Geometry and identity theorems for bicomplex functions and functions of a hyperbolic variable

M. E. E Luna-Elizarrarás, Marco Panza, M. Shapiro, D. C. Struppa

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M.E. Luna–Elizarrarás [*], M.Panza[†], M.Shapiro[‡], D.C. Struppa[§]

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Abstract

Let $\mathbb{D}$ be the two-dimensional real algebra generated by 1 and by a hyperbolic unit k such that $k^2 = 1$. This algebra is often referred to as the algebra of hyperbolic numbers. A function $f : \mathbb{D} \to \mathbb{D}$ is called $\mathbb{D}$-holomorphic in a domain $\Omega \subset \mathbb{D}$ if it admits derivative in the sense that $\lim_{h\to 0} \frac{f(\mathfrak{z}_0+h)-f(\mathfrak{z}_0)}{h}$ exists for every point $\mathfrak{z}_0$ in Ω, and when h is only allowed to be an invertible hyperbolic number. In this paper we prove that $\mathbb{D}$-holomorphic functions satisfy an unexpected limited version of the identity theorem. We will offer two distinct proofs that shed some light on the geometry of $\mathbb{D}$. Since hyperbolic numbers are naturally embedded in the four-dimensional

[*]HIT - Holon Institute of Technology, Golomb St 52, Holon, 5810201 Israel
[†]CNRS, IHPST (CNRS and Univ. of Patis 1, Panthéon-Sorbonne) and Chapman University (Orange, CA)
[‡]HIT - Holon Institute of Technology, Golomb St 52, Holon, 5810201 Israel
[§]The Donald Bren Presidential Chair in Mathematics, Chapman University, Orange, USA

algebra of bicomplex numbers, we use our approach to state and prove an identity theorem for the bicomplex case as well.

1 Introduction

Back in the second half of the nineteenth century, the Irish mathematician W.R.Hamilton introduced what is now called the skew field of quaternions, in a (successful) attempt to describe rotations in the space. His construction is very well known, and consists in building the real algebra on four units, $1, i, j, k = ij$, where i, j, k are imaginary units (i.e their square is equal to -1) which anticommute (i.e. $ij = -ji, ik = -ki, jk = -kj$). The lack of commutativity was an obstacle that hindered Hamilton's progress, until he was able to surrender the comfort of a commutative setting, something he discussed at great length in his famous letters to his son [7].

Concurrently with Hamilton, a much less famous, and much less talented, English mathematician under the name of James Cockle, developed a parallel theory (that he saw as inspired by Hamilton's quaternions), and applied it to new numbers that he called 'tessarines' (see, among others, [2], [3], [4]; on Cockle's results, see also [1]). The collection of works of Cockle will receive a more detailed treatment in a forthcoming work by Panza and Struppa. At this point we will limit ourselves to mentioning that Cockle's tessarines were born almost out of luck, as they are the consequence of a series of ill-conceived ideas. Nevertheless, they are an object worth of study, and that is in fact now the subject of significant research under the name of bicomplex numbers (see [9], for example).

The idea is simple. The algebra of bicomplex numbers, or tessarines as Cockle called them, is the four dimensional real algebra over the units $1, i, j, k = ij$, but this time while i and j are (complex) imaginary units, k is not, since i and j are asked to commute ($ij = ji = k$), and this makes k is what is now referred to as a hyperbolic imaginary unit, namely, a non-real number such that $k^2 = 1$. The resulting algebra is, in some elementary sense, an easier object to study because it preserves the commutative nature of complex numbers, but on the other hand, as we will see shortly, it offers a new set of problems because the commutativity leads to the existence of zero-divisors, a most inconvenient byproduct.

In [2], Cockle also implies that within the real algebra of tessarines, one can identify an interesting subalgebra (he does not quite use such a precise language, but certainly understands the idea), if one considers the two-dimensional real algebra over 1 and k. This algebra is modernly referred to as the real algebra of hyperbolic numbers.

If one now moves to the twentieth century, one can see that many mathematicians (mostly from the Italian algebra and analysis schools) developed a fairly sophisticated theory of holomorphicity for functions defined on various algebras. They understood the subtleties that emerge from the specific properties of each algebra, and possibly the best work in this area, though somewhat forgotten, is the one of Sce, which has recently been translated and commented upon in [5].

In particular, a theory was developed for functions that satisfy holomorphicity conditions on bicomplex functions, as well as those that satisfy similar conditions for functions on the algebra of hyperbolic numbers. The modern theory of those functions for the bicomplex case is described in a fairly complete way in [9], and the beginning of the theory for the hyperbolic case is discussed in [10]. We should also point to reader to [8] where additional work is done on the geometry of the hyperbolic plane.

In this article we consider a natural question that has different answers in the bicomplex and the hyperbolic cases, namely, the question of whether an identity theorem may exist for functions that satisfy holomorphicity conditions. The question was stimulated by a close reading of the original work of Cockle.

His approach in [2] consists in distinguishing "unreal" quantities, depending on the imaginary unit i, from "impossible" ones depending on another non-real unit k (we are using here the modern notation in order to avoid confusion among the units), which commutes with i, and such that $k^2 = 1$. This leads him to a four-dimensional real algebra, generated by 1, i, $j := -ik$, and k. This is the algebra of bicomplex numbers (in modern terms) or of tessarines (in Cockle's terms).

Cockle then proceeds ([2], p. 438) to claim that any "impossible quantity [...] altogether disappears" from the sum $e^{kx} + e^{-kx}$ and from the quotient $\dfrac{e^{kx} - e^{-kx}}{2k}$, where x is any real number. To justify this conclusion he simply assumes that the function of a hyperbolic variable e^{kx} develops as the real exponential, i.e., that

$$e^{kx} = \sum_{n=0}^{\infty} \frac{(kx)^n}{n!},$$

from which it immediately follows, by simple replacement, that

$$e^{kx} + e^{-kx} = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad ; \qquad \frac{e^{kx} - e^{-kx}}{2k} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{1.1}$$

Cockle's process, here, is essentially the same used by Euler's in [6] (vol. 1, chapter VIII), when he derives his classical results about imaginary exponentials and trigonometric functions. In doing that, Euler implicitly assumes that the real exponential function e^x ($x \in \mathbb{R}$) plainly extends to the the new function $e^{ix} : \mathbb{R} \to \mathbb{C}$, having the same power series development as e^x, under the replacement of x with ix. He is silent about how this last function is defined, but we might suppose he was considering it directly defined by this same development (that can be rewritten as $\sum_{h=0}^{\infty}(-1)^h \frac{x^{2h}}{(2h)!} + i\sum_{h=0}^{\infty}(-1)^{2h+1} \frac{x^{2h+1}}{(2h+1)!}$, where the two real series are provably convergent). This would leave still open both the problem of defining a complex function (conveniently denoted by e^z), extending e^x to the whole $\mathbb{C}$, and the question whether this function is unique. We know today how to solve this problem—by simply defining $e^z = e^{x+iy} = e^x e^{iy}$ ($x, y \in \mathbb{R}$), and taking e^{iy} as defined by its power series development, just as Euler presumably did—and answer this question in the affirmative—by appealing to the identity theorem for (complex) holomorphic functions, which Euler, however, could not have proved.

By (apparently) following Euler's approach, Cockle utilizes equalities (1.1) and, is then led to the search for a function $f(x + ky)$ ($x, y \in \mathbb{R}$) of a hyperbolic variable whose restriction to $\mathbb{R}$ is exactly e^x. To obtain this function, it is enough to consider the function $e^x(\cosh y + k \sinh y)$, which clearly satisfies the requirements ([10]). Note however that Cockle had no definition for holomorphicity of functions of hyperbolic variables, and therefore his approach is simply algebraic, with no reference to any analytical properties.

It is therefore natural to ask whether this function is unique, once we impose some holomorphicity property on it. It turns out that the answer is somewhat surprising, in fact quite counterintuitive, and for that reason worthy of a complete discussion.

The plan of the paper is as follows: in Section 2, we give a quick summary on holomorphic functions of bicomplex and hyperbolic variables. Nothing in this section is new, and the reader interested in further details should look at [10] and [9]. Section 3 is the core of the paper. In it we prove an identity theorem for holomorphic functions of a hyperbolic variable. As it will become apparent, this theorem is much weaker than its complex counterpart, but it is still strong enough to ensure the unicity of a holomorphic hyperbolic continuation of the exponential function. After giving the proof of the identity theorem, we explore a bit more the geometry of the hyperbolic plane, and we offer some interesting generalizations of the identity theorem itself. The last section of the paper offers some variants of the identity theorem in the bicomplex setting. Even though it is rather obvious that a more general identity theorem holds for holomorphic functions of a bicomplex variable (almost an immediate consequence of the identithy theorem for holomorphic functions in the complex plane), this last section allows us to make some interesting geometrical considerations on the bicomplex plane.

2 Holomorphic functions on the algebras of bicomplex and hyperbolic numbers

The set $\mathbb{BC}$ of bicomplex numbers is defined by

$$\mathbb{BC} := \{z_1 + jz_2 \mid z_1,\, z_2 \in \mathbb{C}\}$$

where $\mathbb{C} = \{x_1 + ix_2 \,|\, x_1, x_2 \in \mathbb{R}\}$ is the set of complex numbers with the imaginary unit i and where i and $j \neq i$ are commuting imaginary units, i.e., $ij = ji$, $i^2 = j^2 = -1$. The addition and multiplication are defined in a clear way. We will write sometimes $\mathbb{C}(i)$ instead of $\mathbb{C}$ since the set $\mathbb{C}(j) := \{x_1 + jx_2 \mid x_1,\, x_2 \in \mathbb{R}\}$ can be equally called the set of complex numbers; both $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields, and although coexisting inside $\mathbb{BC}$, they are different.

The set of hyperbolic numbers can be defined intrinsically (independently of $\mathbb{BC}$) as

$$\mathbb{D} := \{x + ky \mid x,\, y \in \mathbb{R}\}$$

where k is a hyperbolic imaginary unit, i.e., $k^2 = 1$, $k \neq \pm 1$, commuting with both real numbers x and y. Again, it is clear how to add and to multiply the hyperbolic numbers.

Working with $\mathbb{BC}$, a hyperbolic unit k emerges as the product of the two complex imaginary units: $k := ij$. Thus the ring $\mathbb{BC}$ contains a ring, which is isomorphic to the ring of hyperbolic numbers defined by

$$\mathbb{D} := \{x + ijy \mid x,\, y \in \mathbb{R}\}\,.$$

Let $\mathfrak{S}$ denote the set of zero-divisors in $\mathbb{BC}$. A bicomplex number $z_1 + jz_2$ is a zero-divisor if and only if $z_1^2 + z_2^2 = 0$. There are two very special zero-divisors: $\mathbf{e} := \frac{1}{2}(1+k)$ and $\mathbf{e}^\dagger := \frac{1}{2}(1-k)$; they have the properties: $\mathbf{e}\mathbf{e}^\dagger = 0$, $\mathbf{e}^2 = \mathbf{e}$, $(\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger$, $\mathbf{e} + \mathbf{e}^\dagger = 1$, $\mathbf{e} - \mathbf{e}^\dagger = k$. Finally,

$$\mathfrak{S} = (\mathbb{C} \setminus \{0\})\mathbf{e} \,\cup\, (\mathbb{C} \setminus \{0\})e^\dagger.$$

For any bicomplex number $Z = z_1 + jz_2$ one can write:

$$Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \tag{2.1}$$

where $\beta_1 = z_1 - iz_2$, $\beta_2 = z_1 + iz_2$. It is obvious that $Z = 0$ if and only if $\beta_1 = \beta_2 = 0$. Many operations with bicomplex numbers can be performed term-wise using the idempotent representation.

It is worth noting that $\mathbf{e}$ and $\mathbf{e}^\dagger$ are hyperbolic numbers inside $\mathbb{BC}$ which leads to the idempotent representation for hyperbolic numbers as well; such representation has the same form as (2.1) but with β_1 and β_2 real numbers; if $\mathfrak{z} = x + ky$, then $\beta_1 = x + y$, $\beta_2 = x - y$.

Consider a bicomplex function $F : \Omega \subset \mathbb{BC} \longrightarrow \mathbb{BC}$. The derivative $F'(Z_0)$ of F at a point $Z_0 \in \Omega$ is defined as the limit, if it exists,

$$F'(Z_0) := \lim_{Z \to Z_0} \frac{F(Z) - F(Z_0)}{Z - Z_0} = \lim_{\mathfrak{S}_0 \not\ni H \to 0} \frac{F(Z_0 + H) - F(Z_0)}{H}.$$

Such a derivative maintains many properties of real and complex derivatives; in particular, the arithmetic operations look exactly the same. A function F with derivative at Z_0 is called derivable at Z_0. If F has bicomplex derivative at each point of Ω, then we will say that F is a bicomplex holomorphic, or $\mathbb{BC}$–holomorphic, function.

The following bicomplex Cauchy–Riemann operators are introduced by means of the usual complex derivatives in z and $\bar{z}$:

$$\frac{\partial}{\partial Z} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\right); \qquad \frac{\partial}{\partial Z^\dagger} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}\right);$$
$$\frac{\partial}{\partial \overline{Z}} := \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_1} - j\frac{\partial}{\partial \bar{z}_2}\right); \qquad \frac{\partial}{\partial Z^*} := \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right).$$

Let $F \in C^1(\Omega, \mathbb{BC})$, F is $\mathbb{BC}$–holomorphic if and only if

$$\frac{\partial F}{\partial Z^\dagger}(Z) = \frac{\partial F}{\partial \overline{Z}}(Z) = \frac{\partial F}{\partial Z^*}(Z) = 0 \tag{2.2}$$

hold on Ω. If these identities are satisfied, then

$$F'(Z) = \frac{\partial F}{\partial Z}(Z).$$

Writing $F = F_1 + jF_2$ the identities (2.2) imply that F_1 and F_2 are $\mathbb{C}$–valued holomorphic functions of two complex variables in the classical sense; what is more, they are not independent but they are tied by the Cauchy–Riemann type conditions

$$\frac{\partial F_1}{\partial z_1} = \frac{\partial F_2}{\partial z_2}; \qquad \frac{\partial F_1}{\partial z_2} = -\frac{\partial F_2}{\partial z_1}.$$

How do these formulas look if one uses the idempotent representation of bicomplex numbers? First of all, we write in the idempotent form the bicomplex numbers involved:

$$Z = \beta_1 e + \beta_2 \mathbf{e}^\dagger;$$
$$F(Z) = G_1(Z)\mathbf{e} + G_2(Z)\mathbf{e}^\dagger,$$

for suitable complex valued functions G_1 and G_2. Introduce also the sets $\Omega_1 := \{\beta_1 \mid \beta_1\mathbf{e}+\beta_2\mathbf{e}^\dagger \in \Omega\}$, $\Omega_2 := \{\beta_2 \mid \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \Omega\}$.

In general, the two functions G_1 and G_2 depend on both complex variables β_1 and β_2, namely $G_1(Z) = G_1(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger)$, $G_2(Z) = G_2(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger)$. But the situation is radically different for $\mathbb{BC}$–holomorphic functions. A bicomplex function F is $\mathbb{BC}$–holomorphic if and only if the following conditions hold:

(I) G_1 is a holomorphic function of the variables β_1, β_2 that does not depend on β_2; thus G_1 is a holomorphic function of $\beta_1 \in \Omega_1$.

(II) G_2 is a holomorphic function of the variables β_1, β_2 that does not depend on β_1; thus G_2 is a holomorphic function of $\beta_2 \in \Omega_2$.

This implies that

$$F(Z) = G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^\dagger$$

for $Z \in \Omega$. But the right–hand side is well–defined on $\widetilde{\Omega} = \Omega_1 e + \Omega_2 e^\dagger$, hence F extends to all of $\widetilde{\Omega}$.

Consider now the situation of functions of a hyperbolic variable. Set $\mathfrak{z} = x + ky$, $f(\mathfrak{z}) = u(\mathfrak{z}) + kv(\mathfrak{z}) = u(x, y) + kv(x, y)$. The limit

$$f'(\mathfrak{z}_0) := \lim_{\mathfrak{S}_{\mathbb{D}} \not\ni h \to 0} \frac{f(\mathfrak{z}_0 + h) - f(\mathfrak{z}_0)}{h}$$

is called "derivative of f at $\mathfrak{z}_0$".

The hyperbolic Cauchy–Riemann conditions here are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

The Cauchy–Riemann operators are:

$$\frac{\partial}{\partial \mathfrak{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \mathfrak{z}^\dagger} := \frac{1}{2}\left(\frac{\partial}{\partial x} - k\frac{\partial}{\partial y}\right)$$

and they factorize, up to a constant coefficient, the one–dimensional wave operator on the class $C^2(\Omega)$ with Ω a domain in $\mathbb{D}$.

Using the idempotent representation one gets, with some abuse of notation:

$$f(\mathfrak{z}) = f_1(\mathfrak{z})\mathbf{e} + f_2(\mathfrak{z})\mathbf{e}^\dagger = f_1(x, y)\mathbf{e} + f_2(x, y)\mathbf{e}^\dagger.$$

If we now write $\mathfrak{z} = \nu_1\mathbf{e} + \nu_2\mathbf{e}^\dagger$, with $\nu_1, \nu_2 \in \mathbb{R}$, we have a characterization of $\mathbb{D}$–holomorphicity as follows: a C^1–function of a hyperbolic variable $\mathfrak{z}$ is $\mathbb{D}$–holomorphic if and only if the following conditions hold:

I) f_1 is of class $C^1(\Omega, \mathbb{R})$ and it does not depend on ν_2; thus f_1 is in $C^1(\Omega_1, \mathbb{R})$.

II) f_2 is of class $C^1(\Omega, \mathbb{R})$ and it does not depend on ν_1; thus f_2 is in $C^1(\Omega_2, \mathbb{R})$.

Hence

$$f(\mathfrak{z}) = f_1(\nu_1)\mathbf{e} + f_2(\nu_2)\mathbf{e}^\dagger \tag{2.3}$$

for $\mathfrak{z} \in \Omega$. But the right–hand side of (2.3) is well defined on $\widetilde{\Omega} = \Omega_1\mathbf{e} + \Omega_2\mathbf{e}^\dagger$, hence f extends to all of $\widetilde{\Omega}$. It is important to notice that f_1 and f_2 are here only required to be of class C^1, and no analyticity is expected.

3 An identity theorem and some geometrical considerations

It is well known that holomorphic functions of a complex variable enjoy what is known as the identity theorem. In other words, if $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is holomorphic (i.e., it admits complex derivative, or, equivalently, its components satisfy the Cauchy-Riemann system), Ω is an open and connected set of the complex plane, and f vanishes on a subset of Ω that has an accumulation point in Ω, then it vanishes everywhere in Ω. There are many ways to see, or at least to intuit, why it should be so. To begin with, if a holomorphic function vanishes on a subset of Ω having an accumulation point z_0, then (by continuity) it will vanish at z_0. Since f is holomorphic it is represented (locally) by its Taylor series and from this it forthwith follows that all its derivatives are zero at z_0, implying the identical vanishing of the function on a disk $B(z_0, r) \subset \Omega$. Using the fact that Ω is connected, this vanishing extends onto the whole Ω. A different way to look at this is by noticing that if u and v are, respectively, the real and imaginary parts of f, i.e., $f(x + iy) = u(x, y) + iv(x, y)$, then the Cauchy-Riemann conditions imply that u and v are both harmonic. This makes the Cauchy-Riemann system elliptic, and the identity theorem follows.

One may therefore reasonably ask whether a similar identity theorem holds for $\mathbb{BC}$–holomorphic functions of a bicomplex variable and for $\mathbb{D}$–holomorphic functions of a hyperbolic variable.

We can right away dispose of the bicomplex case by using an argument that essentially replicates the one we have sketched for the complex case, but we prefer to provide our theorem with a proof which is based on the intrinsic properties of bicomplex holomorphic functins.

Theorem 3.1. *$\mathbb{BC}$–holomorphic functions of a bicomplex variable satisfy the identity theorem. Namely, if a function $f : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ is $\mathbb{BC}$–holomorphic on a domain Ω in $\mathbb{BC}$, and if f vanishes identically on a subset Ω^* of Ω with an accumulation point $Z_0 = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$ such that $\beta_1\mathbf{e}$ and $\beta_2\mathbf{e}^\dagger$ are accumulation points of the projections $\mathbf{e}\Omega^*$ and $\mathbf{e}^\dagger\Omega^*$ respectively, then f vanishes identically on Ω.*

Proof. It follows from Section 2 that we can assume that Ω is a product type domain $\Omega = \Omega_1\mathbf{e}+\Omega_2\mathbf{e}^\dagger$. Moreover, a bicomplex holomorphic function F can be written as $F(Z) = G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^\dagger$, with G_1 and G_2 holomorphic functions of the complex variables β_1 and β_2. Since the identity theorem holds for holomorphic functions on domains in the complex plane, the result follows immediately. □

The situation, however, is quite different in the hyperbolic case, since the proof we just gave breaks down because, as we have seen in Section 2, the functions that appear in (2.3) are not necessarily analytic. Again, there are several reasons why one might think that, but maybe the most cogent and straightforward is the fact that if the real and hyperbolic components u, v of a holomorphic function of a hyperbolic variable are of class C^2 then they are both solutions of the wave equation. The equation being hyperbolic it is clear that no general identity theorem can hold for its solutions. We will see, however, that the very special nature of the real algebra of $\mathbb{D}$–holomorphic functions allows at least a limited version of the identity theorem and one, in particular, that ensures the uniqueness of the extension of any continuously differentiable function on $\mathbb{R}$ to all of $\mathbb{D}$, including of course the exponential function.

We have in fact the following theorem:

Theorem 3.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a $\mathbb{D}$–holomorphic function of a hyperbolic variable $x + ky$. If f vanishes identically on the real axis $y = 0$, then f vanishes identically on all of $\mathbb{D}$.*

Proof. We recall that if f is a $\mathbb{D}$–holomorphic function on $\mathbb{D}$, then we can rewrite it as

$$f(x+ky) = f(\nu_1\mathbf{e}+\nu_2\mathbf{e}^\dagger) = f_1(\nu_1)\mathbf{e}+f_2(\nu_2)\mathbf{e}^\dagger$$

where f_1 and f_2 are certain C^1 functions of the variables ν_1 and ν_2 which means that for any x and y in $\mathbb{R}$ there holds:

$$f(x+ky) = f_1(x+y)\mathbf{e}+f_2(x-y)\mathbf{e}^\dagger.$$

If we now assume that f vanishes identically when $y=0$, we obtain

$$0 = f(x) = f_1(x)\mathbf{e}+f_2(x)\mathbf{e}^\dagger.$$

This immediately entails that $f_1(x) = f_2(x) = 0$ for all values of x, which entails, in turn, that

$$f(x+ky) = f_1(x+y)\mathbf{e}+f_2(x-y)\mathbf{e}^\dagger = 0,$$

since both f_1 and f_2 are identically zero as functions of a single variable. This proves the theorem. □

A straightforward consequence of this result is the following:

Corollary 3.3. *Let f and g be two functions from $\mathbb{D}$ to $\mathbb{D}$ that are hyperbolic entire, i.e., they are $\mathbb{D}$–holomorphic in the whole $\mathbb{D}$. If they coincide on the real axis, they will coincide everywhere in $\mathbb{D}$.*

The proof of Theorem 3.2 openly suggests, however, that the real axis is not the only line for which an identity theorem holds. In fact, the same exact proof shows that if a $\mathbb{D}$–holomorphic function f of a hyperbolic variable vanishes identically on a line $y = mx+b$ ($m \neq \pm 1$) or on a line $x = c$, then it vanishes identically on all of $\mathbb{D}$. More generally one has the following result, whose proof is immediate consequence of the proof of Theorem 3.2.

Theorem 3.4. *Let $f : \mathbb{D} \to \mathbb{D}$ be a $\mathbb{D}$–holomorphic function of a hyperbolic variable $x+ky$. If f vanishes identically on a curve $y = g(x)$ such that $x+g(x)$ and $x-g(x)$ assume all real values except, possibly, a discrete set of them, or on a curve $x = g(y)$ such that $y+g(y)$ and $g(y)-y$ assume all real values, then f vanishes identically on $\mathbb{D}$.*

One may wonder whether it is possible to extend this result to functions that vanish on a half-line, for example, of positive real numbers, or what happens more generally when the function f vanishes on a portion of a curve, or even on some generic subset of $\mathbb{D}$. To answer this question we offer a different proof of Theorem 3.2, through an interesting geometric argument that offers a different way to look into the question, and admits an easy generalization.

Theorem 3.5. *Let $f : \mathbb{D} \to \mathbb{D}$ be a $\mathbb{D}$–holomorphic function of a hyperbolic variable $x+ky$. Let f_1 and f_2 be the functions of the variables $\nu_1 = x+y$ and $\nu_2 = x-y$ such that $f(x+ky) = f_1(\nu_1)\mathbf{e}+f_2(\nu_2)\mathbf{e}^\dagger = f_1(x+y)\mathbf{e}+f_2(x-y)\mathbf{e}^\dagger$. If f vanishes at a point x_0+ky_0, then f_1 vanishes identically on the line $y-y_0 = -(x-x_0)$ and f_2 vanishes identically on the line $y-y_0 = x-x_0$. See Figure 1.*

Proof. It is easy to see that if $f(x_0+ky_0) = 0$ then $f_1(x_0+y_0) = f_2(x_0-y_0) = 0$. But then if (x,y) is on the line $y-y_0 = -(x-x_0)$ it is clear that $x+y = x_0+y_0$ and therefore f_1 vanishes identically on that line. Similarly for f_2. □

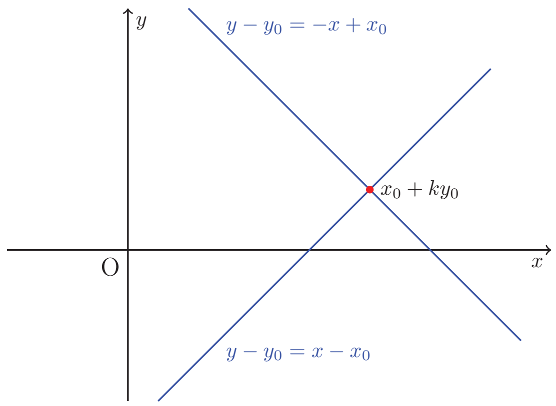

Figure 1: *The lines* $y - y_0 = -x + x_0$ *and* $y - y_0 = x - x_0$.

As a consequence of this theorem, if f satisfies the hypothesis of Theorem 3.2, then $f(x) = 0$ for all $x \in \mathbb{R}$. Take $x_0 \in \mathbb{R}$ arbitrary, then $f_1(x + y) = 0$ for all (x, y) on the line $y = -(x - x_0)$ and $f_2(x - y) = 0$ for all (x, y) on the line $y = x - x_0$. Having moved now x_0 along the whole $\mathbb{R}$ one has that $\mathbb{R}^2 \cong_{\mathbb{R}} \mathbb{D}$ is covered with the straight lines $y = -x + x_0$ and $y = x - x_0$, hence f vanishes on the whole $\mathbb{D}$.

Clearly, Theorems 3.2 and 3.4 are immediate consequences of Theorem 3.5 , since the totality of points in which two lines $y - y_0 = x_0 - x$ and $y - y_0 = x - x_0$, intersect, where x_0, y_0 vary on all the (real) coordinates of the points belonging to a curve as those mentioned in this last theorem, coincide with the whole $\mathbb{D}$. What is more relevant, however, is that both this theorem and its proof merely concern, essentially, a single point of $\mathbb{R}^2$, and the latter reduces, in fact, to nothing but a simple argument pertaining to analytic geometry on $\mathbb{R}^2$. A straightforward consequence of it, following from an obvious argument in real two-dimensional geometry, is, then, that if a $\mathbb{D}$–holomorphic function $f(x + ky)$ vanishes in two whatsoever distinct points $x_0 + ky_0$ and $x_1 + ky_1$, it also vanishes on both the intersection point of the lines $y - y_0 = x_0 - x$ and $y - y_1 = x - x_1$ and the intersection point of the lines $y - y_1 = x_1 - x$ and $y - y_0 = x - x_0$. More in general, from this theorem it follows that, if a $\mathbb{D}$–holomorphic function $f(x + ky)$ vanishes on any subset $\{x + ky\}_{(x,y)\in\Omega}$ of $\mathbb{D}$, where Ω is whatever subset of $\mathbb{R}^2$, then it also vanishes on the subset $\{x + ky\}_{(x,y)\in\Omega^*}$ of $\mathbb{D}$, where Ω^* is the subset of $\mathbb{R}^2$ depending on Ω, which we can call "hyperbolic holomorphicity hull of Ω", defined as follows:

Definition 3.6. Let Ω be a subset of the plane. For each point $P = (x, y)$ in Ω, consider the two lines α_{P+} and α_{P-} passing through that point and having slopes 1 and -1. We define the hyperbolic holomorphicity hull Ω^* of Ω to be the set of points $\{\alpha_{P+} \cap \alpha_{Q-}\}_{P,Q\in\Omega}$.

Figures 2 and 3 are illustrations of this Definition.

We got, then, the following corollary, which constitutes a lemma for a quite general identity theorem for $\mathbb{D}$–holomorphic functions:

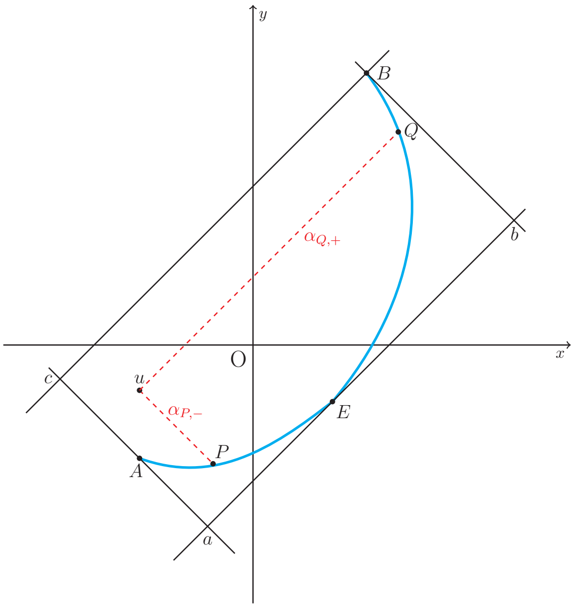

Figure 2: *The set* $\Omega^* = \{u : u \in abBc\}$ *is the hyperbolic holomorphicity hull of the arc of curve* $\Omega = AEB$*. For any point* u *in* Ω^**, there are two points* P, Q *on* Ω *such that the lines* $\alpha_{P,-}$ *and* $\alpha_{Q,+}$ *intersect in* u*.*

Corollary 3.7. *If a* $\mathbb{D}$*–holomorphic function* f *vanishes on a set* Ω *in the plane, it then vanishes identically on the hyperbolic holomorphicity hull of* Ω*.*

We therefore obtain, immediately, the following general result:

Corollary 3.8. *Let* f *and* g *be two* $\mathbb{D}$*–holomorphic functions from* $\mathbb{D}$ *to* $\mathbb{D}$*. If they coincide on a subset* $\{x + ky\}_{(x,y)\in\Omega}$ *of* $\mathbb{D}$*, where* Ω *is a subset of* $\mathbb{R}^2$*, they will also coincide on* $\{x + ky\}_{(x,y)\in\Omega^*}$*, where* Ω^* *is the hyperbolic holomorphicity hull of* Ω*.*

We conclude this section with a remark that is prompted by these apparently simple results. At first sight, when studying $\mathbb{D}$–holomorphic functions of a hyperbolic variable, one is led to an oversimplification when it appears that any such function f is indeed nothing but a pair of real valued functions f_1 and f_2, with no apparent links between them (a similar remark can and has been made for holomorphic functions in the bicomplex setting). Thus it appears that such functions cannot have any special properties, since the functions f_1 and f_2 do not have any particular property. However, since the two functions f_1 and f_2 are defined on two variables connected to each other, $\mathbb{D}$–holomorphicity acts as a separation of variables process. In other words a function is $\mathbb{D}$–holomorphic if and only if it can undergo a separation of variables process that allows it to be written as a pair of one variable functions f_1 and f_2. This process is performed via the change of cartesian basis to the idempotent basis, and is strongly related to the hyperbolic numbers structure in the usual Euclidean real space R^2. This is, indeed, the key property of solutions of the wave equation, which is the differential equation that gives special meaning to the study of $\mathbb{D}$–holomorphic functions.

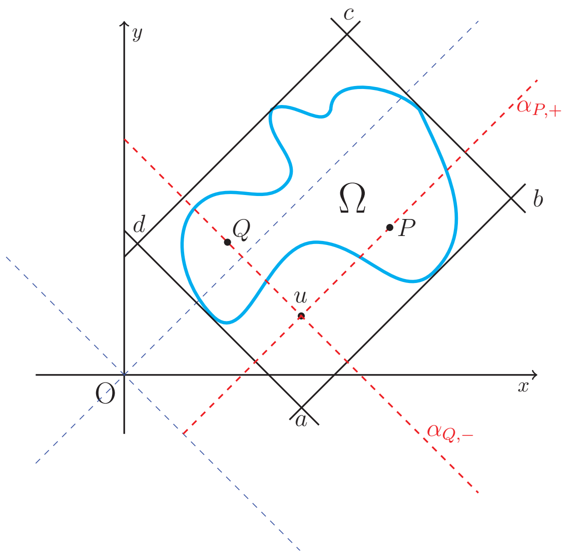

Figure 3: *The set $\Omega^* = \{u : u \in abcd\}$ is the hyperbolic holomorphicity hull of the connected set Ω. For any point u in Ω^*, there are two points P, Q in Ω such that the lines $\alpha_{P,+}$ and $\alpha_{Q,-}$ intersect in u.*

4 Some variants of the Identity Theorem in $\mathbb{BC}$

Although by Theorem 3.1 we know that the Identity Theorem holds for bicomplex holomorphic functions, the proof of Theorem 3.5 offers an inspiration for a different proof of the Identity Theorem for $\mathbb{BC}$–holomorphic functions, based on some simple geometric facts. For this reason we start this section recalling the notion of complex straight lines in $\mathbb{BC}$, [9].

4.1 Complex straight lines in $\mathbb{BC}$

Recall first that any bicomplex number $Z = z_1 + jz_2$ can be identified with a pair of complex numbers (z_1, z_2). This means that, whenever necessary, $\mathbb{BC}$ can be seen as $\mathbb{C}(i)^2$.

By definition, a complex straight line (or simply a complex line) is the set of solutions of the equation

$$a_1 z_1 + a_2 z_2 = b, \tag{4.1}$$

where a_1, a_2, $b \in \mathbb{C}(i)$ are complex coefficients. Since this equation is equivalent to a system of two real linear equations with four real variables, if the rank of the system is 2, the equation defines a 2–dimensional plane in $\mathbb{R}^4$.

Some examples of complex lines are the following.

Taking in (4.1) $a_1 = 0$, $a_2 = 1$ and $b = 0$, we get the equation

$$z_2 = 0,$$

and the respective complex line is the set of complex numbers $\mathbb{C}(i) \subset \mathbb{BC}$.

Taking now $a_1 = 0$, $a_2 = 0$, $b = 0$, then (4.1) becomes

$$z_1 = 0$$

and the complex line in this case is the set $j\mathbb{C}(i) \subset \mathbb{BC}$.

The complex line that passes through a given bicomplex number $Z^0 = z_1^0 + jz_2^0$ and through the origin is the set

$$\mathcal{L}_{Z^0} = \left\{ \lambda Z^0 \mid \lambda \in \mathbb{C}(i) \right\}.$$

Let us represent this complex line as the set of solutions of (4.1). Take $z_1 + jz_2 \in \mathcal{L}_{Z^0}$, then there exists $\lambda \in \mathbb{C}(i)$ such that $z_1 + jz_2 = \lambda Z^0 = \lambda z_1^0 + j\lambda z_2^0$ which leads to the system

$$\begin{cases} z_1 = \lambda z_1^0, \\ \\ z_2 = \lambda z_2^0. \end{cases}$$

If $z_2^0 = 0$, it is clear that $\mathcal{L}_{Z^0} = \mathbb{C}(i)$, thus, assuming that $z_2^0 \neq 0$ one has that $\lambda = \dfrac{z_2}{z_2^0}$ and hence $z_1 = \dfrac{z_2}{z_2^0} z_1^0$, or, equivalently:

$$z_2^0 z_1 - z_1^0 z_2 = 0.$$

Reciprocally, given a homogeneous equation $a_1 z_1 + a_2 z_2 = 0$ with $a_1 \neq 0$, the set of its solutions is $\mathcal{L}_{Z^0}$ with $Z^0 = -a_2 + ja_1$.

Using this notation it is clear that the (real) 2–dimensional planes $\mathbb{BC}_{\mathbf{e}}$ and $\mathbb{BC}_{\mathbf{e}^\dagger}$ are in fact complex lines: $\mathbb{BC}_{\mathbf{e}} = \mathcal{L}_{\mathbf{e}}$ and $\mathbb{BC}_{\mathbf{e}^\dagger} = \mathcal{L}_{\mathbf{e}^\dagger}$. Their equations are, respectively:

$$z_1 + iz_2 = 0, \tag{4.2}$$

$$z_1 - iz_2 = 0. \tag{4.3}$$

Note that the coefficients that appear in equation (4.2) corresponds to the zero-divisor $-2i\mathbf{e}$, but it is clear that $\mathcal{L}_{\mathbf{e}} = \mathcal{L}_{-2i\mathbf{e}}$. Similarly (4.3) corresponds to the zero-divisor $2i\mathbf{e}^\dagger$ and one has that $\mathcal{L}_{\mathbf{e}} = \mathcal{L}_{-2i\mathbf{e}}$.

A complex line that does not pass through the origin can be written as $\mathcal{L}_{Z^0} + W^0$. This means that this line passes through W^0 and it is parallel to $\mathcal{L}_{Z^0}$. Writing $Z^0 = z_1^0 + jz_2^0$ and $W^0 = w_1^0 + jw_2^0$, it is straightforward to prove that $Z = z_1 + jz_2$ belongs to the complex line $\mathcal{L}_{Z^0} + W^0$ if and only if the pair (z_1, z_2) is a solution of the equation

$$z_2^0 z_1 - z_1^0 z_2 = z_2^0 w_1^0 - z_1^0 w_2^0. \tag{4.4}$$

The $\mathbb{C}(j)$ complex lines can be defined in a similar way.

The reader may note that not any (real) two–dimensional plane in $\mathbb{R}^4$ is a $\mathbb{C}(i)$ or a $\mathbb{C}(j)$ complex line. It is in fact immediate to prove that a (real) two–dimensional plane P in $\mathbb{R}^4$ that passes through the origin is a $\mathbb{C}(i)$ complex line if and only if it is closed under the multiplication by i.

4.2 Identity Theorem for $\mathbb{BC}$–holomorphic functions

We are now ready to provide an alternative proof of a special case of the identity theorem for $\mathbb{BC}$–holomorphic functions.

Theorem 4.1. *If F is an entire $\mathbb{BC}$–holomorphic function such that $F(z) = 0$ for all $z \in \mathbb{C}(\mathbf{i})$, then $F(Z) = 0$ for all $Z \in \mathbb{BC}$.*

Proof. Given $Z \in \mathbb{BC}$, write $Z = z_1 + jz_2$ and $F = G_1\mathbf{e} + G_2\mathbf{e}^\dagger$. Since F is $\mathbb{BC}$–holomorphic it satisfies

$$F(Z) = G_1(z_1 - iz_2)\mathbf{e} + G_2(z_1 + iz_2)\mathbf{e}^\dagger.$$

Take $z_0 \in \mathbb{C}(i)$ arbitrary. Since $F(z_0) = 0$ then $G_1(z_0) = 0$ and $G_2(z_0) = 0$. Consider the complex line $\mathcal{L} = \mathcal{L}_\mathbf{e} + z_0$ parallel to $\mathbb{BC}_e$ passing through z_0. From (4.4) we know that its equation is $z_1 + iz_2 = z_0$. Thus, to every $Z = z_1 + jz_2$ that belongs to $\mathcal{L}$ one has

$$G_2(z_1 + iz_2) = G_2(z_0) = 0,$$

i.e., G_2 vanishes in the whole complex line $\mathcal{L}$. Similarly, consider the complex line $\mathcal{L}_{\mathbf{e}^\dagger} + z_0$ given by $z_1 - iz_2 = z_0$. Since $G_1(z_0) = 0$, then G_1 vanishes on the whole complex line $\mathcal{L}_{\mathbf{e}^\dagger} + z_0$. We conclude that the function F vanishes on the union of the lines

$$(\mathcal{L}_\mathbf{e} + z_0) \bigcup (\mathcal{L}_{\mathbf{e}^\dagger} + z_0)\,.$$

It is clear that the whole $\mathbb{BC}$ can be filled with the collection of complex lines:

$$\{\mathcal{L}_\mathbf{e} + z \mid z \in \mathbb{C}(\mathbf{i})\} \quad \text{and} \quad \{\mathcal{L}_{\mathbf{e}^\dagger} + z \mid z \in \mathbb{C}(i)\}\,,$$

hence, moving z_0 along the whole $\mathbb{C}(\mathbf{i})$ we conclude that F vanishes in the whole $\mathbb{BC}$. □

A generalization of the above theorem is

Theorem 4.2. *If F is a bicomplex entire function that vanishes on a complex line $\mathcal{L}_{Z^0} + W^0$, with Z^0 not a zero-divisor, then it vanishes identically on all $\mathbb{BC}$.*

The request on Z^0 to be not a zero-divisor finds its analogue in the hyperbolic case when it was required that the slope m of the line $y = mx + b$ satisfies $m \neq \pm 1$.

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