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The Kato classes revisited with application to self-adjointness

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Abstract

We give a systematic L^1 approach to L^2 form-perturbation theory for many-body convolution type Hamiltonians. The tools behind the construction rely on local equi-integrability in L^1 . In particular, we deepen and extend the Kato classes in different directions.

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1 Introduction

It is well-known that the mathematical theory of Schrödinger operators was born in 1951 with Kato's famous *self-adjointness* theorem [20] for atomic

Hamiltonians in $L^2(\mathbb{R}^{3N})$

$$-\sum_{i=1}^N (2\mu_i)^{-1} \Delta_i + \sum_{i=1}^N V_i(x_i) + \sum_{i<j} V_{ij}(x_i - x_j) \quad (1)$$

on the domain $H^2(\mathbb{R}^{3N})$ provided that $V_i^-, V_{ij}^- \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$; (Δ_i is the Laplacian with respect to the variable $x_i \in \mathbb{R}^3$ and the μ_i 's are positive constants) ⁽¹⁾. In this case, the potential

$$\sum_{i=1}^N V_i(x_i) + \sum_{i<j} V_{ij}(x_i - x_j)$$

(as a multiplication operator) has zero relative operator bound with respect to $\sum_{i=1}^N (2\mu_i)^{-1} \Delta_i$ and the self-adjointness property follows from the Kato-Rellich theorem. Since that time, more systematic results, in terms of Bessel potentials, for more general differential operators, have been obtained; see ([42] Chapter 7). When the natural addition of (unbounded) self-adjoint operators is no longer possible via the Kato-Rellich theorem, there is a possibility to add them (via the KLMN theorem) in the sense of *quadratic forms* [23][13]; (we note also the alternative approach by Trotter limits [6]). Without pretending to be exhaustive, we refer to [43] [44] for the case of positive potentials and to [41] for the general ones while various functional analytic tools and results around form-perturbation theory can be found in [12][36][18].

If we want to stay within the realm of L^1_{loc} potentials, a sufficient (but *not* necessary) condition is that the negative parts of the potentials belong to the *Kato class*. Besides its interest for self-adjointness (see the second famous Kato's paper [22]), this class of potentials turns out to play also a key role in the exploration of "*Schrödinger semigroups*" as shown in the classical paper by M. Aizenman and B. Simon [1] where the connections of the Kato class with Brownian motion, Harnack's inequality and L^p properties of Schrödinger semigroups are analyzed; we refer to the surveys [45][46] and to [48] for more information. We note that this Kato class is attached to the Laplacian but other Kato classes are attached to different generators, see R. Carmona, W. Ch. Masters and B. Simon [4]. Moreover, since the seminal paper [1], the Kato classes have had strong ramifications in different directions, (see e.g. [7][8][25][52][9][16][5] and the numerous references therein).

¹See also in [38] a report by B. Simon on T. Kato's contributions to quantum mechanics.

We are concerned here with the classical problem of self-adjointness only; but we revisit deeply the Kato classes and enlarge them appropriately. *We give new functional analytic developments on both perturbation theory in L^1 spaces and its form-perturbation counterpart in L^2 spaces.* This paper improves, in several directions, some previous results on perturbed convolution semigroups [30] and an unpublished work on multi-particle convolution Hamiltonians [31].

We deal with many-body Hamiltonians on $L^2(\mathbb{R}^{3N})$

$$\mathcal{H} := - \sum_{i=1}^N T_{(i)} + \sum_{i=1}^N V_i(x_i) + \sum_{i < j} V_{ij}(x_i - x_j) \quad (2)$$

where $T_{(i)}$ is a generator (with respect to the variable $x_i \in \mathbb{R}^3$) of a general symmetric convolution semigroup depending a priori of the index i ($1 \leq i \leq N$) and $V_i, V_{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}$ are measurable potentials. The introduction of this general class of Hamiltonians is motivated by both the non-relativistic Hamiltonian (1) and the quasi-relativistic one where

$$T_{(i)} = - \left(\sqrt{-c^2 h^2 \Delta_i + m_i^2 c^4} - m_i c^2 \right)$$

(see [26] Chapter 8) and also by the combination of the two (see [18] Example 2.6). Even if we consider general convolution semigroups, we have in mind the (large) class of subordinate Brownian semigroups. The self-adjointness of \mathcal{H} is of course a prerequisite to build the unitary group $(e^{-it\mathcal{H}})_{t \in \mathbb{R}}$ which solves the Schrödinger equation

$$i \frac{df}{dt} = \mathcal{H}f, \quad f(0) = f_0 \in L^2(\mathbb{R}^{3N}).$$

This work is dedicated to the search of form-bound estimates to build general lower-bounded Hamiltonians \mathcal{H} . To this end, we revisit, deepen and extend in different directions the Kato classes relative to generators of convolution semigroups. We provide a systematic L^1 -functional analytic approach. The construction relies on the analysis of suitable positive semigroups on $L^1(\mathbb{R}^3)$ under the general assumption that the potentials V_-^i and V_-^{ij} (e.g. the negative parts of V^i and V^{ij}) are $T_{(i)}$ -bounded in $L^1(\mathbb{R}^3)$. Indeed, the starting point of our construction is the observation that any potential which is $T_{(i)}$ -bounded in $L^1(\mathbb{R}^3)$ *must be* form-bounded in $L^2(\mathbb{R}^3)$ with respect to the positive operator $-T_{(i)}$. This explains our L^1 approach of L^2 form-perturbation theory. These L^2 bounds turn out to be connected to many

new L^1 results of independent interest. *Our construction relies on L^1 tools only.*

We say that a nonnegative measurable function $W \in L^1_{loc,unif}(\mathbb{R}^3, dx)$ if

$$\sup_{y \in \mathbb{R}^3} \int_{|x| \leq 1} W_y(x) dx < +\infty$$

where

$$W_y(x) := W(y + x).$$

We recall that a nonnegative potential W is $T_{(i)}$ -bounded in $L^1(\mathbb{R}^3)$ if and only if

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} W_y(x) E_\lambda^i(x) dx < +\infty$$

where $E_\lambda^i(x - y)$ is the kernel of $(\lambda - T_{(i)})^{-1}$; in particular

$$W \in L^1_{loc,unif}(\mathbb{R}^3, \mu(dx)) \subset L^1_{loc,unif}(\mathbb{R}^3, dx)$$

where

$$\mu(dx) = E_\lambda(x) dx.$$

Two main statements on many-body Hamiltonians are given.

The first one holds for general convolution semigroups: If the potentials V_-^i (and also the V_-^{ij} 's) are such that

$$\lim_{|y| \rightarrow +\infty} \sup V^i(y) < +\infty \tag{3}$$

and satisfy the *local equi-integrability* condition

$$\lim_{|\Omega| \rightarrow 0, \Omega \subset B} \sup_{|y| \leq c} \int_{\Omega} V_-^i(x) E_\lambda^i(x - y) dx = 0 \tag{4}$$

for any ball $B \subset \mathbb{R}^3$ and any $c > 0$ ($|\Omega|$ is the *Lebesgue* measure of Ω), then the multiplication operator on $L^2(\mathbb{R}^{3N})$ by the potential

$$-\mathcal{V}_-(x_1, \dots, x_N) := - \sum_{i=1}^N V_-^i(x_i) - \sum_{i < j} V_-^{ij}(x_i - x_j)$$

is form-bounded with respect to the positive free Hamiltonian

$$-\mathcal{T} := - \sum_{i=1}^N T_{(i)}$$

with *zero* relative form-bound; (see Theorem 21).

The second statement, for convolution semigroups with spherically symmetric and radially decreasing kernels, gives the same conclusion when we replace (3) by the condition that the V_-^i 's (and also the V_-^{ij} 's) are "*Kato at infinity*" (or K_∞ for short) in the sense

$$\lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow +\infty} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V_-^i(y+z) E_\lambda^i(z) dz = 0, \quad (5)$$

(see Theorem 22). The restriction to this class of convolution semigroups (covering nevertheless e.g. all subordinate Brownian semigroups, see Section 4) is due to the need of a technical result (see Lemma 13 below). Note that Assumption (5) on the potentials is much weaker than Assumption (3) but must be accompanied by a restriction on the class of convolution semigroups.

If the V_-^i 's "*vanish weakly at infinity*" in the sense

$$\limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq 1\}} V_-^i(y+z) dz = 0 \quad (6)$$

then we can replace (5) by the condition

$$\lim_{\lambda \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V_-^i(y+z) E_\lambda^i(z) dz = 0,$$

(see Remark 17). Of course, we could also replace (5) by the stronger (but *more convenient*) condition

$$\lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V_-^i(y+z) E_\lambda^i(z) dz = 0 \quad (7)$$

for some $\lambda > 0$.

For a nonnegative function W on \mathbb{R}^3 , we recall that the membership to the Kato class (relative to $T_{(i)}$) refers to

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^3} \int_{\{|z| \leq \varepsilon\}} W(x+z) E_\lambda^i(z) dz = 0.$$

This property is equivalent to

$$\lim_{\lambda \rightarrow +\infty} \left\| W (\lambda - T_{(i)})^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^3))} = 0 \quad (8)$$

or to the relative operator bounds

$$\|W\varphi\|_{L^1(\mathbb{R}^3)} \leq \varepsilon \|T_{(i)}\varphi\|_{L^1(\mathbb{R}^3)} + c_\varepsilon \|\varphi\|_{L^1(\mathbb{R}^3)}, \quad \varphi \in D(T_{(i)}) \quad (\forall \varepsilon > 0); \quad (9)$$

where $c_\varepsilon > 0$; (see [4]). The membership to the *local* Kato class (relative to $T_{(i)}$) refers to

$$\lim_{\varepsilon \rightarrow 0} \sup_{|x| \leq C} \int_{\{|z| \leq \varepsilon\}} W(x+z) E_\lambda^i(z) dz = 0$$

for any $C > 0$. It turns out that both conditions (4)(7) are satisfied by the Kato class potentials. Moreover, the concept (6) of potential "vanishing weakly at infinity" turns out to be the *right* notion of "smallness at infinity" for spectral problems (see Theorem 45 and Theorem 47).

Theorem 21 and Theorem 22 are new and appear here for the first time. Their proofs are quite involved and are related to many other results of independent interest, most of which also appear here for the first time, see below. We note that Theorem 21 and Theorem 22 are new even when $T_{(i)} = (2\mu_i)^{-1} \Delta_i$; in the latter case, we could also replace the Laplacian by the magnetic Laplacian; (see [2] Theorem 2.5). (*The results which are stated here in \mathbb{R}^3 , in view of many-body-Hamiltonians in $L^2(\mathbb{R}^{3N})$, are actually true in \mathbb{R}^d ($d \in \mathbb{N}$).*)

The starting point of our construction is that any nonnegative potential W which is $T_{(i)}$ -bounded in $L^1(\mathbb{R}^3)$ is form-bounded in $L^2(\mathbb{R}^3)$ with respect to the positive operator $-T_{(i)}$ with relative form-bound less than or equal to

$$\lim_{\lambda \rightarrow +\infty} r_\sigma \left(W (\lambda - T_{(i)})^{-1} \right) \quad (10)$$

where r_σ refers to the spectral radius in $L^1(\mathbb{R}^3)$; (see Theorem 1 and Corollary 2). By defining

$$\delta_i := \lim_{\lambda \rightarrow +\infty} r_\sigma \left[V_-^i (\lambda - T_{(i)})^{-1} \right], \quad \delta_{ij} := \lim_{\lambda \rightarrow +\infty} r_\sigma \left[V_-^{ij} (\lambda - T_{(i)})^{-1} \right],$$

we show that the multiplication operator on $L^2(\mathbb{R}^{3N})$ by $-V_-$ is form-bounded with respect to the positive self-adjoint operator $-T$ with relative form bound less than or equal to

$$\max_{1 \leq i \leq N} (\delta_i) + \max_{1 \leq i \leq N-1} (\widehat{\delta}_i)$$

where $\widehat{\delta}_i = \sum_{j=i+1}^N \delta_{ij}$ ($i \leq N-1$); (see Theorem 4). Thus the study of the *size* of the limit (10) becomes a key issue.

Under the local equi-integrability condition (4), it turns out that the value of the limit (10) is *independent* of the local properties of the potential W ; (see Theorem 6). In particular, if a potential W satisfies (3) then

$$\lim_{\lambda \rightarrow +\infty} r_\sigma \left(W (\lambda - T_{(i)})^{-1} \right) = 0 \quad (11)$$

for general convolution semigroups; (see Theorem 10).

If a priori (3) is not satisfied then (for a suitable class of convolution semigroups) we show that the limit (10) is less than or equal to

$$K_\infty(W) := \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow +\infty} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} W(y+z) E_\lambda^i(z) dz;$$

(see Theorem 16). This explains the interest of the class of potentials W such that $K_\infty(W) = 0$ (i.e. K_∞ potentials). We introduce also the *subclass* \widehat{K}_∞ of potentials W such that

$$K_\infty^\lambda(W) := \lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} W(y+z) E_\lambda^i(z) dz = 0 \quad (12)$$

for some $\lambda > 0$. Indeed, the monotonicity of $K_\infty^\lambda(W)$ in $\lambda > 0$ shows that

$$\widehat{K}_\infty \subset K_\infty. \quad (13)$$

The restricted class of convolution semigroups we alluded to is quite large and contains e.g. all subordinate Brownian semigroups; see Section 4. Locally, the potentials we consider satisfy the local equi-integrability condition (4). At infinity, for general convolution semigroups, we consider the potentials which satisfy (3); but, for the above restricted class of convolution semigroups, we allow the larger class of potentials satisfying (12).

For the sake of clarity, and in order not to slow down the reading of the paper, we postpone to Section 5 a deep scrutinization of our different assumptions. In particular, it turns out that the notions of Kato class, local Kato class or \widehat{K}_∞ class are essentially different incarnations of some hidden *weak compactness assumption*; (see Theorem 27, Theorem 32 and Remark 28). It follows, for example, that a $T_{(i)}$ -bounded potential W belongs to the local Kato class *once*

$$y \ni \mathbb{R}^3 \rightarrow W_y \in L^1(B(0,1); \mu(dx)) \text{ is continuous} \quad (14)$$

where $B(0,1)$ is the unit ball of \mathbb{R}^3 ; (see Corollary 29); this result is reminiscent of an old one ([1] Theorem 4.15) and suggests an interesting *open problem*, (see Remark 31).

We give also a membership criterion to \widehat{K}_∞ in terms of asymptotics of

$$\eta_k := \limsup_{|y| \rightarrow \infty} \int_{2^{-(k+1)} \leq |z| \leq 2^{-k}} W_y(z) dz, \quad (k \in \mathbb{N}).$$

Indeed, $W \in \widehat{K}_\infty$ if

$$\sum_{k=1}^{\infty} \left(\widehat{E}_\lambda(2^{-(k+1)}) \right) \eta_k < +\infty;$$

in particular, if

$$\limsup_{k \rightarrow \infty} (\eta_k)^{\frac{1}{k}} < \left(\limsup_{k \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-k}) \right)^{\frac{1}{k}} \right)^{-1}.$$

where $E_\lambda^i(z) = \widehat{E}_\lambda^i(|z|)$; (see Theorem 34). We have a similar membership criterion for the Kato class by replacing η_k by

$$\widehat{\eta}_k := \sup_{y \in \mathbb{R}^3} \int_{2^{-k} \leq |z| \leq 2^{-(k-1)}} W_y(z) dz;$$

(see Theorem 34). The parameter $\limsup_{k \rightarrow \infty} \left(\widehat{E}_\lambda^i(2^{-k}) \right)^{\frac{1}{k}}$ is also estimated

$$\limsup_{k \rightarrow \infty} \left(\widehat{E}_\lambda^i(2^{-k}) \right)^{\frac{1}{k}} \leq 2^{\frac{3}{s}}$$

where

$$s := \sup \left\{ p \geq 1; \int_{\{|x| \leq 1\}} (E_\lambda^i(x))^p dx < +\infty \right\};$$

(see Lemma 35). It follows that W is a Kato class potential (resp. $W \in \widehat{K}_\infty$) provided that

$$\limsup_{j \rightarrow \infty} (\widehat{\eta}_j)^{\frac{1}{j}} < 2^{-\frac{3}{s}} \quad (\text{resp. } \limsup_{j \rightarrow \infty} (\eta_j)^{\frac{1}{j}} < 2^{-\frac{3}{s}});$$

(see Corollary 37). In the usual examples,

$$\widehat{E}_\lambda^i(\rho) \sim \frac{1}{\rho^{3-\alpha}} \quad (\rho \rightarrow 0) \quad (0 < \alpha \leq 2); \quad (15)$$

($\alpha = 2$ for the heat semigroup, $0 < \alpha < 2$ for the α -stable semigroup and $\alpha = 1$ for the relativistic semigroup) so W is a Kato class potential (resp. $W \in \widehat{K}_\infty$) provided that

$$\limsup_{k \rightarrow \infty} (\widehat{\eta}_k)^{\frac{1}{k}} < 2^{-(3-\alpha)} \quad (\text{resp. } \limsup_{k \rightarrow \infty} (\eta_k)^{\frac{1}{k}} < 2^{-(3-\alpha)});$$

(see Remark 38).

It is well known (see [1] Theorem 1.4 (iii)) that for the Laplacian, W is a Kato class potential provided that $W \in L_{loc,unif}^p$, i.e.

$$\sup_{y \in \mathbb{R}^3} \int_{\{|x| \leq 1\}} (W(y+x))^p dx < +\infty,$$

for some $p > \frac{3}{2}$. Of course, this result can also be formulated, with $p > \frac{3}{\alpha}$, for general generators satisfying (15). We can derive this result from our membership criteria; (see Theorem 39).

If W is such that

$$\lim_{|x| \rightarrow +\infty} \int_{B(x,1) \cap A} W(z) dz = 0$$

for any measurable set A "thin at infinity" in the sense

$$\lim_{|x| \rightarrow +\infty} |B(x,1) \cap A| = 0$$

then W vanishes weakly at infinity in the sense (6) if and only if the super-level sets of W

$$\{W \geq c\}, \quad (c > 0)$$

are "thin at infinity"; (see Theorem 41). This occurs e.g. if W belongs to $L_{loc,unif}^p$, i.e.

$$\sup_{|y| \geq C} \int_{\{|x| \leq 1\}} (W(y+x))^p dx < +\infty,$$

for some $p > 1$ and some large $C > 0$; (see Remark 42).

In contrast to Kato class potentials which are characterized by (8), the more general potentials W we consider here satisfy only

$$\lim_{\lambda \rightarrow +\infty} r_\sigma \left(W (\lambda - T_{(i)})^{-1} \right) = 0.$$

We show that instead of (9)), these potentials W are such that: for any $\varepsilon > 0$ there exists an equivalent norm $\|\cdot\|_{\varepsilon new}$ on $L^1(\mathbb{R}^3)$ (additive on the positive cone) and a constant $c_\varepsilon > 0$ such that

$$\|W\varphi\|_{\varepsilon new} \leq \varepsilon \|T_{(i)}\varphi\|_{\varepsilon new} + c_\varepsilon \|\varphi\|_{\varepsilon new}, \quad \varphi \in D(T_{(i)});$$

(see Theorem 43). These abstract relative operator bounds are not of practical interest and are not used here; but we wonder whether they could be useful in another context.

We note that the form-sum operators on $L^2(\mathbb{R}^{3N})$

$$(-\mathcal{T}) \oplus (-\mathcal{V}_-) \tag{16}$$

we have built admit $C_c^\infty(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ as form-cores; (see Remark 24). In this paper, we have focused on the negative parts of the potentials but, of course, it is quite standard (see e.g. [13] Theorem 4.1, p. 24) to capture (2) as a form-sum of the positive potential

$$\sum_{i=1}^N V_+^i(x_i) + \sum_{i < j} V_+^{ij}(x_i - x_j)$$

and the lower bounded Hamiltonian (16); however, for $V_+^i, V_+^{ij} \in L_{loc}^1(\mathbb{R}^3)$, the question whether $C_c^\infty(\mathbb{R}^{3N})$ is a core of the new form deserves a separate study which is not considered here.

Besides the different results behind the form-bound estimates, we devote Section 6 to two general spectral results in $L^1(\mathbb{R}^3)$; (see Theorem 45 and Theorem 47). The proof of the first one, on stability of essential spectra, uses a result on strictly singular perturbations by T. Kato [21]; (we recall that in L^1 spaces, a strictly singular operator is nothing but a weakly compact operator [35]). The second result provides a sufficient criterion for the existence of spectral gaps.

We point out that the use of local weak compactness arguments is also an efficient tool to understand the impact of *positive* parts of potentials on spectral theory of general substochastic semigroups in abstract L^1 spaces [32]. We mention that a part of this work extends to higher-order elliptic systems where the loss of positivity is compensated by the existence of suitable kernel estimates [33]. Finally, we note that T. Kato [24] considered also *complex* potentials, (see also e.g. [27][34][17] and references therein). In the spirit of [33], by using suitable domination arguments, a great deal of this work could be extended to complex potentials too; we have not tried to elaborate on this point here.

Before giving our results, we recall some facts on the theory of absorption semigroups [48]. Let $(S^p(t))_{t \geq 0}$ be a positive contraction C_0 -semigroup on some $L^p(\Omega; \mu)$ space with generator T^p and let $V : \Omega \rightarrow \mathbb{R}$ be measurable. We assume that $V = V_+ - V_-$ is decomposed as a difference of two nonnegative measurable functions V_\pm (not necessarily the positive and negative parts of V) such that $V_\pm(x) < +\infty$ μ -a.e. To define " $T^p - V_+$ " as a generator (for the time being we are *not* interested in " $T^p - V$ "), we approximate V_+ monotonically from below by $V_+ \wedge j$ ($j \in \mathbb{N}$); the corresponding sequence of

semigroups converges strongly to a semigroup $\left(S_{V_+}^p(t)\right)_{t \geq 0}$ which need not be strongly continuous at $t = 0$. We say that V_+ is admissible if $\left(S_{V_+}^p(t)\right)_{t \geq 0}$ is strongly continuous and denote by $T_{V_+}^p$ its generator. In this case,

$$T_{V_+}^p \supset T^p - V_+$$

([48] Cor 2.7); this occurs e.g. if

$$D(T^p) \cap D(V_+) \text{ is dense in } L^p(\mu)$$

([48] Prop 2.9). Note that if $p = 1$, if $(S^1(t))_{t \geq 0}$ is mass preserving and if $D(T^1) \cap D(V_+)$ is a core for T^1 then $T_{V_+}^1 = T^1 - V_+$ ([48] Cor 4.3 and Prop 4.4). If $(S(t))_{t \geq 0}$ is a symmetric sub-Markov semigroup, i.e. acts in all $L^p(\mu)$ spaces as a positive contraction C_0 -semigroup $(S^p(t))_{t \geq 0}$ with generator T^p and $(S^2(t))_{t \geq 0}$ is self-adjoint then the admissibility of V_+ is p -independent, the dual of $\left(S_{V_+}^p(t)\right)_{t \geq 0}$ is equal to $\left(S_{V_+}^q(t)\right)_{t \geq 0}$ (q is the conjugate exponent) and $\left(S_{V_+}^2(t)\right)_{t \geq 0}$ is self-adjoint ([48] Prop 3.2). Moreover,

$$S_{V_+}^r(t)|_{L^p \cap L^s} = S_{V_+}^s(t)|_{L^p \cap L^s}$$

([48] Prop 3.1). To avoid cumbersome notations, we write $(S_+^p(t))_{t \geq 0}$ instead of $\left(S_{V_+}^p(t)\right)_{t \geq 0}$ and T_+^p instead of $T_{V_+}^p$. We consider now symmetric convolution semigroups

$$S^p(t) : f \in L^p(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^3} f(x-y) m_t(dy) \in L^p(\mathbb{R}^3)$$

where $\{m_t\}_{t \geq 0}$ are (symmetric with respect to the origin) Borel *sub*-probability measures on \mathbb{R}^3 such that $m_0 = \delta_0$ (Dirac measure at zero), $m_t * m_s = m_{t+s}$ and $m_t \rightarrow m_0$ vaguely as $t \rightarrow 0_+$. Such convolution semigroups are related to Lévy processes and cover many examples of practical interest such as Gaussian semigroups, α -stable semigroups, relativistic Schrödinger semigroups etc. Note that $(S^p(t))_{t \geq 0}$ is a positive contraction C_0 -semigroup on $L^p(\mathbb{R}^3)$ ($1 \leq p < +\infty$) with generator

$$T^p : D(T^p) \subset L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3).$$

The sub-probability measures $\{m_t\}_{t \geq 0}$ are characterized by

$$\widehat{m}_t(\zeta) := (2\pi)^{-\frac{N}{2}} \int e^{-i\zeta \cdot x} m_t(dx) = (2\pi)^{-\frac{N}{2}} e^{-tF(\zeta)}, \quad \zeta \in \mathbb{R}^3$$

where F , the so-called characteristic exponent, is a continuous negative definite function (see [19] Definition 3.6.5, p. 122) and has the representation

$$F(\zeta) = c + \zeta.C\zeta + \int_{\mathbb{R}^3 \setminus \{0\}} [1 - \cos(x.\zeta)] \mu(dx)$$

with $c \geq 0$, C is a real symmetric matrix such that $\zeta.C\zeta \geq 0 \forall \zeta \in \mathbb{R}^3$ and μ , the so-called Lévy measure, is a positive (symmetric with respect to the origin) Borel measure on $\mathbb{R}^3 \setminus \{0\}$ such that $\int \min(1, |x|^2) \mu(dx) < +\infty$. Note that $\{m_t\}_{t \geq 0}$ are probability measures if $F(0) = 0$, i.e. $c = 0$. We recall that $F \geq 0$ and $F(\zeta) \leq c_F(1 + |\zeta|^2)$. Note that

$$T^2 \varphi = -(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i\zeta.x} F(\zeta) \widehat{\varphi}(\zeta) d\zeta$$

with domain

$$D(T^2) = \{\varphi \in L^2(\mathbb{R}^3); F\widehat{\varphi} \in L^2(\mathbb{R}^3)\}.$$

We refer e.g. to [18][19] for more information on convolution semigroups. Finally, we recall that $\mathcal{S}(\mathbb{R}^3)$ is a core of T^p ([11] Thm 2.1.15, p. 38). In particular, if $V_+ \in L^1_{loc}(\mathbb{R}^3)$ then $C_c^\infty(\mathbb{R}^3) \subset D(T_1) \cap D(V_+)$ so V_+ is admissible with respect to $\{S^1(t)\}_{t \geq 0}$ and therefore with respect to $\{S^p(t)\}_{t \geq 0}$ for all $p \geq 1$. Note finally that if $m_t(dy)$ is a probability measure then $\{S^1(t)\}_{t \geq 0}$ is mass preserving. It follows that if $\mathcal{S}(\mathbb{R}^3) \subset D(V_+)$ then $D(T_1) \cap D(V_+)$ is a core of T^1 and $T^1_{V_+} = T^1 - V_+$. For the sake of simplicity, we restrict ourselves to convolution semigroups such that

$$\int_{\mathbb{R}^3} e^{-tF(\zeta)} d\zeta < +\infty \quad (t > 0). \quad (17)$$

In this case, m_t ($t > 0$) is absolutely continuous with respect to Lebesgue measure, i.e. $m_t(dx) = k_t(x)dx$ ($t > 0$) where $k_t \in L^1_+(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$ is even. In particular

$$(\lambda - T^p)^{-1} f = \int_{\mathbb{R}^3} E_\lambda(x - y) f(y) dy, \quad f \in L^p(\mathbb{R}^3)$$

where $E_\lambda(z) = \int_0^{+\infty} e^{-\lambda t} k_t(z) dt$ ($\lambda > 0$). Actually, in most of the paper, we just need that $(\lambda - T^p)^{-1}$ is a kernel operator.

2 Form-bounds for one-body Hamiltonians

As noted in the previous section, if

$$V_+ \in L^1_{loc}(\mathbb{R}^3) \quad (18)$$

then we can define an absorption convolution semigroup $\{S_+^p(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}^3)$ with generator T_+^p . We assume that V_- is T_1 -bounded in $L^1(\mathbb{R}^3)$ i.e.

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} V_-(x) E_\lambda(x-y) dx < +\infty \quad (\lambda > 0) \quad (19)$$

where $E_\lambda(x-y)$ is the kernel of $(\lambda - T_1)^{-1}$. Note that (19) is λ -independent. Of course, (19) implies that for any $\delta > 0$

$$\sup_{y \in \mathbb{R}^3} \int_{\{|x-y| \leq \delta\}} V_-(x) E_\lambda(x-y) dx < +\infty \quad (\lambda > 0). \quad (20)$$

We will show (see Proposition 14 below), for a general class of radially symmetric convolution semigroups, that (19) and (20) are equivalent. It is easy to see that (17) implies that $E_\lambda(z)$ is bounded away from zero in the unit ball and therefore (19) implies that $V_- \in L^1_{loc,unif}(\mathbb{R}^3)$, i.e.

$$\sup_{x \in \mathbb{R}^3} \int_{\{|z| \leq 1\}} V_-(x+z) dz < +\infty.$$

According to Desch's theorem [10] (see also [49] or [28] Chapter 8), if

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_1)^{-1}] < 1$$

then $A^1 := T_+^1 + V_-$ with domain $D(A^1) = D(T_+^1)$ generates a positive C_0 -semigroup $\{W^1(t)\}_{t \geq 0}$ on $L^1(\mathbb{R}^3)$. Moreover, $W^1(t)$ maps $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ into itself and, for any $p > 1$,

$$W^1(t) : L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)$$

extends uniquely to a C_0 -semigroup $\{W^p(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}^3)$; see [30]. If we denote by A^p its generator then

$$\{f \in D(T_+^1) \cap L^p(\mathbb{R}^3); T_+^1 f + V_- f \in L^p(\mathbb{R}^3)\}$$

is a core of A^p and A^2 is self-adjoint; see [30] for the details. The following result is already given in ([30] Theorem 21) under the assumption that $V_- \in L^2_{Loc}$. We aim here at removing this L^2_{Loc} assumption by using an approximation argument combined to suitable a priori spectral estimates.

Theorem 1 *Let $\delta := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_+^1)^{-1}] < 1$ and let (17)(18)(19) be satisfied. Then V_- is form-bounded with respect to $-T_+^2$ in $L^2(\mathbb{R}^3)$ with relative form-bound less than or equal to δ .*

Proof. Let

$$s(A^1) = \sup \{ \operatorname{Re} \lambda; \lambda \in \sigma(A^1) \}$$

be the spectral bound of A^1 . We recall that the type of a positive semigroup in L^p spaces coincides with the spectral bound of its generator, see e.g. [51]. We introduce now a nondecreasing sequence of bounded potentials $\{V_-^n\}_n$ converging pointwisely to V_- , e.g. we can choose $V_-^n = V_- \wedge n$. We note the uniform bound

$$r_\sigma [V_-^n(\lambda - T_+^1)^{-1}] \leq r_\sigma [V_-(\lambda - T_+^1)^{-1}] \quad \forall n.$$

Similarly,

$$A_n^1 = T_+^1 + V_-^n : D(T_+^1) \subset L^1(\mathbb{R}^3) \rightarrow L^1(\mathbb{R}^3)$$

is a generator of a positive C_0 -semigroup $\{W_n^1(t)\}_{t \geq 0}$ in $L^1(\mathbb{R}^3)$. Moreover, $V_-^n \leq V_-$ implies $W_n^1(t) \leq W^1(t)$ and then $s(A_n^1) \leq s(A^1) \quad \forall n$. By a symmetry argument, $\{W_n^1(t)\}_{t \geq 0}$ interpolates on all $L^p(\mathbb{R}^3)$ providing C_0 -semigroups $\{W_n^p(t)\}_{t \geq 0}$ in $L^p(\mathbb{R}^3)$ with generator A_n^p where A_n^2 is self-adjoint in $L^2(\mathbb{R}^3)$. Actually, since V_-^n is bounded A_n^2 is nothing but

$$T_+^2 + V_-^n : D(T_+^2) \rightarrow L^2(\mathbb{R}^3).$$

On the other hand, since $s(A_n^1)$ is the type of $\{W_n^1(t)\}_{t \geq 0}$ then for every $\varepsilon > 0$ there exists C_ε such that

$$\|W_n^1(t)\|_{\mathcal{L}(L^1(\mathbb{R}^3))} \leq C_\varepsilon e^{(s(A_n^1) + \varepsilon)t}$$

$$\|(W_n^1(t))'\|_{\mathcal{L}(L^\infty(\mathbb{R}^3))} \leq C_\varepsilon e^{(s(A_n^1) + \varepsilon)t}$$

where $(W_n^1(t))'$ is the dual semigroup on $L^\infty(\mathbb{R}^3)$. Riesz-Thorin's interpolation theorem implies

$$\|W_n^p(t)\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \leq C_\varepsilon e^{(s(A_n^1) + \varepsilon)t}$$

showing thus that $s(A_n^2) \leq s(A_n^1)$ since $\varepsilon > 0$ is arbitrary. Finally we get the uniform estimate

$$s(A_n^2) \leq s(A^1) \quad \forall n.$$

Since V_-^n is bounded then

$$(A_n^2 \varphi, \varphi) \leq s(A^1) \|\varphi\|_{L^2(\mathbb{R}^3)}^2 \quad \forall \varphi \in D(T_+^2) \quad \forall n.$$

If we choose c arbitrarily such that $1 < c < \frac{1}{\delta}$ then

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [cV_-(\lambda - T_1)^{-1}] = c\delta < 1$$

and then, arguing as previously,

$$A_n^{2,c} := T_+^2 + cV_-^n : D(T_+^2) \rightarrow L^2(\mathbb{R}^3)$$

is self-adjoint and

$$(A_n^{2,c} \varphi, \varphi) \leq s(A_1^c) \|\varphi\|_{L^2(\mathbb{R}^3)}^2 \quad \forall \varphi \in D(T_+^2) \quad \forall n$$

where $s(A_1^c)$ denotes the spectral bound of

$$A_1^c = T_+^1 + cV : D(T_1) \subset L^1(\mathbb{R}^3) \rightarrow L^1(\mathbb{R}^3).$$

Thus

$$(A_n^{2,c} \varphi, \varphi) = (T_+^2 \varphi + cV_-^n \varphi, \varphi) = - \left\| \sqrt{-T_+^2} \varphi \right\|^2 + c \int V_-^n |\varphi|^2 dx$$

so that, by the density of $D(T_+^2)$ in $D(\sqrt{-T_+^2})$ (for the graph norm of $\sqrt{-T_+^2}$),

$$c \int V_-^n |\varphi|^2 dx \leq \left\| \sqrt{-T_+^2} \varphi \right\|^2 + s(A_1^c) \|\varphi\|_{L^2(\mathbb{R}^3)}^2 \quad \forall n, \quad \forall \varphi \in D(\sqrt{-T_+^2}).$$

Letting $n \rightarrow \infty$ we get

$$\int V_- |\varphi|^2 dx \leq c^{-1} \left\| \sqrt{-T_+^2} \varphi \right\|^2 + c^{-1} s(A_1^c) \|\varphi\|_{L^2(\mathbb{R}^3)}^2 \quad \forall \varphi \in D(\sqrt{-T_+^2}).$$

This ends the proof since c^{-1} can be chosen as close to δ as we want. \blacksquare

Corollary 2 *If V_- is T_+^1 -bounded in $L^1(\mathbb{R}^3)$ then V_- is form-bounded with respect to $-T_+^2$ in $L^2(\mathbb{R}^3)$.*

Proof. The limit $\delta := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_+^1)^{-1}]$ always exists. Let $\varepsilon > 0$ and $V_-^{\delta+\varepsilon} = (\delta + \varepsilon)^{-1} V_-$. Then

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^{\delta+\varepsilon}(\lambda - T_+^1)^{-1}] = (\delta + \varepsilon)^{-1} \delta < 1$$

so by theorem 1 $(\delta + \varepsilon)^{-1} V_-$ is form-bounded with respect to $-T_+^2$ in $L^2(\mathbb{R}^3)$ with relative form-bound less than or equal to $(\delta + \varepsilon)^{-1} \delta$. ■

According to Theorem 1, we can define

$$(-T_+^2) \oplus (-V_-)$$

(a form-sum operator) via the KLMN theorem (see e.g. [47] Theorem 6. 24, p. 150). A natural conjecture is that this operator coincides with the self-adjoint operator $-A^2$ where A^2 was obtained previously by interpolation arguments from A^1 . Indeed, this is the case.

Theorem 3 *Let $\delta := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_+^1)^{-1}] < 1$ and let (17)(18)(19) be satisfied. Then $-A^2$ is equal to $(-T_+^2) \oplus (-V_-)$.*

Proof. The first observation is

$$(\lambda - T_+^1 - V_-^n)^{-1} \rightarrow (\lambda - T_+^1 - V_-)^{-1} \quad (n \rightarrow +\infty)$$

strongly in $L^1(\mathbb{R}^3)$. Indeed, let $\varepsilon > 0$ and let λ be large enough so that

$$r_\sigma [V_-(\lambda - T_+^1)^{-1}] < \delta + \varepsilon < 1.$$

Then $r_\sigma [V_-^n(\lambda - T_+^1)^{-1}] \leq r_\sigma [V_-(\lambda - T_+^1)^{-1}] \forall n$ and for all $\varphi \in L_+^1(\mathbb{R}^3)$

$$\begin{aligned} (\lambda - T_+^1 - V_-^n)^{-1} \varphi &= (\lambda - T_+^1)^{-1} \sum_{j=0}^{\infty} [V_-^n(\lambda - T_+^1)^{-1}]^j \varphi \\ &\rightarrow (\lambda - T_+^1)^{-1} \sum_{j=0}^{\infty} [V_-(\lambda - T_+^1)^{-1}]^j \varphi = (\lambda - T_+^1 - V_-)^{-1} \varphi \end{aligned}$$

by the monotone convergence theorem and we are done since $L_+^1(\mathbb{R}^3)$ is generating. It follows by Riesz-Thorin's interpolation theorem that

$$(\lambda - A_n^2)^{-1} \rightarrow (\lambda - A^2)^{-1} \text{ strongly in } L^2(\mathbb{R}^3). \quad (21)$$

On the other hand, since V_-^n is a bounded operator then $-A_n^2$ is also the form-sum $-A_n^2 = (-T_+^2) \oplus (-V_-^n)$. A key point is that the resolvent of the form-sum operator $(-T_+^2) \oplus (-V_-^n)$ is given by

$$(\lambda + (-T_+^2) \oplus (-V_-^n))^{-1} = (\lambda - T_+^2)^{-\frac{1}{2}} (I - C_n(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}}$$

(see [47] Theorem 6.25, p. 150) where $C_n(\lambda)$ is the positive bounded self-adjoint operator on $L^2(\mathbb{R}^3)$ defined by the positive bounded quadratic form

$$\varphi \in L^2(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^3} V_-^n \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2$$

with $\|C_n(\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \leq c^{-1}$ for λ large enough (see [47] Theorem 6.25, p. 150). Similarly, the resolvent of the form-sum operator $(-T_+^2) \oplus (-V_-)$ is given by

$$(\lambda + (-T_+^2) \oplus (-V_-))^{-1} = (\lambda - T_+^2)^{-\frac{1}{2}} (I - C(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}}$$

where $C(\lambda)$ is the positive bounded self-adjoint operator on $L^2(\mathbb{R}^3)$ defined by the positive bounded quadratic form

$$\varphi \in L^2(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^3} V_- \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2$$

and $\|C(\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \leq c^{-1} < 1$ for λ large enough (see [47] Theorem 6.25, p. 150). The monotonic convergence of the quadratic forms

$$(C_n(\lambda)\varphi, \varphi) = \int_{\mathbb{R}^3} V_-^n \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2 \rightarrow (C(\lambda)\varphi, \varphi) = \int_{\mathbb{R}^3} V_- \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2$$

implies the strong convergence $(I - C_n(\lambda))^{-1} \rightarrow (I - C(\lambda))^{-1}$ ($n \rightarrow +\infty$) (see e.g. [39] Theorem S. 14, p. 373) and finally the strong convergence

$$(\lambda - T_+^2)^{-\frac{1}{2}} (I - C_n(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}} \rightarrow (\lambda - T_+^2)^{-\frac{1}{2}} (I - C(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}}$$

($n \rightarrow +\infty$) which shows the equality

$$(\lambda - A_2)^{-1} = (\lambda + (-T_+^2) \oplus (-V_-))^{-1}$$

i.e. $-A_2 = (-T_+^2) \oplus (-V_-)$. ■

3 Form-bounds for many-body Hamiltonians

We show now how to define (form-sum) Hamiltonians of the form

$$\left(-\sum_{i=1}^N T_{(i)} \right) \oplus \left(-\sum_{i=1}^N V_-^i(x_i) - \sum_{i < j} V_-^{ij}(x_i - x_j) \right)$$

where, for each i ($1 \leq i \leq N$), $T_{(i)}$ acts on the variable $x_i \in \mathbb{R}^3$ only as a generator of a symmetric convolution semigroup depending a priori on the index i . Thus, we consider a family of symmetric convolution semigroups on $L^2(\mathbb{R}^3)$ indexed by an integer j ($1 \leq j \leq N$) ($N \geq 2$)

$$S_2^j(t) : f \in L^2(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^3} f(x-y)m_t^j(dy) \in L^2(\mathbb{R}^3).$$

Let T_2^j be the generator of $\{S_2^j(t); t \geq 0\}$ and let F_j be the corresponding characteristic exponent. On $L^2((\mathbb{R}^3)^N)$, we define

$$\left\{ \begin{array}{l} T_{(j)}\varphi = -(2\pi)^{-\frac{3N}{2}} \int_{\mathbb{R}^{3N}} e^{i\zeta \cdot x} F_j(\zeta_j) \widehat{\varphi}(\zeta) d\zeta \\ D(T_{(j)}) = \{\varphi \in L^2((\mathbb{R}^3)^N); F_j(\zeta_j) \widehat{\varphi}(\zeta) \in L^2((\mathbb{R}^3)^N)\} \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} \mathcal{T}\varphi = -(2\pi)^{-\frac{3N}{2}} \int_{\mathbb{R}^{3N}} e^{i\zeta \cdot x} \widetilde{F}(\zeta) \widehat{\varphi}(\zeta) d\zeta \\ D(\mathcal{T}) = \{\varphi \in L^2((\mathbb{R}^3)^N); \widetilde{F}(\zeta) \widehat{\varphi}(\zeta) \in L^2((\mathbb{R}^3)^N)\} \end{array} \right\} \quad (22)$$

where

$$\widetilde{F}(\zeta) = \sum_{j=1}^N F_j(\zeta_j)$$

(the ζ_j 's are the component of $\zeta \in (\mathbb{R}^3)^N$). Note that \widetilde{F} is also a continuous negative definite function on $(\mathbb{R}^3)^N$ ([19] Lemma 3.6.7, p. 123). Let

$$V^i, V^{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (i, j \leq N)$$

be measurable and $|V^i(z)| + |V^{ij}(z)| < +\infty$ a.e. Let

$$V^i = V_+^i - V_-^i, \quad V^{ij} = V_+^{ij} - V_-^{ij}$$

be decompositions into differences of nonnegative functions (which need not be the standard positive and negative parts) and let

$$\mathcal{V}_-(x_1, \dots, x_N) := \sum_{i=1}^N V_-^i(x_i) + \sum_{i < j} V_-^{ij}(x_i - x_j). \quad (23)$$

Our first fundamental result is:

Theorem 4 *Let (17) be satisfied by the characteristic exponents. We assume that V_-^i and V_-^{ij} are T_1^i -bounded in $L^1(\mathbb{R}^3)$. Let*

$$\delta_i := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^i(\lambda - T_1^i)^{-1}], \quad \delta_{ij} := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^{ij}(\lambda - T_1^i)^{-1}]$$

(where r_σ refers to spectral radius of bounded operators on $L^1(\mathbb{R}^3)$) and let $\widehat{\delta}_i = \sum_{j=i+1}^N \delta_{ij}$ ($i \leq N-1$). If

$$\delta := \max_{1 \leq i \leq N} (\delta_i) + \max_{1 \leq i \leq N-1} (\widehat{\delta}_i) < 1$$

then the multiplication operator on $L^2(\mathbb{R}^{3N})$ by the potential $-\mathcal{V}_-$ is form-bounded with respect to the positive self-adjoint operator $-\mathcal{T}$ with relative form bound less than or equal to δ .

Proof. According to Theorem 1, V_-^i is form-bounded (with respect to T_2^i) with relative form-bound less than or equal to δ_i . Thus, for each $\varepsilon > 0$ there exists $c_\varepsilon^i > 0$ such that (for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ fixed) and for all $f \in C_c^\infty(\mathbb{R}^{3N})$

$$\begin{aligned} & \int_{\mathbb{R}^3} V_-^i(x_i) |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)|^2 dx_i \\ & \leq (\delta_i + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_2^i} f \right|^2 dx_i + c_\varepsilon^i \int_{\mathbb{R}^3} |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)|^2 dx_i \end{aligned}$$

so that integrating with respect to the remaining variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$

$$\begin{aligned} \int_{\mathbb{R}^{3N}} V_-^i |f|^2 dx & \leq (\delta_i + \varepsilon) \left\| \sqrt{-T_{(i)}} f \right\|_{L^2(\mathbb{R}^{3N})}^2 + c_\varepsilon^i \int_{\mathbb{R}^{3N}} |f|^2 dx \\ & = (\delta_i + \varepsilon) \int_{\mathbb{R}^{3N}} F_i(\zeta_i) \left| \widehat{f}(\zeta) \right|^2 d\zeta + c_\varepsilon^i \int_{\mathbb{R}^{3N}} |f|^2 dx \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^N \int_{\mathbb{R}^{3N}} V_-^i |f|^2 dx & \leq (\max_i \delta_i + \varepsilon) \int_{\mathbb{R}^{3N}} \widetilde{F}(\zeta) \left| \widehat{f}(\zeta) \right|^2 d\zeta + \left(\sum_{i=1}^N c_\varepsilon^i \right) \|f\|_{L^2(\mathbb{R}^{3N})}^2 \\ & = (\max_i \delta_i + \varepsilon) \int_{\mathbb{R}^{3N}} \left| \sqrt{-T} f \right|^2 dx + \left(\sum_{i=1}^N c_\varepsilon^i \right) \|f\|_{L^2(\mathbb{R}^{3N})}^2. \end{aligned}$$

Similarly, V_-^{ij} is form-bounded (with respect to T_2^i) with relative form-bound less than or equal to δ_{ij} . Thus, for each $\varepsilon > 0$ there exists $c_\varepsilon^{ij} > 0$ such that

(for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ fixed)

$$\begin{aligned} & \int_{\mathbb{R}^3} V_{ij}^-(z) |f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \\ & \leq (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_2^i} f \right|^2 dz + c_\varepsilon^{ij} \int_{\mathbb{R}^3} |f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \end{aligned}$$

whence

$$\begin{aligned} & \int_{\mathbb{R}^3} V_{ij}^-(x_i - x_j) |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)|^2 dx_i \tag{24} \\ & = \int_{\mathbb{R}^3} V_{ij}^-(z) |f(x_1, \dots, x_{i-1}, z + x_j, x_{i+1}, \dots, x_N)|^2 dz \\ & \leq (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_2^i} f_{x_j} \right|^2 dz + c_\varepsilon^{ij} \int_{\mathbb{R}^3} |f_{x_j}(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \\ & = (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_2^i} f \right|^2 dz + c_\varepsilon^{ij} \int_{\mathbb{R}^3} |f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \end{aligned}$$

(where $f_{x_j} : z \rightarrow f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)$ is the *translation* by x_j) since the quadratic form is *invariant* by translation. By integrating (24) with respect to the remaining variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ we get

$$\begin{aligned} \int_{\mathbb{R}^{3N}} V_{ij}^- |f|^2 dx & \leq (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^{3N}} \left| \sqrt{-T_{(i)}} f \right|^2 dx + c_\varepsilon^{ij} \|f\|_{L^2(\mathbb{R}^{3N})}^2 \\ & = (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^{3N}} F_i(\zeta_i) \left| \widehat{f}(\zeta) \right|^2 d\zeta + c_\varepsilon^{ij} \int_{\mathbb{R}^{3N}} |f|^2 dx \end{aligned}$$

and

$$\begin{aligned} & \sum_{i < j} \int_{\mathbb{R}^{3N}} V_{ij}^- |f|^2 dx \\ & \leq \sup_i \sum_{j=i+1}^N (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^{3N}} \widetilde{F}(\zeta) \left| \widehat{f}(\zeta) \right|^2 d\zeta + \left(\sum_{i < j} c_\varepsilon^{ij} \right) \int_{\mathbb{R}^{3N}} |f|^2 dx \\ & = \sup_i \sum_{j=i+1}^N (\delta_{ij} + \varepsilon) \left\| \sqrt{-T} f \right\|_{L^2(\mathbb{R}^{3N})}^2 + \left(\sum_{i < j} c_\varepsilon^{ij} \right) \|f\|_{L^2(\mathbb{R}^{3N})}^2. \end{aligned}$$

Finally

$$\begin{aligned}
& \sum_{i=1}^N \int_{\mathbb{R}^{3N}} V_i^- |f|^2 dx + \sum_{i<j} \int_{\mathbb{R}^{3N}} V_{ij}^- |f|^2 dx \\
& \leq \left[(\max_i (\delta_i) + \varepsilon) + \sup_i \sum_{j=i+1}^N (\delta_{ij} + \varepsilon) \right] \left\| \sqrt{-T} f \right\|_{L^2(\mathbb{R}^{3N})}^2 \\
& \quad + \left[\sum_{i=1}^N c_\varepsilon^i + \sum_{i<j} c_\varepsilon^{ij} \right] \|f\|_{L^2(\mathbb{R}^{3N})}^2
\end{aligned}$$

which ends the proof since $\varepsilon > 0$ is arbitrary. ■

Remark 5 *By using Corollary 2, one sees that once the V_-^i 's and the V_-^{ij} 's are T_1^i -bounded in $L^1(\mathbb{R}^3)$ then $-\delta^{-1}\mathcal{V}_-$ is form-bounded with respect to the positive self-adjoint operator $-\mathcal{T}$ with relative form bound < 1 where δ is given by $\max_{1 \leq i \leq N} (\delta_i) + \max_{1 \leq i \leq N-1} (\widehat{\delta}_i)$.*

Our second fundamental result is:

Theorem 6 *We assume that (17) is satisfied by the characteristic exponents. Let $E_\lambda^i(x-y)$ be the kernel of $(\lambda - T_1^i)^{-1}$ and let $E_\lambda^i(\cdot)$ be bounded outside any neighborhood of 0. Let V_-^i be T_1^i -bounded in $L^1(\mathbb{R}^3)$ and set*

$$\alpha_i := \lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V_-^i(x) E_\lambda^i(x-y) dx. \quad (25)$$

We assume that for some $\lambda > 0$, for any ball $B(0, R) \subset \mathbb{R}^3$ and any $c > 0$,

$$\lim_{|\Omega| \rightarrow 0, \Omega \subset B} \sup_{|y| \leq c} \int_{\Omega} V_-^i(x) E_\lambda^i(x-y) dx = 0. \quad (26)$$

Then $\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^i(\lambda - T_1^i)^{-1}] \leq \alpha_i$. We have a similar statement for V_-^{ij} .

Proof. For each $c > 0$, we decompose V_-^i as $V_-^i = V_{-1}^i + V_{-2}^i$ where $V_{-1}^i = V_-^i \mathbf{1}_{\{|x| < c\}}$, $V_{-2}^i = V_-^i \mathbf{1}_{\{|x| \geq c\}}$. Let $\varepsilon > 0$ be fixed. By choosing c and λ large enough

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} V_{-2}^i(x) E_\lambda^i(x-y) dx < \alpha_i + \varepsilon.$$

This implies that the resolvent of the operator

$$T_1^i + (\alpha_i + \varepsilon)^{-1}V_{-2}^i : D(T_1^i) \rightarrow L^1(\mathbb{R}^3)$$

exists (for λ large enough) and is positive which implies that this operator is a generator of positive C_0 -semigroup by Desch's theorem [10] (see also [49] or [28] Chapter 8). Note that (26) implies

$$\lim_{|\Omega| \rightarrow 0} \sup_{\Omega \subset B} \sup_{y \in \mathbb{R}^3} \int_{\Omega} V_{-1}^i(x) E_{\lambda}^i(x-y) dx = 0.$$

Indeed, for $c \geq 2R$ we have $|x-y| \geq R$ for $x \in B$ and $|y| > c$ so that

$$\int_{\Omega} V_{-1}^i(x) E_{\lambda}^i(x-y) dx \leq \sup_{|z| \geq R} E_{\lambda}^i(z) \int_{\Omega} V_{-1}^i(x) dx \rightarrow 0 \quad (|\Omega| \rightarrow 0).$$

This expresses that $V_{-1}^i(\lambda - T_1^i)^{-1}$ is weakly compact, i.e. V_{-1}^i is T_1^i -weakly compact. This is equivalent to

$$V_{-1}^i \text{ is } (T_1^i + (\alpha_i + \varepsilon)^{-1}V_{-2}^i)\text{-weakly compact}$$

or to

$$(\alpha_i + \varepsilon)^{-1}V_{-1}^i \text{ is } (T_1^i + (\alpha_i + \varepsilon)^{-1}V_{-2}^i)\text{-weakly compact}$$

and consequently by ([29] Theorem 6)

$$T_1^i + (\alpha_i + \varepsilon)^{-1}V_{-2}^i + (\alpha_i + \varepsilon)^{-1}V_{-1}^i = T_1^i + (\alpha_i + \varepsilon)^{-1}V_{-1}^i$$

generates a positive c_0 -semigroup. It follows (see [49] Theorem 1.1) that

$$\lim_{\lambda \rightarrow +\infty} r_{\sigma} [(\alpha_i + \varepsilon)^{-1}V_{-1}^i(\lambda - T_1^i)^{-1}] < 1$$

or $\lim_{\lambda \rightarrow +\infty} r_{\sigma} [V_{-1}^i(\lambda - T_1^i)^{-1}] < \alpha_i + \varepsilon$ and this ends the proof since $\varepsilon > 0$ is arbitrary. ■

Remark 7 *The boundedness of $E_{\lambda}^i(\cdot)$ outside any neighborhood of 0 is satisfied by most examples, see Remark 12 below.*

Corollary 8 *Let $E_{\lambda}^i(\cdot)$ be bounded outside any neighborhood of 0. If for any ball $B \subset \mathbb{R}^3$*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in B} \int_{\{|z| \leq \varepsilon\}} V_{-1}^i(x+z) E_{\lambda}^i(z) dz = 0 \quad (27)$$

then the condition (26) is satisfied.

Proof. We note first that

$$\begin{aligned} \int_{\Omega} V_-^i(x) E_{\lambda}^i(x-y) dx &= \int_{\Omega \cap \{|x-y| \leq \varepsilon\}} V_-^i(x) E_{\lambda}^i(x-y) dx \\ &+ \int_{\Omega \cap \{|x-y| > \varepsilon\}} V_-^i(x) E_{\lambda}^i(x-y) dx. \end{aligned}$$

We have

$$\left| \int_{\Omega \cap \{|x-y| > \varepsilon\}} V_-^i(x) E_{\lambda}^i(x-y) dx \right| \leq C_{\varepsilon} \int_{\Omega} V_-^i(x) dx$$

and

$$\begin{aligned} \left| \int_{\Omega \cap \{|x-y| \leq \varepsilon\}} V_-^i(x) E_{\lambda}^i(x-y) dx \right| &= \left| \int_{\{\Omega-y\} \cap \{|z| \leq \varepsilon\}} V_-^i(y+z) E_{\lambda}^i(z) dz \right| \\ &\leq \int_{\{|z| \leq \varepsilon\}} V_-^i(y+z) E_{\lambda}^i(z) dz. \end{aligned}$$

Finally

$$\sup_{|y| \leq c} \int_{\Omega} V_-^i(x) E_{\lambda}^i(x-y) dx \leq C_{\varepsilon} \int_{\Omega} V_-^i(x) dx + \sup_{|y| \leq c} \int_{\{|z| \leq \varepsilon\}} V_-^i(y+z) E_{\lambda}^i(z) dz.$$

Hence it suffices to choose ε small enough to make the last term as small as we want while $C_{\varepsilon} \int_{\Omega} V_-^i(x) dx \rightarrow 0$ as $|\Omega| \rightarrow 0$ for a given ε . ■

Remark 9 Note that (27) expresses the membership to the so-called local Kato class while the Kato class refers to

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^3} \int_{\{|z| \leq \varepsilon\}} V_-^i(x+z) E_{\lambda}^i(z) dz = 0,$$

see [4]. We point out that (27) is not necessary for (26) to hold since (27) (at least for Laplacians) is equivalent to the compactness of

$$V_{-1}^i (\lambda - T_1^i)^{-1} : L^1(\mathbb{R}^3) \rightarrow L_{loc}^1(\mathbb{R}^3) \quad (28)$$

(see [45] Proposition A.2.4 (d)) while (26) expresses just the weak compactness of (28).

A key issue now is to estimate the parameter (25). A first fundamental result in this direction is:

Theorem 10 *We assume that (17) is satisfied by the characteristic exponents. Let $V : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be T_1^i -bounded in $L^1(\mathbb{R}^3)$ and*

$$\lim_{|y| \rightarrow +\infty} \sup V(y) < +\infty.$$

Then

$$\lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_\lambda^i(x-y) dx = 0.$$

Proof. Note that $E_\lambda^i(z) = \int_0^{+\infty} e^{-\lambda t} k_t^i(z) dt$ ($\lambda > 0$) so

$$\|E_\lambda^i\|_{L^1(\mathbb{R}^3)} = \int_0^{+\infty} e^{-\lambda t} \|k_t^i\|_{L^1(\mathbb{R}^3)} dt \leq \frac{1}{\lambda}.$$

Hence

$$\lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_\lambda^i(x-y) dx \leq \frac{\limsup_{|y| \rightarrow +\infty} V(x)}{\lambda}$$

and we are done. ■

Remark 11 *The proof above shows also that if $\limsup_{|y| \rightarrow +\infty} V(y) = 0$ then $\lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_\lambda^i(x-y) dx = 0$ for all $\lambda > 0$.*

The analysis of the parameter (25) is more involved when V is not bounded at infinity. This is the object of the next section.

4 On K_∞ potentials

From now on, we restrict ourselves to the class of convolution semigroups such that for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing, i.e.

$$k_t^i(z) = \widehat{k}_t^i(|z|)$$

where $\widehat{k}_t^i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is *nonincreasing*. In particular $E_\lambda^i(z) = \widehat{E}_\lambda^i(|z|)$ where $\widehat{E}_\lambda^i(|z|) = \int_0^{+\infty} e^{-\lambda t} \widehat{k}_t^i(|z|) dt$. (This property is satisfied in the case of isotropic unimodal Lévy-measures; see [50].) This class contains for instance the usual examples such as the Brownian case

$$T_{(i)} = (2\mu_i)^{-1} \Delta_i$$

(where Δ_i is the Laplacian with respect to the variable $x_i \in \mathbb{R}^3$), the α -stable case

$$T_{(i)} = -(-\Delta_i)^{\frac{\alpha_i}{2}} \quad (0 < \alpha_i < 2),$$

the quasi-relativistic case

$$T_{(i)} = -\left(\sqrt{-c^2 h^2 \Delta_i + m_i^2 c^4} - m_i c^2\right);$$

see [4]. Actually, this property is shared by all subordinate Brownian semigroups. Indeed, a $C^\infty(0, +\infty)$ function f such that $(-1)^k \frac{d^k f}{dx^k} \leq 0$ is called a Bernstein function and is characterized by the representation

$$e^{-tf(x)} = \int_0^{+\infty} e^{-xs} \eta_t(ds)$$

where $(\eta_t)_{t \geq 0}$ is a convolution semigroup of measures on $[0, +\infty)$. If $(S(t))_{t \geq 0}$ is a convolution semigroup on $L^1(\mathbb{R}^3)$ with characteristic exponent $F(\zeta)$ then $f(F(\zeta))$ is a continuous negative definite function and the associated (subordinate) convolution semigroup $(S^f(t))_{t \geq 0}$ operates as

$$S^f(t)\varphi = \int_0^{+\infty} (S(s)\varphi) \eta_t(ds);$$

see ([19] Proposition 3.9.10, p. 179). It follows that if for each $t > 0$, $S(t)$ has a kernel $k_t(\cdot)$ then $S^f(t)$ has also a kernel

$$k_t^f(z) = \int_0^{+\infty} k_s(z) \eta_t(ds).$$

In particular if $k_t(\cdot)$ is spherically symmetric and radially decreasing then so is $k_t^f(\cdot)$.

Remark 12 Note that if $k_t(x) = \widehat{k}_t(|x|)$ with $\rho \rightarrow \widehat{k}_t(\rho)$ non increasing and if $\int_0^{+\infty} e^{-\lambda t} \widehat{k}_t(\rho) dt < +\infty$ ($\rho > 0$) then

$$\widehat{E}_\lambda^f(|z|) := \int_0^{+\infty} e^{-\lambda t} \widehat{k}_t^f(|z|) dt$$

is bounded outside any neighborhood of the origin.

The interest of this class of semigroups lies in Lemma 13 below given in [4]. A measurable function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is said to be $l^1(L^\infty)$ if

$$\|f\|_{l^1(L^\infty)} := \sum_{\alpha \in \mathbb{Z}^3} \sup_{x \in C_\alpha} |f(x)| < +\infty$$

where C_α is the cube centered at $\alpha \in \mathbb{Z}^3$ with sides of length 1.

Lemma 13 ([4] Corollary of Lemma III. 2) *If for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing then $k_t^i(\cdot)$ is $l^1(L^\infty)$ and for each $\delta > 0$ fixed*

$$\sup_{t>0} \left\| \mathbf{1}_{\{|y|\geq\delta\}} k_t^i \right\|_{l^1(L^\infty)} < +\infty.$$

By adapting some calculations from [4] we derive the following estimate of the operator norm $\left\| V(\lambda - T_1^i)^{-1} \right\|$.

Proposition 14 *We assume that for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing. Let $V : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be in $L_{loc,unif}^1(\mathbb{R}^3)$. Then V is T_1^i -bounded in $L^1(\mathbb{R}^3)$ if and only if*

$$\sup_{y \in \mathbb{R}^3} \int_{|z| \leq \delta} V(y+z) E_\lambda^i(z) dz < +\infty \quad (\delta > 0).$$

In this case, $\left\| V(\lambda - T_1^i)^{-1} \right\|$ is less than or equal to

$$\sup_{y \in \mathbb{R}^3} \int_{|z| \leq \delta} V(y+z) E_\lambda^i(z) dz + \frac{\|V\|_{L_{loc,unif}^1}}{\lambda} \sup_{t>0} \left\| \mathbf{1}_{\{|z|>\delta\}} k_t^i(\cdot) \right\|_{l^1(L^\infty)} \quad (29)$$

where $\delta > 0$ is arbitrary and $\|V\|_{L_{loc,unif}^1} := \sup_{y \in \mathbb{R}^3} \int_{y+C_0} V(z) dz$.

Proof. We already know the necessity part. Let us consider the sufficiency part. Since

$$\left\| V(\lambda - T_1^i)^{-1} \right\| = \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} V(y+z) E_\lambda^i(z) dz < +\infty \quad (\lambda > 0)$$

then it suffices to show that

$$\sup_{y \in \mathbb{R}^3} \int_{|z|>\delta} V(y+z) E_\lambda^i(z) dz < +\infty \quad (\lambda > 0).$$

Note that

$$\begin{aligned} \int_{|z|>\delta} V(y+z) E_\lambda^i(z) dz &= \int_{|z|>\delta} V(y+z) \left(\int_0^{+\infty} e^{-\lambda t} k_t^i(z) dt \right) dz \\ &= \int_0^{+\infty} \left(\int_{|z|>\delta} V(y+z) k_t^i(z) dz \right) e^{-\lambda t} dt. \end{aligned}$$

On the other hand, $\int_{|z|>\delta} V(y+z)k_t^i(z)dz$ is equal to

$$\begin{aligned}
& \sum_{\alpha \in \mathbb{Z}^3} \int_{\{|z|>\delta\} \cap C_\alpha} V(y+z)k_t^i(z)dz \\
&= \sum_{\alpha \in \mathbb{Z}^3} \int_{C_\alpha} V(y+z)1_{\{|z|>\delta\}}k_t^i(z)dz \\
&\leq \sum_{\alpha \in \mathbb{Z}^3} \sup_{z \in C_\alpha} (1_{\{|z|>\delta\}}k_t^i(z)) \int_{y+C_\alpha} V(z)dz \\
&\leq \left(\sup_{\alpha \in \mathbb{Z}^3} \int_{y+C_\alpha} V(z)dz \right) \|1_{\{|z|>\delta\}}k_t^i(\cdot)\|_{l^1(L^\infty)} \\
&= \left(\sup_{\alpha \in \mathbb{Z}^3} \int_{y+\alpha+C_0} V(z)dz \right) \|1_{\{|z|>\delta\}}k_t^i(\cdot)\|_{l^1(L^\infty)} \\
&= \left(\sup_{u \in y+\mathbb{Z}^3} \int_{u+C_0} V(z)dz \right) \|1_{\{|z|>\delta\}}k_t^i(\cdot)\|_{l^1(L^\infty)} \\
&\leq \left(\sup_{u \in \mathbb{R}^3} \int_{u+C_0} V(z)dz \right) \|1_{\{|z|>\delta\}}k_t^i(\cdot)\|_{l^1(L^\infty)} \\
&= \|V\|_{L^1_{loc,unif}} \sup_{t>0} \|1_{\{|z|>\delta\}}k_t^i(\cdot)\|_{l^1(L^\infty)}
\end{aligned}$$

so, using Lemma 13,

$$\sup_{y \in \mathbb{R}^3} \int_0^{+\infty} \left(\int_{|z|>\delta} V(y+z)k_t^i(z)dz \right) e^{-\lambda t} dt \leq \frac{\|V\|_{L^1_{loc,unif}}}{\lambda} \sup_{t>0} \|1_{\{|z|>\delta\}}k_t^i(\cdot)\|_{l^1(L^\infty)}$$

and we are done. ■

We say that $V \in L^1_{loc,unif}(\mathbb{R}^3)$ *vanishes weakly at infinity* if

$$\lim_{|y| \rightarrow +\infty} \sup_{C_0} \int_{C_0} V(y+z)dz = 0. \tag{30}$$

This terminology comes from [14]. In Theorem 41 below, we give a characterization of this property in terms of "thinness at infinity" of the superlevel sets $\{V \geq c\}$.

Definition 15 *We assume that (17) is satisfied by the characteristic exponents. Let $V : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be T_1^i -bounded in $L^1(\mathbb{R}^3)$. We define the parameter*

$$K_\infty(V) := \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow +\infty} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V(y+z)E_\lambda^i(z)dz.$$

Another fundamental result is:

Theorem 16 *We assume that (17) is satisfied by the characteristic exponents and for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing. Let $V : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ be T_1^i -bounded in $L^1(\mathbb{R}^3)$. Then*

$$\lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_\lambda^i(x-y) dx \leq K_\infty(V).$$

Proof. We note that

$$\sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_\lambda^i(x-y) dx$$

is nothing but $\|V_c(\lambda - T_1^i)^{-1}\|$ where $V_c(x) := V(x)1_{\{|x| \geq c\}}$. According to (29)

$$\begin{aligned} & \|V_c(\lambda - T_1^i)^{-1}\| \\ & \leq \sup_{y \in \mathbb{R}^3} \int_{|z| \leq \varepsilon} V_c(y+z) E_\lambda^i(z) dz + \frac{\|V_c\|_{L_{loc,unif}^1}}{\lambda} \sup_{t>0} \|1_{\{|z|>\varepsilon\}} k_t^i(\cdot)\|_{l^1(L^\infty)} \quad (\forall \varepsilon > 0). \end{aligned}$$

where

$$\begin{aligned} & \|V_c\|_{L_{loc,unif}^1} \\ & = \sup_{y \in \mathbb{R}^3} \int_{y+C_0} V_c(z) dz = \sup_{y \in \mathbb{R}^3} \int_{\{y+C_0\} \cap \{|z| \geq c\}} V(z) dz \\ & = \sup_{y \in \mathbb{R}^3} \int_{C_0 \cap \{|y+z'| \geq c\}} V(y+z') dz' \leq \sup_{|y| \geq c-1} \int_{C_0} V(y+z') dz'. \quad (31) \end{aligned}$$

Hence, for $0 < \varepsilon < 1$,

$$\begin{aligned} & \lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \|V_c(\lambda - T_1^i)^{-1}\| \\ & \leq \lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|z| \leq \varepsilon} V_c(y+z) E_\lambda^i(z) dz \\ & = \lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{\{|z| \leq \varepsilon\} \cap \{|y+z| \geq c\}} V(y+z) E_\lambda^i(z) dz \\ & \leq \lim_{\lambda \rightarrow +\infty} \lim_{c \rightarrow +\infty} \sup_{|y| \geq c-1} \int_{\{|z| \leq \varepsilon\}} V(y+z) E_\lambda^i(z) dz \\ & = \lim_{\lambda \rightarrow +\infty} \lim_{|y| \rightarrow +\infty} \sup_{\{|z| \leq \varepsilon\}} V(y+z) E_\lambda^i(z) dz \quad (0 < \varepsilon < 1). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ ends the proof. \blacksquare

Remark 17 *If V vanishes weakly at infinity in the sense (30) then we can complement Theorem 16. Indeed, the use of (31) gives an estimate for a given $\lambda > 0$*

$$\lim_{c \rightarrow +\infty} \left\| V_c (\lambda - T_1^i)^{-1} \right\| \leq \lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V(y+z) E_\lambda^i(z) dz$$

so that

$$\begin{aligned} & \lim_{\lambda \rightarrow +\infty} \limsup_{c \rightarrow +\infty} \int_{y \in \mathbb{R}^3} \sup_{|x| \geq c} V(x) E_\lambda^i(x-y) dx \\ & \leq \lim_{\lambda \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V(y+z) E_\lambda^i(z) dz. \end{aligned}$$

Definition 18 *A measurable potential $V : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ which is T_1^i -bounded in $L^1(\mathbb{R}^3)$ is said to be "Kato at infinity" (with respect to T_1^i), or K_∞ for short, if $K_\infty(V) = 0$. We say that V is \widehat{K}_∞ if*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V(y+z) E_\lambda^i(z) dz = 0$$

for some $\lambda > 0$.

Corollary 19 *We assume that (17) is satisfied by the characteristic exponents and for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing. If $V : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ is T_1^i -bounded in $L^1(\mathbb{R}^3)$ and K_∞ then*

$$\lim_{\lambda \rightarrow +\infty} \limsup_{c \rightarrow +\infty} \int_{y \in \mathbb{R}^3} \sup_{|x| \geq c} V(x) E_\lambda^i(x-y) dx = 0.$$

The simple observation

$$K_\infty(V) \leq \lim_{\varepsilon \rightarrow 0} \sup_{y \in \mathbb{R}^3} \int_{\{|z| \leq \varepsilon\}} V(y+z) E_\lambda^i(z) dz$$

implies:

Proposition 20 *We assume that (17) is satisfied by the characteristic exponents and for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing. The Kato class potentials are included in the class of K_∞ potentials.*

We are ready to summarize some consequences of the previous results in two main statements.

Theorem 21 *We assume that (17) is satisfied by the characteristic exponents. Let V_-^i and V_-^{ij} be T_1^i -bounded in $L^1(\mathbb{R}^3)$. If*

$$\limsup_{|y| \rightarrow +\infty} V_-^i(y) + \limsup_{|y| \rightarrow +\infty} V_-^{ij}(y) < +\infty \quad (32)$$

and

$$\lim_{|\Omega| \rightarrow 0, \Omega \subset B} \left(\sup_{|y| \leq c} \int_{\Omega} V_-^i(x) E_{\lambda}^i(x-y) dx + \sup_{|y| \leq c} \int_{\Omega} V_-^{ij}(x) E_{\lambda}^i(x-y) dx \right) = 0$$

(for some $\lambda > 0$) for any ball $B \subset \mathbb{R}^3$ and any constant $c > 0$ then the multiplication operator on $L^2(\mathbb{R}^{3N})$ by the potential $-\mathcal{V}_-$ is form-bounded with respect to $-\mathcal{T}$ with zero relative form bound where \mathcal{T} is given in (22) and \mathcal{V}_- is the potential (23).

Theorem 22 *We assume that (17) is satisfied by the characteristic exponents and for each $t > 0$, $k_t^i(\cdot)$ is spherically symmetric and radially decreasing. Let V_-^i and V_-^{ij} be T_1^i -bounded in $L^1(\mathbb{R}^3)$ and K_{∞} . If*

$$\lim_{|\Omega| \rightarrow 0, \Omega \subset B} \left(\sup_{|y| \leq c} \int_{\Omega} V_-^i(x) E_{\lambda}^i(x-y) dx + \sup_{|y| \leq c} \int_{\Omega} V_-^{ij}(x) E_{\lambda}^i(x-y) dx \right) = 0$$

(for some $\lambda > 0$) for any ball $B \subset \mathbb{R}^3$ and any constant $c > 0$ then the multiplication operator on $L^2(\mathbb{R}^{3N})$ by the potential $-\mathcal{V}_-$ is form-bounded with respect to $-\mathcal{T}$ with zero relative form bound where \mathcal{T} is given in (22) and \mathcal{V}_- is the potential (23).

Remark 23 *Because of (13), we could replace K_{∞} by \widehat{K}_{∞} in Theorem 22.*

Remark 24 *Note that $C_c^{\infty}(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ are cores of $-\mathcal{T}$ (see e.g. [30] Theorem 2 (iii)). Since $D(-\mathcal{T})$ is also core of $\sqrt{-\mathcal{T}}$ then $C_c^{\infty}(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ are cores of $\sqrt{-\mathcal{T}}$, i.e. $C_c^{\infty}(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ are form-cores of $-\mathcal{T}$. It follows from ([39] Theorem X.17, p. 167) that $(-\mathcal{T}) \oplus (-\mathcal{V}_-)$ admits $C_c^{\infty}(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ as form-cores.*

Remark 25 *We note that if for some $i \leq N$*

$$T_{(i)} = - \left(\sqrt{-c^2 h^2 \Delta_i + m_i^2 c^4 - m_i c^2} \right),$$

then we cannot handle Coulomb potentials V_-^i and V_-^{ij} . Indeed, in this case, $E_{\lambda}^i(z)$ behaves like $|z|^{-2}$ near $z = 0$ so that a Coulomb potential is not T_1^i -bounded in $L^1(\mathbb{R}^3)$. In this case, the form-boundedness is obtained via a relativistic Hardy inequality [26].

Remark 26 *The results preceding the last two statements are stated in \mathbb{R}^3 but their proofs work in \mathbb{R}^d ($d \in \mathbb{N}$).*

5 Miscellaneous

From now on, we work in \mathbb{R}^d ($d \in \mathbb{N}$). The object of this section is to scrutinize the different assumptions used in the previous sections. Let

$$V : \mathbb{R}^d \rightarrow \mathbb{R}_+$$

be measurable. We say that $V \in L^1_{loc,unif}(dx)$ if

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} V(y+x) dx < +\infty$$

or

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} V_y(x) dx < +\infty$$

where $\{V_y\}_{y \in \mathbb{R}^d}$ are the translates of V

$$V_y : x \in \mathbb{R}^d \rightarrow V(y+x).$$

Let $(S(t))_{t \geq 0}$ be a convolution C_0 -semigroup on $L^1(\mathbb{R}^d)$

$$S(t) : f \in L^1(\mathbb{R}^d) \rightarrow \int_{\mathbb{R}^d} f(x-y) m_t(dy) \in L^1(\mathbb{R}^d) \quad (33)$$

with generator

$$T : D(T) \subset L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d).$$

We assume that the corresponding characteristic exponent satisfies (17) and $k_t(\cdot)$ is spherically symmetric and radially decreasing. Let $E_\lambda(x-y)$ be the kernel of $(\lambda - T)^{-1}$. By Proposition 14, $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is T -bounded in $L^1(\mathbb{R}^d)$ if and only if

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} V_y(x) E_\lambda(x) dx < +\infty \quad (34)$$

i.e.

$$V \in L^1_{loc,unif}(\mu(dx)).$$

where $\mu(dx) = E_\lambda(x) dx$. The following statement provides us with a different insight into the Kato classes.

Theorem 27 Let $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be T -bounded in $L^1(\mathbb{R}^d)$. Then V is a Kato potential provided that $\{V_y; y \in \mathbb{R}^d\}$ is equi-integrable at the origin with respect to the measure $\mu(dx) = E_\lambda(x)dx$.

Proof. We have

$$\begin{aligned}
\int_{\{|z| \leq \varepsilon\}} V(y+z)E_\lambda(z)dz &= \int_{\{|z| \leq \varepsilon\}} V_y(z)E_\lambda(z)dz \\
&= \int_{\{|z| \leq \varepsilon\} \cap \{V_y(z) \geq j\}} V_y(z)E_\lambda(z)dz \\
&\quad + \int_{\{|z| \leq \varepsilon\} \cap \{V_y(z) < j\}} V_y(z)E_\lambda(z)dz \\
&\leq \int_{\{|z| \leq 1\} \cap \{V_y(z) \geq j\}} V_y(z)E_\lambda(z)dz \\
&\quad + j \int_{\{|z| \leq \varepsilon\}} E_\lambda(z)dz
\end{aligned}$$

so

$$\begin{aligned}
&\sup_{y \in \mathbb{R}^d} \int_{\{|z| \leq \varepsilon\}} V(y+z)E_\lambda(z)dz \\
&\leq \sup_{y \in \mathbb{R}^d} \int_{\{|z| \leq 1\} \cap \{V_y(z) \geq j\}} V_y(z)E_\lambda(z)dz + j \int_{\{|z| \leq \varepsilon\}} E_\lambda(z)dz.
\end{aligned}$$

By the criterion of equi-integrability (see e.g. [3] Theorem 4.7.20, p. 287), we fix j large enough so that $\sup_{y \in \mathbb{R}^d} \int_{\{|z| \leq 1\} \cap \{V_y(z) \geq j\}} V_y(z)E_\lambda(z)dz$ is as small as we want and then let $\varepsilon \rightarrow 0$. ■

Remark 28 Similarly, V is in the local Kato class provided that, for any $c > 0$, $\{V_y; |y| \leq c\}$ is equi-integrable at the origin with respect to the measure $\mu(dx) = E_\lambda(x)dx$.

Corollary 29 Let $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be T -bounded in $L^1(\mathbb{R}^d)$. Then V belongs to the local Kato class once

$$y \ni \mathbb{R}^d \rightarrow V_y \in L^1(B(0, 1); \mu(dx)) \text{ is continuous.} \quad (35)$$

Proof. Indeed, for any compact subset $C \subset \mathbb{R}^d$, $\{V_y\}_{y \in C}$ is a compact subset of

$$L^1(B(0, 1); \mu(dx))$$

and this implies the (local) equi-integrability above. ■

Remark 30 In ([1] Theorem 4.15), we find a characterisation of the local Kato class by a continuity assumption quite similar to (35). We suspect that the continuity assumption (35) characterizes also the local Kato class.

Remark 31 (Open problem) Look for potentials V such that

$$y \ni \mathbb{R}^d \rightarrow V_y \in L^1(B(0,1); \mu(dx)) \text{ is not continuous}$$

while $\{V_y; y \in \mathbb{R}^d\}$ is equi-integrable at the origin with respect to the measure $\mu(dx) = E_\lambda(x)dx$. It seems that such potentials are those which distinguish the local Kato class from the class of potentials satisfying the local equi-integrability (26).

We link also the \widehat{K}_∞ potentials to equi-integrability.

Theorem 32 Let $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be T -bounded in $L^1(\mathbb{R}^d)$. Then \widehat{V} is K_∞ provided that

$$\lim_{j \rightarrow +\infty} \limsup_{|y| \rightarrow \infty} \int_{\{|x| \leq 1\} \cap \{V_y(x) \geq j\}} V_y(x) E_\lambda(x) dx = 0.$$

Proof. Given $\delta > 0$, we fix j large enough so that

$$\limsup_{|y| \rightarrow \infty} \int_{\{|x| \leq 1\} \cap \{V_y(x) \geq j\}} V_y(x) E_\lambda(x) dx \leq \delta.$$

Arguing as previously,

$$\begin{aligned} & \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx \\ &= \limsup_{|y| \rightarrow \infty} \int_{\{|x| \leq \varepsilon\}} V_y(x) E_\lambda(x) dx \\ &\leq \limsup_{|y| \rightarrow \infty} \int_{\{|x| \leq 1\} \cap \{V_y(x) \geq j\}} V_y(x) E_\lambda(x) dx + j \int_{\{|x| \leq \varepsilon\}} E_\lambda(x) dx \\ &\leq \delta + j \int_{\{|x| \leq \varepsilon\}} E_\lambda(x) dx \end{aligned}$$

and then let $\varepsilon \rightarrow 0$. ■

For the Laplacian, it is known that V is a Kato potential provided that $V \in L^p_{loc,unif}$, i.e.

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} (V(y+x))^p dx < +\infty,$$

for some $p > \frac{d}{2}$ (see [1] Theorem 1.4 (iii)). By averaging in angles, it is easy to improve slightly this result (i.e. we gain something in "angles"). Actually, we state this for general rotationally invariant convolution semigroups.

Theorem 33 *Let $E_\lambda(x)$ be rotationally invariant, i.e. $E_\lambda(x) = \widehat{E}_\lambda(|x|)$ and let*

$$h(y, \rho) := \int_{S^{d-1}} V(y + \rho\omega) dS(\omega).$$

Let

$$s_\infty := \sup \left\{ s > 1; \int_{\{|x| \leq 1\}} (E_\lambda(x))^s dx < +\infty \right\}$$

If there exists $s > s_\infty^*$ (the conjugate exponent of s_∞) such that

$$C_s := \sup_{y \in \mathbb{R}^d} \left(\int_0^1 (h(y, \rho))^s \rho^{d-1} d\rho \right) < +\infty$$

then V is a Kato potential.

Proof. Note that V being T -bounded in $L^1(\mathbb{R}^d)$ we have at least

$$\sup_{y \in \mathbb{R}^d} \int_0^1 h(y, \rho) \rho^{d-1} d\rho < +\infty.$$

Since

$$\int_{\{|x| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx = \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho$$

then

$$\begin{aligned} & \sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx \\ & \leq \left[\sup_{y \in \mathbb{R}^d} \left(\int_0^1 (h(y, \rho))^s \rho^{d-1} d\rho \right)^{\frac{1}{s}} \right] \left(\int_0^\varepsilon (\widehat{E}_\lambda(\rho))^{s^*} \rho^{d-1} d\rho \right)^{\frac{1}{s^*}} \end{aligned}$$

and we are done since $\int_0^\varepsilon (\widehat{E}_\lambda(\rho))^{s^*} \rho^{d-1} d\rho \rightarrow 0$ ($\varepsilon \rightarrow 0$). ■

We give now membership criteria to the Kato class and to the \widehat{K}_∞ class in terms of asymptotics of the integrals of W_y over spherical shells. Let

$$\Omega_j = \left\{ z \in \mathbb{R}^d; 2^{-(j+1)} \leq |z| < 2^{-j} \right\}, \quad (j \in \mathbb{N})$$

and

$$\eta_j(y) := \int_{\Omega_j} W_y(z) dz.$$

Let

$$\widehat{\eta}_j := \sup_{y \in \mathbb{R}^d} \int_{\Omega_j} W_y(z) dz \text{ and } \eta_j := \limsup_{|y| \rightarrow \infty} \int_{\Omega_j} W_y(z) dz.$$

Theorem 34 *Let $k_t(\cdot)$ be spherically symmetric and radially decreasing, (i.e. $E_\lambda(z) = \widehat{E}_\lambda(|z|)$ and $\rho \rightarrow \widehat{E}_\lambda(\rho)$ nonincreasing). Let W be locally integrable. Then*

$$\sum_{j=0}^{\infty} \left(\widehat{E}_\lambda(2^{-j}) \right) \eta_j(y) \leq \int_{\{|x| \leq 1\}} W_y(x) E_\lambda(x) dx \leq \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y).$$

Moreover:

(i) W is a Kato class potential if

$$\text{the series } \sum_{j=1}^{\infty} \left(\widehat{E}_\lambda(2^{-(j+1)}) \right) \eta_j(y) \text{ converges uniformly in } y \in \mathbb{R}^d; \quad (36)$$

in particular if

$$\limsup_{j \rightarrow \infty} (\widehat{\eta}_j)^{\frac{1}{j}} < \left(\limsup_{j \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \right)^{-1}.$$

(ii) $W \in \widehat{K}_\infty$ if

$$\sum_{j=1}^{\infty} \left(\widehat{E}_\lambda(2^{-(j+1)}) \right) \eta_j < +\infty;$$

in particular if

$$\limsup_{j \rightarrow \infty} (\eta_j)^{\frac{1}{j}} < \left(\limsup_{j \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \right)^{-1}.$$

Proof. We know that

$$\int_{\{|x| \leq 1\}} W_y(x) E_\lambda(x) dx = \int_0^1 h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho$$

where

$$h(y, \rho) := \int_{S^{d-1}} W_y(\rho\omega) dS(\omega).$$

The fact that

$$\begin{aligned} \int_0^1 h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &= \sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\ &\leq \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-(j+1)}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\ &= \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &= \sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\ &\geq \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-j}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\ &= \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-j}) \eta_j(y) \end{aligned}$$

shows the first claim. We have also

$$\int_{\{|x| \leq \varepsilon\}} W_y(x) E_\lambda(x) dx = \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho.$$

Since $2^{-(j+1)} < \varepsilon$ implies $\frac{\ln \varepsilon^{-1}}{\ln 2} - 1 \leq j$ then

$$\begin{aligned} \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &\leq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 2\}} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\ &\leq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 2\}} \widehat{E}_\lambda(2^{-(j+1)}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\ &= \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 2\}} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y). \end{aligned}$$

Note that $\varepsilon \rightarrow 0$ implies that $\frac{\ln \varepsilon^{-1}}{\ln 2} - 2 \rightarrow +\infty$ so that

$$\sup_{y \in \mathbb{R}^d} \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \rightarrow 0 \quad (\varepsilon \rightarrow 0)$$

provided that the series $\sum_{j \geq 0} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y)$ converges uniformly in $y \in \mathbb{R}^d$. This occurs if

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-(j+1)}) \widehat{\eta}_j \right)^{\frac{1}{j}} < 1.$$

This ends the proof of the second claim since

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-(j+1)}) \eta_j \right)^{\frac{1}{j}} \leq \limsup_{j \rightarrow +\infty} (\eta_j)^{\frac{1}{j}} \limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}}.$$

Finally

$$\limsup_{|y \rightarrow \infty|} \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \leq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 2\}} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j$$

and, similarly,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{|y \rightarrow \infty|} \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho = 0$$

provided that $\sum_{j \geq 0} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j < +\infty$. We end the proof similarly. \blacksquare

We show now how to estimate the parameter

$$\limsup_{j \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}}.$$

Lemma 35 *Let*

$$s := \sup \left\{ p \geq 1; \int_{\{|x| \leq 1\}} (E_\lambda(x))^p dx < +\infty \right\}. \quad (37)$$

Then

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \leq 2^{\frac{d}{s}}.$$

Proof. We already know that $\int_{\{|x|\leq 1\}} E_\lambda(x) dx < +\infty$. Let $s > 1$ and let $1 < p < s$. We have

$$\int_{\{|x|\leq 1\}} (E_\lambda(x))^p dx = |S^{d-1}| \int_0^1 \left(\widehat{E}_\lambda(\rho)\right)^p \rho^{d-1} d\rho$$

and

$$\int_0^1 \left(\widehat{E}_\lambda(\rho)\right)^p \rho^{d-1} d\rho = \sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \left(\widehat{E}_\lambda(\rho)\right)^p \rho^{d-1} d\rho \geq \sum_{j=0}^{\infty} \left(\widehat{E}_\lambda(2^{-j})\right)^p \int_{2^{-(j+1)}}^{2^{-j}} \rho^{d-1} d\rho.$$

Since $\int_{2^{-(j+1)}}^{2^{-j}} \rho^{d-1} d\rho = c_d 2^{-(j+1)d}$ with $c_d = d^{-1} (2^d - 1)$ (see the proof of theorem 39) then

$$\sum_{j=0}^{\infty} 2^{-(j+1)d} \left(\widehat{E}_\lambda(2^{-j})\right)^p < +\infty$$

which implies

$$\limsup_{j \rightarrow +\infty} \left(2^{-(j+1)d} \left(\widehat{E}_\lambda(2^{-j})\right)^p\right)^{\frac{1}{j}} \leq 1$$

i.e.

$$\limsup_{j \rightarrow +\infty} \left(\left(\widehat{E}_\lambda(2^{-j})\right)^p\right)^{\frac{1}{j}} \leq 2^d.$$

Since $\limsup_{j \rightarrow +\infty} \left(\left(\widehat{E}_\lambda(2^{-j})\right)^p\right)^{\frac{1}{j}} = \left(\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j})\right)^{\frac{1}{j}}\right)^p$ then

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j})\right)^{\frac{1}{j}} \leq 2^{\frac{d}{p}} \quad (p < s)$$

Finally, letting $p \rightarrow s$ ends the proof when $s > 1$. If $s = 1$, the above calculations with $p = 1$ end the proof. ■

Remark 36 When $\widehat{E}_\lambda(\rho) \rightarrow +\infty$ ($\rho \rightarrow 0$) (this is generally the case) then $\widehat{E}_\lambda(\rho) \sim \widehat{E}_\beta(\rho)$ ($\rho \rightarrow 0$) ($\lambda, \beta > 0$) (see [4] Lemma III. 3). It follows that the parameter (37) is λ -independent.

As a consequence of Theorem 34 and Lemma 35 we have:

Corollary 37 Let $k_t(\cdot)$ be spherically symmetric and radially decreasing, (i.e. $E_\lambda(z) = \widehat{E}_\lambda(|z|)$ and $\rho \rightarrow \widehat{E}_\lambda(\rho)$ nonincreasing). Let W be locally integrable and let s be given by (37).

(i) W is a Kato class potential provided that

$$\limsup_{j \rightarrow \infty} (\widehat{\eta}_j)^{\frac{1}{j}} < 2^{-\frac{d}{s}}.$$

(ii) $W \in \widehat{K}_\infty$ provided that

$$\limsup_{j \rightarrow \infty} (\eta_j)^{\frac{1}{j}} < 2^{-\frac{d}{s}}.$$

Remark 38 Note that for $d \geq 3$

$$\widehat{E}_0(\rho) \sim \frac{1}{\rho^{d-\alpha}} \quad (\rho \rightarrow 0) \quad (0 < \alpha \leq 2) \quad (38)$$

for the usual examples ($\alpha = 2$ for the heat semigroup, $0 < \alpha < 2$ for the α -stable semigroup and $\alpha = 1$ for the relativistic semigroup; see [4]). Thus $\widehat{E}_0(2^{-(j+1)}) \sim 2^{(j+1)(d-\alpha)}$ and

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_0(2^{-j}) \right)^{\frac{1}{j}} = 2^{d-\alpha}$$

shows that Lemma 35 is optimal. In these cases, W is Kato potential (resp. $W \in \widehat{K}_\infty$) provided that

$$\limsup_{j \rightarrow \infty} (\widehat{\eta}_j)^{\frac{1}{j}} < 2^{-(d-\alpha)} \quad (\text{resp.} \quad \limsup_{j \rightarrow \infty} (\eta_j)^{\frac{1}{j}} < 2^{-(d-\alpha)}).$$

Actually, for subordinate Brownian semigroups (relative to a Bernstein function f), the behaviour of $\widehat{E}_0^f(\rho)$ ($\rho \rightarrow 0$) is determined by the asymptotics of $f(\lambda)$ at infinity. In particular, $\widehat{E}_0^f(\rho) \sim \frac{1}{\rho^{d-\alpha}}$ ($\rho \rightarrow 0$) ($0 < \alpha \leq 2$) if $f(\lambda) \sim \lambda^{\frac{\alpha}{2}}$ ($\lambda \rightarrow +\infty$); (see [37] Theorem 3.1). This is the case of the Bernstein function $f(\lambda) = \left(\lambda + m^{\frac{2}{\alpha}} \right)^{\frac{\alpha}{2}} - m$ ($0 < \alpha < 2$) which defines the relativistic α -stable semigroup with generator

$$T = - \left(-\Delta + m^{\frac{2}{\alpha}} \right)^{\frac{\alpha}{2}} + m$$

and characteristic exponent $F(\zeta) = \left(|\zeta|^2 + m^{\frac{2}{\alpha}} \right)^{\frac{\alpha}{2}} - m$.

We are going to derive Theorem 33 from Theorem 34 under Assumption (38).

Theorem 39 We assume that $E_\lambda(x)$ be rotationally invariant, i.e. $E_\lambda(x) = \widehat{E}_\lambda(|x|)$, and (38) is satisfied. Let $W \in L_{loc,unif}^s$, i.e.

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} (V(y+x))^s dx < +\infty,$$

with $s > \frac{d}{\alpha}$. Then W is Kato class potential.

Proof. We have

$$\begin{aligned} & \int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} W_y(z) dz \\ & \leq \left(\int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} W_y(z)^s dz \right)^{\frac{1}{s}} \left(\int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} dz \right)^{\frac{1}{s^*}} \\ & \leq C_j(s) \left(\int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} dz \right)^{\frac{1}{s^*}} = C_j(s) \left(|S^{d-1}| \int_{2^{-j}}^{2^{-(j-1)}} \rho^{d-1} d\rho \right)^{\frac{1}{s^*}} \end{aligned}$$

Since

$$\int_{2^{-j}}^{2^{-(j-1)}} \rho^{d-1} d\rho = c_d 2^{-jd}$$

(where $c_d = d^{-1} (2^d - 1)$) then

$$\widehat{\eta}_j := \sup_{y \in \mathbb{R}^d} \int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} W_y(z) dz \leq C_j(s) \left(|S^{d-1}| c_d 2^{-jd} \right)^{\frac{1}{s^*}}.$$

Since $\widehat{E}_\lambda(2^{-(j+1)}) \sim 2^{(j+1)(d-\alpha)}$ then

$$\begin{aligned} \limsup_{j \rightarrow \infty} \left(\widehat{\eta}_j \widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} & \leq \limsup_{j \rightarrow \infty} (\widehat{\eta}_j)^{\frac{1}{j}} \limsup_{j \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \\ & \leq \left(\limsup_{j \rightarrow \infty} (C_j(s))^{\frac{1}{j}} \right) 2^{-\frac{d}{s^*}} 2^{(d-\alpha)} \\ & = \left(\limsup_{j \rightarrow \infty} (C_j(s))^{\frac{1}{j}} \right) 2^{\frac{d}{s} - \alpha}. \end{aligned}$$

Note that $\limsup_{j \rightarrow \infty} (C_j(s))^{\frac{1}{j}} \leq 1$ since $W \in L_{loc,unif}^s$. Finally, Theorem 34 ends the proof. ■

We characterize now the property for V to vanish weakly at infinity. We start with a definition.

Definition 40 (i) We say that a measurable set $A \subset \mathbb{R}^d$ is thin at infinity if

$$\lim_{|x| \rightarrow +\infty} |B(x, 1) \cap A| \rightarrow 0.$$

(ii) Let $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be locally integrable. We say that V is locally equi-integrable at infinity if for any measurable set $A \subset \mathbb{R}^d$ thin at infinity we have

$$\lim_{|x| \rightarrow +\infty} \int_{B(x, 1) \cap A} V(z) dz = 0.$$

Theorem 41 Let $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be measurable. Then:

(i) V vanishes weakly at infinity if and only if

$$\int_{B(x, 1) \cap \{V \geq c\}} V(z) dz \rightarrow 0 \quad (|x| \rightarrow +\infty) \quad (\forall c > 0).$$

(ii) If V is locally equi-integrable at infinity then V vanishes weakly at infinity if and only if the superlevel sets $\{V \geq c\}$ are thin at infinity.

Proof. The first claim follows from the inequalities

$$\begin{aligned} \int_{B(x, 1) \cap \{V \geq c\}} V(z) dz &\leq \int_{B(x, 1)} V(z) dz \\ &\leq \int_{B(x, 1) \cap \{V \geq c\}} V(z) dz + \int_{B(x, 1) \cap \{V < c\}} V(z) dz \\ &\leq \int_{B(x, 1) \cap \{V \geq c\}} V(z) dz + c |B(x, 1)| \\ &= \int_{B(x, 1) \cap \{V \geq c\}} V(z) dz + c |B(0, 1)|. \end{aligned} \quad (39)$$

Since $\int_{B(x, 1) \cap \{V \geq c\}} V(z) dz \geq c |B(x, 1) \cap \{V \geq c\}|$ then the necessity part of (ii) is true. Suppose now that $\{V \geq c\}$ is thin at infinity ($\forall c > 0$). Then the local equi-integrability of V at infinity implies that

$$\int_{B(x, 1) \cap \{V \geq c\}} V(z) dz \rightarrow 0 \quad (|x| \rightarrow +\infty) \quad (\forall c > 0).$$

It suffices to choose c small enough and to use (39). ■

Remark 42 Holder's inequality shows easily that if $V \in L_{loc, unif}^p$ at infinity for some $p > 1$ then V is locally equi-integrable at infinity.

We know that a Kato class potential (relative to T) has zero relative operator bound, see (9). We have seen that the potentials considered here satisfy

$$\lim_{\lambda \rightarrow +\infty} r_\sigma \left(W (\lambda - T)^{-1} \right) = 0. \quad (40)$$

We give now a substitute to (9) for such potentials.

Theorem 43 *Let W satisfy (40). Then for any $\varepsilon > 0$ there exists an equivalent norm $\|\cdot\|_{\varepsilon new}$ on $L^1(\mathbb{R}^d)$ (additive on the positive cone) and a constant $c_\varepsilon > 0$ such that*

$$\|W\varphi\|_{\varepsilon new} \leq \varepsilon \|T\varphi\|_{\varepsilon new} + c_\varepsilon \|\varphi\|_{\varepsilon new}, \quad \varphi \in D(T).$$

Proof. We resume an argument given in ([28] lemma 8.3, p. 189). Let $\varepsilon > 0$ be given and let $\hat{\lambda} > 0$ such that $r_\sigma \left(W (\hat{\lambda} - T)^{-1} \right) \leq \frac{\varepsilon}{2}$. There exists $n_0 \in \mathbb{N}$ such that

$$\left\| \left(W (\lambda - T)^{-1} \right)^n \right\| < \varepsilon^n \quad (\forall n \geq n_0).$$

One sees that

$$\|f\|_{\varepsilon new} := \sum_{k=0}^{\infty} \varepsilon^{-k} \left\| \left(W (\lambda - T)^{-1} \right)^k f \right\|$$

is a norm which is additive on $L^1_+(\mathbb{R}^3)$. Moreover, $\|f\|_{\varepsilon new} \geq \|f\|$ and there exists $c > 0$ such that $\|f\|_{\varepsilon new} \leq c \|f\|$. Since

$$\begin{aligned} \left\| W (\hat{\lambda} - T)^{-1} f \right\|_{\varepsilon new} &= \sum_{k=0}^{\infty} \varepsilon^{-k} \left\| \left(W (\lambda - T)^{-1} \right)^{k+1} f \right\| \\ &= \varepsilon \sum_{k=0}^{\infty} \frac{1}{\varepsilon^{k+1}} \left\| \left(W (\lambda - T)^{-1} \right)^{k+1} f \right\| \\ &= \varepsilon \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} \left\| \left(W (\lambda - T)^{-1} \right)^k f \right\| \leq \varepsilon \|f\|_{\varepsilon new} \end{aligned}$$

then $\left\|W\left(\widehat{\lambda}-T\right)^{-1}\right\|_{\varepsilon_{new}} \leq \varepsilon$. Finally, for any $\varphi \in D(T)$

$$\begin{aligned} \|W\varphi\|_{\varepsilon_{new}} &= \left\|W\left(\widehat{\lambda}-T\right)^{-1}\left(\widehat{\lambda}-T\right)\varphi\right\|_{\varepsilon_{new}} \leq \left\|W\left(\widehat{\lambda}-T\right)^{-1}\right\|_{\varepsilon_{new}}\|T\varphi\|_{\varepsilon_{new}} \\ &\quad + \widehat{\lambda}\left\|W\left(\widehat{\lambda}-T\right)^{-1}\right\|_{\varepsilon_{new}}\|\varphi\|_{\varepsilon_{new}} \\ &\leq \varepsilon\|T\varphi\|_{\varepsilon_{new}} + \widehat{\lambda}\left\|W\left(\widehat{\lambda}-T\right)^{-1}\right\|_{\varepsilon_{new}}\|\varphi\|_{\varepsilon_{new}}. \end{aligned}$$

This ends the proof. ■

6 Spectral theory

We end this paper with two spectral results in $L^1(\mathbb{R}^d)$. We start with a preliminary result.

Lemma 44 *We assume that $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is T -bounded in $L^1(\mathbb{R}^d)$ and satisfies*

$$\lim_{|\Omega| \rightarrow 0, \Omega \subset B} \sup_{|y| \leq c} \int_{\Omega} V(x) E_{\lambda}(x-y) dx = 0 \quad (41)$$

(for some $\lambda > 0$) for any ball $B \subset \mathbb{R}^3$ and any constant $c > 0$. If

$$\lim_{c \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_{\lambda}(x-y) dx = 0$$

then V is T -weakly compact on $L^1(\mathbb{R}^d)$. In particular,

$$\sigma_{ess}(T+V) = \sigma_{ess}(T).$$

Proof. Indeed, let $V_c(x) := V(x)1_{\{|x| \geq c\}}$ and $\widehat{V}_c(x) := V(x)1_{\{|x| < c\}}$. Since (26) implies that $\widehat{V}_c(\lambda - T)^{-1}$ is weakly compact then

$$\left\|V(\lambda - T)^{-1} - \widehat{V}_c(\lambda - T)^{-1}\right\| = \|V_c(\lambda - T)^{-1}\| \rightarrow 0 \quad (c \rightarrow +\infty) \quad (42)$$

implies that $V(\lambda - T)^{-1}$ is weakly compact. Since

$$\lim_{\lambda \rightarrow +\infty} r_{\sigma}[V(\lambda - T)^{-1}] = 0$$

then, once $r_\sigma [V(\lambda - T)^{-1}] < 1$, $(\lambda - T - V)^{-1}$ exists and

$$(\lambda - T - V)^{-1} - (\lambda - T)^{-1} = \sum_{j=1}^{\infty} (\lambda - T)^{-1} (V(\lambda - T)^{-1})^j$$

is weakly compact so that $T + V$ and T share the same essential spectrum [21]. ■

It is easy to see that $\sup_{y \in \mathbb{R}^3} \int_{|x| \geq c} V(x) E_\lambda^i(x - y) dx$ is bounded from below by

$$\sup_{y \in \mathbb{R}^3} \int_{\{|x| \geq c\} \setminus \{|x-y| \leq \delta\}} V(x) E_\lambda^i(x - y) dx \geq \widehat{E}_\lambda^i(\delta) \sup_{|y| \geq c + \delta} \int_{\{|z| \leq \delta\}} V(y + z) dz$$

so that (42) implies that V must vanish weakly at infinity.

Our first result is about stability of essential spectra.

Theorem 45 *We assume that $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is T -bounded in $L^1(\mathbb{R}^d)$ and satisfies (41). Let $k_t(\cdot)$ ($t > 0$) be spherically symmetric and radially decreasing. If V vanishes weakly at infinity in the sense (30) and is \widehat{K}_∞ in the sense*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{|y| \rightarrow \infty} \int_{\{|z| \leq \varepsilon\}} V(y + z) E_\lambda(z) dz = 0$$

then V is T -weakly compact. In particular $\sigma_{ess}(T + V) = \sigma_{ess}(T)$. If the C_0 -semigroup $(e^{tT})_{t \geq 0}$ is (operator) norm continuous (in $t > 0$) then $\sigma_{ess}(e^{t(T+V)}) = \sigma_{ess}(e^{tT})$.

Proof. The combination of Remark 17 and Lemma 44 shows that V is T -weakly compact. If additionally $(e^{tT})_{t \geq 0}$ is operator-norm continuous then $e^{t(T+V)} - e^{tT}$ is weakly compact and $\sigma_{ess}(e^{t(T+V)}) = \sigma_{ess}(e^{tT})$, (see [32] Theorem 67). ■

Remark 46 *We note that if a C_0 -semigroup $(S(t))_{t \geq 0}$ is holomorphic and so is the subordinate semigroup $(S^f(t))_{t \geq 0}$ for any Bernstein function f (see e.g. [15]). In particular all Brownian semigroups are holomorphic and consequently norm continuous.*

Our second result is about spectral gaps.

Theorem 47 *Besides the assumptions in Theorem 45, we assume $m_t(dx) = k_t(x) dx$ ($t > 0$) are probability measures. Then:*

(i) $(e^{t(T+V)})_{t \geq 0}$ has spectral gap, i.e.

$$r_{ess}(e^{t(T+V)}) < r_\sigma(e^{t(T+V)}), \quad (t > 0),$$

if

$$\liminf_{\lambda \rightarrow 0_+} \inf_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(y+x) E_\lambda(x) dx > 1. \quad (43)$$

(ii) If $\frac{1}{F} \in L^1_{loc}(\mathbb{R}^d)$ then the potential operator exists; let $E_0(x-y)$ be its kernel. If

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(y+x) E_0(x) dx \leq 1 \quad (44)$$

then the type of $(e^{t(T+V)})_{t \geq 0}$ is equal to zero.

Proof. In this case, $(e^{tT})_{t \geq 0}$ is stochastic, i.e. mass preserving on $L^1_+(\mathbb{R}^d)$ and 0 is the type of $(e^{tT})_{t \geq 0}$. It follows from Theorem 45 that

$$\sigma_{ess}(e^{t(T+V)}) \subset \sigma(e^{tT}) \subset \{\nu \in \mathbb{C}; |\nu| \leq 1\}$$

so $r_{ess}(e^{t(T+V)}) \leq 1$. Note that for any $\varphi \in L^1_+(\mathbb{R}^d)$

$$\begin{aligned} \left\| V(\lambda - T)^{-1} \varphi \right\| &= \int_{\mathbb{R}^d} V(x) dx \int_{\mathbb{R}^d} E_\lambda(x-y) \varphi(y) dy \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} V(x) E_\lambda(x-y) dx \right) \varphi(y) dy \\ &\geq \delta_\lambda \|\varphi\| \end{aligned}$$

where

$$\delta_\lambda := \inf_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(x) E_\lambda(x-y) dx > 0.$$

It follows by iterations that

$$\left\| \left(V(\lambda - T)^{-1} \right)^j \varphi \right\| \geq (\delta_\lambda)^j \|\varphi\| \quad (j \in \mathbb{N})$$

so $\left\| \left(V(\lambda - T)^{-1} \right)^j \right\| \geq (\delta_\lambda)^j$ and

$$r_\sigma(V(\lambda - T)^{-1}) \geq \delta_\lambda \quad (\lambda > 0).$$

Thus (43) implies

$$\liminf_{\lambda \rightarrow 0_+} r_\sigma(V(\lambda - T)^{-1}) > 1.$$

Note that $V(\lambda - T)^{-1}$ is positive so that $r_\sigma(V(\lambda - T)^{-1}) \in \sigma(V(\lambda - T)^{-1})$ (see e.g. [40] Proposition 1). Since $V(\lambda - T)^{-1}$ is weakly compact then $(V(\lambda - T)^{-1})^2$ is compact. Since

$$r_\sigma(V(\lambda - T)^{-1}) > 0$$

then $r_\sigma(V(\lambda - T)^{-1})$ is an isolated eigenvalue of $(V(\lambda - T)^{-1})^{-1}$ associated to a nonnegative eigenfunction φ , (see e.g. [40] Proposition 4). Finally, the continuity of

$$(0, +\infty) \ni \lambda \rightarrow r_\sigma(V(\lambda - T)^{-1})$$

and $\lim_{\lambda \rightarrow +\infty} r_\sigma(V(\lambda - T)^{-1}) = 0$ imply the existence of $\lambda > 0$ such that $r_\sigma(V(\lambda - T)^{-1}) = 1$. Thus, there exists a non-zero $\varphi \in L^1_+(\mathbb{R}^d)$ such that

$$V(\lambda - T)^{-1}\varphi = \varphi \quad (\varphi \in L^1_+(\mathbb{R}^d))$$

so $\psi := (\lambda - T)^{-1}\varphi \in D(T) \cap L^1_+(\mathbb{R}^d)$ and

$$T\psi + V\psi = \lambda\psi, \quad (\psi \neq 0).$$

Finally $e^{\lambda t}$ is an eigenvalue of $(e^{t(T+V)})_{t \geq 0}$ and $r_\sigma(e^{t(T+V)}) > 1$. This shows (i). For the existence of the potential operator, see [19]. Since $V \geq 0$ then a priori, the type of $(e^{t(T+V)})_{t \geq 0}$ is not less than the one of $(e^{tT})_{t \geq 0}$. Note that $\|V(\lambda - T)^{-1}\| < 1$ ($\lambda > 0$) follows from (44) so the spectral bound of $T + V$ is ≤ 0 . This ends the proof of (ii) since this spectral bound must be equal to the type of $(e^{t(T+V)})_{t \geq 0}$ [51]. ■

Remark 48 *In the case (i), $\sigma(e^{t(T+V)}) \cap \{\nu \in \mathbb{C}; |\nu| > 1\}$ consists of a nonempty set of isolated eigenvalues with finite algebraic multiplicities. The same observation holds for $\sigma(T + V) \cap \{\text{Re } \lambda > 0\}$. It is not clear a priori (although very likely) that this spectrum in L^1 is real for general symmetric convolution semigroups $(e^{tT})_{t \geq 0}$. We suspect also that $\sigma_{\text{ess}}(e^{tT}) = \sigma(e^{tT})$ in L^1 so that $r_{\text{ess}}(e^{tT}) = r_{\text{ess}}(e^{t(T+V)}) = 1$.*

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