-1}\theta} = F_\theta + d^\theta a^g + \frac{1}{2}[a^g, a^g].$$

To finish, we check that (4.11) is independent of choices, thus yielding a canonical map. Consider a different choice of representative (P, H', θ') of $[Q]$, corresponding to an isomorphism $\psi_0 \colon Q_0 \to Q_0'$. Explicitly, $\psi_0 = (g_0, B)$ for $g_0 \in \mathcal{G}_P$ and suitable $B \in \Omega^{2,0}$, acting as in (4.3) (we use the notation $a^{g_0} := g_0^{-1}\theta' - \theta$) (see [24, Lem. 2.7]). As before, ψ_0 extends to an isomorphism (given by the same expression)

$$\underline{\psi}_0 \colon E_0 = Q_0 \oplus T^{0,1}X \oplus (T^{0,1}X)^* \to E_0' = Q_0' \oplus T^{0,1}X \oplus (T^{0,1}X)^*,$$

under which an element $(g,\tau) \in \mathrm{Aut}(E_0) \cong \mathbf{Pic}(Q_0)$ transforms by conjugation $(g',\tau') = \underline{\psi}_0(g,\tau)\underline{\psi}_0^{-1} \in \mathrm{Aut}(E_0')$, where

$$\begin{aligned} g' &= g_0 g g_0^{-1} \\ \tau' &= \tau + \langle g^{-1}(a^{g_0}) \wedge a^g\rangle - \langle (g^{-1}g_0^{-1}\theta' - \theta) \wedge a^{g_0}\rangle, \\ &= \tau + \langle g^{-1}(a^{g_0}) \wedge a^g\rangle - \langle (g^{-1}a^{g_0} + a^g) \wedge a^{g_0}\rangle. \end{aligned}$$

From the previous formula, an element $(s,B) \in \mathrm{Lie}\,\mathbf{Pic}(Q_0)$ transforms by

$$\begin{aligned} s' &= \mathrm{Ad}(g_0)s \\ B' &= B + \langle a^{g_0} \wedge d^\theta s\rangle - \langle (-[s, a^{g_0}] + d^\theta s) \wedge a^{g_0}\rangle \\ &= B + 2\langle a^{g_0} \wedge d^\theta s\rangle + \langle [s, a^{g_0}] \wedge a^{g_0}\rangle, \end{aligned}$$

and therefore, arguing as in (4.12), we have $\mathbf{a}_0'(s', B') = \mathbf{a}_0(s, B)$. □

We are now ready to define the normal subgroup $\mathbf{Pic}_A(Q) \subset \mathbf{Pic}(Q)$. Let $\mathbf{Pic}_0(Q)$ denote the component of the identity Id_{E_Q} in $\mathbf{Pic}(Q)$ (see Proposition 4.1). Given an element $\underline{f} \in \mathbf{Pic}_0(Q)$ and a smooth family $\underline{f}_t \in \mathbf{Pic}(Q)$ such that $\underline{f}_0 = \mathrm{Id}_{E_Q}$ and $\underline{f}_1 = \underline{f}$, there exists a unique family $\zeta_t \in \mathrm{Lie}\,\mathbf{Pic}(Q)$ such that

$$\frac{d}{dt}\underline{f}_t = \zeta_t \circ \underline{f}_t.$$

Here, we regard ζ_t as a vector field on the total space of E_Q, following Proposition 4.1.

Definition 4.6. Define $\mathbf{Pic}_A(Q) \subset \mathbf{Pic}(Q)$ as the set of elements $\underline{f} \in \mathbf{Pic}_0(Q)$ such that there exists a smooth family $\underline{f}_t \in \mathbf{Pic}(Q)$ with $t \in [0,1]$, satisfying $\underline{f}_0 = \mathrm{Id}_{E_Q}$, $\underline{f}_1 = \underline{f}$, and

$$\mathbf{a}(\zeta_t) = 0, \quad \text{for all } t. \tag{4.13}$$

By analogy with symplectic geometry, a family $\underline{f}_t \in \mathbf{Pic}(Q)$ satisfying (4.13) will be called a *Hamiltonian isotopy* on $\mathbf{Pic}(Q)$. Notice that any smooth family $\zeta_t \in \operatorname{Lie}\mathbf{Pic}(Q)$ satisfying (4.13) generates a Hamiltonian isotopy.

Proposition 4.7. *The subset* $\boldsymbol{Pic}_A(Q) \subset \boldsymbol{Pic}(Q)$ *defines a normal subgroup of* $\boldsymbol{Pic}(Q)$ *with Lie algebra* $\operatorname{Ker}\mathbf{a}$.

Proof. The proof is a formality, following Lemma 4.5 and [34, Prop. 10.2]. If $\underline{f}^0_t$ is a Hamiltonian isotopy generated by ζ^0_t and $\underline{f}^1_t$ is a Hamiltonian isotopy generated by ζ^1_t, then $\underline{f}^0_t \circ \underline{f}^1_t$ is a Hamiltonian isotopy generated by

$$\zeta^0_t + \operatorname{Ad}(\underline{f}^0_t)\zeta^1_t,$$

and $(\underline{f}^0_t)^{-1}$ is a Hamiltonian isotopy generated by $-\operatorname{Ad}((\underline{f}^0_t)^{-1})\zeta^0_t$. Hence, $\mathbf{Pic}_A(Q)$ is a group. Moreover, if ζ_t generates the Hamiltonian isotopy $\underline{f}_t$, then $\operatorname{Ad}(\underline{f})\zeta_t$ generates the isotopy $\underline{f} \circ \underline{f}_t \circ \underline{f}^{-1}$ for any $\underline{f} \in \mathbf{Pic}(Q)$, and therefore $\mathbf{Pic}_A(Q)$ is a normal subgroup of $\mathbf{Pic}(Q)$. The last part of the statement follows by definition of $\mathbf{Pic}_A(Q)$. □

To finish this section, we comment on the construction of a different normal subgroup

$$\mathbf{Pic}_{dR}(Q) \subset \mathbf{Pic}(Q)$$

associated to the De Rham cohomology $H^2(X, \mathbb{C})$. Even though this alternative construction may look more natural at first sight, the Aeppli cohomology group $H^{1,1}_A(X)$ will play an important role in further developments in Section 6. To define $\mathbf{Pic}_{dR}(Q)$, one considers a Lie algebra homomorphism

$$\mathbf{d}\colon \operatorname{Lie}\mathbf{Pic}(Q) \to H^2(X, \mathbb{C}),$$

given, when choosing an isomorphism $Q \cong Q_0$, by

$$\mathbf{d}_0(s, B) = [B - 2\langle s, F_\theta\rangle] \in H^2(X, \mathbb{C}).$$

The properties of $\mathbf{d}$ are analogue to those of $\mathbf{a}$, and follow as in Lemma 4.5 using equation (4.12). The definition of $\mathbf{Pic}_{dR}(Q)$ is as in Definition 4.6 and its Lie algebra is $\operatorname{Ker}\mathbf{d}$.

Remark 4.8. By analogy with symplectic geometry, it is natural to consider a notion of *flux homomorphism* on the universal cover of $\mathbf{Pic}(Q)$ (see [34, Sec. 10.2]). We leave this interesting perspective for future work.

5. The Chern correspondence

5.1. **Background on Bott-Chern theory.** The goal of this section is to prove an analogue of the classical Chern correspondence–which establishes the existence of a unique compatible connection for any reduction to a maximal compact subgroup on a holomorphic principal bundle–in the context of string algebroids. We first recall some background about Bott-Chern theory which we will need.

Let G be a complex reductive Lie group. Let P be a holomorphic principal G-bundle over a complex manifold X. We fix a maximal compact subgroup

$K \subset G$, and a bi-invariant symmetric complex bilinear form $\langle\,,\rangle$ on the Lie algebra $\mathfrak{g}$ of G. We will assume that it satisfies the reality condition

$$\langle \mathfrak{k} \otimes \mathfrak{k} \rangle \subset \mathbb{R}$$

for the Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ of K. Given a reduction $h \in \Omega^0(P/K)$ of P to K, there is a uniquely defined Chern connection θ^h, whose curvature $F_h := F_{\theta^h}$ satisfies

$$F_h^{0,2} = F_h^{2,0} = 0.$$

We denote by $P_h \subset P$ the corresponding principal K-bundle.

The following result considers secondary characteristic classes introduced by Bott and Chern [11] (see also [10, 16]). We denote by $\Omega^{1,1}_{\mathbb{R}}$ the space of real $(1,1)$-forms on X.

Proposition 5.1 ([10, 16])**.** *For any pair of reductions $h_0, h_1 \in \Omega^0(P/K)$ there is a secondary characteristic class*

$$R(h_1, h_0) \in \Omega^{1,1}_{\mathbb{R}} / \operatorname{Im}(\partial \oplus \bar\partial) \tag{5.1}$$

with the following properties:

(1) $R(h_0, h_0) = 0$, and, for any third metric h_2,

$$R(h_2, h_0) = R(h_2, h_1) + R(h_1, h_0),$$

(2) if h varies in a one-parameter family h_t, then

$$\frac{d}{dt} R(h_t, h_0) = -2i \langle \dot h_t h_t^{-1}, F_{h_t} \rangle, \tag{5.2}$$

(3) the following identity holds

$$dd^c R(h_1, h_0) = \langle F_{h_1} \wedge F_{h_1} \rangle - \langle F_{h_0} \wedge F_{h_0} \rangle.$$

As observed by Donaldson in [16, Prop. 6], the *Bott-Chern class* (5.1) can be defined by integration of (5.2) along a path in the space of reductions of P. More precisely, given h_0 and h_1, one defines

$$\tilde R(h_1, h_0) = -2i \int_0^1 \langle \dot h_t h_t^{-1}, F_{h_t} \rangle dt \in \Omega^{1,1}_{\mathbb{R}}, \tag{5.3}$$

for a choice of path h_t joining h_0 and h_1. For a different choice of path, $\tilde R(h_1, h_0)$ differs by an element in $\operatorname{Im}(\partial \oplus \bar\partial)$, and hence there is a well-defined class $R(h_1, h_0) = [\tilde R(h_1, h_0)]$ in (5.1). Notice that (5.3) implies that

$$R(gh_1, gh_0) = R(h_1, h_0), \tag{5.4}$$

for any automorphism $g \in \mathcal{G}_P$ of P.

The other piece of information which we will need is the following technical lemma from [22]. Given a reduction $h \in \Omega^0(P/K)$, using the polar decomposition

$$G = \exp(i\mathfrak{k}) \cdot K$$

we regard h as a K-equivariant map $h \colon P \to \exp(i\mathfrak{k})$. Recall that given an element $g \in \mathcal{G}_{\underline{P}}$ regarded as an equivariant map $g \colon P \to G$, there is a well-defined covariant derivative

$$d^h g = g^* \omega^L \circ (\theta^h)^\perp \in \Omega^1(\operatorname{ad} \underline{P})$$

where ω^L is the (left-invariant) Maurer-Cartan 1-form on G and $(\theta^h)^\perp$ denotes the horizontal projection with respect to the Chern connection of h.

Lemma 5.2 ([22])**.** *Let h, h' be reductions of P. Define $\tilde{R}(h', h) \in \Omega^{1,1}_{\mathbb{R}}$ as in* (5.3)*, where $h' = e^{iu}h$, for $iu \in \Omega^0(i \operatorname{ad} P_h)$, and $h_t = e^{tiu}h$. Then,*

$$2i\partial\tilde{R}(h', h) + CS(\theta^{h'}) - CS(\theta^h) - d\langle\theta^{h'} \wedge \theta^h\rangle = dB^{2,0},$$

where

$$B^{2,0} = -\int_0^1 \langle a_t \wedge \dot{a}_t\rangle dt \in \Omega^{2,0}$$

and $a_t := \theta^h - \theta^{h_t} = -\partial^h(e^{-2tiu})$ and $\dot{a}_t = 2i\partial^{h_t}u$.

5.2. **Bott-Chern algebroids and compact forms.** Our next goal is to study a special type of string algebroids–known as Bott-Chern algebroids– which appear in the Chern correspondence. These are tight to Bott-Chern secondary characteristic classes and a notion of 'reduction to a maximal compact subgroup' for string algebroids, which we introduce next. We follow the notation in Section 5.1.

A smooth Courant algebroid $(E_{\mathbb{R}}, \langle\,,\rangle, [\,,], \pi)$ over a smooth manifold $\underline{X}$ consists of a smooth vector bundle $E_{\mathbb{R}} \to \underline{X}$ together with a non-degenerate symmetric bilinear form $\langle\,,\rangle$, a vector bundle morphism $\pi : E_{\mathbb{R}} \to TX$ and a bracket $[\,,]$ on sections satisfying the Courant algebroid axioms (see (D1)-(D5) in Section 2.1).

Definition 5.3. A *real string algebroid* with structure group K is a tuple $(P_{\mathbb{R}}, E_{\mathbb{R}}, \rho_{\mathbb{R}})$, where $P_{\mathbb{R}}$ is a smooth principal K-bundle over $\underline{X}$, $E_{\mathbb{R}}$ is a smooth (real) Courant algebroid over $\underline{X}$, and $\rho_{\mathbb{R}}$ is a bracket-preserving morphism inducing a short exact sequence

$$0 \longrightarrow T^*\underline{X} \longrightarrow E_{\mathbb{R}} \xrightarrow{\rho_{\mathbb{R}}} A_{P_{\mathbb{R}}} \longrightarrow 0,$$

such that the induced map of Lie algebroids $\rho_{\mathbb{R}} \colon A_{E_{\mathbb{R}}} \to A_{P_{\mathbb{R}}}$ is an isomorphism restricting to an isomorphism $\operatorname{ad}_{E_{\mathbb{R}}} \cong (\operatorname{ad} P_{\mathbb{R}}, \langle\,,\rangle)$.

Analogously to holomorphic and complex string algebroids, we denote by $H^1(\mathcal{S}_{\mathbb{R}})$ the set of isomorphism classes of real string algebroids on $\underline{X}$ with structure group K. By [24, Prop. A.6], elements in $H^1(\mathcal{S}_{\mathbb{R}})$ are represented by equivalence classes of triples $(P_{\mathbb{R}}, H_{\mathbb{R}}, \theta_{\mathbb{R}})$ satisfying

$$dH_{\mathbb{R}} + \langle F_{\theta_{\mathbb{R}}} \wedge F_{\theta_{\mathbb{R}}}\rangle = 0,$$

where $P_{\mathbb{R}}$ is a principal K-bundle, $H_{\mathbb{R}}$ is a real 3-form on $\underline{X}$, $\theta_{\mathbb{R}}$ is a connection on $P_{\mathbb{R}}$. The triple $(P_{\mathbb{R}}, H_{\mathbb{R}}, \theta_{\mathbb{R}})$ is related to $(P'_{\mathbb{R}}, H'_{\mathbb{R}}, \theta'_{\mathbb{R}})$ if there exists an isomorphism $g \colon P_{\mathbb{R}} \to P'_{\mathbb{R}}$ such that, for some real two-form $B \in \Omega^2$,

$$H'_{\mathbb{R}} = H_{\mathbb{R}} + CS(g\theta_{\mathbb{R}}) - CS(\theta'_{\mathbb{R}}) - d\langle g\theta_{\mathbb{R}} \wedge \theta'_{\mathbb{R}}\rangle + dB.$$

When there is no possibility of confusion, a real string algebroid $(E_{\mathbb{R}}, P_{\mathbb{R}}, \rho_{\mathbb{R}})$ will be denoted simply by $E_{\mathbb{R}}$. Given a principal K-bundle $P_{\mathbb{R}}$, we can induce uniquely a smooth principal G-bundle

$$\underline{P} = P_{\mathbb{R}} \times_K G. \tag{5.5}$$

Similarly, any real string algebroid over $\underline{X}$ induces uniquely a complex string algebroid–in the sense of Definition 2.6. The underlying principal G-bundle is $\underline{P}$ as in (5.5), the complex vector bundle is $E = E_{\mathbb{R}} \otimes \mathbb{C}$, and there is a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*\underline{X} \otimes \mathbb{C} & \longrightarrow & E & \xrightarrow{\rho_c} & A_{\underline{P}} & \longrightarrow & 0 \\ & & \cup\uparrow & & \cup\uparrow & & \uparrow & & \\ 0 & \longrightarrow & T^*\underline{X} & \longrightarrow & E_{\mathbb{R}} & \xrightarrow{\rho_{\mathbb{R}}} & A_{P_{\mathbb{R}}} & \longrightarrow & 0, \end{array} \tag{5.6}$$

where the vertical arrows are canonical, such that the $\mathbb{C}$-linear extension of the bracket, the pairing, and the morphism $\rho_{\mathbb{R}}$ in the bottom sequence induce an isomorphism (this follows by using the universal property of the Atiyah algebroid $A_{\underline{P}}$). Note that the map $A_{P_{\mathbb{R}}} \to A_{\underline{P}}$ is not set-theoretically an inclusion, but a canonical injective map (following the definition of $A_{\underline{P}}$ in (2.3)). This construction will be referred to as the 'complexification' of $E_{\mathbb{R}}$. Conversely, we have the following.

Definition 5.4. Let E be a complex string algebroid. A *compact form* of E is a real string algebroid $E_{\mathbb{R}}$ with structure group K fitting into a diagram (5.6). Compact forms will be denoted simply by $E_{\mathbb{R}} \subset E$.

Example 5.5. Let E_0 be the complex string algebroid given by $(\underline{P}, H_{\mathbb{R}}, \theta_{\mathbb{R}})$ with $H_{\mathbb{R}} \in \Omega^3 \subset \Omega^3_{\mathbb{C}}$ a real three-form and $\theta_{\mathbb{R}}$ a connection on $\underline{P}$ induced by a connection on some reduction $P_{\mathbb{R}} \subset \underline{P}$ to the maximal compact subgroup (cf. Definition 2.13). Then, the tuple $(P_{\mathbb{R}}, H_{\mathbb{R}}, \theta_{\mathbb{R}})$ defines a compact form

$$E_{0,\mathbb{R}} := T\underline{X} \oplus \operatorname{ad} P_{\mathbb{R}} \oplus T^*\underline{X} \subset E_0.$$

Let Q be a string algebroid over a complex manifold X, with underlying smooth manifold $\underline{X}$. From Lemma 2.9, Q has a canonically associated complex string algebroid E_Q.

Definition 5.6. A Bott-Chern algebroid over X is a string algebroid Q such that E_Q admits a compact form $E_{\mathbb{R}} \subset E_Q$.

We provide next a handy characterization of the notion of Bott-Chern algebroid, which recovers the definition given originally in [22]. The proof requires the Bott-Chern classes considered in Proposition 5.1 and Lemma 5.2. We denote by P the holomorphic principal G-bundle underlying Q.

Lemma 5.7. *A string algebroid Q is Bott-Chern if and only if there exists $(\omega, h) \in \Omega^{1,1}_{\mathbb{R}} \times \Omega^0(P/K)$ satisfying*

$$dd^c\omega + \langle F_h \wedge F_h \rangle = 0 \tag{5.7}$$

and $[Q] = [(P, -2i\partial\omega, \theta^h)] \in H^1(\mathcal{S})$.

Proof. Let Q be represented by a tuple $(P, -2i\partial\omega, \theta^h)$. By the equality

$$-2i\partial\omega = d^c\omega - i\partial\omega - i\bar{\partial}\omega = d^c\omega - d(i\omega) \tag{5.8}$$

combined with Proposition 2.11, we have that

$$[E_Q] = [(\underline{P}, -2i\partial\omega, \theta^h)] = [(\underline{P}, d^c\omega, \theta^h)].$$

Let ψ be an isomorphism of E_Q with a standard E_0 given by $(\underline{P}, d^c\omega, \theta^h)$. By Example 5.5, there exists a compact form $E_{0,\mathbb{R}} \subset E_0$. We then have that $\psi^{-1}(E_{0,\mathbb{R}})$ is a compact form of E_Q.

For the converse, let $E_{\mathbb{R}} \subset E_Q$ be a compact form with underlying K-bundle $P_h \subset P$. Then, the isomorphism class of $E_{\mathbb{R}}$ is represented by $(P_h, H_{\mathbb{R}}, \theta^h)$ (we can choose the connection on P_h at will by changing the real three-form accordingly). Let (P, H, θ^h) represent the class of Q (by Proposition 2.3 we can choose the connection on $\mathcal{A}_P$ at will by changing H accordingly). In $H^1(\underline{\mathcal{S}})$, we have

$$[(\underline{P}, H, \theta^h)] = [(\underline{P}, H_{\mathbb{R}}, \theta^h)] \in H^1(\underline{\mathcal{S}}).$$

Therefore, by Proposition 2.11 there exists $g \in \mathcal{G}_{\underline{P}}$ and $B' \in \Omega^2_{\mathbb{C}}$ such that

$$H_{\mathbb{R}} = H + CS(g\theta^h) - CS(\theta^h) - d\langle g\theta^h \wedge \theta^h\rangle + dB'.$$

Notice that $g\theta^{g^{-1}h}$ defines a connection on P_h. Setting

$$H'_{\mathbb{R}} = H_{\mathbb{R}} + CS(\theta^h) - CS(g\theta^{g^{-1}h}) - d\langle \theta^h \wedge g\theta^{g^{-1}h}\rangle$$

we have that

$$[(\underline{P}, H'_{\mathbb{R}}, g\theta^{g^{-1}h})] = [(\underline{P}, H_{\mathbb{R}}, \theta^h)] \in H^1(\underline{\mathcal{S}})$$

and

$$\begin{aligned} H'_{\mathbb{R}} &= H + CS(g\theta^h) - CS(g\theta^{g^{-1}h}) - d\langle g\theta^h \wedge g\theta^{g^{-1}h}\rangle + dB \\ &= H + CS(\theta^h) - CS(\theta^{g^{-1}h}) - d\langle \theta^h \wedge \theta^{g^{-1}h}\rangle + dB \end{aligned}$$

where

$$B = B' - \langle g\theta^h \wedge \theta^h\rangle - \langle \theta^h \wedge g\theta^{g^{-1}h}\rangle + \langle \theta^h \wedge \theta^{g^{-1}h}\rangle \in \Omega^2_{\mathbb{C}}.$$

By Lemma 5.2, there exists a real $(1,1)$-form $R \in \Omega^{1,1}_{\mathbb{R}}$ such that

$$H'_{\mathbb{R}} = H - 2i\partial R + dB,$$

possibly for a different choice of B. Since $H'_{\mathbb{R}}$ is real, its $(3,0)+(2,1)$-part must equal the conjugate of its $(1,2)+(0,3)$-part, so we obtain

$$H - 2i\partial R + dB^{2,0} + \partial B^{1,1} = d\overline{B^{0,2}} + \partial\overline{B^{1,1}},$$

and hence

$$H = -2i\partial(\operatorname{Im} B^{1,1} - R) + d(\overline{B^{0,2}} - B^{2,0}).$$

Therefore, $[(P, H, \theta^h)] = [(P, -2i\partial(\operatorname{Im} B^{1,1} - R), \theta^h)] \in H^1(\mathcal{S})$, as claimed. □

Observe that the complexification of real string algebroids induces a well-defined map

$$c\colon H^1(\mathcal{S}_{\mathbb{R}}) \to H^1(\underline{\mathcal{S}}).$$

Recall also that there is a forgetful map $s\colon H^1(\mathcal{S}) \to H^1(\underline{\mathcal{S}})$ (see Lemma 3.5). We shall use the notation $H^1_{BC}(\mathcal{S})$ for the set of classes of Bott-Chern algebroids inside $H^1(\mathcal{S})$. Then, by Definition 5.6,

$$s(H^1_{BC}(\mathcal{S})) \subseteq c(H^1(\mathcal{S}_{\mathbb{R}})).$$

In the next proposition we show that compact forms on a Bott-Chern algebroid are unique up to isomorphism. Consequently, we can actually define a map

$$r\colon H^1_{BC}(\mathcal{S}) \to H^1(\mathcal{S}_{\mathbb{R}})$$

such that $c \circ r = s$, which sends $[Q]$ to $[E_{\mathbb{R}}]$ for any compact form $E_{\mathbb{R}} \subset E_Q$. This can be thought of as a 'higher Chern character' for Bott-Chern algebroids (cf. Remark 5.9). Our proof will use some results from Section 6.

Proposition 5.8. *Compact forms of a Bott-Chern algebroid are unique up to isomorphism of real string algebroids. Consequently, there exists a unique map $r : H^1_{BC}(\mathcal{S}) \to H^1(\mathcal{S}_{\mathbb{R}})$ fitting into the commuting diagram*

$$\begin{array}{ccc} H^1_{BC}(\mathcal{S}) & \xrightarrow{\quad r \quad} & H^1(\mathcal{S}_{\mathbb{R}}). \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle c} & \\ & H^1(\underline{\mathcal{S}}) & \end{array}$$

Proof. By Lemma 6.5 below, given compact forms $E_{\mathbb{R}}, E'_{\mathbb{R}} \subset E_Q$ there exists $g \in \mathrm{Aut}(E_Q) \cong \mathbf{Pic}(Q)$ such that $g(E_{\mathbb{R}}) = E'_{\mathbb{R}}$. Restricted to $E_{\mathbb{R}}$, g induces an isomorphism of real string algebroids, proving the first part of the statement. By Lemma 3.9 any isomorphism $\psi\colon Q \to Q'$ induces an isomorphism of complex string algebroids $\underline{\psi}\colon E_Q \to E_{Q'}$. Using this, we can define r by $r([Q]) = [E_{\mathbb{R}}]$, for any compact form $E_{\mathbb{R}} \subset E_Q$. Uniqueness follows from the first part, which implies injectivity of c on $r(H^1(\mathcal{S}_{\mathbb{R}}))$.

□

Remark 5.9. When the holomorphic principal bundle $p\colon P \to X$ underlying Q has trivial automorphisms there is a more amenable characterization of the Bott-Chern condition using *real string classes*, in the sense of Redden [38]. To see this, notice that $\mathcal{G}_P = \{1\}$ implies that $[Q] \in H^1(\mathcal{S})$ determines uniquely a De Rham cohomology class

$$[p^*H + CS(\theta)] \in H^3(P, \mathbb{C})$$

for any choice representative $[(P, H, \theta)] = [Q]$ (see Proposition 2.3). Then, Lemma 5.7 implies that Q is Bott-Chern if and only if the pullback of $[p^*H + CS(\theta)]$ to $P_h \subset P$, for any reduction h of P, is a real string class.

5.3. **Chern correspondence for string algebroids.** We start by introducing the type of objects which play the role of the Chern connection in our context.

Definition 5.10. Let $E_{\mathbb{R}}$ be a real string algebroid over X. A *horizontal lift* of $T\underline{X}$ to $E_{\mathbb{R}}$ is given by a subbundle $W \subset E_{\mathbb{R}}$ such that

$$\mathrm{rk}\, W = \dim_{\mathbb{R}} \underline{X}, \qquad \text{and} \qquad W \cap \mathrm{Ker}\, \pi = \{0\}.$$

Following [20, Prop. 3.4], it is not difficult to see that a horizontal lift $W \subset E_{\mathbb{R}}$ is equivalent to a real symmetric 2-tensor σ on $\underline{X}$ and an isotropic splitting $\lambda\colon T\underline{X} \to E_{\mathbb{R}}$ such that

$$W = \{\lambda(V) + \sigma(V) : V \in T\underline{X}\}. \tag{5.9}$$

Recall that λ induces a connection $\theta_{\mathbb{R}}$ on $P_{\mathbb{R}}$, a three-form $H_{\mathbb{R}}$ on $\underline{X}$, and an isomorphism

$$E_{\mathbb{R}} \cong E^0_{\mathbb{R}} := T\underline{X} \oplus \mathrm{ad}\, P_{\mathbb{R}} \oplus T^*\underline{X}, \tag{5.10}$$

so that the string algebroid structure on $E^0_{\mathbb{R}}$ is as in Example 5.5.

Let E be a complex string algebroid with underlying smooth principal G-bundle $\underline{P}$. We assume that G is reductive, and fix a maximal compact subgroup $K \subset G$. Given a compact form $E_{\mathbb{R}} \subset E$ (see definition 5.4), the Cartan involution on $\mathfrak{g}$ determined by the compact Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ combined with the underlying reduction $P_{\mathbb{R}} \subset \underline{P}$ induces a well-defined involution

$$\begin{aligned} \Omega^0(\operatorname{ad} \underline{P}) &\to \Omega^0(\operatorname{ad} \underline{P}) \\ s &\mapsto s^{*_h} \end{aligned} \tag{5.11}$$

whose fixed points are given by $\Omega^0(\operatorname{ad} P_{\mathbb{R}})$.

Lemma 5.11 (Chern correspondence)**.** *Let (E, L) be a pair given by a complex string algebroid E over X and a lifting $L \subset E$ of $T^{0,1}X$. Then, any compact form $E_{\mathbb{R}} \subset E$ determines uniquely a horizontal lift $W \subset \mathrm{E}_{\mathbb{R}}$ such that*

$$L = \{e \in W \otimes \mathbb{C} \mid \pi(e) \in T^{0,1}\underline{X}\} \subset E. \tag{5.12}$$

Proof. We choose an isotropic splitting $\lambda_0 \colon T\underline{X} \to E_{\mathbb{R}}$. We will use the same notation for the $\mathbb{C}$-linear extension of λ_0 to the complexification E. Via the isomorphism (5.10) induced by λ_0, we obtain by complexification an isomorphism of complex string algebroids

$$\underline{f}_0 \colon E_0 \to E$$

inducing the identity on $A_{\underline{P}}$, and such that $\lambda_0 = (\underline{f}_0)_{|T\underline{X}}$. Then, by Lemma 2.15 the lifting L determines uniquely $(\gamma, \beta) \in \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\operatorname{ad} \underline{P})$ such that

$$L_0 := \underline{f}_0^{-1}(L) = (-\gamma, -\beta)(T^{0,1}X).$$

Furthermore, given a horizontal lift $W \subset \mathrm{E}_{\mathbb{R}}$, there exists a uniquely determined pair $(b, a) \in \Omega^2 \oplus \Omega^1(\operatorname{ad} P_{\mathbb{R}})$ and a real symmetric 2-tensor σ on $\underline{X}$ such that

$$W_0 := \underline{f}_0^{-1}(W) = (-b, -a)\{V + \sigma(V) : V \in T\underline{X}\} \subset E_0.$$

The isotropic condition for (5.12) implies that σ is a symmetric tensor of type $(1,1)$. Denote the associated hermitian form by

$$\omega = \sigma(J,) \in \Omega^{1,1}_{\mathbb{R}},$$

where J denotes the almst complex structure of X. Then, condition (5.12) implies

$$\begin{aligned} (-\gamma, -\beta)(T^{0,1}X) &= (i\omega - b, -a)(T^{0,1}X) \\ &= (i\omega - b - \langle a^{0,1} \wedge a^{1,0} \rangle, -a^{0,1})(T^{0,1}X) \end{aligned}$$

and therefore

$$-i\omega + b^{1,1+0,2} + \langle a^{0,1} \wedge a^{1,0} \rangle = \gamma, \qquad a^{0,1} = \beta.$$

From this it follows that

$$\begin{aligned} \omega &= -\operatorname{Im}\left(\gamma^{1,1} - \langle a^{0,1} \wedge a^{1,0} \rangle\right) \\ b &= \operatorname{Re}\left(\gamma^{1,1} - \langle a^{0,1} \wedge a^{1,0} \rangle\right) + \gamma^{0,2} + \overline{\gamma^{0,2}} \\ a &= \beta + \beta^*, \end{aligned} \tag{5.13}$$

where β^* is defined combining the involution (5.11) with the conjugation of complex differential forms. It is not difficult to see that (5.13) is independent of the choice of splitting λ_0. □

Remark 5.12. Similarly as in the classical Chern correspondence for principal bundles, where the integrability of the complex structure on $\underline{P}$ determined by a $(0,1)$-connection plays no role in determining the horizontal subspace of the Chern connection, the involutivity of the lifting $L \subset E$ is not required for the proof of Lemma 5.11.

Let Q be a Bott-Chern algebroid over X with underlying principal G-bundle P. Let (E, L, ψ) be a Morita brick for Q (see Definition 3.2)

$$\begin{array}{ccc} E & & \\ & \searrow & \\ & & Q_L \xrightarrow{\psi} Q. \end{array}$$

Without loss of generality, we will assume that Q and E have the same underlying smooth principal G-bundle $\underline{P}$, with complex gauge group $\mathcal{G}_{\underline{P}}$. Recall that Lemma 3.7 establishes the existence of a unique Morita brick isomorphism $\underline{f}\colon E \to E_Q$ between (E, L, ψ) and $(E_Q, T^{0,1}X, \mathrm{Id}_Q)$. By Definition 5.6, E admits a compact form $E_{\mathbb{R}} \subset E$ with structure group K. Our next result unravels the data determined by $E_{\mathbb{R}} \subset E$ in relation to the fixed Bott-Chern algebroid Q, using the Chern correspondence.

Proposition 5.13. *Let (E, L, ψ) be a Morita brick for Q. Then, any compact form $E_{\mathbb{R}} \subset E$ determines uniquely a triple (ω, h, φ), where*

(1) $\omega \in \Omega^{1,1}_{\mathbb{R}}$ and $h \in \Omega^0(P/K)$ is a reduction of P to $K \subset G$, such that

$$dd^c\omega + \langle F_{gh} \wedge F_{gh} \rangle = 0,$$

where $g \in \mathcal{G}_{\underline{P}}$ is covered by $\underline{f}\colon E \to E_Q$, the unique Morita brick isomorphism between (E, L, ψ) and $(E_Q, T^{0,1}X, \mathrm{Id}_Q)$,

(2) $\varphi\colon Q_0 \to Q$ is an isomorphism of string algebroids given by a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*X & \longrightarrow & Q_0 & \longrightarrow & A_P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & T^*X & \longrightarrow & Q & \longrightarrow & A_P & \longrightarrow & 0, \end{array} \tag{5.14}$$

where the string algebroid structure on Q_0 is given by $(P, -2i\partial\omega, \theta^{gh})$.

Furthermore, the data (ω, h, φ) recovers the flag $W \subset E_{\mathbb{R}} \subset E$, where W is the horizontal lift given by $E_{\mathbb{R}}$ via the Chern correspondence, and the three-form $H_{\mathbb{R}}$ and connection $\theta_{\mathbb{R}}$ induced by W are given by

$$H_{\mathbb{R}} = d^c\omega, \qquad \theta_{\mathbb{R}} = g^{-1}\theta^{gh}. \tag{5.15}$$

Proof. Given a compact form $E_{\mathbb{R}} \subset E$, the principal K-bundle underlying $E_{\mathbb{R}}$ induces a reduction $h \in \Omega^0(P/K)$. Furthermore, the horizontal lift W

determined by L in Lemma 5.11 is equivalent to a pair (ω, λ) where $\omega \in \Omega^{1,1}_{\mathbb{R}}$ and $\lambda\colon T\underline{X} \to E_{\mathbb{R}}$ is an isotropic splitting such that

$$W \otimes \mathbb{C} = e^{-i\omega}\lambda(T^{1,0}X) \oplus e^{i\omega}\lambda(T^{0,1}X)$$

for $L = e^{i\omega}\lambda(T^{0,1}X)$. Recall that λ induces a connection $\theta_{\mathbb{R}}$ on $P_{\mathbb{R}}$, a three-form $H_{\mathbb{R}}$ on $\underline{X}$, and an isomorphism (5.10).

Via (5.10), we obtain by complexification an isomorphism of complex string algebroids

$$\underline{f}_{\lambda}\colon E_0 \to E$$

inducing the identity on $A_{\underline{P}}$, and such that $\lambda = (\underline{f}_{\lambda})_{|T\underline{X}}$ and

$$\underline{f}_{\lambda}^{-1}(W \otimes \mathbb{C}) = e^{-i\omega}(T^{1,0}X) \oplus L_0$$

for $L_0 = \underline{f}_{\lambda}^{-1}(L) = e^{i\omega}(T^{0,1}X)$.

Hence the involutivity of L_0 combined with Lemma 2.15, yields

$$H^{1,2+0,3}_{\mathbb{R}} - i\bar{\partial}\omega = 0, \qquad F^{0,2}_{\theta_{\mathbb{R}}} = F^{2,0}_{\theta_{\mathbb{R}}} = 0.$$

Therefore, using that $H_{\mathbb{R}}$ is real, $H_{\mathbb{R}} = d^c\omega$. The reduction of E_0 by L_0, is the string algebroid $Q_0' = Q_{L_0}$ determined by the triple $(P', -2i\partial\omega, \theta_{\mathbb{R}})$ (see Proposition 2.16), for $P' = (\underline{P}, \theta^{0,1}_{\mathbb{R}})$, where we have used that

$$H^{3,0+2,1}_{\mathbb{R}} - \partial(i\omega) = -2i\partial\omega.$$

We obtain a string algebroid isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*X & \longrightarrow & Q_0' & \longrightarrow & A_{P'} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & T^*X & \longrightarrow & Q_L & \longrightarrow & A_{P'} & \longrightarrow & 0. \end{array}$$

Consider the unique Morita brick isomorphism $\underline{f}\colon E \to E_Q$ between (E, L, ψ) and $(E_Q, T^{0,1}X, \mathrm{Id}_Q)$,

covering $g \in \mathcal{G}_{\underline{P}}$. The condition that $g\colon P' \to P$ is an isomorphism implies now that

$$g\theta_{\mathbb{R}} = \theta^{gh}.$$

We use now the notation $\equiv$ to denote the algebroids, as in Definition 2.12 or Example 5.5, given by a tuple. By Proposition 2.3, g induces an isomorphism

$$\varphi_0\colon Q_0' \equiv (P', -2i\partial\omega, \theta_{\mathbb{R}}) \to Q_0 \equiv (P, -2i\partial\omega, \theta^{gh}) \tag{5.16}$$

and therefore $\varphi := \psi \circ f_\lambda \circ \varphi_0^{-1}\colon Q_0 \to Q$ has the required form (5.14), which proves (1), (2), and (5.15).

Conversely, given (ω, h, φ) as in the statement, we have a real string algebroid $E^0_{\mathbb{R}} \equiv (P_h, d^c\omega, g^{-1}\theta^{gh})$ with complexification E_0 and lifting

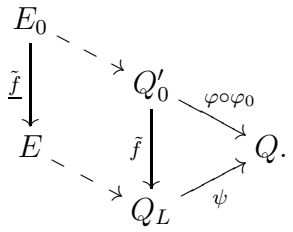
$$L_0 = e^{i\omega}T^{0,1}\underline{X}$$

such that $Q_{L_0} = Q_0' \equiv (P', -2i\partial\omega, g^{-1}\theta^{gh})$.

Consider the isomorphism

$$\varphi_0\colon Q_0' \to Q_0 \equiv (P, -2i\partial\omega, \theta^{gh})$$

induced by g as in (5.16), and the unique isomorphism $\underline{\tilde{f}}\colon E_0 \to E$ such that $\underline{\tilde{f}}(L_0) = L$, given by the diagram



Then, we define the compact form and horizontal lift by

$$E_{\mathbb{R}} := \underline{\tilde{f}}(E^0_{\mathbb{R}}), \qquad W := \underline{\tilde{f}}(\{V + \sigma(V) : V \in T\underline{X}\}),$$

where σ is the symmetric tensor determined by ω. □

Remark 5.14. The notion of *metric* on Q introduced in [22] is recovered from Proposition 5.13 by considering compact forms $E_{\mathbb{R}} \subset E_Q$ and the canonical Morita brick $(E_Q, T^{0,1}X, \mathrm{Id}_Q)$, with the additional assumption that $\omega(\,, J)$ is a Riemannian metric on X. This positivity condition will appear naturally in our moment map construction in Section 7.2.

6. Aeppli classes and Picard orbits

6.1. **$\mathbf{Pic}(Q)$-action on compact forms.** Let Q be a Bott-Chern algebroid over a complex manifold X, with underlying principal G-bundle P. We fix a maximal compact subgroup $K \subset G$. Consider the Picard group $\mathbf{Pic}(Q)$ of Q, as defined in Section 4.1. This section is devoted to the study of the interplay between $\mathbf{Pic}(Q)$ and the space of compact forms with structure group K on the complex string algebroid E_Q (see Lemma 2.9). We introduce the following notation for the space of compact forms on E_Q with structure group K

$$\mathbf{B}_Q = \{E_{\mathbb{R}} \subset E_Q \mid E_{\mathbb{R}} \text{ is a compact form}\}.$$

Via the isomorphism $\mathbf{Pic}(Q) \cong \mathrm{Aut}(E_Q)$ proved in Proposition 4.1–which will be used systematically in this section–there is a natural left action

$$\begin{aligned} \mathbf{Pic}(Q) \times \mathbf{B}_Q &\longrightarrow \mathbf{B}_Q \\ (\underline{f}, E_{\mathbb{R}}) &\longmapsto \underline{f} \cdot E_{\mathbb{R}} := \underline{f}(E_{\mathbb{R}}) \end{aligned} \tag{6.1}$$

which extends the classical action of the complex gauge group $\mathcal{G}_{\underline{P}}$ on the space of reductions $\Omega^0(P/K)$. More precisely, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{Pic}(Q) \times \mathbf{B}_Q & \longrightarrow & \mathbf{B}_Q \\ \downarrow & & \downarrow \\ \mathrm{Ker}\,\sigma_{\underline{P}} \times \Omega^0(P/K) & \longrightarrow & \Omega^0(P/K), \end{array}$$

where $\mathrm{Ker}\,\sigma_{\underline{P}} \subset \mathcal{G}_{\underline{P}}$ is the subgroup defined by (4.1) and the bottom arrow is induced by the left $\mathcal{G}_{\underline{P}}$-action on $\Omega^0(P/K)$. In order to obtain a better understanding of this action, we start by giving a more explicit description of the space $\mathbf{B}_Q$ for the case of a string algebroid given by a triple (P, H, θ) (see

Proposition 2.3 and Definition 2.12). For this, we apply Proposition 5.13 to the canonical Morita brick $(E_Q, T^{0,1}X, \mathrm{Id}_Q)$ for Q.

Lemma 6.1. *Let Q_0 be the string algebroid given by a triple (P, H, θ). Then, $\mathbf{B}_{Q_0}$ can be regarded canonically as the subset*

$$\mathbf{B}_{Q_0} \subset \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^{2,0} \times \Omega^0(P/K)$$

given by

$$\mathbf{B}_{Q_0} = \Big\{(\omega + \upsilon, h) \mid d\upsilon = H + 2i\partial\omega + CS(\theta) - CS(\theta^h) - d\langle\theta \wedge \theta^h\rangle\Big\}. \quad (6.2)$$

Proof. Let $E_{\mathbb{R}} \in \mathbf{B}_{Q_0}$ and consider the triple (ω, h, φ) corresponding to the canonical Morita brick $(E_{Q_0}, T^{0,1}X, \mathrm{Id}_{Q_0})$ for Q_0 via Proposition 5.13. Then, φ is given explicitly by

$$\varphi = (\upsilon, \theta - \theta^h) \quad (6.3)$$

acting as in (2.8), where $\upsilon \in \Omega^{2,0}$ satisfies the condition in (6.2) (see Proposition 2.3), and therefore $E_{\mathbb{R}}$ can be identified with a triple $(\omega + \upsilon, h)$ as in the statement.

Observe that the compact form and horizontal subspace corresponding to a triple $(\omega + \upsilon, h)$ are given by

$$E_{\mathbb{R}} = (\upsilon - i\omega, \theta - \theta^h)(E_{\mathbb{R},h}) \subset E_{Q_0}, \quad (6.4)$$

where

$$E_{\mathbb{R},h} := T\underline{X} \oplus \operatorname{ad} P_h \oplus T^*\underline{X}, \quad (6.5)$$

and

$$W \otimes \mathbb{C} = (\upsilon - 2i\omega, \theta - \theta^h)(T^{1,0}X) \oplus T^{0,1}X. \quad (6.6)$$

□

Our next result provides an explicit formula for the $\mathbf{Pic}(Q)$-action on $\mathbf{B}_Q$ in terms of the model in Lemma 6.1. In the sequel, we will use the notation $\boldsymbol{\omega} = (\omega + \upsilon, h)$ for the elements in $\mathbf{B}_{Q_0}$.

Lemma 6.2. *Let Q_0 be the string algebroid given by a triple (P, H, θ). Let $\underline{f} = (g, \tau) \in \boldsymbol{Pic}(Q_0)$ (see Lemma 4.3) and $\boldsymbol{\omega} = (\omega + \upsilon, h) \in \mathbf{B}_{Q_0}$. Then,*

$$\underline{f} \cdot \boldsymbol{\omega} = (\omega' + \upsilon', gh),$$

where (for $a^g = g^{-1}\theta - \theta$)

$$\begin{aligned} \omega' &= \omega - \mathrm{Im}(\tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle g\theta^h - \theta \wedge \theta^{gh} - g\theta^h\rangle)^{1,1} \\ \upsilon' &= \upsilon + (\tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle g\theta^h - \theta \wedge \theta^{gh} - g\theta^h\rangle)^{2,0} \\ &\quad - \overline{(\tau + \langle g\theta^h - \theta \wedge \theta^{gh} - g\theta^h\rangle)^{0,2}}. \end{aligned} \quad (6.7)$$

Proof. Let $\boldsymbol{\omega} = (\omega + \upsilon, h) \in \mathbf{B}_{Q_0}$ with real form (6.4). Then, for $\underline{f} = (g, \tau)$ we have

$$\begin{aligned} \underline{f}(E_{\mathbb{R}}) &= (\upsilon - i\omega + \tau + \langle a^g \wedge \theta - \theta^h\rangle, g(a^g + \theta - \theta^h))(E_{\mathbb{R},gh}) \\ &= (\upsilon - i\omega + \tau + \langle a^g \wedge \theta - \theta^h\rangle, \theta - g\theta^h)(E_{\mathbb{R},gh}). \end{aligned}$$

where $E_{\mathbb{R},gh}$ is as in (6.5). Using that $(0, g\theta^h - \theta^{gh})(E_{\mathbb{R},gh}) = E_{\mathbb{R},gh}$, we obtain

$$\begin{aligned}\underline{f}(E_{\mathbb{R}}) &= (\upsilon - i\omega + \tau + \langle a^g \wedge \theta - \theta^h\rangle, \theta - g\theta^h)(0, g\theta^h - \theta^{gh})(E_{\mathbb{R},gh}) \\ &= (\upsilon - i\omega + \tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle, \theta - \theta^{gh})(E_{\mathbb{R},gh}).\end{aligned}$$

Let $W' \subset \underline{f}(E_{\mathbb{R}})$ be the horizontal subspace determined by the canonical Morita brick $(E_{Q_0}, T^{0,1}X, \mathrm{Id}_{Q_0})$ via the Chern correspondence. Following the proof of Lemma 5.11, we set

$$(b_0, a_0) = (\upsilon - i\omega + \tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle, \theta - \theta^{gh}).$$

There exists $(b, a) \in \Omega^2 \oplus \Omega^1(\mathrm{ad}\, P_{\mathbb{R},gh})$ and a real symmetric tensor σ' with associated differential form

$$\omega' = \sigma'(J,) \in \Omega^{1,1}_{\mathbb{R}},$$

uniquely determined by the condition

$$W'_0 := (-b_0, -a_0)(W') = (-b, -a)\{V + \sigma'(V) : V \in T\underline{X}\} \subset E_{\mathbb{R},gh} \otimes \mathbb{C}.$$

Next, we define $(\gamma, \beta) \in \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\mathrm{ad}\, \underline{P})$ by

$$(-b_0, -a_0)(T^{0,1}X) = (-\gamma, -\beta)(T^{0,1}X).$$

More explicitly,

$$\gamma = b_0^{1,1+0,2}, \qquad \beta = 0,$$

and therefore (5.13) implies $a = 0$, and

$$\begin{aligned}\omega' &= -\mathrm{Im}\, b_0^{1,1} = \omega - \mathrm{Im}(\tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle)^{1,1} \\ b &= \mathrm{Re}\, b_0^{1,1} + b_0^{0,2} + \overline{b_0^{0,2}} \\ &= \mathrm{Re}\, (\tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle)^{1,1} \\ &\quad + (\tau + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle)^{0,2} + \overline{(\tau + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle)^{0,2}}.\end{aligned} \tag{6.8}$$

The first equation in (6.8) gives the formula for ω' in (6.7). To obtain the formula for υ', notice that (6.6) implies that

$$W' \otimes \mathbb{C} = (\upsilon' - 2i\omega', \theta - \theta^{gh})(T^{1,0}X) \oplus T^{0,1}X$$

and, on the other hand,

$$\begin{aligned}W' \otimes \mathbb{C} &= (b_0, a_0)(W'_0) \\ &= ((b_0 - b)^{1,1+2,0} - i\omega', \theta - \theta^{gh})(T^{1,0}X) \oplus T^{0,1}X.\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}\upsilon' &= (b_0 - b)^{2,0} \\ &= \upsilon + (\tau + \langle a^g \wedge \theta - \theta^h\rangle + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle)^{2,0} \\ &\quad - \overline{(\tau + \langle \theta - g\theta^h \wedge g\theta^h - \theta^{gh}\rangle)^{0,2}},\end{aligned}$$

as claimed. □

Using the previous lemma, we want to calculate a formula for the infinitesimal $\mathbf{Pic}(Q)$-action on $\mathbf{B}_Q$ in terms of the model Q_0. For this, we characterize next the tangent space to $\mathbf{B}_{Q_0}$. Of course, the only difficulty is to show that any solution of the infinitesimal variation of the equation in the right-hand side of (6.2) can be integrated to a curve in $\mathbf{B}_{Q_0}$.

Lemma 6.3. *The tangent space of $\mathbf{B}_{Q_0}$ at $\boldsymbol{\omega} = (\omega + \upsilon, h) \in \mathbf{B}_{Q_0}$ is given by the subspace*

$$T_{\boldsymbol{\omega}}\mathbf{B}_{Q_0} \subset \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^{2,0} \oplus \Omega^0(i \operatorname{ad} P_h).$$

defined by

$$T_{\boldsymbol{\omega}}\mathbf{B}_{Q_0} = \{(\dot\omega + \dot\upsilon, iu) \mid d(\dot\upsilon + 2i\langle \theta^h - \theta \wedge \partial^h u\rangle) = 2i\partial(\dot\omega + 2\langle u, F_h\rangle)\}. \quad (6.9)$$

Proof. Showing that the right-hand side of (6.9) is contained in $T_{\boldsymbol{\omega}}\mathbf{B}_{Q_0}$ is a formality, by taking derivatives along a curve $(\omega_t + \upsilon_t, h_t)$ in $\mathbf{B}_{Q_0}$. To see this, we define

$$C_t = CS(\theta) - CS(\theta^{h_t}) - d\langle \theta \wedge \theta^{h_t}\rangle$$

and use Remark 2.5 combined with Lemma 5.2 to calculate

$$\begin{aligned}
\frac{d}{dt}_{|t=0} C_t &= \frac{d}{dt}_{|t=0}\Big(CS(\theta^h) - CS(\theta^{h_t}) - d\langle \theta \wedge \theta^{h_t}\rangle - d\langle \theta^h \wedge \theta^{h_t}\rangle + d\langle \theta^h \wedge \theta^{h_t}\rangle\Big) \\
&= \frac{d}{dt}_{|t=0}\Big(2\langle \theta^h - \theta^{h_t} \wedge F_h\rangle + d\langle \theta^h - \theta \wedge \theta^{h_t}\rangle\Big) \\
&= 4i\partial\langle u, F_h\rangle - 2id\langle \theta^h - \theta \wedge \partial^h u\rangle.
\end{aligned}$$

Here, we have used the formula for the infinitesimal variation of the Chern connection with respect to $iu \in \Omega^0(i \operatorname{ad} P_h)$ (see Lemma 5.2):

$$\frac{d}{dt}_{|t=0} \theta^{e^{itu}h} = -2i\partial^h u \in \Omega^{1,0}(i \operatorname{ad} P_h).$$

Conversely, given $(\dot\omega + \dot\upsilon, iu)$ satisfying

$$d(\dot\upsilon + 2i\langle \theta^h - \theta \wedge \partial^h u\rangle) = 2i\partial(\dot\omega + 2\langle u, F_h\rangle)$$

we define, for $t \in \mathbb{R}$, $\boldsymbol{\omega}_t = (\omega_t + \upsilon_t, h_t)$ by

$$\begin{aligned}
h_t &= e^{itu}h \\
\omega_t &= \omega + t(\dot\omega + 2\langle u, F_h\rangle) - \tilde R(h_t, h) \\
\upsilon_t &= \upsilon + t(\dot\upsilon + 2i\langle \theta^h - \theta \wedge \partial^h u\rangle) - \int_0^t \langle \theta^{h_s} - \theta^h \wedge 2i\partial^{h_s} u\rangle ds \\
&\quad - \langle \theta \wedge \theta^{h_t}\rangle + \langle \theta \wedge \theta^h\rangle + \langle \theta^h \wedge \theta^{h_t}\rangle
\end{aligned} \quad (6.10)$$

where $\tilde R(h_t, h)$ is defined as in Lemma 5.2. We claim that $\boldsymbol{\omega}_t$ satisfies the equation in the right-hand side of (6.2) and therefore $\boldsymbol{\omega}_t \in \mathbf{B}_{Q_0}$ for all t. To see this, using Lemma 5.2, we calculate

$$\begin{aligned}
d\upsilon_t &= H + 2i\partial\omega_t + CS(\theta) - CS(\theta^h) - d\langle \theta \wedge \theta^{h_t}\rangle + d\langle \theta^h \wedge \theta^{h_t}\rangle \\
&\quad + 2i\partial\tilde R(h_t, h) - d\left(\int_0^t \langle \theta^{h_s} - \theta^h \wedge 2i\partial^{h_s} u\rangle ds\right) \\
&= H + 2i\partial\omega_t + CS(\theta) - CS(\theta^{h_t}) - d\langle \theta \wedge \theta^{h_t}\rangle.
\end{aligned}$$

Finally, using Proposition 5.1 and Lemma 5.2 again,

$$\begin{aligned}\dot\omega_t &:= \frac{d}{dt}\omega_t = \dot\omega + 2\langle u, F_h\rangle - 2\langle u, F_{h_t}\rangle \\ \dot\upsilon_t &:= \frac{d}{dt}\upsilon_t = \dot\upsilon + \langle \theta^h - \theta \wedge 2i\partial^h u\rangle - \langle \theta^{h_t} - \theta \wedge 2i\partial^{h_t} u\rangle.\end{aligned} \tag{6.11}$$

and thus the tangent vector of $\boldsymbol{\omega}_t := (\omega_t + \upsilon_t, h_t)$ at $t = 0$ is $(\dot\omega + \dot\upsilon, iu)$. □

We are ready to calculate the Lie algebra action induced by (6.1)

$$\rho\colon \operatorname{Lie}\mathbf{Pic}(Q) \to \Gamma(T\mathbf{B}_Q). \tag{6.12}$$

Recall that, for any choice of reduction $h \in \Omega^0(P/K)$, the Cartan involution induces a well-defined involution (5.11).

Lemma 6.4. *The Lie algebra action* (6.12) *is surjective. Furthermore, for any choice of representative* $[(P, H, \theta)] = [Q] \in H^1(\mathcal{S})$ *and isomorphism* $Q \cong Q_0$*, the induced action* $\rho_0\colon \operatorname{Lie}\boldsymbol{Pic}(Q_0) \to \Gamma(T\mathbf{B}_{Q_0})$ *is given by*

$$\rho_0(\zeta)_{|\boldsymbol{\omega}} = \Big(\dot\omega + \dot\upsilon, \tfrac{1}{2}(s - s^{*_h})\Big), \tag{6.13}$$

for $\zeta = (s, B) \in \operatorname{Lie}\boldsymbol{Pic}(Q_0)$ *(see Lemma 4.4) and* $\boldsymbol{\omega} = (\omega + \upsilon, h) \in \mathbf{B}_{Q_0}$*, where*

$$\begin{aligned}\dot\omega &= -\mathrm{Im}\,(B^{1,1} + 2\langle \theta^h - \theta \wedge \bar\partial s\rangle) \\ \dot\upsilon &= B^{2,0} - \overline{B^{0,2}} + \langle \theta^h - \theta \wedge \partial^h s^{*_h} + \partial^\theta s\rangle.\end{aligned} \tag{6.14}$$

Proof. We start by proving (6.13). Let $\underline{f}_t = (g_t, \tau_t) \in \mathbf{Pic}(Q_0)$ a one-parameter subgroup and let

$$(s, B) := \frac{d}{dt}_{|t=0}\underline{f}_t.$$

Then, taking derivatives in (6.7) at $t = 0$ we obtain

$$\frac{d}{dt}_{|t=0}\underline{f}_t \cdot \boldsymbol{\omega} = \Big(\dot\omega + \dot\upsilon, \tfrac{1}{2}(s - s^{*_h})\Big)$$

where

$$\begin{aligned}\dot\omega &= -\mathrm{Im}(B + \langle d^\theta s \wedge \theta - \theta^h\rangle + \langle \theta^h - \theta \wedge d^h s\rangle)^{1,1} \\ &= -\mathrm{Im}(B + 2\langle \theta^h - \theta \wedge \bar\partial s\rangle)^{1,1} \\ \dot\upsilon &= (B + \langle d^\theta s \wedge \theta - \theta^h\rangle + \langle \theta^h - \theta \wedge -\partial^h(s - s^{*_h}) + d^h s\rangle)^{2,0} \\ &\quad - \overline{B^{0,2}} - \overline{\langle \theta^h - \theta \wedge -\partial^h(s - s^{*_h}) + d^h s\rangle^{0,2}} \\ &= B^{2,0} - \overline{B^{0,2}} + \langle \theta^h - \theta \wedge \partial^\theta s + \partial^h s^{*_h}\rangle\end{aligned}$$

Finally, we prove the surjectivity of (6.13). Given $(\dot\omega + \dot\upsilon, iu) \in T_{\boldsymbol{\omega}}\mathbf{B}_{Q_0}$, taking imaginary parts in (6.9),

$$d(-\mathrm{Im}\,(\dot\upsilon + 2i\langle \theta^h - \theta \wedge \partial^h u\rangle) + \dot\omega + 2\langle u, F_h\rangle) = 0$$

and therefore

$$\zeta = (iu, -i\dot\omega + i\mathrm{Im}\,(\dot\upsilon + 2i\langle \theta^h - \theta \wedge \partial^h u\rangle) + i\langle \theta - \theta^h \wedge d^\theta u + d^h u\rangle)$$

is an element in $\operatorname{Lie}\mathbf{Pic}(Q_0)$ which satisfies $\rho_0(\zeta)\cdot\boldsymbol{\omega} = (\dot\omega + \dot\upsilon, iu)$. □

6.2. **Aeppli classes and Hamiltonian orbits.** Consider the normal subgroup $\mathbf{Pic}_A(Q) \subset \mathbf{Pic}(Q)$ defined in Proposition 4.7. By (6.1), we obtain an induced left action

$$\mathbf{Pic}_A(Q) \times \mathbf{B}_Q \longrightarrow \mathbf{B}_Q. \tag{6.15}$$

The goal of this section is to provide a cohomological interpretation of the $\mathbf{Pic}_A(Q)$-orbits on $\mathbf{B}_Q$ using Bott-Chern secondary characteristic classes. For this, we use the notion of Aeppli classes on a Bott-Chern algebroid introduced in [22]. In order to achieve our goal, we first prove a structural property of the space of compact forms $\mathbf{B}_Q$.

Lemma 6.5. *The space $\mathbf{B}_Q$ is contractible. Consequently, the $\boldsymbol{Pic}(Q)$-action (6.1) on $\mathbf{B}_Q$ is transitive.*

Proof. We work with a model $Q_0 \cong Q$ as in (6.2), and fix $\boldsymbol{\omega} = (\omega + \upsilon, h) \in \mathbf{B}_{Q_0}$. Using (6.2) and Lemma 5.2, given $\boldsymbol{\omega}' = (\omega' + \upsilon', h') \in \mathbf{B}_{Q_0}$ we have

$$\begin{aligned} d(\upsilon' - \upsilon) - d\langle \theta \wedge \theta^h \rangle + d\langle \theta \wedge \theta^{h'} \rangle &= 2i\partial(\omega' - \omega + \tilde{R}(h', h)) \\ &\quad + d\langle \theta^h \wedge \theta^{h'} \rangle + d\left(\int_0^1 \langle \theta^h - \theta^{h_t} \wedge 2i\partial^{h_t} u \rangle dt \right), \end{aligned} \tag{6.16}$$

where $h_t = e^{itu}$ for $u \in \Omega^0(\operatorname{ad} P_h)$ such that $h' = e^{iu}h$. Therefore, setting

$$\begin{aligned} \dot{\omega} &= \omega' - \omega + \tilde{R}(h', h) - 2\langle u, F_h \rangle \\ \dot{\upsilon} &= \upsilon' - \upsilon - \langle \theta \wedge \theta^h \rangle + \langle \theta \wedge \theta^{h'} \rangle - \langle \theta^h \wedge \theta^{h'} \rangle \\ &\quad - \int_0^1 \langle \theta^h - \theta^{h_t} \wedge 2i\partial^{h_t} u \rangle dt - 2i\langle \theta^h - \theta \wedge \partial^h u \rangle, \end{aligned}$$

it follows $(\dot{\omega} + \dot{\upsilon}, iu) \in T_{\boldsymbol{\omega}} \mathbf{B}_{Q_0}$ (described as in (6.9)). Consider the curve $\boldsymbol{\omega}_t \in \mathbf{B}_{Q_0}$ defined as in (6.10). Explicitly, this is given by

$$\begin{aligned} h_t &= e^{itu} h \\ \omega_t &= \omega + t(\omega' - \omega + \tilde{R}(h', h)) - \tilde{R}(h_t, h) \\ \upsilon_t &= \upsilon + t\left(\upsilon' - \upsilon - \langle \theta \wedge \theta^h \rangle + \langle \theta \wedge \theta^{h'} \rangle - \langle \theta^h \wedge \theta^{h'} \rangle - \int_0^1 \langle \theta^h - \theta^{h_s} \wedge 2i\partial^{h_s} u \rangle ds \right) \\ &\quad - \int_0^t \langle \theta^{h_s} - \theta^h \wedge 2i\partial^{h_s} u \rangle ds - \langle \theta \wedge \theta^{h_t} \rangle + \langle \theta \wedge \theta^h \rangle + \langle \theta^h \wedge \theta^{h_t} \rangle \end{aligned}$$

For $t = 1$ we have $\boldsymbol{\omega}_1 = \boldsymbol{\omega}'$, and therefore a continuous deformation retraction of $\mathbf{B}_{Q_0}$ (in C^∞-topology, say) is defined by

$$\begin{aligned} F\colon [0,1] \times \mathbf{B}_{Q_0} &\to \mathbf{B}_{Q_0} \\ (s, \boldsymbol{\omega}') &\mapsto \boldsymbol{\omega}_{1-s}. \end{aligned}$$

One can then check that this retraction is independent of our choice of model $Q_0 \cong Q$.

The last part of the statement follows from the surjectivity of the infinitesimal action (Lemma 6.4), and the contractibility of the space $\mathbf{B}_Q$ (Lemma 6.5). This is the analogous statement, and is analogously proved, to the uniqueness, up to isomorphism, of the compact form of a holomorphic principal bundle. □

We recall next the notion of Aeppli class on a Bott-Chern algebroid introduced in [22]. The proof of the following result follows from (6.2) and the properties of the Bott-Chern secondary characteristic class in Proposition 5.1 and Lemma 5.2, but we include a short proof for completeness. Observe that the Aeppli cohomology group $H^{1,1}_A(X)$ (4.9) admits a canonical real structure $H^{1,1}_A(X,\mathbb{R})$.

Lemma 6.6. *There is a well-defined map*

$$Ap\colon \mathbf{B}_Q \times \mathbf{B}_Q \to H^{1,1}_A(X,\mathbb{R}),$$

which satisfies the cocycle condition

$$Ap(\boldsymbol{\omega}_2,\boldsymbol{\omega}_0) = Ap(\boldsymbol{\omega}_2,\boldsymbol{\omega}_1) + Ap(\boldsymbol{\omega}_1,\boldsymbol{\omega}_0) \tag{6.17}$$

for any triple of elements in $\mathbf{B}_Q$. More explicitly, given $Q_0 \cong Q$ determined by a triple (P,H,θ) (see Definition 2.12),

$$Ap_0(\boldsymbol{\omega}',\boldsymbol{\omega}) = [\omega' - \omega + \tilde{R}(h',h)] \in H^{1,1}_A(X,\mathbb{R}), \tag{6.18}$$

where $\tilde{R}(h',h)$ is as in Lemma 5.2.

Proof. Given $\boldsymbol{\omega},\boldsymbol{\omega}' \in \mathbf{B}_{Q_0}$, taking imaginary parts in (6.16) we obtain

$$d\,\mathrm{Im}B = d(\omega' - \omega + \tilde{R}(h',h)),$$

where B is given by

$$B = \upsilon' - \upsilon - \langle\theta\wedge\theta^h\rangle + \langle\theta\wedge\theta^{h'}\rangle - \langle\theta^h\wedge\theta^{h'}\rangle - \int_0^1 \langle\theta^h - \theta^{h_t}\wedge 2i\partial^{h_t}u\rangle dt. \tag{6.19}$$

Here, $h_t = e^{itu}$ for $u \in \Omega^0(\mathrm{ad}\,P_h)$ such that $h' = e^{iu}h$. By type decomposition it follows that $\omega' - \omega + \tilde{R}(h',h)$ is $\partial\bar{\partial}$-closed, and hence (6.18) is well-defined. The cocycle condition (6.17) follows from Proposition 5.1. We leave as an exercise to check that Ap_0 is independent of the model $Q_0 \cong Q$. □

As a straightforward consequence of the cocycle condition (6.17), we obtain that the map Ap induces an equivalence relation in $\mathbf{B}_Q$ defined by

$$\boldsymbol{\omega} \sim_A \boldsymbol{\omega}' \quad \textit{if and only if} \quad Ap(\boldsymbol{\omega},\boldsymbol{\omega}') = 0.$$

Definition 6.7. The set of *Aeppli classes* of Q is the quotient

$$\Sigma_A(Q,\mathbb{R}) := \mathbf{B}_Q/\sim_A .$$

The set $\Sigma_A(Q,\mathbb{R})$ has a natural structure of affine space modelled on the kernel of the map

$$\partial\colon H^{1,1}_A(X,\mathbb{R}) \to H^1(\Omega^{2,0}_{cl}) \tag{6.20}$$

induced by the ∂ operator on forms, where $H^1(\Omega^{2,0}_{cl})$ denotes the first Čech cohomology of the sheaf of closed $(2,0)$-forms on X (see [22, Prop. 3.9]).

Our next result shows that the equivalence classes of elements in $\mathbf{B}_Q$ given by the Aeppli classes in Definition 6.7 correspond to $\mathbf{Pic}_A(Q)$-orbits (see (6.15)).

Proposition 6.8. *A pair of $\boldsymbol{\omega},\boldsymbol{\omega}' \in \mathbf{B}_Q$ are in the same $\boldsymbol{Pic}_A(Q)$-orbit if and only if they define the same Aeppli class*

$$[\boldsymbol{\omega}] = [\boldsymbol{\omega}'] \in \Sigma_A(Q,\mathbb{R}).$$

Proof. We fix a model $Q_0 \cong Q$ determined by a triple (P, H, θ) (see Definition 2.12). Let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbf{B}_{Q_0}$ and consider the curve $\boldsymbol{\omega}_t \in \mathbf{B}_{Q_0}$ joining $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, constructed in Lemma 6.5. Then, the Aeppli map along the curve is

$$Ap(\boldsymbol{\omega}_t, \boldsymbol{\omega}) = [t(\omega' - \omega + \tilde{R}(h', h)) - \tilde{R}(h_t, h) + \tilde{R}(h_t, h)] = tAp(\boldsymbol{\omega}', \boldsymbol{\omega}).$$

Assume first that $\boldsymbol{\omega} \sim_A \boldsymbol{\omega}'$, which implies $\boldsymbol{\omega} \sim_A \boldsymbol{\omega}_t$ for all t by the previous equation. Taking derivatives along the curve we obtain (see (6.11))

$$\begin{aligned}
\dot{\omega}_t &:= \frac{d}{dt}\omega_t = \omega' - \omega + \tilde{R}(h', h) - 2\langle u, F_{h_t}\rangle, \\
\dot{\upsilon}_t &:= \frac{d}{dt}\upsilon_t = \upsilon' - \upsilon - \langle \theta \wedge \theta^h\rangle + \langle \theta \wedge \theta^{h'}\rangle - \langle \theta^h \wedge \theta^{h'}\rangle \\
&\qquad - \int_0^1 \langle \theta^h - \theta^{h_s} \wedge 2i\partial^{h_s} u\rangle ds - \langle \theta^{h_t} - \theta \wedge 2i\partial^{h_t} u\rangle,
\end{aligned}$$

which corresponds to the infinitesimal action of

$$\zeta_t = (iu, -i\dot{\omega}_t + i\mathrm{Im}\,(\dot{\upsilon}_t + 2i\langle \theta^{h_t} - \theta \wedge \partial^{h_t} u\rangle) + i\langle \theta - \theta^{h_t} \wedge d^\theta u + d^{h_t} u\rangle).$$

Evaluating in the Lie algebra homomorphism in Lemma 4.5

$$\begin{aligned}
\mathbf{a}_0(\zeta_t) &= [-i\dot{\omega}_t + 2i\langle \theta - \theta^{h_t} \wedge \bar{\partial} u\rangle) - 2i\langle u, F_\theta\rangle] \\
&= -iAp(\boldsymbol{\omega}', \boldsymbol{\omega}) + 2i[\langle u, F_{h_t}\rangle + \langle \theta - \theta^{h_t} \wedge \bar{\partial} u\rangle) - \langle u, F_\theta\rangle] \\
&= 2i[\langle u, F_{h_t} + \bar{\partial}(\theta - \theta^{h_t}) - F_\theta\rangle] = 0,
\end{aligned}$$

where in the last equality we have used that

$$(F_{h_t} - F_\theta)^{1,1} = \bar{\partial}(\theta^{h_t} - \theta). \tag{6.21}$$

Therefore, $\zeta_t \in \mathrm{Lie}\,\mathbf{Pic}_A(Q_0)$ for all t (see Definition 4.6), which proves the 'if part' of the statement.

Conversely, assume that there exists a curve $\boldsymbol{\omega}_t = (\omega_t + \upsilon_t, h_t) \in \mathbf{B}_{Q_0}$ joining $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, and a one-parameter family of Lie algebra elements $\zeta_t = (s_t, B_t) \in \mathrm{Ker}\,\mathbf{a}_0$, such that

$$\rho_0(\zeta_t)_{|\boldsymbol{\omega}_t} := \Big(-\mathrm{Im}\,(B_t^{1,1} + 2\langle \theta^{h_t} - \theta \wedge \bar{\partial} s_t\rangle) + \tilde{\upsilon}_t, \tfrac{1}{2}(s_t - s_t^{*_{h_t}})\Big) = (\dot{\omega}_t + \dot{\upsilon}_t, \dot{h}_t h_t^{-1}),$$

for a suitable $(2,0)$-form $\tilde{\upsilon}_t$ (see (6.14)). Taking derivatives of the Aeppli map along the curve

$$\begin{aligned}
\frac{d}{dt}Ap(\boldsymbol{\omega}_t, \boldsymbol{\omega}) &= [\dot{\omega}_t - 2i\langle \dot{h}_t h_t^{-1}, F_{h_t}\rangle] \\
&= -\mathrm{Im}[B_t^{1,1} + 2\langle \theta^{h_t} - \theta \wedge \bar{\partial} s_t\rangle - 2\langle s_t, F_{h_t}\rangle] \\
&= -\mathrm{Im}[B_t^{1,1} - 2\langle s_t, F_{h_t} - \bar{\partial}(\theta^{h_t} - \theta)\rangle] \\
&= -\mathrm{Im}\ \mathbf{a}_0(\zeta_t) = 0,
\end{aligned}$$

which proves the statement. For the last equality, we have used (6.21) combined with (4.10), while the second equality follows from

$$\mathrm{Im}\ \langle s_t, F_{h_t}\rangle = -i\langle \dot{h}_t h_t^{-1}, F_{h_t}\rangle.$$

□

Remark 6.9. From the proof of Lemma 6.6, it follows that one can define a refined map $\kappa\colon \mathbf{B}_Q \times \mathbf{B}_Q \to H^2(X,\mathbb{R})$, given explicitly by

$$\kappa(\boldsymbol{\omega}', \boldsymbol{\omega}) = [\omega' - \omega + \tilde{R}(h', h) - \mathrm{Im}B] \in H^2(X,\mathbb{R})$$

for any choice of isomorphism $Q_0 \cong Q$, where B is an (6.19). We have not been able to prove that κ satisfies a cocycle condition similar to (6.17), and therefore it is unclear whether it induces an equivalence relation on $\mathbf{B}_Q$. Nonetheless, one can define a De Rham affine space for Bott-Chern algebroids, modelled on $H^2(X,\mathbb{R})$, as the set of $\mathbf{Pic}_{dR}(Q)$-orbits for the group $\mathbf{Pic}_{dR}(Q)$ in Section 4.2.

7. Moment maps

7.1. **Conformally balanced metrics and moment maps.** Let X be a compact complex manifold of dimension n. Consider the space

$$\Omega^{1,1}_{>0} \subset \Omega^{1,1}_{\mathbb{R}}$$

of positive $(1,1)$-forms on X, sitting inside the vector space of real $(1,1)$-forms $\Omega^{1,1}_{\mathbb{R}}$ as an open subspace. We will use the convention that, for $V \in TX$,

$$\omega(V, JV) > 0$$

defines a hermitian metric for any $\omega \in \Omega^{1,1}_{>0}$, where we recall that J is the almost complex structure associated to X. We use the notation (ω, b) for the elements of $T\Omega^{1,1}_{>0}$, the total space of the tangent bundle

$$T\Omega^{1,1}_{>0} \cong \Omega^{1,1}_{>0} \times \Omega^{1,1}_{\mathbb{R}},$$

and the notation $(\dot{\omega}, \dot{b})$ for elements in the tangent bundle of $T\Omega^{1,1}_{>0}$ at (ω, b). The space $T\Omega^{1,1}_{>0}$ has a natural integrable complex structure given by

$$\mathbf{J}(\dot{\omega}, \dot{b}) = (-\dot{b}, \dot{\omega}). \tag{7.1}$$

Consider the partial action of the additive group of complex two-forms

$$\begin{aligned} \Omega^2_{\mathbb{C}} \times T\Omega^{1,1}_{>0} &\to T\Omega^{1,1}_{\mathbb{R}} \\ (B, (\omega, b)) &\mapsto (\omega + \mathrm{Re}\ B^{1,1}, b + \mathrm{Im}\ B^{1,1}), \end{aligned} \tag{7.2}$$

preserving the complex structure $\mathbf{J}$. This section is devoted to the study of a Hamiltonian action of the subgroup of purely imaginary two-forms $i\Omega^2 \subset \Omega^2_{\mathbb{C}}$ for a natural family of Kähler structures on $T\Omega^{1,1}_{>0}$.

To define the family of symplectic structures of our interest, we fix a smooth volume form μ on X compatible with the complex structure. For any $\omega \in \Omega^{1,1}_{>0}$, we define a function f_ω by

$$\frac{\omega^n}{n!} = e^{2f_\omega}\mu. \tag{7.3}$$

We will call f_ω the *dilaton function* of the hermitian metric ω with respect to μ. Note that e^{-f_ω} is the point-wise norm of μ with respect to ω.

Definition 7.1. Given $\ell \in \mathbb{R}\backslash\{2\}$, the *$\ell$-dilaton functional* on $T\Omega^{1,1}_{>0}$ is

$$M_\ell(\omega, b) := \int_X e^{-\ell f_\omega}\frac{\omega^n}{n!}.$$

Associated to the functionals M_ℓ there is a family of exact $(1,1)$-forms defined by

$$\Omega_\ell := -d\mathbf{J}d\log M_\ell. \tag{7.4}$$

The following family of 1-form potentials plays a key role in the present work

$$\lambda_\ell := -\mathbf{J}d\log M_\ell = -\frac{1}{M_\ell}\mathbf{J}dM_\ell. \tag{7.5}$$

Lemma 7.2. *The forms λ_ℓ and Ω_ℓ, evaluated at the tangent vectors $(\dot\omega_j, \dot b_j)$ at the point $(\omega, b) \in T\Omega^{1,1}_{>0}$, are given by*

$$\begin{aligned}
\lambda_\ell &= \frac{\ell-2}{2M_\ell}\int_X \dot b \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!},\\
\Omega_\ell &= \frac{\ell-2}{2M_\ell}\int_X (\dot\omega_1\wedge\dot b_2 - \dot\omega_2\wedge\dot b_1)\wedge e^{-\ell f_\omega}\frac{\omega^{n-2}}{(n-2)!}\\
&\quad + \frac{\ell(\ell-2)}{4M_\ell}\int_X (\Lambda_\omega\dot b_1\Lambda_\omega\dot\omega_2 - \Lambda_\omega\dot b_2\Lambda_\omega\dot\omega_1)e^{-\ell f_\omega}\frac{\omega^n}{n!}\\
&\quad + \left(\frac{\ell-2}{2M_\ell}\right)^2\left(\int_X \Lambda_\omega(\dot\omega_1)e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)\left(\int_X \Lambda_\omega(\dot b_2)e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)\\
&\quad - \left(\frac{\ell-2}{2M_\ell}\right)^2\left(\int_X \Lambda_\omega(\dot\omega_2)e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)\left(\int_X \Lambda_\omega(\dot b_1)e^{-\ell f_\omega}\frac{\omega^n}{n!}\right).
\end{aligned} \tag{7.6}$$

Proof. Let $(\dot\omega, \dot b)$ denote a tangent vector at $(\omega, b) \in \Omega^{1,1}_{>0}$. Using that

$$M_\ell = \int_X e^{(2-\ell)f_\omega}\mu$$

it follows that

$$dM_\ell(\dot\omega, \dot b) = \frac{2-\ell}{2}\int_X \Lambda_\omega(\dot\omega)e^{-\ell f_\omega}\frac{\omega^n}{n!}$$

where we have used that $2\delta f_\omega(\dot\omega) = \Lambda_\omega\dot\omega$ by definition of f_ω. Thus, the first part of (7.6) follows from (7.1). As for the second formula, we calculate

$$\begin{aligned}
&d\mathbf{J}dM_\ell((\dot\omega_1,\dot b_1),(\dot\omega_2,\dot b_2))\\
&\quad= \frac{2-\ell}{2}\int_X (\dot b_2(-\ell(\Lambda_\omega\dot\omega_1)/2) - \dot b_1(-\ell(\Lambda_\omega\dot\omega_2)/2))\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}\\
&\qquad + \frac{2-\ell}{2}\int_X (\dot b_2\wedge\dot\omega_1 - \dot b_1\wedge\dot\omega_2)\wedge e^{-\ell f_\omega}\frac{\omega^{n-2}}{(n-2)!}\\
&\quad= \frac{\ell(2-\ell)}{4}\int_X (\Lambda_\omega\dot b_1\Lambda_\omega\dot\omega_2 - \Lambda_\omega\dot b_2\Lambda_\omega\dot\omega_1)e^{-\ell f_\omega}\frac{\omega^n}{n!}\\
&\qquad + \frac{2-\ell}{2}\int_X (\dot\omega_1\wedge\dot b_2 - \dot\omega_2\wedge\dot b_1)\wedge e^{-\ell f_\omega}\frac{\omega^{n-2}}{(n-2)!},
\end{aligned}$$

and therefore (7.6) follows from

$$\Omega_\ell = -\frac{1}{M_\ell}d\mathbf{J}dM_\ell + \frac{1}{(M_\ell)^2}dM_\ell\wedge\mathbf{J}dM_\ell.$$

□

We provide next a formula for the associated family of symmetric tensors, obtaining Kähler metrics for certain values of the parameter ℓ. Given $(\omega, b) \in T\Omega^{1,1}_{>0}$, we denote by

$$b_0 = b - \frac{\Lambda_\omega b}{n}\omega$$

the primitive part of b with respect to ω.

Lemma 7.3. *The symmetric tensor $g_\ell = \Omega_\ell(, \mathbf{J})$ at (ω, b) is given by*

$$\begin{aligned} g_\ell(\dot\omega, \dot b) = &\frac{2-\ell}{2M_\ell}\int_X (|\dot\omega_0|^2 + |\dot b_0|^2) e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ &+ \frac{2-\ell}{2M_\ell}\left(\frac{\ell}{2} - \frac{n-1}{n}\right)\int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega \dot\omega|^2) e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ &+ \left(\frac{2-\ell}{2M_\ell}\right)^2 \left(\int_X \Lambda_\omega \dot\omega e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2 + \left(\frac{2-\ell}{2M_\ell}\right)^2 \left(\int_X \Lambda_\omega \dot b e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2. \end{aligned} \tag{7.7}$$

In particular, g_ℓ is a Kähler if $2 - \frac{2}{n} < \ell < 2$ and $-g_\ell$ is Kähler if $\ell > 2$.

Proof. The proof of (7.7) is straightforward from (7.1) and (7.6). The Kähler property of $-g_\ell$ for $\ell > 2$ follows from the Cauchy-Schwarz inequality, which implies

$$\frac{1}{M_\ell}\left(\int_X \Lambda_\omega \dot b e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2 \leq \int_X |\Lambda_\omega \dot b|^2 e^{-\ell f_\omega}\frac{\omega^n}{n!}.$$

□

Consider the action of the additive group of purely imaginary two-forms induced by (7.2)

$$\begin{aligned} i\Omega^2 \times T\Omega^{1,1}_{>0} &\to T\Omega^{1,1}_{>0} \\ (iB, (\omega, b)) &\mapsto (\omega, b + B^{1,1}). \end{aligned}$$

Since the $i\Omega^2$-action preserves both $\mathbf{J}$ and M_ℓ, it also preserves the one-form λ_ℓ (see (7.5)). Thus, by (7.4), the action is Hamiltonian and there exists an equivariant moment map, which we calculate in the following result.

Proposition 7.4. *The action of $i\Omega^2$ on $T\Omega^{1,1}_{>0}$ is Hamiltonian, with equivariant moment map*

$$\langle \mu^\ell(\omega, b), B\rangle = \frac{2-\ell}{2M_\ell}\int_X B \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}. \tag{7.8}$$

Upon restriction to the subgroup $i\Omega^2_{ex} \subset i\Omega^2$ of imaginary exact 2-forms on X, zeros of the moment map are given by ℓ-conformally balanced metrics, that is,

$$d(e^{-\ell f_\omega}\omega^{n-1}) = 0.$$

Proof. The $i\Omega^2$-action is Hamiltonian, with moment map

$$\langle \mu^\ell(\omega, b), iB\rangle = -\lambda_\ell(iB \cdot (\omega, b)) = -\lambda_\ell(0, B),$$

where $iB \cdot (\omega, b) \in T_{(\omega,b)} T\Omega^{1,1}_{>0}$ denotes the infinitesimal action of iB on (ω, b). Formula (7.8) follows now from (7.6). The last part of the statement is straightforward and is left to the reader. □

To the knowledge of the authors, the previous result provides the first moment map interpretation of the conformally balanced equation in the literature. In particular, for $\ell = 0$ we obtain a symplectic interpretation of balanced metrics. Similarly, when X admits a holomorphic volume form Ω and we take

$$\mu = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega} \tag{7.9}$$

and $\ell = 1$, Proposition 7.4 characterizes hermitian metrics with holonomy for the Bismut connection contained in $\mathrm{SU}(n)$ as a moment map condition (see e.g. [22]) (cf. Corollary 7.15). Observe that for these two interesting cases we cannot ensure that the metric $\pm g_\ell$ in (7.7) is Kähler.

7.2. **Kähler reduction and the Calabi system.** Let $E_{\mathbb{R}}$ be a real string algebroid with underlying principal K-bundle $P_{\mathbb{R}}$ over our compact complex manifold X. We will assume that the bi-invariant symmetric bilinear form $\langle\, , \rangle$ on the Lie algebra $\mathfrak{k}$ of K is non-degenerate (see Section 5.1). Let G be the complexification of K. Let E be the complexification of $E_{\mathbb{R}}$, with underlying principal G-bundle $\underline{P} = P_{\mathbb{R}} \times_K G$. Given a horizontal lift $W \subset E_{\mathbb{R}}$ of $T\underline{X}$ to $E_{\mathbb{R}}$ (see Definition 5.10) we define

$$L_W := \{e \in W \otimes \mathbb{C} \mid \pi(e) \in T^{0,1}\underline{X}\} \subset E.$$

Consider the set of horizontal lifts of $T\underline{X}$ to $E_{\mathbb{R}}$ such that L_W is isotropic

$$\mathcal{W} := \{W \subset E_{\mathbb{R}} \mid W \text{ is a horizontal lift and } L_W \text{ is isotropic}\}.$$

Recall from Section 5.3 that any $W \in \mathcal{W}$ induces the following data: a real $(1,1)$-form $\omega \in \Omega^{1,1}_{\mathbb{R}}$ on $\underline{X}$, a three-form $H_{\mathbb{R}}$, a connection $\theta_{\mathbb{R}}$ on $P_{\mathbb{R}}$, and an isomorphism $E_{\mathbb{R}} \cong E_{0,\mathbb{R}}$ (see (5.10)), so that the Courant structure on $E_{0,\mathbb{R}}$ is as in Definition 2.13. In particular, there is a well-defined forgetful map

$$\mathcal{W} \longrightarrow \Omega^{1,1}_{\mathbb{R}} \times \mathcal{A}, \tag{7.10}$$

where $\mathcal{A}$ denotes the space of principal connections on $P_{\mathbb{R}}$. Furthermore, via $E_{\mathbb{R}} \cong E_{0,\mathbb{R}}$, we have

$$W = \{V + \sigma(V) : V \in T\underline{X}\},$$

where $\sigma = \omega(\, , J)$. The following result is a straightforward consequence of the Chern correspondence in Lemma 5.11.

Lemma 7.5. *Denote by $\mathcal{L}$ the set of isotropic subbundles $L \subset E$ mapping isomorphically to $T^{0,1}X$ under $\pi\colon E \to T\underline{X} \otimes \mathbb{C}$. Then, there is a bijection*

$$\begin{aligned} \mathcal{W} &\to \mathcal{L} \\ W &\mapsto L_W. \end{aligned} \tag{7.11}$$

The sets $\mathcal{W}$ and $\mathcal{L}$ have natural structures of affine space modelled on the vector spaces $\Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\mathrm{ad}\, P_{\mathbb{R}})$ and $\Omega^{1,1+0,2} \oplus \Omega^{0,1}(\mathrm{ad}\, \underline{P})$, respectively (see Lemma 2.15 and Lemma 5.11). It is not difficult to see that the map (7.11) is affine, and thus the natural complex structure on $\mathcal{L}$ given by multiplication by i induces a complex structure $\mathbf{J}$ on $\mathcal{W}$ making (7.11) holomorphic.

Lemma 7.6. *Any element $W \in \mathcal{W}$ induces a natural bijection $\mathcal{W} \cong \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \mathcal{A}$. Via this identification,* $\mathbf{J}$ *is given by*

$$\mathbf{J}_{|W}(\dot\omega, \dot b, \dot a) = (-\dot b^{1,1}, \dot\omega + ib^{0,2} - i\overline{b^{0,2}}, J\dot a) \tag{7.12}$$

for $(\dot\omega, \dot b, \dot a) \in \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\operatorname{ad} P_{\mathbb{R}})$ and $J\dot a := i\dot a^{0,1} - i\dot a^{1,0}$. Consequently, the forgetful map $\mathcal{W} \longrightarrow \mathcal{A}$ induced by (7.10) *is holomorphic.*

Proof. Without loss of generality, we fix an isotropic splitting $\lambda_0 \colon T\underline{X} \to E_{\mathbb{R}}$, with induced connection θ_0 on $P_{\mathbb{R}}$. Via the isomorphism $E_{\mathbb{R}} \cong E_{0,\mathbb{R}}$ induced by λ_0, as in (5.10), an element $W \in \mathcal{W}$ is given by a triple

$$(\omega, b, \theta_{\mathbb{R}}) \in \Omega^{1,1}_{\mathbb{R}} \times \Omega^2 \times \mathcal{A},$$

with corresponding horizontal lift

$$W = (-b, -a)\{V + \omega(V, J) : V \in T\underline{X}\},$$

for $a = \theta_{\mathbb{R}} - \theta_0$, and isotropic subbundle (see Lemma 5.11)

$$\begin{aligned} L_W &= (i\omega - b, -a)(T^{0,1}X) \\ &= (i\omega - b^{1,1+0,2} - \langle a^{0,1} \wedge a^{1,0}\rangle, -a^{0,1})(T^{0,1}X) \\ &= (i\omega - b^{1,1+0,2} - \tfrac{i}{2}\langle a \wedge Ja\rangle^{1,1}, -a^{0,1})(T^{0,1}X). \end{aligned}$$

Thus, the differential of the map (7.11) at $W \equiv (\omega, b, \theta_{\mathbb{R}})$ can be identified with the linear map

$$\begin{aligned} \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\operatorname{ad} P_{\mathbb{R}}) &\longrightarrow \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\operatorname{ad} \underline{P}) \\ (\dot\omega, \dot b, \dot a) &\longmapsto (\dot b^{1,1+0,2} - i(\dot\omega - \tfrac{1}{2}\langle \dot a \wedge Ja\rangle^{1,1} - \tfrac{1}{2}\langle a \wedge J\dot a\rangle^{1,1}), \dot a^{0,1}), \end{aligned}$$

and the induced complex structure is given by

$$\mathbf{J}_{|W}(\dot\omega, \dot b, \dot a) = (-\dot b^{1,1} + \langle \dot a \wedge a\rangle^{1,1}, \dot\omega + \langle J\dot a \wedge a\rangle^{1,1} + ib^{0,2} - i\overline{b^{0,2}}, J\dot a).$$

Taking now λ_0 to be the isotropic splitting induced by W we have $a = 0$ and the statement follows (for the last part see e.g. [16]). □

Consider now the natural left action of $\operatorname{Aut}(E_{\mathbb{R}})$ on $\mathcal{W}$, given by

$$\begin{aligned} \operatorname{Aut}(E_{\mathbb{R}}) \times \mathcal{W} &\longrightarrow \mathcal{W} \\ (\underline{f}, W) &\longmapsto \underline{f} \cdot W := \underline{f}(W). \end{aligned} \tag{7.13}$$

Our goal is to find a Hamiltonian action on $\mathcal{W}$ induced by (7.13) and study its symplectic reduction. For this, we need a better understanding of the action (7.13). Our next result shows that (7.13) preserves the complex structure $\mathbf{J}$, and furthermore it extends the classical action of the gauge group $\mathcal{G}_{P_{\mathbb{R}}}$ on the space of connections $\mathcal{A}$. Recall from [24, App. A] that there is a well-defined group homomorphism

$$\sigma_{P_{\mathbb{R}}} \colon \mathcal{G}_{P_{\mathbb{R}}} \to H^3(X, \mathbb{R})$$

defined as in Corollary 4.2, inducing an exact sequence

$$0 \longrightarrow \Omega^2 \longrightarrow \operatorname{Aut}(E_{\mathbb{R}}) \longrightarrow \operatorname{Ker} \sigma_{P_{\mathbb{R}}} \longrightarrow \mathcal{G}_{P_{\mathbb{R}}} \xrightarrow{\sigma_{P_{\mathbb{R}}}} H^3(X, \mathbb{R}).$$

Lemma 7.7. *The action* (7.13) *preserves* $\mathbf{J}$. *Furthermore, the forgetful map* (7.10) *jointly with the action* (7.13) *induce a commutative diagram*

$$\begin{array}{ccc} \mathrm{Aut}(E_{\mathbb{R}}) \times \mathcal{W} & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow \\ \mathrm{Ker}\,\sigma_{P_{\mathbb{R}}} \times \Omega^{1,1}_{\mathbb{R}} \times \mathcal{A} & \longrightarrow & \Omega^{1,1}_{\mathbb{R}} \times \mathcal{A}, \end{array}$$

where the bottom arrow is induced by the left $\mathcal{G}_{P_{\mathbb{R}}}$*-action on* $\Omega^{1,1}_{\mathbb{R}} \times \mathcal{A}$, *given by* $g \cdot (\omega, \theta_{\mathbb{R}}) = (\omega, g\theta_{\mathbb{R}})$.

Proof. For the first part, observe that the map (7.11) is equivariant for the action of $\mathrm{Aut}(E_{\mathbb{R}})$ on $\mathcal{L}$, given by

$$\begin{aligned} \mathrm{Aut}(E_{\mathbb{R}}) \times \mathcal{L} &\longrightarrow \mathcal{L} \\ (\underline{f}, L) &\longmapsto \underline{f} \cdot L := \underline{f}(L). \end{aligned} \tag{7.14}$$

Using that (7.14) is induced by the natural complex $\mathrm{Aut}(E)$-action on $\mathcal{L}$ (defined by the same formula), we obtain that $\mathbf{J}$ is preserved by (7.13).

As for the second part, without loss of generality we fix an isotropic splitting $\lambda_0 \colon T\underline{X} \to E_{\mathbb{R}}$, with induced connection θ_0 on $P_{\mathbb{R}}$. Via the induced isomorphism $E_{\mathbb{R}} \cong E_{0,\mathbb{R}}$, as in (5.10), an element $W \in \mathcal{W}$ is given by a triple

$$(\omega, b, \theta_{\mathbb{R}}) \in \Omega^{1,1}_{\mathbb{R}} \times \Omega^2 \times \mathcal{A},$$

with corresponding horizontal lift

$$W = (-b, \theta_0 - \theta_{\mathbb{R}})W_\omega$$

for $W_\omega := \{V + \omega(V, J) : V \in T\underline{X}\}$. An element in $\mathrm{Aut}(E_{\mathbb{R}}) \cong \mathrm{Aut}(E_{0,\mathbb{R}})$ is given by a pair $(g, \tau) \in \mathcal{G}_{P_{\mathbb{R}}} \times \Omega^2$ satisfying (cf. Lemma 4.3)

$$d\tau = CS(g^{-1}\theta_0) - CS(\theta_0) - d\langle g^{-1}\theta_0 \wedge \theta_0 \rangle$$

and the action (7.13) is

$$\begin{aligned} (g, \tau)(W) &= (\tau - b + \langle a^g \wedge \theta_0 - \theta_{\mathbb{R}} \rangle, g(a^g + \theta_0 - \theta_{\mathbb{R}}))(W_\omega) \\ &= (\tau - b + \langle a^g \wedge \theta_0 - \theta_{\mathbb{R}} \rangle, \theta_0 - g\theta_{\mathbb{R}})(W_\omega) \end{aligned}$$

for $a^g = g^{-1}\theta_0 - \theta_0$. Thus, the statement follows. □

Consider the open subset $\Omega^{1,1}_{>0} \subset \Omega^{1,1}_{\mathbb{R}}$ given by the positive $(1,1)$-forms on X. The phase space for our symplectic reduction is the following open subset of $\mathcal{W}$

$$\mathcal{W}_+ = \{W \in \mathcal{W} \mid \omega(\,, J) > 0\} \subset \mathcal{W}.$$

To define our family of symplectic structures, we fix a smooth volume form μ on X compatible with the complex structure. For any $\omega \in \Omega^{1,1}_{>0}$, we define the *dilaton function* $f_\omega \in C^\infty(X)$ as in (7.3).

Definition 7.8. Given $\ell \in \mathbb{R}\backslash\{2\}$, the *$\ell$-dilaton functional* on $\mathcal{W}_+$ is

$$M_\ell(W) := \int_X e^{-\ell f_\omega} \frac{\omega^n}{n!}. \tag{7.15}$$

Observe that M_ℓ is the pullback of the functional in Definition 7.1 by the projection $\mathcal{W} \to \Omega^{1,1}_{\mathbb{R}}$ induced by (7.10). In the sequel we fix $\ell \in \mathbb{R}\backslash\{2\}$. Associated to the functional M_ℓ there is a one-form $\lambda_\ell \in \Omega^1(\mathcal{W}_+)$ on $\mathcal{W}_+$, given by

$$\lambda_\ell := -\mathbf{J}d\log M_\ell = -\frac{1}{M_\ell}\mathbf{J}dM_\ell.$$

Lemma 7.9. *The one-form λ_ℓ is preserved by the* $\mathrm{Aut}(E_{\mathbb{R}})$*-action. Furthermore,*

$$\lambda_{\ell|W}(\dot\omega, \dot b, \dot a) = \frac{\ell-2}{2M_\ell}\int_X \dot b \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}. \tag{7.16}$$

for $(\dot\omega, \dot b, \dot a) \in T_W\mathcal{W}_+ \cong \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\mathrm{ad}\, P_{\mathbb{R}})$.

Proof. The first part of the statement is a direct consequence of Lemma 7.6 and Lemma 7.7. As for formula (7.16), without loss of generality, we fix an isotropic splitting $\lambda_0 \colon T\underline{X} \to E_{\mathbb{R}}$ with induced connection θ_0 on $P_{\mathbb{R}}$. By the proof of Lemma 7.6 combined with Lemma 7.2, the one-form λ_ℓ is

$$\lambda_{\ell|W}(\dot\omega, \dot b, \dot a) = \frac{\ell-2}{2M_\ell}\int_X (\dot b^{1,1} - \langle \dot a \wedge a\rangle^{1,1}) \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}, \tag{7.17}$$

for $a = \theta_{\mathbb{R}} - \theta_0$. Taking now λ_0 to be the isotropic splitting induced by W we have $a = 0$ and the statement follows. □

Similarly as in Section 7.1, we endow $\mathcal{W}_+$ with an $\mathrm{Aut}(E_{\mathbb{R}})$-invariant exact $(1,1)$-form defined by

$$\Omega_\ell := -d\mathbf{J}d\log M_\ell. \tag{7.18}$$

We calculate next a formula for Ω_ℓ and the symmetric two-tensor $g_\ell = \Omega_\ell(, \mathbf{J})$. We use the notation in Lemma 7.3 for the decomposition of two-forms into primitive and non-primitive parts.

Lemma 7.10. *The evaluation of Ω_ℓ and g_ℓ along tangent vectors $(\dot\omega_j, \dot b_j, \dot a_j)$ at the point (ω, b, a) is given by:*

$$\begin{aligned}
\Omega_\ell = {} & \frac{\ell-2}{M_\ell}\int_X \langle \dot a_1 \wedge \dot a_2\rangle \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\
& + \frac{\ell-2}{2M_\ell}\int_X (\dot\omega_1 \wedge \dot b_2 - \dot\omega_2 \wedge \dot b_1) \wedge e^{-\ell f_\omega}\frac{\omega^{n-2}}{(n-2)!} \\
& + \frac{\ell(\ell-2)}{4M_\ell}\int_X (\Lambda_\omega \dot b_1 \Lambda_\omega \dot\omega_2 - \Lambda_\omega \dot b_2 \Lambda_\omega \dot\omega_1) e^{-\ell f_\omega}\frac{\omega^n}{n!} \\
& + \left(\frac{\ell-2}{2M_\ell}\right)^2 \left(\int_X \Lambda_\omega(\dot\omega_1) e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)\left(\int_X \Lambda_\omega(\dot b_2) e^{-\ell f_\omega}\frac{\omega^n}{n!}\right) \\
& - \left(\frac{\ell-2}{2M_\ell}\right)^2 \left(\int_X \Lambda_\omega(\dot\omega_2) e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)\left(\int_X \Lambda_\omega(\dot b_1) e^{-\ell f_\omega}\frac{\omega^n}{n!}\right).
\end{aligned} \tag{7.19}$$

$$
\begin{aligned}
g_\ell(\dot\omega,\dot b,\dot a) = {}& \frac{\ell-2}{M_\ell}\int_X \langle \dot a \wedge J\dot a\rangle \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}\\
&+ \frac{2-\ell}{2M_\ell}\int_X (|\dot\omega_0|^2 + |\dot b_0^{1,1}|^2)e^{-\ell f_\omega}\frac{\omega^n}{n!}\\
&+ \frac{2-\ell}{2M_\ell}\left(\frac{\ell}{2}-\frac{n-1}{n}\right)\int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega \dot\omega|^2)e^{-\ell f_\omega}\frac{\omega^n}{n!}\\
&+ \left(\frac{2-\ell}{2M_\ell}\right)^2\left(\int_X \Lambda_\omega \dot\omega e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2 + \left(\frac{2-\ell}{2M_\ell}\right)^2\left(\int_X \Lambda_\omega \dot b e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2.
\end{aligned}
\tag{7.20}
$$

Proof. We fix an isotropic splitting $\lambda_0\colon T\underline{X} \to E_{\mathbb{R}}$. Formulae (7.19) and (7.20) follow by taking first the exterior derivative in (7.17) and then setting λ_0 to be the splitting induced by W, combined with Lemma 7.2 and Lemma 7.6. □

Remark 7.11. Arguing as in the proof of Lemma 7.3, one can prove that g_ℓ (respectively $-g_\ell$) induces a pseudo-Kähler metric along the subbundle $\Omega^{1,1}_{\mathbb{R}} \oplus \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^1(\operatorname{ad} P_{\mathbb{R}}) \subset T\mathcal{W}_+$ provided that $2-\frac{2}{n} < \ell < 2$ (respectively $\ell > 2$).

By Lemma 7.9, the action of $\operatorname{Aut}(E_{\mathbb{R}})$ on $(\mathcal{W}_+, \Omega_\ell)$ is Hamiltonian, with moment map

$$\langle \mu^\ell(W), \zeta\rangle = -\lambda_\ell(\zeta \cdot W)$$

for $\zeta \in \operatorname{Lie}\operatorname{Aut}(E_{\mathbb{R}})$, where $\zeta\cdot W$ denotes the infinitesimal action. The following explicit formula follows from the proof of Lemma 7.7. Recall that any $W \in \mathcal{W}$ determines an isotropic splitting $\lambda\colon T\underline{X} \to E_{\mathbb{R}}$ with connection $\theta_{\mathbb{R}}$, and via the isomorphism (5.10) the Lie algebra $\operatorname{Lie}\operatorname{Aut}(E_{\mathbb{R}})$ can be identified with (cf. Lemma 4.4

$$\operatorname{Lie}\operatorname{Aut}(E_{\mathbb{R}}) \cong \{(s,B) \mid d(B - 2\langle s, F_{\theta_{\mathbb{R}}}\rangle) = 0\} \subset \Omega^0(\operatorname{ad} P_{\mathbb{R}}) \times \Omega^2. \tag{7.21}$$

Proposition 7.12. *The action of* $\operatorname{Aut}(E_{\mathbb{R}})$ *on* $(\mathcal{W}_+, \Omega_\ell)$ *is Hamiltonian with equivariant moment map*

$$\langle \mu^\ell(W), \zeta\rangle = \frac{\ell-2}{2M_\ell}\int_X B \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}. \tag{7.22}$$

Consider the $\operatorname{Aut}(E_{\mathbb{R}})$-invariant subspace of 'integrable' horizontal lifts

$$\mathcal{W}^0 = \{W \in \mathcal{W} \mid [L_W, L_W] \subset L_W\} \subset \mathcal{W}_+, \tag{7.23}$$

and define $\mathcal{W}^0_+ = \mathcal{W}^0 \cap \mathcal{W}_+$. Via (7.11), $\mathcal{W}^0_+$ maps to an open set of the space of liftings of $T^{0,1}X$ to E, which defines a complex subspace of $\mathcal{L}$. Thus, $\mathcal{W}^0_+ \subset \mathcal{W}_+$ is (formally) a complex submanifold, and inherits an exact $(1,1)$-form denoted also by Ω_ℓ. Similarly as in Section 4.2, we define the following group of 'Hamiltonian' automorphisms of $E_{\mathbb{R}}$. Recall from Lemma 4.5 that there is Lie algebra homomorphism

$$\mathbf{a}\colon \operatorname{Lie}\operatorname{Aut}(E) \to H^{1,1}_A(X),$$

which defines a normal Lie subalgebra $\operatorname{Ker}\mathbf{a} \subset \operatorname{Lie}\operatorname{Aut}(E)$.

Definition 7.13. Define the subgroup $\mathcal{H} \subset \mathrm{Aut}(E_{\mathbb{R}})$ as the set of elements $\underline{f} \in \mathrm{Aut}(E_{\mathbb{R}})$ such that there exists a smooth family $\underline{f}_t \in \mathrm{Aut}(E_{\mathbb{R}})$ with $t \in [0,1]$, satisfying $\underline{f}_0 = \mathrm{Id}_{E_{\mathbb{R}}}$, $\underline{f}_1 = \underline{f}$, and

$$\mathbf{a}(\zeta_t) = 0, \qquad \text{for all } t. \tag{7.24}$$

We are ready to prove the main result of this section.

Proposition 7.14. *The $\mathcal{H}$-action on $(\mathcal{W}^0_+, \Omega_\ell)$ is Hamiltonian, with equivariant moment map induced by (7.22). Furthermore, zeros of the moment map are given by solutions of the Calabi system with level ℓ, defined by*

$$\begin{aligned} F_{\theta_{\mathbb{R}}} \wedge \omega^{n-1} &= 0, & F^{0,2}_{\theta_{\mathbb{R}}} &= 0, \\ d(e^{-\ell f_\omega}\omega^{n-1}) &= 0, & dd^c\omega + \langle F_{\theta_{\mathbb{R}}} \wedge F_{\theta_{\mathbb{R}}} \rangle &= 0. \end{aligned} \tag{7.25}$$

Proof. The integrability condition in the definition of $\mathcal{W}^0_+$ implies that the pair $(\omega, \theta_{\mathbb{R}})$ associated to $W \in \mathcal{W}^0_+$ via (7.10) satisfies the two equations in the right-hand side of (7.25) (see Proposition 5.13). Assume that $\langle \mu_\ell(W), \zeta \rangle = 0$ for all $\zeta \in \mathrm{Lie}\, \mathcal{H}$. Via the identification (7.21), the condition $\mathbf{a}(\zeta) = 0$ implies that

$$B^{1,1} - 2\langle s, F_{\theta_{\mathbb{R}}} \rangle = (d\xi)^{1,1}$$

for some $\xi \in \Omega^1$. Furthermore, for any $\xi \in \Omega^1$ we have

$$(s, d\xi + 2\langle s, F_{\theta_{\mathbb{R}}} \rangle) \in \mathrm{Lie}\, \mathcal{H}.$$

The two equations in the left hand side of (7.25) follow from Proposition 7.12. □

By Proposition 7.14, the coupled system (7.25) can be regarded as a natural analogue of the Hermite-Yang-Mills equations for string algebroids. These equations were originally found in [22] for $\ell = 1$ in a holomorphic setting, that is, fixing the string algebroid and caculating the critical points of the dilaton functional M_ℓ for compact forms in a fixed Aeppli class (see Proposition 6.8). Following [22], we will refer to (7.25) as the *Calabi system*. As a matter of fact, when the structure group K is trivial, the solutions of (7.25) are in correspondence with solutions of the Calabi problem for Kähler metrics on X

$$\frac{\omega^n}{n!} = c\mu, \qquad d\omega = 0, \tag{7.26}$$

for $c \in \mathbb{R}_{>0}$, which motivates the name for these equations (see [22]). Thus, in particular, Proposition 7.14 yields a new moment map interpretation of this classical problem, which shall be compared with [18].

Assume now that X is a (non-necessarily Kähler) Calabi-Yau manifold with holomorphic volume form Ω and we take μ as in (7.9) and $\ell = 1$. In this case, the dilaton function is given by

$$e^{-f_\omega} = \|\Omega\|_\omega,$$

and therefore Proposition 7.14 characterizes solutions of the Hull-Strominger system [31, 40] as a moment map condition (see e.g. [22]).

Corollary 7.15. *Let (X, Ω) be a Calabi-Yau manifold and let μ defined by (7.9). Then, the $\mathcal{H}$-action on $(\mathcal{W}^0_+, \Omega_1)$ is Hamiltonian, with equivariant moment map induced by (7.22). Furthermore, zeros of the moment map are given by solutions of the Hull-Strominger system*

$$\begin{aligned} F_{\theta_{\mathbb{R}}} \wedge \omega^{n-1} = 0, \qquad & F^{0,2}_{\theta_{\mathbb{R}}} = 0, \\ d(\|\Omega\|_\omega \omega^{n-1}) = 0, \qquad & dd^c\omega + \langle F_{\theta_{\mathbb{R}}} \wedge F_{\theta_{\mathbb{R}}} \rangle = 0. \end{aligned} \tag{7.27}$$

To the knowledge of the authors, this result provides the first symplectic interpretation of the Hull-Strominger system in the mathematics literature.

8. Moduli metric and infinitesimal Donaldson-Uhlenbeck-Yau

8.1. **Gauge fixing.** Let X be a compact complex manifold of dimension n. We fix a smooth volume form μ compatible with the orientation. The moduli space of solutions of the *Calabi system with level ℓ* on (X, μ) is defined as the set of classes of 'gauge equivalent' solutions of (7.25). More precisely, it is given by the symplectic quotient

$$\mathcal{M}_\ell := \mu_\ell^{-1}(0)/\mathcal{H},$$

where μ_ℓ is the moment map in Proposition 7.14. In this section we study some basic features of the geometry of $\mathcal{M}_\ell$ and point out some directions for future research. We will proceed formally, ignoring subtleties coming from the theory of infinite dimensional manifolds and Lie groups. In the sequel, the bi-invariant pairing $\langle\, , \rangle$ in the Lie algebra of the maximal compact subgroup $K \subset G$ is assumed to be non-degenerate (see (5.1)). For simplicity, we will also assume that K is semi-simple.

Our first goal is to undertake a *gauge fixing* for solutions of the linearized Calabi system (7.25), whereby the complex structure (7.12) and the symmetric tensor g_ℓ in (7.20) descend to the moduli space via symplectic reduction. Difficulties will arise, due to the fact that g_ℓ is neither a definite pairing nor non-degenerate (see Remark 7.11). Throughout this section, we fix a real string algebroid $E_{\mathbb{R}}$ with principal K-bundle P_h, the level $\ell \in \mathbb{R}$, and $W \in \mathcal{W}^0_+$ solving the Calabi system (7.25), that is, such that $\mu_\ell(W) = 0$. Recall that W determines a holomorphic principal G-bundle P, a conformally balanced hermitian form $\omega \in \Omega^{1,1}_{>0}$, and a Hermite-Yang-Mills Chern connection θ^h on P (via the fixed reduction $P_h \subset P$).

We start by characterizing the tangent space to $\mathcal{M}_\ell$ at $[W]$. By Lemma 7.6, an infinitesimal variation of our horizontal lift W is given by

$$(\dot\omega, \dot b, \dot a) \in \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\operatorname{ad} P_h).$$

Lemma 8.1. *The combined linearization of the Calabi system* (7.25) *and the integrability condition in* (7.23) *is given by the linear equations*

$$\begin{aligned} d^h\dot{a}\wedge\omega^{n-1}+(n-1)F_h\wedge\dot{\omega}\wedge\omega^{n-2}&=0,\\ d\Big(e^{-\ell f_\omega}\Big((n-1)\dot{\omega}\wedge\omega^{n-2}-\frac{\ell}{2}(\Lambda_\omega\dot{\omega})\omega^{n-1}\Big)\Big)&=0,\\ \bar{\partial}\dot{a}^{0,1}&=0,\\ d^c\dot{\omega}+2\langle\dot{a},F_h\rangle-d\dot{b}&=0. \end{aligned} \tag{8.1}$$

Proof. The linearization of (7.25) is

$$\begin{aligned} d^h\dot{a}\wedge\omega^{n-1}+(n-1)F_h\wedge\dot{\omega}\wedge\omega^{n-2}&=0,\\ d\Big(e^{-\ell f_\omega}\Big((n-1)\dot{\omega}\wedge\omega^{n-2}-\frac{\ell}{2}(\Lambda_\omega\dot{\omega})\omega^{n-1}\Big)\Big)&=0,\\ \bar{\partial}\dot{a}^{0,1}&=0,\\ d(d^c\dot{\omega}+2\langle\dot{a},F_h\rangle)&=0, \end{aligned} \tag{8.2}$$

while the integrability condition $[L_W,L_W]\subset L_W$ (see (7.23)) implies at the infinitesimal level that (see Lemma 2.15 and Lemma 5.11)

$$\begin{aligned} \bar{\partial}\dot{a}^{0,1}&=0,\\ d\dot{b}^{0,2}+\bar{\partial}(\dot{b}^{1,1}-i\dot{\omega})-2\langle\dot{a}^{0,1},F_h\rangle&=0. \end{aligned} \tag{8.3}$$

The second equation in (8.3) yields

$$d^c\dot{\omega}=d\dot{b}-2\langle\dot{a},F_h\rangle,$$

and therefore (8.3) implies the last two equations in (8.2). Thus, the tangent to $\mu_\ell^{-1}(0)\subset\mathcal{W}^0_+$ is characterized by the linear equations (8.1). □

We denote by $\mathbf{L}(\dot{\omega},\dot{b},\dot{a})$ the differential operator defined by the left hand side of equations (8.1). We turn next to the study of the infinitesimal action, in order to define a complex. From the proof of Lemma 7.7, we can identify elements $\zeta\in\mathrm{Lie}\mathcal{H}$ with pairs

$$\zeta=(u,B)\in\mathrm{Lie}\,\Omega^0(\mathrm{ad}\,P_h)\oplus\Omega^2$$

satisfying (see Lemma 4.4)

$$d(B-2\langle u,F_h\rangle)=0,\qquad B^{1,1}-2\langle u,F_h\rangle=(d\xi)^{1,1} \tag{8.4}$$

for a real one-form $\xi\in\Omega^1$, and the infinitesimal action at W, denoted $\mathbf{P}(u,B)$, is

$$(u,B)\cdot W=(0,B,d^hu)=(0,(d\xi)^{1,1}+2\langle u,F_h\rangle+B^{0,2}+\overline{B^{0,2}},d^hu). \tag{8.5}$$

Define the vector space

$$\mathcal{R}:=\Omega^{2n}(\mathrm{ad}\,P_h)\oplus\Omega^{2n-1}\oplus\Omega^{0,2}(\mathrm{ad}\,\underline{P})\oplus\Omega^3,$$

given by the domain of the left hand side of (8.1). Then, the operator $\mathbf{L}$ induced by (8.1) jointly with (8.5) define a complex of vector spaces

$$(S^*)\qquad \mathrm{Lie}\mathcal{H}\xrightarrow{\mathbf{P}}\Omega^{1,1}_{\mathbb{R}}\oplus\Omega^2\oplus\Omega^1(\mathrm{ad}\,P_h)\xrightarrow{\mathbf{L}}\mathcal{R}, \tag{8.6}$$

whose cohomology $H^1(S^*) := \frac{\operatorname{Ker}\mathbf{L}}{\operatorname{Im}\mathbf{P}}$ will be formally identified with the tangent space $T_W\mathcal{M}_\ell$. Observe that the elements of $\mathrm{Lie}\mathcal{H}$ do not correspond to sections of a vector bundle, due to the conditions in (8.4), and hence (8.6) is not a complex of differential operators. To circunvent this issue, we consider the Lie subalgebra

$$\{(u, d\xi + 2\langle u, F_h\rangle) \mid \xi \in \Omega^1)\} \subset \mathrm{Lie}\mathcal{H}$$

and define the induced complex

$$(\widehat{S}^*) \qquad \Omega^0(\operatorname{ad} P_h) \oplus \Omega^1 \xrightarrow{\widehat{\mathbf{P}}} \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\operatorname{ad} P_h) \xrightarrow{\mathbf{L}} \mathcal{R}, \tag{8.7}$$

where

$$\widehat{\mathbf{P}}(u, \xi) = (0, d\xi + 2\langle u, F_h\rangle, d^h u).$$

Our next result shows that the moduli space $\mathcal{M}_\ell$ is finite-dimensional. The proof builds on the infinitesimal moduli construction in [23]. Consider the Aeppli cohomology group $H^{0,1}_A(X)$ and the natural map from Dolbeault to Bott-Chern cohomology induced by the ∂ operator:

$$H^{0,2}_{\bar\partial}(X) \xrightarrow{\partial} H^{1,2}_{BC}(X). \tag{8.8}$$

We will denote $h^{0,1}_A(X) = \dim H^{0,1}_A(X)$ and $h^0(\operatorname{ad} P) = \dim H^0(\operatorname{ad} P)$.

Lemma 8.2. *The sequence* (8.7) *is an elliptic complex of differential operators. Consequently, the cohomology $H^1(S^*)$ of* (8.6) *is finite-dimensional. Furthermore, assuming that $h^0(\operatorname{ad} P) = 0$ and $h^{0,1}_A(X) = 0$, there is an exact sequence*

$$0 \longrightarrow \operatorname{Ker}\partial \longrightarrow H^1(\widehat{S}^*) \longrightarrow H^1(S^*) \longrightarrow 0$$

where ∂ is as in (8.8).

Proof. Ellipticity of (8.7) follows as in [23, Prop. 4.4], implying that $H^1(S^*)$ is finite-dimensional due to the existence of a natural surjective map

$$H^1(\widehat{S}^*) \longrightarrow H^1(S^*) \longrightarrow 0. \tag{8.9}$$

The kernel is given by the quotient $\operatorname{Im}\mathbf{P}/\operatorname{Im}\widehat{\mathbf{P}}$, where

$$\operatorname{Im}\mathbf{P} = \{(0, B^{0,2} + \overline{B^{0,2}} + (d\xi)^{1,1} + 2\langle u, F_h\rangle, d^h u) \mid d(B^{0,2} - \bar\partial\xi^{0,1}) = 0\}.$$

We claim that (8.9) induces a well-defined surjective map

$$\begin{array}{rcl} \operatorname{Im}\mathbf{P} & \to & \operatorname{Ker}\partial \subset H^{0,2}_{\bar\partial}(X) \\ (0, \dot b, \dot a) & \mapsto & [B^{0,2} - \bar\partial\xi^{0,1}]. \end{array} \tag{8.10}$$

provided that $h^0(\operatorname{ad} P) = 0$. Firstly, since θ^h is Hermite-Yang-Mills, this condition implies $\dim\ker d^h = 0$. Therefore, if

$$(0, \dot b, \dot a) = (0, B^{0,2}_j + \overline{B^{0,2}_j} + (d\xi_j)^{1,1} + 2\langle u_j, F_h\rangle, d^h u)$$

for $j = 1, 2$ then, $u_1 = u_2$, $B^{0,2}_1 = B^{0,2}_2$ and $\partial\bar\partial(\xi^{0,1}_1 - \xi^{0,1}_2) = 0$, so that (8.10) is well defined. As for surjectivity, if $[\gamma] \in \operatorname{Ker}\partial$, there exists $\xi^{0,1} \in \Omega^{0,1}$ such that

$$d(\gamma - \bar\partial\xi^{0,1}) = 0,$$

and hence $(0, \gamma + \partial\xi^{0,1} + c.c., 0)$ is an element of $\mathrm{Lie}\mathcal{H}$ mapping to $[\gamma]$.

Any element in $\operatorname{Im}\widehat{\mathbf{P}}$ maps to 0 via (8.10), and therefore this induces a well-defined surjection $\operatorname{Im}\mathbf{P}/\operatorname{Im}\widehat{\mathbf{P}} \to \operatorname{Ker}\partial$. We claim that this induced map is injective, provided that $h^{0,1}_A(X) = 0$. To see this, notice that if

$$B^{0,2} - \bar\partial\xi^{0,1} = \bar\partial\xi'^{0,1}$$

it follows that $\partial\bar\partial\xi'^{0,1} = 0$ and hence $\xi'^{0,1}$ is $\bar\partial$-exact. Thus, $B^{0,2} = \bar\partial\xi^{0,1}$ and

$$B^{0,2} + \overline{B^{0,2}} + (d\xi)^{1,1} = d\xi.$$

□

Our strategy to build a complex structure induced by (7.12) on the moduli space is to work orthogonally to the image of the operator $\widehat{\mathbf{P}}$ with respect to the non-definite pairing g_ℓ in (7.20) (cf. [35]). The existence of this complex structure will automatically yield a symmetric tensor of type $(1,1)$, since the two-form Ω_ℓ in (7.18) is well defined on the cohomology $H^1(S^*)$ by Proposition 7.14. Our construction relies on a technical condition already found in [22], which we explain next. Consider the indefinite L^2-pairing on the domain of the operator $\widehat{\mathbf{P}}$ in (8.7) induced by ω and $\langle\,,\rangle$

$$\langle (u,\xi),(u,\xi)\rangle_\ell = \frac{2-\ell}{M_\ell}\left(\int_X \langle u,u\rangle\frac{\omega^n}{n!} + \frac{1}{2}\int_X \xi\wedge J\xi\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}\right), \quad (8.11)$$

where M_ℓ is the value of the functional (7.15) at the solution W.

Lemma 8.3. *The following operator provides an adjoint of $\widehat{\mathbf{P}}$ for the pairings* (8.11) *and* (7.20)

$$\widehat{\mathbf{P}}^*\colon \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\operatorname{ad} P_h) \to \Omega^0(\operatorname{ad} P_h)\oplus\Omega^1$$

where $\widehat{\mathbf{P}}^* = \widehat{\mathbf{P}}^*_0 \oplus \widehat{\mathbf{P}}^*_1$ is defined by

$$\widehat{\mathbf{P}}^*_0(\dot\omega,\dot b,\dot a) = \frac{1}{(n-1)!} * \Big(e^{-\ell f_\omega}\Big(d^h J\dot a\wedge\omega^{n-1} - (n-1)F_h\wedge\dot b\wedge\omega^{n-2}\Big)\Big),$$

$$\widehat{\mathbf{P}}^*_1(\dot\omega,\dot b,\dot a) = \frac{1}{(n-1)!} * d\Big(e^{-\ell f_\omega}\Big((n-1)\dot b^{1,1}\wedge\omega^{n-2} - \frac{\ell}{2}(\Lambda_\omega\dot b)\omega^{n-1}\Big)\Big).$$

Proof. The proof follows from a straightforward calculation using integration by parts. Setting $v = (\dot\omega,\dot b,\dot a)$, $y = (u,\xi)$, and using (7.20) and(7.25) we have

$$\begin{aligned} g_\ell(v,\widehat{\mathbf{P}}y) &= \frac{\ell-2}{M_\ell}\int_X \langle \dot a\wedge Jd^h u\rangle\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\ &\quad - \frac{2-\ell}{2M_\ell}\int_X \dot b^{1,1}\wedge(d\xi + 2\langle u,F_h\rangle)\wedge e^{-\ell f_\omega}\frac{\omega^{n-2}}{(n-2)!} \\ &\quad + \frac{(2-\ell)\ell}{4M_\ell}\int_X (\Lambda_\omega\dot b)d\xi\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\ &= \frac{2-\ell}{M_\ell(n-1)!}\int_X \langle u, d^hJ\dot a\wedge\omega - (n-1)F_h\wedge\dot b\rangle\wedge e^{-\ell f_\omega}\omega^{n-2} \\ &\quad - \frac{2-\ell}{2M_\ell(n-1)!}\int_X \xi\wedge d\Big(e^{-\ell f_\omega}\Big((n-1)\dot b^{1,1}\wedge\omega^{n-2} - \frac{\ell}{2}(\Lambda_\omega\dot b)\omega^{n-1}\Big)\Big). \end{aligned}$$

The statement follows from $*^2_{|\Omega^{2n-1}} = -1$ and the action of the Hodge star operator on one-forms

$$*\xi = Jd\xi \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

□

Consider now the L^2-orthogonal decomposition of Ω^1 induced by the De Rham differential

$$\Omega^1 = \operatorname{Im} d \oplus \operatorname{Im} d^* \oplus \mathcal{H}^1$$

and define a differential operator

$$\begin{array}{rccc} \mathcal{L}: & \Omega^0(\operatorname{ad} P_h) \times \operatorname{Im} d^* & \to & \Omega^0(\operatorname{ad} P_h) \times \operatorname{Im} d^* \\ & (u, \xi) & \mapsto & \widehat{\mathbf{P}}^* \circ \widehat{\mathbf{P}}(u, \xi). \end{array} \tag{8.12}$$

We state next the key condition on the solution W of (7.25) which we need to assume for our argument.

Condition A. The kernel of $\mathcal{L}$ vanishes.

A geometric characterization of Condition A is mentioned in Remark 8.7. On the practical side, this hypothesis will enable us to construct the complex structure on the moduli space under natural cohomological conditions. We build on the following result from [22]. Using ω and a choice of bi-invariant positive-definite bilinear form on $\mathfrak{k}$, we endow the domain of $\mathcal{L}$ with an L^2 norm (possibly different from (8.11), which may be indefinite) and extend the domain of $\mathcal{L}$ to an appropriate Sobolev completion.

Proposition 8.4 ([22])**.** *The operator $\mathcal{L}$ is Fredholm with zero index.*

Assuming Condition A, we obtain a natural gauge fixing via a g_ℓ-orthogonal decomposition

$$\Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\operatorname{ad} P_h) = \operatorname{Im} \widehat{\mathbf{P}} \oplus (\operatorname{Im} \widehat{\mathbf{P}})^{\perp_{g_\ell}}. \tag{8.13}$$

Lemma 8.5. *Assume Condition A. Then, there exists an orthogonal decomposition* (8.13) *for the pairing g_ℓ in* (7.20)*. Consequently, for any element $v \in \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^1(\operatorname{ad} P_h)$ there exists a unique $\Pi v \in \operatorname{Im} \widehat{\mathbf{P}}$ such that $(\dot\omega, \dot b, \dot a) = v - \Pi v$ solves the linear equations*

$$\begin{aligned} d\Big(e^{-\ell f_\omega}\Big((n-1)\dot b^{1,1} \wedge \omega^{n-2} - \frac{\ell}{2}(\Lambda_\omega \dot b)\omega^{n-1}\Big) = 0, \\ d^h J\dot a \wedge \omega^{n-1} - (n-1) * F_h \wedge \dot b \wedge \omega^{n-2} = 0. \end{aligned}$$

Proof. Notice first that from the non-degeneracy of $\langle\,,\rangle$, the pairing given in (8.11) is non-degenerate. Thus

$$\ker \widehat{\mathbf{P}}^* = (\operatorname{Im} \widehat{\mathbf{P}})^{\perp_{g_\ell}}.$$

If $v \in \operatorname{Im} \widehat{\mathbf{P}} \cap (\operatorname{Im} \widehat{\mathbf{P}})^{\perp_{g_\ell}}$, then $v = \widehat{\mathbf{P}}(y)$ for $y \in \Omega^0(\operatorname{ad} P_h) \times \operatorname{Im} d^*$. But then $\widehat{\mathbf{P}}^* \circ \widehat{\mathbf{P}}(y) = 0$ and, by Condition A, $v = 0$. Thus

$$\operatorname{Im} \widehat{\mathbf{P}} \cap (\operatorname{Im} \widehat{\mathbf{P}})^{\perp_{g_\ell}} = \{0\}. \tag{8.14}$$

Let $v \in \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^1(\operatorname{ad} P_h)$. The condition

$$v - \widehat{\mathbf{P}}(y) \in (\operatorname{Im} \widehat{\mathbf{P}})^{\perp_{g_\ell}}$$

for some $y \in \Omega^0(\operatorname{ad} P_h) \times \operatorname{Im} d^*$ is equivalent to

$$\widehat{\mathbf{P}}^*(v) = \widehat{\mathbf{P}}^* \circ \widehat{\mathbf{P}}(y). \tag{8.15}$$

But by Proposition 8.4 and Condition A, $\widehat{\mathbf{P}}^* \circ \widehat{\mathbf{P}}$ is surjective. Then, by elliptic regularity, one can solve (8.15) for $y \in \Omega^0(\operatorname{ad} P_h) \times \operatorname{Im} d^*$. The orthogonal decomposition follows. The last statement of the Lemma comes from the expression of $\widehat{\mathbf{P}}^*$ in Lemma 8.3. □

The above Lemma suggests to define the space of harmonic representatives of the complex (8.7):

$$\mathcal{H}^1(\widehat{S}^*) = \ker \mathbf{L} \cap \ker \widehat{\mathbf{P}}^*.$$

Our next result provides our gauge fixing mechanism for the linearization of the Calabi system (8.1) under natural cohomological assumptions.

Proposition 8.6. *Assume Condition A and the cohomological conditions*

$$h^{0,1}_A(X) = 0, \qquad \operatorname{Ker} \partial = \{0\}, \qquad h^0(\operatorname{ad} P) = 0, \tag{8.16}$$

where ∂ is as in (8.8)*. Then, the inclusion $\mathcal{H}^1(\widehat{S}^*) \subset \ker \mathbf{L}$ induces an isomorphism*

$$\mathcal{H}^1(\widehat{S}^*) \simeq H^1(S^*).$$

More precisely, any class in the cohomology $H^1(S^*)$ of the complex (8.6) *admits a unique representative $(\dot\omega, \dot b, \dot a)$ solving the linear equations*

$$\begin{aligned}
d^h \dot a \wedge \omega^{n-1} + (n-1) F_h \wedge \dot\omega \wedge \omega^{n-2} &= 0,\\
d\Big(e^{-\ell f_\omega} \Big((n-1)\dot\omega \wedge \omega^{n-2} - \frac{\ell}{2} (\Lambda_\omega \dot\omega) \omega^{n-1} \Big) \Big) &= 0,\\
\bar\partial \dot a^{0,1} &= 0,\\
d^c \dot\omega + 2\langle \dot a, F_h \rangle - d\dot b &= 0,\\
d\Big(e^{-\ell f_\omega} \Big((n-1)\dot b^{1,1} \wedge \omega^{n-2} - \frac{\ell}{2} (\Lambda_\omega \dot b) \omega^{n-1} \Big) &= 0,\\
d^h J\dot a \wedge \omega^{n-1} - (n-1) F_h \wedge \dot b \wedge \omega^{n-2} &= 0.
\end{aligned} \tag{8.17}$$

Proof. The correspondence between $H^1(S^*)$ and the space of solutions of (8.17) follows from Lemma 8.1, Lemma 8.2, and Lemma 8.5. □

Remark 8.7. Condition A, the key hypothesis for our gauge fixing mechanism, is secretly a geometric condition. To see this, denote by E the complexification of $E_{\mathbb{R}}$. Recall that $\operatorname{Aut}(E)$ acts on the space of compact forms on the Bott-Chern algebroid $Q := Q_{L_W}$ with surjective infinitesimal action ρ (see Lemma 6.4 and Proposition 4.1). From the proof of Lemma 6.4, there is a partial inverse for ρ which sends an infinitesimal variation $(\dot\omega + \dot\upsilon, iu)$ of $E_{\mathbb{R}}$ to the Lie algebra element $\zeta(\dot\omega + \dot\upsilon, iu) = (iu, -i\dot\omega + i\operatorname{Im} \dot\upsilon) \in \operatorname{Lie} \operatorname{Aut}(E)$. Denote by $\operatorname{Aut}(Q)$ the group of automorphisms of Q, regarded as the isotropy

group of L_W on $\mathrm{Aut}(E)$. Then, one can prove that a solution W of the Calabi system (7.25) with $h^0(\mathrm{ad}\, P) = 0$ satisfies Condition A if and only if the following holds: an infinitesimal variation $(\dot\omega + \dot v, iu)$ of $E_{\mathbb{R}}$ along the Aeppli class $[E_{\mathbb{R}}] \in \Sigma_A(Q, \mathbb{R})$ solves the linearization of the Calabi system (7.25) only if $\zeta(\dot\omega + \dot v, iu) \in \mathrm{Lie}\,\mathrm{Aut}(Q)$. This shall be compared with a classical result in Kähler geometry, which states that solutions of the linearized constant scalar curvature equation, for Kähler metrics in a fixed Kähler class, are in bijective correspondence with Hamiltonian Killing vector fields.

8.2. **The moduli space metric.** We are ready to prove our main result, which shows that the gauge fixing in Proposition 8.6 enables us to descend the complex structure (7.12) and the symmetric tensor g_ℓ in (7.20) to the moduli space $\mathcal{M}_\ell$, via the symplectic reduction in Proposition 7.14.

Theorem 8.8. *Assume Condition A and the cohomological conditions* (8.16)*. Then, the tangent space to $\mathcal{M}_\ell$ at $[W]$, identified with the space of solutions of the gauge fixed linear equations* (8.17)*, inherits a complex structure* $\mathbf{J}$ *and a (possibly degenerate) metric g_ℓ such that $\Omega_\ell = g_\ell(\mathbf{J},)$, given respectively by* (7.12) *and* (7.20)*, and where Ω_ℓ stands for the restriction of* (7.19)*.*

Proof. The fact that $H^1(S^*)$ inherits a complex structure follows from Proposition 8.6, using that $\mathbf{J}$ in (7.12) preserves (8.17). The formula for the metric is a direct consequence of Lemma 7.9 and Proposition 7.12. □

Remark 8.9. Using (8.16), it is not difficult to see that any $[(\dot\omega, \dot b, \dot a)] \in H^1(S^*)$ admits a representative with $\dot b = \dot b^{1,1}$. Thus, relying on Remark 7.11, we expect that (7.20) leads to a non-degenerate metric at least for $\ell > 2 - \frac{2}{n}$.

We study next the structure of the metric (7.20) along the fibres of a natural map from $\mathcal{M}_\ell$ to the moduli space of holomorphic principal G-bundles. As we will see shortly, the moduli space metric constructed in Theorem 8.8 is 'semi-topological', in the sense that fibre-wise it can be expressed in terms of classical cohomological quantities associated to a gauge-fixed variation of the solution. Denote by

$$\mathcal{A}^0 = \{\theta_{\mathbb{R}} \in \mathcal{A} \mid F^{0,2}_{\theta_{\mathbb{R}}} = 0\}$$

the space of integrable connections on $P_h = P_{\mathbb{R}}$. Via the classical Chern correspondence, we can identify $\mathcal{A}^0$ with the space of structures of holomorphic principal G-bundle on $\underline{P} := P_h \times_K G$, which we denote by $\mathcal{C}^0$, obtaining a well-defined map

$$\mathcal{M}_\ell \to \mathcal{C}^0/\mathcal{G}_{\underline{P}}. \tag{8.18}$$

By standard theory, $\mathcal{C}^0/\mathcal{G}_{\underline{P}}$ is the well-studied moduli space of holomorphic principal G-bundles over X with fixed topological bundle $\underline{P}$. As before, we fix a solution W of (7.25) and consider the corresponding point

$$[P] \in \mathcal{C}^0/\mathcal{G}_{\underline{P}}.$$

We start by characterizing the tangent space to the fibre of (8.18) over the class $[P]$, using the gauge fixing in Proposition 8.6.

Lemma 8.10. *Assume Condition A and the cohomological conditions* (8.16). *Then, any infinitesimal variation in the fibre of* (8.18) *over* $[P]$ *at* $[W]$ *admits a unique representative of its class in* $H^1(S^*)$ *of the form* $(\dot\omega, \dot b, -Jd^h s + d^h s')$, *for* $s, s' \in \Omega^0(\mathrm{ad}\, P_h)$, *solving the linear equations*

$$\begin{aligned} -d^h Jd^h s \wedge \omega^{n-1} + (n-1)F_h \wedge \dot\omega \wedge \omega^{n-2} &= 0,\\ d\Big(e^{-\ell f_\omega}\Big((n-1)\dot\omega \wedge \omega^{n-2} - \frac{\ell}{2}(\Lambda_\omega \dot\omega)\omega^{n-1}\Big)\Big) &= 0,\\ d^c(\dot\omega - 2\langle s, F_h\rangle) - d(\dot b - 2\langle s', F_h\rangle) &= 0,\\ d\Big(e^{-\ell f_\omega}\Big((n-1)\dot b^{1,1} \wedge \omega^{n-2} - \frac{\ell}{2}(\Lambda_\omega \dot b)\omega^{n-1}\Big) &= 0,\\ -d^h Jd^h s' \wedge \omega^{n-1} + (n-1)F_h \wedge \dot b \wedge \omega^{n-2} &= 0. \end{aligned} \tag{8.19}$$

Proof. Let $(\dot\omega, \dot b, \dot a) \in \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\mathrm{ad}\, P_h)$ be an infinitesimal variation of the solution W of (7.25). Assuming that it is tangent to the fibre over $[P]$, there exists $r \in \Omega^0(\mathrm{ad}\, \underline{P})$ such that

$$\dot a^{0,1} = \bar\partial r.$$

Then we can write uniquely

$$\dot a = -Jd^h s + d^h s'$$

for $s, s' \in \Omega^0(\mathrm{ad}\, P_h)$. The statement follows from Proposition 8.6 using that $(d^h)^2 s \wedge \omega^{n-1} = [F_h, s] \wedge \omega^{n-1} = 0$ by (7.25). □

Remark 8.11. Using that θ^h is Hermite-Yang-Mills and that $h^0(\mathrm{ad}\, P)$ vanishes, by the first and last equations in (8.19) the elements s and s' are uniquely determined by $\dot\omega$ and $\dot b$.

The gauge fixed system (8.19) for variations along the fibres of (8.18) allows us to define Aeppli and Bott-Chern cohomology classes. Let $(\dot\omega, \dot b, -Jd^h s + d^h s')$ be, as in Lemma 8.10, a solution of (8.19). From the third equation in (8.19) we obtain

$$dd^c(\dot\omega - 2\langle s, F_h\rangle) = 0, \qquad dd^c(\dot b - 2\langle s', F_h\rangle) = 0,$$

and we can define the variation of the 'complexified Aeppli class' of the solution (cf. Proposition 6.8) by

$$\begin{aligned} \dot{\mathfrak{a}} &= \mathrm{Re}\, \dot{\mathfrak{a}} + i\mathrm{Im}\, \dot{\mathfrak{a}}\\ &= [\dot\omega - 2\langle s, F_h\rangle] + i[\dot b - 2\langle s', F_h\rangle] \in H^{1,1}_A(X). \end{aligned}$$

Notice that, by Lemma 7.7, the balanced class

$$\mathfrak{b} = \frac{1}{(n-1)!}[e^{-\ell f_\omega}\omega^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$$

is independent of the representative in $[W] \in \mathcal{M}_\ell$. Thus, using the second and fourth equations in (8.19), we define the variations of the 'complexified balanced class' by

$$\begin{aligned} \dot{\mathfrak{b}} &= \mathrm{Re}\, \dot{\mathfrak{b}} + i\mathrm{Im}\, \dot{\mathfrak{b}}\\ &= [\mathrm{Re}\, \dot\nu] + i[\mathrm{Im}\, \dot\nu] \in H^{n-1,n-1}_{BC}(X), \end{aligned}$$

where $\dot\nu \in \Omega^{n-1,n-1}$ is defined by

$$(n-1)!\mathrm{Re}\,\dot\nu := e^{-\ell f_\omega}(n-1)\dot\omega_0\wedge\omega^{n-2} + \frac{n(2-\ell)-2}{2n}e^{-\ell f_\omega}(\Lambda_\omega\dot\omega)\omega^{n-1},$$

$$(n-1)!\mathrm{Im}\,\dot\nu := e^{-\ell f_\omega}(n-1)\dot b_0\wedge\omega^{n-2} + \frac{n(2-\ell)-2}{2n}e^{-\ell f_\omega}(\Lambda_\omega\dot b)\omega^{n-1}.$$

The subscript 0 stands for the primitive $(1,1)$-forms

$$\dot\omega_0 = \dot\omega - \frac{1}{n}(\Lambda_\omega\dot\omega)\omega, \qquad \dot b_0 = \dot b - \frac{1}{n}(\Lambda_\omega\dot b)\omega.$$

The variation of the balanced class $\mathfrak{b}$ of ω corresponds in our notation to $\mathrm{Re}\,\dot{\mathfrak{b}}$. For the next result, we use the duality pairing $H^{1,1}_A(X) \cong H^{n-1,n-1}_{BC}(X)^*$ between the Aeppli and Bott-Chern cohomologies.

Lemma 8.12. *The pairing between* $\mathrm{Re}\,\dot{\mathfrak{b}}$ *and* $\mathrm{Re}\,\dot{\mathfrak{a}}$ *is given by:*

$$\begin{aligned}\mathrm{Re}\,\dot{\mathfrak{b}}\cdot\mathrm{Re}\,\dot{\mathfrak{a}} = &-\int_X |\dot\omega_0|^2 e^{-\ell f_\omega}\frac{\omega^n}{n!} + \frac{n(2-\ell)-2}{2n}\int_X e^{-\ell f_\omega}|\Lambda_\omega\dot\omega|^2\frac{\omega^n}{n!}\\ &+ 2\int_X \langle d^h s\wedge Jd^h s\rangle\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}.\end{aligned}$$

Proof. Define $\tilde\omega = e^{-\ell f_\omega/n-1}\omega$. Using that $\tilde\omega$ is balanced, we have

$$\begin{aligned}\Delta_{\tilde\omega}\langle s,s\rangle :&= 2i\Lambda_{\tilde\omega}\bar\partial\partial\langle s,s\rangle\\ &= 4i\langle\Lambda_{\tilde\omega}\bar\partial\partial^h s,s\rangle + 2\Lambda_{\tilde\omega}\langle(d^h)^c s\wedge d^h s\rangle\\ &= 4i\langle\Lambda_{\tilde\omega}\bar\partial\partial^h s,s\rangle + 2\Lambda_{\tilde\omega}\langle Jd^h s\wedge d^h s\rangle.\end{aligned}$$

By equation $F_h\wedge\omega^{n-1}=0$, we can express d^hJd^h as follows

$$-(d^hJd^hs)\wedge\omega^{n-1} = (2i\bar\partial\partial^h s)\wedge\omega^{n-1} - [F_h,s]\wedge\omega^{n-1} = (2i\bar\partial\partial^h s)\wedge\omega^{n-1}$$

and hence the first equation in (8.19) gives

$$\Delta_{\tilde\omega}\langle s,s\rangle\frac{\tilde\omega^n}{n!} = -2\langle F_h,s\rangle\wedge\dot\omega_0\wedge\frac{e^{-\ell f_\omega}\omega^{n-2}}{(n-2)!} + 2\Lambda_{\tilde\omega}\langle Jd^h s\wedge d^h s\rangle\frac{\tilde\omega^n}{n!}.$$

Finally, we calculate

$$\begin{aligned}\mathrm{Re}\,\dot{\mathfrak{b}}\cdot\mathrm{Re}\,\dot{\mathfrak{a}} &= \int_X \mathrm{Re}\,\dot\nu\wedge(\dot\omega_0 + (\Lambda_\omega\dot\omega)\omega/n - 2\langle s,F_h\rangle)\\ &= \int_X \dot\omega_0\wedge\dot\omega_0\wedge e^{-\ell f_\omega}\frac{\omega^{n-2}}{(n-2)!} + \frac{n(2-\ell)-2}{2n}\int_X e^{-\ell f_\omega}|\Lambda_\omega\dot\omega|^2\frac{\omega^n}{n!}\\ &\quad - 2\int_X\langle s,F_h\rangle\wedge e^{-\ell f_\omega}\dot\omega_0\wedge\frac{\omega^{n-2}}{(n-2)!}\\ &= -\int_X|\dot\omega_0|^2 e^{-\ell f_\omega}\frac{\omega^n}{n!} + \frac{n(2-\ell)-2}{2n}\int_X e^{-\ell f_\omega}|\Lambda_\omega\dot\omega|^2\frac{\omega^n}{n!}\\ &\quad + 2\int_X\Lambda_{\tilde\omega}\langle d^h s\wedge Jd^h s\rangle\frac{\tilde\omega^n}{n!}.\end{aligned}$$

□

Note that we have a similar formula for the pairing $\operatorname{Im}\dot{\mathfrak{b}}\cdot\operatorname{Im}\dot{\mathfrak{a}}$. We calculate next our formula for the metric in the fibres of (8.18).

Theorem 8.13. *Assume Condition A and* (8.16). *Let* $(\dot\omega,\dot b,-Jd^hs+d^hs')$ *be an element in the tangent of the fiber of* (8.18) *solving equations* (8.19). *Denote by* $\dot{\mathfrak{b}}$ *and* $\dot{\mathfrak{a}}$ *the associated variations of complex Bott-Chern class and Aeppli class. Then*

$$g_\ell=\frac{2-\ell}{2M_\ell}\left(\frac{2-\ell}{2M_\ell}(\operatorname{Re}\dot{\mathfrak{a}}\cdot\mathfrak{b})^2-\operatorname{Re}\dot{\mathfrak{a}}\cdot\operatorname{Re}\dot{\mathfrak{b}}+\frac{2-\ell}{2M_\ell}(\operatorname{Im}\dot{\mathfrak{a}}\cdot\mathfrak{b})^2-\operatorname{Im}\dot{\mathfrak{a}}\cdot\operatorname{Im}\dot{\mathfrak{b}}\right) \tag{8.20}$$

Proof. The proof follows from Theorem 8.8 and Lemma 8.10 by a straightforward calculation. E.g., for $v=(\dot\omega,0,-Jd^hs)$ we have

$$\begin{aligned}
g_\ell(v,v)&=\frac{\ell-2}{M_\ell}\int_X\langle d^hs\wedge Jd^hs\rangle\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}+\frac{2-\ell}{2M_\ell}\int_X|\dot\omega_0^{1,1}|^2e^{-\ell f_\omega}\frac{\omega^n}{n!}\\
&\quad+\frac{2-\ell}{2M_\ell}\left(\frac{\ell}{2}-\frac{n-1}{n}\right)\int_X|\Lambda_\omega\dot\omega|^2e^{-\ell f_\omega}\frac{\omega^n}{n!}+\left(\frac{2-\ell}{2M_\ell}\right)^2(\operatorname{Re}\dot{\mathfrak{a}}\cdot\mathfrak{b})^2\\
&=\frac{2-\ell}{2M_\ell}\left(-2\int_X\langle d^hs\wedge Jd^hs\rangle\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!}+\int_X|\dot\omega_0^{1,1}|^2e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)\\
&\quad+\frac{2-\ell}{2M_\ell}\left(-\frac{n(2-\ell)-2}{2n}\int_X|\Lambda_\omega\dot\omega|^2e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)+\left(\frac{2-\ell}{2M_\ell}\right)^2(\operatorname{Re}\dot{\mathfrak{a}}\cdot\mathfrak{b})^2\\
&=\frac{2-\ell}{2M_\ell}(-\operatorname{Re}\dot{\mathfrak{b}}\cdot\operatorname{Re}\dot{\mathfrak{a}})+\left(\frac{2-\ell}{2M_\ell}\right)^2(\operatorname{Re}\dot{\mathfrak{a}}\cdot\mathfrak{b})^2.
\end{aligned}$$

□

When the structure group K is trivial, the solutions of (7.25) are in correspondence with solutions of the Calabi problem for Kähler metrics on X (see (7.26)). By Yau's solution of the Calabi Conjecture [44], when $\ell<2$ formula (8.20) defines a positive definite Kähler metric on the 'complexified Kähler moduli space' of Kähler metrics on X with prescribed volume form, as obtained via symplectic reduction in Corollary 7.15. Observe that we have an isomorphism $H^{1,1}_A(X)\cong H^{1,1}(X)$, and therefore the moduli metric is positive-definite by the Lefschetz decomposition.

A case of special interest is when X admits a holomorphic volume form Ω and we take μ as in (7.9) and $\ell=1$. In this case, (7.25) is equivalent to the condition of $\mathrm{SU}(n)$-holonomy for the metric and equation (8.20) matches (up to homothety) Strominger's formula for the special Kähler metric on the 'complexified Kähler moduli' for X [13, Equation (4.1)]. As a consequence of our framework, this classical moduli space is recovered, along with its special metric, via pseudo-Kähler reduction in Theorem 8.8. It is interesting to observe that the formula for the *holomorphic prepotential* on a Calabi-Yau threefold, given by the natural cubic form on $H^{1,1}(X)$, breaks as soon as we split the Kähler class into the Aeppli and Bott-Chern parameters $\mathfrak{a}$ and $\mathfrak{b}$.

On a (non necessarily Kähler) Calabi-Yau threefold (X, Ω) and for a suitable choice of the structure group K, the equations (7.25) are equivalent to the Hull-Strominger system [31, 40] provided that $\ell = 1$ and we take μ as in (7.9) (see Corollary 7.15). For this interesting system of equations, the physics of string theory predicts that the moduli space metric (7.20) should be positive definite along the fibres of (8.18). This follows from our formula for the moduli space Kähler potential, given in this case by

$$K = -\log \int_X \|\Omega\|_\omega \frac{\omega^3}{6}, \tag{8.21}$$

and the following *physical conjecture*, explained in Appendix A.3. Our formula for the Kähler potential (8.21), with $\ell = 1$, shall be compared with [14, Eq. (1.3)], which puts forward the case $\ell = 0$.

Conjecture 8.14. *Formula* (8.21) *defines the Kähler potential for a Kähler metric in the moduli space of solutions of the Hull-Strominger system, for fixed bundle P and fixed Calabi-Yau threefold (X, Ω).*

Combined with Theorem 8.13, we obtain an interesting physical prediction relating the variations of the Aeppli classes and balanced classes of solutions in the special case of the Hull-Strominger system on a Calabi-Yau threefold.

Conjecture 8.15. *If (X, Ω, P) admits a solution of the Hull-Strominger system, then* (8.20) *is positive definite. In particular, the variations of the Aeppli and balanced classes of nearby solutions must satisfy*

$$\mathrm{Re}\, \dot{\mathfrak{a}} \cdot \mathrm{Re}\, \dot{\mathfrak{b}} < \frac{1}{2\int_X \|\Omega\|_\omega \frac{\omega^3}{6}} (\mathrm{Re}\, \dot{\mathfrak{a}} \cdot \mathfrak{b})^2. \tag{8.22}$$

Formula (8.22) provides a potential obstruction to the existence of solutions of the Hull-Strominger system around a given solution. For example, if we fix $\mathrm{Re}\, \dot{\mathfrak{a}}$, the possible variations in the balanced class $\mathrm{Re}\, \dot{\mathfrak{b}}$ are constrained by the duality pairing $\mathrm{Re}\, \dot{\mathfrak{a}} \cdot \mathrm{Re}\, \dot{\mathfrak{b}}$, via an affective bound in terms of the balanced class of the given solution and the value of the dilaton functional. We expect this phenomenon to be related to some global obstruction to the existence of solutions. It would be interesting to obtain a physical explanation for the inequality (8.22).

8.3. **Infinitesimal Donaldson-Uhlenbeck-Yau.** We discuss next the relation between $\mathcal{M}_\ell$ and the moduli space of string algebroids Q over X with fixed class $[E_Q] = [E] \in H^1(\underline{\mathcal{S}})$ (see Lemma 3.5). This relation is suggested by the correspondence between the moduli space of solutions of the Hermite-Yang-Mills equations and the moduli space of polystable principal bundles, given by the Donaldson-Uhlenbeck-Yau Theorem [16, 41]. In the case of our main interest, E is the complexification of $E_{\mathbb{R}}$, our string algebroids are Bott-Chern, and $[E_Q] = [E]$ is equivalent to $r(Q) = [E_{\mathbb{R}}]$ (see Proposition 5.8).

In order to establish this relation, notice that the proof of Lemma 7.7 shows that (7.13) extends to a left $\mathrm{Aut}(E)$-action

$$\begin{aligned} \mathrm{Aut}(E) \times \mathcal{W} &\longrightarrow \mathcal{W} \\ (\underline{f}, W) &\longmapsto \underline{f} \cdot W := \underline{f}(W') \end{aligned} \tag{8.23}$$

where $W' := W(\underline{f}^{-1}(E_{\mathbb{R}}), L_W) \subset \underline{f}^{-1}(E_{\mathbb{R}})$ is the horizontal subspace induced by the Chern correspondence in Lemma 5.11. Similarly as in Lemma 7.7, the forgetful map (7.10) jointly with the action (8.23) induce a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}(E) \times \mathcal{W} & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow \\ \mathrm{Ker}\, \sigma_{\underline{P}} \times \mathcal{A} & \longrightarrow & \mathcal{A}, \end{array}$$

where $\mathrm{Ker}\, \sigma_{\underline{P}} \subset \mathcal{G}_{\underline{P}}$ is as in Corollary 4.2 and the bottom arrow is induced by the action of the complex gauge group $\mathcal{G}_{\underline{P}}$ on $\mathcal{A}$ (see e.g. [16]). Consider the isomorphism $\mathcal{A}^0 \cong \mathcal{C}^0$ between the space of integrable connections $\mathcal{A}^0$ on P_h and the space $\mathcal{C}^0$ of holomorphic principal G-bundle structures on $\underline{P}$, given by the classical Chern correspondence (cf. Section 8.2). Consider the subgroup $\mathrm{Aut}_A(E) \subset \mathrm{Aut}(E)$ as in Definition 4.6. Then, the set-theoretical Chern correspondence in Lemma 7.5 induces a diagram

$$\begin{array}{ccccc} \mathcal{M}_\ell & \longrightarrow & \mathcal{W}^0/\mathrm{Aut}_A(E) & \xrightarrow{\cong} & \mathcal{L}^0/\mathrm{Aut}_A(E) \\ & & \downarrow & & \downarrow \\ & & \mathcal{W}^0/\mathrm{Aut}(E) & \xrightarrow{\cong} & \mathcal{L}^0/\mathrm{Aut}(E) \\ & & \downarrow & & \downarrow \\ & & \mathcal{A}^0/\mathrm{Ker}\, \sigma_{\underline{P}} & \xrightarrow{\cong} & \mathcal{C}^0/\mathrm{Ker}\, \sigma_{\underline{P}} \\ & & \downarrow & & \downarrow \\ & & \mathcal{A}^0/\mathcal{G}_{\underline{P}} & \xrightarrow{\cong} & \mathcal{C}^0/\mathcal{G}_{\underline{P}}, \end{array} \tag{8.24}$$

where $\mathcal{L}^0$ denotes the space of liftings of $T^{0,1}X$ to E and $\sigma_{\underline{P}}$ is as in (4.1).

Let us analyse briefly the tower of moduli spaces on the right hand side of the diagram (8.24). Firstly, $\mathcal{C}^0/\mathcal{G}_{\underline{P}}$ is the moduli space of holomorphic principal G-bundles over X with fixed topological bundle $\underline{P}$, as considered in Section 8.2. The fibre of the map

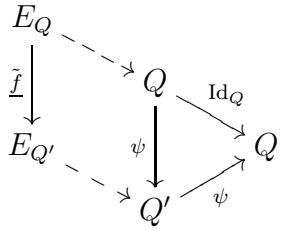
$$\mathcal{C}^0/\mathrm{Ker}\, \sigma_{\underline{P}} \to \mathcal{C}^0/\mathcal{G}_{\underline{P}}$$

over $[P]$ is discrete (see (4.8)). Assuming that the automorphism group $\mathcal{G}_P$ of P is trivial, the fibre is parametrized by $\mathrm{Im}\, \sigma_{\underline{P}} \subset H^3(X, \mathbb{C})$. As for the moduli space $\mathcal{L}^0/\mathrm{Aut}(E)$, we have the following.

Lemma 8.16. *The set $\mathcal{L}^0/\mathrm{Aut}(E)$ parametrizes isomorphism classes of string algebroids Q over X with $[E_Q] = [E] \in H^1(\underline{\mathcal{S}})$ (see Lemma 3.5).*

Proof. Any element $L \in \mathcal{L}_0$ determines a string algebroid Q_L with $[E_{Q_L}] = [E] \in H^1(\underline{\mathcal{S}})$ (see Lemma 3.7). If L and L' are in the same $\mathrm{Aut}(E)$-orbit then by (3.2) it follows that Q_L and $Q_{L'}$ are isomorphic. Conversely, given a string algebroid Q with $[E_Q] = [E]$, then any choice of isomorphism $\underline{f}\colon E_Q \to E$ determines $L = \underline{f}(T^{0,1}X) \in \mathcal{L}^0$. For a different choice of isomorphism $\underline{f'}\colon E_Q \to E$, we have $L' = \underline{f'} \circ \underline{f}^{-1} \cdot L$ which lies in the same $\mathrm{Aut}(E)$-orbit.

Finally, if $\psi\colon Q \to Q'$ is an isomorphism of string algebroids, then there exists a unique isomorphism $\underline{\tilde{f}}\colon E_Q \to E_{Q'}$ in a diagram



which determines $\tilde{L} = \underline{f} \circ \underline{\tilde{f}}^{-1}(T^{0,1}X) \in \mathcal{L}^0$. By Lemma 3.9, $\tilde{L} \in \mathrm{Aut}(E)\cdot L$. $\square$

Remark 8.17. The local geometry of the bigger moduli space of string algebroids over X with varying $[E_Q] \in H^1(\underline{\mathcal{S}})$ has been recently understood in [24] via the construction of a Kuranishi slice theorem.

By the previous lemma, $\mathcal{L}^0/\operatorname{Aut}(E)$ is the moduli space of string algebroids Q over X with fixed complex string algebroid E, while $\mathcal{L}^0/\operatorname{Aut}_A(E)$ is a *Teichmüller space* for string algebroids. We analyse next in detail the infinitesimal structure of the Teichmüller space when E is the complexification of $E_{\mathbb{R}}$, towards a Donaldson-Uhlenbeck-Yau type theorem for the Calabi system. For this, we fix a solution W of (7.25) and consider the associated element $L_W \in \mathcal{L}_0$ via the Chern correspondence in Lemma 7.5. Relying on Lemma 2.15 and Section 8.2, the tangent space of $\mathcal{L}^0/\operatorname{Aut}_A(E)$ at $[L_W]$ is given (formally) by the cohomology of the complex

$$(C^*) \quad \operatorname{Lie}\operatorname{Aut}_A(E) \xrightarrow{\mathbf{P}^c} \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\operatorname{ad}\underline{P}) \xrightarrow{\mathbf{L}^c} \Omega^{1,2+0,3} \oplus \Omega^{0,2}(\operatorname{ad}\underline{P}), \quad (8.25)$$

where $\operatorname{Lie}\operatorname{Aut}_A(E) \subset \Omega^0(\operatorname{ad}\underline{P}) \oplus \Omega^2_{\mathbb{C}}$ is as in Proposition 4.7 and

$$\begin{aligned}\mathbf{P}^c(r,B) &= (B^{1,1+0,2}, \bar\partial r),\\ \mathbf{L}^c(\dot\gamma,\dot\beta) &= (d\dot\gamma^{0,2} + \bar\partial\dot\gamma^{1,1} - 2\langle\dot\beta, F_h\rangle, \bar\partial\dot\beta).\end{aligned}$$

Similarly as for (8.6), (8.25) is not a complex of differential operators (as $\operatorname{Lie}\operatorname{Aut}_A(E)$ is not the space of sections of a vector bundle) and we consider the Lie subalgebra

$$\{(d\xi + 2\langle r, F_h\rangle \mid \xi \in \Omega^1_{\mathbb{C}})\} \subset \operatorname{Lie}\operatorname{Aut}_A(E)$$

and define the induced complex

$$(\widehat{C}^*) \qquad \Omega^0(\operatorname{ad}\underline{P}) \oplus \Omega^1_{\mathbb{C}} \xrightarrow{\widehat{\mathbf{P}}^c} \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\operatorname{ad}\underline{P}) \xrightarrow{\mathbf{L}^c} \Omega^{1,2+0,3} \oplus \Omega^{0,2}(\operatorname{ad}\underline{P}), \quad (8.26)$$

where

$$\widehat{\mathbf{P}}^c(r,\xi) = (d\xi^{0,1} + \bar\partial\xi^{1,0} + 2\langle r, F_h\rangle, \bar\partial r).$$

We show next that the Teichmüller space $\mathcal{L}^0/\operatorname{Aut}_A(E)$ is finite-dimensional.

Lemma 8.18. *The sequence* (8.26) *is an elliptic complex of differential operators. Consequently, the cohomology* $H^1(C^*)$ *of* (8.6) *is finite-dimensional. Furthermore, assuming that* $h^0(\operatorname{ad}P) = 0$ *and* $h^{0,1}_A(X=0)$*, there is an exact sequence*

$$0 \longrightarrow \operatorname{Ker}\partial \longrightarrow H^1(\widehat{C}^*) \longrightarrow H^1(C^*) \longrightarrow 0$$

where ∂ is as in (8.8).

The proof is analogue to that of Lemma 8.2 and it is therefore omitted. Ellipticity of the complex $\widehat{C}^*$ can be easily obtained via comparison with the Dolbeault complex of the holomorphic vector bundle underlying Q_{L_W} (cf. [3, 15] and arXiv version 1503.07562v1 of reference [23]).

Our strategy to compare $H^1(C^*)$ with the tangent to the moduli space of solutions of the Calabi system, given by $H^1(S^*)$ as in Lemma 8.2, is to work orthogonally to the image of the operator $\widehat{\mathbf{P}}^c$ with respect to the non-definite pairing g_ℓ in (7.20). Notice here that the Chern correspondence in Lemma 7.5 induces an isomorphism

$$\begin{aligned}\Upsilon\colon \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\mathrm{ad}\, P_h) &\longrightarrow \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\mathrm{ad}\, \underline{P})\\ (\dot\omega, \dot b, \dot a) &\longmapsto (\dot b^{1,1+0,2} - i\dot\omega, \dot a^{0,1}),\end{aligned}$$

which we use to define the pairing g_ℓ on $\Omega^{1,1+0,2} \oplus \Omega^{0,1}(\mathrm{ad}\, \underline{P})$.

Theorem 8.19. *Assume Condition A and the cohomological conditions* (8.16). *Then, the cohomology of the complexes* (8.6) *and* (8.25) *are canonically isomorphic $H^1(S^*) \cong H^1(C^*)$.*

Proof. Using the conditions $h^{0,1}_A(X) = h^0(\mathrm{ad}\, P) = 0$ one can easily prove that

$$\mathrm{Im}\, \widehat{\mathbf{P}} \cap \mathbf{J}\mathrm{Im}\, \widehat{\mathbf{P}} = \{0\}.$$

Then, via the isomorphism Υ, we have equalities

$$\begin{aligned}\Upsilon^{-1}(\mathrm{Im}\, \widehat{\mathbf{P}}^c) &= \mathrm{Im}\, \widehat{\mathbf{P}} \oplus \mathbf{J}\mathrm{Im}\, \widehat{\mathbf{P}}\\ \Upsilon^{-1}(\mathrm{Ker}\, \mathbf{L}^c) &= \{(\dot\omega, \dot b, \dot a) \mid \bar\partial \dot a^{0,1} = 0, d^c\dot\omega + 2\langle \dot a, F_h\rangle - d\dot b = 0\}\end{aligned}$$

Assuming Condition A, there is a g_ℓ-orthogonal projection Π as in Lemma 8.5 and we consider the map

$$\begin{aligned}\Pi^c\colon \Omega^{1,1}_{\mathbb{R}} \oplus \Omega^2 \oplus \Omega^1(\mathrm{ad}\, P_h) &\longrightarrow (\mathrm{Im}\, \widehat{\mathbf{P}} \oplus \mathbf{J}\mathrm{Im}\, \widehat{\mathbf{P}})^{\perp_{g_\ell}}\\ v &\longmapsto \Pi^c v := -\mathbf{J}(\mathrm{Id} - \Pi)\mathbf{J}(\mathrm{Id} - \Pi)v.\end{aligned}$$

We take $y_j \in \Omega^0(\mathrm{ad}\, P_h) \oplus \Omega^1$, for $j = 1, 2$, and check that it is well-defined

$$\begin{aligned}g_\ell(\Pi^c v, \widehat{\mathbf{P}}(y_1) + \mathbf{J}\widehat{\mathbf{P}}(y_2)) &= g_\ell(v - \Pi v, \mathbf{J}\widehat{\mathbf{P}}(y_2))\\ &\quad + \Omega_\ell(\Pi\mathbf{J}(v - \Pi v), \widehat{\mathbf{P}}(y_1)) + g_\ell(\Pi\mathbf{J}(v - \Pi v), \widehat{\mathbf{P}}(y_2))\\ &= g_\ell(v - \Pi v, \mathbf{J}\widehat{\mathbf{P}}(y_2)) + g_\ell(\mathbf{J}(v - \Pi v), \widehat{\mathbf{P}}(y_2)) = 0.\end{aligned}$$

For the second equality we used that $\mathrm{Im}\, \Pi \subset \mathrm{Im}\, \widehat{\mathbf{P}}$, that μ_ℓ is equivariant, and also $\mu_\ell(W) = 0$. By Proposition 8.6, there is an equality

$$\mathcal{H}^1(\widehat{S}^*) := \ker \mathbf{L} \cap \ker \widehat{\mathbf{P}}^* = (\mathrm{Im}\, \widehat{\mathbf{P}} \oplus \mathbf{J}\mathrm{Im}\, \widehat{\mathbf{P}})^{\perp_{g_\ell}} \cap \Upsilon^{-1}(\mathrm{Ker}\, \mathbf{L}^c)$$

and therefore, since Π^c preserves $\Upsilon^{-1}(\mathrm{Ker}\, \mathbf{L}^c)$, it induces a well-defined surjective map

$$\Pi^c\colon \Upsilon^{-1}(\mathrm{Ker}\, \mathbf{L}^c) \longrightarrow \mathcal{H}^1(\widehat{S}^*).$$

We claim that this map induces an isomorphism $H^1(\widehat{C}^*) \cong \mathcal{H}^1(\widehat{S}^*)$. To see this, notice that (8.14) implies that $\Pi\mathbf{J}\mathrm{Im}\,\widehat{\mathbf{P}} = 0$, as

$$g_\ell(\widehat{\mathbf{P}}(y_1), \Pi\mathbf{J}\widehat{\mathbf{P}}(y_2)) = -\Omega_\ell(\widehat{\mathbf{P}}(y_1), \widehat{\mathbf{P}}(y_2)) = 0.$$

for any y_1, y_2. Then, if $v = \widehat{\mathbf{P}}(y_1) + \mathbf{J}\widehat{\mathbf{P}}(y_2)$ it follows that

$$\begin{aligned}\Pi^c v &= v - \Pi v + \mathbf{J}\Pi\mathbf{J}(v - \Pi v)\\ &= (\mathrm{Id} - \Pi)\mathbf{J}\widehat{\mathbf{P}}(y_2) + \mathbf{J}\Pi\mathbf{J}(\mathrm{Id} - \Pi)\mathbf{J}\widehat{\mathbf{P}}(y_2) = \mathbf{J}\widehat{\mathbf{P}}(y_2) - \mathbf{J}\Pi\widehat{\mathbf{P}}(y_2) = 0.\end{aligned}$$

Conversely, if $\Pi^c v = 0$:

$$v = \Pi v - \mathbf{J}\Pi\mathbf{J}(v - \Pi v) \in \mathrm{Im}\,\widehat{\mathbf{P}} \oplus \mathbf{J}\mathrm{Im}\,\widehat{\mathbf{P}},$$

and therefore $H^1(\widehat{C}^*) \cong \mathcal{H}^1(\widehat{S}^*)$, as claimed. The proof follows combining Lemma 8.2 with Lemma 8.18. □

Our Theorem 8.19 can be regarded as an infinitesimal Donaldson-Uhlenbeck-Yau type theorem, relating the moduli space of solutions of the Calabi system with the Teichmüller space $\mathcal{L}^0/\operatorname{Aut}_A(E)$ for string algebroids. This strongly suggests that–if we shift our perspective and consider the Calabi system as equations

$$\begin{aligned} F_h \wedge \omega^{n-1} &= 0,\\ d(e^{-\ell f_\omega}\omega^{n-1}) &= 0,\end{aligned} \tag{8.27}$$

for a compact form on fixed string algebroid Q along a fixed Aeppli class $\mathfrak{a} \in \Sigma_A(Q, \mathbb{R})$ (see Proposition 6.8)–the existence of solutions should be related to a stability condition in the sense of Geometric Invariant Theory. This was essentially the point of view taken in [22]. The precise relation with stability in our context is still unclear, as the balanced class $\mathfrak{b} \in H^{n-1,n-1}(X, \mathbb{R})$ of the solution varies in the moduli space $\mathcal{M}_\ell$. Recall that $\mathfrak{b}$ is required to measure slope stability of the holomorphic bundle in the classical Donaldson-Uhlenbeck-Yau Theorem [16, 41] (see also [33]). The conjectural stability condition which characterizes the existence of solutions of (8.27) should be for pairs given by string algebroid Q of Bott-Chern type equipped with a 'complexified Aeppli class' (see Appendix B). It must be closely related to the properties of the *integral of the moment map* μ_ℓ for compact forms in a fixed Aeppli class, given by the ℓ-dilaton functional (cf. [22])

$$M_\ell \colon \mathbf{B}_Q \to \mathbb{R}.$$

We speculate that there is a relation between this new form of stability and the conjectural inequality (8.22). This may lead to an obstruction to the global existence which goes beyond the slope stability of the bundle and the balanced property of the manifold (cf. [45]).

8.4. **Examples.** We present an interesting class of examples of solutions of the Calabi system where Condition A holds, and Theorem 8.8, Theorem 8.13 and Theorem 8.19 apply. These examples are non-Kähler solutions of (7.25) obtained by deformation of a solution of the Calabi problem for Kähler metrics, as in (7.26), equipped with a polystable vector bundle. Our method is analogue

to the one used in [2] to find solutions of the Hull-Strominger system on Kähler Calabi-Yau manifolds.

Let X be a compact Kähler manifold equipped with smooth volume form μ compatible with the orientation and a Kähler class $\mathbf{k} \in H^{1,1}(X,\mathbb{R})$. Let V_0 and V_1 be $\mathbf{k}$-stable holomorphic vector bundles over X with vanishing first Chern class and the same second Chern character

$$ch_2(V_0) = ch_2(V_1) \in H^{2,2}(X,\mathbb{R}).$$

Given $\ell, \epsilon \in \mathbb{R}$, consider the system of equations

$$\begin{aligned} F_{h_0} \wedge \omega^{n-1} &= 0, \\ F_{h_1} \wedge \omega^{n-1} &= 0, \\ d(e^{-\ell f_\omega}\omega^{n-1}) &= 0, \\ dd^c\omega - \epsilon \operatorname{tr}_0 F_{h_0} \wedge F_{h_0} + \epsilon \operatorname{tr}_1 F_{h_1} \wedge F_{h_1} &= 0. \end{aligned} \tag{8.28}$$

for a hermitian form ω on X and hermitian metrics h_j in V_j. Taking P to be the bundle of split frames of $V_0 \oplus V_1$, any solution of (8.28) provides a solution of the Calabi system (7.25) for

$$\langle\, , \rangle_\epsilon = -\epsilon \operatorname{tr}_0 + \epsilon \operatorname{tr}_1 .$$

Combining the Donaldson-Uhlenbeck-Yau Theorem [16, 41] with Yau's solution of the Calabi Conjecture [44], there exists a unique solution $(\omega_0, h_{0,0}, h_{1,0})$ of (8.28) for $\epsilon = 0$ with $[\omega_0] = \mathbf{k}$. Notice here that such solution must be necessarily Kähler (see [22]), that is, $d\omega_0 = 0$.

Proposition 8.20. *Assume $\ell > 2 - \frac{2}{n}$ and $h^{0,1}(X) = 0$, and let (X, V_0, V_1) be as above. Then, there exists $\epsilon_0 > 0$ and a smooth family $(\omega_\epsilon, h_{0,\epsilon}, h_{1,\epsilon})$ of solutions of (8.28) parametrized by $[0, \epsilon_0[$ such that Condition A holds for sufficiently small $\epsilon > 0$. Furthermore, $(\omega_\epsilon, h_{0,\epsilon}, h_{1,\epsilon})$ converge uniformly in C^∞ norm to $(\omega_0, h_{0,0}, h_{1,0})$ as $\epsilon \to 0$.*

Proof. Existence of the family of solutions $(\omega_\epsilon, h_{0,\epsilon}, h_{1,\epsilon})$ follows as in [2] by application of an implicit function theorem argument (cf. [22, Lem. 5.17]). We prove now that any such solution satisfies Condition A for sufficiently small ϵ. Denote by P_{h_ϵ} the bundle of split unitary frames for $h_\epsilon = h_{0,\epsilon} \times h_{1,\epsilon}$. For $\mathcal{L}_\epsilon$ as in (8.12) and $(u, \xi) \in \Omega^0(\operatorname{ad} P_{h_\epsilon}) \times \operatorname{Im} d^*$, the condition $\mathcal{L}_\epsilon(u,\xi) = 0$ is equivalent to

$$\begin{aligned} d\Big(e^{-\ell f_{\omega_\epsilon}}\Big((n-1)((d\xi)^{1,1} + 2\langle u, F_{h_\epsilon}\rangle_\epsilon) \wedge \omega_\epsilon^{n-2} - \frac{\ell}{2}(\Lambda_{\omega_\epsilon} d\xi)\omega_\epsilon^{n-1}\Big)\Big) &= 0, \\ d^{h_\epsilon} J d^{h_\epsilon} u \wedge \omega_\epsilon^{n-1} - (n-1) F_{h_\epsilon} \wedge (d\xi + 2\langle u, F_{h_\epsilon}\rangle_\epsilon) \wedge \omega_\epsilon^{n-2} &= 0. \end{aligned}$$

Consider the family of elliptic operator

$$U_{\epsilon,0} : \quad \Omega^0(\operatorname{ad} P_{h_\epsilon}) \quad \to \quad \Omega^{2n}(\operatorname{ad} P_{h_\epsilon})$$

defined by

$$U_{\epsilon,0}(u) = d^{h_\epsilon} J d^{h_\epsilon} u \wedge \omega_\epsilon^{n-1} - (n-1) F_{h_\epsilon} \wedge (2\langle u, F_{h_\epsilon}\rangle_\epsilon) \wedge \omega_\epsilon^{n-2}.$$

By hypothesis, $\tilde{U}_{0,0}$ is elliptic with zero kernel, and therefore $U_{\epsilon,0}$ has vanishing kernel for sufficiently small ϵ by upper semi-continuity of $\dim \operatorname{Ker} U_{\epsilon,0}$. Notice

that $U_{\epsilon,0}$ can be regarded as an operator $\Omega^0(\operatorname{ad} P_{h_0}) \to \Omega^0(\operatorname{ad} P_{h_0})$ by a gauge transformation depending only on h_ϵ. Let $\epsilon > 0$ such that $\operatorname{Ker} U_{\epsilon,0} = \{0\}$, and assume that $(u_\epsilon, \xi_\epsilon) \in \operatorname{Ker} \mathcal{L}_\epsilon$. Given $\lambda \in \mathbb{R}$, consider the family of elliptic operators

$$U_{\epsilon,\lambda} : \quad \Omega^0(\operatorname{ad} P_{h_\epsilon}) \quad \to \quad \Omega^0(\operatorname{ad} P_{h_\epsilon})$$

defined by

$$U_{\epsilon,\lambda}(u) = d^{h_\epsilon} J d^{h_\epsilon} u \wedge \omega_\epsilon^{n-1} - (n-1) F_{h_\epsilon} \wedge (\lambda d\xi_\epsilon + 2\langle u, F_{h_\epsilon}\rangle_\epsilon) \wedge \omega_\epsilon^{n-2}.$$

By upper semi-continuity of $\dim \operatorname{Ker} U_{\epsilon,\lambda}$ we have that $\operatorname{Ker} U_{\epsilon,\lambda} = \{0\}$ for sufficiently small λ. Since $\lambda u_\epsilon \in \operatorname{Ker} U_{\epsilon,\lambda}$, it follows that $u_\epsilon = 0$. Using now Lemma 8.3 and setting $v = (0, d\xi_\epsilon, 0)$, we have $g_\ell(v, v) = 0$ and therefore

$$\int_X |((d\xi)^{1,1})_0|^2 e^{-\ell f_{\omega_\epsilon}} \frac{\omega_\epsilon^n}{n!} + \left(\frac{\ell}{2} - \frac{n-1}{n}\right) \int_X |\Lambda_{\omega_\epsilon} d\xi|^2 e^{-\ell f_{\omega_\epsilon}} \frac{\omega_\epsilon^n}{n!} = 0.$$

For $\ell > 2 - \frac{2}{n}$ this implies $(d\xi)^{1,1} = 0$, and therefore $\partial\bar\partial \xi^{0,1} = 0$. Finally, using that $h^{0,1}(X) = 0$ we conclude $\xi^{0,1} = \bar\partial\phi$ for some complex valued function ϕ, and hence $d\xi = 0$. □

We finish with concrete examples where the hypothesis of Proposition 8.20 are satisfied. We will take X to be a Calabi-Yau threefold with holomorphic volume form Ω, and μ as in (7.9). We choose a Kähler class $\mathbf{k}$, and $\mathbf{k}$-stable bundles V_0 and V_1 such that

$$c_1(V_j) = 0, \qquad c_2(V_j) = c_2(X)$$

(see [2, 21] and references therein for constructions of such bundles). In this setup, $h^{0,1}(X) = h^{0,2}(X) = 0$ and $h^0(\operatorname{End} V_0) = h^0(\operatorname{End} V_1) = 0$. Hence, the hypothesis of Proposition 8.20 hold, and Theorem 8.8, Theorem 8.13 and Theorem 8.19 apply.

Our choice of bundles V_0, V_1 can be interpreted, geometrically, as a deformation of the special Kähler metric on the 'complexified Kähler moduli' for the Calabi-Yau manifold X (see Section 8.2). More precisely, Proposition 8.20 combined with Theorem 8.13 gives a family of pseudo-Kähler metrics $g_{\ell,\epsilon}$ (see (8.20)) in a non-empty open subset of $H^{1,1}(X) \cong H^{1,1}_A(X)$, for $(\ell, \epsilon) \in]\frac{4}{3}, 2[\times[0, \epsilon_0[$. Here, the fibre of (8.18) over $[P]$ (for P the bundle of split frames of $V_0 \oplus V_1$) is regarded as a subset of $H^{1,1}(X)$ via [22, Corollary 5.14]. The special Kähler metric in the 'complexified Kähler moduli' of X is recovered (up to homothety) in the $\epsilon \to 0$ limit of this family. The case of the Hull-Strominger equations corresponds to $\ell = 1$, and it is not covered by our result.

Example 8.21. Let X be a complete intersection Calabi-Yau threefold. By [32, Cor. 2.2], TX has unobstructed deformations parametrized by $H^1(\operatorname{End} TX)$. Since TX is stable for any Kähler class, any pair of small deformations V_0 and V_1 of TX are also stable. For the quintic hypersurface $h^1(\operatorname{End} TX) = 224$ and we obtain a family of deformations of the special Kähler metric on $H^{1,1}(X)$ of dimension 450, parametrized by a non-empty open subset of

$$H^1(\operatorname{End} TX) \times H^1(\operatorname{End} TX) \times]\tfrac{4}{3}, 2[\times[0, \epsilon_0[.$$

Appendix A. Moduli Kähler potential and the gravitino mass

A.1. **The gravitino equation.** In this section we explain the physical argument which leads to Conjecture 8.14 and to the formula (8.21) for the Kähler potential on the moduli space of solutions of the Hull-Strominger system. This provides further motivation for Conjecture 8.15. In addition, we hope that this addendum makes the present work more accessible to physicists.

We start with a brief detour which shows that both string algebroids of Bott-Chern type and the Hull-Strominger system, as considered in the present work, appear naturally via variational principles in string theory. Consider a flux compactification of the heterotic string from 10 to 4 dimensions. Spacetime is assumed to be topologically of the form

$$\mathbb{R}^4 \times \underline{X},$$

where $\underline{X}$ is the internal (spin) compact manifold. The 10-dimensional metric is a warped product [31, 40]

$$g_{10} = e^{2D}(g_{1,3} + g_6),$$

where $g_{1,3}$ is the flat Minkowski metric, g_6 is a Riemannian metric, and D is a conformal factor which only depends on $\underline{X}$.

Assume that this geometry satisfies the 10-dimensional *gravitino equation*, that is, there exists a covariantly constant spinor for the spin connection associated to

$$\nabla^{g_{10}} - \frac{1}{2}H_{10},$$

for $\nabla^{g_{10}}$ the Levi-Civita connection and H_{10} the 10-dimensional three-form flux. Then, we have the integrability condition [40]

$$|d(D - \phi^{10})|^2 = 0$$

where ϕ_{10} is the 10-dimensional dilaton, which implies $D = \phi^{10}$. Assuming a natural compactification ansatz for the spinor, we further obtain an SU(3)-structure (Ψ, ω) on $\underline{X}$ satisfying

$$H_{\mathbb{R}} = -N + (d^c\omega)^{2,1+1,2}. \tag{A.1}$$

Here N denotes the Nijenhuis tensor of the almost complex structure induced by Ψ and $H_{\mathbb{R}} = H_{10}$, depending only on the internal manifold.

Under these assumptions, we would like to characterize compactification backgrounds with $N = 1$ supersymmetry in 4-dimensions. In other words, we want to understand which solutions of the 10-dimensional *gravitino equation* also satisfy the *dilatino equation* and the *gaugino equation*, which at this point can be written simply as

$$(H_{\mathbb{R}} + 2d\phi) \cdot \eta = 0, \qquad F_A \cdot \eta = 0.$$

Here, η is a spinor on $(\underline{X}, g_6)$ associated to (Ψ, ω), A is the gauge field, and $\phi = \phi^{10}$, depending only on the internal manifold. Strikingly, this question can be turned into a variational problem for two natural physical quantities: the heterotic superpotential and the dilaton of the 4-dimensional effective theory.

A.2. **The heterotic superpotential.** Consider the heterotic superpotential, defined on solutions of the gravitino equation as follows [29, 28]

$$\mathcal{W} = \int_{\underline{X}} e^{-2\phi}(H_{\mathbb{R}} - id\omega) \wedge \Psi. \tag{A.2}$$

The variation of $\mathcal{W}$ with respect to Ψ implies that

$$(H_{\mathbb{R}} - id\omega)^{1,2+0,3} = 0$$

since the variation of Ψ lies in $\Omega^{3,0} \oplus \Omega^{2,1}$, and therefore

$$N = 0, \qquad H_{\mathbb{R}} = d^c\omega$$

by (A.1) (in particular, the almost complex structure induced by Ψ is integrable). The variation of $\mathcal{W}$ with respect to ω implies that

$$d(e^{-2\phi}\Psi) = 0,$$

and therefore $\Omega = e^{-2\phi}\Psi$ is a holomorphic volume form on $\underline{X}$. Note that the previous conditions already imply $\mathcal{W} = 0$.

The variation of $\mathcal{W}$ with respect to $H_{\mathbb{R}}$ requires a special treatment due to the Green-Schwarz mechanism for anomaly cancellation, relating $H_{\mathbb{R}}$ with the gauge field A and an auxiliary connection ∇ on the tangent bundle via the Bianchi identity

$$dH_{\mathbb{R}} = \alpha' \operatorname{tr} R_\nabla \wedge R_\nabla - \alpha' \operatorname{tr} F_A \wedge F_A. \tag{A.3}$$

One way of understanding mathematically this variation is to regard the data ω, $H_{\mathbb{R}}$, ∇, and A as induced by a horizontal lift on a real string algebroid

$$W \subset E_{\mathbb{R}}$$

(see Section 5.3) and impose a Dirac quantization condition on the isomorphism class (see Section 5.2)

$$[E_{\mathbb{R}}] \in H^1(\mathcal{S}_{\mathbb{R}}).$$

Choose an SU(3)-structure on $\underline{X}$ and consider the principal bundle $P_{\mathbb{R}}$ given by the product of the bundle of special unitary frames on $\underline{X}$ with the gauge bundle. The set $H^1(\mathcal{S}_{\mathbb{R}})$ fits naturally in a exact sequence of pointed sets

$$H^1(\mathcal{S}_{\mathbb{R}}) \xrightarrow{\jmath} H^1(\mathcal{C}_K) \xrightarrow{p_1} H^4(\underline{X}, \mathbb{R}),$$

(see [24, Prop. A.4]) where $H^1(\mathcal{C}_K)$ is the set of isomorphism classes of principal K-bundles and p_1 stands for the first Pontryagin class of the bundle with respect to (cf. (5.1))

$$\langle\, ,\rangle = -\alpha' \operatorname{tr}_{T\underline{X}} + \alpha' \operatorname{tr}.$$

The fibre $\jmath^{-1}([P_{\mathbb{R}}])$ is a quotient of the $H^3(\underline{X}, \mathbb{R})$-torsor of *real string classes* [38] (see [24, Prop. A.8]). Integral elements in $H^1(\mathcal{S}_{\mathbb{R}})$ are given by (classes of) isotopy classes of lifts of $P_{\mathbb{R}} \colon X \to BK$ to the classifying space of the corresponding *string group* [42]. Dirac quantization of $[E_{\mathbb{R}}]$ implies that a variation $\dot{H}_{\mathbb{R}}$ of $H_{\mathbb{R}}$ must satisfy

$$\dot{H}_{\mathbb{R}} = db - 2\alpha' \operatorname{tr} \dot{\nabla} \wedge R_\nabla + 2\alpha' \operatorname{tr} \dot{A} \wedge F_A,$$

for a two-form b on $\underline{X}$. Taking this into account, the variation of the superpotential $\mathcal{W}$ with respect to $H_{\mathbb{R}}$ implies

$$F_A \wedge \Psi = 0, \qquad R_\nabla \wedge \Psi = 0,$$

or equivalently $F_A^{0,2} = 0 = R_\nabla^{0,2}$.

The upshot of the previous discussion is the following: by imposing the condition of critical point for the heterotic superpotential (on top of the gravitino equation)

$$\delta \mathcal{W} = 0,$$

we have obtained a familiar geometry discussed in Section 5.3, namely, a Calabi-Yau threefold (X, Ω) and a real string algebroid $E_{\mathbb{R}}$ equipped with a horizontal lift $W \subset E_{\mathbb{R}}$ inducing a lifting of $T^{0,1}X$ (see Lemma 7.5)

$$L_W \subset E_{\mathbb{R}} \otimes \mathbb{C}.$$

In particular, by Lemma 2.15 and Proposition 5.8 we obtain a string algebroid $Q = Q_{L_W}$ over the the Calabi-Yau threefold (X, Ω) endowed with a compact form (hence, Q is Bott-Chern by Definition 5.6).

A.3. **The dilaton functional and the gravitino mass.** Consider the universal formula for the 4-dimensional dilaton in the effective field theory induced by a string compactification [4]

$$e^{-2\phi_4} = \int_{\underline{X}} e^{-2\phi_{10}} \mathrm{vol}_{g_6}. \tag{A.4}$$

Imposing the gravitino equation and $\delta \mathcal{W} = 0$, we obtain the alternative expression

$$e^{-2\phi_4} = \int_{\underline{X}} \|\Omega\|_\omega \frac{\omega^3}{6},$$

which is precisely the formula for the *dilaton functional* with level $\ell = 1$ and volume form (7.9). We shift our perspective and regard $e^{-2\phi_4}$ as a functional for compact forms on a fixed Bott-Chern algebroid Q, given by a critical point of the superpotential. Fixing now the Aeppli class $[E_{\mathbb{R}}] \in \Sigma_A(Q, \mathbb{R})$ (see Section 6.2), the variation of the dilaton functional is given by

$$\frac{1}{4} \int_{\underline{X}} (2i\langle h^{-1}\dot{h}, F_h\rangle + \partial\xi^{0,1} + \overline{\partial\xi^{0,1}}) \wedge \|\Omega\|_\omega \omega^2,$$

and we obtain the desired variational characterization of the remaning $N = 1$ supersymmetry conditions in four dimensions, as observed originally in [22].

Proposition A.1 ([22])**.** *The critical points of the dilaton functional for compact forms on Q in a fixed Aeppli class are the solutions of the equations*

$$d(\|\Omega\|_\omega \omega^2) = 0, \qquad F_h \wedge \omega^2 = 0.$$

Let us now turn to Conjecture 8.14 and formula (8.21) for the moduli Kähler potential. We build on a universal relation between the Kähler potential, the superpotential, and the *gravitino mass*. In the context of $N = 1$ four-dimensional supergravity the gravitino mass $m_{3/2}$ can be written as

$$m_{3/2} = c_0 e^{K/2} \mathcal{W}$$

for some universal constant $c_0 \in \mathbb{R}$. By the standard supersymmetry lore, $N = 1$ supersymmetry in four dimensions imposes that the scalar manifold is a Kähler Hodge manifold, and K is the Kähler potential. For a compactification of 10-dimensional heterotic supergravity to 4-dimensions, the superpotential is given by (A.2), and the scalar manifold corresponds to the moduli space of solutions of the Hull-Strominger system.

A Gukov-type formula [29] for the gravitino mass in 4-dimensional heterotic flux compatifications was derived in [30] (valid to first order in α' expansion), namely,

$$m_{3/2} = \frac{\sqrt{8}e^{\phi_4}\mathcal{W}}{4\int_{\underline{X}} \|\Omega\|_\omega \frac{\omega^3}{6}},$$

where $e^{-2\phi_4}$ is the four-dimensional dilaton (A.4). The previous two formulae need to be understood off-shell, that is, without imposing the supersymmetry conditions coming from $\delta\mathcal{W} = 0$, nor the equations of motion of the ten-dimensional theory. Comparing the two formula for the gravitino mass, it follows that

$$e^K = \frac{e^{2\phi_4}}{2c_0^2(\int_{\underline{X}} \|\Omega\|_\omega \frac{\omega^3}{6})^2}.$$

By the discussion in Section A.2, imposing now $\delta\mathcal{W} = 0$, we have $e^{-2\phi_4} = \int_{\underline{X}} \|\Omega\|_\omega \frac{\omega^3}{6}$, and therefore we obtain the following off-shell formula for the moduli Kähler potential

$$K = -3\log \int_{\underline{X}} \|\Omega\|_\omega \frac{\omega^3}{6} - 2\log c_0 - \log 2. \tag{A.5}$$

This physical prediction from the heterotic string must be handled very carefully. In the physical analysis, the connection ∇ on $T\underline{X}$ is a complicated function of the hermitian form ω and the parameter α' in the Bianchi identity (A.3). Thus, a comparison with our mathematical study of the metric in Section 8.2 only seems to be valid if we fix the holomorphic principal bundle underlying the Bott-Chern algebroid Q. This motivates Conjecture 8.14. Notice that formula (A.5) for the Kähler potential agrees with [14, Eq. (1.3)] to first order in α' expansion. We thank J. McOrist for this interesting observation.

Appendix B. Complexified Aeppli classes

In this section we dwell further into the geometry of the sequence of moduli spaces on the right hand side of (8.24). Our goal is to find an explanation for the variations of 'complexified Aeppli classes' appearing in formula (8.20) for the fibre-wise moduli metric, via the study of the Teichmüller space $\mathcal{L}^0/\operatorname{Aut}_A(E)$ for string algebroids. Recall here that the infinitesimal Donaldson-Uhlenbeck-Yau type Theorem 8.19 identifies the tangent to the moduli space $\mathcal{M}_\ell$ with the tangent to the Teichmüller space. We follow the notation in Section 8.3.

By Lemma 8.16, the fibre over $[P] \in \mathcal{C}^0/\operatorname{Ker}\sigma_{\underline{P}}$ of the natural map

$$\mathcal{L}^0/\operatorname{Aut}(E) \to \mathcal{C}^0/\operatorname{Ker}\sigma_{\underline{P}} \tag{B.1}$$

parametrizes isomorphism classes of string algebroids with underlying principal G-bundle P. To give a cohomological interpretation of this fibre, denote by $\Omega^{2,0}_{cl}$ the sheaf of (holomorphic) closed $(2,0)$-forms on X. Recall from [23, Lem. 2.10] that there is a group homomorphism

$$\sigma_P \colon \mathcal{G}_P \to H^1(\Omega^{2,0}_{cl})$$

defined by

$$\sigma_P(g) = [CS(g\theta^h) - CS(\theta^h) - d\langle g\theta^h \wedge \theta^h\rangle] \in H^1(\Omega^{2,0}_{cl}),$$

for any choice of reduction $h \in \Omega^0(P/K)$. Here we use [27] (see also [23, Lem. 3.3]) to identify

$$H^1(\Omega^{2,0}_{cl}) \cong \frac{\text{Ker } d\colon \Omega^{3,0} \oplus \Omega^{2,1} \to \Omega^{4,0} \oplus \Omega^{3,1} \oplus \Omega^{2,2}}{\text{Im } d\colon \Omega^{2,0} \to \Omega^{3,0} \oplus \Omega^{2,1}}. \tag{B.2}$$

The quotient

$$H^1(\Omega^{2,0}_{cl})/\text{Im } \sigma_P.$$

can be identified with the set of isomorphism classes of string algebroids with underlying holomorphic principal G-bundle P (see [24, Prop. 3.11]). We want to describe the fibre of (B.1) as a natural subspace of $H^1(\Omega^{2,0}_{cl})/\text{Im } \sigma_P$. Using (4.9) and the isomorphism (B.2), we define a map

$$\partial \colon H^{1,1}_A(X) \to H^1(\Omega^{2,0}_{cl}), \tag{B.3}$$

induced by the ∂ operator on forms acting on representatives. We consider also the the natural map from Aeppli to Bott-Chern cohomology induced by the $\bar\partial$ operator:

$$H^{1,1}_A(X) \xrightarrow{\bar\partial} \overline{H^1(\Omega^{2,0}_{cl})} := \frac{\text{Ker } d\colon \Omega^{1,2} \oplus \Omega^{0,3} \to \Omega^{2,2} \oplus \Omega^{1,3} \oplus \Omega^{0,4}}{\text{Im } d\colon \Omega^{0,2} \to \Omega^{1,2} \oplus \Omega^{0,3}} \tag{B.4}$$

Lemma B.1. *The fibre of* (B.1) *over* $[P]$ *is an affine space modelled on the image of the map*

$$\partial \colon \ker \bar\partial \to H^1(\Omega^{2,0}_{cl})/\text{Im } \sigma_P \tag{B.5}$$

induced by (B.3)*, where* $\ker \bar\partial \subset H^{1,1}_A(X)$ *is defined by* (B.4)

Proof. Fix a lifting $L_0 \in \mathcal{L}^0$ and denote by P the induced holomorphic principal G-bundle structure on $\underline{P}$. Without loss of generality, we fix an isotropic splitting $\lambda_0 \colon T\underline{X} \to E$ and regard

$$\mathcal{L}^0 \subset \Omega^{1,1+0,2} \oplus \Omega^{0,1}(\text{ad } \underline{P}).$$

We can choose λ_0 such that $L_0 = (0,0)$, with induced three-form $H \in \Omega^{3,0+2,1}$ and connection θ^h, for some choice of reduction $h \in \Omega^0(P/K)$. Then, by Proposition 2.16, if $L = (\gamma, \beta) \in \mathcal{L}^0$ induces $[P] \in \mathcal{C}^0/\operatorname{Ker} \sigma_{\underline{P}}$ it follows that β is in the $\operatorname{Ker} \sigma_{\underline{P}}$-orbit of 0. By (4.1), we can 'gauge' β and assume that $(\gamma, \beta) = (\gamma, 0)$. Hence,

$$d\gamma^{0,2} + \bar\partial \gamma^{1,1} = 0$$

and γ induces a class

$$[\gamma^{1,1}] \in \ker \bar\partial \subset H^{1,1}_A(X).$$

The change in the isomorphism class of the string algebroid, from L_0 to L, is (see Proposition 2.16)

$$\partial([\gamma^{1,1}]) := [\partial\gamma^{1,1}] \in H^1(\Omega^{2,0}_{cl}).$$

An element $(\gamma', 0) \in \mathcal{L}^0$ is in the same $\mathrm{Aut}(E)$-orbit as $(\gamma, 0)$ if and only if the corresponding string algebroids are isomorphic (see Lemma 8.16). This is equivalent to the existence of $g \in \mathcal{G}_P$ and $B \in \Omega^{2,0}$ such that (see Proposition 2.3)

$$\partial\gamma'^{1,1} = \partial\gamma^{1,1} + CS(g\theta^h) - CS(\theta^h) - d\langle g\theta^h \wedge \theta^h\rangle + dB.$$

Thus, the induced map from the fibre of (B.1) over $[P]$ to $H^1(\Omega^{2,0}_{cl})/\mathrm{Im}\ \sigma_P$ is well-defined and injective. Surjectivity onto the image of (B.5) follows from Proposition B. □

We turn next to the study of the map

$$\mathcal{L}^0/\operatorname{Aut}_A(E) \to \mathcal{L}^0/\operatorname{Aut}(E). \tag{B.6}$$

Set-theoretically, the fibre over $[L] \in \mathcal{L}^0/\operatorname{Aut}(E)$ is given by the double quotient $\operatorname{Aut}_A(E)\backslash\operatorname{Aut}(E)/\operatorname{Aut}(Q_L)$, where $\operatorname{Aut}(Q_L)$ is regarded as the isotropy group of L in $\operatorname{Aut}(E)$. Following Proposition 4.7 and Lemma B.1, this quotient should have an intepretation in terms of the Aeppli cohomology group $H^{1,1}_A(X)$. The precise relation goes beyond the scope of the present work, and shall be compared with the link between the group of symplectomorphisms and the first cohomology of a symplectic manifold via the flux homomorphism (see Remark 4.8). Our modest goal here is to characterize the tangent to the fibre of (B.6). Strikingly, this infinitesimal study requires the classical Futaki invariant for the principal bundle P [19] (see also [1]). Let $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$ be a Bott-Chern class. Then, the Futaki invariant of P is given by a Lie algebra homomorp hism

$$\mathcal{F}_{\mathfrak{b}} \colon \operatorname{Lie} \mathcal{G}_P \to \mathbb{C}$$

which provides an obstruction to the existence of solutions of the Hermite-Yang-Mills equations for a given balanced metric on X with class $\mathfrak{b}$ (and hence in particular of (7.25)). Using the duality pairing $H^{1,1}_A(X) \cong H^{n-1,n-1}_{BC}(X)^*$ between the Aeppli and Bott-Chern cohomologies, the Futaki invariant can be regarded as the Lie algebra homomorphism

$$\begin{aligned} \mathcal{F} \colon \operatorname{Lie} \mathcal{G}_P &\to H^{1,1}_A(X) \\ s &\mapsto [\langle s, F_h\rangle] \end{aligned}$$

for any choice of reduction $h \in \Omega^0(P/K)$. Using Lemma 5.2, it is not difficult to see that (B.5) induces a well-defined map

$$\partial \colon \ker\bar\partial/\mathrm{Im}\ \mathcal{F} \to H^1(\Omega^{2,0}_{cl})/\mathrm{Im}\ d\sigma_P, \tag{B.7}$$

where $\ker\bar\partial \subset H^{1,1}_A(X)$ is defined by (B.4).

Lemma B.2. *Let $L \in \mathcal{L}^0$ with induced principal bundle P. Then, the tangent to the fibre of* (B.6) *over $[L] \in \mathcal{L}^0/\operatorname{Aut}(E)$ is isomorphic to the kernel of* (B.7).

Proof. We build on the proof of Lemma B.1, following the same notation. We fix a lifting $L_0 \in \mathcal{L}^0$ and an isotropic splitting $\lambda_0 \colon T\underline{X} \to E$. If $(\gamma, 0), (\gamma', 0) \in \mathcal{L}^0$ represent elements over $[L_0] \in \mathcal{L}^0/\operatorname{Aut}(E)$ there exists $(g, \tau) \in \operatorname{Aut}(E)$ (see Lemma 4.3) such that $g \in \mathcal{G}_P \cap \operatorname{Ker} \mathcal{G}_{\underline{P}}$ and

$$\gamma' = \gamma - \tau^{1,1+0,2}.$$

Therefore, if $(\dot\gamma, 0), (\dot\gamma', 0)$ are tangent to the fibre over $[L_0]$ we have (see Definition 4.6)

$$\dot\gamma'^{1,1} - \dot\gamma^{1,1} - 2\langle s, F_h\rangle \in \operatorname{Im} \partial \oplus \bar\partial$$

for $s \in \operatorname{Lie} \mathcal{G}_P$. Thus, the map

$$[(\dot\gamma, 0)] \mapsto [\dot\gamma^{1,1}] \in \ker\, \tilde\partial \subset \ker \bar\partial / \operatorname{Im} \mathcal{F}$$

is well-defined an injective. Surjectivity follows from Lemma 2.15. □

As a straightforward consequence of Lemma B.1 and Lemma B.2, we obtain the following cohomological interpretation of the tangent space to the fibres of the map between moduli spaces

$$\mathcal{L}^0/\operatorname{Aut}_A(E) \to \mathcal{C}^0/\mathcal{G}_{\underline{P}} \tag{B.8}$$

induced by (8.24). Relying on Theorem 8.19, this provides the desired explanation for the 'complexified Aeppli classes' appearing in formula (8.20) for the fibre-wise moduli metric.

Proposition B.3. *The tangent space to the fibre of* (B.8) *over* $[P]$ *is isomorphic to* $\ker \bar\partial / \operatorname{Im} \mathcal{F} \subset H^{1,1}_A(X)/\operatorname{Im} \mathcal{F}$*, where* $\bar\partial$ *is as in* (B.4).

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