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Cohesive fracture in 1D: quasi-static evolution and derivation from static phase-field models

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Abstract

In this paper we propose a notion of irreversibility for the evolution of cracks in presence of cohesive forces, which allows for different responses in the loading and unloading processes, motivated by a variational approximation with damage models. We investigate its applicability to the construction of a quasi-static evolution in a simple one-dimensional model. The cohesive fracture model arises naturally via Γ -convergence from a phase-field model of the generalized Ambrosio-Tortorelli type, which may be used as regularization for numerical simulations.

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1 Introduction

The variational approach to fracture, as formulated by Francfort and Marigo [34] (see also [13]), is based on Griffith's idea [40] that the crack growth is determined by the competition between the energy spent to increase the crack and the corresponding release in the bulk elastic energy. By representing the crack as a $(n - 1)$ -dimensional surface Γ and assuming for simplicity that the material behaves as a linearly elastic body in the unbroken part $\Omega \setminus \Gamma$ of the reference configuration, for pristine materials one is led to consider an energy of the form

$$E(u, \Gamma) := \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} g_0(|[u](x)|) d\mathcal{H}^{n-1}, \quad (1.1)$$

where $[u](x) := u^+(x) - u^-(x)$ is the difference of the traces of the displacement $u : \Omega \rightarrow \mathbb{R}$ on the two sides of Γ . In the original Griffith's theory for *brittle fracture* the energy associated with the crack is proportional to the measure of the surface of the crack itself, and g_0 is assumed not to depend on the crack opening $[u]$. It was however early recognized (Barenblatt [10]) that in many situations fracture should be regarded as a gradual process, and that the presence of cohesive forces between the lips of the crack should be taken into account: in *cohesive fracture* models Griffith's energy is replaced by various surface energies depending on the actual opening of the crack; in this case the function g_0 in (1.1) is typically assumed to be an increasing and concave function, with $g_0(0) = 0$, with positive and finite slope $g_0'(0) \in (0, \infty)$ at the origin, and asymptotically converging to the value of the fracture toughness. Variants of (1.1) with g_0 not necessarily concave have been used to study the formation of micro-cracks [30, 31].

The notion of *irreversible quasi-static evolution* in the brittle case (g_0 constant), proposed in [34], under the action of time-dependent loads (for instance, in the form of a prescribed boundary displacement $b(t)$), requires three essential ingredients: one looks for a map $t \mapsto (u(t), \Gamma(t))$ satisfying

- (i) (*irreversibility*) $\Gamma(t_1) \subset \Gamma(t_2)$ for $t_1 < t_2$;
- (ii) (*static equilibrium*) $(u(t), \Gamma(t))$ is a global minimizer of the energy (1.1) with respect to the boundary condition $b(t)$;
- (iii) (*energy balance*) the increment in stored energy plus the energy spent in crack increase equals the work of external forces.

The common approach in showing the existence of a variational evolution with the properties above is based on a time-discretization algorithm, in which one selects at each time step a global minimizer of the total energy, and then recovers the time-continuous limit by sending to zero the discretization parameter. This program has been successfully accomplished in the brittle case in a series of contributions: the existence of a quasi-static evolution was first proven by Dal Maso and Toader [28] in a two-dimensional, antiplane shear setting and with a uniform bound on the number of connected components of the crack, then extended to the case of planar elasticity by Chambolle [19]. The general n -dimensional case, without restrictions on the number of connected components, was studied by Francfort and Larsen [35] (antiplane case) and by Dal Maso, Francfort and Toader [24] (finite elasticity). Further results in this direction were obtained in [42, 26, 43, 5].

In the cohesive case the picture is less clear and there is no agreement on a single notion of irreversibility. Indeed, while in the brittle case the irreversibility constraint is a purely geometric condition, in the cohesive context one has to take into account also the amplitude of the cracks, through the inclusion of some internal variable which keeps track of the complete history of the

fracture process. In particular, it would be desirable to model the possibility of having different responses of the material to loading and unloading processes. Up to now, several plausible choices have been proposed in the mathematical literature under the assumption that the crack path is prescribed (see also the discussion in [13, Chapter 5]): Dal Maso and Zanini [29] adopt as internal variable the maximal opening of the crack, postulating that the energy is dissipated only once for a given value of the jump, and additional dissipation can only occur if the distance between the crack lips is further increased; Bourdin, Francfort and Marigo [13] propose a different model, which adopts as the memory variable the cumulated opening of the crack and allows for the phenomenon of fatigue, as the surface energy evolves during each loading phase; in the model by Cagnetti and Toader [18] part of the dissipated energy is recovered when the crack opening is decreased; finally also Crismale, Lazzaroni and Orlando [21] include fatigue assuming that some energy is dissipated also during the unloading phase. In these cases the existence of a quasi-static evolution (encoding in the definition the relevant notion of irreversibility) is established. In the cohesive fracture context, further results on evolutions of local minimizers or critical points (rather than global minimizers) are obtained in [17, 4, 9, 45].

In the first part of this paper (Sections 2–4) we discuss the construction of a quasi-static evolution for a one-dimensional model of cohesive fracture, which allows for a different response of the material to loading and unloading. This is achieved by introducing a cohesive energy density $g(s, s')$ which depends both on the crack opening s and on the internal variable s' , which keeps track of the maximum previous opening (see Section 2 for the detailed assumptions on g). It is the goal of the second part of the paper (Sections 6–8) to show how those assumptions on the energy density g can be naturally derived by introducing an irreversibility condition on a phase-field approximation.

The construction of a quasi-static evolution follows a rather standard approach, which we now outline. We let $\Omega = (0, 1)$ represent an elastic bar, on which we prescribe a time-dependent boundary displacement $t \mapsto (b(t, 0), b(t, 1))$. We also include in the energy a lower-order penalization of the form $\gamma \|u - w(t)\|_{L^2(0,1)}^2$, where $\gamma > 0$ and $t \mapsto w(t)$ is a given function which prescribes the average behavior of the material. This foundation-type term, as usual in one-dimensional simplifications of models in solid mechanics, represents the boundary conditions in the directions that have been eliminated. A simple way to understand its physical significance is to imagine solving for $\tilde{u} : (0, 1) \times (-H, H) \rightarrow \mathbb{R}$ with boundary conditions $\tilde{u}(x, \pm H) = w(x)$ on the top and bottom boundaries, and then focusing on $u(x) := \tilde{u}(x, 0)$. The difference $u - w$ would be penalized by a $\partial_2 \tilde{u}$ term in the energy. If dealing with the Dirichlet functional, the $H^{1/2}$ norm of the difference would be the most natural way to model this effect; for general energy densities there is no clear natural form. For simplicity we use the L^2 norm. Due to the presence of this term the bar can be fractured at multiple points, while if $\gamma = 0$ one can prove that minimizers of the energy have at most one jump.

We keep track of the state of the fracture at each time $t \in [0, T]$ by means of a pair of internal variables $(\Gamma(t), s(t))$, where $\Gamma(t)$ is a discrete set of points (representing the sites of the crack points), and $s(t) : \Gamma(t) \rightarrow (0, \infty)$ is the maximal amplitude of the jump at the points of $\Gamma(t)$ for all the previous times. For technical reasons, we fix a (arbitrarily small) positive threshold $\bar{s} > 0$ and we keep track only of the jumps that overcome this value; in other words, fracture points where the jump is smaller than \bar{s} are not affected by the irreversibility condition, and the opening can be reduced with a complete recover of the dissipated energy. From the mathematical point of view this guarantees a uniform bound on the total number of crack points (this assumption can be easily removed in the case $\gamma = 0$, that is if we do not include the lower order penalization). It is often convenient to switch to the equivalent formulation in which s is first extended by 0 outside Γ , and Γ is implicitly replaced by the set $\{s > 0\}$.

One main novelty in the problem we consider here is that a new energy density $g(s, s')$ appears at the points where the bar was previously broken. The first argument s of g represents the current opening of the crack at a given point; the second argument s' represents the maximal amplitude of the jump reached at that point at the previous times, in particular $g(s, 0) = g_0(s)$ represents the fracture energy in a pristine material. More in general, when $s \geq s'$ the additional constraint is

irrelevant and one has $g(s, s') = g_0(s)$: this means that, in the process of crack opening, the energy per unit area of the fracture at a point x is given by $g_0(|[u(x)]|)$, the “original” cohesive energy density. The behaviour of $g(s, s')$ is instead different when $s < s'$: in this case $s \mapsto g(s, s')$ is monotone nondecreasing and, generally, $g(s, s') > g_0(s)$. Hence, when $|[u](x)|$ is smaller than the maximal opening reached at previous times, the energy density follows a curve which is above the graph of g_0 , see Figure 1. If a fracture with opening s is closed, a certain amount of energy $g(0, s)$ is permanently dissipated, so that a successive opening only has the smaller cost $g(s, s) - g(0, s)$. We refer to Section 6.1 and Section 7.1 below for the construction of specific examples for the functions g_0 and g , respectively.

We consider a time discretization $0 = t_0 < t_1 < \dots < t_N = T$ and we initially select the displacement u_0 by minimizing the energy

$$\int_0^1 |u'|^2 dx + \sum_{x \in J_u} g_0(|[u](x)|) + \gamma \int_0^1 |u - w(t_0)|^2 dx \quad (1.2)$$

among all $u \in \text{SBV}(0, 1)$ attaining the boundary conditions $b(t_0)$ at the endpoints of the bar. To be more precise, we need to consider the lower semicontinuous envelope of the energy and a minimizer will belong in general to $\text{BV}(0, 1)$: indeed the presence of the lower-order term allows for the possibility of having more than one jump, while some regularity can still be proved, see Proposition 3.2. In the case $\gamma = 0$ it is well-known that minimizers in dimension one are in SBV with a single jump, see [15]; see also [25, 16] for further regularity results for cohesive energies.

We then define

$$\Gamma_0 := \{x \in J_{u_0}^{b(t_0)} : |[u_0](x)| > \bar{s}\}, \quad s_0(x) := |[u_0](x)| \quad \text{for } x \in \Gamma_0$$

(where the superscript $b(t_0)$ in the jump set of u_0 indicates that we consider as part of the crack also the points at which the boundary conditions are not attained). Iteratively, assuming to have constructed (u_i, Γ_i, s_i) for $i = 0, \dots, k-1$, we select the displacement u_k at time t_k by minimizing (the relaxation of) the energy

$$\int_0^1 |u'|^2 dx + \sum_{x \in \Gamma_{k-1}} g(|[u](x)|, s_{k-1}(x)) + \sum_{x \in J_u \setminus \Gamma_{k-1}} g_0(|[u](x)|) + \gamma \int_0^1 |u - w(t_k)|^2 dx \quad (1.3)$$

with respect to the boundary condition $b(t_k)$, and we update the state of the fracture by setting

$$\Gamma_k := \Gamma_{k-1} \cup \{x \in J_{u_k}^{b(t_k)} : |[u_k](x)| > \bar{s}\}, \quad s_k(x) := |[u_k](x)| \vee s_{k-1}(x) \quad \text{for } x \in \Gamma_k.$$

Passing to the limit as the time-step goes to zero we obtain in Theorem 4.1 the existence of a quasi-static evolution $(u(t), \Gamma(t), s(t))$ satisfying the by now classical unilateral minimality and energy conservation conditions of quasi-static energy minimizing evolutions, see [44]. The irreversibility condition is reflected in the fact that the sets $\Gamma(t)$ and the maps $s(t)$ are monotonically increasing in time.

The second part of the paper is devoted to explain how the notion of irreversibility, used in the construction of the quasi-static evolution in the first part, is related to a corresponding irreversibility constraint on approximating damage models. In particular, one of the goals of the second part is to derive the properties of the cohesive energy density g , stated in Section 2, from a phase-field approximation of the cohesive functional (1.3) in the spirit of [20].

In order to explain the main idea, it is first convenient to refer once again to the brittle case, and to recall the classical Ambrosio-Tortorelli approximation [7, 8] of the Mumford-Shah functional

$$E(u) := \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u), \quad u \in \text{SBV}(\Omega),$$

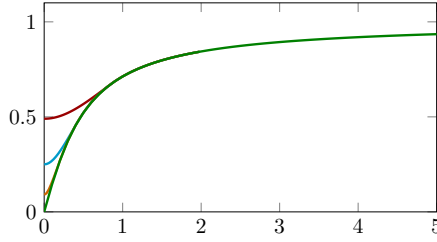


Figure 1: Graph of $g_0(s)$ (bottom curve) and $g(s, s')$ for $m_{s'} = 0.3, 0.5$ and 0.7 (from bottom to top) using the function $f_1^b(s)$ defined in (6.5) with $\ell = 1.5$, $\ell_1 = 0.2$. See Section 6.1 and in particular Figure 5 below for further details.

by the regularizing energies

$$E_\varepsilon(u_\varepsilon, v_\varepsilon) := \int_{\Omega} (v_\varepsilon^2 + o(\varepsilon)) |\nabla u_\varepsilon|^2 dx + \frac{1}{2} \int_{\Omega} \left(\varepsilon |\nabla v_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{\varepsilon} \right) dx, \quad (u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times H^1(\Omega),$$

with $0 \leq v_\varepsilon \leq 1$. Here the function u_ε has to be understood as a regularization of the displacement u , possibly creating jumps in the limit as $\varepsilon \rightarrow 0$, while v_ε can be interpreted as a damage variable which concentrates in the regions where the singularities of u are created: more precisely, $v_\varepsilon \rightarrow 1$ almost everywhere in Ω , and $v_\varepsilon \rightarrow 0$ on the $(n - 1)$ -dimensional surface where the limit jump will appear. The notion of quasi-static evolution in the regular context of the Ambrosio-Tortorelli functionals can be formulated including an irreversibility condition in the form of a global monotonicity constraint on v_ε (the damage can only increase in time). It is proved in [38] that this evolution converges as $\varepsilon \rightarrow 0$ to a quasi-static evolution for brittle fracture.

Returning to the cohesive setting, in the spirit of Ambrosio and Tortorelli it is shown in [20] (see also [36] for a numerical insight) that, for a suitable choice of the function f_ε , functionals of the form

$$F_\varepsilon(u_\varepsilon, v_\varepsilon) := \int_{\Omega} \left(f_\varepsilon^2(v_\varepsilon) |\nabla u_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2 \right) dx, \quad (u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times H^1(\Omega), \quad (1.4)$$

with $0 \leq v_\varepsilon \leq 1$, Γ -converge as $\varepsilon \rightarrow 0$ to the relaxation of an energy of the form (1.1) (see Section 6.3 for the precise statement). The function g_0 obtained as limit surface energy density can be characterized as the solution to a minimum problem,

$$g_0(s) = \inf \left\{ \int_{-\infty}^{\infty} \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt : (\alpha, \beta) \in H_{\text{loc}}^1(\mathbb{R}) \times H_{\text{loc}}^1(\mathbb{R}), \alpha' \in L^1(\mathbb{R}), \right. \\ \left. \left| \int_{-\infty}^{\infty} \alpha'(t) dt \right| = 1, 0 \leq \beta \leq 1, \lim_{|t| \rightarrow \infty} \beta(t) = 1 \right\}, \quad (1.5)$$

and satisfies the natural requirements for a cohesive energy (see Figure 1). As before, u_ε is a regularization of u developing singularities in the limit, and v_ε converges to the value 1 almost everywhere and the measure $|\nabla v_\varepsilon| \mathcal{L}^n$ concentrates on the limit jump. The crucial observation is that, in contrast with the Ambrosio-Tortorelli case, v_ε does not tend to zero everywhere on the jump, but reaches a value depending on the amplitude of the limit jump of u . Indeed, the one-dimensional blow-up analysis of the behaviour of recovery sequences $(u_\varepsilon, v_\varepsilon) \rightarrow (u, 1)$ around a jump point $\bar{x} \in J_u$, which we perform in Section 6.3, shows that the rescaled functions $z_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon x)$, $w_\varepsilon(x) := v_\varepsilon(x_\varepsilon + \varepsilon x)$ converge to an optimal profile (α_s, β_s) for the minimum problem (1.5) defining $g_0(s)$, for the value $s = |[u](\bar{x})|$ (Theorem 6.5); moreover the minimum value reached by v_ε in a small neighbourhood of \bar{x} , $m_s := \min_{\mathbb{R}} \beta_s$, is in one-to-one correspondence with the limit amplitude s of the jump.

Therefore the relevant information about the amplitude of the limit jump $[u]$ is carried only by the value of the minimum of the damage variable v_ε . This suggests to impose, at level ε , an

irreversibility condition in the form of a monotonicity constraint on the value of local minima of v_ε : in the points where v_ε has a local minimum, its value can only decrease. By looking at the Γ -limit of the functionals F_ε in (1.4) with this additional constraint (see Section 7.3), one finds that the limit energy density has the form

$$g(s, s') = \inf \left\{ \int_{-\infty}^{\infty} \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : (\alpha, \beta) \in H_{\text{loc}}^1(\mathbb{R}) \times H_{\text{loc}}^1(\mathbb{R}), \alpha' \in L^1(\mathbb{R}), \right. \\ \left. \left| \int_{-\infty}^{\infty} \alpha'(t) dt \right| = 1, 0 \leq \beta \leq 1, \lim_{|t| \rightarrow \infty} \beta(t) = 1, \inf_{\mathbb{R}} \beta \leq m_{s'} \right\}, \quad (1.6)$$

see Figure 1. In other words, we restrict the class of admissible profiles by imposing that the minimum value of the profile β does not exceed the minimum value $m_{s'}$ of the optimal profile $\beta_{s'}$ corresponding to the amplitude s' .

We then show that the function g derived in this way has exactly the properties stated in Section 2. Notice that when $s \geq s'$ the additional constraint in (1.6) is irrelevant and one has $g_0(s) = g(s, s')$. At the same time, $g(0, s')$ is (at least for s' not too large, see Section 8 for details) strictly smaller than $g_0(s') = g(s', s')$. Indeed, upon closure of the fracture, the part of the energy that originates from the elastic degrees of freedom (corresponding to the $f^2(\beta)|\alpha'|^2$ term in (1.6)) is recovered, whereas the part that originates from the internal variable (corresponding to the two terms $\frac{(1-\beta)^2}{4}$ and $|\beta'|^2$ in (1.6), with $\beta(\pm\infty) = 1$ and $\beta(0) \leq m_{s'}$) is permanently dissipated. This behaviour is consistent with models where some of the dissipated energy is recovered when the crack opening is decreased [18, 4, 45, 46] and our analysis shows that it can be recovered as the limit of damage models.

It is worth to briefly comment on the connection of this approach with other results in the mathematical literature. As remarked above, in our model the response of the material when the crack opening decreases is similar to that considered in [18], that is some dissipated energy is recovered. However, the memory of the fracture is permanent, and further iterations of opening and closing the same fracture are reversible. In particular, the phenomenon of fatigue is completely absent. Fatigue effects are instead taken into account in [13] and in [21] (see also [1]): in particular, in [21] also when the crack opening is reduced some energy is dissipated, and oscillations of jumps can produce a complete fracture even if the maximal crack amplitude remains small. This behaviour is consistent with models where cohesive effects result from the interaction between damage and plasticity (see [2, 3, 27]); on the contrary, our approach is based on the approximation [20], where the cohesive behaviour is seen as the effect of the interplay between elasticity and damage. We expect that our approach could be extended to models of fatigue, by replacing the maximal amplitude of the opening by the cumulated opening as an internal variable. This extension is, however, beyond the scope of this paper.

We further remark that the functionals (1.4), when a monotonicity constraint on the minimum values of v_ε is included, provide a variational approximation of the limit energy (1.3) in a static setting (see Section 7.3 for the rigorous Γ -convergence result). It remains an open question whether a result in the spirit of Giacomini [38] holds also in this case, that is if one can construct a variational evolution for the functionals F_ε and show its convergence, as $\varepsilon \rightarrow 0$, to a quasi-static evolution for the limit cohesive model.

We believe that this approach is also suitable for numerical implementation. An appropriate algorithm should identify the regions where the damage v_ε is concentrated, and impose the monotonicity constraint only at the local minimum points. This might be numerically less expensive than imposing the monotonicity constraint at any point. However, it is unclear how to handle the interplay between the asymptotic process of Γ -convergence and evolution. This problem is closely related to the difficulties that arise in studying the interplay of relaxation and evolution, and which may lead to the definition of complex concepts of solutions [37, 33].

The generalization to the higher dimensional case poses several challenges. A first step in this direction has been obtained in [41], where the Γ -convergence result in Section 7.3 has been extended to general dimensions, under minimal regularity assumptions on the set Γ on which the

irreversibility constraint is imposed. The next step would consist in the construction of a quasi-static evolution under the assumption of a prescribed crack path. We stress, however, that in the present 1D analysis the position of the crack is not prescribed a priori.

Structure of the paper. The paper is mainly divided into two parts. The first part (Sections 2–5) is devoted to the proof of the existence of a quasi-static evolution in Theorem 4.1. In particular, in Section 2 we state the main assumptions on the cohesive energy density g and we set up the evolution problem; then the construction of a quasi-static evolution for the one-dimensional model is discussed in Section 3 (time-discretization) and Section 4 (continuous limit); finally in Section 5 we compute the relaxation of the cohesive energy.

In the second part of the paper (Sections 6–8) we explain the connection of the model with the phase-field approximation. In Section 6 we report the approximation result from [20] and we discuss the main properties of the cohesive energy density $g_0(s) = g(s, 0)$ of the pristine material; the Γ -convergence result in [20] is also improved with the analysis of the behaviour of recovery sequences in Section 6.3. In Section 7 we introduce the new surface energy density g , and we show in Section 7.3 how it can be recovered, via Γ -convergence, from damage models which include an irreversibility constraint. The proofs of the properties of g_0 and g in Sections 6 and 7 are rather technical and are postponed to Section 8.

Notation. Along the paper we work in dimension one and for simplicity in the interval $\Omega := (0, 1)$. For a function of bounded variation $u \in \text{BV}(0, 1)$ we denote by Du its distributional derivative, which is a bounded Radon measure on $(0, 1)$, and by $|Du|$ its total variation. The standard decomposition of Du is given by

$$Du = u' \mathcal{L}^1 + D^c u + \sum_{x \in J_u} [u](x) \delta_x,$$

where $u' \in L^1(0, 1)$ denotes the density of the absolutely continuous part of Du with respect to the Lebesgue measure \mathcal{L}^1 , $D^c u$ is the Cantor part of Du , J_u is the jump set of u (sometimes also denoted by $J(u)$ in the following), $[u](x) := u^+(x) - u^-(x)$, and $u^+(x)$ and $u^-(x)$ are the approximate limits from the right and from the left of u at x respectively. We denote by $\text{SBV}(0, 1)$ the space of functions $u \in \text{BV}(0, 1)$ such that $D^c u \equiv 0$. For the properties of functions of bounded variation we refer to the monograph [6]. We use standard notation for Γ -convergence, see [14, 22].

2 Cohesive quasi-static evolution: general setting and assumptions of the model

We start by introducing the main assumptions on the cohesive energy densities $g(s, s')$ considered in this paper. We refer to the Introduction for the interpretation of the meaning of the two variables of g , and to Figure 1 for a visualization of the qualitative behaviour of g .

We let $g: [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ satisfy the following assumptions:

- (i) g is continuous and monotone nondecreasing in both variables, $g(s, s') = g(s, 0)$ if $s \geq s'$;
- (ii) $g(s, s') \leq 1$ and $\lim_{s \rightarrow +\infty} g(s, s') = 1$ for any s' ;
- (iii) there exist $\ell, \tilde{\ell} > 0$ and $1 < p < 2$ such that

$$g(s, 0) = \ell s - \tilde{\ell} s^p + o(s^p) \quad \text{as } s \rightarrow 0^+; \tag{2.1}$$

- (iv) for every $s_1, s_2, s' \geq 0$,

$$g(s_1 + s_2, s') \leq g(s_1, 0) + g(s_2, s'); \tag{2.2}$$

if in addition $s_1 > 0$ and $s_2 \vee s' > 0$, the inequality is strict.

Notice that for fixed s' , the map $g(\cdot, s')$ is Lipschitz continuous with Lipschitz constant ℓ due to its monotonicity, its subadditivity, and the bound (2.1); in particular $g(s, 0) \leq \ell s$. Moreover, the global continuity and the strict subadditivity (iv) of g imply that for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$, depending on ε , such that

$$g(s_1 + s_2, 0) - g(s_1, 0) - g(s_2, 0) \leq -c_\varepsilon, \quad \text{for every } s_1, s_2 \geq \varepsilon, \quad (2.3)$$

$$g(s_1 + s_2, s') - g(s_1, 0) - g(s_2, s') \leq -c_\varepsilon, \quad \text{for every } s_2 \geq 0 \text{ and } s_1, s' \geq \varepsilon. \quad (2.4)$$

Remark 2.1. *We briefly discuss the role of the previous assumptions. We first observe that the function $g_0(s) = g(s, 0)$ (that is, the cohesive energy of a “new” jump) has the expected behaviour of a cohesive energy in the sense of Barenblatt. The assumption on the behaviour of g at infinity in (ii) is just a normalization condition and can be assumed without loss of generality. The assumption (2.1) is instead fundamental and guarantees that minimizers of the associated energy (see (2.8) below) in one dimension have no Cantor part (see Proposition 3.2); we expect that the range of exponents $p \in (1, 2)$ is not optimal. Finally, the subadditivity condition (iv) is a natural assumption and is related to the lower semicontinuity of the associated functional (2.8). The condition of strict subadditivity, which means that one larger jump is energetically more efficient than two small jumps which are spatially very close to each other, is important for obtaining regularity of minimizers, see for example Proposition 3.1(iii) and Theorem 4.1(i). We further observe that monotonicity and strict subadditivity imply that any fracture which is opened and then closed again dissipates a positive amount of energy, $g(0, s) > 0$ for all $s > 0$.*

We next prepare the setting for the construction of a quasi-static evolution. Let $T > 0$ be the final time. The state of the fracture at each time $t \in [0, T]$ is modeled by a discrete set of points (representing the sites where the crack is localized), to each of which is attached a scalar quantity (representing the maximal amplitude of the jump at all the previous times):

$$\Gamma \subset [0, 1] \text{ finite set}, \quad s : \Gamma \rightarrow (\bar{s}, \infty). \quad (2.5)$$

Here $\bar{s} > 0$ is a fixed positive threshold (which can be taken arbitrarily small but strictly positive, for technical reasons). The effect of the irreversibility condition affects only the fracture points where the jump amplitude exceeds the threshold \bar{s} , since only these points will be included in the crack set Γ ; in other words, for small fractures the process is completely reversible. The internal variable s in (2.5) will be always implicitly extended as $s(x) = 0$ for $x \notin \Gamma$.

The bar is subject to a time-dependent boundary displacement

$$b \in H^1([0, T]; \mathbb{R}^2), \quad \text{with } b(t) = (b(t, 0), b(t, 1)). \quad (2.6)$$

It is also convenient to denote, for t fixed, by $b(t) : [0, 1] \rightarrow \mathbb{R}$ the affine function interpolating between the boundary values $b(t, 0)$ and $b(t, 1)$. We will usually write $b(t, x) := b(t)(x)$ and we will denote by \dot{b} the time-derivative of b , by $b'(t)$ the spatial derivative of the function $b(t)$.

We would then iteratively minimize a functional of the form

$$\int_0^1 |u'|^2 dx + \sum_{x \in \Gamma \cup J_u} g(|[u](x)|, s(x)) \quad (2.7)$$

among all possible displacements u attaining the boundary conditions $u(0) = b(t, 0)$, $u(1) = b(t, 1)$. The surface energy in the new fracture points $J_u \setminus \Gamma$, where the bar was previously pristine, is $g(|[u]|, 0)$; at the points of Γ , where the fracture was present at previous times, an irreversibility condition appears: the surface energy density is given by $g(|[u]|, s)$, with $s > \bar{s}$, which takes into account the previous work made on Γ .

Since the function $g(\cdot, 0)$ has finite slope at the origin, the minimization of the functional (2.7) is in principle not well-posed in SBV, and we have to consider its relaxation with respect to the weak*-topology of BV, which is given by

$$\Phi(u; \Gamma, s, b) := \int_0^1 h(|u'|) dx + \sum_{x \in J_u^b \cup \Gamma} g(|[u](x)|, s(x)) + \ell |D^c u|(0, 1), \quad (2.8)$$

where we highlighted the dependence of Φ on the state of the fracture (described by the pair (Γ, s)) and on the boundary datum b . The rigorous proof of the relaxation result will be given in Section 5. Here the elastic energy density h is

$$h(\xi) := \begin{cases} \xi^2 & \text{if } |\xi| \leq \frac{\ell}{2}, \\ \ell|\xi| - \frac{\ell^2}{4} & \text{if } |\xi| > \frac{\ell}{2}, \end{cases} \quad (2.9)$$

and to take into account the boundary conditions, we use the following notation:

$$J_u^b := J_u \cup \{x \in \{0, 1\} : u(x) \neq b(x)\}, \quad (2.10)$$

$$[u](0) := u^+(0) - b(0), \quad [u](1) := b(1) - u^-(1). \quad (2.11)$$

Finally, we also add to the total energy a lower-order term of the form $\gamma \|u - w(t)\|_{L^2(0,1)}^2$, where $\gamma \geq 0$ is a fixed constant and

$$w \in AC([0, T]; L^\infty(0, 1)) \quad (2.12)$$

is a given map, absolutely continuous in time.

3 Cohesive quasi-static evolution: the time-discrete evolution

We construct in this section a time-discrete evolution for the one-dimensional cohesive fracture model, relative to a given time-dependent boundary displacement $b(t)$. We assume in the following that the boundary displacement b satisfying the assumption (2.6) is given, as well as the lower-order term w satisfying (2.12) and that a positive threshold $\bar{s} > 0$ for the minimal opening which leads to irreversibility is fixed.

We fix a discretization step $\tau > 0$ and we consider a subdivision of $[0, T]$ of the form

$$0 = t_0 < t_1 < \dots < t_{N_\tau} < t_{N_\tau+1} = T,$$

where N_τ is the largest integer such that $\tau N_\tau < T$, and $t_k := k\tau$ for $k \in \{0, \dots, N_\tau\}$. We also set $b_k^\tau = b(t_k)$, $w_k^\tau = w(t_k)$.

The time-discrete evolution in this setting is defined as follows. For $k = 0$, select a solution $u_0^\tau \in \text{BV}(0, 1)$ of the minimum problem

$$\min \left\{ \Phi(u; b_0^\tau) + \gamma \int_0^1 |u - w_0^\tau|^2 dx : u \in \text{BV}(0, 1) \right\} \quad (3.1)$$

(with the convention that we do not indicate the dependence on (Γ, s) in the functional Φ defined in (2.8) when $\Gamma = \emptyset$). We set

$$\Gamma_0^\tau := \{x \in J^{b_0^\tau}(u_0^\tau) : |[u_0^\tau](x)| > \bar{s}\}, \quad s_0^\tau(x) := |[u_0^\tau](x)|, \quad (3.2)$$

where $\bar{s} > 0$ is the threshold fixed at the beginning of this section. Assume now to have constructed $u_i^\tau \in \text{BV}(0, 1)$ and pairs $(\Gamma_i^\tau, s_i^\tau)$ in the form (2.5), for $i = 0, \dots, k-1$. We let $u_k^\tau \in \text{BV}(0, 1)$ be a minimum point of

$$\min \left\{ \Phi(u; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u - w_k^\tau|^2 dx : u \in \text{BV}(0, 1) \right\}, \quad (3.3)$$

and we set

$$\begin{aligned} \Gamma_k^\tau &:= \Gamma_{k-1}^\tau \cup \{x \in J^{b_k^\tau}(u_k^\tau) : |[u_k^\tau](x)| > \bar{s}\}, \\ s_k^\tau(x) &:= \begin{cases} |[u_k^\tau](x)| \vee s_{k-1}^\tau(x) & \text{if } x \in \Gamma_{k-1}^\tau, \\ |[u_k^\tau](x)| & \text{if } x \in \Gamma_k^\tau \setminus \Gamma_{k-1}^\tau. \end{cases} \end{aligned} \quad (3.4)$$

Notice in particular that all the sets Γ_k^τ are finite, since at each step the minimizer u_k^τ has at most a finite number of jump points where the amplitude of the jump is larger than the fixed threshold \bar{s} . The well-posedness of this construction is proved in the following proposition. We show below that $u_k^\tau \in \text{SBV}(0, 1)$.

Proposition 3.1. *For all $k = 0, 1, \dots, N_\tau$ there exists $u_k^\tau \in \text{BV}(0, 1)$ such that, by defining $(\Gamma_k^\tau, s_k^\tau)$ as in (3.4) (with $\Gamma_{-1}^\tau = \emptyset$), the following hold:*

- (i) u_k^τ is a minimizer of problem (3.3);
- (ii) $\|u_k^\tau\|_\infty \leq \max\{\|b_k^\tau\|_\infty, \|w_k^\tau\|_\infty\}$;
- (iii) $\inf_\tau \inf_k \min\{|x - y| : x, y \in \Gamma_k^\tau, x \neq y\} > 0$.

Proof. As this is a static result, we fix τ and k and drop them from the notation. We need to show that, given a pair (Γ, s) as in (2.5), an affine function $b : [0, 1] \rightarrow \mathbb{R}$, and $w \in L^\infty(0, 1)$, the minimum problem

$$\min \left\{ \Phi(u; \Gamma, s, b) + \gamma \int_0^1 |u - w|^2 dx : u \in \text{BV}(0, 1) \right\} \quad (3.5)$$

has a solution. Let $(u_n)_n$ be a minimizing sequence for problem (3.5). By a truncation argument we can take $\|u_n\|_\infty \leq \max\{\|b\|_\infty, \|w\|_\infty\}$. By comparing u_n with the function b (and possibly replacing u_n with b) we obtain the uniform bound

$$\Phi(u_n; \Gamma, s, b) + \gamma \int_0^1 |u_n - w|^2 dx \leq \int_0^1 h(|b'|) dx + \sum_{x \in \Gamma} g(0, s(x)) + \gamma \int_0^1 |b - w|^2 dx \leq C.$$

Let now $s_* > 0$ be such that $g(s_*, 0) = \frac{\ell}{2}$, and let $\ell_* \in (0, \ell)$ be such that $g(s, 0) \geq \ell_* s$ for all $s \in [0, s_*]$. Then the lower estimates $h(t) \geq \ell t - \frac{\ell^2}{4}$, $g(t, s) \geq g(t, 0)$ yield

$$\begin{aligned} C &\geq \Phi(u_n; \Gamma, s, b) \geq \ell \int_0^1 |u_n'| dx - \frac{\ell^2}{4} + \ell |D^c u_n|(0, 1) + \sum_{x \in J_{u_n}^b} g(|[u_n](x)|, 0) \\ &\geq \ell \int_0^1 |u_n'| dx - \frac{\ell^2}{4} + \ell |D^c u_n|(0, 1) + \sum_{|[u_n](x)| \leq s_*} \ell_* |[u_n](x)| + \frac{1}{2} \mathcal{H}^0(\{|[u_n](x)| > s_*\}) \quad (3.6) \\ &\geq \ell \int_0^1 |u_n'| dx - \frac{\ell^2}{4} + \ell |D^c u_n|(0, 1) + \sum_{|[u_n](x)| \leq s_*} \ell_* |[u_n](x)| + \frac{1}{4\|u_n\|_\infty} \sum_{|[u_n](x)| > s_*} |[u_n](x)|. \end{aligned}$$

Therefore the sequence $(u_n)_n$ is bounded in $\text{BV}(0, 1)$ and, up to subsequences, it converges weakly* in BV to some function $u \in \text{BV}(0, 1)$, which is a minimizer of (3.5) by the lower semicontinuity of the functional Φ , proved in Theorem 5.1 below. In particular, it also follows that $\|u\|_\infty \leq \max\{\|b\|_\infty, \|w\|_\infty\}$.

It remains to prove (iii). We first show that there is a minimal distance between any two new fracture points in $\Gamma_k^\tau \setminus \Gamma_{k-1}^\tau$, independent of k and τ . Consider $x_1, x_2 \in J^{b_k^\tau}(u_k^\tau) \setminus \Gamma_{k-1}^\tau$, with $s_i := [u_k^\tau](x_i)$ and $|s_i| > \bar{s}$, $i = 1, 2$. Suppose also $x_1 < x_2$. Let $v := u_k^\tau + s_2 \chi_{(x_1, x_2)}$: by minimality of u_k^τ in problem (3.3) we have

$$0 \leq g(|s_1 + s_2|, 0) - g(|s_1|, 0) - g(|s_2|, 0) + 2\gamma s_2 \int_{x_1}^{x_2} (u_k^\tau - w_k^\tau) dx + \gamma s_2^2 |x_1 - x_2| \leq -c_{\bar{s}} + C|x_1 - x_2|,$$

where the last inequality follows from (2.3) and the uniform L^∞ -bound on u_k^τ and w . This proves that the distance between x_1 and x_2 is larger than a uniform positive constant.

Similarly, we prove that any new fracture point of u_k^τ cannot be too close to the points of Γ_{k-1}^τ . Consider $x_1 \in J^{b_k^\tau}(u_k^\tau) \setminus \Gamma_{k-1}^\tau$ with $s_1 := [u_k^\tau](x_1)$, $|s_1| > \bar{s}$. Let $x_2 \in \Gamma_{k-1}^\tau$ and let $s_2 := [u_k^\tau](x_2)$.

Assuming without loss of generality that $x_1 < x_2$ (the construction in the other case is symmetric), by comparing the energy of u_k^τ and $v := u_k^\tau - s_1 \chi_{(x_1, x_2)}$ we have

$$\begin{aligned} 0 &\leq g(|s_1 + s_2|, s_{k-1}^\tau(x_2)) - g(|s_1|, 0) - g(|s_2|, s_{k-1}^\tau(x_2)) - 2\gamma s_1 \int_{x_1}^{x_2} (u_k^\tau - w_k^\tau) dx + \gamma s_1^2 |x_1 - x_2| \\ &\leq -c_{\bar{s}} + C|x_1 - x_2|, \end{aligned}$$

where the last inequality follows from (2.4) and the uniform L^∞ -bound on u_k^τ and w . \square

Proposition 3.2. *Any minimizer u of (3.3) obeys $D^c u = 0$ and $|u'| \leq \frac{\ell}{2}$ almost everywhere.*

Proof. We define the measure $\mu := (u' - \frac{\ell}{2})_+ + (D^c u)_+$. We shall show that $\mu = 0$, the argument with $(u' + \frac{\ell}{2})_- + (D^c u)_-$ is identical.

Assume that there are $a \neq b \in (0, 1)$ such that $\mu((a - \varepsilon, a + \varepsilon)) > 0$ and $\mu((b - \varepsilon, b + \varepsilon)) > 0$ for all $\varepsilon > 0$ sufficiently small. If such a pair does not exist, then the measure μ is concentrated on $\{0, 1\}$ and at most one additional point, and since μ does not contain any Dirac measure we obtain $\mu = 0$ and we are done. Let $I_\varepsilon := (a - \varepsilon, a + \varepsilon)$, $J_{\varepsilon'} := (b - \varepsilon', b + \varepsilon')$, where $\varepsilon, \varepsilon'$ are positive numbers such that these two intervals are disjoint subsets of $(0, 1)$. We set

$$\psi_{\varepsilon, \varepsilon'}(x) := \nu_{\varepsilon, \varepsilon'}((0, x)), \quad \text{with} \quad \nu_{\varepsilon, \varepsilon'} := \frac{1}{\mu(I_\varepsilon)} \mu \llcorner I_\varepsilon - \frac{1}{\mu(J_{\varepsilon'})} \mu \llcorner J_{\varepsilon'}.$$

We compare u with $u + \rho \psi_{\varepsilon, \varepsilon'}$ for some ρ chosen below. This gives, since $\psi_{\varepsilon, \varepsilon'} \in \text{BV}(0, 1)$ with $\psi_{\varepsilon, \varepsilon'}(0) = \psi_{\varepsilon, \varepsilon'}(1) = 0$ and $J_{\psi_{\varepsilon, \varepsilon'}} = \emptyset$,

$$\begin{aligned} 0 &\leq \int_0^1 [h(u' + \rho \psi'_{\varepsilon, \varepsilon'}) - h(u')] dx + \ell |D^c(u + \rho \psi_{\varepsilon, \varepsilon'})|(0, 1) - \ell |D^c u|(0, 1) \\ &\quad + \gamma \int_0^1 [(u - w_k^\tau + \rho \psi_{\varepsilon, \varepsilon'})^2 - (u - w_k^\tau)^2] dx. \end{aligned}$$

We observe that

$$u' + \rho \psi'_{\varepsilon, \varepsilon'} - \frac{\ell}{2} = u' - \frac{\ell}{2} + \frac{\rho}{\mu(I_\varepsilon)} \chi_{I_\varepsilon} (u' - \frac{\ell}{2})_+ - \frac{\rho}{\mu(J_{\varepsilon'})} \chi_{J_{\varepsilon'}} (u' - \frac{\ell}{2})_+$$

where χ_{I_ε} and $\chi_{J_{\varepsilon'}}$ are the characteristic functions of the two intervals. Therefore for any $\rho \in (-\mu(I_\varepsilon), \mu(J_{\varepsilon'}))$ we have $u' + \rho \psi'_{\varepsilon, \varepsilon'} - \frac{\ell}{2} \geq 0$ whenever $\rho \psi'_{\varepsilon, \varepsilon'} \neq 0$, so that $h(u' + \rho \psi'_{\varepsilon, \varepsilon'}) - h(u') = \ell \rho \psi'_{\varepsilon, \varepsilon'}$. Analogously, for the same set of values of ρ we have

$$(D^c u)_+ + \rho D^c \psi_{\varepsilon, \varepsilon'} = \left(1 + \frac{\rho}{\mu(I_\varepsilon)} \chi_{I_\varepsilon} - \frac{\rho}{\mu(J_{\varepsilon'})} \chi_{J_{\varepsilon'}} \right) (D^c u)_+ \geq 0$$

which implies $|D^c(u + \rho \psi_{\varepsilon, \varepsilon'})|(0, 1) = |D^c u|(0, 1) + \rho D^c \psi_{\varepsilon, \varepsilon'}(0, 1)$. The above expression then becomes

$$0 \leq \int_0^1 \ell \rho \psi'_{\varepsilon, \varepsilon'} dx + \ell \rho D^c \psi_{\varepsilon, \varepsilon'}(0, 1) + \gamma \int_0^1 [2\rho(u - w_k^\tau) \psi_{\varepsilon, \varepsilon'} + \rho^2 \psi_{\varepsilon, \varepsilon'}^2] dx.$$

The sum of the first two terms is $\ell \rho \nu_{\varepsilon, \varepsilon'}(0, 1) = 0$, and since ρ can still be chosen arbitrarily small we conclude that $\int_0^1 (u - w_k^\tau) \psi_{\varepsilon, \varepsilon'} dx = 0$ for all admissible $\varepsilon, \varepsilon'$. By Fubini's theorem we have

$$0 = \int_0^1 (u - w_k^\tau)(x) \psi_{\varepsilon, \varepsilon'}(x) dx = \int_0^1 (u - w_k^\tau)(x) \int_0^x d\nu_{\varepsilon, \varepsilon'}(y) dx = \int_0^1 F(y) d\nu_{\varepsilon, \varepsilon'}(y)$$

for all $\varepsilon, \varepsilon'$, where $F(y) := \int_y^1 (u - w_k^\tau)(x) dx$ is continuous.

We now consider the measure $\hat{\nu}_\varepsilon := \frac{1}{\mu(I_\varepsilon)} \mu \llcorner I_\varepsilon - \delta_b$ and set $\hat{\psi}_\varepsilon(x) := \hat{\nu}_\varepsilon((0, x))$. Since $\nu_{\varepsilon, \varepsilon'} \rightarrow \hat{\nu}_\varepsilon$ as $\varepsilon' \rightarrow 0$, and F is continuous, we obtain $\int_0^1 F d\hat{\nu}_\varepsilon = 0$. We compare u with $u + \rho \hat{\psi}_\varepsilon$ and obtain

$$0 \leq \int_0^1 \left[h(u' + \rho \hat{\psi}'_\varepsilon) - h(u') \right] dx + \ell |D^c(u + \rho \hat{\psi}_\varepsilon)|(0, 1) - \ell |D^c u|(0, 1) \\ + \gamma \int_0^1 \left[(u - w_k^\tau + \rho \hat{\psi}_\varepsilon)^2 - (u - w_k^\tau)^2 \right] dx + g(|[u](b) - \rho|, s_{k-1}^\tau(b)) - g(|[u](b)|, s_{k-1}^\tau(b))$$

where as usual $s_{k-1}^\tau(b) = 0$ if $b \notin \Gamma_{k-1}^\tau$. By the same computation as above, using that $\hat{\nu}_\varepsilon(I_\varepsilon) = 1$,

$$0 \leq \rho \ell + 2\gamma \rho \int_0^1 F d\hat{\nu}_\varepsilon + \gamma \rho^2 \int_0^1 \hat{\psi}_\varepsilon^2 dx + g(|[u](b) - \rho|, s_{k-1}^\tau(b)) - g(|[u](b)|, s_{k-1}^\tau(b)).$$

By (2.2) and (2.1) we have for $|\rho|$ small

$$g(|[u](b) - \rho|, s_{k-1}^\tau(b)) \leq g(|[u](b)|, s_{k-1}^\tau(b)) + g(|\rho|, 0) \leq g(|[u](b)|, s_{k-1}^\tau(b)) + \ell |\rho| - \tilde{\ell} |\rho|^p + o(|\rho|^p).$$

Recalling $\int_0^1 F d\hat{\nu}_\varepsilon = 0$ we have

$$0 \leq \rho \ell + \ell |\rho| - \tilde{\ell} |\rho|^p + o(|\rho|^p) + \gamma \rho^2 \int_0^1 \hat{\psi}_\varepsilon^2 dx$$

for all ρ sufficiently small (positive or negative). This is a contradiction, hence we conclude that $\mu = 0$. \square

Proposition 3.1 allows to construct a *piecewise constant evolution*, relative to the discrete boundary values $b_k^\tau = b(t_k)$ and to $w_k^\tau := w(t_k)$,

$$t \mapsto (u_\tau(t), \Gamma_\tau(t), s_\tau(t)) \quad \text{for } t \in [0, T], \quad (3.7)$$

where the piecewise constant interpolations in time are defined as

$$u_\tau(t) := u_k^\tau, \quad \Gamma_\tau(t) := \Gamma_k^\tau, \quad s_\tau(t) := s_k^\tau \quad \text{for } t \in [t_k, t_{k+1}), \quad (3.8)$$

with $k = 0, 1, \dots, N_\tau$, with $\Gamma_{-1}^\tau = \emptyset$, $s_{-1}^\tau = 0$. We also consider the piecewise constant and the piecewise affine interpolations in time of the boundary data, which are respectively defined as

$$b_\tau(t) := b_k^\tau, \quad b^\tau(t) := b_k^\tau + \frac{t - t_k}{\tau} (b_{k+1}^\tau - b_k^\tau) \quad \text{for } t \in [t_k, t_{k+1}), \quad (3.9)$$

and similarly

$$w_\tau(t) := w_k^\tau, \quad w^\tau(t) := w_k^\tau + \frac{t - t_k}{\tau} (w_{k+1}^\tau - w_k^\tau) \quad \text{for } t \in [t_k, t_{k+1}). \quad (3.10)$$

In the following we use the notation $u_\tau(t, x) := (u_\tau(t))(x)$ (and similarly for the other functions introduced above). We recall that we extend s_k^τ by 0 outside Γ_k^τ .

We observe that for any k

$$\Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) = \Phi(u_k^\tau; \Gamma_k^\tau, s_k^\tau, b_k^\tau). \quad (3.11)$$

To see this, it suffices to show that

$$g(|[u_k^\tau](x)|, s_k^\tau(x)) = g(|[u_k^\tau](x)|, s_{k-1}^\tau(x)) \quad \text{for any } x \in J^{b_k^\tau}(u_k^\tau) \cup \Gamma_k^\tau.$$

This is true, since in (3.4) we defined $s_k^\tau(x) = s_{k-1}^\tau(x) \vee |[u_k^\tau](x)|$. If $s_{k-1}^\tau(x) \geq |[u_k^\tau](x)|$ then $s_k^\tau(x) = s_{k-1}^\tau(x)$ and the two terms are identical. If not, then $s_k^\tau(x) = |[u_k^\tau](x)|$ and, recalling the assumption (i) on g in Section 2, we have $g(|[u_k^\tau](x)|, s_k^\tau(x)) = g(|[u_k^\tau](x)|, 0) = g(|[u_k^\tau](x)|, s_{k-1}^\tau(x))$. Therefore (3.11) holds.

We are interested in the evolution of the discrete total energy at time t , with timestep τ , which is defined for $t \in [0, T]$ by

$$\mathcal{E}_\tau(t) := \Phi(u_\tau(t); \Gamma_\tau(t), s_\tau(t), b_\tau(t)) + \gamma \int_0^1 |u_\tau(t) - w_\tau(t)|^2 dx. \quad (3.12)$$

Lemma 3.3. For any $t \in [0, T]$, letting t_k be the discretization point such that $t \in [t_k, t_{k+1})$, we have

$$\mathcal{E}_\tau(t) \leq \mathcal{E}_\tau(0) + \int_0^{t_k} \int_0^1 \left(h'((u_\tau(r))')(\dot{b}^\tau(r))' + 2\gamma(u_\tau(r) - w_\tau(r))(\dot{b}^\tau(r) - \dot{w}^\tau(r)) \right) dx dr + R_\tau(t), \quad (3.13)$$

where

$$\begin{aligned} R_\tau(t) := & \int_0^{t_k} \left(\tau \int_0^1 |(\dot{b}^\tau(r))'|^2 dx + \gamma \tau \int_0^1 |\dot{b}^\tau(r)|^2 dx \right. \\ & \left. + \gamma \left| \int_0^1 (w_\tau(r + \tau) - w_\tau(r)) \dot{w}^\tau(r) dx \right| + 2\gamma \left| \int_0^1 (w_\tau(r + \tau) - w_\tau(r)) \dot{b}^\tau(r) dx \right| \right) dr. \end{aligned} \quad (3.14)$$

Proof. We compare u_k^τ with $v := u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau$: by minimality of u_k^τ in problem (3.3) we have by Proposition 3.2

$$\begin{aligned} & \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx \\ & \leq \int_0^1 h(|(u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau)'|) dx + \gamma \int_0^1 |u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau - w_k^\tau|^2 dx \\ & \quad + \sum_{x \in \Gamma_{k-1}^\tau \cup J^{b_k^\tau - b_{k-1}^\tau}(u_{k-1}^\tau)} g(|[u_{k-1}^\tau](x)|, s_{k-1}^\tau(x)). \end{aligned}$$

If $x \in \{0, 1\}$, we remark that $v(x) - b_k^\tau(x) = u_{k-1}^\tau(x) - b_{k-1}^\tau(x)$, so that the notation $[u_{k-1}^\tau](x)$ is appropriate. Therefore, recalling (3.11),

$$\begin{aligned} & \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx \\ & \leq \Phi(u_{k-1}^\tau; \Gamma_{k-2}^\tau, s_{k-2}^\tau, b_{k-1}^\tau) + \gamma \int_0^1 |u_{k-1}^\tau - w_{k-1}^\tau|^2 dx \\ & \quad + \int_0^1 \left[h(|(u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau)'|) - h(|(u_{k-1}^\tau)'|) \right] dx \\ & \quad + \gamma \int_0^1 \left(|u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau - w_k^\tau|^2 - |u_{k-1}^\tau - w_{k-1}^\tau|^2 \right) dx. \end{aligned} \quad (3.15)$$

Since h' is a Lipschitz function with $|h''| \leq 2$, the mean-value theorem gives

$$h(x + y) \leq h(x) + yh'(x) + y^2 \quad \text{for all } x, y \in \mathbb{R}.$$

In particular,

$$h((u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau)') - h((u_{k-1}^\tau)') \leq h'((u_{k-1}^\tau)')(b_k^\tau - b_{k-1}^\tau)' + |(b_k^\tau - b_{k-1}^\tau)'|^2.$$

Similarly,

$$\begin{aligned} & |u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau - w_k^\tau|^2 - |u_{k-1}^\tau - w_{k-1}^\tau|^2 \\ & = (w_k^\tau + w_{k-1}^\tau - 2u_{k-1}^\tau)(w_k^\tau - w_{k-1}^\tau) + 2(u_{k-1}^\tau - w_k^\tau)(b_k^\tau - b_{k-1}^\tau) + |b_k^\tau - b_{k-1}^\tau|^2. \end{aligned}$$

By inserting these inequalities into (3.15) and iterating we find

$$\begin{aligned} \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx &\leq \Phi(u_0^\tau; b_0^\tau) + \gamma \int_0^1 |u_0^\tau - w_0^\tau|^2 dx \\ &+ \sum_{i=1}^k \left(\int_0^1 h'((u_{i-1}^\tau)')(b_i^\tau - b_{i-1}^\tau)' dx + \int_0^1 |(b_i^\tau - b_{i-1}^\tau)'|^2 dx \right. \\ &\quad + \gamma \int_0^1 (w_i^\tau + w_{i-1}^\tau - 2u_{i-1}^\tau)(w_i^\tau - w_{i-1}^\tau) dx \\ &\quad \left. + 2\gamma \int_0^1 (u_{i-1}^\tau - w_i^\tau)(b_i^\tau - b_{i-1}^\tau) dx + \gamma \int_0^1 |b_i^\tau - b_{i-1}^\tau|^2 dx \right). \end{aligned}$$

We can rewrite this inequality in terms of the interpolants defined in (3.8), (3.9), (3.10): for any $t \in [0, T]$, denoting by t_k the discretization point such that $t \in [t_k, t_{k+1})$, and observing that (3.11) and (3.12) give

$$\mathcal{E}_\tau(t) = \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx,$$

we have

$$\begin{aligned} \mathcal{E}_\tau(t) &\leq \mathcal{E}_\tau(0) + \int_0^{t_k} \left(\int_0^1 h'((u_\tau(r))')(b^\tau(r))' dx + \tau \int_0^1 |(b^\tau(r))'|^2 dx \right. \\ &\quad + \gamma \int_0^1 (w_\tau(r + \tau) + w_\tau(r))\dot{w}^\tau(r) dx - 2\gamma \int_0^1 w_\tau(r + \tau)\dot{b}^\tau(r) dx \\ &\quad \left. + 2\gamma \int_0^1 u_\tau(r)(\dot{b}^\tau(r) - \dot{w}^\tau(r)) dx + \gamma\tau \int_0^1 |\dot{b}^\tau(r)|^2 dx \right) dr. \end{aligned}$$

This concludes the proof. \square

4 Cohesive quasi-static evolution: the time-continuous evolution

The goal of this section is to pass to the limit in the time-discrete evolution as the time-step τ goes to zero. As in the previous section, we fix a positive threshold $\bar{s} > 0$ for the jumps, the final time $T > 0$, a time-dependent boundary displacement $b \in H^1([0, T]; \mathbb{R}^2)$, and a lower order term $w \in \text{AC}([0, T]; L^\infty(0, 1))$. Let $(\tau_n)_{n \in \mathbb{N}}$ be a decreasing sequence of time-discretization steps with $\tau_n \rightarrow 0$ as $n \rightarrow \infty$. Correspondingly, let

$$0 = t_0^n < t_1^n < \dots < t_{N_n}^n < t_{N_n+1}^n = T$$

be the subdivision of $[0, T]$ with $t_k^n := k\tau_n$ for $k \in \{0, \dots, N_n\}$. Let

$$t \mapsto (u_n(t) := u_{\tau_n}(t), \Gamma_n(t) := \Gamma_{\tau_n}(t), s_n(t) := s_{\tau_n}(t)), \quad \text{for } t \in [0, T], \quad (4.1)$$

be the piecewise constant interpolation, defined in (3.8), of a time-discrete evolution relative to the boundary data $b_k^n = b(t_k^n)$ and to $w_k^n = w(t_k^n)$, as constructed in the previous section. Let also

$$b_n(t) := b_{\tau_n}(t), \quad b^n(t) := b^{\tau_n}(t), \quad w_n(t) := w_{\tau_n}(t), \quad w^n(t) := w^{\tau_n}(t) \quad (4.2)$$

be the piecewise constant and the piecewise affine interpolants of the maps b and w , according to (3.9)–(3.10). We also consider the associated energy $\mathcal{E}_n(t) := \mathcal{E}_{\tau_n}(t)$, see (3.12). Our main result is the following.

Theorem 4.1 (Existence of a quasi-static evolution). *Under the assumptions of Section 2, with b and w as in (2.6) and (2.12), there exists $(u(t), \Gamma(t), s(t))$, for $t \in [0, T]$, with the following properties:*

- (i) $u(t) \in \text{BV}(0, 1)$, $\sup_{t \in [0, T]} \|u(t)\|_\infty \leq \sup_{t \in [0, T]} \max\{\|b(t)\|_\infty, \|w(t)\|_\infty\}$, $\Gamma(t) \subset [0, 1]$ is a finite set, $s(t) : \Gamma(t) \rightarrow [\bar{s}, \infty)$, $\sup_{t \in [0, T]} \|s(t)\|_\infty \leq 2 \sup_{t \in [0, T]} \max\{\|b(t)\|_\infty, \|w(t)\|_\infty\}$;
- (ii) (irreversibility) $\Gamma(t_1) \subset \Gamma(t_2)$ and $s(t_1) \leq s(t_2)$ for all $0 \leq t_1 \leq t_2 \leq T$;
- (iii) $u(0)$ minimizes the functional $\Phi(v; b(0)) + \gamma \|v - w(0)\|_{L^2(0,1)}^2$ among all $v \in \text{BV}(0, 1)$, and $\Gamma(0) = \{x \in J^{b(0)}(u(0)) : |[u(0)](x)| > \bar{s}\}$, $s(0) = |[u](0)|$ on $\Gamma(0)$;
- (iv) $\{x \in J^{b(t)}(u(t)) : |[u(t)](x)| > \bar{s}\} \subset \Gamma(t)$ and $[u](t) \leq s(t)$ on $\Gamma(t)$, for all $t \in (0, T]$;
- (v) (static equilibrium) for all $t \in (0, T]$, $u(t)$ minimizes the functional

$$\Phi(v; \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |v - w(t)|^2 dx$$

among all $v \in \text{BV}(0, 1)$;

- (vi) (non-dissipativity) the total energy

$$\mathcal{E}(t) := \Phi(u(t); \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |u(t) - w(t)|^2 dx$$

satisfies for every $t \in [0, T]$

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_0^1 \left(h'((u(r))')(\dot{b}(r)))' + 2\gamma(u(r) - w(r))(\dot{b}(r) - \dot{w}(r)) \right) dx dr. \quad (4.3)$$

Corollary 4.2. *Theorem 4.1 implies that $(u(t), \Gamma(t), s(t))$ minimizes the functional*

$$\Phi(v; \Gamma, s, b(t)) + \gamma \int_0^1 |v - w(t)|^2 dx$$

among all (v, Γ, s) such that Γ is a finite set with $\Gamma(t) \subset \Gamma$, $s : \Gamma \rightarrow [\bar{s}, \infty)$ satisfies $s \geq s(t)$ on $\Gamma(t)$, and $v \in \text{BV}(0, 1)$ is such that $\{x \in J^{b(t)}(v) : |[v](x)| > \bar{s}\} \subset \Gamma$.

Proof. By Theorem 4.1(i) and (iv), $(u(t), \Gamma(t), s(t))$ is admissible. Let now (v, Γ, s) be admissible. By monotonicity of g we have $\Phi(v; \Gamma(t), s(t), b(t)) \leq \Phi(v; \Gamma, s, b(t))$, and Theorem 4.1(v) then concludes the proof. \square

Proof of Theorem 4.1. We divide the proof into several steps. In the following, we will denote by C a generic constant, possibly depending on b , w , and γ , but independent of n and t , which might change from line to line. As a preliminary remark, we observe that $u_n(t)$ is a minimizer of problem (3.3), for the value of k such that $t \in [t_k^n, t_{k+1}^n)$; then

$$\begin{aligned} & \Phi(u_n(t); \Gamma_n(t - \tau_n), s_n(t - \tau_n), b_n(t)) + \gamma \int_0^1 |u_n(t) - w_n(t)|^2 dx \\ & \leq \Phi(v; \Gamma_n(t - \tau_n), s_n(t - \tau_n), b_n(t)) + \gamma \int_0^1 |v - w_n(t)|^2 dx \quad \text{for all } v \in \text{BV}(0, 1). \end{aligned}$$

Using (3.11) on the left-hand side, and monotonicity on the right-hand side, this implies

$$\begin{aligned} & \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |u_n(t) - w_n(t)|^2 dx \\ & \leq \Phi(v; \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |v - w_n(t)|^2 dx \quad \text{for all } v \in \text{BV}(0, 1). \end{aligned} \quad (4.4)$$

Step 1: compactness. The first goal is to prove a uniform bound on $\mathcal{E}_n(t)$, independent of n and t , by using the energy inequality (3.13). To this aim, notice that by using $v = b(0)$ as test function

in (3.1) we easily obtain a uniform bound on the initial energy $\mathcal{E}_n(0)$. Then (3.13) yields, as $|h'(\xi)| \leq \ell$ for all ξ ,

$$\begin{aligned} \mathcal{E}_n(t) &\leq \mathcal{E}_n(0) + \int_0^{t_k^n} \int_0^1 \left(h'((u_n(r))') (\dot{b}^n(r))' + 2\gamma(u_n(r) - w_n(r))(\dot{b}^n(r) - \dot{w}^n(r)) \right) dx dr + R_n(t) \\ &\leq C + \ell \int_0^T \int_0^1 |(\dot{b}^n(r))'| dx dr \\ &\quad + 2\gamma \sup_{r \in [0, T]} \left(\|u_n(r)\|_\infty + \|w_n(r)\|_\infty \right) \int_0^T \int_0^1 \left(|\dot{b}^n(r)| + |\dot{w}^n(r)| \right) dx dr + R_n(t). \end{aligned}$$

For the remainder $R_n(t)$, defined in (3.14), we have:

$$R_n(t) \leq C\tau_n \int_0^T \|\dot{b}^n(r)\|_{H^1(0,1)}^2 dr + C \sup_{r \in [0, T]} \|w_n(r)\|_\infty \int_0^T \left(\|\dot{w}^n(r)\|_{L^1(0,1)} + \|\dot{b}^n(r)\|_{L^1(0,1)} \right) dr.$$

In view of the assumptions on b and w and of Proposition 3.1(ii), all the previous quantities are uniformly bounded, therefore we obtain a uniform estimate on the energies $\mathcal{E}_n(t)$ and in turn, similarly to (3.6), on the BV-norm of the functions $u_n(t)$:

$$\sup_{n,t} \mathcal{E}_n(t) < \infty, \quad \sup_{n,t} |Du_n(t)|(0,1) < \infty. \quad (4.5)$$

Fix now a countable dense set $D \subset [0, T]$, with $0 \in D$. By a diagonal argument and by the uniform bounds (4.5) we can find a subsequence (which we denote by the same symbol) such that

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly* in BV}(0,1), \quad \text{for all } t \in D, \quad (4.6)$$

for some $u(t) \in \text{BV}(0,1)$, with $\|u(t)\|_\infty \leq \max_{r \in [0, T]} \max\{\|b(r)\|_\infty, \|w(r)\|_\infty\}$. This implies in particular that $\sup_t \sup_n \sup_x |s_n(t)(x)| < \infty$.

Thanks to Proposition 3.1(iii), it is clear that we have a uniform bound on the number of points of $\Gamma_n(t)$:

$$\sup_{n,t} \mathcal{H}^0(\Gamma_n(t)) < \infty. \quad (4.7)$$

Up to further subsequences, we can therefore assume that for any $t \in D$ and n large enough $\Gamma_n(t) = \{\bar{x}_1^n(t), \dots, \bar{x}_{N(t)}^n(t)\}$ (with the points $\bar{x}_i^n(t)$ distinct and $N(t)$ independent of n), and that each sequence $\bar{x}_i^n(t)$ converges as $n \rightarrow \infty$; we denote by $\Gamma(t) = \{\bar{x}_1(t), \dots, \bar{x}_{N(t)}(t)\}$ the set of limit points of these sequences,

$$\Gamma(t) := \left\{ \bar{x}_i(t) := \lim_{n \rightarrow \infty} \bar{x}_i^n(t), i = 1, \dots, N(t) \right\} \subset [0, 1] \quad \text{for } t \in D. \quad (4.8)$$

Notice that all the points $\bar{x}_i(t)$ are distinct, since two different sequences $(\bar{x}_i^n(t))_n, (\bar{x}_j^n(t))_n$ cannot converge to the same limit point by Proposition 3.1(iii). Moreover, in view of the monotonicity property $\Gamma_n(t_1) \subset \Gamma_n(t_2)$ for $t_1 < t_2$, we also have

$$\Gamma(t_1) \subset \Gamma(t_2) \quad \text{for all } t_1, t_2 \in D, t_1 < t_2. \quad (4.9)$$

Finally, we also have compactness for the maps $s_n(t)$: by possibly extracting another subsequence, for every $t \in D$ there exists a map $s(t) : \Gamma(t) \rightarrow [\bar{s}, \infty)$, such that

$$s_n(t)(\bar{x}_i^n(t)) \rightarrow s(t)(\bar{x}_i(t)) \quad \text{for all } i = 1, \dots, N(t), t \in D. \quad (4.10)$$

Moreover $s(t_1) \leq s(t_2)$ on $\Gamma(t_1)$ for all $t_1, t_2 \in D, t_1 < t_2$, since every map $s_n(t)$ is nondecreasing in time by construction. As usual, we extend s by 0 on $[0, 1] \setminus \Gamma(t)$.

Step 2. We now prove the following claims:

$$\{x \in J^{b(t)}(u(t)) : |[u(t)](x)| > \bar{s}\} \subset \Gamma(t) \quad \text{for all } t \in D, \quad (4.11)$$

and

$$|[u(t)](x)| \leq s(t, x) \quad \text{for all } x \in \Gamma(t) \text{ and } t \in D. \quad (4.12)$$

We first recall that by Proposition 3.2 we have $u_n \in SBV(0, 1)$ and $|u'_n| \leq \frac{\ell}{2}$ almost everywhere.

In order to prove (4.11), suppose by contradiction that for some $t \in D$ there exists $\bar{x} \in J^{b(t)}(u(t)) \setminus \Gamma(t)$ with $|[u(t)](\bar{x})| \geq \bar{s} + \varepsilon$, for some $\varepsilon > 0$. We also assume that $\bar{x} \in (0, 1)$, as the boundary case follows by a similar argument. Let $I_\delta := (\bar{x} - \delta, \bar{x} + \delta)$ be such that $I_\delta \cap \Gamma_n(t) = \emptyset$ for all sufficiently large n ; then, by definition of $\Gamma_n(t)$, all the jumps of $u_n(t)$ inside I_δ are smaller than the threshold \bar{s} : $|[u_n(t)](x)| \leq \bar{s}$ for every $x \in J_{u_n(t)} \cap I_\delta$. Moreover, as $\Gamma_n(t - \tau_n) \subset \Gamma_n(t)$, $u_n(t)$ solves the minimum problem

$$\min \left\{ \int_{I_\delta} h(|v'|) dx + \sum_{x \in J_v \cap I_\delta} g(|[v](x)|, 0) + \ell |D^c v|(I_\delta) + \gamma \int_{I_\delta} |v - w_n(t)|^2 dx : v \in \text{BV}(I_\delta), \right. \\ \left. \{v \neq u_n(t)\} \subset\subset I_\delta \right\}. \quad (4.13)$$

By (4.6) and lower semicontinuity of the total variation, we have for every $\delta > 0$

$$\liminf_{n \rightarrow \infty} |Du_n(t)|(I_\delta) \geq |Du(t)|(I_\delta) \geq \bar{s} + \varepsilon.$$

In view of Proposition 3.2, it is not possible that the limit jump of $u(t)$ at \bar{x} is created by a nonzero contribution from the absolutely continuous part of the measures $Du_n(t)$: more precisely, we can assert that

$$\text{for every } \delta > 0 \text{ sufficiently small} \quad \liminf_{n \rightarrow \infty} \sum_{x \in J_{u_n(t)} \cap I_\delta} |[u_n(t)](x)| \geq \bar{s} + \frac{\varepsilon}{2},$$

and each $u_n(t)$ has at least two jumps in I_δ since $|[u_n(t)](x)| \leq \bar{s}$. In particular, for n large we can find a partition $J_{u_n(t)} \cap I_\delta = A_n \cup B_n$ such that

$$\sum_{x \in A_n} |[u_n(t)](x)| \geq \frac{\varepsilon}{4}, \quad \sum_{x \in B_n} |[u_n(t)](x)| \geq \frac{\varepsilon}{4}. \quad (4.14)$$

We construct a competitor for the minimum problem (4.13) by moving all the jumps of $u_n(t)$ in a single point $x_n \in I_\delta \setminus J_{u_n(t)}$: then the minimality of $u_n(t)$ in (4.13) and the subadditivity of g yield

$$0 \leq g \left(\left| \sum_{x \in J_{u_n(t)} \cap I_\delta} [u_n(t)](x) \right|, 0 \right) - \sum_{x \in J_{u_n(t)} \cap I_\delta} g(|[u_n(t)](x)|, 0) + C\delta \\ \leq g \left(\sum_{x \in J_{u_n(t)} \cap I_\delta} |[u_n(t)](x)|, 0 \right) - g \left(\sum_{x \in A_n} |[u_n(t)](x)|, 0 \right) - g \left(\sum_{x \in B_n} |[u_n(t)](x)|, 0 \right) + C\delta \\ \leq -c_\varepsilon/4 + C\delta,$$

the last inequality following by (4.14) and (2.3). This is a contradiction for δ small enough and completes the proof of (4.11).

We next show (4.12). Let $\bar{x}_i(t)$ be any point in $\Gamma(t)$. Recalling (4.10) and that by construction $|[u_n(t)](\bar{x}_i^n(t))| \leq s_n(t, \bar{x}_i^n(t))$, it is sufficient to prove that

$$|[u(t)](\bar{x}_i(t))| \leq \liminf_{n \rightarrow \infty} |[u_n(t)](\bar{x}_i^n(t))|. \quad (4.15)$$

Let us momentarily omit the dependence on t . Suppose by contradiction that for some $\varepsilon > 0$ one has $|[u](\bar{x}_i)| \geq \liminf_{n \rightarrow \infty} |[u_n](\bar{x}_i^n)| + \varepsilon$. Then, by Proposition 3.2, (4.6), and lower semicontinuity of the total variation, we have for $I_\delta := (\bar{x}_i - \delta, \bar{x}_i + \delta)$ and sufficiently small δ

$$\limsup_{n \rightarrow \infty} \sum_{x \in J_{u_n} \cap I_\delta \setminus \{\bar{x}_i^n\}} |[u_n](x)| \geq \frac{\varepsilon}{2}.$$

We construct a competitor by moving all the jumps of u_n at the point \bar{x}_i^n : the minimality of u_n in (4.4) yields for n large enough

$$\begin{aligned} 0 &\leq g\left(\left|\sum_{x \in J_{u_n} \cap I_\delta} [u_n](x)\right|, s_n(\bar{x}_i^n)\right) - g(|[u_n](\bar{x}_i^n)|, s_n(\bar{x}_i^n)) - \sum_{x \in J_{u_n} \cap I_\delta \setminus \{\bar{x}_i^n\}} g(|[u_n](x)|, 0) + C\delta \\ &\leq g\left(\sum_{x \in J_{u_n} \cap I_\delta} |[u_n](x)|, s_n(\bar{x}_i^n)\right) - g(|[u_n](\bar{x}_i^n)|, s_n(\bar{x}_i^n)) - g\left(\sum_{x \in J_{u_n} \cap I_\delta \setminus \{\bar{x}_i^n\}} |[u_n](x)|, 0\right) + C\delta \\ &\leq -c_\varepsilon/2 + C\delta, \end{aligned}$$

where the last inequality follows by (2.4). This is a contradiction for δ small enough, and proves that (4.15) holds.

Step 3: lower semicontinuity of the energies. We now claim that for every $t \in D$

$$\Phi(u(t); \Gamma(t), s(t), b(t)) \leq \liminf_{n \rightarrow \infty} \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)). \quad (4.16)$$

First observe that, in view of the continuity and monotonicity properties of g , (4.15), and (4.10), we have

$$\sum_{x \in \Gamma(t)} g(|[u(t)](x)|, s(t, x)) \leq \liminf_{n \rightarrow \infty} \sum_{x \in \Gamma_n(t)} g(|[u_n(t)](x)|, s_n(t, x)). \quad (4.17)$$

Consider any relatively open set $A \subset\subset [0, 1] \setminus \Gamma(t)$; by lower semicontinuity of the functional

$$\begin{aligned} &\int_A h(|u'(t)|) dx + \sum_{x \in J^{b(t)}(u(t)) \cap A} g(|[u(t)](x)|, 0) + \ell |D^c u(t)|(A) \\ &\leq \liminf_{n \rightarrow \infty} \left[\int_A h(|u'_n(t)|) dx + \sum_{x \in J^{b_n(t)}(u_n(t)) \cap A} g(|[u_n(t)](x)|, 0) + \ell |D^c u_n(t)|(A) \right] \end{aligned}$$

so that taking the supremum over all A and recalling (4.17) we obtain (4.16).

Step 4: static equilibrium. We now show that the minimality condition (v) holds for every $t \in D$. In order to do this, we first show that for every $v \in \text{BV}(0, 1)$ we can construct $v_n \in \text{BV}(0, 1)$ such that $v_n \rightarrow v$ in $L^2(0, 1)$ as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \left[\Phi(v_n; \Gamma_n(t), s_n(t), b_n(t)) - \Phi(v; \Gamma(t), s(t), b(t)) \right] \leq 0. \quad (4.18)$$

Recalling (4.8), this can be done by considering the (possible) jumps of v on the points of $\Gamma(t)$ and moving them to the corresponding points of $\Gamma_n(t)$: more precisely, assuming for simplicity that $\Gamma(t) \subset (0, 1)$ (the construction can be straightforwardly adapted if one of the boundary points belongs to $\Gamma(t)$), we define v_n by the conditions

$$v_n^+(0) := v^+(0), \quad Dv_n := Dv - \sum_{i=1}^{N(t)} [v](\bar{x}_i(t)) \delta_{\bar{x}_i(t)} + \sum_{i=1}^{N(t)} [v](\bar{x}_i^n(t)) \delta_{\bar{x}_i^n(t)}.$$

Then $v_n \rightarrow v$ in $L^2(0, 1)$. By (4.8), if n is sufficiently large then $\bar{x}_i^n(t) \neq \bar{x}_j(t)$ for $i \neq j$. We set

$\chi_i^n = 1$ if $\bar{x}_i^n(t) \neq \bar{x}_i(t)$, and $\chi_i^n = 0$ if $\bar{x}_i^n(t) = \bar{x}_i(t)$. We estimate

$$\begin{aligned}
& \Phi(v_n; \Gamma_n(t), s_n(t), b_n(t)) - \Phi(v; \Gamma(t), s(t), b(t)) \\
&= \sum_{i=1}^{N(t)} \chi_i^n \left[g(|[v](\bar{x}_i^n(t)) + [v](\bar{x}_i(t))|, s_n(t, \bar{x}_i^n(t))) - g(|[v](\bar{x}_i(t))|, s(t, \bar{x}_i(t))) - g(|[v](\bar{x}_i^n(t))|, 0) \right] \\
&+ \sum_{i=1}^{N(t)} (1 - \chi_i^n) \left[g(|[v](\bar{x}_i(t))|, s_n(t, \bar{x}_i^n(t))) - g(|[v](\bar{x}_i(t))|, s(t, \bar{x}_i(t))) \right] \\
&+ g(|v^+(0) - b_n(t, 0)|, 0) - g(|v^+(0) - b(t, 0)|, 0) + g(|v^-(1) - b_n(t, 1)|, 0) - g(|v^-(1) - b(t, 1)|, 0) \\
&\stackrel{(2.2)}{\leq} \sum_{i=1}^{N(t)} \chi_i^n \left[g(|[v](\bar{x}_i(t))|, s_n(t, \bar{x}_i^n(t))) - g(|[v](\bar{x}_i(t))|, s(t, \bar{x}_i(t))) \right] \\
&+ \sum_{i=1}^{N(t)} (1 - \chi_i^n) \left[g(|[v](\bar{x}_i(t))|, s_n(t, \bar{x}_i^n(t))) - g(|[v](\bar{x}_i(t))|, s(t, \bar{x}_i(t))) \right] \\
&+ g(|v^+(0) - b_n(t, 0)|, 0) - g(|v^+(0) - b(t, 0)|, 0) + g(|v^-(1) - b_n(t, 1)|, 0) - g(|v^-(1) - b(t, 1)|, 0).
\end{aligned}$$

Hence (4.18) follows by taking into account (4.10), the continuity of g , and that $b_n(t) \rightarrow b(t)$.

We are now in position to conclude the proof of (v) for $t \in D$. Let $v \in \text{BV}(0, 1)$ and let v_n be the sequence constructed before. Then, using the convergence of $u_n(t) \rightarrow u(t)$ and $w_n(t) \rightarrow w(t)$ in $L^2(0, 1)$,

$$\begin{aligned}
& \Phi(u(t); \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |u(t) - w(t)|^2 dx \\
&\stackrel{(4.16)}{\leq} \liminf_{n \rightarrow \infty} \left[\Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |u_n(t) - w_n(t)|^2 dx \right] \\
&\stackrel{(4.4)}{\leq} \liminf_{n \rightarrow \infty} \left[\Phi(v_n; \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |v_n - w_n(t)|^2 dx \right] \\
&\stackrel{(4.18)}{\leq} \Phi(v; \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |v - w(t)|^2 dx.
\end{aligned}$$

Notice in particular that, by taking $v = u(t)$, the previous chain of inequalities yields the convergence of the energies:

$$\Phi(u(t); \Gamma(t), s(t), b(t)) = \lim_{n \rightarrow \infty} \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)). \quad (4.19)$$

Step 5: definition of the evolution for $t \notin D$. By the monotonicity property (4.9), the uniform bound on the number $N(t)$ of points in $\Gamma(t)$, and the monotonicity of $t \mapsto s(t)$, there exists a set $D' \subset [0, T] \setminus D$, at most countable, such that

$$\bigcap_{t' \in D, t' \geq t} \Gamma(t') = \bigcup_{t' \in D, t' \leq t} \Gamma(t'), \quad \inf_{t' \in D, t' \geq t} s(t') = \sup_{t' \in D, t' \leq t} s(t') \quad \text{for all } t \in [0, T] \setminus D'. \quad (4.20)$$

We then define $\Gamma(t)$ and $s(t)$ to be equal to the common values in (4.20) for all $t \in [0, T] \setminus D'$. Moreover, by repeating the construction in Steps 1–4 for the points $t \in D'$, and up to a further subsequence, we obtain a triple $(u(t), \Gamma(t), s(t))$ for all $t \in D \cup D'$ such that the conclusions of the previous steps hold for every $t \in D \cup D'$.

It remains to define $u(t)$ for $t \in [0, T] \setminus (D \cup D')$. For $t \in [0, T]$, we introduce the quantity

$$\theta_n(t) := \int_0^1 \left(h'((u_n(t))')(\dot{b}(t))' + 2\gamma(u_n(t) - w_n(t))(\dot{b}(t) - \dot{w}(t)) \right) dx, \quad (4.21)$$

as well as

$$\theta_\infty(t) := \limsup_{n \rightarrow \infty} \theta_n(t). \quad (4.22)$$

Notice that $|\theta_n(t)| \leq C(\|\dot{b}(t)\|_{W^{1,1}(0,1)} + \|\dot{w}(t)\|_{L^1(0,1)}) \in L^1(0, T)$, therefore $\theta_\infty \in L^1(0, T)$ and by Fatou's Lemma

$$\limsup_{n \rightarrow \infty} \int_0^t \theta_n(r) dr \leq \int_0^t \theta_\infty(r) dr \quad \text{for all } t \in [0, T]. \quad (4.23)$$

For a given $t \in [0, T] \setminus (D \cup D')$, we can find a subsequence $(n_j)_j$, dependent on t , such that $\theta_{n_j}(t) \rightarrow \theta_\infty(t)$, and $u_{n_j}(t)$ converges weakly* in $BV(0, 1)$ to a function $u(t)$ as $j \rightarrow \infty$. We choose this limit function to define the triple $(u(t), \Gamma(t), s(t))$ for $t \in [0, T] \setminus (D \cup D')$. Notice that the arguments in Steps 2–4 can be repeated at the point t for this subsequence: therefore the evolution $t \mapsto (u(t), \Gamma(t), s(t))$ satisfies the properties (i)–(v) in the statement for all $t \in [0, T]$.

Step 6: non-dissipativity. To complete the proof, it only remains to show the condition (vi). Setting

$$\theta(t) := \int_0^1 \left(h'((u(t))')(\dot{b}(t))' + 2\gamma(u(t) - w(t))(\dot{b}(t) - \dot{w}(t)) \right) dx, \quad (4.24)$$

we first claim that

$$\theta(t) = \theta_\infty(t) \quad \text{for almost every } t \in [0, T]. \quad (4.25)$$

In particular, this will give the measurability and integrability of $\theta(t)$ in $[0, T]$. The claim (4.25) can be proved by an argument similar to [24, Lemma 4.11]: fix $t \in [0, T]$ and consider any sequence of positive numbers $\delta_i \rightarrow 0$ and any $d \in \mathbb{R}$. By the definition of $u(t)$ there is a sequence $n_j \rightarrow \infty$ such that $u_{n_j}(t)(x) + d\delta_i x \rightharpoonup u(t)(x) + d\delta_i x$ weakly* in $BV(0, 1)$; by arguing as in Step 3 to prove the lower semicontinuity of the energy along the sequence $u_{n_j}(t)(x) + d\delta_i x$, and recalling (4.19), we find

$$\begin{aligned} & \frac{1}{\delta_i} \left(\Phi(u(t) + d\delta_i x; \Gamma(t), s(t), b(t) + d\delta_i x) - \Phi(u(t); \Gamma(t), s(t), b(t)) \right) \\ & \leq \liminf_{j \rightarrow \infty} \frac{1}{\delta_i} \left(\Phi(u_{n_j}(t) + d\delta_i x; \Gamma_{n_j}(t), s_{n_j}(t), b_{n_j}(t) + d\delta_i x) - \Phi(u_{n_j}(t); \Gamma_{n_j}(t), s_{n_j}(t), b_{n_j}(t)) \right). \end{aligned}$$

By writing the explicit expressions of the previous quantities, we find

$$\frac{1}{\delta_i} \int_0^1 \left(h(|u'(t) + \delta_i d|) - h(|u'(t)|) \right) dx \leq \liminf_{j \rightarrow \infty} \frac{1}{\delta_i} \int_0^1 \left(h(|u'_{n_j}(t) + \delta_i d|) - h(|u'_{n_j}(t)|) \right) dx,$$

therefore there exists an increasing sequence of integers $j_i \geq i$ such that

$$\frac{1}{\delta_i} \int_0^1 \left(h(u'(t) + \delta_i d) - h(u'(t)) \right) dx \leq \frac{1}{\delta_i} \int_0^1 \left(h(u'_{n_{j_i}}(t) + \delta_i d) - h(u'_{n_{j_i}}(t)) \right) dx + \frac{1}{i}.$$

Taking the limit $i \rightarrow \infty$ we obtain proceeding as in [24, Lemma 4.11],

$$\begin{aligned} d \int_0^1 h'(u'(t)) dx &= \liminf_{i \rightarrow \infty} \int_0^1 \frac{h(u'(t) + \delta_i d) - h(u'(t))}{\delta_i} dx \\ &\leq \liminf_{i \rightarrow \infty} \int_0^1 \frac{h(u'_{n_{j_i}}(t) + \delta_i d) - h(u'_{n_{j_i}}(t))}{\delta_i} dx \\ &= \liminf_{i \rightarrow \infty} d \int_0^1 h'(u'_{n_{j_i}}(t) + \tau_i d) dx = \liminf_{i \rightarrow \infty} d \int_0^1 h'(u'_{n_{j_i}}(t)) dx \end{aligned}$$

for suitable $\tau_i : (0, 1) \rightarrow [0, \delta_i]$. Taking $d = 1$ and $d = -1$ we see that this is actually an equality and that the limit does not depend on the subsequence j_i , which implies

$$\lim_{j \rightarrow \infty} \int_0^1 h'((u_{n_j}(t))') dx = \int_0^1 h'((u(t))') dx.$$

By the strong convergence of $u_{n_j}(t) \rightarrow u(t)$ and $w_{n_j}(t) \rightarrow w(t)$ in $L^2(0, 1)$, we then conclude that $\theta_{n_j}(t) \rightarrow \theta(t)$ as $j \rightarrow \infty$ and, since θ_∞ was the limit of the subsequence θ_{n_j} , this shows (4.25).

By Lemma 3.3 we have

$$\begin{aligned} \mathcal{E}_{n_j}(t) &\leq \mathcal{E}(0) + \int_0^{t_k^{n_j}} \theta_{n_j}(r) dr + R_{n_j}(t) + \int_0^{t_k^{n_j}} \int_0^1 h'((u_{n_j}(r))')(\dot{b}^{n_j}(r) - \dot{b}(r))' dx dr \\ &\quad + 2\gamma \int_0^{t_k^{n_j}} \int_0^1 (u_{n_j}(r) - w_{n_j}(r))(\dot{b}^{n_j}(r) - \dot{b}(r) - \dot{w}^{n_j}(r) + \dot{w}(r)) dx dr, \end{aligned} \quad (4.26)$$

where $t_k^{n_j}$ is the discretization point such that $t \in [t_k^{n_j}, t_{k+1}^{n_j})$. One can now check that the last two terms in the previous expression are actually equal to zero, and that $R_{n_j}(t) \rightarrow 0$ as $j \rightarrow \infty$, thanks to the assumptions on b and w . Therefore, recalling (4.23), (4.25), and that $\mathcal{E}_{n_j}(t) \rightarrow \mathcal{E}(t)$ by (4.19), we conclude that

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \int_0^t \theta(r) dr. \quad (4.27)$$

To conclude the proof it remains to show the opposite inequality. Let us stress that the subsequence n_j depends on t , so that we do not have a unique subsequence converging pointwise almost everywhere in $[0, T]$ to $\theta(t)$. This prevents to take directly the lower limit in the opposite of inequality (4.26), to get the opposite of inequality (4.27). In order to overcome this difficulty, we will first approximate the Lebesgue integral of θ by Riemann sums.

We fix $t \in (0, T]$. We first observe that there exists a sequence of subdivisions of $[0, t]$ of the form

$$0 = \rho_0^m < \rho_1^m < \dots < \rho_{i_m-1}^m < \rho_{i_m}^m = t, \quad \text{with} \quad \lim_{m \rightarrow \infty} \max_{i=1, \dots, i_m} |\rho_i^m - \rho_{i-1}^m| = 0,$$

with the property that

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{i_m} \left| (\rho_i^m - \rho_{i-1}^m) \theta(\rho_i^m) - \int_{\rho_{i-1}^m}^{\rho_i^m} \theta(r) dr \right| = 0, \quad (4.28)$$

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{i_m} \int_0^1 \left| (\rho_i^m - \rho_{i-1}^m) \dot{b}(\rho_i^m) - \int_{\rho_{i-1}^m}^{\rho_i^m} \dot{b}(r) dr \right| dx = 0, \quad (4.29)$$

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{i_m} \int_0^1 \left| (\rho_i^m - \rho_{i-1}^m) \dot{w}(\rho_i^m) - \int_{\rho_{i-1}^m}^{\rho_i^m} \dot{w}(r) dr \right| dx = 0 \quad (4.30)$$

(see [24, Lemma 4.12], and also [23, Lemma 4.12] for a proof, adapting the arguments of [32, page 63]). We now exploit the global stability of $u(\rho_{i-1}^m)$ (property (v)), taking the competitor $u(\rho_i^m) + b(\rho_{i-1}^m) - b(\rho_i^m)$ and adopting an iteration argument similar to that in the proof of Lemma 3.3. We repeatedly use that $b \in H^1([0, T], \mathbb{R}^2)$, with $b(t) = (b(t, 0), b(t, 1))$, and that $b(t) : [0, 1] \rightarrow \mathbb{R}$ denotes the affine function interpolating between the boundary values $b(t, 0)$ and $b(t, 1)$. Then, using also the monotonicity property (ii), we find

$$\begin{aligned} \mathcal{E}(t) &\geq \mathcal{E}(0) + \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 h'((u(\rho_i^m))')(\dot{b}(r))' dx dr \\ &\quad + 2\gamma \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 (w(\rho_i^m) - u(\rho_i^m))(\dot{w}(r) - \dot{b}(r)) dx dr - \mathcal{S}_m(t), \end{aligned} \quad (4.31)$$

where

$$\begin{aligned} \mathcal{S}_m(t) := & \gamma \sum_{i=1}^{i_m} \int_0^1 (w(\rho_i^m) - w(\rho_{i-1}^m))^2 dx - 2\gamma \sum_{i=1}^{i_m} \int_0^1 (w(\rho_i^m) - w(\rho_{i-1}^m))(b(\rho_i^m) - b(\rho_{i-1}^m)) \\ & + \sum_{i=1}^{i_m} \int_0^1 |b'(\rho_i^m) - b'(\rho_{i-1}^m)|^2 dx + \gamma \sum_{i=1}^{i_m} \int_0^1 |b(\rho_i^m) - b(\rho_{i-1}^m)|^2 dx. \end{aligned}$$

Notice that, in view of the assumptions on w and b , we have $\mathcal{S}_m(t) \rightarrow 0$ as $m \rightarrow \infty$. Recalling the definition (4.24) of the map $\theta(t)$, we further obtain from (4.31)

$$\mathcal{E}(t) \geq \mathcal{E}(0) + \sum_{i=1}^{i_m} (\rho_i^m - \rho_{i-1}^m) \theta(\rho_i^m) - \mathcal{R}_m(t) - \mathcal{S}_m(t), \quad (4.32)$$

with the position

$$\begin{aligned} \mathcal{R}_m(t) := & \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 h'((u(\rho_i^m))') (\dot{b}(\rho_i^m) - \dot{b}(r))' dx dr \\ & + 2\gamma \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 (u(\rho_i^m) - w(\rho_i^m)) (\dot{b}(\rho_i^m) - \dot{b}(r) - \dot{w}(\rho_i^m) + \dot{w}(r)) dx dr. \end{aligned}$$

Using the definition of h and the uniform bound in L^∞ on u and w we can estimate

$$\begin{aligned} |\mathcal{R}_m(t)| \leq & \ell \sum_{i=1}^{i_m} \left| (\rho_i^m - \rho_{i-1}^m) (\dot{b}(\rho_i^m))' - \int_{\rho_{i-1}^m}^{\rho_i^m} (\dot{b}(r))' dr \right| \\ & + C \sum_{i=1}^{i_m} \int_0^1 \left| (\rho_i^m - \rho_{i-1}^m) (\dot{b}(\rho_i^m) - \dot{w}(\rho_i^m)) - \int_{\rho_{i-1}^m}^{\rho_i^m} (\dot{b}(r) - \dot{w}(r)) dr \right| dx, \end{aligned}$$

and the previous quantity vanishes in the limit as $m \rightarrow \infty$ in view of (4.29)–(4.30). Eventually, by passing to the limit as $m \rightarrow \infty$ in (4.32), recalling (4.28) and that $\mathcal{R}_m(t) \rightarrow 0$, $\mathcal{S}_m(t) \rightarrow 0$, we conclude that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + \int_0^t \theta(r) dr,$$

which completes the proof of (vi). \square

Remark 4.3. Notice that in the proof of Theorem 4.1 we do not obtain a unique subsequence $(n_j)_j$, independent of t , such that the time-discrete evolution $(u_{n_j}(t), \Gamma_{n_j}(t), s_{n_j}(t))$ converges to the limit evolution $(u(t), \Gamma(t), s(t))$ for all $t \in [0, T]$. This in general holds only for t in a countable dense set. It would be possible to prove convergence of a subsequence for every $t \in [0, T]$ if we knew that the minimum problem solved by $u(t)$ has a unique solution.

5 Relaxation of the cohesive energy

In order to construct a time-discrete evolution in the cohesive setting including our notion of irreversibility, we made use in Sections 2 and 3 of a relaxation result in the spirit of [11, 12]. The main difference is that the surface part of our energy contains also information on points representing a preexisting crack, and therefore is not considered in the existing literature; however, the proof is a small variant of the standard theory, and we present it in this section. A similar problem was discussed by Giacomini in [39].

Let $b : \{0, 1\} \rightarrow \mathbb{R}$ be a given boundary datum. Let also (Γ, s) be a given pair with $\Gamma \subset [0, 1]$ countable and $s : \Gamma \rightarrow (0, \infty)$. Let $\Phi_0 : \text{BV}(0, 1) \rightarrow [0, \infty]$ be the functional defined by

$$\Phi_0(u) := \begin{cases} \int_0^1 |u'|^2 dx + \sum_{x \in J_u^b \cup \Gamma} g(|[u](x)|, s(x)) & \text{if } u \in \text{SBV}(0, 1), \\ \infty & \text{otherwise,} \end{cases} \quad (5.1)$$

where J_u^b and the jump $[u](x)$ at the boundary points $x \in \{0, 1\}$ are defined in (2.10) and (2.11) respectively.

Theorem 5.1. *Assume that the surface energy density g satisfies assumptions (i)-(iv) of Section 2. The relaxation of the functional Φ_0 with respect to the weak*-topology of $\text{BV}(0, 1)$ is given by $\Phi : \text{BV}(0, 1) \rightarrow [0, \infty]$,*

$$\Phi(u) := \int_0^1 h(|u'|) dx + \sum_{x \in J_u^b \cup \Gamma} g(|[u](x)|, s(x)) + \ell |D^c u|(0, 1), \quad (5.2)$$

where the elastic energy density h is defined in (2.9).

Proof. In order to handle the boundary conditions, we work in a larger open interval I containing $[0, 1]$, for instance $I := (-1, 2)$, and we extend the boundary values as $b(x) = b(0)$ for $x \leq 0$, $b(x) = b(1)$ for $x \geq 1$. We consider the functional

$$\Psi_0(u) := \begin{cases} \int_0^1 |u'|^2 dx + \sum_{x \in J_u^b \cup \Gamma} g(|[u](x)|, s(x)) & \text{if } u \in \text{SBV}(I), \\ \infty & \text{otherwise in } \text{BV}(I). \end{cases} \quad \begin{matrix} u = b \text{ on } I \setminus (0, 1), \\ \end{matrix}$$

The statement is equivalent to proving that the relaxation of the functional Ψ_0 with respect to the weak*-topology of $\text{BV}(I)$ is given by

$$\Psi(u) := \int_0^1 h(|u'|) dx + \sum_{x \in J_u^b \cup \Gamma} g(|[u](x)|, s(x)) + \ell |D^c u|(0, 1) \quad (5.3)$$

if $u \in \text{BV}(I)$ with $u = b$ on $I \setminus (0, 1)$, $\Psi(u) = \infty$ otherwise in $\text{BV}(I)$.

Step 1: lower semicontinuity of Ψ . Let $u_n \in \text{BV}(I)$ be a sequence converging weakly* in BV to some function u ; in particular, $u_n \rightarrow u$ in $L^1(I)$ and $Du_n \xrightarrow{*} Du$ in the sense of measures. We can also assume without loss of generality that $u_n = b$ on $I \setminus (0, 1)$ for every n , and that $\Psi(u_n)$ has a limit as $n \rightarrow \infty$. Since Ψ can be obtained as the supremum of functionals corresponding to a finite set Γ , we may also assume that $\Gamma = \{x_1, \dots, x_m\}$, with $s_i = s(x_i)$.

We now isolate the possible jumps of u_n at the points $x_i \in \Gamma$: namely, we consider the atomic measures

$$\mu_n := \sum_{i=1}^m [u_n](x_i) \delta_{x_i},$$

and we can assume that $[u_n](x_i) \rightarrow a_i$ for suitable values $a_i \in \mathbb{R}$, or equivalently

$$\mu_n \xrightarrow{*} \mu := \sum_{i=1}^m a_i \delta_{x_i}. \quad (5.4)$$

We then consider the function $v_n \in \text{BV}(I)$ defined by $v_n(x) := b(0) + (Du_n - \mu_n)((-1, x))$. Notice in particular that v_n is continuous at the points $x_i \in \Gamma$, $i = 1, \dots, m$. Moreover $v_n \rightarrow v$ weakly*

in BV, with $v \in \text{BV}(I)$ satisfying $Du = Dv + \mu$. With this decomposition we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Psi(u_n) &\geq \liminf_{n \rightarrow \infty} \left[\int_0^1 h(|v'_n|) dx + \sum_{x \in J_{v_n}} g(|[v_n](x)|, 0) + \ell|D^c v_n|(0, 1) \right] \\ &\quad + \liminf_{n \rightarrow \infty} \sum_{i=1}^m g(|[u_n](x_i)|, s_i). \end{aligned}$$

By standard results the first term on the right-hand side is lower semicontinuous with respect to weak*-convergence in BV, see for instance [6, Theorem 5.2]; therefore, using also (5.4) and the continuity of g ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Psi(u_n) &\geq \left[\int_0^1 h(|v'|) dx + \sum_{x \in J_v} g(|[v](x)|, 0) + \ell|D^c v|(0, 1) \right] + \sum_{i=1}^m g(|a_i|, s_i) \\ &= \int_0^1 h(|u'|) dx + \sum_{x \in J_u \setminus \Gamma} g(|[u](x)|, 0) + \ell|D^c u|(0, 1) \\ &\quad + \sum_{i=1}^m \left(g(|[v](x_i)|, 0) + g(|a_i|, s_i) \right). \end{aligned}$$

Finally, as $[u](x_i) = [v](x_i) + a_i$, using the subadditivity property of g , we can bound from below the last sum by $\sum_{i=1}^m g(|[u](x_i)|, s_i)$, and in turn we recover the desired inequality $\liminf_{n \rightarrow \infty} \Psi(u_n) \geq \Psi(u)$.

Step 2: relaxation. For all open sets $A \subset I$ we consider the localized versions $\Psi_0(\cdot; A)$ and $\Psi(\cdot; A)$ of Ψ_0 and Ψ respectively. We denote by $\bar{\Psi}_0(\cdot; A)$ the relaxation of $\Psi_0(\cdot; A)$ with respect to the weak*-topology of BV. Notice that this coincides with the lower semicontinuous envelope of $\Psi_0(\cdot; A)$ with respect to the strong topology of L^1 , that is

$$\bar{\Psi}_0(u; A) = \inf \left\{ \liminf_{n \rightarrow \infty} \Psi_0(u_n; A) : u_n \rightarrow u \text{ in } L^1(A) \right\}$$

(see Remark 5.3 below). The thesis amounts to show that $\bar{\Psi}_0 = \Psi$.

We further remark that since g is nondecreasing in the first argument, $\sum_{x \in \Gamma} g(0, s(x)) \leq \Psi(u) \leq \Psi_0(u)$ for all u . In particular, we can assume that $\sum_{x \in \Gamma} g(0, s(x)) < \infty$ for the rest of the proof.

By the lower semicontinuity of Ψ proved in the previous step, the inequality $\Psi \leq \bar{\Psi}_0$ is immediate. We therefore show the opposite inequality. By the same argument as in [12, Proposition 3.3] we have that for every fixed $u \in \text{BV}(I)$ the set function $\bar{\Psi}_0(u; \cdot)$ is the restriction to the family of open subsets of I of a regular Borel measure.

Notice that for every $u \in \text{SBV}(I)$ with $u = b$ on $I \setminus (0, 1)$ and every open set $A \subset I$ we have, using the subadditivity of g ,

$$\Psi_0(u; A) \leq \int_A |u'|^2 dx + \sum_{x \in J_u \cap A} g(|[u](x)|, 0) + \sum_{x \in \Gamma \cap A} g(0, s(x)).$$

As a consequence of [12, Theorem 3.1] we obtain for every $u \in \text{BV}(I)$ with $u = b$ on $I \setminus (0, 1)$ and every open set $A \subset I$

$$\bar{\Psi}_0(u; A) \leq \int_A h(|u'|) dx + \ell|D^c u|(A) + \sum_{x \in J_u \cap A} g(|[u](x)|, 0) + \sum_{x \in \Gamma \cap A} g(0, s(x)).$$

In particular, from the previous inequality it follows that

$$\bar{\Psi}_0(u; \cdot) \llcorner (I \setminus \Gamma) \leq h(|u'|) \mathcal{L}^1 + \ell|D^c u| + \sum_{x \in J_u \setminus \Gamma} g(|[u](x)|, 0) \delta_x. \quad (5.5)$$

It remains to evaluate $\bar{\Psi}_0(u; \cdot) \llcorner \Gamma$. Without loss of generality we may assume that $u \in \text{BV}(I)$ with $u = b$ on $I \setminus (0, 1)$. Let $K \subset \Gamma$ be any finite subset of Γ . For $\varepsilon > 0$ we take an open set $A_\varepsilon \subset I$ with $K \subset A_\varepsilon$ and

$$|Du|(A_\varepsilon \setminus K) < \varepsilon, \quad \sum_{x \in A_\varepsilon \cap \Gamma \setminus K} g(0, s(x)) < \varepsilon.$$

We can construct a sequence of functions $u_h \in \text{SBV}(I)$, with $u_h = b$ on $I \setminus (0, 1)$, such that u_h is piecewise constant, $|Du_h|(A_\varepsilon \setminus K) \leq |Du|(A_\varepsilon \setminus K) < \varepsilon$, and $u_h \rightarrow u$ strongly in $L^\infty(I)$. Then

$$\begin{aligned} \bar{\Psi}_0(u; A_\varepsilon) &\leq \liminf_{h \rightarrow \infty} \Psi_0(u_h; A_\varepsilon) = \liminf_{h \rightarrow \infty} \sum_{x \in A_\varepsilon \cap (J_{u_h} \cup \Gamma)} g(|[u_h](x)|, s(x)) \\ &\leq \liminf_{h \rightarrow \infty} \left(\sum_{x \in K} g(|[u_h](x)|, s(x)) + \sum_{x \in A_\varepsilon \cap \Gamma \setminus K} g(0, s(x)) + \sum_{x \in (J_{u_h} \setminus K) \cap A_\varepsilon} g(|[u_h](x)|, 0) \right) \\ &\leq \sum_{x \in K} g(|[u](x)|, s(x)) + \sum_{x \in A_\varepsilon \cap \Gamma \setminus K} g(0, s(x)) + \ell \limsup_{h \rightarrow 0} |Du_h|(A_\varepsilon \setminus K) \\ &\leq \sum_{x \in K} g(|[u](x)|, s(x)) + (1 + \ell)\varepsilon, \end{aligned}$$

where we used the subadditivity of g in the second line, and the inequality $g(s, 0) \leq \ell s$ in the third line. By letting $\varepsilon \rightarrow 0$ we obtain

$$\bar{\Psi}_0(u; K) \leq \sum_{x \in K} g(|[u](x)|, s(x)),$$

and since K is an arbitrary finite subset of Γ we conclude that

$$\bar{\Psi}_0(u; \cdot) \llcorner \Gamma \leq \sum_{x \in \Gamma} g(|[u](x)|, s(x)) \delta_x. \quad (5.6)$$

The combination of (5.5) and (5.6) yields the inequality $\bar{\Psi}_0 \leq \Psi$ and concludes the proof of the theorem. \square

Remark 5.2. Observe that the functional Φ is also the lower semicontinuous envelope of

$$\tilde{\Phi}_0(u) := \begin{cases} \int_0^1 |u'|^2 dx + \sum_{x \in J_u^0 \cup \Gamma} g(|[u](x)|, s(x)) & \text{if } u \in \text{SBV}(0, 1) \\ & \text{with } \mathcal{H}^0(J_u) < \infty, \\ \infty & \text{otherwise,} \end{cases}$$

which is finite only on functions with a finite number of jumps. Indeed, the functions of the recovery sequence constructed in the second step of the proof of Theorem 5.1 are piecewise constant and therefore they satisfy the additional constraint.

Remark 5.3. Notice that the relaxation $\bar{\Phi}$ of Φ_0 with respect to the weak*-topology of BV coincides with the relaxation of Φ_0 in the L^1 -topology, which is given by

$$\Phi^1(u) := \inf \left\{ \liminf_{n \rightarrow \infty} \Phi_0(u_n) : u_n \rightarrow u \text{ in } L^1(0, 1) \right\}. \quad (5.7)$$

Indeed, the inequality $\Phi^1 \leq \bar{\Phi}$ is obvious. For the converse, first notice that the same estimate as in (3.6) gives that for every sequence $(u_n)_n$ with $\sup_n \|u_n\|_\infty < \infty$ and $\sup_n \Phi_0(u_n) < \infty$ one also has $\sup_n |Du_n|(0, 1) < \infty$. Then, given any sequence $u_n \rightarrow u$ in $L^1(0, 1)$ with $\sup_n \Phi_0(u_n) < \infty$, consider the truncation $u_n^M := (-M) \vee (u_n \wedge M)$, for $M > 0$ sufficiently large. By the previous observation we have $u_n^M \rightarrow u^M$ in the weak*-topology of BV, and therefore

$$\bar{\Phi}(u^M) \leq \liminf_{n \rightarrow \infty} \Phi_0(u_n^M) \leq \liminf_{n \rightarrow \infty} \Phi_0(u_n).$$

By passing to the limit as $M \rightarrow \infty$, the inequality $\bar{\Phi} \leq \Phi^1$ follows.

6 Static phase-field approximation: the cohesive energy of pristine material

The goal of this and the next section is to derive a cohesive energy density $g(s, s')$ from a (static) phase-field approximation, and to show that the function obtained in this way satisfies the assumptions in Section 2. In this section we focus on the cohesive energy g_0 of “pristine” materials, defined in (6.6): this corresponds to the function $g(s, 0)$ in the notation of Section 2, i.e. it is the energy at points which were not previously fractured. The definition of g_0 is based on the phase-field approximation proved in [20], which we state in Section 6.3.

6.1 A class of cohesive energies g_0 for pristine material

Let f be a function with the following properties:

$$f \in C^1([0, 1]; [0, \infty)) \text{ is nondecreasing, } f^{-1}(0) = \{0\}, \quad (6.1)$$

$$f_1(s) := sf(1-s) \text{ is strictly decreasing, and } f_1(\sqrt{\cdot}) \text{ is convex,} \quad (6.2)$$

$$f_1(s) = \ell - \ell_1 s + o(s) \quad \text{as } s \rightarrow 0^+, \quad (6.3)$$

for some $\ell, \ell_1 > 0$, where $\lim_{s \rightarrow 0^+} o(s)/s = 0$. The assumptions (6.2)–(6.3) are of technical nature and are slightly stronger than the corresponding assumption in [20]: we need to include them in order to guarantee further properties of the surface energy density g_0 defined below. For instance, prototype pairs (f, f_1) with these properties are

$$f^a(s) := \frac{\ell s}{1-s} \quad \text{and} \quad f_1^a(s) := \ell(1-s), \quad \text{for any } \ell > 0, \quad (6.4)$$

(with $\ell_1 = \ell$ and $s \mapsto f_1^a(\sqrt{s}) = \ell(1 - \sqrt{s})$ convex) as well as

$$f^b(s) := \frac{(\ell + b(1-s))s^2}{1-s} \quad \text{and} \quad f_1^b(s) := (\ell + bs)(1-s)^2, \quad \text{for any } \ell > 0, b \in (-\ell, 2\ell), \quad (6.5)$$

(with $\ell_1 = 2\ell - b$ and $s \mapsto f_1^b(\sqrt{s}) = (\ell + b\sqrt{s})(1 - \sqrt{s})^2$ convex).

We now introduce a function g_0 , depending on f , which will turn out to be a cohesive energy density for a pristine material, i.e., using the notation of Section 2, will satisfy $g_0(\cdot) = g(\cdot, 0)$, for a suitable g . The relation of the functions f and g_0 to the phase-field model is discussed in Section 6.3 below, see in particular (6.16), (6.17) and (6.18). We define $g_0 : [0, \infty) \rightarrow [0, \infty)$ as follows (see Figure 2):

$$g_0(s) := \inf_{(\alpha, \beta) \in \mathcal{U}_1} \mathcal{G}_s(\alpha, \beta), \quad (6.6)$$

where

$$\mathcal{G}_s(\alpha, \beta) := \int_{-\infty}^{\infty} \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt, \quad (6.7)$$

$$\mathcal{U}_1 := \left\{ (\alpha, \beta) \in H_{\text{loc}}^1(\mathbb{R}) \times H_{\text{loc}}^1(\mathbb{R}) : \alpha' \in L^1(\mathbb{R}), \left| \int_{-\infty}^{\infty} \alpha'(t) dt \right| = 1, \right. \\ \left. 0 \leq \beta \leq 1, \lim_{|t| \rightarrow \infty} \beta(t) = 1 \right\}. \quad (6.8)$$

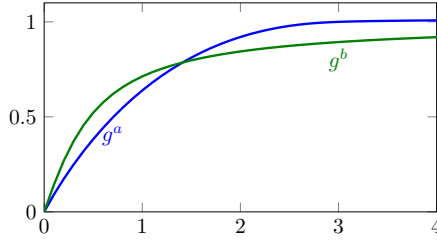


Figure 2: Graph of the function g_0 for two different choices of f . Specifically, g_0^a is generated using the function f^a defined in (6.4) with $\ell = 1$, and obeys $g_0^a(s) = 1$ for $s \geq s_{\text{frac}}$ with $s_{\text{frac}} \sim 3$; g_0^b is generated using the function f^b defined in (6.5) with $\ell = 1.5$, $\ell_1 = 0.2$ (hence $b = 2.8$), and obeys $g_0^b < 1$ for all $s \in [0, \infty)$.

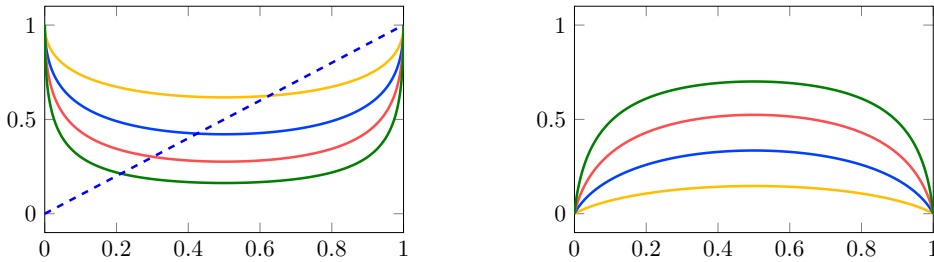


Figure 3: Left: Graph of an optimal pair (α, β) from the characterization of g_0 in (6.10), for $f_1^a(s) := \ell(1 - s)$, $\ell = 1$, as in (6.4). The dashed line is $\alpha(t) = t$, the full curves are β with (from top to bottom) $s = 0.5$, $s = 1$, $s = 1.5$, $s = 2$. Right: Graph of an optimal function γ from the characterization of g_0 in (6.11). From bottom to top, $s = 0.5$, $s = 1$, $s = 1.5$, $s = 2$.

6.2 Main properties of g_0

In this section we summarize the main properties satisfied by the function g_0 , defined by (6.6). Since the proofs are rather technical, we postpone all of them to Section 8.1.

We first observe that the following equivalent characterizations hold for g_0 :

$$g_0(s) = \inf_{T>0} \inf \left\{ \int_{-T}^T \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : \alpha, \beta \in H^1(-T, T), \right. \\ \left. \alpha(-T) = 0, \alpha(T) = 1, 0 \leq \beta \leq 1, \beta(\pm T) = 1 \right\} \quad (6.9)$$

$$= \inf \left\{ \int_0^1 |1 - \beta| \sqrt{s^2 f^2(\beta) |\alpha'|^2 + |\beta'|^2} dt : (\alpha, \beta) \in H^1(0, 1) \times H^1(0, 1), \right. \\ \left. \alpha(0) = 0, \alpha(1) = 1, 0 \leq \beta \leq 1, \beta(0) = \beta(1) = 1 \right\} \quad (6.10)$$

$$= \inf \left\{ \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt : \gamma \in W_0^{1,1}([0, 1], [0, 1]) \right\}. \quad (6.11)$$

For a proof, see Proposition 8.1 and Proposition 8.3. See Figure 3 for the qualitative behaviour of the optimal profiles in the minimum problems (6.10) and (6.11). We remark that $(1 - \beta)f(\beta) = f_1(1 - \beta)$ by (6.3) is continuous at $\beta = 1$, so that the integrand in (6.10) is interpreted as $s\ell|\alpha'|$ on the set $\{\beta = 1\}$. Notice also that the infimum in (6.10) is invariant under reparametrization of the interval $(0, 1)$.

The next theorem shows that g_0 enjoys the same properties as the function $g(\cdot, 0)$ in Section 2.

The interpretation of g_0 as a function of type $g(\cdot, 0)$, for a suitable function g , will be clear in Section 7.

Theorem 6.1. *Let g_0 be defined by (6.6) with a function f satisfying properties (6.1), (6.2), (6.3). Then g_0 satisfies the following properties:*

- (i) g_0 is monotone nondecreasing;
- (ii) $g_0(s) \leq 1 \wedge \ell s$ and $\lim_{s \rightarrow \infty} g_0(s) = 1$;
- (iii) there exists $\tilde{\ell} > 0$ such that

$$g_0(s) = \ell s - \tilde{\ell} s^{5/3} + o(s^{5/3}) \quad \text{as } s \rightarrow 0^+; \quad (6.12)$$

- (iv) for every $s_1, s_2 \geq 0$,

$$g_0(s_1 + s_2) \leq g_0(s_1) + g_0(s_2); \quad (6.13)$$

if in addition $s_1, s_2 > 0$, the inequality is strict;

- (v) g_0 is Lipschitz continuous with Lipschitz constant ℓ .

The function g_0 satisfies $g_0(s) \in [0, 1]$ for all s . Many properties of g_0 are different if $g_0(s) < 1$ or $g_0(s) = 1$. Therefore we define

$$s_{\text{frac}} := \sup\{s : g_0(s) < 1\} \in (0, \infty]. \quad (6.14)$$

The existence and the properties of optimal profiles in the minimum problem (6.6) are discussed in the following theorem. In particular we show that, in the case $g_0(s) < 1$, the infimum in (6.6) is attained by an optimal pair (α_s, β_s) , and that the minimum value m_s of the optimal profile β_s is uniquely determined by s , and is a monotone function of s .

Theorem 6.2. *Let g_0 be defined by (6.6) with a function f satisfying properties (6.1), (6.2), (6.3), and let s_{frac} be defined as in (6.14). Then the following hold:*

- (i) For all $s < s_{\text{frac}}$ one has $g_0(s) < 1$, and there exists an optimal pair $(\alpha_s, \beta_s) \in \mathcal{U}_1$ such that $g_0(s) = \mathcal{G}_s(\alpha_s, \beta_s)$; the minimizer of (6.6) is unique up to translations, in the sense that if (α_s, β_s) and $(\hat{\alpha}_s, \hat{\beta}_s)$ are minimizers then there are $a_1, t_1 \in \mathbb{R}$ such that $\alpha_s(t) = a_1 + \hat{\alpha}_s(t - t_1)$, $\beta_s(t) = \hat{\beta}_s(t - t_1)$.
- (ii) For all $s \geq s_{\text{frac}}$ one has $g_0(s) = 1$, and $g_0(s) = \mathcal{G}_s(0, \beta_s)$, where $\beta_s(t) := 1 - e^{-\frac{|t|}{2}}$.

Moreover, the value of the minimum of the optimal profile β_s ,

$$m_s := \min_{t \in \mathbb{R}} \beta_s(t), \quad (6.15)$$

is uniquely determined by s . The map $s \mapsto m_s$ is continuous, strictly decreasing in $[0, s_{\text{frac}})$, with $m_0 = 1$ and $m_s = 0$ for $s \geq s_{\text{frac}}$.

Fig. 4 shows the behavior of an optimal pair (α, β) in the sense of Theorem 6.2(i).

In the following proposition we discuss under which conditions $s_{\text{frac}} < \infty$.

Proposition 6.3. *Let $\eta := \liminf_{t \uparrow 1} \frac{f_1(t)}{1-t^2}$, where f_1 is the function defined in (6.2). If $\eta = 0$ then $g_0(s) < 1$ for all s , so that $s_{\text{frac}} = \infty$. If $\eta > 0$ and $s\eta \geq 6$ then $g_0(s) = 1$, so that $s_{\text{frac}} \leq 6/\eta$.*

We observe that the function f^a defined in (6.4) has $\eta = \frac{1}{2}\ell$ and therefore $s_{\text{frac}} \in (0, 12/\ell]$, whereas the function f^b defined in (6.5) has $\eta = 0$ and therefore $s_{\text{frac}} = \infty$. See also Figure 2.

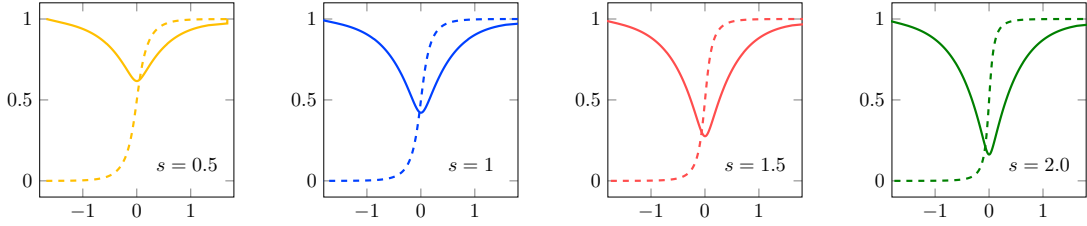


Figure 4: Graph of an optimal pair (α, β) from the definition of g_0 in (6.6) with $f_1^a(s) := \ell(1-s)$, $\ell = 1$, as in (6.4), for $s = 0.5$, $s = 1$, $s = 1.5$ and $s = 2$ (from left to right). In each graph α is dashed, β is the full curve.

6.3 Phase-field approximation and blow-up around jump points

We now clarify in which sense the function g_0 , defined in (6.6), is related to a phase-field approximation. Following [20], we introduce a family of functionals $\mathcal{F}_\varepsilon : L^1(0, 1) \times L^1(0, 1) \rightarrow [0, \infty]$, depending on a real parameter $\varepsilon > 0$, by setting

$$\mathcal{F}_\varepsilon(u, v) := \begin{cases} \int_0^1 \left(f_\varepsilon^2(v) |u'|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon |v'|^2 \right) dx & \text{if } u, v \in H^1(0, 1), \\ \infty & \text{otherwise.} \end{cases} \quad (6.16)$$

Here the function $f_\varepsilon : [0, 1] \rightarrow \mathbb{R}$ is defined as

$$f_\varepsilon(s) := 1 \wedge \varepsilon^{1/2} f(s), \quad f_\varepsilon(1) = 1, \quad (6.17)$$

where f is a fixed function satisfying the assumptions (6.1)–(6.3).

We also introduce the limit functional $\mathcal{F} : L^1(0, 1) \times L^1(0, 1) \rightarrow [0, \infty]$:

$$\mathcal{F}(u, v) := \begin{cases} \int_0^1 h(|u'|) dx + \int_{J_u} g_0(|[u]|) d\mathcal{H}^0 + \ell |D^c u|(0, 1) & \text{if } u \in \text{BV}(0, 1), \\ \infty & \text{otherwise.} \end{cases} \quad (6.18)$$

The density $h : \mathbb{R} \rightarrow [0, \infty)$ is defined as in (2.9),

$$h(\xi) := \begin{cases} \xi^2 & \text{if } 0 \leq |\xi| \leq \frac{\ell}{2}, \\ \ell |\xi| - \frac{\ell^2}{4} & \text{if } |\xi| > \frac{\ell}{2} \end{cases} \quad (6.19)$$

and $g_0 : [0, \infty) \rightarrow [0, \infty)$ is the density defined in (6.6).

We will also consider the localized versions of the functionals above, by writing $\mathcal{F}_\varepsilon(u, v; A)$ and $\mathcal{F}(u, v; A)$ when the domain $(0, 1)$ is replaced by an open set $A \subset (0, 1)$. The following Γ -convergence result is proved in [20].

Theorem 6.4 (Conti-Focardi-Iurlano [20]). *Let f obey (6.1) and (6.2). The functionals \mathcal{F}_ε defined in (6.16) Γ -converge as $\varepsilon \rightarrow 0^+$ to the functional \mathcal{F} defined in (6.18) in $L^1(0, 1) \times L^1(0, 1)$. Moreover, if $(u_\varepsilon, v_\varepsilon)$ satisfies the uniform bound*

$$\sup_\varepsilon \left(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon\|_{L^1(0,1)} \right) < \infty, \quad (6.20)$$

then there exists a subsequence $(u_{\varepsilon_k}, v_{\varepsilon_k})_k$ and a function $u \in \text{BV}(0, 1)$ such that $u_{\varepsilon_k} \rightarrow u$ almost everywhere in $(0, 1)$ and $v_{\varepsilon_k} \rightarrow 1$ in $L^1(0, 1)$.

We next discuss the behavior of a recovery sequence of \mathcal{F}_ε near to a limit jump point. In such points the material, originally pristine, develops a fracture. This investigation will motivate the introduction, in Section 7, of an irreversibility constraint on the functionals \mathcal{F}_ε , which amounts to consider the case when a fracture develops in a material that is not pristine, but already pre-fractured in some points.

We show that the blow-ups of a recovery sequence $(u_\varepsilon, v_\varepsilon)$ of \mathcal{F}_ε around a jump point \bar{x} of the limit u converge to an optimal profile for $g_0(|[u](\bar{x})|)$, that is, to an optimal pair for the problem (6.6). Motivated by Proposition 3.2, we will restrict to the case $u \in \text{SBV}(0, 1)$, with $|u'| \leq \frac{\ell}{2}$, and for simplicity we will further assume $\mathcal{H}^0(J_u) < \infty$. The proof of the theorem is given in Section 8.2.

Theorem 6.5 (Behaviour of recovery sequences). *Let f obey (6.1)–(6.3) and let \mathcal{F}_ε be defined as in (6.16)–(6.17). Let $u \in \text{SBV}(0, 1)$ have a finite number of jumps and satisfy $|u'| \leq \frac{\ell}{2}$ almost everywhere in $(0, 1)$. Let $(u_\varepsilon, v_\varepsilon)$ be a recovery sequence for \mathcal{F}_ε corresponding to u , that is, $(u_\varepsilon, v_\varepsilon) \in H^1(0, 1) \times H^1(0, 1)$, $0 \leq v_\varepsilon \leq 1$ almost everywhere, and*

$$u_\varepsilon \rightarrow u \text{ in } L^1(0, 1), \quad v_\varepsilon \rightarrow 1 \text{ in } L^1(0, 1), \quad \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow \mathcal{F}(u, 1) \quad (6.21)$$

as $\varepsilon \rightarrow 0^+$. Suppose in addition that $\sup_\varepsilon \|u_\varepsilon\|_\infty < \infty$. Then there exists a subsequence $\varepsilon_k \rightarrow 0$ with the following properties:

- (i) For every jump point $\bar{x} \in J_u$ there exist $x_k \rightarrow \bar{x}$ such that

$$w_k(x) := v_{\varepsilon_k}(x_k + \varepsilon_k x) \rightarrow \beta_{\bar{s}}(x) \quad \text{strongly in } H_{\text{loc}}^1(\mathbb{R}), \quad (6.22)$$

where $\bar{s} := |[u](\bar{x})|$ and $\beta_{\bar{s}}$ is an optimal profile for the definition of $g_0(\bar{s})$ in (6.6), in the sense of Theorem 6.2. Moreover, in the case $g_0(|[u](\bar{x})|) < 1$ we also have that

$$z_k(x) := u_{\varepsilon_k}(x_k + \varepsilon_k x) \rightarrow \bar{s}\alpha_{\bar{s}}(x) \quad \text{strongly in } H_{\text{loc}}^1(\mathbb{R}), \quad (6.23)$$

where $\alpha_{\bar{s}}$ is such that $(\alpha_{\bar{s}}, \beta_{\bar{s}}) \in \mathcal{U}_1$ and $g_0(\bar{s}) = \mathcal{G}_{\bar{s}}(\alpha_{\bar{s}}, \beta_{\bar{s}})$.

- (ii) For every $\eta > 0$, setting $I_\eta := \bigcup_{x \in J_u} (x - \eta, x + \eta)$ we have that as $\varepsilon \rightarrow 0$

$$\int_{(0,1) \setminus I_\eta} f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 dx \rightarrow \int_{(0,1) \setminus I_\eta} |u'|^2 dx, \quad \int_{(0,1) \setminus I_\eta} \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon_k} + \varepsilon |v'_\varepsilon|^2 \right) dx \rightarrow 0. \quad (6.24)$$

7 Static phase-field approximation: the cohesive energy of pre-fractured material

7.1 A class of cohesive energies g for pre-fractured material

We now introduce a new surface energy density g , depending on two variables s, s' , and detail its connection with the phase-field approximating energies. We will show in particular that the function g defined below satisfies all the assumptions in Section 2, so that for such an energy density a quasi-static evolution can be constructed, as described in Section 4.

We let f satisfy the assumptions (6.1)–(6.3), as in Section 6.1. For $s, s' \geq 0$ we set

$$g(s, s') := \inf_{(\alpha, \beta) \in \mathcal{V}_{s'}} \mathcal{G}_s(\alpha, \beta), \quad (7.1)$$

where

$$\mathcal{V}_{s'} := \left\{ (\alpha, \beta) \in \mathcal{U}_1 : \inf \beta \leq m_{s'} \right\}, \quad (7.2)$$

\mathcal{G}_s and \mathcal{U}_1 have been defined in (6.7) and (6.8) respectively, and $m_{s'}$ denotes the value of the minimum of an optimal profile for $g_0(s')$, see (6.15). The effect of the second variable s' is to introduce a “memory effect” that takes into account the maximal amplitude of the jumps at previous times. We remark that, when $s' = 0$, the constraint is not active since $m_0 = 1$, and therefore

$$g(s, 0) = g_0(s), \quad (7.3)$$

where g_0 is the function defined in (6.6). We refer to Figure 5 and Figure 6 for an illustration.

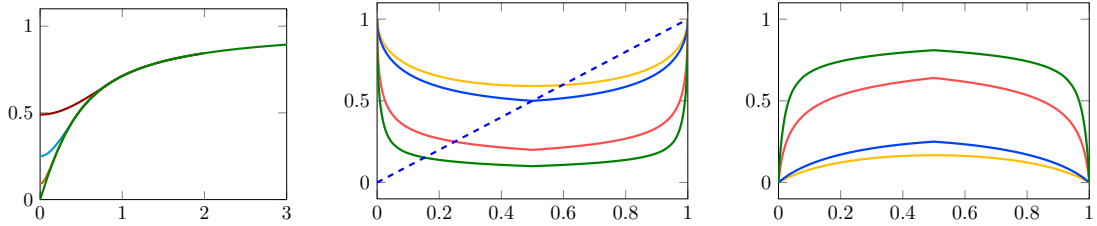


Figure 5: Left: Graph of $g_0(s)$ and $g(s, s')$ for $m_{s'} = 0.3, 0.5$ and 0.7 (from bottom to top) and $f_1^b(s)$ in (6.5) with $\ell = 1.5$, $\ell_1 = 0.2$ as in Figure 1 and Figure 2. Middle and right: Graph of (α, β) and γ from (7.5) and (7.6) for $s = 0.3$ and $m_{s'} = 0.1, 0.2, 0.5, 0.7$ and the same f_1^b corresponding to Figure 3. For $m_{s'} = 0.7$ the curves are the same as for g_0 . See also Figure 6.

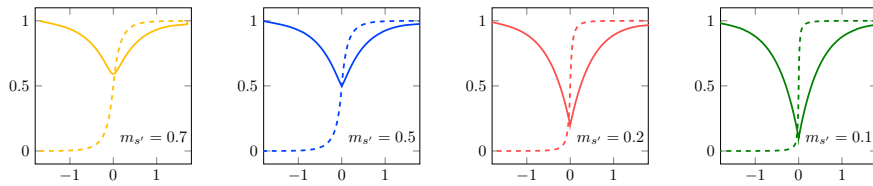


Figure 6: Graph of the optimal (α, β) entering (7.4) with a very large T , for $s = 0.3$ and $m_{s'} = 0.7, 0.5, 0.2, 0.1$, as in Figure 5. For the largest value of $m_{s'}$ one sees that the constraint is not active and β is smooth at $t = 0$; for the others the constraint is active and the derivative of β jumps. Correspondingly, the profile of α becomes more concentrated.

7.2 Main properties of g

In this section we state the main properties of g , whose definition is given in (7.1). The corresponding proofs are postponed to Section 8.3. The next theorem establishes that g enjoys all assumptions listed in Section 2.

Theorem 7.1. *Let g be defined by (7.1) with a function f satisfying properties (6.1), (6.2), (6.3). Then g satisfies hypotheses (i)–(iv) of Section 2.*

Analogously to what happens for g_0 (see Section 6.2), the following equivalent characterizations hold for g :

$$g(s, s') = \inf_{T>0} \inf \left\{ \int_{-T}^T \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : \alpha, \beta \in H^1(-T, T), \right. \\ \left. |\alpha(T) - \alpha(-T)| = 1, 0 \leq \beta \leq 1, \beta(\pm T) = 1, \inf \beta \leq m_{s'} \right\} \quad (7.4)$$

$$= \inf \left\{ \int_0^1 |1 - \beta| \sqrt{s^2 f^2(\beta) |\alpha'|^2 + |\beta'|^2} dt : (\alpha, \beta) \in H^1(0, 1) \times H^1(0, 1), \right. \\ \left. \alpha(0) = 0, \alpha(1) = 1, 0 \leq \beta \leq 1, \beta(0) = \beta(1) = 1, \inf \beta \leq m_{s'} \right\} \quad (7.5)$$

$$= \inf \left\{ \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt : \gamma \in W_0^{1,1}([0, 1], [0, 1]), \sup \gamma \geq (1 - m_{s'})^2 \right\}. \quad (7.6)$$

Extending α and β by constants it is easy to show that the inf in the first line of (7.4) is non-increasing in T . Finally, the following technical lemma holds.

Lemma 7.2. For $\mu > 0$, let $g^{(\mu)} : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be defined by

$$g^{(\mu)}(s, s') := \inf_{T>0} \inf \left\{ \int_{-T}^T \left(f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : \alpha, \beta \in H^1(-T, T), \right. \\ \left. |\alpha(T) - \alpha(-T)| = s, 0 \leq \beta \leq 1, \beta(\pm T) = 1 - \mu, \inf \beta \leq m_{s'} \right\}. \quad (7.7)$$

Then we have

$$|g(s, s') - g^{(\mu)}(s, s')| \leq 3\mu^2, \quad (7.8)$$

for every $s, s' \geq 0$.

7.3 Phase-field approximation

In this subsection we show that the new energy density g defined in (7.1) appears in the Γ -limit of the phase-field energies \mathcal{F}_ε introduced in (6.16) when we include a suitable irreversibility constraint. Precisely, below we first introduce an irreversibility constraint at level ε on the functionals \mathcal{F}_ε , in the form of a monotonicity condition on the *minimum values* of the damage variable v_ε . This choice is motivated by the blow-up analysis performed in Theorem 6.5. Then, we prove that the Γ -limit of such constrained functionals is of the form (2.8), with g given by (7.1).

We assume in the following that a pair (Γ, s) as in (2.5) is given: Γ is a finite subset of $[0, 1]$, and $s : \Gamma \rightarrow (0, \infty)$. We also fix a boundary datum $b : \{0, 1\} \rightarrow \mathbb{R}$. In order to deal with the boundary conditions, it is convenient to work in a larger open interval Ω containing $[0, 1]$, for instance $\Omega := (-1, 2)$. For $\varepsilon > 0$ let $L_\varepsilon > 0$ be such that $L_\varepsilon \rightarrow 0$ and $\frac{L_\varepsilon}{\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$. We introduce a constrained functional, defined on $L^1(\Omega) \times L^1(\Omega)$, by setting

$$\hat{\mathcal{F}}_\varepsilon(u, v; \Gamma, s, b) := \begin{cases} \mathcal{F}_\varepsilon(u, v; \Omega) & \text{if } v(x) \leq m_{s(x)} \text{ for every } x \in \Gamma, \\ & u(x) = b(0) \text{ for } x < -L_\varepsilon, u(x) = b(1) \text{ for } x > 1 + L_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \quad (7.9)$$

Here m_s denotes the minimum value of an optimal profile β_s for $g_0(s)$, see Theorem 6.2 and in particular (6.15). Recalling the definition (2.8) of the relaxed functional $\Phi(\cdot; \Gamma, s, b)$, the limit functional is defined on $L^1(\Omega) \times L^1(\Omega)$ by

$$\hat{\mathcal{F}}(u, v; \Gamma, s, b) := \begin{cases} \Phi(u; \Gamma, s, b) & \text{if } u \in \text{BV}(\Omega), v = 1 \text{ a.e.,} \\ & u(x) = b(0) \text{ for } x < 0, u(x) = b(1) \text{ for } x > 1, \\ \infty & \text{otherwise.} \end{cases} \quad (7.10)$$

The main result of this section is the following.

Theorem 7.3. Let a finite set $\Gamma \subset [0, 1]$, a map $s : \Gamma \rightarrow (0, \infty)$, and a boundary Dirichlet datum $b : \{0, 1\} \rightarrow \mathbb{R}$ be given. Then the functionals $\hat{\mathcal{F}}_\varepsilon(\cdot; \Gamma, s, b)$ Γ -converge as $\varepsilon \rightarrow 0^+$ to $\hat{\mathcal{F}}(\cdot; \Gamma, s, b)$ in $L^1(\Omega) \times L^1(\Omega)$.

In the rest of this section we drop the dependence on (Γ, s, b) in the functionals $\hat{\mathcal{F}}_\varepsilon$ and $\hat{\mathcal{F}}$ to lighten the notation, as these quantities are fixed. In order to prove the theorem we introduce the following standard notions:

$$\hat{\mathcal{F}}'(u, v) := \Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u, v) \\ := \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \text{ in } L^1(\Omega) \times L^1(\Omega) \right\}, \\ \hat{\mathcal{F}}''(u, v) := \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u, v) \\ := \inf \left\{ \limsup_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \text{ in } L^1(\Omega) \times L^1(\Omega) \right\}.$$

The proof of Theorem 7.3 follows by combining Proposition 7.4 and Proposition 7.5 below.

Proposition 7.4 (Liminf inequality). *For every $(u, v) \in L^1(\Omega) \times L^1(\Omega)$ it holds*

$$\hat{\mathcal{F}}(u, v) \leq \hat{\mathcal{F}}'(u, v). \quad (7.11)$$

Proof. The proof follows the lines of [20, Proposition 5.1], with the natural modifications required to include the additional constraint. We denote $\Gamma = \{\bar{x}_1, \dots, \bar{x}_k\}$, and $\bar{s}_i = s(\bar{x}_i)$. Fix any sequence $(u_\varepsilon, v_\varepsilon) \rightarrow (u, v)$ in $L^1(\Omega) \times L^1(\Omega)$ with $\sup_\varepsilon \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$. The proof will be achieved by showing that $u \in \text{BV}(\Omega)$, $v = 1$ almost everywhere, and

$$\hat{\mathcal{F}}(u, 1) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega). \quad (7.12)$$

By possibly passing to a subsequence, we can assume without loss of generality that the liminf in (7.12) is in fact a limit, and that the convergence of u_ε and v_ε is also pointwise almost everywhere. The uniform bound on the energy of $(u_\varepsilon, v_\varepsilon)$ gives $v = 1$ almost everywhere, and $v_\varepsilon(\bar{x}_i) \leq m_{\bar{s}_i} < 1$. Moreover, $u(x) = b(0)$ for $x < 0$, $u(x) = b(1)$ for $x > 1$.

By repeating the construction in the first part of the proof of [20, Proposition 5.1], given any $\delta > 0$ one can determine a finite number of points $S = \{t_1, \dots, t_L\} \subset \Omega$, with $\Gamma \subset S$, with the following property: for all $\eta > 0$ sufficiently small, $\eta \ll \delta$ and $\eta < 1 - m_{\bar{s}_i}$ for all i , setting

$$S_\eta := \bigcup_{i=1}^L (t_i - \eta, t_i + \eta),$$

one has for all ε sufficiently small (depending on η)

$$(1 - \omega(\delta)) \int_{\Omega \setminus S_\eta} h(|u'_\varepsilon|) dx \leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega \setminus S_\eta) \quad (7.13)$$

where $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ is a modulus of continuity.

The uniform bound on the energies $\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon)$, together with (7.13), yields

$$\sup_\eta \sup_\varepsilon \int_{\Omega \setminus S_\eta} |u'_\varepsilon| dx < \infty,$$

and therefore $u \in \text{BV}(\Omega \setminus S_\eta)$ for all $\eta > 0$; in turn we have $u \in \text{BV}(\Omega)$ by the finiteness of S , and by lower semicontinuity we obtain

$$(1 - \omega(\delta)) \hat{\mathcal{F}}(u, 1; \Omega \setminus S_\eta) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega \setminus S_\eta). \quad (7.14)$$

We now estimate the contribution to the energy coming from the region S_η . We first consider the points $\bar{x}_i \in \Gamma$, $i = 1, \dots, k$, which in particular belong to S (recall that $\Gamma \subset S$). Let $J_\eta^i := (\bar{x}_i - \eta, \bar{x}_i + \eta)$. We introduce another small parameter $\mu > 0$ and we choose $x_1, x_2 \in J_\eta^i$, $x_1 < \bar{x}_i < x_2$, with the following properties:

$$v_\varepsilon(x_1) \rightarrow 1, \quad v_\varepsilon(x_2) \rightarrow 1, \quad (7.15)$$

$$u_\varepsilon(x_1) \rightarrow u(x_1), \quad u_\varepsilon(x_2) \rightarrow u(x_2), \quad (7.16)$$

$$|u(x_1) - u^-(\bar{x}_i)| < \mu, \quad |u(x_2) - u^+(\bar{x}_i)| < \mu. \quad (7.17)$$

We define $I := (x_1, x_2)$ and denote by C_ε the connected component of the set

$$\{x \in I : v_\varepsilon(x) < 1 - \eta\} \quad (7.18)$$

containing \bar{x}_i , and by C_ε^j the connected components (different from C_ε) of the same set (7.18) in which v_ε achieves the value $1 - \delta$ (recall that $\eta \ll \delta$, $\eta < 1 - m_{\bar{s}_i}$ and (7.15)). In each such

component C_ε^j we have, denoting by y its first endpoint and by z an interior point in which $v_\varepsilon(z) = 1 - \delta$,

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; C_\varepsilon^j) &\geq \int_y^z \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|v'_\varepsilon|^2 \right) dx \geq \int_y^z (1-v_\varepsilon)|v'_\varepsilon| dx \\ &\geq \frac{1}{2}(1-v_\varepsilon(z))^2 - \frac{1}{2}(1-v_\varepsilon(y))^2 = \frac{\delta^2 - \eta^2}{2} \geq \frac{1}{4}\delta^2.\end{aligned}$$

Therefore the number N_ε of the components C_ε^j is uniformly bounded by $\frac{c}{\delta^2}$, where c is a constant independent of ε . Moreover in each set C_ε^j and in C_ε we have $f_\varepsilon(v_\varepsilon) = \sqrt{\varepsilon}f(v_\varepsilon)$ for ε small, as $v_\varepsilon < 1 - \eta$. Recalling that $\bar{x}_i \in C_\varepsilon$ and $v_\varepsilon(\bar{x}_i) \leq m_{\bar{s}_i}$, we can now compute using the rescaling $\alpha_\varepsilon(t) := u_\varepsilon(\bar{x}_i + \varepsilon t)$, $\beta_\varepsilon(t) := v_\varepsilon(\bar{x}_i + \varepsilon t)$

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; C_\varepsilon) &= \int_{\frac{1}{\varepsilon}(C_\varepsilon - \bar{x}_i)} \left(f^2(\beta_\varepsilon)|\alpha'_\varepsilon|^2 + \frac{(1-\beta_\varepsilon)^2}{4} + |\beta'_\varepsilon|^2 \right) dt \\ &\geq g^{(\eta)} \left(\left| \int_{C_\varepsilon} u'_\varepsilon dx \right|, \bar{s}_i \right) \geq g \left(\left| \int_{C_\varepsilon} u'_\varepsilon dx \right|, \bar{s}_i \right) - 3\eta^2,\end{aligned}\quad (7.19)$$

where $g^{(\eta)}$ is the function defined in (7.7) and the last inequality follows by Lemma 7.2. On the other components C_ε^j the same argument gives

$$\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; C_\varepsilon^j) \geq g \left(\left| \int_{C_\varepsilon^j} u'_\varepsilon dx \right|, 0 \right) - 3\eta^2 = g_0 \left(\left| \int_{C_\varepsilon^j} u'_\varepsilon dx \right| \right) - 3\eta^2,\quad (7.20)$$

where g_0 has been defined in (6.6) and satisfies (7.3). Finally, outside the selected components, that is in the set $\tilde{C}_\varepsilon := I \setminus (C_\varepsilon \cup \bigcup_{j=1}^{N_\varepsilon} C_\varepsilon^j)$, one has $v_\varepsilon \geq 1 - \delta$ and therefore an estimate analogous to (7.13) holds:

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \tilde{C}_\varepsilon) &\geq (1 - \omega(\delta)) \int_{\tilde{C}_\varepsilon} h(|u'_\varepsilon|) dx \geq (1 - \omega(\delta)) \left[\ell \int_{\tilde{C}_\varepsilon} |u'_\varepsilon| dx - \frac{\ell^2}{4} |\tilde{C}_\varepsilon| \right] \\ &\geq (1 - \omega(\delta)) g_0 \left(\left| \int_{\tilde{C}_\varepsilon} u'_\varepsilon dx \right| \right) - \frac{\ell^2}{2} \eta.\end{aligned}\quad (7.21)$$

where we used the definition (6.19) of h and Theorem 6.1(ii). By collecting (7.19)–(7.21)

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; I) &\geq g \left(\left| \int_{C_\varepsilon} u'_\varepsilon dx \right|, \bar{s}_i \right) - 3\eta^2 + \sum_{j=1}^{N_\varepsilon} g_0 \left(\left| \int_{C_\varepsilon^j} u'_\varepsilon dx \right| \right) - 3N_\varepsilon \eta^2 \\ &\quad + (1 - \omega(\delta)) g_0 \left(\left| \int_{\tilde{C}_\varepsilon} u'_\varepsilon dx \right| \right) - \frac{\ell^2}{2} \eta \\ &\geq g \left(\left| \int_{C_\varepsilon} u'_\varepsilon dx \right|, \bar{s}_i \right) + (1 - \omega(\delta)) g_0 \left(\left| \int_{I \setminus C_\varepsilon} u'_\varepsilon dx \right| \right) - \left(3 + \frac{c}{\delta^2} \right) \eta^2 - \frac{\ell^2}{2} \eta \\ &\geq (1 - \omega(\delta)) g(|u_\varepsilon(x_2) - u_\varepsilon(x_1)|, \bar{s}_i) - \frac{c}{\delta^2} \eta^2 - \frac{\ell^2}{2} \eta,\end{aligned}$$

where we used (6.13) in the second passage and (2.2) in the third one. Hence (7.16) and the continuity of g yield

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; J_\eta^i) \geq (1 - \omega(\delta)) g(|u(x_2) - u(x_1)|, \bar{s}_i) - \frac{c}{\delta^2} \eta^2 - \frac{\ell^2}{2} \eta,$$

and finally letting $\mu \rightarrow 0$

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; J_\eta^i) \geq (1 - \omega(\delta)) g(|[u]|(\bar{x}_i), \bar{s}_i) - \frac{c}{\delta^2} \eta^2 - \frac{\ell^2}{2} \eta.\quad (7.22)$$

The inequality (7.22) gives an estimate of the contribution to the energy coming from the points in Γ . For the points $t_i \in S \setminus \Gamma$, we can reproduce the argument above just removing the component C_ε (in this case the argument is the same as in the proof of [20, Proposition 5.1]) and obtain that, for $I_\eta^i = (t_i - \eta, t_i + \eta)$,

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; I_\eta^i) \geq (1 - \omega(\delta))g_0(|[u]|(\bar{x}_i)) + O(\eta). \quad (7.23)$$

Eventually we collect (7.14), (7.22), and (7.23) and we let $\eta \rightarrow 0$:

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) \geq (1 - \omega(\delta))\hat{\mathcal{F}}(u, 1).$$

The conclusion (7.12) follows by letting $\delta \rightarrow 0$. \square

Proposition 7.5 (Limsup inequality). *For every $(u, v) \in L^1(\Omega) \times L^1(\Omega)$ it holds*

$$\hat{\mathcal{F}}''(u, v) \leq \hat{\mathcal{F}}(u, v). \quad (7.24)$$

Proof. Let us consider first the case $v \equiv 1$ and $u \in \text{SBV}(0, 1)$, with $u' \in L^2(0, 1)$ and $\mathcal{H}^0(J_u) < \infty$, and recall that by assumption also $\mathcal{H}^0(\Gamma) < \infty$. By a localization argument we can further assume that Γ consists of a single point $x_0 \in [0, 1]$, and that u has at most one jump point, also located at x_0 . Indeed, in any interval which does not contain any point of Γ the conclusion follows directly by Theorem 6.4.

Let therefore $\Gamma = \{x_0\}$, $x_0 \in [0, 1]$, and let $s_0 := s(x_0)$. Let us also assume for the moment that u only takes the two values $u^\pm(x_0)$ in a neighbourhood of x_0 . Given any $\sigma > 0$, using the characterization (7.4) of g (and rescaling α) we can find $T > 0$ and $\alpha, \beta \in H^1(-T, T)$ such that $\alpha(-T) = u^-(x_0)$, $\alpha(T) = u^+(x_0)$, $0 \leq \beta \leq 1$, $\beta(\pm T) = 1$, $\beta(0) \leq m_{s_0}$, and

$$\int_{-T}^T \left(f^2(\beta)|\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt \leq g(|[u](x_0)|, s_0) + \sigma.$$

We then take as recovery sequence

$$u_\varepsilon(x) := \begin{cases} \alpha\left(\frac{x-x_0}{\varepsilon}\right) & \text{if } x \in A_\varepsilon, \\ u(x) & \text{if } x \in \Omega \setminus A_\varepsilon, \end{cases} \quad v_\varepsilon(x) := \begin{cases} \beta\left(\frac{x-x_0}{\varepsilon}\right) & \text{if } x \in A_\varepsilon, \\ 1 & \text{if } x \in \Omega \setminus A_\varepsilon, \end{cases}$$

where $A_\varepsilon := (x_0 - \varepsilon T, x_0 + \varepsilon T)$. Notice that, since $\frac{L_\varepsilon}{\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, we have $A_\varepsilon \subset (x_0 - L_\varepsilon, x_0 + L_\varepsilon)$ for ε small enough, and therefore u_ε satisfies the boundary conditions as in (7.9) in the case x_0 is a boundary point. It is easily seen that $u_\varepsilon \rightarrow u$ and $v_\varepsilon \rightarrow 1$ in $L^1(0, 1)$. Moreover, as $v_\varepsilon(x_0) = \beta(0) \leq m_{s_0}$ we have

$$\begin{aligned} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) &= \int_{\Omega \setminus A_\varepsilon} |u'|^2 dx + \int_{A_\varepsilon} \left(f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|v'_\varepsilon|^2 \right) dx \\ &\leq \int_0^1 |u'|^2 dx + \int_{-T}^T \left(f^2(\beta)|\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt \\ &\leq \int_0^1 |u'|^2 dx + g(|[u](x_0)|, s_0) + \sigma. \end{aligned}$$

Since σ is arbitrary we obtain

$$\hat{\mathcal{F}}''(u, 1) \leq \limsup_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \int_0^1 |u'|^2 dx + g(|[u](x_0)|, s_0). \quad (7.25)$$

In order to remove the assumption that u is locally piecewise constant in a neighbourhood of x_0 , we consider the sequence

$$u_j(x) := \begin{cases} u(x_0 - \frac{1}{j}) & \text{for } x_0 - \frac{1}{j} < x < x_0, \\ u(x_0 + \frac{1}{j}) & \text{for } x_0 < x < x_0 + \frac{1}{j}, \\ u(x) & \text{for } x \in \Omega \setminus (x_0 - \frac{1}{j}, x_0 + \frac{1}{j}). \end{cases}$$

In view of the previous discussion inequality (7.25) holds with u replaced by u_j ; since $u_j \rightarrow u$ in $L^1(0, 1)$ and $u'_j \rightarrow u'$ in $L^2(0, 1)$ as $j \rightarrow \infty$, by lower semicontinuity of $\hat{\mathcal{F}}''$ we conclude that (7.25) is still satisfied by u .

The inequality (7.25), together with Theorem 6.4 and a localization argument, proves that for every $u \in \text{SBV}(0, 1)$ with $u' \in L^2(0, 1)$ and $\mathcal{H}^0(J_u) < \infty$ we have

$$\hat{\mathcal{F}}''(u, 1) \leq \int_0^1 |u'|^2 dx + \sum_{x \in \Gamma} g(|[u](x)|, s(x)) + \sum_{x \in J^b(u) \setminus \Gamma} g_0(|[u](x)|).$$

Then the conclusion (7.24) follows since $\hat{\mathcal{F}}$ is the lower semicontinuous envelope of the right-hand side, by Theorem 5.1 and Remark 5.2. \square

Remark 7.6. *From Theorem 7.3, in the particular case $\Gamma = \emptyset$, it follows that the Γ -convergence result of [20] (see Theorem 6.4) continues to hold if we include Dirichlet boundary conditions.*

8 Static phase-field approximation: proofs

We prove in this section all properties stated in Sections 6.2 and 7.2 of the fracture energy densities g_0 and g , defined in (6.6) and (7.1) respectively. We also give the proof of the blow-up result in Section 6.3. Along this section, we work under the assumptions (6.1)–(6.3) on f .

8.1 Proof of the statements of Section 6.2

We detail a series of propositions the properties of the function g_0 , defined in (6.6). The proofs of the results stated in Section 6.2 follow by combining the propositions below.

We start by proving the two alternative representation formulas (6.9) and (6.10) for g_0 , and by discussing the existence and the properties of optimal profiles for the minimum problem (6.6).

Proposition 8.1. *The following properties hold.*

- (i) *The equivalent characterizations (6.9) and (6.10) hold.*
- (ii) *For $g_0(s) < 1$ the problem in (6.6) has a minimizer $(\alpha_s, \beta_s) \in \mathcal{U}_1$.*
- (iii) *Any minimizer obeys $\min \beta_s \geq 1 - \sqrt{g_0(s)}$.*
- (iv) *If $0 \leq s' < s$ and $g_0(s) < 1$, then $g_0(s') < g_0(s)$.*
- (v) *For any minimizer (α_s, β_s) there is $t_* \in \mathbb{R}$ such that β_s is nonincreasing in $(-\infty, t_*)$ and nondecreasing in (t_*, ∞) .*
- (vi) *If $g_0(s) < 1$, then for any minimizer (α_s, β_s) there are $T_-, T_+ \in \mathbb{R} \cup \{\pm\infty\}$ such that $\beta_s \in C^1((T_-, T_+); [0, 1])$ and $\alpha_s \in C^1(\mathbb{R})$, with $\beta_s = 1$ and $\alpha'_s = 0$ on $\mathbb{R} \setminus (T_-, T_+)$. The map $\alpha_s - \alpha_s(-\infty)$ is a C^1 bijection of (T_-, T_+) onto $(0, 1)$.*
- (vii) *Any minimizer (α_s, β_s) obeys*

$$s^2 f^2(\beta_s) |\alpha'_s|^2 + |\beta'_s|^2 = \frac{(1 - \beta_s)^2}{4} \text{ pointwise in } \mathbb{R}. \quad (8.1)$$

Proof. To prove (i), we show the equivalence of (6.6) and (6.9); the equivalence of (6.9) and (6.10) is proved in [20, Proposition 4.3]. Let \hat{g} denote the infimum in (6.9). Notice that the minimum problems in (6.9) are decreasing with respect to T ; the inequality $g_0(s) \leq \hat{g}(s)$ is trivial, since if $\alpha, \beta \in H^1(-T, T)$ are admissible functions in (6.9) they can be extended as constants outside $(-T, T)$ to obtain a pair in \mathcal{U}_1 .

To prove the opposite inequality, we then have to show that for every $(\alpha, \beta) \in \mathcal{U}_1$ the inequality $\hat{g}(s) \leq \mathcal{G}_s(\alpha, \beta)$ holds. Since translations and truncations of α do not increase the energy, as well

as the symmetric reflection with respect to the origin of α and β , we can assume without loss of generality that $0 \leq \alpha \leq 1$, $\lim_{t \rightarrow -\infty} \alpha(t) = 0$, $\lim_{t \rightarrow \infty} \alpha(t) = 1$.

Fix $T > 0$ and let

$$M_T := 1 - \sqrt{1 - \alpha(T)}.$$

Notice that $0 \leq M_T \leq 1$ and $\lim_{T \rightarrow \infty} M_T = 1$. We define a new pair (α_T, β_T) , admissible for the minimum problem (6.9), by modifying the functions (α, β) outside the interval $(-T, T)$ as follows:

$$\alpha_T(t) := \begin{cases} \alpha(t) & \text{if } t \in (-T, T), \\ \alpha(T) & \text{if } t \in [T, T+1), \\ \text{linear interpolation} & \text{if } t \in [T+1, T+2), \\ 1 & \text{if } t \in [T+2, \infty), \end{cases}$$

$$\beta_T(t) := \begin{cases} \beta(t) & \text{if } t \in (-T, T), \\ \text{linear interpolation} & \text{if } t \in [T, T+1), \\ M_T & \text{if } t \in [T+1, T+2), \\ \text{linear interpolation} & \text{if } t \in [T+2, T+3), \\ 1 & \text{if } t \in [T+3, \infty) \end{cases}$$

(in the interval $(-\infty, -T)$ we do a symmetric construction, with the value M_T replaced by $M'_T := 1 - \sqrt{\alpha(-T)}$). Then we have $\alpha_T, \beta_T \in H^1(-T-3, T+3)$, $\alpha_T(-T-3) = 0$, $\alpha_T(T+3) = 1$, $0 \leq \beta_T \leq 1$, $\beta_T(-T-3) = \beta_T(T+3) = 1$, and the pair (α_T, β_T) is therefore admissible in the minimum problem (6.9). Furthermore

$$\begin{aligned} \omega_+(T) &:= \int_T^{T+3} \left(s^2 f^2(\beta_T) |\alpha'_T|^2 + \frac{(1 - \beta_T)^2}{4} + |\beta'_T|^2 \right) dt \\ &= \int_{(T, T+1) \cup (T+2, T+3)} \left(\frac{(1 - \beta_T)^2}{4} + |\beta'_T|^2 \right) dt + s^2 f^2(M_T) |1 - \alpha(T)|^2 + \frac{(1 - M_T)^2}{4} \\ &= \frac{1}{4} \int_0^1 (1 - \beta(T) - (M_T - \beta(T))t)^2 dt + \frac{1}{4} \int_0^1 (1 - M_T - (1 - M_T)t)^2 dt \\ &\quad + |\beta(T) - M_T|^2 + |1 - M_T|^2 + s^2 (f(M_T)(1 - M_T))^2 (1 - \alpha(T)) + \frac{(1 - M_T)^2}{4}, \end{aligned}$$

and from the fact that $\alpha(T), \beta(T), M_T \rightarrow 1$ as $T \rightarrow \infty$ we obtain, using assumption (6.3),

$$\omega_+(T) \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Similarly

$$\omega_-(T) := \int_{-T-3}^{-T} \left(s^2 f^2(\beta_T) |\alpha'_T|^2 + \frac{(1 - \beta_T)^2}{4} + |\beta'_T|^2 \right) dt \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Therefore

$$\begin{aligned} \mathcal{G}_s(\alpha, \beta) &\geq \int_{-T}^T \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt \\ &= \int_{-T-3}^{T+3} \left(s^2 f^2(\beta_T) |\alpha'_T|^2 + \frac{(1 - \beta_T)^2}{4} + |\beta'_T|^2 \right) dt - \omega_+(T) - \omega_-(T) \\ &\geq \hat{g}(s) - \omega_+(T) - \omega_-(T) \end{aligned}$$

and the conclusion follows by sending $T \rightarrow \infty$. This concludes the proof of (i).

We now observe that for any β with $1 - \beta \in H^1(\mathbb{R})$ and any $t_0 \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} \left(\frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt \geq - \int_{-\infty}^{t_0} (1 - \beta) \beta' dt + \int_{t_0}^{\infty} (1 - \beta) \beta' dt = (1 - \beta(t_0))^2, \quad (8.2)$$

with equality if and only if $\beta(t) = 1 - (1 - \beta(t_0))e^{-|t-t_0|/2}$. Assertion (iii) follows immediately.

Let us prove (ii). Assume now that $g_0(s) < 1$, which is equivalent to $s < s_{\text{frac}}$. By equation (8.2), in the minimization it is sufficient to consider functions β which obey $\beta > 0$ almost everywhere. For a fixed β , (6.6) is a linear problem in α . One can check that the minimizer is such that $f^2(\beta)\alpha'$ is constant on the set $\{\beta < 1\}$, and that $\alpha' = 0$ on $\{\beta = 1\}$ (up to null sets). Therefore the minimizer takes the form $\alpha'(t) = cf^{-2}(\beta(t))$ for some $c = c(s, \beta) \in \mathbb{R}$, where to shorten notation we write $f^{-2}(1) = 0$. Integrating shows that $1 = c \int_{-\infty}^{\infty} f^{-2}(\beta) dt$, so that

$$\hat{\mathcal{G}}_s(\beta) := \min_{\alpha} \mathcal{G}_s(\alpha, \beta) = s^2 \left(\int_{-\infty}^{\infty} f^{-2}(\beta) dt \right)^{-1} + \int_{-\infty}^{\infty} \left(\frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt.$$

Let now $\beta_k \in H_{\text{loc}}^1(\mathbb{R}; [0, 1])$ be a minimizing sequence for $\hat{\mathcal{G}}_s$. Since $\inf \hat{\mathcal{G}}_s = g_0(s) < 1$ we can assume that there is $\delta > 0$ such that $\hat{\mathcal{G}}_s(\beta_k) \leq 1 - \delta$ for all k . By (8.2) we then obtain $1 - \delta \geq (1 - \beta_k(t))^2$ and therefore $\beta_k(t) \geq \frac{1}{2}\delta$ for all k and all t .

The sequence $1 - \beta_k$ is bounded in $H^1(\mathbb{R})$ and has therefore a subsequence converging weakly to some function $1 - \beta_s$. Since the second term in $\hat{\mathcal{G}}_s$ is convex, if we prove that $\int_{-\infty}^{\infty} f^{-2}(\beta_k) dt \rightarrow \int_{-\infty}^{\infty} f^{-2}(\beta_s) dt$ then β_s is a minimizer of $\hat{\mathcal{G}}_s$, and by the formula stated above we can reconstruct α . To prove continuity we observe that (possibly after extracting a further subsequence) $\beta_k \rightarrow \beta_s$ pointwise almost everywhere, and that (6.2) and $\beta_k(t) \geq \frac{1}{2}\delta$ imply $f(\beta_k) = f_1(1 - \beta_k)/(1 - \beta_k) \geq f_1(1 - \frac{1}{2}\delta)/(1 - \beta_k)$ and therefore $0 \leq f^{-2}(\beta_k) \leq C_s(1 - \beta_k)^2$ pointwise. Since $\int_{-\infty}^{\infty} (1 - \beta_k)^2 dt \leq 4$ and $1 - \beta_k \rightarrow 1 - \beta$ in $L^2(\mathbb{R})$, by Fatou's Lemma we obtain

$$\int_{-\infty}^{\infty} f^{-2}(\beta_s) dt = \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f^{-2}(\beta_k) dt.$$

Therefore β_s is a minimizer of $\hat{\mathcal{G}}_s$. This concludes the proof of (ii).

Assume now that $s \in (0, \infty)$, $g_0(s) < 1$ and $s' \in [0, s)$. Let β_s be a minimizer of $\hat{\mathcal{G}}_s$. By the above estimates $\int_{-\infty}^{\infty} f^{-2}(\beta_s) dt \leq C_s \int_{-\infty}^{\infty} (1 - \beta_s)^2 dt < \infty$ and therefore $g_0(s') \leq \hat{\mathcal{G}}_{s'}(\beta_s) < \hat{\mathcal{G}}_s(\beta_s) = g_0(s)$. This concludes the proof of (iv).

In order to prove (v), it suffices to show that for any $m \in [0, 1]$ the set $I_m := \{t : \beta_s(t) \leq m\}$ is an interval. This will imply the stated monotonicity properties, with t_* any point in the intersection of all I_m for $m > \inf \beta_s$. If I_m were not an interval, there would be $t_1 < t_2 \in I_m$ with $m = \beta_s(t_1) = \beta_s(t_2)$ and $\beta_s(t) > m$ for $t \in (t_1, t_2)$. We can construct an admissible competitor $(\alpha, \beta) \in \mathcal{U}_1$ by modifying (α_s, β_s) as follows:

$$\alpha(t) := \begin{cases} \alpha_s(t) & \text{if } t \leq t_1, \\ \alpha_s(t_1) + \frac{\alpha_s(t_2) - \alpha_s(t_1)}{\delta}(t - t_1) & \text{if } t_1 < t < t_1 + \delta, \\ \alpha_s(t + t_2 - t_1 - \delta) & \text{if } t \geq t_1 + \delta, \end{cases}$$

$$\beta(t) := \begin{cases} \beta_s(t) & \text{if } t \leq t_1, \\ m & \text{if } t_1 < t < t_1 + \delta, \\ \beta_s(t + t_2 - t_1 - \delta) & \text{if } t \geq t_1 + \delta, \end{cases}$$

where $\delta > 0$ is to be chosen. Then, by using Young's inequality and by (6.2),

$$\begin{aligned} \mathcal{G}_s(\alpha_s, \beta_s) - \mathcal{G}_s(\alpha, \beta) &= \int_{t_1}^{t_2} \left(s^2 f^2(\beta_s) |\alpha'_s|^2 + \frac{(1 - \beta_s)^2}{4} + |\beta'_s|^2 \right) dt \\ &\quad - \int_{t_1}^{t_1 + \delta} \left(s^2 f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt \\ &> \int_{t_1}^{t_2} s f(\beta_s) (1 - \beta_s) |\alpha'_s| dt - \frac{s^2 f^2(m) |\alpha_s(t_2) - \alpha_s(t_1)|^2}{\delta} - \frac{(1 - m)^2}{4} \delta \\ &\geq s f(m) (1 - m) |\alpha_s(t_2) - \alpha_s(t_1)| - \frac{s^2 f^2(m) |\alpha_s(t_2) - \alpha_s(t_1)|^2}{\delta} - \frac{(1 - m)^2}{4} \delta. \end{aligned}$$

By optimizing in δ we see that the right-hand side in the previous inequality is zero, and this contradicts the minimality of (α_s, β_s) . This concludes the proof of (v).

Let us prove (vi). Let $g_0(s) < 1$ and choose an optimal pair $(\alpha_s, \beta_s) \in \mathcal{U}_1$ for $g_0(s)$. By (iii) and (6.1) we have $f(\beta_s) \geq f(1 - \sqrt{g_0(s)}) > 0$ on \mathbb{R} . Let $T_- := \inf\{t : \beta_s(t) < 1\} \in \mathbb{R} \cup \{-\infty\}$, $T_+ := \sup\{t : \beta_s(t) < 1\} \in \mathbb{R} \cup \{\infty\}$. By (v) we have $\beta_s < 1$ in (T_-, T_+) , by finiteness of the integral we have $\alpha'_s = 0$ in $\mathbb{R} \setminus (T_-, T_+)$. By taking variations in the variable α in the minimum problem (6.6) we find, as in the proof of (ii),

$$f^2(\beta_s)\alpha'_s = \text{const.} \quad \text{almost everywhere in } (T_-, T_+).$$

Therefore $\alpha'_s > 0$ in (T_-, T_+) with $\lim_{t \uparrow T_+} = \lim_{t \downarrow T_-} \alpha'_s(t) = 0$, and in particular $\alpha_s : (T_-, T_+) \rightarrow (\alpha_s(T_-), \alpha_s(T_+))$ is a C^1 , bijective map (with $\alpha_s(T_{\pm})$ interpreted as the limit of $\alpha_s(t)$ for $t \rightarrow \pm\infty$ if $T_{\pm} = \pm\infty$). By taking variations in the variable β we obtain instead

$$\int_{-\infty}^{\infty} \left(s^2 f(\beta_s) f'(\beta_s) |\alpha'_s|^2 \varphi - \frac{(1 - \beta_s)\varphi}{4} + \beta'_s \varphi' \right) dt = 0 \quad \text{for all } \varphi \in C_c^\infty(T_-, T_+),$$

from which it follows by standard arguments that $\beta_s \in C^1(T_-, T_+)$. Hence (vi) is proved.

Finally, taking internal variations (in the sense of considering competitors of the type $(\alpha_s(t + \varepsilon\varphi(t)), \beta_s(t + \varepsilon\varphi(t)))$, for $\varphi \in C_c^1(\mathbb{R})$) one also obtains the usual equipartition result, in the sense that the minimizer fulfills (8.1). This proves (vii). \square

We next list in the following proposition some basic properties of the function g_0 , already observed in [20, Proposition 4.1] for the exception of (v), which guarantees that the function g_0 is strictly below the linear function ℓs : for the proof of this last property, we require the additional assumption (6.3) (which is not needed for the Γ -convergence result in [20]). Notice that this condition, which is used in the proof of the strict subadditivity of g_0 (Proposition 8.4), will be significantly improved in Proposition 8.5.

Proposition 8.2. *The function g_0 defined in (6.6) enjoys the following properties:*

- (i) $g_0(0) = 0$, and g_0 is subadditive, i.e. $g_0(s_1 + s_2) \leq g_0(s_1) + g_0(s_2)$ for every $s_1, s_2 \in \mathbb{R}^+$;
- (ii) g_0 is nondecreasing, $g_0(s) \leq 1 \wedge \ell s$, and g_0 is Lipschitz continuous with Lipschitz constant ℓ ;
- (iii) $\lim_{s \rightarrow \infty} g_0(s) = 1$;
- (iv) $\lim_{s \rightarrow 0^+} \frac{g_0(s)}{s} = \ell$;
- (v) $g_0(s) < \ell s$ for all $s > 0$.

Proof. Only the statement (v) requires a new proof, the others follow from [20, Proposition 4.1] using Proposition 8.1(i). With fixed $s > 0$, let $\sigma \in (0, 1)$ and set $\alpha(t) := 0$ in $[0, \frac{1}{3}]$, $\alpha(t) := 1$ in $[\frac{2}{3}, 1]$, and the linear interpolation between these two values in $[\frac{1}{3}, \frac{2}{3}]$; set also $\beta(t) := \sigma$ in $[\frac{1}{3}, \frac{2}{3}]$, and the linear interpolation between the values σ and 1 in each of the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. By using the pair (α, β) as a competitor in the characterization (6.10) of g_0 we find

$$\begin{aligned} g_0(s) &\leq \inf_{\sigma \in (0,1)} [s(1 - \sigma)f(\sigma) + (1 - \sigma)^2] \\ &= \inf_{\sigma \in (0,1)} [s\ell - s\ell_1(1 - \sigma) + o(1 - \sigma) + (1 - \sigma)^2], \end{aligned} \tag{8.3}$$

where we used the assumption (6.3). The strict inequality $g_0(s) < \ell s$ follows from (8.3), by choosing a σ sufficiently close to 1. \square

In the following proposition we prove the representation formula (6.11) for g_0 , which removes the invariance under reparametrization in (6.10) and has a unique minimizer. We recall the definition (6.14) of the threshold s_{frac} .

Proposition 8.3. *The following properties hold:*

- (i) *The characterization (6.11) of g_0 holds.*
- (ii) *For $s \in (0, s_{\text{frac}})$ the variational problem (6.11) has a unique minimizer γ_s . It obeys $\gamma_s \in C^1(0, 1)$, $\gamma_s(\frac{1}{2}) = \max \gamma_s = (1 - \min \beta_s)^2$, where β_s is any minimizer from Proposition 8.1(ii).*
- (iii) *If $0 < s_1 < s_2 < s_{\text{frac}}$ then $0 < \gamma_{s_1}(t) < \gamma_{s_2}(t)$ for all $t \in (0, 1)$.*
- (iv) *The function*

$$m_s := \begin{cases} 1 - \sqrt{\gamma_s(\frac{1}{2})}, & \text{if } s \in [0, s_{\text{frac}}), \\ 0, & \text{if } s > s_{\text{frac}}, \end{cases} \quad (8.4)$$

is continuous and strictly decreasing in $[0, s_{\text{frac}})$.

- (v) *For $s < s_{\text{frac}}$, the minimizer of (6.6) is unique up to translations, in the sense that if (α_s, β_s) and $(\hat{\alpha}_s, \hat{\beta}_s)$ are minimizers then there are $a_1, t_1 \in \mathbb{R}$ such that $\alpha_s(t) = a_1 + \hat{\alpha}_s(t - t_1)$, $\beta_s(t) = \hat{\beta}_s(t - t_1)$.*

Proof. (i): The identity (6.11) follows from (6.10) by reparametrization: indeed, first observe that by a density argument the infimum in (6.10) can be written as

$$\inf \left\{ \int_0^1 \sqrt{s^2(1-\beta)^2 f^2(\beta) |\alpha'|^2 + \frac{1}{4} \left| \frac{d}{dt}(1-\beta)^2 \right|^2} dt : (1-\beta)^2 \in C_c^1(0, 1), 0 \leq \beta \leq 1, \right. \\ \left. \alpha \in C^1([0, 1]), \alpha(0) = 0, \alpha(1) = 1, \alpha' > 0 \right\}. \quad (8.5)$$

Similarly, the infimum in (6.11) can be considered on the class of $\gamma \in C_c^1(0, 1)$ with $0 \leq \gamma \leq 1$. Then, for (α, β) admissible in (8.5), the reparametrization

$$\gamma(t) := (1 - \beta)^2 \circ \alpha^{-1}(t) \quad (8.6)$$

gives an admissible profile for (6.11) with

$$\int_0^1 \sqrt{s^2(1-\beta)^2 f^2(\beta) |\alpha'|^2 + \frac{1}{4} \left| \frac{d}{dt}(1-\beta)^2 \right|^2} dt = \int_0^1 \sqrt{s^2(f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt. \quad (8.7)$$

Conversely, given $\gamma \in C_c^1(0, 1)$ with $0 \leq \gamma \leq 1$, we consider the pair (α, β) defined by $\alpha(t) := t$, $\beta(t) := 1 - \sqrt{\gamma(t)}$, which is admissible in (8.5) and satisfies (8.7). This proves (i).

(ii): Assume now $g_0(s) < 1$ and let $(\alpha_s, \beta_s) \in \mathcal{U}_1$ be a minimizer for (6.6) with $\alpha_s(-\infty) = 0$, which exists by Proposition 8.1(ii). We consider the rescaled profile $\gamma_s : [0, 1] \rightarrow [0, 1]$, $\gamma_s(t) := (1 - \beta_s \circ \alpha_s^{-1}(t))^2$: we have $\gamma_s \in C^1(0, 1)$, $0 \leq \gamma_s \leq 1$, $\gamma_s(0) = \gamma_s(1) = 0$, and, with T_-, T_+ as in Proposition 8.1(vi),

$$g_0(s) = \mathcal{G}_s(\alpha_s, \beta_s) \geq \int_{T_-}^{T_+} |1 - \beta_s| \sqrt{s^2 f^2(\beta_s) |\alpha_s'|^2 + |\beta_s'|^2} dt = \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{|\gamma_s'|^2}{4}} dt,$$

which shows that γ_s is a minimizer in problem (6.11). We have $1 - \min \beta_s = \max \sqrt{\gamma_s}$ and by symmetrization the maximum is attained at the point $t = \frac{1}{2}$. Thanks to the convexity assumption (6.2), the minimizer γ_s of (6.11) is unique.

(iii) and (iv): We will exploit further properties of the optimal profile γ_s for $s < s_{\text{frac}}$: by computing the Euler-Lagrange equation satisfied by a C^1 -minimizer of problem (6.11) we find

$$\left(\frac{\gamma_s'}{2\sqrt{s^2(f_1(\sqrt{\gamma_s}))^2 + \frac{1}{4}|\gamma_s'|^2}} \right)' = \frac{s^2 f_1(\sqrt{\gamma_s}) f_1'(\sqrt{\gamma_s})}{\sqrt{\gamma_s} \sqrt{s^2(f_1(\sqrt{\gamma_s}))^2 + \frac{1}{4}|\gamma_s'|^2}} \quad \text{in } (0, 1),$$

which yields the C^2 -regularity of γ_s in $(0, 1)$ and, after elementary computations,

$$\gamma_s'' = \frac{f_1(\sqrt{\gamma_s})f_1'(\sqrt{\gamma_s})}{\sqrt{\gamma_s}} \left(2s^2 + \frac{|\gamma_s'|^2}{(f_1(\sqrt{\gamma_s}))^2} \right) \quad \text{in } (0, 1). \quad (8.8)$$

Furthermore, the equation (8.8) has a first integral:

$$(f_1(\sqrt{\gamma_s}))^2 = c \sqrt{s^2(f_1(\sqrt{\gamma_s}))^2 + \frac{1}{4}|\gamma_s'|^2} \quad \text{in } (0, 1)$$

for a constant $c > 0$, which can be computed by imposing that γ_s has a maximum at the point $t = \frac{1}{2}$ with value $(1 - m_s)^2$: this gives $c = \frac{1}{s}f_1(1 - m_s)$ and in turn

$$\gamma_s' = \pm 2sf_1(\sqrt{\gamma_s}) \sqrt{\left(\frac{f_1(\sqrt{\gamma_s})}{f_1(1 - m_s)} \right)^2 - 1} \quad \text{in } (0, 1) \quad (8.9)$$

(with positive sign in $(0, \frac{1}{2})$, and negative sign in $(\frac{1}{2}, 1)$).

By using the previous equations we can now show the continuity of the map $s \mapsto m_s$. Indeed, consider a sequence $s_n \rightarrow s \in [0, s_{\text{frac}})$ and the corresponding optimal profiles γ_{s_n} . In view of (8.9), the first derivatives γ_{s_n}' are uniformly bounded in $L^\infty(0, 1)$, hence possibly extracting a subsequence we have $\gamma_{s_n} \rightarrow \gamma$ weakly* in $W^{1,\infty}(0, 1)$ (and uniformly), for some γ vanishing at the boundary. By continuity of g_0 we find that γ is the unique minimizer in problem (6.11) for $g_0(s)$; in particular $(1 - m_s)^2 = \gamma(\frac{1}{2}) = \lim_n \gamma_{s_n}(\frac{1}{2}) = \lim_n (1 - m_{s_n})^2$, which proves the continuity of m_s .

We then show that the map $s \mapsto m_s$ is injective (then also the strict monotonicity follows). Assume by contradiction that for two different values $s_1 < s_2$ one has $m_{s_1} = m_{s_2}$, and let $\gamma := \gamma_{s_2} - \gamma_{s_1}$ be the difference of the corresponding optimal profiles. By (8.9) we have $\gamma'(0) > 0$, and hence $\gamma(t) > 0$ for all sufficiently small t . On the other hand, at the point $t = \frac{1}{2}$ by assumption $\gamma(\frac{1}{2}) = \gamma_{s_1}'(\frac{1}{2}) = \gamma_{s_2}'(\frac{1}{2}) = 0$, therefore we find using (8.8)

$$\gamma''(\frac{1}{2}) = 2(s_2^2 - s_1^2) \frac{f_1(1 - m_{s_1})f_1'(1 - m_{s_1})}{1 - m_{s_1}} < 0$$

(by the assumption (6.2)). We conclude that $\gamma(t) < 0$ for t sufficiently close to $\frac{1}{2}$. Let then $\bar{t} \in (0, \frac{1}{2})$ be the smallest value with $\gamma(\bar{t}) = 0$, that is $\gamma_{s_1}(\bar{t}) = \gamma_{s_2}(\bar{t}) =: \sigma$. By (8.9)

$$\gamma'(\bar{t}) = 2(s_2 - s_1)f_1(\sqrt{\sigma}) \sqrt{\left(\frac{f_1(\sqrt{\sigma})}{f_1(1 - m_{s_1})} \right)^2 - 1} > 0,$$

which is a contradiction.

At this point we conclude the proof of (iii). Fix two values $s_1 < s_2 < s_{\text{frac}}$ and let $\gamma := \gamma_{s_2} - \gamma_{s_1}$ be the difference of the corresponding optimal profiles. Then $m_{s_2} < m_{s_1}$, hence $f_1(1 - m_{s_2}) < f_1(1 - m_{s_1})$ and by (8.9) we have $\gamma'(0) > 0$, and hence $\gamma(t) > 0$ for all sufficiently small t . If $\gamma(t) > 0$ for all $t \in (0, 1)$, we are done. Otherwise, let $\bar{t} \in (0, 1)$ be the smallest value with $\gamma(\bar{t}) = 0$. Then (8.9) implies $\gamma'(\bar{t}) > 0$, which is a contradiction.

(v): Let (α_s, β_s) and $(\hat{\alpha}_s, \hat{\beta}_s) \in \mathcal{U}_1$ be minimizers for (6.6). Without loss of generality we can assume $\alpha_s(-\infty) = \hat{\alpha}_s(-\infty) = 0$. By Proposition 8.1(vi), α_s is a C^1 bijection from (T_-, T_+) onto $(0, 1)$; α_s and β_s are constant on $(-\infty, T_-)$ and (T_+, ∞) (if these intervals are nonempty). We define $\gamma \in C^1(0, 1)$ by $\gamma := (1 - \beta_s)^2 \circ \alpha_s^{-1}$, so that $\gamma(t) \rightarrow 0$ as $t \rightarrow 0$ and $t \rightarrow 1$. Then γ is a competitor in (6.11). Using first equipartition, that was proven in (8.1), and then the change of

variables $r := \alpha_s(t)$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left(s^2 f^2(\beta_s) |\alpha'_s|^2 + \frac{(1-\beta_s)^2}{4} + |\beta'_s|^2 \right) dt &= \int_{-\infty}^{\infty} \sqrt{(1-\beta_s)^2 s^2 f^2(\beta_s) |\alpha'_s|^2 + (1-\beta_s)^2 |\beta'_s|^2} dt \\ &= \int_{T_-}^{T_+} \sqrt{s^2 f_1^2(\sqrt{\gamma} \circ \alpha_s) |\alpha'_s|^2 + \frac{1}{4} |\gamma'|^2 \circ \alpha_s |\alpha'_s|^2} dt \\ &= \int_0^1 \sqrt{s^2 f_1^2(\sqrt{\gamma}) + \frac{1}{4} |\gamma'|^2} dr, \end{aligned}$$

so that γ is a minimizer of (6.11). By uniqueness, the pair $(\hat{\alpha}_s, \hat{\beta}_s)$ has to produce the same γ , so that $\beta_s \circ \alpha_s^{-1} = \hat{\beta}_s \circ \hat{\alpha}_s^{-1}$. The function $\varphi := \hat{\alpha}_s^{-1} \circ \alpha_s$ is a C^1 bijection from (T_-, T_+) onto (\hat{T}_-, \hat{T}_+) . Then $\beta_s = \hat{\beta}_s \circ \varphi$, $\alpha_s = \hat{\alpha}_s \circ \varphi$. Recalling the equipartition condition (8.1),

$$\begin{aligned} 0 &= s^2 f^2(\beta_s) |\alpha'_s|^2 + |\beta'_s|^2 - \frac{(1-\beta_s)^2}{4} \\ &= [s^2 f^2(\hat{\beta}_s) |\hat{\alpha}'_s|^2 + |\hat{\beta}'_s|^2] \circ \varphi |\varphi'|^2 - \frac{(1-\hat{\beta}_s)^2}{4} \circ \varphi = \frac{(1-\hat{\beta}_s)^2}{4} \circ \varphi [|\varphi'|^2 - 1] \end{aligned}$$

where in the last step we used the equipartition condition (8.1) for $(\hat{\alpha}_s, \hat{\beta}_s)$. We conclude that $|\varphi'| = 1$ wherever $\hat{\beta}_s(\varphi) < 1$, which implies $\varphi(t) = t + t_1$ for some $t_1 \in \mathbb{R}$ and concludes the proof. \square

We remark that (α, β) minimizing (6.10) is not unique, as the variational problem (6.10) is invariant under reparametrization. However, if one fixes α , for example by setting $\alpha(t) = t$, then for $s < s_{\text{frac}}$ one easily obtains uniqueness of the corresponding β . Indeed, any minimizer β produces an optimal γ via (8.6). Analogously, once α is fixed, monotonicity of $s \mapsto \gamma_s(t)$ implies monotonicity of β with respect to s .

Proposition 8.4 (Strict subadditivity). *For every $s > 0$ and $t \in (0, 1)$ the function g_0 satisfies the inequality $g_0(ts) > tg_0(s)$. In particular g_0 is strictly subadditive: for every $s_1, s_2 > 0$*

$$g_0(s_1 + s_2) < g_0(s_1) + g_0(s_2).$$

Proof. Let $s > 0$, $t \in (0, 1)$, $\bar{s} = ts$. If $g_0(\bar{s}) = 1$, then $tg_0(s) < 1 = g_0(\bar{s})$ and the first assertion is proven. Otherwise, by Proposition 8.1(ii) there is $(\alpha_{\bar{s}}, \beta_{\bar{s}}) \in \mathcal{U}_1$ such that $\mathcal{G}_{\bar{s}}(\alpha_{\bar{s}}, \beta_{\bar{s}}) = g_0(\bar{s})$. Then by rescaling $\tilde{\alpha}(y) := \alpha_{\bar{s}}(ty)$, $\tilde{\beta}(y) := \beta_{\bar{s}}(ty)$ we have $(\tilde{\alpha}, \tilde{\beta}) \in \mathcal{U}_1$ and therefore

$$\begin{aligned} g_0(\bar{s}) &= \int_{-\infty}^{\infty} \left(t^2 s^2 f^2(\beta_{\bar{s}}) |\alpha'_{\bar{s}}|^2 + \frac{(1-\beta_{\bar{s}})^2}{4} + |\beta'_{\bar{s}}|^2 \right) dx \\ &= \int_{-\infty}^{\infty} \left(s^2 f^2(\tilde{\beta}) |\tilde{\alpha}'|^2 + \frac{(1-\tilde{\beta})^2}{4} + \frac{|\tilde{\beta}'|^2}{t^2} \right) t dy \\ &= t \mathcal{G}_s(\tilde{\alpha}, \tilde{\beta}) + \left(\frac{1}{t} - t \right) \int_{-\infty}^{\infty} |\tilde{\beta}'|^2 dy \\ &\geq tg_0(s) + \left(\frac{1}{t} - t \right) t \int_{-\infty}^{\infty} |\beta'_s|^2 dy > tg_0(s), \end{aligned}$$

where the last inequality follows by $\beta_{\bar{s}} \neq 0$. This proves the first part of the statement. By writing

$$g_0(s_1) + g_0(s_2) = g_0\left((s_1 + s_2) \frac{s_1}{s_1 + s_2}\right) + g_0\left((s_1 + s_2) \frac{s_2}{s_1 + s_2}\right) > g_0(s_1 + s_2)$$

the strict subadditivity of g_0 follows. \square

In the following proposition we perform a careful asymptotic analysis of the function $g_0(s)$ as $s \rightarrow 0^+$. Notice that, in view of Proposition 8.2, g_0 is differentiable at the origin, with positive and finite slope; we now compute the second order correction by means of a Γ -convergence expansion in the minimum problems (6.11).

Proposition 8.5. *There exists $\tilde{\ell} > 0$ such that*

$$g_0(s) = \ell s - \tilde{\ell} s^{5/3} + o(s^{5/3}) \quad \text{as } s \rightarrow 0^+. \quad (8.10)$$

Moreover, if γ_s is a minimizer in problem (6.11), then the sequence $s^{-4/3}\gamma_s$ converges uniformly in $[0, 1]$ as $s \rightarrow 0^+$ to the unique solution $\bar{\eta} \in H_0^1(0, 1)$ of the minimum problem

$$\min \left\{ H(\eta) : \eta \in H_0^1(0, 1), \eta \geq 0 \right\} \quad (8.11)$$

where

$$H(\eta) := \int_0^1 \left(-\frac{\ell_1}{\ell} \sqrt{\eta} + \frac{|\eta'|^2}{8\ell^2} \right) dt. \quad (8.12)$$

In particular

$$(1 - m_s)^2 = \max_{t \in [0, 1]} \gamma_s(t) \leq cs^{4/3} \quad (8.13)$$

for all $s > 0$ small enough and for a uniform constant $c > 0$.

We remark that the minimum in (8.11) can be found explicitly, and leads to

$$\tilde{\ell} = -\ell \min H(\bar{\eta}) = \ell_1^{4/3} \ell^{1/3} \frac{3}{10} \left(\frac{3}{2} \right)^{2/3} \simeq 0.4 \ell_1^{4/3} \ell^{1/3}. \quad (8.14)$$

The computation is straightforward but somewhat cumbersome, we do not report it for brevity since this result is not needed for what follows.

Proof. To guess the order of the second term in the expansion of g_0 as $s \rightarrow 0$, we perform a Taylor expansion in the energy of a minimizer γ_s for (6.11). Let γ_s be an optimal profile for problem (6.11), as in Proposition 8.3(ii). Using $\max \gamma_s = \gamma_s(\frac{1}{2}) = (1 - m_s)^2$ and Proposition 8.2(ii) gives

$$\ell s \geq g_0(s) \geq \frac{1}{2} \int_0^1 |\gamma_s'| dt = (1 - m_s)^2,$$

and therefore

$$0 \leq \gamma_s(t) \leq \ell s \quad \text{for all } t \in [0, 1]. \quad (8.15)$$

Furthermore, by the Euler-Lagrange equation (8.9) satisfied by γ_s we further deduce for s small, using (8.15) and (6.3), the bound on the derivative

$$\begin{aligned} |\gamma_s'(t)| &= 2s f_1(\sqrt{\gamma_s}) \sqrt{\left(\frac{f_1(\sqrt{\gamma_s})}{f_1(1 - m_s)} \right)^2 - 1} \\ &\leq 2s \frac{f_1(\sqrt{\gamma_s})}{f_1(1 - m_s)} \sqrt{2\ell \ell_1 (1 - m_s - \sqrt{\gamma_s}) + o(1 - m_s)} \leq c_0 s^{\frac{5}{4}} \end{aligned} \quad (8.16)$$

for all $t \in [0, 1]$, for some positive constant c_0 , independent of s . Since γ_s converges to zero uniformly as $s \rightarrow 0$, we can formally expand the energy of γ_s by using (6.3) and neglecting higher order terms:

$$\begin{aligned} g_0(s) &= \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{|\gamma_s'|^2}{4}} dt \sim \int_0^1 \sqrt{s^2 \ell^2 - 2s^2 \ell \ell_1 \sqrt{\gamma_s} + \frac{|\gamma_s'|^2}{4}} dt \\ &\sim s\ell + s\ell \int_0^1 \left(-\frac{\ell_1}{\ell} \sqrt{\gamma_s} + \frac{|\gamma_s'|^2}{8s^2 \ell^2} \right) dt. \end{aligned}$$

For s small, the two terms in the last integral are of the same order if $\gamma_s \sim s^{4/3}$, so that, in turn, we might expect $g_0(s) - \ell s \sim s^{5/3}$.

The previous formal considerations can be made rigorous by means of an asymptotic development by Γ -convergence. The expected decay of minimizers of (6.11) suggests to consider the rescaling $\eta(t) := s^{-\frac{4}{3}}\gamma(t)$: with this position the minimum problem (6.11) can be reformulated as

$$g_0(s) = \inf \left\{ \int_0^1 \sqrt{s^2(f_1(s^{2/3}\eta^{1/2}))^2 + \frac{s^{8/3}|\eta'|^2}{4}} : \eta \in W_0^{1,1}(0,1), 0 \leq \eta \leq s^{-4/3} \right\}. \quad (8.17)$$

We compute the Γ -limit as $s \rightarrow 0$, with respect to the strong topology of $L^1(0,1)$, of the family of functionals defined by

$$H_s(\eta) := \frac{1}{s^{2/3}} \left(\frac{1}{s\ell} \int_0^1 \sqrt{s^2(f_1(s^{2/3}\eta^{1/2}))^2 + \frac{s^{8/3}|\eta'|^2}{4}} dt - 1 \right) \quad (8.18)$$

if $\eta \in W_0^{1,1}(0,1)$, $0 \leq \eta \leq \ell s^{-1/3}$, and $|\eta'| \leq c_0 s^{-1/12}$, and $H_s(\eta) := \infty$ otherwise in $W_0^{1,1}(0,1)$. Notice that the previous bounds are satisfied by the (rescaled) minimizer γ_s , in view of (8.15)–(8.16). We will prove that the Γ -limit is given by the functional H defined as the expression in (8.12) for $\eta \in H_0^1(0,1)$, $\eta \geq 0$, and $H(\eta) := \infty$ otherwise in $W_0^{1,1}(0,1)$.

We first prove the Γ -liminf inequality, together with the compactness of sequences with bounded energy with respect to the weak topology of $H^1(0,1)$. Without loss of generality, we consider a subsequence $\eta_s \in W_0^{1,1}(0,1)$, not relabeled, such that $0 \leq \eta_s \leq \ell s^{-1/3}$, $|\eta'_s| \leq c_0 s^{-1/12}$, and $\lim_{s \rightarrow 0} H_s(\eta_s) < \infty$. Since $s^{2/3}\eta_s^{1/2} \leq \ell^{1/2}s^{1/2}$, by (6.3) we have

$$f_1(s^{2/3}\eta_s^{1/2}) = \ell - \ell_1 s^{2/3}\eta_s^{1/2} + o(s^{2/3}\eta_s^{1/2}),$$

and therefore

$$\begin{aligned} H_s(\eta_s) &= \frac{1}{s^{2/3}} \left(\frac{1}{s\ell} \int_0^1 \sqrt{s^2(\ell - \ell_1 s^{2/3}\eta_s^{1/2} + o(s^{2/3}\eta_s^{1/2}))^2 + \frac{s^{8/3}|\eta'_s|^2}{4}} dt - 1 \right) \\ &= \frac{1}{s^{2/3}} \left(\int_0^1 \sqrt{1 - 2\frac{\ell_1}{\ell} s^{2/3}\eta_s^{1/2} + o(s^{2/3}\eta_s^{1/2}) + \frac{s^{2/3}|\eta'_s|^2}{4\ell^2}} dt - 1 \right). \end{aligned}$$

Now, since

$$\left| -2\frac{\ell_1}{\ell} s^{2/3}\eta_s^{1/2} + o(s^{2/3}\eta_s^{1/2}) + \frac{s^{2/3}|\eta'_s|^2}{4\ell^2} \right| \leq cs^{1/2},$$

a Taylor's expansion gives

$$H_s(\eta_s) = \int_0^1 \left(-\frac{\ell_1}{\ell} \eta_s^{1/2} + \frac{|\eta'_s|^2}{8\ell^2} \right) dt + \frac{1}{s^{2/3}} \int_0^1 o(s^{2/3}\eta_s^{1/2} + s^{2/3}|\eta'_s|^2) dt. \quad (8.19)$$

Notice that, for s small enough, the second term in the right-hand side of (8.19) can be controlled by the first one:

$$H_s(\eta_s) \geq \int_0^1 \left(-\frac{2\ell_1}{\ell} \eta_s^{1/2} + \frac{|\eta'_s|^2}{16\ell^2} \right) dt.$$

Now, for any $\varepsilon > 0$ we have $\eta_s^{1/2} \leq \varepsilon \eta_s^2 + c_\varepsilon$ by Young's inequality; therefore, using also Poincaré's inequality we end up with

$$H_s(\eta_s) \geq -\frac{2\varepsilon\ell_1}{\ell} \int_0^1 \eta_s^2 dt - \frac{2c_\varepsilon\ell_1}{\ell} + \frac{1}{16\ell^2} \int_0^1 |\eta'_s|^2 dt \geq \left(\frac{1}{16\ell^2} - \frac{2\varepsilon\ell_1}{\ell} \right) \int_0^1 |\eta'_s|^2 dt - \frac{2c_\varepsilon\ell_1}{\ell},$$

and choosing ε small enough we deduce that the sequence $(\eta_s)_s$ is uniformly bounded in $H_0^1(0,1)$. Up to subsequences, $\eta_s \rightharpoonup \eta$ weakly in $H^1(0,1)$ (and uniformly) for some $\eta \in H_0^1(0,1)$, $\eta \geq 0$. By (8.19) we also deduce that

$$\liminf_{s \rightarrow 0} H_s(\eta_s) \geq \int_0^1 \left(-\frac{\ell_1}{\ell} \eta^{1/2} + \frac{|\eta'|^2}{8\ell^2} \right) dt = H(\eta).$$

As for the Γ -limsup inequality, fixed $\eta \in C_c^1(0, 1)$, $\eta \geq 0$, we simply define $\eta_s := \eta$. Hence, (8.19) gives for s small enough

$$H_s(\eta_s) = \int_0^1 \left(-\frac{\ell_1}{\ell} \eta^{1/2} + \frac{|\eta'|^2}{8\ell^2} \right) dt + o(1) = H(\eta) + o(1),$$

where $o(1)$ depends on η and tends to 0 as $s \rightarrow 0$. The Γ -limsup inequality follows by density of $C_c^1(0, 1)$ in $H_0^1(0, 1)$.

The existence of a minimizer $\bar{\eta} \in H_0^1(0, 1)$ for the limit functional H follows by a standard application of the direct method of the Calculus of Variations, and its uniqueness by strict convexity of the functional H . We can now check that the value of the minimum of H is strictly negative: by considering $\lambda \sin(\pi t)$ as a competitor, with $\lambda > 0$, we have

$$\min_{\eta \in W_0^{1,1}(0,1)} H(\eta) \leq -\frac{\ell_1}{\ell} \lambda^{1/2} \int_0^1 \sin^{1/2}(\pi t) dt + \frac{\lambda^2}{8\ell^2} \int_0^1 \pi^2 \cos^2(\pi t) dt,$$

which is strictly negative if λ is small enough. We denote $\tilde{\ell} := -\ell \min H > 0$. By (8.17) and standard properties of Γ -convergence we obtain the convergence of the minimizers of H_s to $\bar{\eta}$ and

$$\frac{g_0(s) - \ell s}{\ell s^{5/3}} = \min_{\eta \in W_0^{1,1}(0,1)} H_s(\eta) \rightarrow \min_{\eta \in W_0^{1,1}(0,1)} H(\eta) = -\frac{\tilde{\ell}}{\ell} \quad \text{as } s \rightarrow 0^+,$$

so that (8.10) and (8.13) follow. \square

We are now in position to give the proofs of the results in Section 6.2, which essentially collect the main properties of g_0 from the previous statements.

Proof of Theorem 6.1. The result follows by combining Proposition 8.2, Proposition 8.4, and Proposition 8.5. \square

Proof of Theorem 6.2. The first assertion follows from Proposition 8.1(ii) and Proposition 8.3(v). The second follows from monotonicity (Proposition 8.2(ii)) and an explicit computation (see also (8.2)). The remaining assertions follow from Proposition 8.3. \square

Proof of Proposition 6.3. First we observe that $\eta = \liminf_{t \uparrow 1} \frac{f_1(\sqrt{t})}{1-t}$, that by convexity of $f_1(\sqrt{\cdot})$ the lim inf is actually a limit, and that, using convexity, $f_1(0) = \ell$ and $f_1(1) = 0$, we have

$$\eta(1-t) \leq f_1(\sqrt{t}) \leq \ell(1-t) \quad \text{for all } t \in [0, 1],$$

which implies $\eta \in [0, \ell]$ and $f_1(1-b) \geq \eta(1-(1-b)^2) = \eta(2b-b^2)$ for all $b \in [0, 1]$. We also observe that, using (6.2), the integrand in (6.10) can be estimated by

$$\begin{aligned} (1-\beta) \sqrt{s^2 f^2(\beta)(\alpha')^2 + (\beta')^2} &= \sqrt{s^2 f_1^2(1-\beta)(\alpha')^2 + \frac{1}{4} \left(\frac{d}{dt} (1-\beta)^2 \right)^2} \\ &\geq \sqrt{s^2 \eta^2 (2\beta - \beta^2)^2 (\alpha')^2 + \frac{1}{4} \left(\frac{d}{dt} (1-\beta)^2 \right)^2}. \end{aligned} \quad (8.20)$$

To prove the first assertion we construct a competitor for the characterization of g_0 in (6.10). Assume $\eta = 0$ and fix $s > 0$. By definition of η we can find $q_s \in (0, 1)$ such that $s f_1(q_s) \leq \frac{1}{2}(1-q_s^2)$. We let $\beta = 1 - q_s$ in $(\frac{1}{3}, \frac{2}{3})$, $\beta = 1$ in $\{0, 1\}$, and the linear interpolation in between; we let $\alpha = 0$ in $(0, \frac{1}{3})$, $\alpha = 1$ in $(\frac{2}{3}, 1)$, and the linear interpolation in between. A straightforward computation shows that

$$g_0(s) \leq \int_0^1 \sqrt{s^2 f_1^2(1-\beta)(\alpha')^2 + \frac{1}{4} \left(\frac{d}{dt} (1-\beta)^2 \right)^2} dt = q_s^2 + s f_1(q_s) \leq \frac{1}{2}(1+q_s^2) < 1.$$

We now turn to the second assertion. We assume $s\eta \geq 6$, let $\alpha \in H^1(0, 1)$, $\beta \in H^1(0, 1)$ be test functions in the characterization (6.10) of g_0 , denote by $I(\alpha, \beta)$ the integral in (6.10), and prove $I(\alpha, \beta) \geq 1$.

Assume first that there is $q \in (0, \frac{1}{2}]$ such that $s\eta \int_{\{q \leq \beta < 2q\}} |\alpha'| dt \geq 2$. Then (8.20) yields

$$\begin{aligned} \int_0^1 (1 - \beta) \sqrt{s^2 f^2(\beta) (\alpha')^2 + (\beta')^2} dt &\geq \int_{\{q \leq \beta < 2q\}} s\eta (2\beta - \beta^2) |\alpha'| dt + \int_{\{2q \leq \beta \leq 1\}} \frac{1}{2} \left| \frac{d}{dt} (1 - \beta)^2 \right| dt \\ &\geq 2(2q - q^2) + (1 - 2q)^2 = 1 + 2q^2 > 1 \end{aligned}$$

and the proof is concluded.

We let $\beta_{\min} := \min \beta$. If $\beta_{\min} = 0$ then necessarily $I(\alpha, \beta) \geq 1$ and we are done. We can therefore assume $\beta_{\min} > 0$. We also define, for $j \in \mathbb{N}$, $\beta_j := \min\{2^j \beta_{\min}, 1\}$ and $E_j := \{t : \beta_j \leq \beta < \beta_{j+1}\}$. We let J be the largest index with $\beta_j < 1$ and observe that (E_j) , $j = 0, \dots, J$, is a partition of $(0, 1)$, with $\beta(\inf E_j) = \beta(\sup E_j) = \beta_{j+1}$ and $\min \beta(E_j) = \beta_j$. We define $a_j := \int_{E_j} |\alpha'| dt$, which obey $\sum_j a_j \geq 1$, and

$$g_j := \int_{E_j} (1 - \beta) \sqrt{s^2 f^2(\beta) (\alpha')^2 + (\beta')^2} dt \geq \int_{E_j} \sqrt{s^2 \eta^2 (2\beta_j - \beta_j^2)^2 (\alpha')^2 + \frac{1}{4} \left(\frac{d}{dt} (1 - \beta)^2 \right)^2} dt.$$

We first treat g_J . We observe that

$$I(\alpha, \beta) \geq g_J \geq \int_{E_J} s\eta (2\beta_J - \beta_J^2) |\alpha'| dt = s\eta (2\beta_J - \beta_J^2) a_J \geq s\eta \frac{3}{4} a_J,$$

so that if $s\eta a_J \geq \frac{4}{3}$ we are done. Therefore we can assume $s\eta a_J < \frac{4}{3}$ in the following.

By Jensen's inequality, for any functions $h, k : E_j \rightarrow \mathbb{R}$ we have $|E_j|^{-1} \int_{E_j} \sqrt{h^2 + k^2} dt \geq \sqrt{(|E_j|^{-1} \int_{E_j} h dt)^2 + (|E_j|^{-1} \int_{E_j} k dt)^2}$ and therefore $\int_{E_j} \sqrt{h^2 + k^2} dt \geq \sqrt{(\int_{E_j} |h| dt)^2 + (\int_{E_j} |k| dt)^2}$. For $j < J$, $\int_{E_j} \frac{1}{2} \left| \frac{d}{dt} (1 - \beta)^2 \right| dt \geq (1 - \beta_j)^2 - (1 - \beta_{j+1})^2 = (\beta_{j+1} - \beta_j)(2 - \beta_{j+1} - \beta_j) = \beta_j(2 - 3\beta_j)$. Therefore, again for $j < J$,

$$g_j \geq \sqrt{s^2 \eta^2 \beta_j^2 (2 - \beta_j)^2 a_j^2 + (\beta_j(2 - 3\beta_j))^2} = \beta_j(2 - 3\beta_j) \sqrt{1 + x_j^2}$$

with $x_j := (s\eta a_j)(2 - \beta_j)/(2 - 3\beta_j)$. Recalling $s\eta a_j \leq 2$ and $\beta_j \leq \frac{1}{2}$ we obtain $x_j \in [0, 6]$. We use the fact that $\sqrt{1 + x^2} \geq 1 + \frac{1}{8}x^2$ for all $x \in [0, 6]$ and obtain, inserting first the definition of x_j and then the one of β_j ,

$$g_j \geq \beta_j(2 - 3\beta_j) \left(1 + \frac{1}{8} \frac{s^2 \eta^2 a_j^2 (2 - \beta_j)^2}{(2 - 3\beta_j)^2} \right) \geq (1 - \beta_j)^2 - (1 - \beta_{j+1})^2 + \beta_{\min} \frac{1}{8} \frac{(2 - \beta_j)^2}{2 - 3\beta_j} 2^j (s\eta a_j)^2.$$

We observe that $\frac{(2-x)^2}{2-3x} \geq 2$ for $x \in [0, \frac{1}{2}]$, sum over j , and recall that $g_J \geq (1 - \beta_J)^2$. This gives

$$\sum_j g_j \geq (1 - \beta_{\min})^2 + \beta_{\min} \frac{1}{4} \sum_{j < J} 2^j (s\eta a_j)^2.$$

We recall that we are working under the assumption that $\sum_j s\eta a_j \geq s\eta \geq 6$ and $s\eta a_J \leq \frac{4}{3}$, which imply $4 \leq \sum_{j < J} s\eta a_j$. By Hölder's inequality,

$$16 \leq \left(\sum_{j < J} s\eta a_j \right)^2 \leq \sum_{j < J} 2^j (s\eta a_j)^2 \cdot \sum_{j < J} 2^{-j} \leq 2 \sum_{j < J} 2^j (s\eta a_j)^2.$$

Therefore $\sum_{j < J} 2^j (s\eta a_j)^2 \geq 8$, so that $I(\alpha, \beta) = \sum_j g_j \geq (1 - \beta_{\min})^2 + 2\beta_{\min} = 1 + \beta_{\min}^2 > 1$. This concludes the proof. \square

8.2 Proof of Theorem 6.5 in Section 6.3

This section is devoted to the proof of Theorem 6.5, which will be achieved through a sequence of lemmas. In the following we assume that u , u_ε and v_ε satisfy the assumption of Theorem 6.5. We start by (i). Since this assertion is local, we can assume without loss of generality that the jump set of u consists of a single point $\bar{x} \in (0, 1)$, $J_u = \{\bar{x}\}$. We let

$$I_\eta := (\bar{x} - \eta, \bar{x} + \eta),$$

where $\eta > 0$ is always supposed to be small enough so that $I_\eta \subset (0, 1)$. Notice that, since the endpoints of I_η are not in the jump set of u , one has the localized convergence

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; I_\eta) = \mathcal{F}(u, 1; I_\eta).$$

In the following, we will work with a fixed subsequence $\varepsilon_k \rightarrow 0^+$, $k \rightarrow \infty$; to lighten the notation we will replace the subscript ε_k by k (thus, for instance, we set $u_k := u_{\varepsilon_k}$, $v_k := v_{\varepsilon_k}, \dots$). We further assume that for this subsequence $u_k \rightarrow u$ and $v_k \rightarrow 1$ almost everywhere in $(0, 1)$, and choose a continuous representative for u_k and v_k .

Lemma 8.6. *For \mathcal{L}^1 -a.e. η sufficiently small we have $u_k(\bar{x} \pm \eta) \rightarrow u(\bar{x} \pm \eta)$, $v_k(\bar{x} \pm \eta) \rightarrow 1$, and, if $x_k \in \bar{I}_\eta$ is a minimum point of v_k in the interval I_η , in the sense that*

$$m_k := \inf_{x \in I_\eta} v_k(x) = v_k(x_k), \quad (8.21)$$

then the following properties hold:

- (i) $f_k(v_k(x_k)) = f_k(m_k) \rightarrow 0$ as $k \rightarrow \infty$;
- (ii) $\limsup_{k \rightarrow \infty} m_k < 1$;
- (iii) $x_k \rightarrow \bar{x}$ as $k \rightarrow \infty$;
- (iv) if $g_0(|[u](\bar{x})|) < 1$, then $\liminf_{k \rightarrow \infty} m_k > 0$.

Proof. Since we are assuming that $u_k \rightarrow u$ and $v_k \rightarrow 1$ almost everywhere in $(0, 1)$, the convergence at the endpoints of the interval I_η is automatically satisfied for almost every η .

(i): If for a subsequence we had $f_k(m_k) \geq \sigma > 0$, then as the energies $\mathcal{F}_k(u_k, v_k)$ are equi-bounded we would get

$$C \geq \int_{I_\eta} f_k^2(v_k) |u'_k|^2 dx \geq \sigma^2 \int_{I_\eta} |u'_k|^2 dx.$$

Therefore $u \in H^1(I_\eta)$, which is a contradiction since $\bar{x} \in J_u \cap I_\eta$.

(ii): Suppose by contradiction that for a subsequence $m_k \rightarrow 1$. For a given $\delta > 0$, we have $v_k(x) \geq 1 - \delta$ in I_η for k sufficiently large; by assumption (6.3) this implies that

$$|(1 - v_k)f(v_k) - \ell| \leq \ell\omega(\delta)$$

for some modulus of continuity $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Therefore for k large enough

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; I_\eta) &\geq \int_{I_\eta} \left(f_k^2(v_k) |u'_k|^2 + \frac{(1 - v_k)^2}{4\varepsilon_k} \right) dx \geq \int_{I_\eta} |u'_k|^2 \wedge \left(\varepsilon_k f^2(v_k) |u'_k|^2 + \frac{(1 - v_k)^2}{4\varepsilon_k} \right) dx \\ &\geq \int_{I_\eta} \left(|u'_k|^2 \wedge (1 - v_k)f(v_k) |u'_k| \right) dx \geq (1 - \omega(\delta)) \int_{I_\eta} \left(|u'_k|^2 \wedge \ell |u'_k| \right) dx \\ &\geq (1 - \omega(\delta)) \left(\ell \int_{I_\eta} |u'_k| dx - \frac{\ell^2}{4} 2\eta \right). \end{aligned} \quad (8.22)$$

Notice now that in view of (6.12) we have $g_0(s) \leq \ell s - \tilde{g}(s)$ for s small and for some continuous and strictly positive function $\tilde{g}(s) > 0$; inserting this inequality into (8.22) we find

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; I_\eta) &\geq (1 - \omega(\delta)) \left[g_0 \left(\left| \int_{I_\eta} u'_k \, dx \right| \right) + \tilde{g} \left(\left| \int_{I_\eta} u'_k \, dx \right| \right) - \frac{\ell^2 \eta}{2} \right] \\ &= (1 - \omega(\delta)) \left[g_0(|u_k(\bar{x} + \eta) - u_k(\bar{x} - \eta)|) + \tilde{g}(|u_k(\bar{x} + \eta) - u_k(\bar{x} - \eta)|) - \frac{\ell^2 \eta}{2} \right]. \end{aligned}$$

By passing to the limit as $k \rightarrow \infty$, since the left-hand side converges to the limit energy in I_η and by assumption $u_k(\bar{x} \pm \eta) \rightarrow u(\bar{x} \pm \eta)$ we have

$$\begin{aligned} \int_{I_\eta} h(|u'|) \, dx + g_0(|[u](\bar{x})|) &= \mathcal{F}(u, 1; I_\eta) \\ &\geq (1 - \omega(\delta)) \left[g_0(|u(\bar{x} + \eta) - u(\bar{x} - \eta)|) + \tilde{g}(|u(\bar{x} + \eta) - u(\bar{x} - \eta)|) - \frac{\ell^2 \eta}{2} \right]. \end{aligned}$$

Therefore, letting first $\delta \rightarrow 0$ and then in turn $\eta \rightarrow 0$, we conclude that

$$g_0(|[u](\bar{x})|) \geq g_0(|[u](\bar{x})|) + \tilde{g}(|[u](\bar{x})|) > g_0(|[u](\bar{x})|),$$

which is a contradiction.

(iii): By contradiction, assume that for a (not relabeled) subsequence $x_k \rightarrow \tilde{x} \neq \bar{x}$. Fix $\delta \in (0, \eta)$ such that $v_k(\tilde{x} + \delta) \rightarrow 1$ and $\bar{x} \notin [\tilde{x} - \delta, \tilde{x} + \delta]$. For k large enough, $x_k \in (\tilde{x} - \delta, \tilde{x} + \delta)$ and

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; (\tilde{x} - \delta, \tilde{x} + \delta)) &\geq \int_{\tilde{x} - \delta}^{\tilde{x} + \delta} \left(\frac{(1 - v_k)^2}{4\varepsilon_k} + \varepsilon_k |v'_k|^2 \right) \, dx \geq \int_{x_k}^{\tilde{x} + \delta} (1 - v_k) v'_k \, dx \\ &= \frac{1}{2} (1 - m_k)^2 - \frac{1}{2} (1 - v_k(\tilde{x} + \delta))^2. \end{aligned}$$

Notice that the liminf as $k \rightarrow \infty$ of the right-hand side is a strictly positive quantity, in view of property (ii), which does not depend on δ . On the other hand

$$\lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; (\tilde{x} - \delta, \tilde{x} + \delta)) = \mathcal{F}(u, 1; (\tilde{x} - \delta, \tilde{x} + \delta)) = \int_{\tilde{x} - \delta}^{\tilde{x} + \delta} h(|u'|) \, dx,$$

and therefore we conclude that $\int_{\tilde{x} - \delta}^{\tilde{x} + \delta} h(|u'|) \, dx \geq \frac{1}{2} \liminf_{k \rightarrow \infty} (1 - m_k)^2 > 0$. This is a contradiction since δ can be chosen arbitrarily small.

(iv): By arguing as in the previous step and using the fact that $x_k \in I_\eta$ for k large by (iii) we have the inequality

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; I_\eta) &\geq \int_{\bar{x} - \eta}^{\bar{x} + \eta} (1 - v_k) |v'_k| \, dx \geq \frac{1}{2} \int_{\bar{x} - \eta}^{x_k} [(1 - v_k)^2]' \, dx - \frac{1}{2} \int_{x_k}^{\bar{x} + \eta} [(1 - v_k)^2]' \, dx \\ &\geq (1 - m_k)^2 - \frac{1}{2} (1 - v_k(\bar{x} - \eta))^2 - \frac{1}{2} (1 - v_k(\bar{x} + \eta))^2. \end{aligned}$$

By letting $k \rightarrow \infty$, and using that $v_k(\bar{x} \pm \eta) \rightarrow 1$, we have

$$\limsup_{k \rightarrow \infty} (1 - m_k)^2 \leq \int_{I_\eta} h(|u'|) \, dx + g_0(|[u](\bar{x})|),$$

and the right-hand side is strictly smaller than one provided η is small enough. \square

In the following we fix $\eta > 0$ such that the conclusions of Lemma 8.6 hold, and we define x_k and m_k as in (8.21). Since these quantities depend on the choice of η , in the arguments below we will always let $k \rightarrow \infty$ and $\eta \rightarrow 0$ in this order. The following formula, obtained by a simple

change of variables, will be often useful: for every fixed $T > 0$ and for $k \geq k_T$, for some k_T large enough,

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; (x_k - \varepsilon_k T, x_k + \varepsilon_k T)) &= \int_{x_k - \varepsilon_k T}^{x_k + \varepsilon_k T} \left(f_k^2(v_k) |u'_k|^2 + \frac{(1 - v_k)^2}{4\varepsilon_k} + \varepsilon_k |v'_k|^2 \right) dx \\ &= \int_{-T}^T \left(\frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 + \frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx, \end{aligned} \quad (8.23)$$

where w_k and z_k are the functions defined in (6.22) and (6.23) respectively.

Lemma 8.7 (Compactness of w_k). *Up to subsequences, $w_k \rightharpoonup w$ weakly in $H^1_{\text{loc}}(\mathbb{R})$ and therefore locally uniformly for some function w such that $1 - w \in H^1(\mathbb{R})$. In particular w satisfies*

$$\lim_{|x| \rightarrow \infty} w(x) = 1, \quad w(0) = \lim_{k \rightarrow \infty} v_k(x_k) = \lim_{k \rightarrow \infty} m_k < 1.$$

Proof. The result is an immediate consequence of the uniform bound on $\|1 - w_k\|_{H^1(-T, T)}$ for every $T > 0$, which follows from (8.23) and the fact that (u_k, v_k) is a recovery sequence and has therefore equibounded energy. The fact that $w(0) < 1$ follows from Lemma 8.6(ii). \square

In order to proceed with the proof of Theorem 6.5, it is convenient to distinguish between the cases $w(0) = 0$ and $w(0) > 0$. Indeed, we will see that these conditions correspond to $g_0(|[u](\bar{x})|) = 1$ and $g_0(|[u](\bar{x})|) < 1$ respectively; according to Theorem 6.2 the behaviour of minimizing sequences for the minimum problem defining g_0 is different in the two cases.

We first consider the (easier) case $w(0) = 0$.

Lemma 8.8. *If $w(0) = 0$, then Theorem 6.5(i) holds.*

Proof. By (iv) of Lemma 8.6 it has to be $g_0(|[u](\bar{x})|) = 1$. Therefore for $\eta > 0$ and $T > 0$ we have by (8.23)

$$\begin{aligned} \mathcal{F}(u, 1; I_\eta) &= \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) \\ &\geq \liminf_{k \rightarrow \infty} \int_{-T}^T \left(\frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx \geq \int_{-T}^T \left(\frac{(1 - w)^2}{4} + |w'|^2 \right) dx, \end{aligned}$$

where the last inequality follows by lower semicontinuity with respect to the weak convergence of w_k to w in $H^1(-T, T)$, proved in Lemma 8.7. By letting first $T \rightarrow \infty$ and then $\eta \rightarrow 0$ we obtain

$$1 = g_0(|[u](\bar{x})|) = \lim_{\eta \rightarrow 0} \mathcal{F}(u, 1; I_\eta) \geq \int_{-\infty}^{\infty} \left(\frac{(1 - w)^2}{4} + |w'|^2 \right) dx.$$

As $w(0) = 0$, by Young inequality we conclude that the right-hand side of the previous expression is exactly 1 and then $w(x) = 1 - e^{-\frac{|x|}{2}}$ is the optimal profile for $g_0(|[u](\bar{x})|)$.

It remains to prove the strong convergence in $H^1_{\text{loc}}(\mathbb{R})$. To this aim, it is sufficient to show convergence of the energies: suppose by contradiction that for some $M > 0$ and $\sigma > 0$

$$\limsup_{k \rightarrow \infty} \int_{-M}^M \left(\frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx = \int_{-M}^M \left(\frac{(1 - w)^2}{4} + |w'|^2 \right) dx + \sigma. \quad (8.24)$$

Then for every $\eta > 0$ and $T > M$ one has

$$\begin{aligned} \mathcal{F}(u, 1; I_\eta) &= \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) \geq \limsup_{k \rightarrow \infty} \int_{-T}^T \left(\frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx \\ &\geq \limsup_{k \rightarrow \infty} \int_{-M}^M \left(\frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx + \liminf_{k \rightarrow \infty} \int_{[-T, T] \setminus [-M, M]} \left(\frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx \\ &\geq \int_{-T}^T \left(\frac{(1 - w)^2}{4} + |w'|^2 \right) dx + \sigma, \end{aligned}$$

so that by letting first $T \rightarrow \infty$ and then $\eta \rightarrow 0$ we end up with

$$g_0(|[u](\bar{x})|) \geq \int_{-\infty}^{\infty} \left(\frac{(1-w)^2}{4} + |w'|^2 \right) dx + \sigma,$$

which is a contradiction. This completes the proof. \square

Lemma 8.9. *If $w(0) > 0$, there is a subsequence such that z_k converges to some $z \in H_{\text{loc}}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ strongly in $H_{\text{loc}}^1(\mathbb{R})$, and w_k converges to $w \in H_{\text{loc}}^1(\mathbb{R})$ strongly in $H_{\text{loc}}^1(\mathbb{R})$. The pair $(\bar{s}^{-1}z, w)$, where $\bar{s} := [u](\bar{x})$, is a minimizer of the functional $\mathcal{G}_{|\bar{s}|}$, introduced in (6.7), in the class \mathcal{U}_1 .*

Proof. Preliminaries. We first prove the weak compactness of the sequence z_k . For every fixed $T > 0$ and for $k \geq k_T$ one has

$$\min_{[-T, T]} w_k = \min_{[x_k - \varepsilon_k T, x_k + \varepsilon_k T]} v_k = v_k(x_k) = w_k(0) \rightarrow w(0) > 0,$$

so that

$$\inf_{k \geq k_T} \min_{[-T, T]} \frac{1}{\varepsilon_k} f_k^2(w_k) = \inf_{k \geq k_T} \frac{1}{\varepsilon_k} \wedge f^2(w_k(0)) > 0.$$

By (8.23) it follows that for every $T > 0$

$$\sup_{k \geq k_T} \int_{-T}^T |z'_k|^2 dx < \infty. \quad (8.25)$$

Since by assumption the sequence z_k is also uniformly bounded in L^∞ , we can extract a further subsequence such that

$$z_k \rightharpoonup z \quad \text{weakly in } H_{\text{loc}}^1(\mathbb{R}) \quad (8.26)$$

for some function $z \in H_{\text{loc}}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Possibly after extracting a subsequence, by (8.25) we can assume that $|z'_k|^2 \mathcal{L}^1$ converges weakly in measures to a Radon measure μ . By lower semicontinuity, this implies

$$|z'|^2 \mathcal{L}^1 \leq \mu \quad \text{as measures,} \quad (8.27)$$

with $\mu(-T, T) = \int_{-T}^T |z'|^2 dx$ if and only if $z_k \rightarrow z$ strongly in $H^1(-T, T)$.

Step 1. We prove that $\mu(\{w = 1\}) = 0$ and

$$g_0(|[u](\bar{x})|) \geq \int_{\mathbb{R}} f^2(w) d\mu + \int_{-\infty}^{\infty} \left(\frac{(1-w)^2}{4} + |w'|^2 \right) dx.$$

To see this, we first observe that for almost any $T > 0$ we have $\mu(\{-T, T\}) = 0$ and, by (8.23),

$$\mathcal{F}(u, 1; I_\eta) = \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) \geq \limsup_{k \rightarrow \infty} \int_{-T}^T \left(\frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 + \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx.$$

For $\delta > 0$, we let $A_\delta := \{x \in (-T, T) : w(x) < 1 - \delta\}$. For almost every δ we have $\mu(\partial A_\delta) = 0$. By uniform convergence (Lemma 8.7), for sufficiently large k we have $w_k(x) < 1 - \frac{1}{2}\delta$ on A_δ , therefore $f(w_k) \rightarrow f(w)$ uniformly on A_δ . Since $\mu(\partial A_\delta) = 0$, $|z'_k|^2 \mathcal{L}^1 \rightharpoonup \mu$, and $f(w) \in C^0(\bar{A}_\delta)$,

$$\int_{A_\delta} f^2(w) d\mu = \lim_{k \rightarrow \infty} \int_{A_\delta} f^2(w) |z'_k|^2 dx = \lim_{k \rightarrow \infty} \int_{A_\delta} f^2(w_k) |z'_k|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{-T}^T \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 dx.$$

By monotone convergence, this implies

$$\int_{(-T, T) \cap \{w < 1\}} f^2(w) d\mu = \lim_{\delta \rightarrow 0} \int_{A_\delta} f^2(w) d\mu \leq \liminf_{k \rightarrow \infty} \int_{-T}^T \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 dx. \quad (8.28)$$

Also by uniform convergence of w_k to w , for k sufficiently large we have $1 - 2\delta \leq w_k(x)$ on $B_\delta := (-T, T) \setminus A_\delta$. Recalling (6.1), for large k we have $f^2(1 - 2\delta) \leq \varepsilon_k^{-1} \wedge f^2(w_k) = \varepsilon_k^{-1} f_k^2(w_k)$ on B_δ . Since $\mu(\partial B_\delta) = 0$,

$$f^2(1 - 2\delta)\mu(B_\delta) = f^2(1 - 2\delta) \lim_{k \rightarrow \infty} \int_{B_\delta} |z'_k|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{B_\delta} \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 dx \leq C.$$

By monotone convergence,

$$\mu((-T, T) \cap \{w = 1\}) = \lim_{\delta \rightarrow 0} \mu(B_\delta) \leq \lim_{\delta \rightarrow 0} \frac{C}{f^2(1 - 2\delta)} = 0.$$

Since w_k converges to w weakly in $H^1(-T, T)$ (Lemma 8.7), (8.28) yields

$$\int_{(-T, T)} f^2(w) d\mu + \int_{(-T, T)} \left(\frac{(1-w)^2}{4} + |w'|^2 \right) dx \leq \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) = \mathcal{F}(u, 1; I_\eta)$$

for all T and all η . Sending first $T \rightarrow \infty$ and then $\eta \rightarrow 0$,

$$\int_{\mathbb{R}} f^2(w) d\mu + \int_{\mathbb{R}} \left(\frac{(1-w)^2}{4} + |w'|^2 \right) dx \leq g_0(|[u](\bar{x})|). \quad (8.29)$$

Step 2. By Lemma 8.7 and (8.26), for every $j \in \mathbb{N}$ we can pick $T_j > j$ such that $|w(\pm T_j) - 1|^2 \leq 2^{-j}$ and $z_k(\pm T_j) \rightarrow z(\pm T_j)$ as $k \rightarrow \infty$. Since u_k and z_k are bounded in L^∞ , possibly passing to a subsequence we have

$$z_+ := \lim_{j \rightarrow \infty} z(T_j), \quad z_- := \lim_{j \rightarrow \infty} z(-T_j). \quad (8.30)$$

We intend to show that $z_+ - z_- = [u](\bar{x})$.

To do so, we separate the part of the profile in the inner interval $(a, b) := (a_k^j, b_k^j) := (x_k - \varepsilon_k T_j, x_k + \varepsilon_k T_j)$ from the part outside (for sufficiently large k we can assume $[a - \varepsilon_k, b + \varepsilon_k] \subset I_\eta$). Specifically, we set

$$u_{k,j}^{\text{in}}(x) := \begin{cases} u_k(a) & \text{if } x \leq a, \\ u_k(x) & \text{if } a < x \leq b, \\ u_k(b) & \text{if } x > b, \end{cases} \quad u_{k,j}^{\text{out}}(x) := \begin{cases} u_k(x) - u_k(a) & \text{if } x \leq a, \\ 0 & \text{if } a \leq x \leq b, \\ u_k(x) - u_k(b) & \text{if } x > b \end{cases}$$

which implies, since $u_k(a) = z_k(-T_j) \rightarrow z(-T_j)$ and $u_k(b) = z_k(T_j) \rightarrow z(T_j)$, that

$$u_{k,j}^{\text{in}}(x) \rightarrow u_j^{\text{in}}(x) := \begin{cases} z(-T_j) & \text{if } x \leq \bar{x}, \\ z(T_j) & \text{if } x > \bar{x}, \end{cases} \quad u_{k,j}^{\text{out}} \rightarrow u_j^{\text{out}} := u - u_j^{\text{in}}$$

in $L^1(I_\eta)$ (as $k \rightarrow \infty$). Correspondingly, we define

$$v_{k,j}^{\text{in}}(x) := \begin{cases} 1 & \text{if } x \leq a - \varepsilon_k, \\ v_k(x) & \text{if } a \leq x \leq b, \\ 1 & \text{if } x \geq b + \varepsilon_k, \\ \text{affine interp.} & \text{in between,} \end{cases} \quad v_{k,j}^{\text{out}}(x) := \begin{cases} v_k(x) & \text{if } x \leq a, \\ 1 & \text{if } a + \varepsilon_k \leq x \leq b - \varepsilon_k, \\ v_k(x) & \text{if } x \geq b, \\ \text{affine interp.} & \text{in between.} \end{cases}$$

We observe that $v_{k,j}^{\text{in}} \rightarrow 1$ with $\min v_{k,j}^{\text{in}} = v_{k,j}^{\text{in}}(x_k) = m_k \rightarrow w(0) < 1$; $v_{k,j}^{\text{out}} \rightarrow 1$. By Lemma 8.7 and $|w(\pm T_j) - 1| \leq 2^{-j}$, for k sufficiently large (on a scale depending on j) we have $|v_k(a_k^j) - 1| \leq 2^{-j+1}$ and $|v_k(b_k^j) - 1| \leq 2^{-j+1}$. The energy then obeys

$$\begin{aligned} \mathcal{F}_k(u_{k,j}^{\text{in}}, v_{k,j}^{\text{in}}; I_\eta) &\leq \mathcal{F}_k(u_k, v_k; (a, b)) + c2^{-j}, \\ \mathcal{F}_k(u_{k,j}^{\text{out}}, v_{k,j}^{\text{out}}; I_\eta) &\leq \mathcal{F}_k(u_k, v_k; I_\eta \setminus (a, b)) + c2^{-j}, \end{aligned}$$

so that

$$\mathcal{F}_k(u_{k,j}^{\text{in}}, v_{k,j}^{\text{in}}; I_\eta) + \mathcal{F}_k(u_{k,j}^{\text{out}}, v_{k,j}^{\text{out}}; I_\eta) \leq \mathcal{F}_k(u_k, v_k; I_\eta) + c2^{-j}.$$

Passing to the limit and using (6.21),

$$\mathcal{F}(u_j^{\text{in}}, 1; I_\eta) + \mathcal{F}(u_j^{\text{out}}, 1; I_\eta) \leq \liminf_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) + c2^{-j} = \mathcal{F}(u, 1; I_\eta) + c2^{-j}$$

for all j and η . Using the explicit form of u_j^{in} , u_j^{out} and \mathcal{F} , we obtain

$$g_0(|[u_j^{\text{in}}]|) + g_0(|[u_j^{\text{out}}]|) \leq g_0(|[u]|) + c2^{-j}, \quad (8.31)$$

where all jumps are evaluated at \bar{x} . Recalling $\min v_{k,j}^{\text{in}} = v_{k,j}^{\text{in}}(x_k) = m_k \rightarrow w(0) < 1$ and $v_{k,j}^{\text{in}} = 1$ on ∂I_η ,

$$(1 - m_k)^2 \leq \int_{I_\eta} |1 - v_{k,j}^{\text{in}}| |(v_{k,j}^{\text{in}})'| dx \leq \int_{I_\eta} \left(\frac{(1 - v_{k,j}^{\text{in}})^2}{4\varepsilon_k} + \varepsilon_k |(v_{k,j}^{\text{in}})'|^2 \right) dx \leq \mathcal{F}_k(u_{k,j}^{\text{in}}, v_{k,j}^{\text{in}}; I_\eta)$$

which gives, by the same argument, first

$$(1 - w(0))^2 + \mathcal{F}(u_j^{\text{out}}, 1; I_\eta) \leq \mathcal{F}(u, 1; I_\eta) + c2^{-j}$$

and then

$$(1 - w(0))^2 + g_0(|[u_j^{\text{out}}]|) \leq g_0(|[u]|) + c2^{-j}. \quad (8.32)$$

Taking $j \rightarrow \infty$ in (8.31) and (8.32) we obtain, with (8.30),

$$g_0(|z^+ - z^-|) + g_0(|[u] - (z^+ - z^-)|) \leq g_0(|[u]|) \quad \text{and} \quad (1 - w(0))^2 + g_0(|[u] - (z^+ - z^-)|) \leq g_0(|[u]|).$$

Recalling that g_0 is strictly subadditive on $(0, \infty)$, we see that one of the two terms in the first inequality must vanish. The second inequality excludes the case $z^+ = z^-$, therefore $z^+ - z^- = [u]$.

In order to show that $z' \in L^1(\mathbb{R})$, we prove that z is monotone. Assume for definiteness that $z^- \leq z^+$. We define

$$\bar{z} := z_- \vee (z_+ \wedge z) \quad \text{and} \quad \hat{z}(x) := \begin{cases} \min \bar{z}([x, 0]) & \text{if } x \leq 0, \\ \max \bar{z}([0, x]) & \text{if } x > 0. \end{cases}$$

Then $z_- \leq \hat{z} \leq \bar{z}$ on $(-\infty, 0)$ and $\bar{z} \leq \hat{z} \leq z_+$ on $(0, \infty)$, hence $\hat{z}(\pm t) \rightarrow z_\pm$ for $t \rightarrow \infty$. Obviously \hat{z} is monotone. Further, $\hat{z} \in H_{\text{loc}}^1(\mathbb{R})$, $0 \leq \hat{z}' \leq |z'|$, $\int_{\mathbb{R}} \hat{z}' dx = z^+ - z^-$, and

$$\int_{-\infty}^{\infty} f^2(w) |\hat{z}'|^2 dx \leq \int_{-\infty}^{\infty} f^2(w) |z'|^2 dx,$$

with equality if and only if $\hat{z} = z$ almost everywhere, which would imply that z is also monotone. In particular, $(\bar{s}^{-1} \hat{z}, w) \in \mathcal{U}_1$ for $\bar{s} = [u](\bar{x})$. Using (6.6), $|\hat{z}'| \leq |z'|$, (8.27) and (8.29) we obtain

$$\begin{aligned} g_0(|[u](\bar{x})|) &\leq \int_{-\infty}^{\infty} \left(f^2(w) |\hat{z}'|^2 + \frac{(1-w)^2}{4} + |w'|^2 \right) dx \\ &\leq \int_{-\infty}^{\infty} \left(f^2(w) |z'|^2 + \frac{(1-w)^2}{4} + |w'|^2 \right) dx \\ &\leq \int_{\mathbb{R}} f^2(w) d\mu + \int_{-\infty}^{\infty} \left(\frac{(1-w)^2}{4} + |w'|^2 \right) dx \leq g_0(|[u](\bar{x})|). \end{aligned}$$

Therefore equality holds throughout. This implies in particular $\hat{z} = z$, so that z is monotone, $(\bar{s}^{-1} z, w) \in \mathcal{U}_1$, and $\mu = |z'|^2 \mathcal{L}^1$, which gives strong convergence of z_k to z in $H^1(-T, T)$ for all T .

Step 3. In order to complete the proof it only remains to show the strong convergence of w_k to w . We have for all $T > 0$

$$\liminf_{k \rightarrow \infty} \int_{-T}^T \left(\frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx \geq \int_{-T}^T \left(\frac{(1-w)^2}{4} + |w'|^2 \right) dx \quad (8.33)$$

and

$$\liminf_{k \rightarrow \infty} \int_{-T}^T \frac{1}{\varepsilon_k} f_{\varepsilon_k}^2(w_k) |z'_k|^2 dx \geq \int_{-T}^T f^2(w) |z'|^2 dx.$$

Then arguing as in the case $w(0) = 0$ and assuming by contradiction that (8.24) holds, it is easily seen that one gets the contradiction

$$g_0(|[u](\bar{x})|) \geq \int_{-\infty}^{\infty} \left(f^2(w) |z'|^2 + \frac{(1-w)^2}{4} + |w'|^2 \right) dx + \sigma,$$

proving that (8.33) actually holds with an equality and a limit on the left-hand side. This gives the strong convergence of w_k to w . \square

Proof of Theorem 6.5. The first assertion follows from Lemma 8.8 and Lemma 8.9. We remark that the uniqueness result of Theorem 6.2(i) implies convergence of the entire sequence (after suitable translations).

It remains to prove the second assertion of Theorem 6.5. We fix $\eta > 0$ and we let $A := (0, 1) \setminus \bigcup_{x \in J_u} [x - \eta, x + \eta]$. We first remark that

$$v_\varepsilon \rightarrow 1 \quad \text{uniformly on } A.$$

Indeed, this follows by the same argument used to prove Lemma 8.6(iii). Therefore by assumption (6.3) we can write

$$(1 - v_\varepsilon) f(v_\varepsilon) \geq (1 - \omega(\varepsilon)) \ell \quad \text{on } A,$$

for some modulus of continuity $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We introduce the set $A_\varepsilon := \{x \in A : f_\varepsilon(v_\varepsilon)(x) < 1\}$ and we observe that, since $f_\varepsilon(v_\varepsilon) = \sqrt{\varepsilon} f(v_\varepsilon)$ on A_ε ,

$$\begin{aligned} \int_A \left(f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} \right) dx &\geq \int_{A_\varepsilon} \left(\varepsilon f^2(v_\varepsilon) |u'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} \right) dx + \int_{A_\varepsilon^c} f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 dx \\ &\geq \int_{A_\varepsilon} (1 - v_\varepsilon) f(v_\varepsilon) |u'_\varepsilon| dx + \int_{A_\varepsilon^c} |u'_\varepsilon|^2 dx \\ &\geq \ell(1 - \omega(\varepsilon)) \int_{A_\varepsilon} |u'_\varepsilon| dx + \int_{A_\varepsilon^c} |u'_\varepsilon|^2 dx. \end{aligned} \quad (8.34)$$

The sequence $u'_\varepsilon \chi_{A_\varepsilon^c}$ is bounded in $L^2(A)$, and therefore, possibly passing to a subsequence, has a weak limit $p \in L^2(A)$. The sequence $u'_\varepsilon \chi_{A_\varepsilon}$ is bounded in $L^1(A)$, and analogously we can assume that $u'_\varepsilon \chi_{A_\varepsilon} \rightharpoonup \mu$ and $|u'_\varepsilon| \chi_{A_\varepsilon} \rightharpoonup \nu$ weakly in measures, with

$$|\mu|(A) \leq \nu(A) \leq \liminf_{\varepsilon \rightarrow 0} \int_A |u'_\varepsilon| \chi_{A_\varepsilon} dx.$$

Since $u'_\varepsilon = u'_\varepsilon \chi_{A_\varepsilon} + u'_\varepsilon \chi_{A_\varepsilon^c}$ converges distributionally to $u' \in L^\infty(A)$, we have $\mu + p \mathcal{L}^1 = u' \mathcal{L}^1$, which implies that $\mu = (u' - p) \mathcal{L}^1$ and $|\mu|(A) = \int_A |u' - p| dx$. Thus (8.34) gives

$$\liminf_{\varepsilon \rightarrow 0} \int_A \left(f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} \right) dx \geq \int_A (\ell |u' - p| + p^2) dx \geq \int_A (|u'|^2 + |u' - p|^2) dx,$$

where in the last step we used that for any $t \in [-\frac{\ell}{2}, \frac{\ell}{2}]$ and any $y \in \mathbb{R}$ one has

$$\ell |t - y| + y^2 \geq t^2 + (t - y)^2.$$

Indeed, this inequality is the same as $\ell|t - y| \geq 2t(t - y)$, which is true since $\ell \geq 2|t|$.

Since we know that

$$\limsup_{\varepsilon \rightarrow 0} \int_A \left(f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx \leq \mathcal{F}(u, 1; A) = \int_A |u'|^2 dx,$$

we conclude that $p = u'$. In particular

$$\begin{aligned} \int_A |u'|^2 dx &\leq \liminf_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int_A f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 dx \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) = \mathcal{F}(u, 1; A) = \int_A |u'|^2 dx, \end{aligned}$$

hence equality holds throughout and (6.24) is proven. This completes the proof. \square

8.3 Proof of the statements of Section 7.2

Let g be defined as in (7.1). First, let us observe that $g(s, 0) = g_0(s)$ and that the characterizations of g given in (7.4)–(7.6) follow by the same arguments used to prove the corresponding characterizations of g_0 in (6.9)–(6.11). We collect the main properties of the function g in the following proposition.

Proposition 8.10. *The function g enjoys the following properties:*

(i) g is monotone nondecreasing in both variables, $g(s, s') = g_0(s)$ if $s \geq s'$;

(ii) for every $s_1, s_2, s' \geq 0$,

$$g(s_1 + s_2, s') \leq g_0(s_1) + g(s_2, s'); \quad (8.35)$$

(iii) $g(0, s') = (1 - m_{s'})^2$, $(1 - m_{s'})^2 \leq g(s, s') \leq 1 \wedge ((1 - m_{s'})^2 + \ell s)$;

(iv) g is continuous in both variables and, for fixed s' , the map $g(\cdot, s')$ is Lipschitz continuous with Lipschitz constant ℓ (in particular g is globally continuous);

(v) $\lim_{s \uparrow \infty} g(s, s') = 1$; for $s' > 0$ $\lim_{s \downarrow 0} \frac{g(s, s') - g(0, s')}{s} = 0$.

Proof. (i): The monotonicity of g in the first variable follows from $\mathcal{G}_{s_1}(\alpha, \beta) \leq \mathcal{G}_{s_2}(\alpha, \beta)$ for $s_1 < s_2$, while the monotonicity in the second variable follows since the map $s' \mapsto m_{s'}$ is decreasing. If $s \geq s'$, the optimal profile β_s for $g_0(s)$ is admissible in the minimum problem which defines $g(s, s')$, by monotonicity of $m_{s'}$, and therefore the values of the two functions coincide. This completes the proof of (i).

(ii): Fix $\sigma > 0$ and let $T > 0$, $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in H^1(-T, T) \times H^1(-T, T)$ be such that $\alpha_i(-T) = 0$, $\alpha_i(T) = s_i$, $0 \leq \beta_i \leq 1$, $\beta_i(\pm T) = 1$, $\inf \beta_2 \leq \beta_{s'}(0)$, and

$$\begin{aligned} \int_{-T}^T \left(f^2(\beta_1) |\alpha'_1|^2 + \frac{(1 - \beta_1)^2}{4} + |\beta'_1|^2 \right) dt &\leq g_0(s_1) + \sigma, \\ \int_{-T}^T \left(f^2(\beta_2) |\alpha'_2|^2 + \frac{(1 - \beta_2)^2}{4} + |\beta'_2|^2 \right) dt &\leq g(s_2, s') + \sigma, \end{aligned}$$

according to the representation formulas (6.9) and (7.4) after rescaling α . By defining

$$\alpha(t) = \begin{cases} \alpha_1(t) & \text{if } t \in [-T, T], \\ \alpha_2(t - 2T) + s_1 & \text{if } t \in [T, 3T], \end{cases} \quad \beta(t) = \begin{cases} \beta_1(t) & \text{if } t \in [-T, T], \\ \beta_2(t - 2T) & \text{if } t \in [T, 3T], \end{cases}$$

we have $\alpha, \beta \in H^1(-T, 3T)$, with $\alpha(-T) = 0$, $\alpha(3T) = s_1 + s_2$, $\beta(-T) = \beta(3T) = 1$, $0 \leq \beta \leq 1$, $\inf \beta \leq \beta_{s'}(0)$. By using the pair $(\frac{1}{s_1 + s_2} \alpha, \beta)$ as a competitor in (7.4) we obtain

$$g(s_1 + s_2, s') \leq g_0(s_1) + g(s_2, s') + 2\sigma,$$

and the conclusion follows as σ is arbitrary.

(iii): The value of $g(0, s')$ can be computed by observing that in this case the minimum problem (7.1) defining g is independent of α , and the optimal profile is given by the function

$$\bar{\beta}_{s'}(t) := 1 - (1 - m_{s'})e^{-\frac{|t|}{2}}. \quad (8.36)$$

The lower bound on $g(s, s')$ follows directly by monotonicity. To derive the upper bound with the constant 1, one can simply consider the admissible pair $(\alpha, \beta) \in \mathcal{V}_{s'}$ given by

$$\alpha(t) = \begin{cases} 0 & \text{if } t \in (-\infty, -1), \\ \frac{t}{2} + \frac{1}{2} & \text{if } t \in [-1, 1], \\ 1 & \text{if } t \in (1, \infty), \end{cases} \quad \beta(t) = \begin{cases} 1 - e^{-\frac{t+1}{2}} & \text{if } t \in (-\infty, -1), \\ 0 & \text{if } t \in [-1, 1], \\ 1 - e^{-\frac{t-1}{2}} & \text{if } t \in (1, \infty). \end{cases}$$

The inequality $g(s, s') \leq g(0, s') + \ell s$ follows directly from (8.35) and $g_0(s) \leq \ell s$.

(iv): The continuity of $s' \mapsto g(s, s')$ can be proved by taking into account the continuity of the map $s' \mapsto m_{s'}$, proved in Theorem 6.2. The Lipschitz continuity of $s \mapsto g(s, s')$ is a consequence of the monotonicity of this map, of the subadditivity inequality (8.35), and of the bound $g_0(s) \leq \ell s$.

(v): The first limit is a trivial consequence of $g(s, s') = g_0(s)$ for $s \geq s'$, and of Theorem 6.1. To compute the slope of $g(\cdot, s')$ at the origin, notice that given any α such that α' has compact support and the pair $(\alpha, \bar{\beta}_{s'})$ belongs to $\mathcal{V}_{s'}$, where $\bar{\beta}_{s'}$ is the optimal profile defined in (8.36), we have

$$g(s, s') \leq \int_{-\infty}^{\infty} \left(s^2 f^2(\bar{\beta}_{s'}) |\alpha'|^2 + \frac{(1 - \bar{\beta}_{s'})^2}{4} + |\bar{\beta}_{s'}|^2 \right) dt = \int_{-\infty}^{\infty} s^2 f^2(\bar{\beta}_{s'}) |\alpha'|^2 dt + (1 - m_{s'})^2.$$

Dividing by s and letting $s \rightarrow 0^+$ we obtain the inequality $\limsup_{s \rightarrow 0^+} \frac{g(s, s') - g(0, s')}{s} \leq 0$. The lower bound $g(s, s') \geq g(0, s')$ concludes the proof. \square

The next proposition shows that the inequality (8.35) is strict.

Proposition 8.11 (Strict subadditivity). *For every $s_1 > 0$, $s_2 \geq 0$, and $s' > 0$, we have*

$$g(s_1 + s_2, s') < g_0(s_1) + g(s_2, s'). \quad (8.37)$$

Proof. If $s_1 + s_2 \geq s'$, then $g(s_1 + s_2, s') = g_0(s_1 + s_2)$ and (8.37) holds since g_0 is strictly subadditive by Theorem 6.1. If $g_0(s_1) = 1$ the assertion follows from $g(s_1 + s_2, s') \leq 1$ and $g(s_2, s') > 0$. Analogously if $g(s_2, s') = 1$. Let us assume now that $s_1 + s_2 < s'$, $s_1 < s_{\text{frac}}$ and $g(s_2, s') < 1$. Fixed $\eta \in (0, \frac{1}{2}(1 - g(s_2, s')))$, let $\gamma_1, \gamma_2 \in W_0^{1,1}([0, 1], [0, 1])$ be such that $\max \gamma_2 \geq (1 - m_{s'})^2$ and

$$1 > g_0(s_1) = \int_0^1 \sqrt{s_1^2 (f_1(\sqrt{\gamma_1}))^2 + \frac{|\gamma_1'|^2}{4}} dt, \quad (8.38)$$

$$1 > g(s_2, s') + \eta \geq \int_0^1 \sqrt{s_2^2 (f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt. \quad (8.39)$$

Notice that this is possible by the characterization (7.6) and Proposition 8.3(ii). In particular γ_1 is the unique minimizer of the problem defining $g_0(s_1)$, and (8.39) implies $0 \leq \gamma_2 \leq \frac{1}{2}(1 + g(s_2, s')) < 1$ as well as $\int_0^1 |\gamma_2'| dt \leq 2$.

The function $\gamma := (\gamma_1 + \gamma_2) \wedge 1$ is admissible for $g(s_1 + s_2, s')$, being $\max \gamma \geq \max \gamma_2 \geq (1 - m_{s'})^2$. Therefore,

$$\begin{aligned} g(s_1 + s_2, s') &\leq \int_0^1 \sqrt{(s_1 + s_2)^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt \\ &\leq \int_0^1 \sqrt{s_1^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_1'|^2}{4}} dt + \int_0^1 \sqrt{s_2^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_2'|^2}{4}} dt. \end{aligned}$$

In order to estimate the first integral, we observe that $\gamma \geq \gamma_1$ by definition and that by (6.2) $f_1(\sqrt{\cdot})$ is strictly decreasing, hence

$$\int_0^1 \sqrt{s_1^2(f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_1'|^2}{4}} dt \leq \int_0^1 \sqrt{s_1^2(f_1(\sqrt{\gamma_1}))^2 + \frac{|\gamma_1'|^2}{4}} dt. \quad (8.40)$$

As for the second integral, we first assume $s_2 > 0$ and show that there exists a constant $c_{s_1, s_2, s'} > 0$, only depending on s_1, s_2 and s' , such that

$$\int_0^1 \sqrt{s_2^2(f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_2'|^2}{4}} dt \leq \int_0^1 \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt - c_{s_1, s_2, s'}. \quad (8.41)$$

This would lead to

$$\begin{aligned} g(s_1 + s_2, s') &\leq \int_0^1 \sqrt{s_1^2(f_1(\sqrt{\gamma_1}))^2 + \frac{|\gamma_1'|^2}{4}} dt + \int_0^1 \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt - c_{s_1, s_2, s'} \\ &\leq g_0(s_1) + g(s_2, s') + \eta - c_{s_1, s_2, s'}, \end{aligned}$$

and then to (8.37) as $\eta \rightarrow 0$.

In order to check (8.41), we let $\rho := \frac{1-g(s_2, s')}{2} \wedge \frac{1}{2}\gamma_1(\frac{1}{2}) > 0$. Then there is $\delta > 0$, only depending on s_1, s_2 and s' , such that $\gamma_1 \geq \rho$ in $J := (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$. This implies $\gamma = 1 \wedge (\gamma_1 + \gamma_2) \geq \gamma_2 + \rho$ in J . Convexity and strict monotonicity of $f_1(\sqrt{\cdot})$ imply convexity and strict monotonicity of $(f_1(\sqrt{\cdot}))^2$, and hence monotonicity of its difference quotients. Therefore

$$f_1^2(\sqrt{\gamma_2}) - f_1^2(\sqrt{\gamma}) \geq f_1^2(\sqrt{\gamma_2}) - f_1^2(\sqrt{\gamma_2 + \rho}) \geq f_1^2(\sqrt{1 - \rho}) - f_1^2(1) =: \tilde{c} > 0 \text{ in } J.$$

We remark that \tilde{c} only depends on ρ and therefore on s_1, s_2, s' . Let $I_\eta := \{t \in J : |\gamma_2'(t)| \leq 2\delta^{-1}\}$. From $\int_0^1 |\gamma_2'| dt \leq 2$ and $\mathcal{L}^1(J) = 2\delta$ we obtain $\mathcal{L}^1(I_\eta) \geq \delta$. Since $\sqrt{A - \varepsilon} \leq \sqrt{A} - \frac{\varepsilon}{2\sqrt{A}}$ whenever $0 \leq \varepsilon \leq A$ and $f_1 \leq \ell$,

$$\begin{aligned} \int_{I_\eta} \sqrt{s_2^2(f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_2'|^2}{4}} dt &\leq \int_{I_\eta} \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} - s_2^2 \tilde{c} dt \\ &\leq \int_{I_\eta} \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt - \frac{s_2^2 \tilde{c} \delta}{2\sqrt{s_2^2 \ell^2 + \delta^{-2}}}, \end{aligned}$$

where the last term is nonzero and only depends on s_1, s_2 and s' . Hence (8.41) follows.

It remains to deal with the case $s_2 = 0$. In this situation we can take $\gamma_2(t) = 2(t \wedge (1 - t))(1 - m_{s'})^2$ and $\eta = 0$ in (8.39), then $\gamma > \gamma_1$ pointwise in $(0, 1)$ and (8.40) is a strict inequality; by monotonicity of f_1 (8.41) holds with $c_{s_1, s_2, s'} = 0$. The proof is then concluded as above. \square

Proof of Theorem 7.1. All the properties listed in Section 2 follow by combining Theorem 6.1, Proposition 8.10 and Proposition 8.11, and observing that by definition one has $g(s, 0) = g_0(s)$. \square

Proof of Lemma 7.2. Given $T > 0$ and an admissible pair (α_μ, β_μ) for $g^{(\mu)}(s, s')$, we construct an admissible pair (α, β) for $g(s, s')$ in $(-T - 1, T + 1)$ by setting $\alpha \equiv \alpha_\mu(-T)$ in $(-T - 1, -T)$, $\alpha = \alpha_\mu$ in $(-T, T)$, $\alpha \equiv \alpha_\mu(T)$ in $(T, T + 1)$, and by setting $\beta = \beta_\mu$ in $(-T, T)$ and linearly linked to the value 1 in $(-T - 1, -T)$ and $(T, T + 1)$. Then by (7.4), scaling α ,

$$\begin{aligned} g(s, s') &\leq \int_{-T-1}^{T+1} \left(f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt \\ &\leq \int_{-T}^T \left(f^2(\beta_\mu) |\alpha'_\mu|^2 + \frac{(1 - \beta_\mu)^2}{4} + |\beta'_\mu|^2 \right) dt + 3\mu^2, \end{aligned}$$

from which it follows that $g(s, s') \leq g^{(\mu)}(s, s') + 3\mu^2$. The other inequality follows by an analogous construction, reversing the roles of g and $g^{(\mu)}$. \square

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