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# A phase-field approach to quasi-static evolution for a cohesive fracture model

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## Abstract

In this paper we propose a notion of irreversibility for the evolution of cracks in presence of cohesive forces, which allows for different responses in the loading and unloading processes, motivated by a variational approximation with damage models. We investigate its applicability to the construction of a quasi-static evolution in a simple one-dimensional model. The cohesive fracture model arises naturally via  $\Gamma$ -convergence from a phase-field model of the generalized Ambrosio-Tortorelli type, which may be used as regularization for numerical simulations.

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# 1 Introduction

In this paper we propose a notion of irreversibility for the evolution of cracks in presence of cohesive forces, which allows for different responses in the loading and unloading processes, motivated by a variational approximation with damage models. We investigate its applicability to the construction of a quasi-static evolution in a simple one-dimensional model.

The variational approach to fracture, as formulated by Francfort and Marigo [31] (see also [13]), is based on Griffith's idea [36] that the crack growth is determined by the competition between the energy spent to increase the crack and the corresponding release in the bulk elastic energy. By representing the crack as a  $(n - 1)$ -dimensional surface  $\Gamma$  and assuming for simplicity that the material behaves as a linearly elastic body in the unbroken part  $\Omega \setminus \Gamma$  of the reference configuration, one is led to consider an energy of the form

$$E(u, \Gamma) := \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} g(|[u](x)|) d\mathcal{H}^{n-1}, \quad (1.1)$$

where  $[u](x) := u^+(x) - u^-(x)$  is the difference of the traces of the displacement  $u : \Omega \rightarrow \mathbb{R}$  on the two sides of  $\Gamma$ . In the original Griffith's theory for *brittle fracture* the energy associated with the crack is proportional to the measure of the surface of the crack itself, and  $g$  is assumed not to depend on the crack opening  $[u]$ . It was however early recognized (Barenblatt [10]) that in many situations fracture should be regarded as a gradual process, and that the presence of cohesive forces between the lips of the crack should be taken into account: in *cohesive fracture* models Griffith's energy is replaced by various surface energies depending on the actual opening of the crack; in this case the function  $g$  in (1.1) is typically assumed to be an increasing and concave function, with  $g(0) = 0$ , with positive and finite slope  $g'(0) \in (0, \infty)$  at the origin, and asymptotically converging to the value of the fracture toughness. Variants of (1.1) with  $g$  not necessarily concave have been used to study the formation of micro-cracks [28, 29].

The notion of *irreversible quasi-static evolution* in the brittle case ( $g$  constant), proposed in [31], under the action of time-dependent loads (for instance, in the form of a prescribed boundary displacement  $b(t)$ ), requires three essential ingredients: one looks for a map  $t \mapsto (u(t), \Gamma(t))$  satisfying

- (i) (*irreversibility*)  $\Gamma(t_1) \subset \Gamma(t_2)$  for  $t_1 < t_2$ ;
- (ii) (*static equilibrium*)  $(u(t), \Gamma(t))$  is a global minimizer of the energy (1.1) with respect to the boundary condition  $b(t)$ ;
- (iii) (*energy balance*) the increment in stored energy plus the energy spent in crack increase equals the work of external forces.

The common approach in showing the existence of a variational evolution with the properties above is based on a time-discretization algorithm, in which one selects at each time step a global minimizer of the total energy, and then recovers the time-continuous limit by sending to zero the discretization parameter. This program has been successfully accomplished in the brittle case in a series of contributions: the existence of a quasi-static evolution was first proven by Dal Maso and Toader [26] in a two-dimensional, antiplane shear setting and with a uniform bound on the number of connected components of the crack, then extended to the case of planar elasticity by Chambolle [18]. The general  $n$ -dimensional case, without restrictions on the number of connected components, was studied by Francfort and Larsen [32] (antiplane case) and by Dal Maso, Francfort and Toader [22] (finite elasticity). Further results in this direction were obtained in [24, 38, 5].

In the cohesive case the picture is less clear and there is no agreement on a single notion of irreversibility. Indeed, while in the brittle case the irreversibility constraint is a purely geometric condition, in the cohesive context one has to take into account also the amplitude of the cracks, through the inclusion of some internal variable which keeps track of the complete history of the fracture process. In particular, it would be desirable to model the possibility of having different responses of the material to loading and unloading processes. Up to now, several plausible choices

have been proposed in the mathematical literature: Dal Maso and Zanini [27] adopt as internal variable the maximal opening of the crack, postulating that the energy is dissipated only once for a given value of the jump, and additional dissipation can only occur if the distance between the crack lips is further increased; in the model by Cagnetti and Toader [17] part of the dissipated energy is recovered when the crack opening is decreased; Crismale, Lazzaroni and Orlando [20] include in the mathematical model the phenomenon of fatigue, that is some energy is dissipated also during the unloading phase. In each of these cases the existence of a quasi-static evolution (encoding in the definition the relevant notion of irreversibility) is established under the assumption that the crack path is prescribed. In the cohesive fracture context, further results on evolutions of local minimizers or critical point (rather than global minimizers) are obtained in [16, 4, 9, 39].

In this paper we propose the study of a notion of irreversibility which originates from the variational approximation of the (relaxation of the) cohesive functional (1.1) obtained in [19]. In order to explain the main idea, it is first convenient to refer once again to the brittle case, and to recall the classical Ambrosio-Tortorelli approximation [7, 8] of the Mumford-Shah functional

$$E(u) := \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u), \quad u \in \text{SBV}(\Omega),$$

by the regularizing energies

$$E_{\varepsilon}(u_{\varepsilon}, v_{\varepsilon}) := \int_{\Omega} (v_{\varepsilon}^2 + o(\varepsilon)) |\nabla u_{\varepsilon}|^2 dx + \frac{1}{2} \int_{\Omega} \left( \varepsilon |\nabla v_{\varepsilon}|^2 + \frac{(1 - v_{\varepsilon})^2}{\varepsilon} \right) dx, \quad (u_{\varepsilon}, v_{\varepsilon}) \in H^1(\Omega) \times H^1(\Omega),$$

with  $0 \leq v_{\varepsilon} \leq 1$ . Here the function  $u_{\varepsilon}$  has to be understood as a regularization of the displacement  $u$ , possibly creating jumps in the limit as  $\varepsilon \rightarrow 0$ , while  $v_{\varepsilon}$  can be interpreted as a damage variable which concentrates in the regions where the singularities of  $u$  are created: more precisely,  $v_{\varepsilon} \rightarrow 1$  almost everywhere in  $\Omega$ , and  $v_{\varepsilon} \rightarrow 0$  on the  $(n - 1)$ -dimensional surface where the limit jump will appear. The notion of quasi-static evolution in the regular context of the Ambrosio-Tortorelli functionals can be formulated including an irreversibility condition in the form of a global monotonicity constraint on  $v_{\varepsilon}$  (the damage can only increase in time). It is proved in [34] that this evolution converges as  $\varepsilon \rightarrow 0$  to a quasi-static evolution for brittle fracture.

Returning to the cohesive setting, in the spirit of Ambrosio and Tortorelli it is shown in [19] (see also [33] for a numerical insight) that, for a suitable choice of the function  $f_{\varepsilon}$ , functionals of the form

$$F_{\varepsilon}(u_{\varepsilon}, v_{\varepsilon}) := \int_{\Omega} \left( f_{\varepsilon}^2(v_{\varepsilon}) |\nabla u_{\varepsilon}|^2 + \frac{(1 - v_{\varepsilon})^2}{4\varepsilon} + \varepsilon |\nabla v_{\varepsilon}|^2 \right) dx, \quad (u_{\varepsilon}, v_{\varepsilon}) \in H^1(\Omega) \times H^1(\Omega), \quad (1.2)$$

with  $0 \leq v_{\varepsilon} \leq 1$ ,  $\Gamma$ -converge as  $\varepsilon \rightarrow 0$  to the relaxation of an energy of the form (1.1) (see Section 2 for the precise statement). The function  $g$  obtained as limit surface energy density can be characterized as the solution to a minimum problem,

$$g(s) = \inf \left\{ \int_{-\infty}^{\infty} \left( f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt : (\alpha, \beta) \in H_{\text{loc}}^1(\mathbb{R}) \times H_{\text{loc}}^1(\mathbb{R}), \alpha' \in L^1(\mathbb{R}), \right. \\ \left. \left| \int_{-\infty}^{\infty} \alpha'(t) dt \right| = s, 0 \leq \beta \leq 1, \lim_{|t| \rightarrow \infty} \beta(t) = 1 \right\}, \quad (1.3)$$

and satisfies the natural requirements for a cohesive energy (see Figure 1). As before,  $u_{\varepsilon}$  is a regularization of  $u$  developing singularities in the limit, and  $v_{\varepsilon}$  converges to the value 1 almost everywhere and concentrates on the limit jump. The crucial observation is that, in contrast with the Ambrosio-Tortorelli case,  $v_{\varepsilon}$  does not tend to zero everywhere on the jump, but reaches a value depending on the amplitude of the limit jump of  $u$ . Indeed, the one-dimensional blow-up analysis of the behaviour of recovery sequences  $(u_{\varepsilon}, v_{\varepsilon}) \rightarrow (u, 1)$  around a jump point  $\bar{x} \in J_u$ , which we perform in Section 7, shows that the rescaled functions  $z_{\varepsilon}(x) := u_{\varepsilon}(x_{\varepsilon} + \varepsilon x)$ ,  $w_{\varepsilon}(x) := v_{\varepsilon}(x_{\varepsilon} + \varepsilon x)$  converge to an optimal profile  $(\alpha_s, \beta_s)$  for the minimum problem (1.3) defining  $g(s)$ , for the value

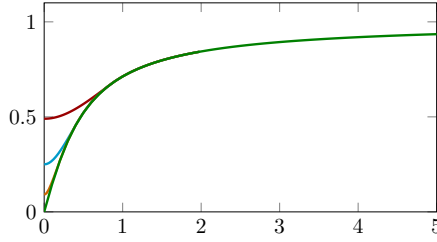


Figure 1: Graph of  $g(s)$  (bottom curve) and  $\bar{g}(s, s')$  for  $m_{s'} = 0.3, 0.5$  and  $0.7$  (from bottom to top) using the function  $f_1^b(s)$  defined in (2.7) with  $\ell = 1.5$ ,  $\ell_1 = 0.2$ . See Figure 5 below for further details.

$s = |[u](\bar{x})|$  (Theorem 7.1); moreover the minimum value reached by  $v_\varepsilon$  in a small neighbourhood of  $\bar{x}$ ,  $m_s := \min_{\mathbb{R}} \beta_s$ , is in one-to-one correspondence with the limit amplitude  $s$  of the jump.

Therefore the relevant information about the amplitude of the limit jump  $[u]$  is carried only by the value of the minimum of the damage variable  $v_\varepsilon$ . This suggests to impose, at level  $\varepsilon$ , an irreversibility condition in the form of a monotonicity constraint on the value of local minima of  $v_\varepsilon$ : in the points where  $v_\varepsilon$  has a local minimum, its value can only decrease. By looking at the  $\Gamma$ -limit of the functionals  $F_\varepsilon$  in (1.2) with this additional constraint (see Section 6), one finds that the limit energy density has the form

$$\bar{g}(s, s') = \inf \left\{ \int_{-\infty}^{\infty} \left( f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : (\alpha, \beta) \in H_{\text{loc}}^1(\mathbb{R}) \times H_{\text{loc}}^1(\mathbb{R}), \alpha' \in L^1(\mathbb{R}), \right. \\ \left. \left| \int_{-\infty}^{\infty} \alpha'(t) dt \right| = s, 0 \leq \beta \leq 1, \lim_{|t| \rightarrow \infty} \beta(t) = 1, \inf_{\mathbb{R}} \beta \leq m_{s'} \right\}, \quad (1.4)$$

see Figure 1. In other words, we restrict the class of admissible profiles by imposing that the minimum value of the profile  $\beta$  does not exceed the minimum value  $m_{s'}$  of the optimal profile  $\beta_{s'}$  corresponding to the amplitude  $s'$ . The second variable  $s'$  in  $\bar{g}$  represents the maximal amplitude of the jump reached at the previous times, while  $s$  stands for the current opening of the crack. When  $s \geq s'$  the additional constraint in (1.4) is irrelevant and one has  $g(s) = \bar{g}(s, s')$ : this means that, in the process of crack opening, the energy per unit area of the fracture at a point  $x$  is given by  $g(|[u](x)|)$ . The behaviour of  $\bar{g}(s, s')$  is instead different when  $s < s'$ : in this case  $s \mapsto \bar{g}(s, s')$  is monotone nondecreasing and, generally,  $\bar{g}(s, s') > g(s)$ . Hence, when  $[u](x)$  is smaller than the maximal opening reached at previous times, the energy density follows a curve which is above the graph of  $g$ . At the same time,  $\bar{g}(0, s')$  is (at least for small openings, see Section 3 for details) strictly smaller than  $g(s') = \bar{g}(s', s')$ . Indeed, upon closure of the fracture, the part of the energy that originates from the elastic degrees of freedom (corresponding to the  $f^2(\beta) |\alpha'|^2$  term in (1.4)) is recovered, whereas the part that originates from the internal variable (corresponding to the two terms  $\frac{(1-\beta)^2}{4}$  and  $|\beta'|^2$  in (1.4), with  $\beta(\pm\infty) = 1$  and  $\beta(0) \leq m_{s'}$ ) is permanently dissipated. This behaviour is consistent with models where some of the dissipated energy is recovered when the crack opening is decreased [17, 4, 39, 40], and our analysis shows that it can be recovered as the limit of damage models.

We now discuss how to implement this idea in the construction of a quasi-static evolution for a one-dimensional model of cohesive fracture. We let  $\Omega = (0, 1)$  represent an elastic bar, on which we prescribe a time-dependent boundary displacement  $t \mapsto (b(t, 0), b(t, 1))$ . We also include in the energy a lower-order penalization of the form  $\gamma \|u - w(t)\|_{L^2(0,1)}^2$ , where  $\gamma > 0$  and  $t \mapsto w(t)$  is a given function which prescribes the average behavior of the material. This foundation-type term, as usual in one-dimensional simplifications of models in solid mechanics, represents the boundary conditions in the directions that have been eliminated. A simple way to understand its physical significance is to imagine solving for  $\tilde{u} : (0, 1) \times (-H, H) \rightarrow \mathbb{R}$  with boundary conditions

$\tilde{u}(x, \pm H) = w(x)$  on the top and bottom boundaries, and then focusing on  $u(x) := \tilde{u}(x, 0)$ . The difference  $u - w$  would be penalized by a  $\partial_2 \tilde{u}$  term in the energy. If dealing with the Dirichlet functional, the  $H^{1/2}$  norm of the difference would be the most natural way to model this effect; for general energy densities there is no clear natural form. For simplicity we use the  $L^2$  norm. Due to the presence of this term the bar can be fractured at multiple points, while if  $\gamma = 0$  one can prove that minimizers of the energy have at most one jump.

We keep track of the state of the fracture at each time  $t \in [0, T]$  by means of a pair of internal variables  $(\Gamma(t), s(t))$ , where  $\Gamma(t)$  is a discrete set of points (representing the sites of the crack points), and  $s(t) : \Gamma(t) \rightarrow (0, \infty)$  is the maximal amplitude of the jump at the points of  $\Gamma(t)$  for all the previous times. For technical reasons, we fix a (arbitrarily small) positive threshold  $\bar{s} > 0$  and we keep track only of the jumps that overcome this value; in other words, fracture points where the jump is smaller than  $\bar{s}$  are not affected by the irreversibility condition, and the opening can be reduced with a complete recover of the dissipated energy. From the mathematical point of view this guarantees a uniform bound on the total number of crack points (this assumption can be easily removed in the case  $\gamma = 0$ , that is if we do not include the lower order penalization). It is often convenient to switch to the equivalent formulation in which  $s$  is first extended by 0 outside  $\Gamma$ , and  $\Gamma$  is implicitly replaced by the set  $\{s > 0\}$ .

We consider a time discretization  $0 = t_0 < t_1 < \dots < t_N = T$  and we initially select the displacement  $u_0$  by minimizing the energy

$$\int_0^1 |u'|^2 dx + \sum_{x \in J_u} g(|[u](x)|) + \gamma \int_0^1 |u - w(t_0)|^2 dx \quad (1.5)$$

among all  $u \in \text{SBV}(0, 1)$  attaining the boundary conditions  $b(t_0)$  at the endpoints of the bar. To be more precise, we need to consider the lower semicontinuous envelope of the energy and a minimizer will belong in general to  $\text{BV}(0, 1)$ : indeed the presence of the lower-order term allows for the possibility of having more than one jump, while some regularity can still be proved, see Proposition 4.2. In the case  $\gamma = 0$  it is well-known that minimizers in dimension one are in  $\text{SBV}$  with a single jump, see [14]; see also [23, 15] for further regularity results for cohesive energies.

We then define

$$\Gamma_0 := \{x \in J_{u_0}^{b(t_0)} : |[u_0](x)| > \bar{s}\}, \quad s_0(x) := |[u_0](x)| \quad \text{for } x \in \Gamma_0$$

(where the superscript  $b(t_0)$  in the jump set of  $u_0$  indicates that we consider as part of the crack also the points at which the boundary conditions are not attained). Iteratively, assuming to have constructed  $(u_i, \Gamma_i, s_i)$  for  $i = 0, \dots, k-1$ , we select the displacement  $u_k$  at time  $t_k$  by minimizing (the relaxation of) the energy

$$\int_0^1 |u'|^2 dx + \sum_{x \in \Gamma_{k-1}} \bar{g}(|[u](x)|, s_{k-1}(x)) + \sum_{x \in J_u \setminus \Gamma_{k-1}} g(|[u](x)|) + \gamma \int_0^1 |u - w(t_k)|^2 dx \quad (1.6)$$

with respect to the boundary condition  $b(t_k)$ , and we update the state of the fracture by setting

$$\Gamma_k := \{x \in J_{u_k}^{b(t_k)} : |[u_k](x)| > \bar{s}\}, \quad s_k(x) := |[u_k](x)| \vee s_{k-1}(x) \quad \text{for } x \in \Gamma_k.$$

The energy (1.6) is that obtained as the  $\Gamma$ -limit of the functionals  $F_\varepsilon$  with the irreversibility constraint discussed before, and one can see that the new energy density  $\bar{g}$  appears at the points where the bar was previously broken.

By passing to the limit as the time-step goes to zero, we obtain in Theorem 5.1 the existence of a quasi-static evolution  $(u(t), \Gamma(t), s(t))$  satisfying the usual properties of static equilibrium and energy balance. The irreversibility condition is reflected in the fact that the sets  $\Gamma(t)$  and the maps  $s(t)$  are monotonically increasing in time.

It is worth to briefly comment on the connection of this approach with other results in the mathematical literature. As remarked above, in our model the response of the material when

the crack opening decreases is similar to that considered in [17], that is some dissipated energy is recovered. In particular, the phenomenon of fatigue is completely absent. Fatigue effects are instead taken into account in [20] (see also [1]): here also when the crack opening is reduced some energy is dissipated, and oscillations of jumps can produce a complete fracture even if the maximal crack amplitude remains small. This behaviour is consistent with models where cohesive effects result from the interaction between damage and plasticity (see [2, 3, 25]); on the contrary, our approach is based on the approximation [19], where the cohesive behaviour is seen as the effect of the interplay between elasticity and damage.

We further remark that the functionals (1.2), when a monotonicity constraint on the minimum values of  $v_\varepsilon$  is included, provide a variational approximation of the limit energy (1.6) in a static setting (see Section 6 for the rigorous  $\Gamma$ -convergence result). It remains an open question whether a result in the spirit of Giacomini [34] holds also in this case, that is if one can construct a variational evolution for the functionals  $F_\varepsilon$  and show its convergence, as  $\varepsilon \rightarrow 0$ , to a quasi-static evolution for the limit cohesive model.

We believe that this approach is also suitable for numerical implementation. An appropriate algorithm should identify the regions where the damage  $v_\varepsilon$  is concentrated, and impose the monotonicity constraint only at the local minimum points. This might be numerically less expensive than imposing the monotonicity constraint at any point.

The generalization to the higher dimensional case poses several challenges. A first step in this direction has been obtained in [37], where the  $\Gamma$ -convergence result in Section 6 has been extended to general dimensions, under minimal regularity assumptions on the set  $\Gamma$  on which the irreversibility constraint is imposed. The next step would consist in the construction of a quasi-static evolution under the assumption of a prescribed crack path.

**Structure of the paper.** The paper is organized as follows. In Section 2 we collect some preliminary definitions and we report the main approximation result from [19]. In Section 3 we discuss in details the properties of the fracture energy density  $g$  obtained in the  $\Gamma$ -limit of the damage models; we then introduce a new surface energy density  $\bar{g}$ , depending on an internal variable which contains the memory of the maximum value reached by the opening of the fracture. The construction of a quasi-static evolution for the one-dimensional model is discussed in Section 4 (time-discretization) and Section 5 (continuous limit). In Section 6 we show how the energy that we consider can be approximated, in the spirit of [19], by damage models which include an irreversibility constraint. The behaviour of recovery sequences is studied in Section 7. In the concluding Appendix A we compute the relaxation of the energy.

## 2 General setting, definitions, and main assumptions

We work in dimension one and for simplicity in the interval  $\Omega := (0, 1)$ . For a function of bounded variation  $u \in \text{BV}(0, 1)$  we denote by  $Du$  its distributional derivative, which is a bounded Radon measure on  $(0, 1)$ , and by  $|Du|$  its total variation. The standard decomposition of  $Du$  is given by

$$Du = u' \mathcal{L}^1 + D^c u + \sum_{x \in J_u} [u](x) \delta_x,$$

where  $u' \in L^1(0, 1)$  denotes the density of the absolutely continuous part of  $Du$  with respect to the Lebesgue measure  $\mathcal{L}^1$ ,  $D^c u$  is the Cantor part of  $Du$ ,  $J_u$  is the jump set of  $u$  (sometimes also denoted by  $J(u)$  in the following),  $[u](x) := u^+(x) - u^-(x)$ , and  $u^+(x)$  and  $u^-(x)$  are the approximate limits from the right and from the left of  $u$  at  $x$  respectively. We denote by  $\text{SBV}(0, 1)$  the space of functions  $u \in \text{BV}(0, 1)$  such that  $D^c u \equiv 0$ . For the properties of functions of bounded variation we refer to the monograph [6].

Following [19], we introduce a family of functionals  $\mathcal{F}_\varepsilon : L^1(0, 1) \times L^1(0, 1) \rightarrow [0, \infty]$ , depending

on a real parameter  $\varepsilon > 0$ , by setting

$$\mathcal{F}_\varepsilon(u, v) := \begin{cases} \int_0^1 \left( f_\varepsilon^2(v) |u'|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon |v'|^2 \right) dx & \text{if } u, v \in H^1(0, 1), \\ \infty & \text{otherwise.} \end{cases} \quad \begin{matrix} 0 \leq v \leq 1 \text{ } \mathcal{L}^1\text{-a.e. in } (0, 1), \\ \end{matrix} \quad (2.1)$$

Here the function  $f_\varepsilon : [0, 1] \rightarrow \mathbb{R}$  is defined as

$$f_\varepsilon(s) := 1 \wedge \varepsilon^{1/2} f(s), \quad f_\varepsilon(1) = 1, \quad (2.2)$$

where  $f$  is a fixed function with the following properties:

$$f \in C^1([0, 1]; [0, \infty)) \text{ is nondecreasing, } f^{-1}(0) = \{0\}, \quad (2.3)$$

$$f_1(s) := sf(1-s) \text{ is strictly decreasing, and } f_1(\sqrt{\cdot}) \text{ is convex,} \quad (2.4)$$

$$f_1(s) = \ell - \ell_1 s + o(s) \quad \text{as } s \rightarrow 0^+, \quad (2.5)$$

for some  $\ell, \ell_1 > 0$ , where  $\lim_{s \rightarrow 0^+} o(s)/s = 0$ . The assumptions (2.4)–(2.5) are of technical nature and are slightly stronger than the corresponding assumption in [19]: we need to include them in order to guarantee further properties of the surface energy density  $g$  defined below (see Section 3 for details). For instance, prototype pairs  $(f, f_1)$  with these properties are

$$f^a(s) := \frac{\ell s}{1-s} \quad \text{and} \quad f_1^a(s) := \ell(1-s), \quad \text{for any } \ell > 0, \quad (2.6)$$

(with  $\ell_1 = \ell$  and  $s \mapsto f_1^a(\sqrt{s}) = \ell(1 - \sqrt{s})$  convex) as well as

$$f^b(s) := \frac{(\ell + b(1-s))s^2}{1-s} \quad \text{and} \quad f_1^b(s) := (\ell + bs)(1-s)^2, \quad \text{for any } \ell > 0, b \in (-\ell, 2\ell), \quad (2.7)$$

(with  $\ell_1 = 2\ell - b$  and  $s \mapsto f_1^b(\sqrt{s}) = (\ell + b\sqrt{s})(1 - \sqrt{s})^2$  convex).

We also introduce the limit functional  $\mathcal{F} : L^1(0, 1) \times L^1(0, 1) \rightarrow [0, \infty]$ :

$$\mathcal{F}(u, v) := \begin{cases} \int_0^1 h(|u'|) dx + \int_{J_u} g(|[u]|) d\mathcal{H}^0 + \ell |D^c u|(0, 1) & \text{if } u \in \text{BV}(0, 1), \\ \infty & \text{otherwise.} \end{cases} \quad \begin{matrix} v = 1 \text{ } \mathcal{L}^1\text{-a.e. in } (0, 1), \\ \end{matrix} \quad (2.8)$$

The densities  $h, g : [0, \infty) \rightarrow [0, \infty)$  are defined as follows:

$$h(s) := \begin{cases} s^2 & \text{if } 0 \leq s \leq \frac{\ell}{2}, \\ \ell s - \frac{\ell^2}{4} & \text{if } s > \frac{\ell}{2} \end{cases} \quad (2.9)$$

(extended to all  $\mathbb{R}$  as even function), and

$$g(s) := \inf \left\{ \int_0^1 |1 - \beta| \sqrt{s^2 f^2(\beta) |\alpha'|^2 + |\beta'|^2} dt : (\alpha, \beta) \in H^1(0, 1) \times H^1(0, 1), \right. \\ \left. \alpha(0) = 0, \alpha(1) = 1, 0 \leq \beta \leq 1, \beta(0) = \beta(1) = 1 \right\}, \quad (2.10)$$

see Figure 2 and Figure 3. We remark that  $(1 - \beta)f(\beta) = f_1(1 - \beta)$  by (2.5) is continuous at  $\beta = 1$ , so that the integrand is interpreted as  $s\ell|\alpha'|$  on the set  $\{\beta = 1\}$ . Notice that the infimum in (2.10) is invariant under reparametrization of the interval  $(0, 1)$ . The properties of the surface energy density  $g$  will be discussed in Section 3. We will also consider the localized versions of the functionals above, by writing  $\mathcal{F}_\varepsilon(u, v; A)$  and  $\mathcal{F}(u, v; A)$  when the domain  $(0, 1)$  is replaced by an open set  $A \subset (0, 1)$ . The following  $\Gamma$ -convergence result is proved in [19].



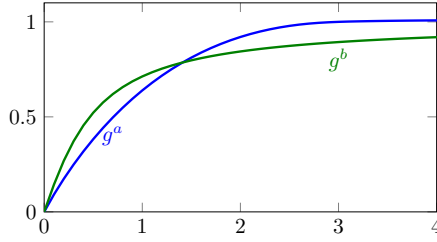


Figure 2: Graph of the function  $g$  for two different choices of  $f$ . Specifically,  $g^a$  is generated using the function  $f^a$  defined in (2.6) with  $\ell = 1$ , and obeys  $g^a(s) < 1$  for  $s \in [0, s_{\text{frac}})$  with  $s_{\text{frac}} \sim 3$ ;  $g^b$  is generated using the function  $f^b$  defined in (2.7) with  $\ell = 1.5$ ,  $\ell_1 = 0.2$  (hence  $b = 2.8$ ), and obeys  $g^b(s) < \infty$  for all  $s \in [0, \infty)$ .

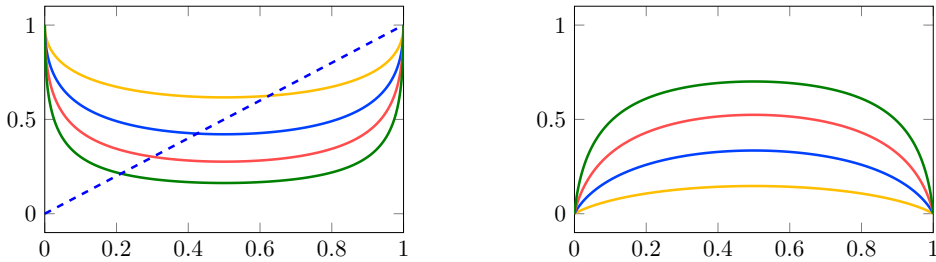


Figure 3: Left: Graph of an optimal pair  $(\alpha, \beta)$  from the definition of  $g$  in (2.10) for  $f_1^a(s) := \ell(1-s)$ ,  $\ell = 1$ , as in (2.6). The dashed line is  $\alpha(t) = t$ , the full curves are  $\beta$  with (from top to bottom)  $s = 0.5$ ,  $s = 1$ ,  $s = 1.5$ ,  $s = 2$ . Right: Graph of an optimal function  $\gamma$  from the equivalent characterization of  $g$  in Proposition 3.3. From bottom to top,  $s = 0.5$ ,  $s = 1$ ,  $s = 1.5$ ,  $s = 2$ .

**Theorem 2.1** (Conti-Focardi-Iurlano [19]). *Let  $f$  obey (2.3) and (2.4). The functionals  $\mathcal{F}_\varepsilon$  defined in (2.1)–(2.2)  $\Gamma$ -converge as  $\varepsilon \rightarrow 0^+$  to the functional  $\mathcal{F}$  defined in (2.8)–(2.10) in  $L^1(0,1) \times L^1(0,1)$ . Moreover, if  $(u_\varepsilon, v_\varepsilon)$  satisfies the uniform bound*

$$\sup_\varepsilon \left( \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon\|_{L^1(0,1)} \right) < \infty, \quad (2.11)$$

*then there exists a subsequence  $(u_{\varepsilon_k}, v_{\varepsilon_k})_k$  and a function  $u \in \text{BV}(0,1)$  such that  $u_{\varepsilon_k} \rightarrow u$  almost everywhere in  $(0,1)$  and  $v_{\varepsilon_k} \rightarrow 1$  in  $L^1(0,1)$ .*

The definitions and assumptions stated in this Section are assumed to hold, without explicit mention, in the rest of the paper.

### 3 The fracture energy density

We discuss in this section the features of the fracture energy density  $g$ , defined in (2.10), and we introduce a new surface energy density  $\bar{g}$  (see (3.33)), which appears in the  $\Gamma$ -limit of the phase-field energies  $\mathcal{F}_\varepsilon$  when we include an irreversibility constraint.

#### 3.1 The function $g$ and the cohesive energy of pristine material

We detail in a series of propositions several properties of the function  $g$ , defined in (2.10). Most of them are of technical nature and can be skipped at a first reading; we advise the reader to focus on Proposition 3.2 and Theorem 3.4, which contain the main ideas needed in the rest of the paper.

We shall show below (Proposition 3.1(ii)) that  $g(s) \in [0, 1]$  for all  $s$ . Many properties of  $g$  are different if  $g(s) < 1$  or  $g(s) = 1$ . Therefore we define

$$s_{\text{frac}} := \sup\{s : g(s) < 1\} \in (0, \infty]. \quad (3.1)$$

In Proposition 3.5 we discuss under which conditions  $s_{\text{frac}} < \infty$ .

We start by listing in the following proposition some basic properties of the function  $g$ , already observed in [19, Proposition 4.1] for the exception of (v), which guarantees that the function  $g$  is strictly below the linear function  $\ell s$ : for the proof of this last property, we require the additional assumption (2.5) (which is not needed for the  $\Gamma$ -convergence result in [19]). Notice that this condition, which is used in the proof of the strict subadditivity of  $g$  (Proposition 3.6), will be significantly improved in Proposition 3.7.

**Proposition 3.1.** *The function  $g$  defined in (2.10) enjoys the following properties:*

- (i)  $g(0) = 0$ , and  $g$  is subadditive, i.e.  $g(s_1 + s_2) \leq g(s_1) + g(s_2)$  for every  $s_1, s_2 \in \mathbb{R}^+$ ;
- (ii)  $g$  is nondecreasing,  $g(s) \leq 1 \wedge \ell s$ , and  $g$  is Lipschitz continuous with Lipschitz constant  $\ell$ ;
- (iii)  $\lim_{s \rightarrow \infty} g(s) = 1$ ;
- (iv)  $\lim_{s \rightarrow 0^+} \frac{g(s)}{s} = \ell$ ;
- (v)  $g(s) < \ell s$  for all  $s > 0$ .

*Proof.* Only the statement (v) requires a new proof, the others are contained in [19, Proposition 4.1]. With fixed  $s > 0$ , let  $\sigma \in (0, 1)$  and set  $\alpha(t) := 0$  in  $[0, \frac{1}{3}]$ ,  $\alpha(t) := s$  in  $[\frac{2}{3}, 1]$ , and the linear interpolation between these two values in  $[\frac{1}{3}, \frac{2}{3}]$ ; set also  $\beta(t) := \sigma$  in  $[\frac{1}{3}, \frac{2}{3}]$ , and the linear interpolation between the values  $\sigma$  and 1 in each of the two intervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$ . By using the pair  $(\alpha, \beta)$  as a competitor in the definition (2.10) of  $g$  we find

$$\begin{aligned} g(s) &\leq \inf_{\sigma \in (0,1)} [s(1-\sigma)f(\sigma) + (1-\sigma)^2] \\ &= \inf_{\sigma \in (0,1)} [s\ell - s\ell_1(1-\sigma) + o(1-\sigma) + (1-\sigma)^2], \end{aligned} \quad (3.2)$$

where we used the assumption (2.5). The strict inequality  $g(s) < \ell s$  follows from (3.2), by choosing a  $\sigma$  sufficiently close to 1.  $\square$

We now prove two alternative representation formulas for  $g$ . The first one, (3.3), is expressed in terms of the minimization of a functional on the whole real line that resembles more closely the form of the energy  $\mathcal{F}_\varepsilon$ . The second representation formula (3.9) instead removes the invariance under reparametrization in (2.10) and has a unique minimizer.

**Proposition 3.2.** *The following properties hold:*

- (i) *We have*

$$g(s) = \inf_{(\alpha, \beta) \in \mathcal{U}_1} \mathcal{G}_s(\alpha, \beta), \quad (3.3)$$

where

$$\mathcal{U}_1 := \left\{ (\alpha, \beta) \in H_{\text{loc}}^1(\mathbb{R}) \times H_{\text{loc}}^1(\mathbb{R}) : \alpha' \in L^1(\mathbb{R}), \left| \int_{-\infty}^{\infty} \alpha'(t) dt \right| = 1, \right. \\ \left. 0 \leq \beta \leq 1, \lim_{|t| \rightarrow \infty} \beta(t) = 1 \right\}, \quad (3.4)$$

$$\mathcal{G}_s(\alpha, \beta) := \int_{-\infty}^{\infty} \left( s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt. \quad (3.5)$$

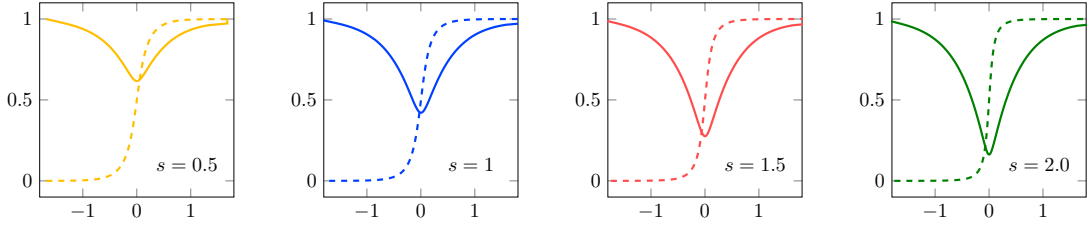


Figure 4: Graph of an optimal pair  $(\alpha, \beta)$  from the characterization of  $g$  in Proposition 3.2 with for  $f_1^\alpha(s) := \ell(1-s)$ ,  $\ell = 1$ , as in (2.6), for  $s = 0.5$ ,  $s = 1$ ,  $s = 1.5$  and  $s = 2$  (from left to right). In each graph  $\alpha$  is dashed,  $\beta$  is the full curve.

- (ii) For  $g(s) < 1$  the problem in (3.3) has a minimizer  $(\alpha_s, \beta_s) \in \mathcal{U}_1$ .
- (iii) Any minimizer obeys  $\min \beta_s \geq 1 - \sqrt{g(s)}$ .
- (iv) If  $0 \leq s' < s$  and  $g(s) < 1$ , then  $g(s') < g(s)$ .
- (v) For any minimizer  $(\alpha_s, \beta_s)$  there is  $t_* \in \mathbb{R}$  such that  $\beta_s$  is nonincreasing in  $(-\infty, t_*)$  and nondecreasing in  $(t_*, \infty)$ .
- (vi) If  $g(s) < 1$ , then for any minimizer  $(\alpha_s, \beta_s)$  there are  $T_-, T_+ \in \overline{\mathbb{R}}$  such that  $\beta_s \in C^1(T_-, T_+)$  and  $\alpha_s \in C^1(\mathbb{R})$ , with  $\beta_s = 1$  and  $\alpha'_s = 0$  on  $\mathbb{R} \setminus (T_-, T_+)$ . The map  $\alpha_s - \alpha_s(-\infty)$  is a  $C^1$  bijection of  $(T_-, T_+)$  onto  $(0, 1)$ .
- (vii) Any minimizer  $(\alpha_s, \beta_s)$  obeys

$$s^2 f^2(\beta_s) |\alpha'_s|^2 + |\beta'_s|^2 = \frac{(1 - \beta_s)^2}{4} \text{ pointwise in } \mathbb{R}. \quad (3.6)$$

This in particular implies that  $\alpha'_s = 0$  on the set  $\{\beta_s = 1\}$ .

*Proof.* The two infima in (3.3) coincide by a simple rescaling of  $\alpha$ ; therefore we have only to prove the first equality in (3.3). We recall that in [19, Proposition 4.3] the following representation formula for  $g$  was proved:

$$g(s) = \inf_{T>0} \inf \left\{ \int_{-T}^T \left( s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : \alpha, \beta \in H^1(-T, T), \right. \\ \left. \alpha(-T) = 0, \alpha(T) = 1, 0 \leq \beta \leq 1, \beta(-T) = \beta(T) = 1 \right\}. \quad (3.7)$$

Notice that the minimum problems in (3.7) are decreasing with respect to  $T$ ; the inequality  $\inf_{(\alpha, \beta) \in \mathcal{U}_1} \mathcal{G}_s(\alpha, \beta) \leq g(s)$  is trivial, since if  $\alpha, \beta \in H^1(-T, T)$  are admissible functions in (3.7) they can be extended as constants outside  $(-T, T)$  to obtain a pair in  $\mathcal{U}_1$ .

We therefore have to show that for every  $(\alpha, \beta) \in \mathcal{U}_1$  the inequality  $g(s) \leq \mathcal{G}_s(\alpha, \beta)$  holds. Since translations and truncations of  $\alpha$  do not increase the energy, as well as the symmetric reflection with respect to the origin of  $\alpha$  and  $\beta$ , we can assume without loss of generality that  $0 \leq \alpha \leq 1$ ,  $\lim_{t \rightarrow -\infty} \alpha(t) = 0$ ,  $\lim_{t \rightarrow \infty} \alpha(t) = 1$ .

Fix  $T > 0$  and let

$$M_T := 1 - \sqrt{1 - \alpha(T)}.$$

Notice that  $0 \leq M_T \leq 1$  and  $\lim_{T \rightarrow \infty} M_T = 1$ . We define a new pair  $(\alpha_T, \beta_T)$ , admissible for the minimum problem (3.7), by modifying the functions  $(\alpha, \beta)$  outside the interval  $(-T, T)$  as follows:

$$\alpha_T(t) := \begin{cases} \alpha(t) & \text{if } t \in (-T, T), \\ \alpha(T) & \text{if } t \in [T, T+1), \\ \text{linear interpolation} & \text{if } t \in [T+1, T+2), \\ 1 & \text{if } t \in [T+2, \infty), \end{cases}$$

$$\beta_T(t) := \begin{cases} \beta(t) & \text{if } t \in (-T, T), \\ \text{linear interpolation} & \text{if } t \in [T, T+1), \\ M_T & \text{if } t \in [T+1, T+2), \\ \text{linear interpolation} & \text{if } t \in [T+2, T+3), \\ 1 & \text{if } t \in [T+3, \infty) \end{cases}$$

(in the interval  $(-\infty, -T)$  we do a symmetric construction, with the value  $M_T$  replaced by  $M'_T := 1 - \sqrt{\alpha(-T)}$ ). Then we have  $\alpha_T, \beta_T \in H^1(-T-3, T+3)$ ,  $\alpha_T(-T-3) = 0$ ,  $\alpha_T(T+3) = 1$ ,  $0 \leq \beta_T \leq 1$ ,  $\beta_T(-T-3) = \beta_T(T+3) = 1$ , and the pair  $(\alpha_T, \beta_T)$  is therefore admissible in the minimum problem (3.7). Furthermore

$$\begin{aligned} \omega_+(T) &:= \int_T^{T+3} \left( s^2 f^2(\beta_T) |\alpha'_T|^2 + \frac{(1-\beta_T)^2}{4} + |\beta'_T|^2 \right) dt \\ &= \int_{(T, T+1) \cup (T+2, T+3)} \left( \frac{(1-\beta_T)^2}{4} + |\beta'_T|^2 \right) dt + s^2 f^2(M_T) |1 - \alpha(T)|^2 + \frac{(1-M_T)^2}{4} \\ &= \frac{1}{4} \int_0^1 (1 - \beta(T) - (M_T - \beta(T))t)^2 dt + \frac{1}{4} \int_0^1 (1 - M_T - (1 - M_T)t)^2 dt \\ &\quad + |\beta(T) - M_T|^2 + |1 - M_T|^2 + s^2 (f(M_T)(1 - M_T))^2 (1 - \alpha(T)) + \frac{(1 - M_T)^2}{4}, \end{aligned}$$

and from the fact that  $\alpha(T), \beta(T), M_T \rightarrow 1$  as  $T \rightarrow \infty$  we obtain, using assumption (2.5),

$$\omega_+(T) \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Similarly

$$\omega_-(T) := \int_{-T-3}^{-T} \left( s^2 f^2(\beta_T) |\alpha'_T|^2 + \frac{(1-\beta_T)^2}{4} + |\beta'_T|^2 \right) dt \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Therefore

$$\begin{aligned} \mathcal{G}_s(\alpha, \beta) &\geq \int_{-T}^T \left( s^2 f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt \\ &= \int_{-T-3}^{T+3} \left( s^2 f^2(\beta_T) |\alpha'_T|^2 + \frac{(1-\beta_T)^2}{4} + |\beta'_T|^2 \right) dt - \omega_+(T) - \omega_-(T) \\ &\geq g(s) - \omega_+(T) - \omega_-(T) \end{aligned}$$

and the conclusion follows by sending  $T \rightarrow \infty$ . This concludes the proof of (i).

We now observe that for any  $\beta$  with  $1 - \beta \in H^1(\mathbb{R})$  and any  $t_0 \in \mathbb{R}$  we have

$$\int_{-\infty}^{\infty} \left( \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt \geq - \int_{-\infty}^{t_0} (1-\beta)\beta' dt + \int_{t_0}^{\infty} (1-\beta)\beta' dt = (1-\beta(t_0))^2, \quad (3.8)$$

with equality if and only if  $\beta(t) = 1 - (1 - \beta(t_0))e^{-|t-t_0|/2}$ . Assertion (iii) follows immediately.

Let us prove (ii). Assume now that  $g(s) < 1$ , which is equivalent to  $s < s_{\text{frac}}$ . By equation (3.8), in the minimization it is sufficient to consider functions  $\beta$  which obey  $\beta > 0$  almost everywhere. For a fixed  $\beta$ , (3.3) is a linear problem in  $\alpha$ . One can check that the minimizer is such that  $f^2(\beta)\alpha'$  is constant on the set  $\{\beta < 1\}$ , and that  $\alpha' = 0$  on  $\{\beta = 1\}$  (up to null sets). Therefore the minimizer takes the form  $\alpha'(t) = cf^{-2}(\beta(t))$  for some  $c = c(s, \beta) \in \mathbb{R}$ , where to shorten notation we write  $f^{-2}(1) = 0$ . Integrating shows that  $1 = c \int_{-\infty}^{\infty} f^{-2}(\beta) dt$ , so that

$$\hat{\mathcal{G}}_s(\beta) := \min_{\alpha} \mathcal{G}_s(\alpha, \beta) = s^2 \left( \int_{-\infty}^{\infty} f^{-2}(\beta) dt \right)^{-1} + \int_{-\infty}^{\infty} \left( \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt.$$

Let now  $\beta_k \in H_{\text{loc}}^1(\mathbb{R}; [0, 1])$  be a minimizing sequence for  $\hat{\mathcal{G}}_s$ . Since  $\inf \hat{\mathcal{G}}_s = g(s) < 1$  we can assume that there is  $\delta > 0$  such that  $\hat{\mathcal{G}}_s(\beta_k) \leq 1 - \delta$  for all  $k$ . By (3.8) we then obtain  $1 - \delta \geq (1 - \beta_k(t))^2$  and therefore  $\beta_k(t) \geq \frac{1}{2}\delta$  for all  $k$  and all  $t$ .

The sequence  $1 - \beta_k$  is bounded in  $H^1(\mathbb{R})$  and has therefore a subsequence converging weakly to some function  $1 - \beta_s$ . Since the second term in  $\hat{\mathcal{G}}_s$  is convex, if we prove that  $\int_{-\infty}^{\infty} f^{-2}(\beta_k) dt \rightarrow \int_{-\infty}^{\infty} f^{-2}(\beta_s) dt$  then  $\beta_s$  is a minimizer of  $\hat{\mathcal{G}}_s$ , and by the formula stated above we can reconstruct  $\alpha$ . To prove continuity we observe that (possibly after extracting a further subsequence)  $\beta_k \rightarrow \beta_s$  pointwise almost everywhere, and that (2.4) and  $\beta_k(t) \geq \frac{1}{2}\delta$  imply  $f(\beta_k) = f_1(1 - \beta_k)/(1 - \beta_k) \geq f_1(1 - \frac{1}{2}\delta)/(1 - \beta_k)$  and therefore  $0 \leq f^{-2}(\beta_k) \leq C_s(1 - \beta_k)^2$  pointwise. Since  $\int_{-\infty}^{\infty} (1 - \beta_k)^2 dt \leq 4$ , by Fatou's Lemma we obtain

$$\int_{-\infty}^{\infty} f^{-2}(\beta_s) dt = \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f^{-2}(\beta_k) dt.$$

Therefore  $\beta_s$  is a minimizer of  $\hat{\mathcal{G}}_s$ . This concludes the proof of (ii).

Assume now that  $s \in (0, \infty)$ ,  $g(s) < 1$  and  $s' \in (0, s)$ . Let  $\beta_s$  be a minimizer of  $\hat{\mathcal{G}}_s$ . By the above estimates  $\int_{-\infty}^{\infty} f^{-2}(\beta_s) dt \leq C_s \int_{-\infty}^{\infty} (1 - \beta_s)^2 dt < \infty$  and therefore  $g(s') \leq \hat{\mathcal{G}}_{s'}(\beta_s) < \hat{\mathcal{G}}_s(\beta_s) = g(s)$ . This concludes the proof of (iv).

In order to prove (v), it suffices to show that for any  $m \in [0, 1]$  the set  $I_m := \{t : \beta_s(t) \leq m\}$  is an interval. This will imply the stated monotonicity properties, with  $t_*$  any point in the intersection of all  $I_m$  for  $m > \inf \beta_s$ . If  $I_m$  were not an interval, there would be  $t_1 < t_2 \in I_m$  with  $m = \beta_s(t_1) = \beta_s(t_2)$  and  $\beta_s(t) > m$  for  $t \in (t_1, t_2)$ . We can construct an admissible competitor  $(\alpha, \beta) \in \mathcal{U}_1$  by modifying  $(\alpha_s, \beta_s)$  as follows:

$$\alpha(t) := \begin{cases} \alpha_s(t) & \text{if } t \leq t_1, \\ \alpha_s(t_1) + \frac{\alpha_s(t_2) - \alpha_s(t_1)}{\delta}(t - t_1) & \text{if } t_1 < t < t_1 + \delta, \\ \alpha_s(t + t_2 - t_1 - \delta) & \text{if } t \geq t_1 + \delta, \end{cases}$$

$$\beta(t) := \begin{cases} \beta_s(t) & \text{if } t \leq t_1, \\ m & \text{if } t_1 < t < t_1 + \delta, \\ \beta_s(t + t_2 - t_1 - \delta) & \text{if } t \geq t_1 + \delta, \end{cases}$$

where  $\delta > 0$  is to be chosen. Then, by using Young's inequality and by (2.4),

$$\begin{aligned} \mathcal{G}_s(\alpha_s, \beta_s) - \mathcal{G}_s(\alpha, \beta) &= \int_{t_1}^{t_2} \left( s^2 f^2(\beta_s) |\alpha'_s|^2 + \frac{(1 - \beta_s)^2}{4} + |\beta'_s|^2 \right) dt \\ &\quad - \int_{t_1}^{t_1 + \delta} \left( s^2 f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt \\ &> \int_{t_1}^{t_2} s f(\beta_s) (1 - \beta_s) |\alpha'_s| dt - \frac{s^2 f^2(m) |\alpha_s(t_2) - \alpha_s(t_1)|^2}{\delta} - \frac{(1 - m)^2}{4} \delta \\ &\geq s f(m) (1 - m) |\alpha_s(t_2) - \alpha_s(t_1)| - \frac{s^2 f^2(m) |\alpha_s(t_2) - \alpha_s(t_1)|^2}{\delta} - \frac{(1 - m)^2}{4} \delta. \end{aligned}$$

By optimizing in  $\delta$  we see that the right-hand side in the previous inequality is zero, and this contradicts the minimality of  $(\alpha_s, \beta_s)$ . This concludes the proof of (v).

Let us prove (vi). Let  $g(s) < 1$  and choose an optimal pair  $(\alpha_s, \beta_s) \in \mathcal{U}_1$  for  $g(s)$ . By (iii) and (2.3) we have  $f(\beta_s) \geq f(1 - \sqrt{g(s)}) > 0$  on  $\mathbb{R}$ . Let  $T_- := \inf\{t : \beta_s(t) < 1\} \in \mathbb{R} \cup \{-\infty\}$ ,  $T_+ := \sup\{t : \beta_s(t) < 1\} \in \mathbb{R} \cup \{\infty\}$ . By (v) we have  $\beta_s < 1$  in  $(T_-, T_+)$ , by finiteness of the integral we have  $\alpha'_s = 0$  in  $\mathbb{R} \setminus (T_-, T_+)$ . By taking variations in the variable  $\alpha$  in the minimum problem (3.3) we find

$$f^2(\beta_s)\alpha'_s = \text{const.} \quad \text{almost everywhere in } (T_-, T_+).$$

Therefore  $\alpha'_s > 0$  in  $(T_-, T_+)$  with  $\lim_{t \uparrow T_+} \alpha'_s(t) = \lim_{t \downarrow T_-} \alpha'_s(t) = 0$ , and in particular  $\alpha_s : (T_-, T_+) \rightarrow (\alpha_s(T_-), \alpha_s(T_+))$  is a  $C^1$ , bijective map (with  $\alpha_s(T_\pm)$  interpreted as the limit of  $\alpha_s(t)$  for  $t \rightarrow \pm\infty$  if  $T_\pm = \pm\infty$ ). By taking variations in the variable  $\beta$  we obtain instead

$$\int_{-\infty}^{\infty} \left( s^2 f(\beta_s) f'(\beta_s) |\alpha'_s|^2 \varphi - \frac{(1 - \beta_s)\varphi}{4} + \beta'_s \varphi' \right) dt = 0 \quad \text{for all } \varphi \in C_c^\infty(T_-, T_+),$$

from which it follows by standard arguments that  $\beta_s \in C^1(T_-, T_+)$ . Hence (vi) is proved.

Finally, taking internal variations (in the sense of considering competitors of the type  $(\alpha_s(t + \varepsilon\varphi(t)), \beta_s(t + \varepsilon\varphi(t)))$ , for  $\varphi \in C_c^1(\mathbb{R})$ ) one also obtains the usual equipartition result, in the sense that the minimizer fulfills (3.6). This proves (vii).  $\square$

**Proposition 3.3.** *The following properties hold:*

(i) *For every  $s \geq 0$  the function  $g$  defined in (2.10) can be obtained as*

$$g(s) = \inf \left\{ \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt : \gamma \in W_0^{1,1}(0,1), 0 \leq \gamma \leq 1 \right\}, \quad (3.9)$$

where  $f_1$  is the function defined in (2.4).

(ii) *For  $s \in (0, s_{\text{frac}})$  the variational problem (3.9) has a unique minimizer  $\gamma_s$ . It obeys  $\gamma_s \in C^1(0,1)$ ,  $\gamma_s(\frac{1}{2}) = \max \gamma_s = (1 - \min \beta_s)^2$ , where  $\beta_s$  is any minimizer from Proposition 3.2(ii).*

(iii) *If  $0 < s_1 < s_2 < s_{\text{frac}}$  then  $0 < \gamma_{s_1}(t) < \gamma_{s_2}(t)$  for all  $t \in (0,1)$ .*

(iv) *The function*

$$m_s := \begin{cases} 1 - \sqrt{\gamma_s(\frac{1}{2})}, & \text{if } s \in [0, s_{\text{frac}}), \\ 0, & \text{if } s > s_{\text{frac}}, \end{cases} \quad (3.10)$$

is continuous and strictly decreasing in  $[0, s_{\text{frac}})$ .

(v) *For  $s < s_{\text{frac}}$ , the minimizer of (3.3) is unique up to translations, in the sense that if  $(\alpha_s, \beta_s)$  and  $(\hat{\alpha}_s, \hat{\beta}_s)$  are minimizers then there are  $a_1, t_1 \in \mathbb{R}$  such that  $\alpha_s(t) = a_1 + \hat{\alpha}_s(t - t_1)$ ,  $\beta_s(t) = \hat{\beta}_s(t - t_1)$ .*

*Proof.* (i): The identity (3.9) follows by reparametrization: indeed, first observe that by a density argument the infimum in (2.10) can be written as

$$\inf \left\{ \int_0^1 \sqrt{s^2 (1 - \beta)^2 f^2(\beta) |\alpha'|^2 + \frac{1}{4} \left| \frac{d}{dt} (1 - \beta)^2 \right|^2} dt : (1 - \beta)^2 \in C_c^1(0,1), 0 \leq \beta \leq 1, \right. \\ \left. \alpha \in C^1([0,1]), \alpha(0) = 0, \alpha(1) = 1, \alpha' > 0 \right\}. \quad (3.11)$$

Similarly, the infimum in (3.9) can be considered on the class of  $\gamma \in C_c^1(0,1)$  with  $0 \leq \gamma \leq 1$ . Then, for  $(\alpha, \beta)$  admissible in (3.11), the reparametrization

$$\gamma(t) := (1 - \beta)^2 \circ \alpha^{-1}(t) \quad (3.12)$$

gives an admissible profile for (3.9) with

$$\int_0^1 \sqrt{s^2(1-\beta)^2 f^2(\beta) |\alpha'|^2 + \frac{1}{4} \left| \frac{d}{dt} (1-\beta)^2 \right|^2} dt = \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt. \quad (3.13)$$

Conversely, given  $\gamma \in C_c^1(0,1)$  with  $0 \leq \gamma \leq 1$ , we consider the pair  $(\alpha, \beta)$  defined by  $\alpha(t) := t$ ,  $\beta(t) := 1 - \sqrt{\gamma(t)}$ , which is admissible in (3.11) and satisfies (3.13). This proves (i).

(ii): Assume now  $g(s) < 1$  and let  $(\alpha_s, \beta_s) \in \mathcal{U}_1$  be a minimizer for (3.3) with  $\alpha_s(-\infty) = 0$ , which exists by Proposition 3.2(ii). We consider the rescaled profile  $\gamma_s : [0,1] \rightarrow [0,1]$ ,  $\gamma_s(t) := (1 - \beta_s \circ \alpha_s^{-1}(t))^2$ : we have  $\gamma_s \in C^1(0,1)$ ,  $0 \leq \gamma_s \leq 1$ ,  $\gamma_s(0) = \gamma_s(1) = 0$ , and, with  $T_-, T_+$  as in Proposition 3.2(vi),

$$g(s) = \mathcal{G}_s(\alpha_s, \beta_s) \geq \int_{T_-}^{T_+} |1 - \beta_s| \sqrt{s^2 f^2(\beta_s) |\alpha_s'|^2 + |\beta_s'|^2} dt = \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{|\gamma_s'|^2}{4}} dt,$$

which shows that  $\gamma_s$  is a minimizer in problem (3.9). We have  $1 - \min \beta_s = \max \sqrt{\gamma_s}$  and by symmetrization the maximum is attained at the point  $t = \frac{1}{2}$ . Thanks to the convexity assumption (2.4), the minimizer  $\gamma_s$  of (3.9) is unique.

(iii) and (iv): We will exploit further properties of the optimal profile  $\gamma_s$  for  $s < s_{\text{frac}}$ : by computing the Euler-Lagrange equation satisfied by a  $C^1$ -minimizer of problem (3.9) we find

$$\left( \frac{\gamma_s'}{2\sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{1}{4} |\gamma_s'|^2}} \right)' = \frac{s^2 f_1(\sqrt{\gamma_s}) f_1'(\sqrt{\gamma_s})}{\sqrt{\gamma_s} \sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{1}{4} |\gamma_s'|^2}} \quad \text{in } (0,1),$$

which yields the  $C^2$ -regularity of  $\gamma_s$  in  $(0,1)$  and, after elementary computations,

$$\gamma_s'' = \frac{f_1(\sqrt{\gamma_s}) f_1'(\sqrt{\gamma_s})}{\sqrt{\gamma_s}} \left( 2s^2 + \frac{|\gamma_s'|^2}{(f_1(\sqrt{\gamma_s}))^2} \right) \quad \text{in } (0,1). \quad (3.14)$$

Furthermore, the equation (3.14) has a first integral:

$$(f_1(\sqrt{\gamma_s}))^2 = c \sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{1}{4} |\gamma_s'|^2} \quad \text{in } (0,1)$$

for a constant  $c > 0$ , which can be computed by imposing that  $\gamma_s$  has a maximum at the point  $t = \frac{1}{2}$  with value  $(1 - m_s)^2$ : this gives  $c = \frac{1}{s} f_1(1 - m_s)$  and in turn

$$\gamma_s' = \pm 2s f_1(\sqrt{\gamma_s}) \sqrt{\left( \frac{f_1(\sqrt{\gamma_s})}{f_1(1 - m_s)} \right)^2 - 1} \quad \text{in } (0,1) \quad (3.15)$$

(with positive sign in  $(0, \frac{1}{2})$ , and negative sign in  $(\frac{1}{2}, 1)$ ).

By using the previous equations we can now show the continuity of the map  $s \mapsto m_s$ . Indeed, consider a sequence  $s_n \rightarrow s \in [0, s_{\text{frac}})$  and the corresponding optimal profiles  $\gamma_{s_n}$ . In view of (3.15), the first derivatives  $\gamma_{s_n}'$  are uniformly bounded in  $L^\infty(0,1)$ , hence possibly extracting a subsequence we have  $\gamma_{s_n} \rightharpoonup \gamma$  weakly\* in  $W^{1,\infty}(0,1)$  (and uniformly), for some  $\gamma$  vanishing at the boundary. By continuity of  $g$  we find that  $\gamma$  is the unique minimizer in problem (3.9) for  $g(s)$ ; in particular  $(1 - m_s)^2 = \gamma(\frac{1}{2}) = \lim_n \gamma_{s_n}(\frac{1}{2}) = \lim_n (1 - m_{s_n})^2$ , which proves the continuity of  $m_s$ .

We then show that the map  $s \mapsto m_s$  is injective (then also the strict monotonicity follows). Assume by contradiction that for two different values  $s_1 < s_2$  one has  $m_{s_1} = m_{s_2}$ , and let  $\gamma := \gamma_{s_2} - \gamma_{s_1}$  be the difference of the corresponding optimal profiles. By (3.15) we have  $\gamma'(0) > 0$ , and hence  $\gamma(t) > 0$  for all sufficiently small  $t$ . On the other hand, at the point  $t = \frac{1}{2}$  by assumption  $\gamma(\frac{1}{2}) = \gamma_{s_1}'(\frac{1}{2}) = \gamma_{s_2}'(\frac{1}{2}) = 0$ , therefore we find using (3.14)

$$\gamma''(\frac{1}{2}) = 2(s_2^2 - s_1^2) \frac{f_1(1 - m_{s_1}) f_1'(1 - m_{s_1})}{1 - m_{s_1}} < 0$$

(by the assumption (2.4)). We conclude that  $\gamma(t) < 0$  for  $t$  sufficiently close to  $\frac{1}{2}$ . Let then  $\bar{t} \in (0, \frac{1}{2})$  be the smallest value with  $\gamma(\bar{t}) = 0$ , that is  $\gamma_{s_1}(\bar{t}) = \gamma_{s_2}(\bar{t}) =: \sigma$ . By (3.15)

$$\gamma'(\bar{t}) = 2(s_2 - s_1)f_1(\sqrt{\sigma})\sqrt{\left(\frac{f_1(\sqrt{\sigma})}{f_1(1 - m_{s_1})}\right)^2 - 1} > 0,$$

which is a contradiction.

At this point we conclude the proof of (iii). Fix two values  $s_1 < s_2 < s_{\text{frac}}$  and let  $\gamma := \gamma_{s_2} - \gamma_{s_1}$  be the difference of the corresponding optimal profiles. Then  $m_{s_2} < m_{s_1}$ , hence  $f_1(1 - m_{s_2}) < f_1(1 - m_{s_1})$  and by (3.15) we have  $\gamma'(0) > 0$ , and hence  $\gamma(t) > 0$  for all sufficiently small  $t$ . If  $\gamma(t) > 0$  for all  $t \in (0, 1)$ , we are done. Otherwise, let  $\bar{t} \in (0, 1)$  be the smallest value with  $\gamma(\bar{t}) = 0$ . Then (3.15) implies  $\gamma'(\bar{t}) > 0$ , which is a contradiction.

(v): Let  $(\alpha_s, \beta_s)$  and  $(\hat{\alpha}_s, \hat{\beta}_s) \in \mathcal{U}_1$  be minimizers for (3.3). Without loss of generality we can assume  $\alpha_s(-\infty) = \hat{\alpha}_s(-\infty) = 0$ . By Proposition 3.2(vi),  $\alpha_s$  is a  $C^1$  bijection from  $(T_-, T_+)$  onto  $(0, 1)$ ;  $\alpha_s$  and  $\beta_s$  are constant on  $(-\infty, T_-)$  and  $(T_+, \infty)$  (if these intervals are nonempty). We define  $\gamma \in C^1(0, 1)$  by  $\gamma := (1 - \beta_s)^2 \circ \alpha_s^{-1}$ , so that  $\gamma(t) \rightarrow 0$  as  $t \rightarrow 0$  and  $t \rightarrow 1$ . Then  $\gamma$  is a competitor in (3.9). Using first equipartition, that was proven in (3.6), and then the change of variables  $r := \alpha_s(t)$ , we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left( s^2 f^2(\beta_s) |\alpha'_s|^2 + \frac{(1 - \beta_s)^2}{4} + |\beta'_s|^2 \right) dt &= \int_{-\infty}^{\infty} \sqrt{(1 - \beta_s)^2 s^2 f^2(\beta_s) |\alpha'_s|^2 + (1 - \beta_s)^2 |\beta'_s|^2} dt \\ &= \int_{T_-}^{T_+} \sqrt{s^2 f_1^2(\sqrt{\gamma} \circ \alpha_s) |\alpha'_s|^2 + \frac{1}{4} |\gamma'|^2 \circ \alpha_s |\alpha'_s|^2} dt \\ &= \int_0^1 \sqrt{s^2 f_1^2(\sqrt{\gamma}) + \frac{1}{4} |\gamma'|^2} dr, \end{aligned}$$

so that  $\gamma$  is a minimizer of (3.9). By uniqueness, the pair  $(\hat{\alpha}_s, \hat{\beta}_s)$  has to produce the same  $\gamma$ , so that  $\beta_s \circ \alpha_s^{-1} = \hat{\beta}_s \circ \hat{\alpha}_s^{-1}$ . The function  $\varphi := \hat{\alpha}_s^{-1} \circ \alpha_s$  is a  $C^1$  bijection from  $(T_-, T_+)$  onto  $(\hat{T}_-, \hat{T}_+)$ . Then  $\beta_s = \hat{\beta}_s \circ \varphi$ ,  $\alpha_s = \hat{\alpha}_s \circ \varphi$ . Recalling the equipartition condition (3.6),

$$\begin{aligned} 0 &= s^2 f^2(\beta_s) |\alpha'_s|^2 + |\beta'_s|^2 - \frac{(1 - \beta_s)^2}{4} \\ &= [s^2 f^2(\hat{\beta}_s) |\hat{\alpha}'_s|^2 + |\hat{\beta}'_s|^2] \circ \varphi |\varphi'|^2 - \frac{(1 - \hat{\beta}_s)^2}{4} \circ \varphi = \frac{(1 - \hat{\beta}_s)^2}{4} \circ \varphi [|\varphi'|^2 - 1] \end{aligned}$$

where in the last step we used the equipartition condition (3.6) for  $(\hat{\alpha}_s, \hat{\beta}_s)$ . We conclude that  $|\varphi'| = 1$  wherever  $\hat{\beta}_s(\varphi) < 1$ , which implies  $\varphi(t) = t + t_1$  for some  $t_1 \in \mathbb{R}$  and concludes the proof.  $\square$

We remark that  $(\alpha, \beta)$  minimizing (2.10) is not unique, as the variational problem (2.10) is invariant under reparametrization. However, if one fixes  $\alpha$ , for example by setting  $\alpha(t) = t$ , then for  $s < s_{\text{frac}}$  one easily obtains uniqueness of the corresponding  $\beta$ . Indeed, any minimizer  $\beta$  produces an optimal  $\gamma$  via (3.12). Analogously, once  $\alpha$  is fixed, monotonicity of  $s \mapsto \gamma_s(t)$  implies monotonicity of  $\beta$  with respect to  $s$ .

The main result of this section about the surface energy density  $g$  is contained in the following theorem, which collects the main results from the previous statements. In particular we show that, in the case  $g(s) < 1$ , the infimum in (3.3) is attained by an optimal pair  $(\alpha_s, \beta_s)$ , and that the minimum value  $m_s$  of the optimal profile  $\beta_s$  is uniquely determined by  $s$ , and is a monotone function of  $s$ .

**Theorem 3.4.** *Let  $g$  be defined in (2.10), with a function  $f$  with the properties (2.3), (2.4), (2.5). Let  $s_{\text{frac}}$  be as in (3.1). Then the following holds.*



(i) For all  $s < s_{\text{frac}}$  one has  $g(s) < 1$ , and there exists an optimal pair  $(\alpha_s, \beta_s) \in \mathcal{U}_1$  such that  $g(s) = \mathcal{G}_s(\alpha_s, \beta_s)$ ;

(ii) For all  $s \geq s_{\text{frac}}$  one has  $g(s) = 1$ , and  $g(s) = \mathcal{G}_s(0, \beta_s)$ , where  $\beta_s(t) := 1 - e^{-\frac{t}{s}}$ .

Moreover, the value of the minimum of the optimal profile  $\beta_s$ ,

$$m_s := \min_{t \in \mathbb{R}} \beta_s(t), \quad (3.16)$$

is uniquely determined by  $s$ . The map  $s \mapsto m_s$  is continuous, strictly decreasing in  $[0, s_{\text{frac}})$ , with  $m_0 = 1$  and  $m_s = 0$  for  $s \geq s_{\text{frac}}$ .

*Proof.* The first assertion follows from Proposition 3.2(i) and (ii). The second follows from monotonicity (Proposition 3.1(ii)) and an explicit computation (see also (3.8)). The remaining assertions follow from Proposition 3.3.  $\square$

**Proposition 3.5.** Let  $\eta := \liminf_{t \uparrow 1} \frac{f_1(t)}{1-t^2}$ . If  $\eta = 0$  then  $g(s) < 1$  for all  $s$ , so that  $s_{\text{frac}} = \infty$ . If  $\eta > 0$  and  $s\eta \geq 6$  then  $g(s) = 1$ , so that  $s_{\text{frac}} \leq 6/\eta$ .

We observe that the function  $f^a$  defined in (2.6) has  $\eta = \frac{1}{2}\ell$  and therefore  $s_{\text{frac}} \in (0, 12/\ell)$ , whereas the function  $f^b$  defined in (2.7) has  $\eta = 0$  and therefore  $s_{\text{frac}} = \infty$ . See also Figure 2.

*Proof.* First we observe that  $\eta = \liminf_{t \uparrow 1} \frac{f_1(\sqrt{t})}{1-t}$ , that by convexity of  $f_1(\sqrt{\cdot})$  the lim inf is actually a limit, and that, using convexity,  $f_1(0) = \ell$  and  $f_1(1) = 0$ , we have

$$\eta(1-t) \leq f_1(\sqrt{t}) \leq \ell(1-t) \quad \text{for all } t \in [0, 1],$$

which implies  $\eta \in [0, \ell]$  and  $f_1(1-b) \geq \eta(1-(1-b)^2) = \eta(2b-b^2)$  for all  $b \in [0, 1]$ . We also observe that, using (2.4), the integrand in (2.10) can be estimated by

$$\begin{aligned} (1-\beta)\sqrt{s^2 f_1^2(\beta)(\alpha')^2 + (\beta')^2} &= \sqrt{s^2 f_1^2(1-\beta)(\alpha')^2 + \frac{1}{4}\left(\frac{d}{dt}(1-\beta)^2\right)^2} \\ &\geq \sqrt{s^2 \eta^2 (2\beta - \beta^2)^2 (\alpha')^2 + \frac{1}{4}\left(\frac{d}{dt}(1-\beta)^2\right)^2}. \end{aligned} \quad (3.17)$$

To prove the first assertion we construct a competitor for the definition of  $g$  in (2.10). Assume  $\eta = 0$  and fix  $s > 0$ . By definition of  $\eta$  we can find  $q_s \in (0, 1)$  such that  $s f_1(q_s) \leq \frac{1}{2}(1-q_s^2)$ . We let  $\beta = 1 - q_s$  in  $(\frac{1}{3}, \frac{2}{3})$ ,  $\beta = 1$  in  $\{0, 1\}$ , and the linear interpolation in between; we let  $\alpha = 0$  in  $(0, \frac{1}{3})$ ,  $\alpha = 1$  in  $(\frac{2}{3}, 1)$ , and the linear interpolation in between. A straightforward computation shows that

$$g(s) \leq \int_0^1 \sqrt{s^2 f_1^2(1-\beta)(\alpha')^2 + \frac{1}{4}\left(\frac{d}{dt}(1-\beta)^2\right)^2} dt = q_s^2 + s f_1(q_s) \leq \frac{1}{2}(1+q_s^2) < 1.$$

We now turn to the second assertion. We assume  $s\eta \geq 6$ , let  $\alpha \in H^1(0, 1)$ ,  $\beta \in H^1(0, 1)$  be test functions in the definition of  $g$  in (2.10), denote by  $I(\alpha, \beta)$  the integral in (2.10), and prove  $I(\alpha, \beta) \geq 1$ .

Assume first that there is  $q \in (0, \frac{1}{2}]$  such that  $s\eta \int_{\{q \leq \beta < 2q\}} |\alpha'| dt \geq 2$ . Then (3.17) yields

$$\begin{aligned} \int_0^1 (1-\beta)\sqrt{s^2 f_1^2(\beta)(\alpha')^2 + (\beta')^2} dt &\geq \int_{\{q \leq \beta < 2q\}} s\eta(2\beta - \beta^2)|\alpha'| dt + \int_{\{2q \leq \beta \leq 1\}} \frac{1}{2} \left| \frac{d}{dt}(1-\beta)^2 \right| dt \\ &\geq 2(2q - q^2) + (1-2q)^2 = 1 + 2q^2 > 1 \end{aligned}$$

and the proof is concluded.

We let  $\beta_{\min} := \min \beta$ . If  $\beta_{\min} = 0$  then necessarily  $I(\alpha, \beta) \geq 1$  and we are done. We can therefore assume  $\beta_{\min} > 0$ . We also define, for  $j \in \mathbb{N}$ ,  $\beta_j := \min\{2^j \beta_{\min}, 1\}$  and  $E_j := \{t : \beta_j \leq$

$\beta < \beta_{j+1}$ . We let  $J$  be the largest index with  $\beta_J < 1$  and observe that  $(E_j)$ ,  $j = 0, \dots, J$ , is a partition of  $(0, 1)$ , with  $\beta(\inf E_j) = \beta(\sup E_j) = \beta_{j+1}$  and  $\min \beta(E_j) = \beta_j$ . We define  $a_j := \int_{E_j} |\alpha'| dt$ , which obey  $\sum_j a_j \geq 1$ , and

$$g_j := \int_{E_j} (1 - \beta) \sqrt{s^2 f^2(\beta) (\alpha')^2 + (\beta')^2} dt \geq \int_{E_j} \sqrt{s^2 \eta^2 (2\beta_j - \beta_j^2)^2 (\alpha')^2 + \frac{1}{4} \left( \frac{d}{dt} (1 - \beta)^2 \right)^2} dt.$$

We first treat  $g_J$ . We observe that

$$I(\alpha, \beta) \geq g_J \geq \int_{E_J} s\eta(2\beta_J - \beta_J^2) |\alpha'| dt = s\eta(2\beta_J - \beta_J^2) a_J \geq s\eta \frac{3}{4} a_J,$$

so that if  $s\eta a_J \geq \frac{4}{3}$  we are done. Therefore we can assume  $s\eta a_J < \frac{4}{3}$  in the following.

By Jensen's inequality, for any functions  $h, k : E_j \rightarrow \mathbb{R}$  we have  $|E_j|^{-1} \int_{E_j} \sqrt{h^2 + k^2} dt \geq \sqrt{(|E_j|^{-1} \int_{E_j} h dt)^2 + (|E_j|^{-1} \int_{E_j} k dt)^2}$  and therefore  $\int_{E_j} \sqrt{h^2 + k^2} dt \geq \sqrt{(\int_{E_j} |h| dt)^2 + (\int_{E_j} |k| dt)^2}$ . For  $j < J$ ,  $\int_{E_j} \frac{1}{2} \left| \frac{d}{dt} (1 - \beta)^2 \right| dt \geq (1 - \beta_j)^2 - (1 - \beta_{j+1})^2 = (\beta_{j+1} - \beta_j)(2 - \beta_{j+1} - \beta_j) = \beta_j(2 - 3\beta_j)$ . Therefore, again for  $j < J$ ,

$$g_j \geq \sqrt{s^2 \eta^2 \beta_j^2 (2 - \beta_j)^2 a_j^2 + (\beta_j(2 - 3\beta_j))^2} = \beta_j(2 - 3\beta_j) \sqrt{1 + x_j^2}$$

with  $x_j := (s\eta a_j)(2 - \beta_j)/(2 - 3\beta_j)$ . Recalling  $s\eta a_j \leq 2$  and  $\beta_j \leq \frac{1}{2}$  we obtain  $x_j \in [0, 6]$ . We use the fact that  $\sqrt{1 + x^2} \geq 1 + \frac{1}{8}x^2$  for all  $x \in [0, 6]$  and obtain, inserting first the definition of  $x_j$  and then the one of  $\beta_j$ ,

$$g_j \geq \beta_j(2 - 3\beta_j) \left( 1 + \frac{1}{8} \frac{s^2 \eta^2 a_j^2 (2 - \beta_j)^2}{(2 - 3\beta_j)^2} \right) \geq (1 - \beta_j)^2 - (1 - \beta_{j+1})^2 + \beta_{\min} \frac{1}{8} \frac{(2 - \beta_j)^2}{2 - 3\beta_j} 2^j (s\eta a_j)^2.$$

We observe that  $\frac{(2-x)^2}{2-3x} \geq 2$  for  $x \in [0, \frac{1}{2}]$ , sum over  $j$ , and recall that  $g_J \geq (1 - \beta_J)^2$ . This gives

$$\sum_j g_j \geq (1 - \beta_{\min})^2 + \beta_{\min} \frac{1}{4} \sum_{j < J} 2^j (s\eta a_j)^2.$$

We recall that we are working under the assumption that  $\sum_j s\eta a_j \geq s\eta \geq 6$  and  $s\eta a_J \leq \frac{4}{3}$ , which imply  $4 \leq \sum_{j < J} s\eta a_j$ . By Hölder's inequality,

$$16 \leq \left( \sum_{j < J} s\eta a_j \right)^2 \leq \sum_{j < J} 2^j (s\eta a_j)^2 \cdot \sum_{j < J} 2^{-j} \leq 2 \sum_{j < J} 2^j (s\eta a_j)^2.$$

Therefore  $\sum_{j < J} 2^j (s\eta a_j)^2 \geq 8$ , so that  $I(\alpha, \beta) = \sum_j g_j \geq (1 - \beta_{\min})^2 + 2\beta_{\min} = 1 + \beta_{\min}^2 > 1$ . This concludes the proof.  $\square$

**Proposition 3.6** (Strict subadditivity). *For every  $s > 0$  and  $t \in (0, 1)$  the function  $g$  satisfies the inequality  $g(ts) > tg(s)$ . In particular  $g$  is strictly subadditive: for every  $s_1, s_2 > 0$*

$$g(s_1 + s_2) < g(s_1) + g(s_2).$$

*Proof.* Let  $s > 0$ ,  $t \in (0, 1)$ ,  $\bar{s} = ts$ . By Proposition 3.2(ii) there is  $(\alpha_{\bar{s}}, \beta_{\bar{s}}) \in \mathcal{U}_1$  such that  $\mathcal{G}_{\bar{s}}(\alpha_{\bar{s}}, \beta_{\bar{s}}) = g(\bar{s})$ . Then by rescaling  $\tilde{\alpha}(y) := \alpha_{\bar{s}}(ty)$ ,  $\tilde{\beta}(y) := \beta_{\bar{s}}(ty)$  we have  $(\tilde{\alpha}, \tilde{\beta}) \in \mathcal{U}_1$  and

therefore

$$\begin{aligned}
g(\bar{s}) &= \int_{-\infty}^{\infty} \left( t^2 s^2 f^2(\beta_{\bar{s}}) |\alpha'_{\bar{s}}|^2 + \frac{(1 - \beta_{\bar{s}})^2}{4} + |\beta'_{\bar{s}}|^2 \right) dx \\
&= \int_{-\infty}^{\infty} \left( s^2 f^2(\tilde{\beta}) |\tilde{\alpha}'|^2 + \frac{(1 - \tilde{\beta})^2}{4} + \frac{|\tilde{\beta}'|^2}{t^2} \right) t dy \\
&= t \mathcal{G}_s(\tilde{\alpha}, \tilde{\beta}) + \left( \frac{1}{t} - t \right) \int_{-\infty}^{\infty} |\tilde{\beta}'|^2 dy \\
&\geq t g(s) + \left( \frac{1}{t} - t \right) t \int_{-\infty}^{\infty} |\beta'_{\bar{s}}|^2 dy > t g(s),
\end{aligned}$$

where the last inequality follows by  $\beta_{\bar{s}} \neq 0$ . This proves the first part of the statement. By writing

$$g(s_1) + g(s_2) = g\left((s_1 + s_2) \frac{s_1}{s_1 + s_2}\right) + g\left((s_1 + s_2) \frac{s_2}{s_1 + s_2}\right) > g(s_1 + s_2)$$

the strict subadditivity of  $g$  follows.  $\square$

We remark for later use that, by Proposition 3.6 and continuity, given any  $s > 0$  there exists a constant  $c_s > 0$ , depending on  $s$ , such that

$$g(s_1 + s_2) - g(s_1) - g(s_2) \leq -c_s \quad \text{for every } s_1, s_2 \geq s. \quad (3.18)$$

In the following two propositions we perform a careful asymptotic analysis of the function  $g(s)$  as  $s \rightarrow 0^+$ . Notice that, in view of Proposition 3.1,  $g$  is differentiable at the origin, with positive and finite slope; we now compute the second order correction by means of a  $\Gamma$ -convergence expansion in the minimum problems (3.9). This result is used in Proposition 3.8 to prove a quantitative version of the subadditivity of  $g$  in a neighbourhood of the origin.

**Proposition 3.7.** *There exists  $\tilde{\ell} > 0$  such that*

$$g(s) = \ell s - \tilde{\ell} s^{5/3} + o(s^{5/3}) \quad \text{as } s \rightarrow 0^+. \quad (3.19)$$

Moreover, if  $\gamma_s$  is a minimizer in problem (3.9), then the sequence  $s^{-4/3} \gamma_s$  converges uniformly in  $[0, 1]$  as  $s \rightarrow 0^+$  to the unique solution  $\bar{\eta} \in H_0^1(0, 1)$  of the minimum problem

$$\min \left\{ H(\eta) : \eta \in H_0^1(0, 1), \eta \geq 0 \right\} \quad (3.20)$$

where

$$H(\eta) := \int_0^1 \left( -\frac{\ell_1}{\ell} \sqrt{\eta} + \frac{|\eta'|^2}{8\ell^2} \right) dt. \quad (3.21)$$

In particular

$$(1 - m_s)^2 = \max_{t \in [0, 1]} \gamma_s(t) \leq c s^{4/3} \quad (3.22)$$

for all  $s > 0$  small enough and for a uniform constant  $c > 0$ .

We remark that the minimum in (3.20) can be found explicitly, and leads to

$$\tilde{\ell} = -\ell \min H(\bar{\eta}) = \ell_1^{4/3} \ell^{1/3} \frac{3}{10} \left( \frac{3}{2} \right)^{2/3} \simeq 0.4 \ell_1^{4/3} \ell^{1/3}. \quad (3.23)$$

The computation is straightforward but somewhat cumbersome, we do not report it for brevity since this result is not needed for what follows.

*Proof.* To guess the order of the second term in the expansion of  $g$  as  $s \rightarrow 0$ , we perform a Taylor expansion in the energy of a minimizer  $\gamma_s$  for (3.9). Let  $\gamma_s$  be an optimal profile for problem (3.9), as in Proposition 3.3(ii). Using  $\max \gamma_s = \gamma_s(\frac{1}{2}) = (1 - m_s)^2$  and Proposition 3.1(ii) gives

$$\ell s \geq g(s) \geq \frac{1}{2} \int_0^1 |\gamma'_s| dt = (1 - m_s)^2,$$

and therefore

$$0 \leq \gamma_s(t) \leq \ell s \quad \text{for all } t \in [0, 1]. \quad (3.24)$$

Furthermore, by the Euler-Lagrange equation (3.15) satisfied by  $\gamma_s$  we further deduce for  $s$  small, using (3.24) and (2.5), the bound on the derivative

$$\begin{aligned} |\gamma'_s(t)| &= 2s f_1(\sqrt{\gamma_s}) \sqrt{\left(\frac{f_1(\sqrt{\gamma_s})}{f_1(1 - m_s)}\right)^2 - 1} \\ &\leq 2s \frac{f_1(\sqrt{\gamma_s})}{f_1(1 - m_s)} \sqrt{2\ell\ell_1(1 - m_s - \sqrt{\gamma_s}) + o(1 - m_s)} \leq c_0 s^{\frac{5}{4}} \end{aligned} \quad (3.25)$$

for all  $t \in [0, 1]$ , for some positive constant  $c_0$ , independent of  $s$ . Since  $\gamma_s$  converges to zero uniformly as  $s \rightarrow 0$ , we can formally expand the energy of  $\gamma_s$  by using (2.5) and neglecting higher order terms:

$$\begin{aligned} g(s) &= \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma_s}))^2 + \frac{|\gamma'_s|^2}{4}} dt \sim \int_0^1 \sqrt{s^2 \ell^2 - 2s^2 \ell \ell_1 \sqrt{\gamma_s} + \frac{|\gamma'_s|^2}{4}} dt \\ &\sim s\ell + s\ell \int_0^1 \left( -\frac{\ell_1}{\ell} \sqrt{\gamma_s} + \frac{|\gamma'_s|^2}{8s^2 \ell^2} \right). \end{aligned}$$

For  $s$  small, the two terms in the last integral are of the same order if  $\gamma_s \sim s^{4/3}$ , so that, in turn, we might expect  $g(s) - \ell s \sim s^{5/3}$ .

The previous formal considerations can be made rigorous by means of an asymptotic development by  $\Gamma$ -convergence. The expected decay of minimizers of (3.9) suggests to consider the rescaling  $\eta(t) := s^{-\frac{4}{3}} \gamma(t)$ : with this position the minimum problem (3.9) can be reformulated as

$$g(s) = \inf \left\{ \int_0^1 \sqrt{s^2 (f_1(s^{2/3} \eta^{1/2}))^2 + \frac{s^{8/3} |\eta'|^2}{4}} : \eta \in W_0^{1,1}(0, 1), 0 \leq \eta \leq s^{-4/3} \right\}. \quad (3.26)$$

We compute the  $\Gamma$ -limit as  $s \rightarrow 0$ , with respect to the strong topology of  $L^1(0, 1)$ , of the family of functionals defined by

$$H_s(\eta) := \frac{1}{s^{2/3}} \left( \frac{1}{s\ell} \int_0^1 \sqrt{s^2 (f_1(s^{2/3} \eta^{1/2}))^2 + \frac{s^{8/3} |\eta'|^2}{4}} dt - 1 \right) \quad (3.27)$$

if  $\eta \in W_0^{1,1}(0, 1)$ ,  $0 \leq \eta \leq \ell s^{-1/3}$ , and  $|\eta'| \leq c_0 s^{-1/12}$ , and  $H_s(\eta) := \infty$  otherwise in  $W_0^{1,1}(0, 1)$ . Notice that the previous bounds are satisfied by the (rescaled) minimizer  $\gamma_s$ , in view of (3.24)–(3.25). We will prove that the  $\Gamma$ -limit is given by the functional  $H$  defined as the expression in (3.21) for  $\eta \in H_0^1(0, 1)$ ,  $\eta \geq 0$ , and  $H(\eta) := \infty$  otherwise in  $W_0^{1,1}(0, 1)$ .

We first prove the  $\Gamma$ -liminf inequality, together with the compactness of sequences with bounded energy with respect to the weak topology of  $H^1(0, 1)$ . Without loss of generality, we consider a subsequence  $\eta_s \in W_0^{1,1}(0, 1)$ , not relabeled, such that  $0 \leq \eta_s \leq \ell s^{-1/3}$ ,  $|\eta'_s| \leq c_0 s^{-1/12}$ , and  $\lim_{s \rightarrow 0} H_s(\eta_s) < \infty$ . Since  $s^{2/3} \eta_s^{1/2} \leq \ell^{1/2} s^{1/2}$ , by (2.5) we have

$$f_1(s^{2/3} \eta_s^{1/2}) = \ell - \ell_1 s^{2/3} \eta_s^{1/2} + o(s^{2/3} \eta_s^{1/2}),$$

and therefore

$$\begin{aligned} H_s(\eta_s) &= \frac{1}{s^{2/3}} \left( \frac{1}{s\ell} \int_0^1 \sqrt{s^2 (\ell - \ell_1 s^{2/3} \eta_s^{1/2} + o(s^{2/3} \eta_s^{1/2}))^2 + \frac{s^{8/3} |\eta'_s|^2}{4}} dt - 1 \right) \\ &= \frac{1}{s^{2/3}} \left( \int_0^1 \sqrt{1 - 2\frac{\ell_1}{\ell} s^{2/3} \eta_s^{1/2} + o(s^{2/3} \eta_s^{1/2}) + \frac{s^{2/3} |\eta'_s|^2}{4\ell^2}} dt - 1 \right). \end{aligned}$$

Now, since

$$\left| -2\frac{\ell_1}{\ell}s^{2/3}\eta_s^{1/2} + o(s^{2/3}\eta_s^{1/2}) + \frac{s^{2/3}|\eta'_s|^2}{4\ell^2} \right| \leq cs^{1/2},$$

a Taylor's expansion gives

$$H_s(\eta_s) = \int_0^1 \left( -\frac{\ell_1}{\ell}\eta_s^{1/2} + \frac{|\eta'_s|^2}{8\ell^2} \right) dt + \frac{1}{s^{2/3}} \int_0^1 o(s^{2/3}\eta_s^{1/2} + s^{2/3}|\eta'_s|^2) dt. \quad (3.28)$$

Notice that, for  $s$  small enough, the second term in the right-hand side of (3.28) can be controlled by the first one:

$$H_s(\eta_s) \geq \int_0^1 \left( -\frac{2\ell_1}{\ell}\eta_s^{1/2} + \frac{|\eta'_s|^2}{16\ell^2} \right) dt.$$

Now, for any  $\varepsilon > 0$  we have  $\eta_s^{1/2} \leq \varepsilon\eta_s^2 + c_\varepsilon$  by Young's inequality; therefore, using also Poincaré's inequality we end up with

$$H_s(\eta_s) \geq -\frac{2\varepsilon\ell_1}{\ell} \int_0^1 \eta_s^2 dt - \frac{2c_\varepsilon\ell_1}{\ell} + \frac{1}{16\ell^2} \int_0^1 |\eta'_s|^2 dt \geq \left( \frac{1}{16\ell^2} - \frac{2\varepsilon\ell_1}{\ell} \right) \int_0^1 |\eta'_s|^2 dt - \frac{2c_\varepsilon\ell_1}{\ell},$$

and choosing  $\varepsilon$  small enough we deduce that the sequence  $(\eta_s)_s$  is uniformly bounded in  $H_0^1(0, 1)$ . Up to subsequences,  $\eta_s \rightharpoonup \eta$  weakly in  $H^1(0, 1)$  (and uniformly) for some  $\eta \in H_0^1(0, 1)$ ,  $\eta \geq 0$ . By (3.28) we also deduce that

$$\liminf_{s \rightarrow 0} H_s(\eta_s) \geq \int_0^1 \left( -\frac{\ell_1}{\ell}\eta^{1/2} + \frac{|\eta'|^2}{8\ell^2} \right) dt = H(\eta).$$

As for the  $\Gamma$ -limsup inequality, fixed  $\eta \in C_c^1(0, 1)$ ,  $\eta \geq 0$ , we simply define  $\eta_s := \eta$ . Hence, (3.28) gives for  $s$  small enough

$$H_s(\eta_s) = \int_0^1 \left( -\frac{\ell_1}{\ell}\eta^{1/2} + \frac{|\eta'|^2}{8\ell^2} \right) dt + o(1) = H(\eta) + o(1),$$

where  $o(1)$  depends on  $\eta$  and tends to 0 as  $s \rightarrow 0$ . The  $\Gamma$ -limsup inequality follows by density of  $C_c^1(0, 1)$  in  $H_0^1(0, 1)$ .

The existence of a minimizer  $\bar{\eta} \in H_0^1(0, 1)$  for the limit functional  $H$  follows by a standard application of the direct method of the Calculus of Variations, and its uniqueness by strict convexity of the functional  $H$ . We can now check that the value of the minimum of  $H$  is strictly negative: by considering  $\lambda \sin(\pi t)$  as a competitor, with  $\lambda > 0$ , we have

$$\min_{\eta \in W_0^{1,1}(0,1)} H(\eta) \leq -\frac{\ell_1}{\ell} \lambda^{1/2} \int_0^1 \sin^{1/2}(\pi t) dt + \frac{\lambda^2}{8\ell^2} \int_0^1 \pi^2 \cos^2(\pi t) dt,$$

which is strictly negative if  $\lambda$  is small enough. We denote  $\tilde{\ell} := -\ell \min H > 0$ . By (3.26) and standard properties of  $\Gamma$ -convergence we obtain the convergence of the minimizers of  $H_s$  to  $\bar{\eta}$  and

$$\frac{g(s) - \ell s}{\ell s^{5/3}} = \min_{\eta \in W_0^{1,1}(0,1)} H_s(\eta) \rightarrow \min_{\eta \in W_0^{1,1}(0,1)} H(\eta) = -\frac{\tilde{\ell}}{\ell} \quad \text{as } s \rightarrow 0^+,$$

so that (3.19) and (3.22) follow.  $\square$

**Proposition 3.8.** *There exists  $\tilde{s} > 0$  and  $\tilde{c} > 0$  such that for every  $s_1, s_2 \in (0, \tilde{s})$*

$$g(s_1 + s_2) \leq g(s_1) + g(s_2) - \tilde{c} \min\{s_1, s_2\} \max\{s_1^{2/3}, s_2^{2/3}\}. \quad (3.29)$$

*Proof.* We assume without loss of generality that  $s_1 \leq s_2 < s_{\text{frac}}$ . Let  $\gamma_1$  and  $\gamma_2$  be optimal profiles in the minimum problem (3.9) for  $g(s_1)$  and  $g(s_2)$ , respectively. By Proposition 3.3(iii) we have  $\gamma_1 \leq \gamma_2$ ; furthermore, by (3.19), (2.5) and (3.22) we have

$$\ell s_i - \tilde{\ell} s_i^{5/3} + o(s_i^{5/3}) = g(s_i) \geq s_i \int_0^1 f_1(\sqrt{\gamma_i}) dt = \ell s_i - \ell_1 s_i \int_0^1 \sqrt{\gamma_i} dt + o(s_i^{5/3}),$$

that is,

$$\int_0^1 \sqrt{\gamma_i} dt \geq \frac{\tilde{\ell}}{2\ell_1} s_i^{2/3} \quad (3.30)$$

provided that  $\tilde{s}$  is small enough. We also observe preliminarily that by (2.5) and (3.22)

$$f_1(\sqrt{\gamma_1 + \gamma_2}) - f_1(\sqrt{\gamma_1}) = -\ell_1(\sqrt{\gamma_1 + \gamma_2} - \sqrt{\gamma_1}) + o(s_2^{2/3}) \leq -\frac{\ell_1}{4}\sqrt{\gamma_2} + o(s_2^{2/3}) \quad (3.31)$$

(where we used the elementary inequality  $\sqrt{x+y} - \sqrt{x} \geq \frac{1}{4}\sqrt{y}$  for  $y \geq x$ ).

We consider the competitor  $\gamma := \gamma_1 + \gamma_2$  for  $g(s_1 + s_2)$ :

$$\begin{aligned} g(s_1 + s_2) &\leq \int_0^1 \sqrt{(s_1 + s_2)^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt = \int_0^1 \left| \left( (s_1 + s_2) f_1(\sqrt{\gamma}), \frac{1}{2}(\gamma'_1 + \gamma'_2) \right) \right| dt \\ &\leq \int_0^1 \left| \left( s_1 f_1(\sqrt{\gamma}), \frac{1}{2}\gamma'_1 \right) \right| dt + \int_0^1 \left| \left( s_2 f_1(\sqrt{\gamma}), \frac{1}{2}\gamma'_2 \right) \right| dt. \end{aligned} \quad (3.32)$$

To estimate the first term we set

$$v := \left( s_1 f_1(\sqrt{\gamma}), \frac{1}{2}\gamma'_1 \right), \quad w := \left( s_1 f_1(\sqrt{\gamma_1}), \frac{1}{2}\gamma'_1 \right),$$

and obtain, using  $|v|^2 - |w|^2 = s_1^2(f_1^2(\sqrt{\gamma}) - f_1^2(\sqrt{\gamma_1})) \leq 0$  and (3.31)

$$\begin{aligned} |v| &= |w| + s_1^2(f_1(\sqrt{\gamma}) - f_1(\sqrt{\gamma_1})) \frac{(f_1(\sqrt{\gamma}) + f_1(\sqrt{\gamma_1}))}{|v| + |w|} \\ &\leq |w| - s_1 \left( \frac{\ell_1}{4}\sqrt{\gamma_2} + o(s_2^{2/3}) \right) \frac{s_1 f_1(\sqrt{\gamma_1})}{2|w|} \\ &\leq |w| - cs_1\sqrt{\gamma_2} + cs_1 o(s_2^{2/3}) \end{aligned}$$

for some uniform constant  $c > 0$ . In the last estimate we used  $|\gamma'_1| \leq 2s_1 f_1(\sqrt{\gamma_1})$ , which follows (for small  $\tilde{s}$ ) from the first equality in (3.25) using (2.5) and (3.22). For the second term in (3.32), noting that  $f_1(\sqrt{\gamma_1 + \gamma_2}) \leq f_1(\sqrt{\gamma_2})$  by (2.4), we have

$$\left| \left( s_2 f_1(\sqrt{\gamma}), \frac{1}{2}\gamma'_2 \right) \right| \leq \left| \left( s_2 f_1(\sqrt{\gamma_2}), \frac{1}{2}\gamma'_2 \right) \right|.$$

Then inserting these estimates into (3.32) we end up with

$$\begin{aligned} g(s_1 + s_2) &\leq \int_0^1 \left| \left( s_1 f_1(\sqrt{\gamma_1}), \frac{1}{2}\gamma'_1 \right) \right| dt + \int_0^1 \left| \left( s_2 f_1(\sqrt{\gamma_2}), \frac{1}{2}\gamma'_2 \right) \right| dt \\ &\quad - cs_1 \int_0^1 \sqrt{\gamma_2} dt + cs_1 o(s_2^{2/3}) \\ &= g(s_1) + g(s_2) - \frac{c\tilde{\ell}}{2\ell_1} s_1 s_2^{2/3} + cs_1 o(s_2^{2/3}) \end{aligned}$$

where we used (3.30) in the last inequality. Then (3.29) follows, provided that  $\tilde{s}$  is sufficiently small.  $\square$

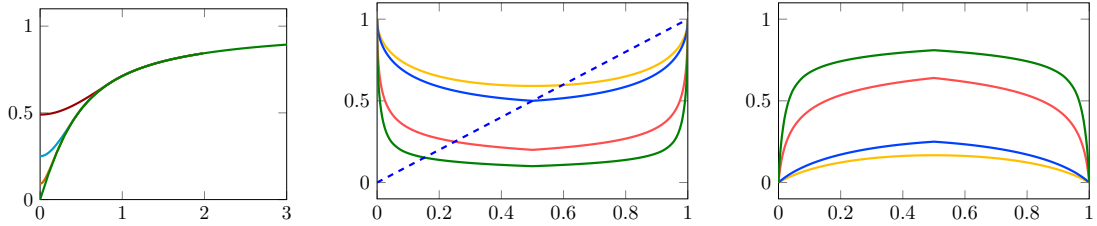


Figure 5: Left: Graph of  $g(s)$  and  $\bar{g}(s, s')$  for  $m_{s'} = 0.3, 0.5$  and  $0.7$  (from bottom to top) and  $f_1^b(s)$  in (2.7) with  $\ell = 1.5, \ell_1 = 0.2$  as in Figure 1 and Figure 2. Middle and right: Graph of  $(\alpha, \beta)$  and  $\gamma$  from (3.36) and (3.37) for  $s = 0.3$  and  $m_{s'} = 0.1, 0.2, 0.5, 0.7$  and the same  $f_1^b$  corresponding to Figure 3. For  $m_{s'} = 0.7$  the curves are the same as for  $g$ . See also Figure 6.

### 3.2 The function $\bar{g}$ and the cohesive energy of pre-damaged material

We now introduce a new surface energy density, which appears in the  $\Gamma$ -limit of the phase-field energies when we include an irreversibility constraint (see Section 6). We set, for  $s, s' \geq 0$ ,

$$\bar{g}(s, s') := \inf_{(\alpha, \beta) \in \mathcal{V}_{s'}} \mathcal{G}_s(\alpha, \beta), \quad (3.33)$$

where  $\mathcal{G}_s$  has been defined in (3.5),

$$\mathcal{V}_{s'} := \left\{ (\alpha, \beta) \in \mathcal{U}_1 : \inf \beta \leq m_{s'} \right\}, \quad (3.34)$$

and  $m_{s'}$  denotes the value of the minimum of an optimal profile for  $g(s')$ , see (3.16). The effect of the second variable  $s'$  is to introduce a “memory effect” that takes into account the maximal amplitude of the jumps at previous times. We refer to Figure 5 and Figure 6 for an illustration.

We remark that by the same arguments used in Proposition 3.3 and in [19, Proposition 4.3] one has the following characterizations of  $\bar{g}$ :

$$\begin{aligned} \bar{g}(s, s') &= \inf_{T>0} \inf \left\{ \int_{-T}^T \left( f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt : \alpha, \beta \in H^1(-T, T), \right. \\ &\quad \left. |\alpha(T) - \alpha(-T)| = s, 0 \leq \beta \leq 1, \beta(\pm T) = 1, \inf \beta \leq m_{s'} \right\} \end{aligned} \quad (3.35)$$

$$\begin{aligned} &= \inf \left\{ \int_0^1 |1 - \beta| \sqrt{f^2(\beta) |\alpha'|^2 + |\beta'|^2} dt : (\alpha, \beta) \in H^1(0, 1) \times H^1(0, 1), \right. \\ &\quad \left. \alpha(0) = 0, \alpha(1) = s, 0 \leq \beta \leq 1, \beta(0) = \beta(1) = 1, \inf \beta \leq m_{s'} \right\} \end{aligned} \quad (3.36)$$

$$= \inf \left\{ \int_0^1 \sqrt{s^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt : \gamma \in W_0^{1,1}([0, 1], [0, 1]), \sup \gamma \geq (1 - m_{s'})^2 \right\}. \quad (3.37)$$

Extending  $\alpha$  and  $\beta$  by constants it is easy to show that the inf in the first line of (3.35) is nonincreasing in  $T$ . We collect the main properties of the function  $\bar{g}$  in the following proposition.

**Proposition 3.9.** *The function  $\bar{g}$  enjoys the following properties:*

(i)  $\bar{g}$  is monotone nondecreasing in both variables,  $\bar{g}(s, s') = g(s)$  if  $s \geq s'$ ,  $\bar{g}(s, s') \geq g(s)$  if  $s < s'$ ;

(ii) for every  $s_1, s_2, s' \geq 0$ ,

$$\bar{g}(s_1 + s_2, s') \leq g(s_1) + \bar{g}(s_2, s'); \quad (3.38)$$

(iii)  $\bar{g}(0, s') = (1 - m_{s'})^2$ ,  $(1 - m_{s'})^2 \leq \bar{g}(s, s') \leq 1 \wedge ((1 - m_{s'})^2 + \ell s)$ ;

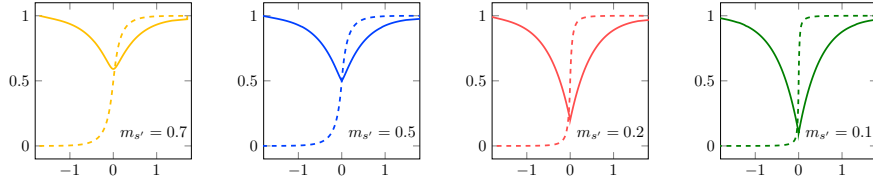


Figure 6: Graph of the optimal  $(\alpha, \beta)$  entering (3.35) with a very large  $T$ , for  $s = 0.3$  and  $m_{s'} = 0.7, 0.5, 0.2, 0.1$ , as in Figure 5. For the largest value of  $m_{s'}$  one sees that the constraint is not active and  $\beta$  is smooth at  $t = 0$ ; for the others the constraint is active and the derivative of  $\beta$  jumps. Correspondingly, the profile of  $\alpha$  becomes more concentrated.

(iv)  $\bar{g}$  is continuous in both variables and, for fixed  $s'$ , the map  $\bar{g}(\cdot, s')$  is Lipschitz continuous with Lipschitz constant  $\ell$  (in particular  $\bar{g}$  is globally continuous);

(v)  $\lim_{s \uparrow \infty} \bar{g}(s, s') = 1$ ,  $\lim_{s \downarrow 0} \frac{\bar{g}(s, s') - \bar{g}(0, s')}{s} = 0$ .

*Proof.* (i): The monotonicity of  $\bar{g}$  in the first variable follows from  $\mathcal{G}_{s_1}(\alpha, \beta) \leq \mathcal{G}_{s_2}(\alpha, \beta)$  for  $s_1 < s_2$ , while the monotonicity in the second variable follows since the map  $s' \mapsto m_{s'}$  is decreasing. The inequality  $\bar{g}(s, s') \geq g(s)$  is trivial by (3.3) and (3.33). If  $s \geq s'$ , the optimal profile  $\beta_s$  for  $g(s)$  is admissible in the minimum problem which defines  $\bar{g}(s, s')$ , by monotonicity of  $m_{s'}$ , and therefore the values of the two functions coincide. This completes the proof of (i).

(ii): Fix  $\sigma > 0$  and let  $T > 0$ ,  $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in H^1(-T, T) \times H^1(-T, T)$  be such that  $\alpha_i(-T) = 0$ ,  $\alpha_i(T) = s_i$ ,  $0 \leq \beta_i \leq 1$ ,  $\beta_i(\pm T) = 1$ ,  $\inf \beta_2 \leq \beta_{s'}(0)$ , and

$$\begin{aligned} \int_{-T}^T \left( f^2(\beta_1) |\alpha_1'|^2 + \frac{(1 - \beta_1)^2}{4} + |\beta_1'|^2 \right) dt &\leq g(s_1) + \sigma, \\ \int_{-T}^T \left( f^2(\beta_2) |\alpha_2'|^2 + \frac{(1 - \beta_2)^2}{4} + |\beta_2'|^2 \right) dt &\leq \bar{g}(s_2, s') + \sigma, \end{aligned}$$

according to the representation formulas (3.7) and (3.35). By defining

$$\alpha(t) = \begin{cases} \alpha_1(t) & \text{if } t \in [-T, T], \\ \alpha_2(t - 2T) + s_1 & \text{if } t \in [T, 3T], \end{cases} \quad \beta(t) = \begin{cases} \beta_1(t) & \text{if } t \in [-T, T], \\ \beta_2(t - 2T) & \text{if } t \in [T, 3T], \end{cases}$$

we have  $\alpha, \beta \in H^1(-T, 3T)$ , with  $\alpha(-T) = 0$ ,  $\alpha(3T) = s_1 + s_2$ ,  $\beta(-T) = \beta(3T) = 1$ ,  $0 \leq \beta \leq 1$ ,  $\inf \beta \leq \beta_{s'}(0)$ . By using the pair  $(\alpha, \beta)$  as a competitor in (3.35) we obtain

$$\bar{g}(s_1 + s_2, s') \leq g(s_1) + \bar{g}(s_2, s') + 2\sigma,$$

and the conclusion follows as  $\sigma$  is arbitrary.

(iii): The value of  $\bar{g}(0, s')$  can be computed by observing that in this case the minimum problem (3.33) defining  $\bar{g}$  is independent of  $\alpha$ , and the optimal profile is given by the function

$$\bar{\beta}_{s'}(t) := 1 - (1 - m_{s'})e^{-\frac{|t|}{2}}. \quad (3.39)$$

The lower bound on  $\bar{g}(s, s')$  follows directly by monotonicity. To derive the upper bound with the constant 1, one can simply consider the admissible pair  $(\alpha, \beta) \in \mathcal{V}_{s'}$  given by

$$\alpha(t) = \begin{cases} 0 & \text{if } t \in (-\infty, -1), \\ \frac{t}{2} + \frac{1}{2} & \text{if } t \in [-1, 1], \\ 1 & \text{if } t \in (1, \infty), \end{cases} \quad \beta(t) = \begin{cases} 1 - e^{-\frac{t+1}{2}} & \text{if } t \in (-\infty, -1), \\ 0 & \text{if } t \in [-1, 1], \\ 1 - e^{-\frac{t-1}{2}} & \text{if } t \in (1, \infty). \end{cases}$$



The inequality  $\bar{g}(s, s') \leq \bar{g}(0, s') + \ell s$  follows directly from (3.38) and  $g(s) \leq \ell s$ .

(iv): The continuity of  $s' \mapsto \bar{g}(s, s')$  can be proved by taking into account the continuity of the map  $s' \mapsto m_{s'}$ , proved in Theorem 3.4. The Lipschitz continuity of  $s \mapsto \bar{g}(s, s')$  is a consequence of the monotonicity of this map, of the subadditivity inequality (3.38), and of the bound  $g(s) \leq \ell s$ .

(v): The first limit is a trivial consequence of  $\bar{g}(s, s') = g(s)$  for  $s \geq s'$ , and of Proposition 3.1. To compute the slope of  $\bar{g}(\cdot, s')$  at the origin, notice that given any  $\alpha$  such that  $\alpha'$  has compact support and the pair  $(\alpha, \bar{\beta}_{s'})$  belongs to  $\mathcal{V}_{s'}$ , where  $\bar{\beta}_{s'}$  is the optimal profile defined in (3.39), we have

$$\bar{g}(s, s') \leq \int_{-\infty}^{\infty} \left( s^2 f^2(\bar{\beta}_{s'}) |\alpha'|^2 + \frac{(1 - \bar{\beta}_{s'})^2}{4} + |\bar{\beta}'_{s'}|^2 \right) dt = \int_{-\infty}^{\infty} s^2 f^2(\bar{\beta}_{s'}) |\alpha'|^2 dt + (1 - m_{s'})^2.$$

Dividing by  $s$  and letting  $s \rightarrow 0^+$  we obtain the inequality  $\limsup_{s \rightarrow 0^+} \frac{\bar{g}(s, s') - \bar{g}(0, s')}{s} \leq 0$ . The lower bound  $\bar{g}(s, s') \geq \bar{g}(0, s')$  concludes the proof.  $\square$

The next proposition shows that the inequality (3.38) is strict.

**Proposition 3.10** (Strict subadditivity). *For every  $s_1 > 0$ ,  $s_2 \geq 0$ , and  $s' > 0$ , we have*

$$\bar{g}(s_1 + s_2, s') < g(s_1) + \bar{g}(s_2, s'). \quad (3.40)$$

In particular, for every  $\varepsilon > 0$  there exists a constant  $c_\varepsilon > 0$  such that for every  $s_2 \geq 0$  and  $s_1, s' \geq \varepsilon$  one has

$$\bar{g}(s_1 + s_2, s') - g(s_1) - \bar{g}(s_2, s') \leq -c_\varepsilon. \quad (3.41)$$

*Proof.* If  $s_1 + s_2 \geq s'$ , then  $\bar{g}(s_1 + s_2, s') = g(s_1 + s_2)$  and (3.40) holds since  $g$  is strictly subadditive by Proposition 3.6. If  $g(s_1) = 1$  the assertion follows from  $\bar{g}(s_1 + s_2, s') \leq 1$  and  $\bar{g}(s_2, s') > 0$ . Let us assume now that  $s_1 + s_2 < s'$  and  $s_1 < s_{\text{frac}}$ . Fixed  $\eta > 0$ , let  $\gamma_1, \gamma_2 \in W_0^{1,1}([0, 1], [0, 1])$  be such that  $\max \gamma_1 = (1 - m_{s_1})^2$ ,  $\max \gamma_2 \geq (1 - m_{s'})^2$ , and

$$g(s_1) = \int_0^1 \sqrt{s_1^2 (f_1(\sqrt{\gamma_1}))^2 + \frac{|\gamma_1'|^2}{4}} dt, \quad (3.42)$$

$$\bar{g}(s_2, s') + \eta > \int_0^1 \sqrt{s_2^2 (f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt. \quad (3.43)$$

Notice that this is possible by the characterization (3.37) and Proposition 3.3(ii). In particular  $\gamma_1$  is the unique minimizer of the problem defining  $g(s_1)$ .

The function  $\gamma := (\gamma_1 + \gamma_2) \wedge 1$  is admissible for  $\bar{g}(s_1 + s_2, s')$ , being  $\gamma(1/2) \geq (1 - m_{s'})^2$ . Therefore,

$$\begin{aligned} \bar{g}(s_1 + s_2, s') &\leq \int_0^1 \sqrt{(s_1 + s_2)^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma'|^2}{4}} dt \\ &\leq \int_0^1 \sqrt{s_1^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_1'|^2}{4}} dt + \int_0^1 \sqrt{s_2^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_2'|^2}{4}} dt. \end{aligned}$$

In order to estimate the first integral, we observe that  $\gamma \geq \gamma_1$  by definition, that  $\gamma(1/2) > \gamma_1(1/2)$  (since  $\gamma_2(1/2) > 0$  by  $s' > 0$  and  $\gamma_1(1/2) < 1$  by  $s_1 < s_{\text{frac}}$ ), and that by (2.4)  $f_1(\sqrt{\cdot})$  is strictly decreasing, hence

$$\int_0^1 \sqrt{s_1^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_1'|^2}{4}} dt < \int_0^1 \sqrt{s_1^2 (f_1(\sqrt{\gamma_1}))^2 + \frac{|\gamma_1'|^2}{4}} dt.$$

As for the second integral, we first assume  $s_2 > 0$  and show that there exists a constant  $c_{s_1, s_2} > 0$ , only depending on  $s_1$  and  $s_2$ , such that

$$\int_0^1 \sqrt{s_2^2 (f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_2'|^2}{4}} dt \leq \int_0^1 \sqrt{s_2^2 (f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt - c_{s_1, s_2}. \quad (3.44)$$

This would lead to

$$\begin{aligned}\bar{g}(s_1 + s_2, s') &\leq \int_0^1 \sqrt{s_1^2(f_1(\sqrt{\gamma_1}))^2 + \frac{|\gamma_1'|^2}{4}} dt + \int_0^1 \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt - c_{s_1, s_2} \\ &\leq g(s_1) + \bar{g}(s_2, s') + \eta - c_{s_1, s_2},\end{aligned}$$

and then to (3.40) as  $\eta \rightarrow 0$ .

In order to check (3.44), let us first observe that there exists  $\delta_{s_1} > 0$ , only depending on  $\gamma_1$ , and therefore only depending on  $s_1$ , such that  $\gamma_1 \geq (1 - m_{s_1})^2/2$  in  $(\frac{1}{2} - \delta_{s_1}, \frac{1}{2} + \delta_{s_1})$ . Since  $f_1$  is strictly monotone, there exists a constant  $\tilde{c}_{s_1} > 0$ , only depending on  $s_1$ , such that

$$(f_1(\sqrt{\gamma_2}))^2 - \left( f_1 \left( \sqrt{\frac{(1 - m_{s_1})^2}{2} + \gamma_2} \right) \right)^2 \geq \tilde{c}_{s_1}.$$

Then,

$$\begin{aligned}\int_{\frac{1}{2} - \delta_{s_1}}^{\frac{1}{2} + \delta_{s_1}} \sqrt{s_2^2(f_1(\sqrt{\gamma}))^2 + \frac{|\gamma_2'|^2}{4}} dt &\leq \int_{\frac{1}{2} - \delta_{s_1}}^{\frac{1}{2} + \delta_{s_1}} \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4} - s_2^2 \tilde{c}_{s_1}} dt \\ &\leq \int_{\frac{1}{2} - \delta_{s_1}}^{\frac{1}{2} + \delta_{s_1}} \sqrt{s_2^2(f_1(\sqrt{\gamma_2}))^2 + \frac{|\gamma_2'|^2}{4}} dt - 2\delta_{s_1} \hat{c}_{s_1, s_2},\end{aligned}$$

where  $\hat{c}_{s_1, s_2} > 0$  only depends on  $s_1$  and  $s_2$ . Hence (3.44) follows.

It remains to deal with the case  $s_2 = 0$ . In this situation we can take  $\eta = 0$  in (3.43), and correspondingly we have that (3.44) holds with  $c_{s_1, s_2} = 0$ . The proof is then concluded as above.  $\square$

We will also need the following technical result.

**Lemma 3.11.** For  $\mu > 0$  let  $\bar{g}^{(\mu)} : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  be defined by

$$\begin{aligned}\bar{g}^{(\mu)}(s, s') &:= \inf_{T > 0} \inf \left\{ \int_{-T}^T \left( f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt : \alpha, \beta \in H^1(-T, T), \right. \\ &\quad \left. |\alpha(T) - \alpha(-T)| = s, 0 \leq \beta \leq 1, \beta(\pm T) = 1 - \mu, \inf \beta \leq m_{s'} \right\}.\end{aligned}\tag{3.45}$$

Then  $|\bar{g}(s, s') - \bar{g}^{(\mu)}(s, s')| \leq 3\mu^2$  for every  $s, s' \geq 0$ .

*Proof.* Given  $T > 0$  and an admissible pair  $(\alpha_\mu, \beta_\mu)$  for  $\bar{g}^{(\mu)}(s, s')$ , we construct an admissible pair  $(\alpha, \beta)$  for  $\bar{g}(s, s')$  in  $(-T - 1, T + 1)$  by setting  $\alpha \equiv \alpha_\mu(-T)$  in  $(-T - 1, -T)$ ,  $\alpha = \alpha_\mu$  in  $(-T, T)$ ,  $\alpha \equiv \alpha_\mu(T)$  in  $(T, T + 1)$ , and by setting  $\beta = \beta_\mu$  in  $(-T, T)$  and linearly linked to the value 1 in  $(-T - 1, -T)$  and  $(T, T + 1)$ . Then by (3.35)

$$\begin{aligned}\bar{g}(s, s') &\leq \int_{-T-1}^{T+1} \left( f^2(\beta) |\alpha'|^2 + \frac{(1 - \beta)^2}{4} + |\beta'|^2 \right) dt \\ &\leq \int_{-T}^T \left( f^2(\beta_\mu) |\alpha'_\mu|^2 + \frac{(1 - \beta_\mu)^2}{4} + |\beta'_\mu|^2 \right) dt + 3\mu^2,\end{aligned}$$

from which it follows that  $\bar{g}(s, s') \leq \bar{g}^{(\mu)}(s, s') + 3\mu^2$ . The other inequality follows by an analogous construction, reversing the roles of  $\bar{g}$  and  $\bar{g}^{(\mu)}$ .  $\square$

## 4 The time-discrete cohesive evolution

We construct in this section a time-discrete evolution for our one-dimensional model for cohesive fracture, relative to a given time-dependent boundary displacement  $b(t)$ . This construction is based on the minimization of a modification of the functional (2.8), in which we introduce a new irreversibility condition. The state of the fracture at each time is modeled by a discrete set of points (representing the sites where the crack is localized), to each of which is attached a scalar quantity (representing the maximal amplitude of the jump at all the previous times):

$$\Gamma \subset [0, 1] \text{ finite set}, \quad s : \Gamma \rightarrow (\bar{s}, \infty). \quad (4.1)$$

Here  $\bar{s} > 0$  is a fixed positive threshold (which can be taken arbitrarily small but strictly positive, for technical reasons). The effect of the irreversibility condition affects only the fracture points where the jump amplitude exceeds the threshold  $\bar{s}$ , since only these points will be included in the crack set  $\Gamma$ ; in other words, for small fractures the process is completely reversible. The internal variable  $s$  in (4.1) will be always implicitly extended as  $s(x) = 0$  for  $x \notin \Gamma$ .

The bar is subject to a time-dependent boundary displacement

$$b \in H^1([0, T]; \mathbb{R}^2), \quad \text{with } b(t) = (b(t, 0), b(t, 1)). \quad (4.2)$$

It is also convenient to denote, for  $t$  fixed, by  $b(t) : [0, 1] \rightarrow \mathbb{R}$  the affine function interpolating between the boundary values  $b(t, 0)$  and  $b(t, 1)$ . We will usually write  $b(t, x) := b(t)(x)$  and we will denote by  $\dot{b}$  the time-derivative of  $b$ , by  $b'(t)$  the spatial derivative of the function  $b(t)$ .

We would then iteratively minimize a functional of the form

$$\int_0^1 |u'|^2 dx + \sum_{x \in \Gamma} \bar{g}(|[u](x)|, s(x)) + \sum_{x \in J_u \setminus \Gamma} g(|[u](x)|) \quad (4.3)$$

among all possible displacements  $u$  attaining the boundary conditions  $u(0) = b(t, 0)$ ,  $u(1) = b(t, 1)$ . The surface energy in the new fracture points  $J_u \setminus \Gamma$ , where the bar was previously undamaged, is exactly the same as in (2.8); however, at the points of  $\Gamma$ , where the fracture was present at previous times, an irreversibility condition appears: the surface energy density is modified and is given by the function  $\bar{g}$ , defined in (3.33), which takes into account the previous work made on  $\Gamma$ .

Since the function  $g$  has finite slope at the origin, the minimization of the functional (4.3) is in principle not well-posed in SBV, and we have to consider its relaxation with respect to the weak\*-topology of BV, which is given by

$$\Phi(u; \Gamma, s, b) := \int_0^1 h(|u'|) dx + \sum_{x \in \Gamma} \bar{g}(|[u](x)|, s(x)) + \sum_{x \in J_u^b \setminus \Gamma} g(|[u](x)|) + \ell |D^c u|(0, 1), \quad (4.4)$$

where we highlighted the dependence of  $\Phi$  on the state of the fracture (described by the pair  $(\Gamma, s)$ ) and on the boundary datum  $b$ . Here the elastic energy density  $h$  is the same as in (2.9); to take into account the boundary conditions, we used the following notation:

$$J_u^b := J_u \cup \{x \in \{0, 1\} : u(x) \neq b(x)\}, \quad (4.5)$$

$$[u](0) := u^+(0) - b(0), \quad [u](1) := b(1) - u^-(1). \quad (4.6)$$

The rigorous proof of the relaxation result will be given in Appendix A.

We now discuss the construction of a time-discrete evolution. Let  $T > 0$  be the final time. We assume that the boundary displacement  $b$  satisfying the assumption (4.2) is given. We also add to the total energy a lower-order term of the form  $\gamma \|u - w(t)\|_{L^2(0,1)}^2$ , where  $\gamma \geq 0$  is a fixed constant and

$$w \in AC([0, T]; L^\infty(0, 1)) \quad (4.7)$$

is a given map, absolutely continuous in time.

We fix a discretization step  $\tau > 0$  and we consider a subdivision of  $[0, T]$  of the form

$$0 = t_0 < t_1 < \dots < t_{N_\tau} < t_{N_\tau+1} = T,$$

where  $N_\tau$  is the largest integer such that  $\tau N_\tau < T$ , and  $t_k := k\tau$  for  $k \in \{0, \dots, N_\tau\}$ . We also set  $b_k^\tau = b(t_k)$ ,  $w_k^\tau = w(t_k)$ .

The time-discrete evolution in this setting is defined as follows. For  $k = 0$ , select a solution  $u_0^\tau \in \text{BV}(0, 1)$  of the minimum problem

$$\min \left\{ \Phi(u; b_0^\tau) + \gamma \int_0^1 |u - w_0^\tau|^2 dx : u \in \text{BV}(0, 1) \right\} \quad (4.8)$$

(with the convention that we do not indicate the dependence on  $(\Gamma, s)$  in the functional  $\Phi$  defined in (4.4) when  $\Gamma = \emptyset$ ). We set

$$\Gamma_0^\tau := \{x \in J^{b_0^\tau}(u_0^\tau) : |[u_0^\tau](x)| > \bar{s}\}, \quad s_0^\tau(x) := |[u_0^\tau](x)|, \quad (4.9)$$

where  $\bar{s} > 0$  is the threshold fixed at the beginning of this section. Assume now to have constructed  $u_i^\tau \in \text{BV}(0, 1)$  and pairs  $(\Gamma_i^\tau, s_i^\tau)$  in the form (4.1), for  $i = 0, \dots, k-1$ . We let  $u_k^\tau \in \text{BV}(0, 1)$  be a minimum point of

$$\min \left\{ \Phi(u; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u - w_k^\tau|^2 dx : u \in \text{BV}(0, 1) \right\}, \quad (4.10)$$

and we set

$$\begin{aligned} \Gamma_k^\tau &:= \Gamma_{k-1}^\tau \cup \{x \in J^{b_k^\tau}(u_k^\tau) : |[u_k^\tau](x)| > \bar{s}\}, \\ s_k^\tau(x) &:= \begin{cases} |[u_k^\tau](x)| \vee s_{k-1}^\tau(x) & \text{if } x \in \Gamma_{k-1}^\tau, \\ |[u_k^\tau](x)| & \text{if } x \in \Gamma_k^\tau \setminus \Gamma_{k-1}^\tau. \end{cases} \end{aligned} \quad (4.11)$$

Notice in particular that all the sets  $\Gamma_k^\tau$  are finite, since at each step the minimizer  $u_k^\tau$  has at most a finite number of jump points where the amplitude of the jump is larger than the fixed threshold  $\bar{s}$ . The well-posedness of this construction is proved in the following proposition. We show below that  $u_k^\tau \in \text{SBV}(0, 1)$ .

**Proposition 4.1.** *For all  $k = 0, 1, \dots, N_\tau$  there exists  $u_k^\tau \in \text{BV}(0, 1)$  such that, by defining  $(\Gamma_k^\tau, s_k^\tau)$  as in (4.11) (with  $\Gamma_{-1}^\tau = \emptyset$ ), the following hold:*

- (i)  $u_k^\tau$  is a minimizer of problem (4.10);
- (ii)  $\|u_k^\tau\|_\infty \leq \max\{\|b_k^\tau\|_\infty, \|w_k^\tau\|_\infty\}$ ;
- (iii)  $\inf_\tau \inf_k \min\{|x - y| : x, y \in \Gamma_k^\tau, x \neq y\} > 0$ .

*Proof.* We need to show that, given a pair  $(\Gamma, s)$  as in (4.1), an affine function  $b : [0, 1] \rightarrow \mathbb{R}$ , and  $w \in L^\infty(0, 1)$ , the minimum problem

$$\min \left\{ \Phi(u; \Gamma, s, b) + \gamma \int_0^1 |u - w|^2 dx : u \in \text{BV}(0, 1) \right\} \quad (4.12)$$

has a solution. Let  $(u_n)_n$  be a minimizing sequence for problem (4.12). By a truncation argument we can take  $\|u_n\|_\infty \leq \max\{\|b\|_\infty, \|w\|_\infty\}$ . By comparing  $u_n$  with the function  $b$  we obtain the uniform bound

$$\Phi(u_n; \Gamma, s, b) + \gamma \int_0^1 |u_n - w|^2 dx \leq \int_0^1 h(|b'|) dx + \sum_{x \in \Gamma} \bar{g}(0, s(x)) + \gamma \int_0^1 |b - w|^2 dx - \frac{1}{n} \leq C.$$

Let now  $s' > 0$  be such that  $g(s') = \frac{1}{2}$ , and let  $\ell' \in (0, \ell)$  be such that  $g(s) \geq \ell' s$  for all  $s \in [0, s']$ . Then the lower estimates  $h(t) \geq \ell t - \frac{\ell^2}{4}$ ,  $\bar{g}(t, s) \geq g(t)$  yield

$$\begin{aligned} C &\geq \Phi(u_n; \Gamma, s, b) \geq \ell \int_0^1 |u'_n| dx - \frac{\ell^2}{4} + \ell |D^c u_n|(0, 1) + \sum_{x \in J_{u_n}} g(|[u_n](x)|) \\ &\geq \ell \int_0^1 |u'_n| dx - \frac{\ell^2}{4} + \ell |D^c u_n|(0, 1) + \sum_{|[u_n](x)| \leq s'} \ell' |[u_n](x)| + \frac{1}{2} \mathcal{H}^0(\{|[u_n](x)| > s'\}) \quad (4.13) \\ &\geq \ell \int_0^1 |u'_n| dx - \frac{\ell^2}{4} + \ell |D^c u_n|(0, 1) + \sum_{|[u_n](x)| \leq s'} \ell' |[u_n](x)| + \frac{1}{4 \|u_n\|_\infty} \sum_{|[u_n](x)| > s'} |[u_n](x)|. \end{aligned}$$

Therefore the sequence  $(u_n)_n$  is bounded in  $BV(0, 1)$  and, up to subsequences, it converges weakly\* in  $BV$  to some function  $u \in BV(0, 1)$ , which is a minimizer of (4.12) by the lower semicontinuity of the functional  $\Phi$ , proved in Theorem A.1. In particular, it also follows that  $\|u\|_\infty \leq \max\{\|b\|_\infty, \|w\|_\infty\}$ .

It remains to prove (iii). We first show that there is a minimal distance between any two new fracture points in  $\Gamma_k^\tau \setminus \Gamma_{k-1}^\tau$ , independent of  $k$  and  $\tau$ . Consider  $x_1, x_2 \in J^{b_k}(u_k^\tau) \setminus \Gamma_{k-1}^\tau$ , with  $s_i := [u_k^\tau](x_i)$  and  $|s_i| > \bar{s}$ ,  $i = 1, 2$ . Suppose also  $x_1 < x_2$ . Let  $v := u_k^\tau + s_2 \chi_{(x_1, x_2)}$ : by minimality of  $u_k^\tau$  in problem (4.10) we have

$$0 \leq g(|s_1 + s_2|) - g(|s_1|) - g(|s_2|) + 2\gamma s_2 \int_{x_1}^{x_2} (u_k^\tau - w_k^\tau) dx + \gamma s_2^2 |x_1 - x_2| \leq -c_{\bar{s}} + C|x_1 - x_2|,$$

where the last inequality follows from (3.18) and the uniform  $L^\infty$ -bound on  $u_k^\tau$  and  $w$ . This proves that the distance between  $x_1$  and  $x_2$  is larger than a uniform positive constant.

Similarly, we prove that any new fracture point of  $u_k^\tau$  can not be too close to the points of  $\Gamma_{k-1}^\tau$ . Consider  $x_1 \in J^{b_k}(u_k^\tau) \setminus \Gamma_{k-1}^\tau$  with  $s_1 := [u_k^\tau](x_1)$ ,  $|s_1| > \bar{s}$ . Let  $x_2 \in \Gamma_{k-1}^\tau$  and let  $s_2 := [u_k^\tau](x_2)$ . Assuming without loss of generality that  $x_1 < x_2$  (the construction in the other case is symmetric), by comparing the energy of  $u_k^\tau$  and  $v := u_k^\tau - s_1 \chi_{(x_1, x_2)}$  we have

$$\begin{aligned} 0 &\leq \bar{g}(|s_1 + s_2|, s_{k-1}^\tau(x_2)) - g(|s_1|) - \bar{g}(|s_2|, s_{k-1}^\tau(x_2)) - 2\gamma s_1 \int_{x_1}^{x_2} (u_k^\tau - w_k^\tau) dx + \gamma s_1^2 |x_1 - x_2| \\ &\leq -c_{\bar{s}} + C|x_1 - x_2|, \end{aligned}$$

where the last inequality follows from (3.41) and the uniform  $L^\infty$ -bound on  $u_k^\tau$  and  $w$ .  $\square$

**Proposition 4.2.** *Any minimizer  $u$  of (4.10) obeys  $D^c u = 0$  and  $|u'| \leq \frac{\ell}{2}$  almost everywhere.*

*Proof.* We define the measure  $\mu := (u' - \frac{\ell}{2})_+ + (D^c u)_+$ . We shall show that  $\mu = 0$ , the argument with  $(u' + \frac{\ell}{2})_- + (D^c u)_-$  is identical.

Assume that there are  $a \neq b \in (0, 1)$  such that  $\mu((a - \varepsilon, a + \varepsilon)) > 0$  and  $\mu((b - \varepsilon, b + \varepsilon)) > 0$  for all  $\varepsilon > 0$  sufficiently small. If such a pair does not exist, then the measure  $\mu$  is concentrated on  $\{0, 1\}$  and at most one additional point, and since  $\mu$  does not contain any Dirac measure we obtain  $\mu = 0$  and we are done. Let  $I_\varepsilon := (a - \varepsilon, a + \varepsilon)$ ,  $J_{\varepsilon'} := (b - \varepsilon', b + \varepsilon')$ , where  $\varepsilon, \varepsilon'$  are positive numbers such that these two intervals are disjoint subsets of  $(0, 1)$ . We set

$$\psi_{\varepsilon, \varepsilon'}(x) := \nu_{\varepsilon, \varepsilon'}((0, x)), \quad \text{with} \quad \nu_{\varepsilon, \varepsilon'} := \frac{1}{\mu(I_\varepsilon)} \mu \llcorner I_\varepsilon - \frac{1}{\mu(J_{\varepsilon'})} \mu \llcorner J_{\varepsilon'}.$$

We compare  $u$  with  $u + \rho \psi_{\varepsilon, \varepsilon'}$  for some  $\rho$  chosen below. This gives, since  $\psi_{\varepsilon, \varepsilon'} \in BV(0, 1)$  with  $\psi_{\varepsilon, \varepsilon'}(0) = \psi_{\varepsilon, \varepsilon'}(1) = 0$  and  $J_{\psi_{\varepsilon, \varepsilon'}} = \emptyset$ ,

$$\begin{aligned} 0 &\leq \int_0^1 [h(u' + \rho \psi'_{\varepsilon, \varepsilon'}) - h(u')] dx + \ell |D^c(u + \rho \psi_{\varepsilon, \varepsilon'})|(0, 1) - \ell |D^c u|(0, 1) \\ &\quad + \gamma \int_0^1 [(u - w_k^\tau + \rho \psi_{\varepsilon, \varepsilon'})^2 - (u - w_k^\tau)^2] dx. \end{aligned}$$

We observe that

$$u' + \rho\psi'_{\varepsilon,\varepsilon'} - \frac{\ell}{2} = u' - \frac{\ell}{2} + \frac{\rho}{\mu(I_\varepsilon)}\chi_{I_\varepsilon}(u' - \frac{\ell}{2})_+ - \frac{\rho}{\mu(J_{\varepsilon'})}\chi_{J_{\varepsilon'}}(u' - \frac{\ell}{2})_+$$

where  $\chi_{I_\varepsilon}$  and  $\chi_{J_{\varepsilon'}}$  are the characteristic functions of the two intervals. Therefore for any  $\rho \in (-\mu(I_\varepsilon), \mu(J_{\varepsilon'}))$  we have  $u' + \rho\psi'_{\varepsilon,\varepsilon'} - \frac{\ell}{2} \geq 0$  whenever  $\rho\psi'_{\varepsilon,\varepsilon'} \neq 0$ , so that  $h(u' + \rho\psi'_{\varepsilon,\varepsilon'}) - h(u') = \ell\rho\psi'_{\varepsilon,\varepsilon'}$ . Analogously, for the same set of values of  $\rho$  we have

$$(D^c u)_+ + \rho D^c \psi_{\varepsilon,\varepsilon'} = \left(1 + \frac{\rho}{\mu(I_\varepsilon)}\chi_{I_\varepsilon} - \frac{\rho}{\mu(J_{\varepsilon'})}\chi_{J_{\varepsilon'}}\right) (D^c u)_+ \geq 0$$

which implies  $|D^c(u + \rho\psi_{\varepsilon,\varepsilon'})|(0,1) = |D^c u|(0,1) + \rho D^c \psi_{\varepsilon,\varepsilon'}(0,1)$ . The above expression then becomes

$$0 \leq \int_0^1 \ell\rho\psi'_{\varepsilon,\varepsilon'} dx + \ell\rho D^c \psi_{\varepsilon,\varepsilon'}(0,1) + \gamma \int_0^1 [2\rho(u - w_k^\tau)\psi_{\varepsilon,\varepsilon'} + \rho^2\psi_{\varepsilon,\varepsilon'}^2] dx.$$

The sum of the first two terms is  $\ell\rho\nu_{\varepsilon,\varepsilon'}(0,1) = 0$ , and since  $\rho$  can still be chosen arbitrarily small we conclude that  $\int_0^1 (u - w_k^\tau)\psi_{\varepsilon,\varepsilon'} dx = 0$  for all admissible  $\varepsilon, \varepsilon'$ . By Fubini's theorem we have

$$0 = \int_0^1 (u - w_k^\tau)(x)\psi_{\varepsilon,\varepsilon'}(x) dx = \int_0^1 (u - w_k^\tau)(x) \int_0^x d\nu_{\varepsilon,\varepsilon'}(y) dx = \int_0^1 F(y) d\nu_{\varepsilon,\varepsilon'}(y)$$

for all  $\varepsilon, \varepsilon'$ , where  $F(y) := \int_y^1 (u - w_k^\tau)(x) dx$  is continuous.

We now consider the measure  $\hat{\nu}_\varepsilon := \frac{1}{\mu(I_\varepsilon)}\mu \llcorner I_\varepsilon - \delta_b$  and set  $\hat{\psi}_\varepsilon(x) := \hat{\nu}_\varepsilon((0,x))$ . Since  $\nu_{\varepsilon,\varepsilon'} \rightarrow \hat{\nu}_\varepsilon$  as  $\varepsilon' \rightarrow 0$ , and  $F$  is continuous, we obtain  $\int_0^1 F d\hat{\nu}_\varepsilon = 0$ . We compare  $u$  with  $u + \rho\hat{\psi}_\varepsilon$  and obtain

$$\begin{aligned} 0 &\leq \int_0^1 [h(u' + \rho\hat{\psi}'_\varepsilon) - h(u')] dx + \ell|D^c(u + \rho\hat{\psi}_\varepsilon)|(0,1) - \ell|D^c u|(0,1) \\ &\quad + \gamma \int_0^1 [(u - w_k^\tau + \rho\hat{\psi}_\varepsilon)^2 - (u - w_k^\tau)^2] dx + \bar{g}(|[u](b) - \rho|, s_{k-1}^\tau(b)) - \bar{g}(|[u](b)|, s_{k-1}^\tau(b)) \end{aligned}$$

where as usual  $s_{k-1}^\tau(b) = 0$  if  $b \notin \Gamma_{k-1}^\tau$ . By the same computation as above, using that  $\hat{\nu}_\varepsilon(I_\varepsilon) = 1$ ,

$$0 \leq \rho\ell + 2\gamma\rho \int_0^1 F d\hat{\nu}_\varepsilon + \gamma\rho^2 \int_0^1 \hat{\psi}_\varepsilon^2 dx + \bar{g}(|[u](b) - \rho|, s_{k-1}^\tau(b)) - \bar{g}(|[u](b)|, s_{k-1}^\tau(b)).$$

By Proposition 3.9(ii) and Proposition 3.7 we have for  $|\rho|$  small

$$\bar{g}(|[u](b) - \rho|, s_{k-1}^\tau(b)) \leq \bar{g}(|[u](b)|, s_{k-1}^\tau(b)) + g(|\rho|) \leq \bar{g}(|[u](b)|, s_{k-1}^\tau(b)) + \ell|\rho| - \tilde{\ell}|\rho|^{5/3} + o(|\rho|^{5/3}).$$

Recalling  $\int_0^1 F d\hat{\nu}_\varepsilon = 0$  we have

$$0 \leq \rho\ell + \ell|\rho| - \tilde{\ell}|\rho|^{5/3} + o(|\rho|^{5/3}) + \gamma\rho^2 \int_0^1 \hat{\psi}_\varepsilon^2 dx$$

for all  $\rho$  sufficiently small (positive or negative). This is a contradiction, hence we conclude that  $\mu = 0$ .  $\square$

Proposition 4.1 allows to construct a *piecewise constant evolution*, relative to the discrete boundary values  $b_k^\tau = b(t_k)$  and to  $w_k^\tau := w(t_k)$ ,

$$t \mapsto (u_\tau(t), \Gamma_\tau(t), s_\tau(t)) \quad \text{for } t \in [0, T], \quad (4.14)$$

where the piecewise constant interpolations in time are defined as

$$u_\tau(t) := u_k^\tau, \quad \Gamma_\tau(t) := \Gamma_k^\tau, \quad s_\tau(t) := s_k^\tau \quad \text{for } t \in [t_k, t_{k+1}), \quad (4.15)$$

with  $k = 0, 1, \dots, N_\tau$ , with  $\Gamma_{-1}^\tau = \emptyset$ ,  $s_{-1}^\tau = 0$ . We also consider the piecewise constant and the piecewise affine interpolations in time of the boundary data, which are respectively defined as

$$b_\tau(t) := b_k^\tau, \quad b^\tau(t) := b_k^\tau + \frac{t - t_k}{\tau}(b_{k+1}^\tau - b_k^\tau) \quad \text{for } t \in [t_k, t_{k+1}), \quad (4.16)$$

and similarly

$$w_\tau(t) := w_k^\tau, \quad w^\tau(t) := w_k^\tau + \frac{t - t_k}{\tau}(w_{k+1}^\tau - w_k^\tau) \quad \text{for } t \in [t_k, t_{k+1}). \quad (4.17)$$

In the following we use the notation  $u_\tau(t, x) := (u_\tau(t))(x)$  (and similarly for the other functions introduced above). We recall that we extend  $s_k^\tau$  by 0 outside  $\Gamma_k^\tau$ .

We observe that for any  $k$

$$\Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) = \Phi(u_k^\tau; \Gamma_k^\tau, s_k^\tau, b_k^\tau). \quad (4.18)$$

To see this, it suffices to show that

$$\bar{g}(|[u_k^\tau](x)|, s_k^\tau(x)) = \bar{g}(|[u_k^\tau](x)|, s_{k-1}^\tau(x)) \quad \text{for any } x \in J^{b_k^\tau}(u_k^\tau) \cup \Gamma_k^\tau.$$

This is true, since in (4.11) we defined  $s_k^\tau(x) = s_{k-1}^\tau(x) \vee |[u_k^\tau](x)|$ . If  $s_{k-1}^\tau(x) \geq |[u_k^\tau](x)$  then  $s_{k-1}^\tau(x) = s_k^\tau(x)$  and the two terms are identical. If not, then  $s_k^\tau(x) = |[u_k^\tau](x)$  and, recalling Proposition 3.9(i), we have  $\bar{g}(|[u_k^\tau](x)|, s_k^\tau(x)) = g(|[u_k^\tau](x)|) = \bar{g}(|[u_k^\tau](x)|, s_{k-1}^\tau(x))$ . Therefore (4.18) holds.

We are interested in the evolution of the discrete total energy at time  $t$ , with timestep  $\tau$ , which is defined for  $t \in [0, T]$  by

$$\mathcal{E}_\tau(t) := \Phi(u_\tau(t); \Gamma_\tau(t), s_\tau(t), b_\tau(t)) + \gamma \int_0^1 |u_\tau(t) - w_\tau(t)|^2 dx. \quad (4.19)$$

**Lemma 4.3.** *For any  $t \in [0, T]$ , letting  $t_k$  be the discretization point such that  $t \in [t_k, t_{k+1})$ , we have*

$$\mathcal{E}_\tau(t) \leq \mathcal{E}_\tau(0) + \int_0^{t_k} \int_0^1 \left( h'((u_\tau(r))')(\dot{b}^\tau(r))' + 2\gamma(u_\tau(r) - w_\tau(r))(\dot{b}^\tau(r) - \dot{w}^\tau(r)) \right) dx dr + R_\tau(t), \quad (4.20)$$

where

$$\begin{aligned} R_\tau(t) := & \int_0^{t_k} \left( \tau \int_0^1 |(\dot{b}^\tau(r))'|^2 dx + \gamma \tau \int_0^1 |\dot{b}^\tau(r)|^2 dx \right. \\ & \left. + \gamma \left| \int_0^1 (w_\tau(r + \tau) - w_\tau(r)) \dot{w}^\tau(r) dx \right| + 2\gamma \left| \int_0^1 (w_\tau(r + \tau) - w_\tau(r)) \dot{b}^\tau(r) dx \right| \right) dr. \end{aligned} \quad (4.21)$$

*Proof.* We compare  $u_k^\tau$  with  $v := u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau$ : by minimality of  $u_k^\tau$  in problem (4.10) we have by Proposition 4.2

$$\begin{aligned} & \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx \\ & \leq \int_0^1 h(|(u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau)'|) dx + \gamma \int_0^1 |u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau - w_k^\tau|^2 dx \\ & \quad + \sum_{x \in \Gamma_{k-1}^\tau \cup J^{b_k^\tau - b_{k-1}^\tau}(u_{k-1}^\tau)} \bar{g}(|[u_{k-1}^\tau](x)|, s_{k-1}^\tau(x)). \end{aligned}$$

If  $x \in \{0, 1\}$ , we remark that  $v(x) - b_k^\tau(x) = u_{k-1}^\tau(x) - b_{k-1}^\tau(x)$ , so that the notation  $[u_{k-1}^\tau](x)$  is appropriate. Therefore, recalling (4.18),

$$\begin{aligned}
& \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx \\
& \leq \Phi(u_{k-1}^\tau; \Gamma_{k-2}^\tau, s_{k-2}^\tau, b_{k-1}^\tau) + \gamma \int_0^1 |u_{k-1}^\tau - w_{k-1}^\tau|^2 dx \\
& \quad + \int_0^1 \left[ h(|(u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau)'|) - h(|(u_{k-1}^\tau)'|) \right] dx \\
& \quad + \gamma \int_0^1 \left( |u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau - w_k^\tau|^2 - |u_{k-1}^\tau - w_{k-1}^\tau|^2 \right) dx. \tag{4.22}
\end{aligned}$$

Since  $h'$  is a Lipschitz function with  $|h''| \leq 2$ , the mean-value theorem gives

$$h(x+b) \leq h(x) + bh'(x) + b^2 \quad \text{for all } x, b \in \mathbb{R}.$$

In particular,

$$h((u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau)') - h((u_{k-1}^\tau)') \leq h'((u_{k-1}^\tau)')(b_k^\tau - b_{k-1}^\tau)' + |(b_k^\tau - b_{k-1}^\tau)'|^2.$$

Similarly,

$$\begin{aligned}
& |u_{k-1}^\tau + b_k^\tau - b_{k-1}^\tau - w_k^\tau|^2 - |u_{k-1}^\tau - w_{k-1}^\tau|^2 \\
& = (w_k^\tau + w_{k-1}^\tau - 2u_{k-1}^\tau)(w_k^\tau - w_{k-1}^\tau) + 2(u_{k-1}^\tau - w_k^\tau)(b_k^\tau - b_{k-1}^\tau) + |b_k^\tau - b_{k-1}^\tau|^2.
\end{aligned}$$

By inserting these inequalities into (4.22) and iterating we find

$$\begin{aligned}
& \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx \leq \Phi(u_0^\tau; b_0^\tau) + \gamma \int_0^1 |u_0^\tau - w_0^\tau|^2 dx \\
& \quad + \sum_{i=1}^k \left( \int_0^1 h'((u_{i-1}^\tau)')(b_i^\tau - b_{i-1}^\tau)' dx + \int_0^1 |(b_i^\tau - b_{i-1}^\tau)'|^2 dx \right. \\
& \quad \quad + \gamma \int_0^1 (w_i^\tau + w_{i-1}^\tau - 2u_{i-1}^\tau)(w_i^\tau - w_{i-1}^\tau) dx \\
& \quad \quad \left. + 2\gamma \int_0^1 (u_{i-1}^\tau - w_i^\tau)(b_i^\tau - b_{i-1}^\tau) dx + \gamma \int_0^1 |b_i^\tau - b_{i-1}^\tau|^2 dx \right).
\end{aligned}$$

We can rewrite this inequality in terms of the interpolants defined in (4.15), (4.16), (4.17): for any  $t \in [0, T]$ , denoting by  $t_k$  the discretization point such that  $t \in [t_k, t_{k+1})$ , and observing that (4.18) and (4.19) give

$$\mathcal{E}_\tau(t) = \Phi(u_k^\tau; \Gamma_{k-1}^\tau, s_{k-1}^\tau, b_k^\tau) + \gamma \int_0^1 |u_k^\tau - w_k^\tau|^2 dx,$$

we have

$$\begin{aligned}
\mathcal{E}_\tau(t) & \leq \mathcal{E}_\tau(0) + \int_0^{t_k} \left( \int_0^1 h'((u_\tau(r))')(b^\tau(r))' dx + \tau \int_0^1 |(b^\tau(r))'|^2 dx \right. \\
& \quad + \gamma \int_0^1 (w_\tau(r+\tau) + w_\tau(r))\dot{w}^\tau(r) dx - 2\gamma \int_0^1 w_\tau(r+\tau)\dot{b}^\tau(r) dx \\
& \quad \left. + 2\gamma \int_0^1 u_\tau(r)(\dot{b}^\tau(r) - \dot{w}^\tau(r)) dx + \gamma\tau \int_0^1 |\dot{b}^\tau(r)|^2 dx \right) dr.
\end{aligned}$$

This concludes the proof.  $\square$



## 5 The time-continuous cohesive evolution

The goal of this section is to pass to the limit in the time-discrete evolution as the time-step  $\tau$  goes to zero. As in the previous section, we fix a positive threshold  $\bar{s} > 0$  for the jumps, the final time  $T > 0$ , a time-dependent boundary displacement  $b \in H^1([0, T]; \mathbb{R}^2)$ , and a lower order term  $w \in \text{AC}([0, T]; L^\infty(0, 1))$ . Let  $(\tau_n)_{n \in \mathbb{N}}$  be a decreasing sequence of time-discretization steps with  $\tau_n \rightarrow 0$  as  $n \rightarrow \infty$ . Correspondingly, let

$$0 = t_0^n < t_1^n < \dots < t_{N_n}^n < t_{N_n+1}^n = T$$

be the subdivision of  $[0, T]$  with  $t_k^n := k\tau_n$  for  $k \in \{0, \dots, N_n\}$ . Let

$$t \mapsto (u_n(t) := u_{\tau_n}(t), \Gamma_n(t) := \Gamma_{\tau_n}(t), s_n(t) := s_{\tau_n}(t)), \quad \text{for } t \in [0, T], \quad (5.1)$$

be the piecewise constant interpolation, defined in (4.15), of a time-discrete evolution relative to the boundary data  $b_k^n = b(t_k^n)$  and to  $w_k^n = w(t_k^n)$ , as constructed in the previous section. Let also

$$b_n(t) := b_{\tau_n}(t), \quad b^n(t) := b^{\tau_n}(t), \quad w_n(t) := w_{\tau_n}(t), \quad w^n(t) := w^{\tau_n}(t) \quad (5.2)$$

be the piecewise constant and the piecewise affine interpolants of the maps  $b$  and  $w$ , according to (4.16)–(4.17). We also consider the associated energy  $\mathcal{E}_n(t) := \mathcal{E}_{\tau_n}(t)$ , see (4.19). Our main result is the following.

**Theorem 5.1** (Existence of a quasi-static cohesive evolution). *Under the assumptions of Section 2, with  $b$  and  $w$  as in (4.2) and (4.7), there exists  $(u(t), \Gamma(t), s(t))$ , for  $t \in [0, T]$ , with the following properties:*

- (i)  $u(t) \in \text{BV}(0, 1)$ ,  $\sup_{t \in [0, T]} \|u(t)\|_\infty \leq \sup_{t \in [0, T]} \max\{\|b(t)\|_\infty, \|w(t)\|_\infty\}$ ,  $\Gamma(t) \subset [0, 1]$  is a finite set,  $s(t) : \Gamma(t) \rightarrow [\bar{s}, \infty)$ ,  $\sup_{t \in [0, T]} \|s(t)\|_\infty \leq 2 \sup_{t \in [0, T]} \max\{\|b(t)\|_\infty, \|w(t)\|_\infty\}$ ;
- (ii) (irreversibility)  $\Gamma(t_1) \subset \Gamma(t_2)$  and  $s(t_1) \leq s(t_2)$  for all  $0 \leq t_1 \leq t_2 \leq T$ ;
- (iii)  $u(0)$  minimizes the functional  $\Phi(v; b(0)) + \gamma \|v - w(0)\|_{L^2(0, 1)}^2$  among all  $v \in \text{BV}(0, 1)$ , and  $\Gamma(0) = \{x \in J^{b(0)}(u(0)) : |[u(0)](x)| > \bar{s}\}$ ,  $s(0) = |[u](0)|$  on  $\Gamma(0)$ ;
- (iv)  $\{x \in J^{b(t)}(u(t)) : |[u(t)](x)| > \bar{s}\} \subset \Gamma(t)$  and  $[u](t) \leq s(t)$  on  $\Gamma(t)$ , for all  $t \in (0, T]$ ;
- (v) (static equilibrium) for all  $t \in (0, T]$ ,  $u(t)$  minimizes the functional

$$\Phi(v; \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |v - w(t)|^2 dx$$

among all  $v \in \text{BV}(0, 1)$ ;

- (vi) (non-dissipativity) the total energy

$$\mathcal{E}(t) := \Phi(u(t); \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |u(t) - w(t)|^2 dx$$

satisfies for every  $t \in [0, T]$

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_0^1 \left( h'((u(r))')(\dot{b}(r)))' + 2\gamma(u(r) - w(r))(\dot{b}(r) - \dot{w}(r)) \right) dx dr. \quad (5.3)$$

**Corollary 5.2.** *Theorem 5.1 implies that  $(u(t), \Gamma(t), s(t))$  minimizes the functional*

$$\Phi(v; \Gamma, s, b(t)) + \gamma \int_0^1 |v - w(t)|^2 dx$$

among all  $(v, \Gamma, s)$  such that  $\Gamma$  is a finite set with  $\Gamma(t) \subset \Gamma$ ,  $s : \Gamma \rightarrow [\bar{s}, \infty)$  satisfies  $s \geq s(t)$  on  $\Gamma(t)$ , and  $v \in \text{BV}(0, 1)$  is such that  $\{x \in J^{b(t)}(v) : |[v](x)| > \bar{s}\} \subset \Gamma$ .

*Proof.* By (i) and (iv),  $(u(t), \Gamma(t), s(t))$  is admissible. Let now  $(v, \Gamma, s)$  be admissible. By monotonicity of  $\bar{g}$  we have  $\Phi(v; \Gamma(t), s(t), b(t)) \leq \Phi(v; \Gamma, s, b(t))$ , and (v) then concludes the proof.  $\square$

*Proof of Theorem 5.1.* We divide the proof into several steps. In the following, we will denote by  $C$  a generic constant, possibly depending on  $b, w$ , and  $\gamma$ , but independent of  $n$  and  $t$ , which might change from line to line. As a preliminary remark, we observe that  $u_n(t)$  is a minimizer of problem (4.10), for the value of  $k$  such that  $t \in [t_k^n, t_{k+1}^n)$ ; then

$$\begin{aligned} & \Phi(u_n(t); \Gamma_n(t - \tau_n), s_n(t - \tau_n), b_n(t)) + \gamma \int_0^1 |u_n(t) - w_n(t)|^2 dx \\ & \leq \Phi(v; \Gamma_n(t - \tau_n), s_n(t - \tau_n), b_n(t)) + \gamma \int_0^1 |v - w_n(t)|^2 dx \quad \text{for all } v \in \text{BV}(0, 1). \end{aligned}$$

Using (4.18) on the left-hand side, and monotonicity on the right-hand side, this implies

$$\begin{aligned} & \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |u_n(t) - w_n(t)|^2 dx \\ & \leq \Phi(v; \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |v - w_n(t)|^2 dx \quad \text{for all } v \in \text{BV}(0, 1). \end{aligned} \tag{5.4}$$

*Step 1: compactness.* The first goal is to prove a uniform bound on  $\mathcal{E}_n(t)$ , independent of  $n$  and  $t$ , by using the energy inequality (4.20). To this aim, notice that by using  $v = b(0)$  as test function in (4.8) we easily obtain a uniform bound on the initial energy  $\mathcal{E}_n(0)$ . Then (4.20) yields, as  $|h'(s)| \leq \ell$ ,

$$\begin{aligned} \mathcal{E}_n(t) & \leq \mathcal{E}_n(0) + \int_0^{t_k^n} \int_0^1 \left( h'((u_n(r))') (\dot{b}^n(r))' + 2\gamma(u_n(r) - w_n(r))(\dot{b}^n(r) - \dot{w}^n(r)) \right) dx dr + R_n(t) \\ & \leq C + \ell \int_0^T \int_0^1 |(\dot{b}^n(r))'| dx dr \\ & \quad + 2\gamma \sup_{r \in [0, T]} \left( \|u_n(r)\|_\infty + \|w_n(r)\|_\infty \right) \int_0^T \int_0^1 \left( |\dot{b}^n(r)| + |\dot{w}^n(r)| \right) dx dr + R_n(t). \end{aligned}$$

For the remainder  $R_n(t)$ , defined in (4.21), we have:

$$R_n(t) \leq C\tau_n \int_0^T \|\dot{b}^n(r)\|_{H^1(0,1)}^2 dr + C \sup_{r \in [0, T]} \|w_n(r)\|_\infty \int_0^T \left( \|\dot{w}^n(r)\|_{L^1(0,1)} + \|\dot{b}^n(r)\|_{L^1(0,1)} \right) dr.$$

In view of the assumptions on  $b$  and  $w$  and of Proposition 4.1(ii), all the previous quantities are uniformly bounded, therefore we obtain a uniform estimate on the energies  $\mathcal{E}_n(t)$  and in turn, similarly to (4.13), on the BV-norm of the functions  $u_n(t)$ :

$$\sup_{n,t} \mathcal{E}_n(t) < \infty, \quad \sup_{n,t} |Du_n(t)|(0, 1) < \infty. \tag{5.5}$$

Fix now a countable dense set  $D \subset [0, T]$ , with  $0 \in D$ . By a diagonal argument and by the uniform bounds (5.5) we can find a subsequence (which we denote by the same symbol) such that

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly* in } \text{BV}(0, 1), \quad \text{for all } t \in D, \tag{5.6}$$

for some  $u(t) \in \text{BV}(0, 1)$ , with  $\|u(t)\|_\infty \leq \max_{r \in [0, T]} \max\{\|b(r)\|_\infty, \|w(r)\|_\infty\}$ . This implies in particular that  $\sup_t \sup_n \sup_x |s_n(t)(x)| < \infty$ .

Thanks to Proposition 4.1(iii), it is clear that we have a uniform bound on the number of points of  $\Gamma_n(t)$ :

$$\sup_{n,t} \mathcal{H}^0(\Gamma_n(t)) < \infty. \tag{5.7}$$

Up to further subsequences, we can therefore assume that for any  $t \in D$  and  $n$  large enough  $\Gamma_n(t) = \{\bar{x}_1^n(t), \dots, \bar{x}_{N(t)}^n(t)\}$  (with the points  $\bar{x}_i^n(t)$  distinct and  $N(t)$  independent of  $n$ ), and that each sequence  $\bar{x}_i^n(t)$  converges as  $n \rightarrow \infty$ ; we denote by  $\Gamma(t) = \{\bar{x}_1(t), \dots, \bar{x}_{N(t)}(t)\}$  the set of limit points of these sequences,

$$\Gamma(t) := \left\{ \bar{x}_i(t) := \lim_{n \rightarrow \infty} \bar{x}_i^n(t), i = 1, \dots, N(t) \right\} \subset [0, 1] \quad \text{for } t \in D. \quad (5.8)$$

Notice that all the points  $\bar{x}_i(t)$  are distinct, since two different sequences  $(\bar{x}_i^n(t))_n, (\bar{x}_j^n(t))_n$  cannot converge to the same limit point by Proposition 4.1(iii). Moreover, in view of the monotonicity property  $\Gamma_n(t_1) \subset \Gamma_n(t_2)$  for  $t_1 < t_2$ , we also have

$$\Gamma(t_1) \subset \Gamma(t_2) \quad \text{for all } t_1, t_2 \in D, t_1 < t_2. \quad (5.9)$$

Finally, we also have compactness for the maps  $s_n(t)$ : by possibly extracting another subsequence, for every  $t \in D$  there exists a map  $s(t) : \Gamma(t) \rightarrow [\bar{s}, \infty)$ , such that

$$s_n(t)(\bar{x}_i^n(t)) \rightarrow s(t)(\bar{x}_i(t)) \quad \text{for all } i = 1, \dots, N(t), t \in D. \quad (5.10)$$

Moreover  $s(t_1) \leq s(t_2)$  on  $\Gamma(t_1)$  for all  $t_1, t_2 \in D, t_1 < t_2$ , since every map  $s_n(t)$  is nondecreasing in time by construction. As usual, we extend  $s$  by 0 on  $[0, 1] \setminus \Gamma(t)$ .

*Step 2.* We now prove the following claims:

$$\{x \in J^{b(t)}(u(t)) : |[u(t)](x)| > \bar{s}\} \subset \Gamma(t) \quad \text{for all } t \in D, \quad (5.11)$$

and

$$|[u(t)](x)| \leq s(t, x) \quad \text{for all } x \in \Gamma(t) \text{ and } t \in D. \quad (5.12)$$

We first recall that by Proposition 4.2 we have  $u_n \in SBV(0, 1)$  and  $|u_n'| \leq \frac{\ell}{2}$  almost everywhere.

In order to prove (5.11), suppose by contradiction that for some  $t \in D$  there exists  $\bar{x} \in J^{b(t)}(u(t)) \setminus \Gamma(t)$  with  $|[u(t)](\bar{x})| \geq \bar{s} + \varepsilon$ , for some  $\varepsilon > 0$ . We also assume that  $\bar{x} \in (0, 1)$ , as the boundary case follows by a similar argument. Let  $I_\delta := (\bar{x} - \delta, \bar{x} + \delta)$  be such that  $I_\delta \cap \Gamma_n(t) = \emptyset$  for all sufficiently large  $n$ ; then, by definition of  $\Gamma_n(t)$ , all the jumps of  $u_n(t)$  inside  $I_\delta$  are smaller than the threshold  $\bar{s}$ :  $|[u_n(t)](x)| \leq \bar{s}$  for every  $x \in J_{u_n(t)} \cap I_\delta$ . Moreover, as  $\Gamma_n(t - \tau_n) \subset \Gamma_n(t)$ ,  $u_n(t)$  solves the minimum problem

$$\min \left\{ \int_{I_\delta} h(|v'|) dx + \sum_{x \in J_v \cap I_\delta} g(|[v](x)|) + \ell |D^c v|(I_\delta) + \gamma \int_{I_\delta} |v - w_n(t)|^2 dx : v \in BV(I_\delta), \right. \\ \left. \{v \neq u_n(t)\} \subset\subset I_\delta \right\}. \quad (5.13)$$

By (5.6) and lower semicontinuity of the total variation, we have for every  $\delta > 0$

$$\liminf_{n \rightarrow \infty} |Du_n(t)|(I_\delta) \geq |Du(t)|(I_\delta) \geq \bar{s} + \varepsilon.$$

In view of Proposition 4.2, it is not possible that the limit jump of  $u(t)$  at  $\bar{x}$  is created by a nonzero contribution from the absolutely continuous part of the measures  $Du_n(t)$ : more precisely, we can assert that

$$\text{for every } \delta > 0 \text{ sufficiently small} \quad \liminf_{n \rightarrow \infty} \sum_{x \in J_{u_n(t)} \cap I_\delta} |[u_n(t)](x)| \geq \bar{s} + \frac{\varepsilon}{2},$$

and each  $u_n(t)$  has at least two jumps in  $I_\delta$  since  $|[u_n(t)](x)| \leq \bar{s}$ . In particular, for  $n$  large we can find a partition  $J_{u_n(t)} \cap I_\delta = A_n \cup B_n$  such that

$$\sum_{x \in A_n} |[u_n(t)](x)| \geq \frac{\varepsilon}{4}, \quad \sum_{x \in B_n} |[u_n(t)](x)| \geq \frac{\varepsilon}{4}. \quad (5.14)$$

We construct a competitor for the minimum problem (5.13) by moving all the jumps of  $u_n(t)$  in a single point  $x_n \in I_\delta \setminus J_{u_n(t)}$ : then the minimality of  $u_n(t)$  in (5.13) and the subadditivity of  $g$  yield

$$\begin{aligned} 0 &\leq g\left(\left|\sum_{x \in J_{u_n(t)} \cap I_\delta} [u_n(t)](x)\right|\right) - \sum_{x \in J_{u_n(t)} \cap I_\delta} g(|[u_n(t)](x)|) + C\delta \\ &\leq g\left(\sum_{x \in J_{u_n(t)} \cap I_\delta} |[u_n(t)](x)|\right) - g\left(\sum_{x \in A_n} |[u_n(t)](x)|\right) - g\left(\sum_{x \in B_n} |[u_n(t)](x)|\right) + C\delta \\ &\leq -c_\varepsilon/4 + C\delta, \end{aligned}$$

the last inequality following by (5.14) and (3.18). This is a contradiction for  $\delta$  small enough and completes the proof of (5.11).

We next show (5.12). Let  $\bar{x}_i(t)$  be any point in  $\Gamma(t)$ . Recalling (5.10) and that by construction  $|[u_n(t)](\bar{x}_i^n(t))| \leq s_n(t, \bar{x}_i^n(t))$ , it is sufficient to prove that

$$|[u(t)](\bar{x}_i(t))| \leq \liminf_{n \rightarrow \infty} |[u_n(t)](\bar{x}_i^n(t))|. \quad (5.15)$$

Let us momentarily omit the dependence on  $t$ . Suppose by contradiction that for some  $\varepsilon > 0$  one has  $|[u](\bar{x}_i)| \geq \liminf_{n \rightarrow \infty} |[u_n](\bar{x}_i^n)| + \varepsilon$ . Then, by Proposition 4.2, (5.6), and lower semicontinuity of the total variation, we have for  $I_\delta := (\bar{x}_i - \delta, \bar{x}_i + \delta)$  and sufficiently small  $\delta$

$$\limsup_{n \rightarrow \infty} \sum_{x \in J_{u_n} \cap I_\delta \setminus \{\bar{x}_i^n\}} |[u_n](x)| \geq \frac{\varepsilon}{2}.$$

We construct a competitor by moving all the jumps of  $u_n$  at the point  $\bar{x}_i^n$ : the minimality of  $u_n$  in (5.4) yields for  $n$  large enough

$$\begin{aligned} 0 &\leq \bar{g}\left(\left|\sum_{x \in J_{u_n} \cap I_\delta} [u_n](x)\right|, s_n(\bar{x}_i^n)\right) - \bar{g}(|[u_n](\bar{x}_i^n)|, s_n(\bar{x}_i^n)) - \sum_{x \in J_{u_n} \cap I_\delta \setminus \{\bar{x}_i^n\}} g(|[u_n](x)|) + C\delta \\ &\leq \bar{g}\left(\sum_{x \in J_{u_n} \cap I_\delta} |[u_n](x)|, s_n(\bar{x}_i^n)\right) - \bar{g}(|[u_n](\bar{x}_i^n)|, s_n(\bar{x}_i^n)) - g\left(\sum_{x \in J_{u_n} \cap I_\delta \setminus \{\bar{x}_i^n\}} |[u_n](x)|\right) + C\delta \\ &\leq -c_\varepsilon/2 + C\delta, \end{aligned}$$

where the last inequality follows by (3.41). This is a contradiction for  $\delta$  small enough, and proves that (5.15) holds.

*Step 3: lower semicontinuity of the energies.* We now claim that for every  $t \in D$

$$\Phi(u(t); \Gamma(t), s(t), b(t)) \leq \liminf_{n \rightarrow \infty} \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)). \quad (5.16)$$

First observe that, in view of the continuity and monotonicity properties of  $\bar{g}$ , (5.15), and (5.10), we have

$$\sum_{x \in \Gamma(t)} \bar{g}(|[u(t)](x)|, s(t, x)) \leq \liminf_{n \rightarrow \infty} \sum_{x \in \Gamma_n(t)} \bar{g}(|[u_n(t)](x)|, s_n(t, x)). \quad (5.17)$$

Consider any relatively open set  $A \subset\subset [0, 1] \setminus \Gamma(t)$ ; by lower semicontinuity of the functional

$$\begin{aligned} \int_A h(|u'(t)|) dx + \sum_{x \in J^{b(t)}(u(t)) \cap A} g(|[u(t)](x)|) + \ell |D^c u(t)|(A) \\ \leq \liminf_{n \rightarrow \infty} \left[ \int_A h(|u'_n(t)|) dx + \sum_{x \in J^{b_n(t)}(u_n(t)) \cap A} g(|[u_n(t)](x)|) + \ell |D^c u_n(t)|(A) \right] \end{aligned}$$

so that taking the supremum over all  $A$  and recalling (5.17) we obtain (5.16).

*Step 4: static equilibrium.* We now show that the minimality condition (v) holds for every  $t \in D$ . In order to do this, we first show that for every  $v \in \text{BV}(0, 1)$  we can construct  $v_n \in \text{BV}(0, 1)$  such that  $v_n \rightarrow v$  in  $L^2(0, 1)$  as  $n \rightarrow \infty$  and

$$\limsup_{n \rightarrow \infty} \left[ \Phi(v_n; \Gamma_n(t), s_n(t), b_n(t)) - \Phi(v; \Gamma(t), s(t), b(t)) \right] \leq 0. \quad (5.18)$$

Recalling (5.8), this can be done by considering the (possible) jumps of  $v$  on the points of  $\Gamma(t)$  and moving them to the corresponding points of  $\Gamma_n(t)$ : more precisely, assuming for simplicity that  $\Gamma(t) \subset (0, 1)$  (the construction can be straightforwardly adapted if one of the boundary points belongs to  $\Gamma(t)$ ), we define  $v_n$  by the conditions

$$v_n^+(0) := v^+(0), \quad Dv_n := Dv - \sum_{i=1}^{N(t)} [v](\bar{x}_i(t)) \delta_{\bar{x}_i(t)} + \sum_{i=1}^{N(t)} [v](\bar{x}_i^n(t)) \delta_{\bar{x}_i^n(t)}.$$

Then  $v_n \rightarrow v$  in  $L^2(0, 1)$ . By (5.8), if  $n$  is sufficiently large then  $\bar{x}_i^n(t) \neq \bar{x}_j(t)$  for  $i \neq j$ . We set  $\chi_i^n = 1$  if  $\bar{x}_i^n(t) \neq \bar{x}_i(t)$ , and  $\chi_i^n = 0$  if  $\bar{x}_i^n(t) = \bar{x}_i(t)$ . We estimate

$$\begin{aligned} & \Phi(v_n; \Gamma_n(t), s_n(t), b_n(t)) - \Phi(v; \Gamma(t), s(t), b(t)) \\ &= \sum_{i=1}^{N(t)} \chi_i^n \left[ \bar{g}(|[v](\bar{x}_i^n(t)) + [v](\bar{x}_i(t))|, s_n(t), \bar{x}_i^n(t)) - \bar{g}(|[v](\bar{x}_i(t))|, s(t), \bar{x}_i(t)) - g(|[v](\bar{x}_i^n(t))|) \right] \\ &+ \sum_{i=1}^{N(t)} (1 - \chi_i^n) \left[ \bar{g}(|[v](\bar{x}_i(t))|, s_n(t), \bar{x}_i^n(t)) - \bar{g}(|[v](\bar{x}_i(t))|, s(t), \bar{x}_i(t)) \right] \\ &+ g(|v^+(0) - b_n(t, 0)|) - g(|v^+(0) - b(t, 0)|) + g(|v^-(1) - b_n(t, 1)|) - g(|v^-(1) - b(t, 1)|) \\ &\stackrel{(3.38)}{\leq} \sum_{i=1}^{N(t)} \chi_i^n \left[ \bar{g}(|[v](\bar{x}_i(t))|, s_n(t), \bar{x}_i^n(t)) - \bar{g}(|[v](\bar{x}_i(t))|, s(t), \bar{x}_i(t)) \right] \\ &+ \sum_{i=1}^{N(t)} (1 - \chi_i^n) \left[ \bar{g}(|[v](\bar{x}_i(t))|, s_n(t), \bar{x}_i^n(t)) - \bar{g}(|[v](\bar{x}_i(t))|, s(t), \bar{x}_i(t)) \right] \\ &+ g(|v^+(0) - b_n(t, 0)|) - g(|v^+(0) - b(t, 0)|) + g(|v^-(1) - b_n(t, 1)|) - g(|v^-(1) - b(t, 1)|). \end{aligned}$$

Hence (5.18) follows by taking into account (5.10), continuity of  $g$  and  $\bar{g}$  and that  $b_n(t) \rightarrow b(t)$ .

We are now in position to conclude the proof of (v) for  $t \in D$ . Let  $v \in \text{BV}(0, 1)$  and let  $v_n$  be the sequence constructed before. Then, using the convergence of  $u_n(t) \rightarrow u(t)$  and  $w_n(t) \rightarrow w(t)$  in  $L^2(0, 1)$ ,

$$\begin{aligned} & \Phi(u(t); \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |u(t) - w(t)|^2 dx \\ & \stackrel{(5.16)}{\leq} \liminf_{n \rightarrow \infty} \left[ \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |u_n(t) - w_n(t)|^2 dx \right] \\ & \stackrel{(5.4)}{\leq} \liminf_{n \rightarrow \infty} \left[ \Phi(v_n; \Gamma_n(t), s_n(t), b_n(t)) + \gamma \int_0^1 |v_n - w_n(t)|^2 dx \right] \\ & \stackrel{(5.18)}{\leq} \Phi(v; \Gamma(t), s(t), b(t)) + \gamma \int_0^1 |v - w(t)|^2 dx. \end{aligned}$$

Notice in particular that, by taking  $v = u(t)$ , the previous chain of inequalities yields the convergence of the energies:

$$\Phi(u(t); \Gamma(t), s(t), b(t)) = \lim_{n \rightarrow \infty} \Phi(u_n(t); \Gamma_n(t), s_n(t), b_n(t)). \quad (5.19)$$

*Step 5: definition of the evolution for  $t \notin D$ .* By the monotonicity property (5.9), the uniform bound on the number  $N(t)$  of points in  $\Gamma(t)$ , and the monotonicity of  $t \mapsto s(t)$ , there exists a set  $D' \subset [0, T] \setminus D$ , at most countable, such that

$$\bigcap_{t' \in D, t' \geq t} \Gamma(t') = \bigcup_{t' \in D, t' \leq t} \Gamma(t'), \quad \inf_{t' \in D, t' \geq t} s(t') = \sup_{t' \in D, t' \leq t} s(t') \quad \text{for all } t \in [0, T] \setminus D'. \quad (5.20)$$

We then define  $\Gamma(t)$  and  $s(t)$  to be equal to the common values in (5.20) for all  $t \in [0, T] \setminus D'$ . Moreover, by repeating the construction in Steps 1–4 for the points  $t \in D'$ , and up to a further subsequence, we obtain a triple  $(u(t), \Gamma(t), s(t))$  for all  $t \in D \cup D'$  such that the conclusions of the previous steps hold for every  $t \in D \cup D'$ .

It remains to define  $u(t)$  for  $t \in [0, T] \setminus (D \cup D')$ . For  $t \in [0, T]$ , we introduce the quantity

$$\theta_n(t) := \int_0^1 \left( h'((u_n(t))')(\dot{b}(t))' + 2\gamma(u_n(t) - w_n(t))(\dot{b}(t) - \dot{w}(t)) \right) dx, \quad (5.21)$$

as well as

$$\theta_\infty(t) := \limsup_{n \rightarrow \infty} \theta_n(t). \quad (5.22)$$

Notice that  $|\theta_n(t)| \leq C(\|\dot{b}(t)\|_{W^{1,1}(0,1)} + \|\dot{w}(t)\|_{L^1(0,1)}) \in L^1(0, T)$ , therefore  $\theta_\infty \in L^1(0, T)$  and by Fatou's Lemma

$$\limsup_{n \rightarrow \infty} \int_0^t \theta_n(r) dr \leq \int_0^t \theta_\infty(r) dr \quad \text{for all } t \in [0, T]. \quad (5.23)$$

For a given  $t \in [0, T] \setminus (D \cup D')$ , we can find a subsequence  $(n_j)_j$ , dependent on  $t$ , such that  $\theta_{n_j}(t) \rightarrow \theta_\infty(t)$ , and  $u_{n_j}(t)$  converges weakly\* in  $BV(0, 1)$  to a function  $u(t)$  as  $j \rightarrow \infty$ . We choose this limit function to define the triple  $(u(t), \Gamma(t), s(t))$  for  $t \in [0, T] \setminus (D \cup D')$ . Notice that the arguments in Steps 2–4 can be repeated at the point  $t$  for this subsequence: therefore the evolution  $t \mapsto (u(t), \Gamma(t), s(t))$  satisfies the properties (i)–(v) in the statement for all  $t \in [0, T]$ .

*Step 6: non-dissipativity.* To complete the proof, it only remains to show the condition (vi). Setting

$$\theta(t) := \int_0^1 \left( h'((u(t))')(\dot{b}(t))' + 2\gamma(u(t) - w(t))(\dot{b}(t) - \dot{w}(t)) \right) dx, \quad (5.24)$$

we first claim that

$$\theta(t) = \theta_\infty(t) \quad \text{for almost every } t \in [0, T]. \quad (5.25)$$

In particular, this will give the measurability and integrability of  $\theta(t)$  in  $[0, T]$ . The claim (5.25) can be proved by an argument similar to [22, Lemma 4.11]: fix  $t \in [0, T]$  and consider any sequence of positive numbers  $\delta_i \rightarrow 0$  and any  $d \in \mathbb{R}$ . By the definition of  $u(t)$  there is a sequence  $n_j \rightarrow \infty$  such that  $u_{n_j}(t)(x) + d\delta_i x \rightharpoonup u(t)(x) + d\delta_i x$  weakly\* in  $BV(0, 1)$ ; by arguing as in Step 3 to prove the lower semicontinuity of the energy along the sequence  $u_{n_j}(t)(x) + d\delta_i x$ , and recalling (5.19), we find

$$\begin{aligned} & \frac{1}{\delta_i} \left( \Phi(u(t) + d\delta_i x; \Gamma(t), s(t), b(t) + d\delta_i x) - \Phi(u(t); \Gamma(t), s(t), b(t)) \right) \\ & \leq \liminf_{j \rightarrow \infty} \frac{1}{\delta_i} \left( \Phi(u_{n_j}(t) + d\delta_i x; \Gamma_{n_j}(t), s_{n_j}(t), b_{n_j}(t) + d\delta_i x) - \Phi(u_{n_j}(t); \Gamma_{n_j}(t), s_{n_j}(t), b_{n_j}(t)) \right). \end{aligned}$$

By writing the explicit expressions of the previous quantities, we find

$$\frac{1}{\delta_i} \int_0^1 \left( h(|u'(t) + \delta_i d|) - h(|u'(t)|) \right) dx \leq \liminf_{j \rightarrow \infty} \frac{1}{\delta_i} \int_0^1 \left( h(|u'_{n_j}(t) + \delta_i d|) - h(|u'_{n_j}(t)|) \right) dx,$$

therefore there exists an increasing sequence of integers  $j_i \geq i$  such that

$$\frac{1}{\delta_i} \int_0^1 \left( h(u'(t) + \delta_i d) - h(u'(t)) \right) dx \leq \frac{1}{\delta_i} \int_0^1 \left( h(u'_{n_{j_i}}(t) + \delta_i d) - h(u'_{n_{j_i}}(t)) \right) dx + \frac{1}{i}.$$

Taking the limit  $i \rightarrow \infty$  we obtain proceeding as in [22, Lemma 4.11],

$$\begin{aligned} d \int_0^1 h'(u'(t)) \, dx &= \liminf_{i \rightarrow \infty} \int_0^1 \frac{h(u'(t) + \delta_i d) - h(u'(t))}{\delta_i} \, dx \\ &\leq \liminf_{i \rightarrow \infty} \int_0^1 \frac{h(u'_{n_{j_i}}(t) + \delta_i d) - h(u'_{n_{j_i}}(t))}{\delta_i} \, dx \\ &= \liminf_{i \rightarrow \infty} d \int_0^1 h'(u'_{n_{j_i}}(t) + \tau_i d) \, dx = \liminf_{i \rightarrow \infty} d \int_0^1 h'(u'_{n_{j_i}}(t)) \, dx \end{aligned}$$

for suitable  $\tau_i : (0, 1) \rightarrow [0, \delta_i]$ . Taking  $d = 1$  and  $d = -1$  we see that this is actually an equality and that the limit does not depend on the subsequence  $j_i$ , which implies

$$\lim_{j \rightarrow \infty} \int_0^1 h'((u_{n_j}(t))') \, dx = \int_0^1 h'((u(t))') \, dx.$$

By the strong convergence of  $u_{n_j}(t) \rightarrow u(t)$  and  $w_{n_j}(t) \rightarrow w(t)$  in  $L^2(0, 1)$ , we then conclude that  $\theta_{n_j}(t) \rightarrow \theta(t)$  as  $j \rightarrow \infty$  and, since  $\theta_\infty$  was the limit of the subsequence  $\theta_{n_j}$ , this shows (5.25).

By Lemma 4.3 we have

$$\begin{aligned} \mathcal{E}_{n_j}(t) &\leq \mathcal{E}(0) + \int_0^{t_k^{n_j}} \theta_{n_j}(r) \, dr + R_{n_j}(t) + \int_0^{t_k^{n_j}} \int_0^1 h'((u_{n_j}(r))') (\dot{b}^{n_j}(r) - \dot{b}(r))' \, dx \, dr \\ &\quad + 2\gamma \int_0^{t_k^{n_j}} \int_0^1 (u_{n_j}(r) - w_{n_j}(r)) (\dot{b}^{n_j}(r) - \dot{b}(r) - \dot{w}^{n_j}(r) + \dot{w}(r)) \, dx \, dr, \end{aligned} \quad (5.26)$$

where  $t_k^{n_j}$  is the discretization point such that  $t \in [t_k^{n_j}, t_{k+1}^{n_j})$ . One can now check that the last two terms in the previous expression are actually equal to zero, and that  $R_{n_j}(t) \rightarrow 0$  as  $j \rightarrow \infty$ , thanks to the assumptions on  $b$  and  $w$ . Therefore, recalling (5.23), (5.25), and that  $\mathcal{E}_{n_j}(t) \rightarrow \mathcal{E}(t)$  by (5.19), we conclude that

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \int_0^t \theta(r) \, dr. \quad (5.27)$$

To conclude the proof it remains to show the opposite inequality. Let us stress that the subsequence  $n_j$  depends on  $t$ , so that we do not have a unique subsequence converging pointwise almost everywhere in  $[0, T]$  to  $\theta(t)$ . This prevents to take directly the lower limit in the opposite of inequality (5.26), to get the opposite of inequality (5.27). In order to overcome this difficulty, we will first approximate the Lebesgue integral of  $\theta$  by Riemann sums.

We fix  $t \in (0, T]$ . We first observe that there exists a sequence of subdivisions of  $[0, t]$  of the form

$$0 = \rho_0^m < \rho_1^m < \dots < \rho_{i_m-1}^m < \rho_{i_m}^m = t, \quad \text{with} \quad \lim_{m \rightarrow \infty} \max_{i=1, \dots, i_m} |\rho_i^m - \rho_{i-1}^m| = 0,$$

with the property that

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{i_m} \left| (\rho_i^m - \rho_{i-1}^m) \theta(\rho_i^m) - \int_{\rho_{i-1}^m}^{\rho_i^m} \theta(r) \, dr \right| = 0, \quad (5.28)$$

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{i_m} \int_0^1 \left| (\rho_i^m - \rho_{i-1}^m) \dot{b}(\rho_i^m) - \int_{\rho_{i-1}^m}^{\rho_i^m} \dot{b}(r) \, dr \right| \, dx = 0, \quad (5.29)$$

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{i_m} \int_0^1 \left| (\rho_i^m - \rho_{i-1}^m) \dot{w}(\rho_i^m) - \int_{\rho_{i-1}^m}^{\rho_i^m} \dot{w}(r) \, dr \right| \, dx = 0 \quad (5.30)$$

(see [22, Lemma 4.12], and also [21, Lemma 4.12] for a proof, adapting the arguments of [30, page 63]). We now exploit the global stability of  $u(\rho_{i-1}^m)$  (property (v)), taking the competitor  $u(\rho_i^m) + b(\rho_{i-1}^m) - b(\rho_i^m)$  and adopting an iteration argument similar to that in the proof of Lemma 4.3. We repeatedly use that  $b \in H^1([0, T], \mathbb{R}^2)$ , with  $b(t) = (b(t, 0), b(t, 1))$ , and that  $b(t) : [0, 1] \rightarrow \mathbb{R}$  denotes the affine function interpolating between the boundary values  $b(t, 0)$  and  $b(t, 1)$ . Then, using also the monotonicity property (ii), we find

$$\begin{aligned} \mathcal{E}(t) &\geq \mathcal{E}(0) + \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 h'((u(\rho_i^m))')(\dot{b}(r))' dx dr \\ &\quad + 2\gamma \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 (w(\rho_i^m) - u(\rho_i^m))(\dot{w}(r) - \dot{b}(r)) dx dr - \mathcal{S}_m(t), \end{aligned} \quad (5.31)$$

where

$$\begin{aligned} \mathcal{S}_m(t) &:= \gamma \sum_{i=1}^{i_m} \int_0^1 (w(\rho_i^m) - w(\rho_{i-1}^m))^2 dx - 2\gamma \sum_{i=1}^{i_m} \int_0^1 (w(\rho_i^m) - w(\rho_{i-1}^m))(b(\rho_i^m) - b(\rho_{i-1}^m)) \\ &\quad + \sum_{i=1}^{i_m} \int_0^1 |b'(\rho_i^m) - b'(\rho_{i-1}^m)|^2 dx + \gamma \sum_{i=1}^{i_m} \int_0^1 |b(\rho_i^m) - b(\rho_{i-1}^m)|^2 dx. \end{aligned}$$

Notice that, in view of the assumptions on  $w$  and  $b$ , we have  $\mathcal{S}_m(t) \rightarrow 0$  as  $m \rightarrow \infty$ . Recalling the definition (5.24) of the map  $\theta(t)$ , we further obtain from (5.31)

$$\mathcal{E}(t) \geq \mathcal{E}(0) + \sum_{i=1}^{i_m} (\rho_i^m - \rho_{i-1}^m) \theta(\rho_i^m) - \mathcal{R}_m(t) - \mathcal{S}_m(t), \quad (5.32)$$

with the position

$$\begin{aligned} \mathcal{R}_m(t) &:= \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 h'((u(\rho_i^m))')(\dot{b}(\rho_i^m) - \dot{b}(r))' dx dr \\ &\quad + 2\gamma \sum_{i=1}^{i_m} \int_{\rho_{i-1}^m}^{\rho_i^m} \int_0^1 (u(\rho_i^m) - w(\rho_i^m))(\dot{b}(\rho_i^m) - \dot{b}(r) - \dot{w}(\rho_i^m) + \dot{w}(r)) dx dr. \end{aligned}$$

Using the definition of  $h$  and the uniform bound in  $L^\infty$  on  $u$  and  $w$  we can estimate

$$\begin{aligned} |\mathcal{R}_m(t)| &\leq \ell \sum_{i=1}^{i_m} \left| (\rho_i^m - \rho_{i-1}^m) (\dot{b}(\rho_i^m))' - \int_{\rho_{i-1}^m}^{\rho_i^m} (\dot{b}(r))' dr \right| \\ &\quad + C \sum_{i=1}^{i_m} \int_0^1 \left| (\rho_i^m - \rho_{i-1}^m) (\dot{b}(\rho_i^m) - \dot{w}(\rho_i^m)) - \int_{\rho_{i-1}^m}^{\rho_i^m} (\dot{b}(r) - \dot{w}(r)) dr \right| dx, \end{aligned}$$

and the previous quantity vanishes in the limit as  $m \rightarrow \infty$  in view of (5.29)–(5.30). Eventually, by passing to the limit as  $m \rightarrow \infty$  in (5.32), recalling (5.28) and that  $\mathcal{R}_m(t) \rightarrow 0$ ,  $\mathcal{S}_m(t) \rightarrow 0$ , we conclude that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + \int_0^t \theta(r) dr,$$

which completes the proof of (vi).  $\square$

**Remark 5.3.** Notice that in the proof of Theorem 5.1 we do not obtain a unique subsequence  $(n_j)_j$ , independent of  $t$ , such that the time-discrete evolution  $(u_{n_j}(t), \Gamma_{n_j}(t), s_{n_j}(t))$  converges to the limit evolution  $(u(t), \Gamma(t), s(t))$  for all  $t \in [0, T]$ . This in general holds only for  $t$  in a countable dense set. It would be possible to prove convergence of a subsequence for every  $t \in [0, T]$  if we knew that the minimum problem solved by  $u(t)$  has a unique solution.



## 6 $\Gamma$ -convergence of the constrained problems

In this section we show that the functional (4.4), which contains additional internal variables describing the current state of the fracture, can be approximated by regular functionals, in the spirit of [19]. This amounts to introduce an irreversibility constraint at level  $\varepsilon$  on the functionals  $\mathcal{F}_\varepsilon$ , defined in (2.1), in the form of a monotonicity condition on the *minimum values* of the damage variable  $v_\varepsilon$ .

We assume in the following that a pair  $(\Gamma, s)$  as in (4.1) is given:  $\Gamma$  is a finite subset of  $[0, 1]$ , and  $s : \Gamma \rightarrow (0, \infty)$ . We also fix a boundary datum  $b : \{0, 1\} \rightarrow \mathbb{R}$ . In order to deal with the boundary conditions, it is convenient to work in a larger open interval  $\Omega$  containing  $[0, 1]$ , for instance  $\Omega := (-1, 2)$ . For  $\varepsilon > 0$  let  $L_\varepsilon > 0$  be such that  $L_\varepsilon \rightarrow 0$  and  $\frac{L_\varepsilon}{\varepsilon} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . We introduce a constrained functional, defined on  $L^1(\Omega) \times L^1(\Omega)$ , by setting

$$\hat{\mathcal{F}}_\varepsilon(u, v; \Gamma, s, b) := \begin{cases} \mathcal{F}_\varepsilon(u, v; \Omega) & \text{if } v(x) \leq m_{s(x)} \text{ for every } x \in \Gamma, \\ & u(x) = b(0) \text{ for } x < -L_\varepsilon, u(x) = b(1) \text{ for } x > 1 + L_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \quad (6.1)$$

Here  $m_s$  denotes the minimum value of an optimal profile  $\beta_s$  for  $g(s)$ , see Theorem 3.4 and in particular (3.16). Recalling the definition (4.4) of the relaxed functional  $\Phi(\cdot; \Gamma, s, b)$ , the limit functional is defined on  $L^1(\Omega) \times L^1(\Omega)$  by

$$\hat{\mathcal{F}}(u, v; \Gamma, s, b) := \begin{cases} \Phi(u; \Gamma, s, b) & \text{if } u \in \text{BV}(\Omega), v = 1 \text{ a.e.,} \\ & u(x) = b(0) \text{ for } x < 0, u(x) = b(1) \text{ for } x > 1, \\ \infty & \text{otherwise.} \end{cases} \quad (6.2)$$

The main result of this section is the following.

**Theorem 6.1.** *Let a finite set  $\Gamma \subset [0, 1]$ , a map  $s : \Gamma \rightarrow (0, \infty)$ , and a boundary Dirichlet datum  $b : \{0, 1\} \rightarrow \mathbb{R}$  be given. Then the functionals  $\hat{\mathcal{F}}_\varepsilon(\cdot; \Gamma, s, b)$   $\Gamma$ -converge as  $\varepsilon \rightarrow 0^+$  to  $\hat{\mathcal{F}}(\cdot; \Gamma, s, b)$  in  $L^1(\Omega) \times L^1(\Omega)$ .*

In the rest of this section we drop the dependence on  $(\Gamma, s, b)$  in the functionals  $\hat{\mathcal{F}}_\varepsilon$  and  $\hat{\mathcal{F}}$  to lighten the notation, as these quantities are fixed. In order to prove the theorem we introduce the following standard notions:

$$\begin{aligned} \hat{\mathcal{F}}'(u, v) &:= \Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u, v) \\ &:= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \text{ in } L^1(\Omega) \times L^1(\Omega) \right\}, \\ \hat{\mathcal{F}}''(u, v) &:= \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u, v) \\ &:= \inf \left\{ \limsup_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \text{ in } L^1(\Omega) \times L^1(\Omega) \right\}. \end{aligned}$$

The proof of Theorem 6.1 follows by combining Proposition 6.2 and Proposition 6.3 below.

**Proposition 6.2** (Liminf inequality). *For every  $(u, v) \in L^1(\Omega) \times L^1(\Omega)$  it holds*

$$\hat{\mathcal{F}}(u, v) \leq \hat{\mathcal{F}}'(u, v). \quad (6.3)$$

*Proof.* The proof follows the lines of [19, Proposition 5.1], with the natural modifications required to include the additional constraint. We denote  $\Gamma = \{\bar{x}_1, \dots, \bar{x}_k\}$ , and  $\bar{s}_i = s(\bar{x}_i)$ . Fix any sequence  $(u_\varepsilon, v_\varepsilon) \rightarrow (u, v)$  in  $L^1(\Omega) \times L^1(\Omega)$  with  $\sup_\varepsilon \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$ . The proof will be achieved by showing that  $u \in \text{BV}(\Omega)$ ,  $v = 1$  almost everywhere, and

$$\hat{\mathcal{F}}(u, 1) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega). \quad (6.4)$$

By possibly passing to a subsequence, we can assume without loss of generality that the  $\liminf$  in (6.4) is in fact a limit, and that the convergence of  $u_\varepsilon$  and  $v_\varepsilon$  is also pointwise almost everywhere. The uniform bound on the energy of  $(u_\varepsilon, v_\varepsilon)$  gives  $v = 1$  almost everywhere, and  $v_\varepsilon(\bar{x}_i) \leq m_{\bar{s}_i} < 1$ . Moreover,  $u(x) = b(0)$  for  $x < 0$ ,  $u(x) = b(1)$  for  $x > 1$ .

By repeating the construction in the first part of the proof of [19, Proposition 5.1], given any  $\delta > 0$  one can determine a finite number of points  $S = \{t_1, \dots, t_L\} \subset \Omega$ , with  $\Gamma \subset S$ , with the following property: for all  $\eta > 0$  sufficiently small,  $\eta \ll \delta$  and  $\eta < 1 - m_{\bar{s}_i}$  for all  $i$ , setting

$$S_\eta := \bigcup_{i=1}^L (t_i - \eta, t_i + \eta),$$

one has for all  $\varepsilon$  sufficiently small (depending on  $\eta$ )

$$(1 - \omega(\delta)) \int_{\Omega \setminus S_\eta} h(|u'_\varepsilon|) dx \leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega \setminus S_\eta) \quad (6.5)$$

where  $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  is a modulus of continuity.

The uniform bound on the energies  $\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon)$ , together with (6.5), yields

$$\sup_\eta \sup_\varepsilon \int_{\Omega \setminus S_\eta} |u'_\varepsilon| dx < \infty,$$

and therefore  $u \in BV(\Omega \setminus S_\eta)$  for all  $\eta > 0$ ; in turn we have  $u \in BV(\Omega)$  by the finiteness of  $S$ , and by lower semicontinuity we obtain

$$(1 - \omega(\delta)) \hat{\mathcal{F}}(u, 1; \Omega \setminus S_\eta) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega \setminus S_\eta). \quad (6.6)$$

We now estimate the contribution to the energy coming from the region  $S_\eta$ . We first consider the points  $\bar{x}_i \in \Gamma$ ,  $i = 1, \dots, k$ , which in particular belong to  $S$  (recall that  $\Gamma \subset S$ ). Let  $J_\eta^i = (\bar{x}_i - \eta, \bar{x}_i + \eta)$ . We introduce another small parameter  $\mu > 0$  and we choose  $x_1, x_2 \in J_\eta^i$ ,  $x_1 < \bar{x}_i < x_2$ , with the following properties:

$$v_\varepsilon(x_1) \rightarrow 1, \quad v_\varepsilon(x_2) \rightarrow 1, \quad (6.7)$$

$$u_\varepsilon(x_1) \rightarrow u(x_1), \quad u_\varepsilon(x_2) \rightarrow u(x_2), \quad (6.8)$$

$$|u(x_1) - u^-(\bar{x}_i)| < \mu, \quad |u(x_2) - u^+(\bar{x}_i)| < \mu. \quad (6.9)$$

We define  $I := (x_1, x_2)$  and denote by  $C_\varepsilon$  the connected component of the set

$$\{x \in I : v_\varepsilon(x) < 1 - \eta\} \quad (6.10)$$

containing  $\bar{x}_i$ , and by  $C_\varepsilon^j$  the connected components (different from  $C_\varepsilon$ ) of the same set (6.10) in which  $v_\varepsilon$  achieves the value  $1 - \delta$  (recall that  $\eta \ll \delta$ ,  $\eta < 1 - m_{\bar{s}_i}$  and (6.7)). In each of such component  $C_\varepsilon^j$  we have, denoting by  $y$  its first endpoint and by  $z$  an interior point in which  $v_\varepsilon(z) \leq 1 - \delta$ ,

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; C_\varepsilon^j) &\geq \int_y^z \left( \frac{(1 - v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) dx \geq \int_y^z (1 - v_\varepsilon) |v'_\varepsilon| dx \\ &\geq \frac{1}{2} (1 - v_\varepsilon(z))^2 - \frac{1}{2} (1 - v_\varepsilon(y))^2 = \frac{\delta^2 - \eta^2}{2} \geq \frac{1}{4} \delta^2. \end{aligned}$$

Therefore the number  $N_\varepsilon$  of the components  $C_\varepsilon^j$  is uniformly bounded by  $\frac{c}{\delta^2}$ , where  $c$  is a constant independent of  $\varepsilon$ . Moreover in each set  $C_\varepsilon^j$  and in  $C_\varepsilon$  we have  $f_\varepsilon(v_\varepsilon) = \sqrt{\varepsilon} f(v_\varepsilon)$  for  $\varepsilon$  small, as  $v_\varepsilon < 1 - \eta$ . Recalling that  $\bar{x}_i \in C_\varepsilon$  and  $v_\varepsilon(\bar{x}_i) \leq m_{\bar{s}_i}$ , we can now compute using the rescaling  $\alpha_\varepsilon(t) := u_\varepsilon(\bar{x}_i + \varepsilon t)$ ,  $\beta_\varepsilon(t) := v_\varepsilon(\bar{x}_i + \varepsilon t)$

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; C_\varepsilon) &= \int_{\frac{1}{\varepsilon}(C_\varepsilon - \bar{x}_i)} \left( f^2(\beta_\varepsilon) |\alpha'_\varepsilon|^2 + \frac{(1 - \beta_\varepsilon)^2}{4} + |\beta'_\varepsilon|^2 \right) dt \\ &\geq \bar{g}^{(n)} \left( \left| \int_{C_\varepsilon} u'_\varepsilon dx \right|, \bar{s}_i \right) \geq \bar{g} \left( \left| \int_{C_\varepsilon} u'_\varepsilon dx \right|, \bar{s}_i \right) - 3\eta^2, \end{aligned} \quad (6.11)$$

where  $\bar{g}^{(\eta)}$  is the function defined in (3.45) and the last inequality follows by Lemma 3.11. On the other components  $C_\varepsilon^j$  the same argument gives

$$\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; C_\varepsilon^j) \geq \bar{g}\left(\left|\int_{C_\varepsilon^j} u'_\varepsilon dx\right|, 0\right) - 3\eta^2 = g\left(\left|\int_{C_\varepsilon^j} u'_\varepsilon dx\right|\right) - 3\eta^2. \quad (6.12)$$

Finally, outside the selected components, that is in the set  $\tilde{C}_\varepsilon := \Gamma \setminus (C_\varepsilon \cup \bigcup_{j=1}^{N_\varepsilon} C_\varepsilon^j)$ , one has  $v_\varepsilon \geq 1 - \delta$  and therefore an estimate analogous to (6.5) holds:

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \tilde{C}_\varepsilon) &\geq (1 - \omega(\delta)) \int_{\tilde{C}_\varepsilon} h(|u'_\varepsilon|) dx \geq (1 - \omega(\delta)) \left[ \ell \int_{\tilde{C}_\varepsilon} |u'_\varepsilon| dx - \frac{\ell^2}{4} |\tilde{C}_\varepsilon| \right] \\ &\geq (1 - \omega(\delta)) g\left(\left|\int_{\tilde{C}_\varepsilon} u'_\varepsilon dx\right|\right) - \frac{\ell^2}{2} \eta. \end{aligned} \quad (6.13)$$

where we used the definition (2.9) of  $h$  and Proposition 3.1(ii). By collecting (6.11)–(6.13)

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; I) &\geq \bar{g}\left(\left|\int_{C_\varepsilon} u'_\varepsilon dx\right|, \bar{s}_i\right) - 3\eta^2 + \sum_{j=1}^{N_\varepsilon} g\left(\left|\int_{C_\varepsilon^j} u'_\varepsilon dx\right|\right) - 3N_\varepsilon \eta^2 \\ &\quad + (1 - \omega(\delta)) g\left(\left|\int_{\tilde{C}_\varepsilon} u'_\varepsilon dx\right|\right) - \frac{\ell^2}{2} \eta \\ &\geq \bar{g}\left(\left|\int_{C_\varepsilon} u'_\varepsilon dx\right|, \bar{s}_i\right) + (1 - \omega(\delta)) g\left(\left|\int_{\Gamma \setminus C_\varepsilon} u'_\varepsilon dx\right|\right) - \left(3 + \frac{c}{\delta^2}\right) \eta^2 - \frac{\ell^2}{2} \eta \\ &\geq (1 - \omega(\delta)) \bar{g}(|u_\varepsilon(x_2) - u_\varepsilon(x_1)|, \bar{s}_i) - \frac{c}{\delta^2} \eta^2 - \frac{\ell^2}{2} \eta, \end{aligned}$$

where we used the subadditivity of  $g$  in the second passage, and (3.38) in the third one. Hence (6.8) and continuity of  $\bar{g}$  yield

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; J_\eta^i) \geq (1 - \omega(\delta)) \bar{g}(|u(x_2) - u(x_1)|, \bar{s}_i) - \frac{c}{\delta^2} \eta^2 - \frac{\ell^2}{2} \eta,$$

and finally letting  $\mu \rightarrow 0$

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; J_\eta^i) \geq (1 - \omega(\delta)) \bar{g}(|[u]|(\bar{x}_i), \bar{s}_i) - \frac{c}{\delta^2} \eta^2 - \frac{\ell^2}{2} \eta. \quad (6.14)$$

The inequality (6.14) gives an estimate of the contribution to the energy coming from the points in  $\Gamma$ . For the points  $t_i \in S \setminus \Gamma$ , we can reproduce the argument above just removing the component  $C_\varepsilon$  (in this case the argument is the same as in the proof of [19, Proposition 5.1]) and obtain that, for  $I_\eta^i = (t_i - \eta, t_i + \eta)$ ,

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; I_\eta^i) \geq (1 - \omega(\delta)) g(|[u]|(\bar{x}_i)) + O(\eta). \quad (6.15)$$

Eventually we collect (6.6), (6.14), and (6.15) and we let  $\eta \rightarrow 0$ :

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) \geq (1 - \omega(\delta)) \hat{\mathcal{F}}(u, 1).$$

The conclusion (6.4) follows by letting  $\delta \rightarrow 0$ .  $\square$

**Proposition 6.3** (Limsup inequality). *For every  $(u, v) \in L^1(\Omega) \times L^1(\Omega)$  it holds*

$$\hat{\mathcal{F}}''(u, v) \leq \hat{\mathcal{F}}(u, v). \quad (6.16)$$

*Proof.* Let us consider first the case  $v \equiv 1$  and  $u \in \text{SBV}(0, 1)$ , with  $u' \in L^2(0, 1)$  and  $\mathcal{H}^0(J_u) < \infty$ , and recall that by assumption also  $\mathcal{H}^0(\Gamma) < \infty$ . By a localization argument we can further assume that  $\Gamma$  consists of a single point  $x_0 \in [0, 1]$ , and that  $u$  has at most one jump point, also located at  $x_0$ . Indeed, in any interval which does not contain any point of  $\Gamma$  the conclusion follows directly by Theorem 2.1.

Let therefore  $\Gamma = \{x_0\}$ ,  $x_0 \in [0, 1]$ , and let  $s_0 := s(x_0)$ . Let us also assume for the moment that  $u$  only takes the two values  $u^\pm(x_0)$  in a neighbourhood of  $x_0$ . Given any  $\sigma > 0$ , using the characterization (3.35) of  $\bar{g}$  we can find  $T > 0$  and  $\alpha, \beta \in H^1(-T, T)$  such that  $\alpha(-T) = u^-(x_0)$ ,  $\alpha(T) = u^+(x_0)$ ,  $0 \leq \beta \leq 1$ ,  $\beta(\pm T) = 1$ ,  $\beta(0) \leq m_{s_0}$ , and

$$\int_{-T}^T \left( f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt \leq \bar{g}(|[u](x_0)|, s_0) + \sigma.$$

We then take as recovery sequence

$$u_\varepsilon(x) := \begin{cases} \alpha\left(\frac{x-x_0}{\varepsilon}\right) & \text{if } x \in A_\varepsilon, \\ u(x) & \text{if } x \in \Omega \setminus A_\varepsilon, \end{cases} \quad v_\varepsilon(x) := \begin{cases} \beta\left(\frac{x-x_0}{\varepsilon}\right) & \text{if } x \in A_\varepsilon, \\ 1 & \text{if } x \in \Omega \setminus A_\varepsilon, \end{cases}$$

where  $A_\varepsilon := (x_0 - \varepsilon T, x_0 + \varepsilon T)$ . Notice that, since  $\frac{L_\varepsilon}{\varepsilon} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , we have  $A_\varepsilon \subset (x_0 - L_\varepsilon, x_0 + L_\varepsilon)$  for  $\varepsilon$  small enough, and therefore  $u_\varepsilon$  satisfies the boundary conditions as in (6.1) in the case  $x_0$  is a boundary point. It is easily seen that  $u_\varepsilon \rightarrow u$  and  $v_\varepsilon \rightarrow 1$  in  $L^1(0, 1)$ . Moreover, as  $v_\varepsilon(x_0) = \beta(0) \leq m_{s_0}$  we have

$$\begin{aligned} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) &= \int_{\Omega \setminus A_\varepsilon} |u'|^2 dx + \int_{A_\varepsilon} \left( f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) dx \\ &\leq \int_0^1 |u'|^2 dx + \int_{-T}^T \left( f^2(\beta) |\alpha'|^2 + \frac{(1-\beta)^2}{4} + |\beta'|^2 \right) dt \\ &\leq \int_0^1 |u'|^2 dx + \bar{g}(|[u](x_0)|, s_0) + \sigma. \end{aligned}$$

Since  $\sigma$  is arbitrary we obtain

$$\hat{\mathcal{F}}''(u, 1) \leq \limsup_{\varepsilon \rightarrow 0} \hat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \int_0^1 |u'|^2 dx + \bar{g}(|[u](x_0)|, s_0). \quad (6.17)$$

In order to remove the assumption that  $u$  is locally piecewise constant in a neighbourhood of  $x_0$ , we consider the sequence

$$u_j(x) := \begin{cases} u(x_0 - \frac{1}{j}) & \text{for } x_0 - \frac{1}{j} < x < x_0, \\ u(x_0 + \frac{1}{j}) & \text{for } x_0 < x < x_0 + \frac{1}{j}, \\ u(x) & \text{for } x \in \Omega \setminus (x_0 - \frac{1}{j}, x_0 + \frac{1}{j}). \end{cases}$$

In view of the previous discussion inequality (6.17) holds with  $u$  replaced by  $u_j$ ; since  $u_j \rightarrow u$  in  $L^1(0, 1)$  and  $u'_j \rightarrow u'$  in  $L^2(0, 1)$  as  $j \rightarrow \infty$ , by lower semicontinuity of  $\hat{\mathcal{F}}''$  we conclude that (6.17) is still satisfied by  $u$ .

The inequality (6.17), together with Theorem 2.1 and a localization argument, proves that for every  $u \in \text{SBV}(0, 1)$  with  $u' \in L^2(0, 1)$  and  $\mathcal{H}^0(J_u) < \infty$  we have

$$\hat{\mathcal{F}}''(u, 1) \leq \int_0^1 |u'|^2 dx + \sum_{x \in \Gamma} \bar{g}(|[u](x)|, s(x)) + \sum_{x \in J^b(u) \setminus \Gamma} g(|[u](x)|).$$

Then the conclusion (6.16) follows since  $\hat{\mathcal{F}}$  is the lower semicontinuous envelope of the right-hand side, by Theorem A.1 and Remark A.2.  $\square$

**Remark 6.4.** From Theorem 6.1, in the particular case  $\Gamma = \emptyset$ , it follows that the  $\Gamma$ -convergence result of [19] (see Theorem 2.1) continues to hold if we include Dirichlet boundary conditions.

## 7 Detailed behavior of recovery sequences around jumps

The goal of this section is to explain and justify the choice of the irreversibility condition  $v(x) \leq m_{\bar{s}(x)}$  in (6.1). We will refine the  $\Gamma$ -convergence result proved in [19] (see Theorem 2.1), providing new information about the behaviour of recovery sequences  $(u_\varepsilon, v_\varepsilon) \rightarrow (u, 1)$ . We show in particular that the blow-ups of the functions  $(u_\varepsilon, v_\varepsilon)$  around a jump point  $\bar{x}$  of the limit  $u$  converge to the optimal profiles for  $g(|[u](\bar{x})|)$ , that is to an optimal pair for the problem (3.3). Motivated by Proposition 4.2, we will restrict to the case  $u \in SBV(0, 1)$ , with  $|u'| \leq \frac{\ell}{2}$ , and for simplicity we will further assume  $\mathcal{H}^0(J_u) < +\infty$ .

**Theorem 7.1** (Behaviour of recovery sequences). *Under the assumptions of Section 2, let  $u \in SBV(0, 1)$  have a finite number of jumps and satisfy  $|u'| \leq \frac{\ell}{2}$  almost everywhere in  $(0, 1)$ . Let  $(u_\varepsilon, v_\varepsilon)$  be a recovery sequence for  $u$ , that is  $(u_\varepsilon, v_\varepsilon) \in H^1(0, 1) \times H^1(0, 1)$ ,  $0 \leq v_\varepsilon \leq 1$  almost everywhere, and*

$$u_\varepsilon \rightarrow u \text{ in } L^1(0, 1), \quad v_\varepsilon \rightarrow 1 \text{ in } L^1(0, 1), \quad \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow \mathcal{F}(u, 1) \quad (7.1)$$

as  $\varepsilon \rightarrow 0^+$ . Suppose in addition that  $\sup_\varepsilon \|u_\varepsilon\|_\infty < \infty$ . Then there exists a subsequence  $\varepsilon_k \rightarrow 0$  with the following properties:

(i) For every jump point  $\bar{x} \in J_u$  there exist  $x_k \rightarrow \bar{x}$  such that

$$w_k(x) := v_{\varepsilon_k}(x_k + \varepsilon_k x) \rightarrow \beta_{\bar{s}}(x) \quad \text{strongly in } H_{\text{loc}}^1(\mathbb{R}), \quad (7.2)$$

where  $\bar{s} := |[u](\bar{x})|$  and  $\beta_{\bar{s}}$  is an optimal profile for  $g(\bar{s})$ , according to Theorem 3.4. Moreover, in the case  $g(|[u](\bar{x})|) < 1$  we also have that

$$z_k(x) := u_{\varepsilon_k}(x_k + \varepsilon_k x) \rightarrow \bar{s}\alpha_{\bar{s}}(x) \quad \text{strongly in } H_{\text{loc}}^1(\mathbb{R}), \quad (7.3)$$

where  $\alpha_{\bar{s}}$  is such that  $(\alpha_{\bar{s}}, \beta_{\bar{s}}) \in \mathcal{U}_1$  and  $g(\bar{s}) = \mathcal{G}_{\bar{s}}(\alpha_{\bar{s}}, \beta_{\bar{s}})$ .

(ii) For every  $\eta > 0$ , setting  $I_\eta := \bigcup_{x \in J_u} (x - \eta, x + \eta)$  we have that as  $\varepsilon \rightarrow 0$

$$\int_{(0,1) \setminus I_\eta} f_\varepsilon^2(v_\varepsilon) |u'_\varepsilon|^2 dx \rightarrow \int_{(0,1) \setminus I_\eta} |u'|^2 dx, \quad \int_{(0,1) \setminus I_\eta} \left( \frac{(1-v_\varepsilon)^2}{4\varepsilon_k} + \varepsilon |v'_\varepsilon|^2 \right) dx \rightarrow 0. \quad (7.4)$$

The rest of this section is devoted to the proof of this result, which will be achieved through a sequence of lemmas. In the following we assume that  $u$ ,  $u_\varepsilon$  and  $v_\varepsilon$  satisfy the assumption of Theorem 7.1. We start by (i). Since this assertion is local, we can assume without loss of generality that the jump set of  $u$  consists of a single point  $\bar{x} \in (0, 1)$ ,  $J_u = \{\bar{x}\}$ . We let

$$I_\eta := (\bar{x} - \eta, \bar{x} + \eta),$$

where  $\eta > 0$  is always supposed to be small enough so that  $I_\eta \subset (0, 1)$ . Notice that, since the endpoints of  $I_\eta$  are not in the jump set of  $u$ , one has the localized convergence

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; I_\eta) = \mathcal{F}(u, 1; I_\eta).$$

In the following, we will work with a fixed subsequence  $\varepsilon_k \rightarrow 0^+$ ,  $k \rightarrow \infty$ ; to lighten the notation we will replace the subscript  $\varepsilon_k$  by  $k$  (thus, for instance, we set  $u_k := u_{\varepsilon_k}$ ,  $v_k := v_{\varepsilon_k}, \dots$ ). We further assume that for this subsequence  $u_k \rightarrow u$  and  $v_k \rightarrow 1$  almost everywhere in  $(0, 1)$ , and choose a continuous representative for  $u_k$  and  $v_k$ .

**Lemma 7.2.** *For  $\mathcal{L}^1$ -a.e.  $\eta$  sufficiently small we have  $u_k(\bar{x} \pm \eta) \rightarrow u(\bar{x} \pm \eta)$ ,  $v_k(\bar{x} \pm \eta) \rightarrow 1$ , and, if  $x_k \in \bar{I}_\eta$  is a minimum point of  $v_k$  in the interval  $I_\eta$ :*

$$m_k := \inf_{x \in I_\eta} v_k(x) = v_k(x_k), \quad (7.5)$$

then the following properties hold:

- (i)  $f_k(v_k(x_k)) = f_k(m_k) \rightarrow 0$  as  $k \rightarrow \infty$ ;
- (ii)  $\limsup_{k \rightarrow \infty} m_k < 1$ ;
- (iii)  $x_k \rightarrow \bar{x}$  as  $k \rightarrow \infty$ ;
- (iv) if  $g(|[u](\bar{x})|) < 1$ , then  $\liminf_{k \rightarrow \infty} m_k > 0$ .

*Proof.* Since we are assuming that  $u_k \rightarrow u$  and  $v_k \rightarrow 1$  almost everywhere in  $(0, 1)$ , the convergence at the endpoints of the interval  $I_\eta$  is automatically satisfied for almost every  $\eta$ .

(i): If for a subsequence we had  $f_k(m_k) \geq \sigma > 0$ , then as the energies  $\mathcal{F}_k(u_k, v_k)$  are equi-bounded we would get

$$C \geq \int_{I_\eta} f_k^2(v_k) |u'_k|^2 dx \geq \sigma^2 \int_{I_\eta} |u'_k|^2 dx.$$

Therefore  $u \in H^1(I_\eta)$ , which is a contradiction since  $\bar{x} \in J_u \cap I_\eta$ .

(ii): Suppose by contradiction that for a subsequence  $m_k \rightarrow 1$ . For a given  $\delta > 0$ , we have  $v_k(x) \geq 1 - \delta$  in  $I_\eta$  for  $k$  sufficiently large; by assumption (2.5) this implies that

$$|(1 - v_k)f(v_k) - \ell| \leq \ell\omega(\delta)$$

for some modulus of continuity  $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Therefore for  $k$  large enough

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; I_\eta) &\geq \int_{I_\eta} \left( f_k^2(v_k) |u'_k|^2 + \frac{(1 - v_k)^2}{4\varepsilon_k} \right) dx \geq \int_{I_\eta} |u'_k|^2 \wedge \left( \varepsilon_k f^2(v_k) |u'_k|^2 + \frac{(1 - v_k)^2}{4\varepsilon_k} \right) dx \\ &\geq \int_{I_\eta} \left( |u'_k|^2 \wedge (1 - v_k)f(v_k) |u'_k| \right) dx \geq (1 - \omega(\delta)) \int_{I_\eta} \left( |u'_k|^2 \wedge \ell |u'_k| \right) dx \\ &\geq (1 - \omega(\delta)) \left( \ell \int_{I_\eta} |u'_k| dx - \frac{\ell^2}{4} 2\eta \right). \end{aligned} \quad (7.6)$$

Notice now that in view of Proposition 3.7 we have  $g(s) \leq \ell s - \tilde{g}(s)$  for  $s$  small and for some continuous and strictly positive function  $\tilde{g}(s) > 0$ ; inserting this inequality into (7.6) we find

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; I_\eta) &\geq (1 - \omega(\delta)) \left[ g \left( \left| \int_{I_\eta} u'_k dx \right| \right) + \tilde{g} \left( \left| \int_{I_\eta} u'_k dx \right| \right) - \frac{\ell^2 \eta}{2} \right] \\ &= (1 - \omega(\delta)) \left[ g(|u_k(\bar{x} + \eta) - u_k(\bar{x} - \eta)|) + \tilde{g}(|u_k(\bar{x} + \eta) - u_k(\bar{x} - \eta)|) - \frac{\ell^2 \eta}{2} \right]. \end{aligned}$$

By passing to the limit as  $k \rightarrow \infty$ , since the left-hand side converges to the limit energy in  $I_\eta$  and by assumption  $u_k(\bar{x} \pm \eta) \rightarrow u(\bar{x} \pm \eta)$  we have

$$\begin{aligned} \int_{I_\eta} h(|u'|) dx + g(|[u](\bar{x})|) &= \mathcal{F}(u, 1; I_\eta) \\ &\geq (1 - \omega(\delta)) \left[ g(|u(\bar{x} + \eta) - u(\bar{x} - \eta)|) + \tilde{g}(|u(\bar{x} + \eta) - u(\bar{x} - \eta)|) - \frac{\ell^2 \eta}{2} \right]. \end{aligned}$$

Therefore, letting first  $\delta \rightarrow 0$  and then in turn  $\eta \rightarrow 0$ , we conclude that

$$g(|[u](\bar{x})|) \geq g(|[u](\bar{x})|) + \tilde{g}(|[u](\bar{x})|) > g(|[u](\bar{x})|),$$

which is a contradiction.

(iii): By contradiction, assume that for a (not relabeled) subsequence  $x_k \rightarrow \tilde{x} \neq \bar{x}$ . Fix  $\delta \in (0, \eta)$  such that  $v_k(\tilde{x} + \delta) \rightarrow 1$  and  $\bar{x} \notin [\tilde{x} - \delta, \tilde{x} + \delta]$ . For  $k$  large enough,  $x_k \in (\tilde{x} - \delta, \tilde{x} + \delta)$  and

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; (\tilde{x} - \delta, \tilde{x} + \delta)) &\geq \int_{\tilde{x} - \delta}^{\tilde{x} + \delta} \left( \frac{(1 - v_k)^2}{4\varepsilon_k} + \varepsilon_k |v'_k|^2 \right) dx \geq \int_{x_k}^{\tilde{x} + \delta} (1 - v_k) v'_k dx \\ &= \frac{1}{2} (1 - m_k)^2 - \frac{1}{2} (1 - v_k(\tilde{x} + \delta))^2. \end{aligned}$$

Notice that the  $\liminf$  as  $k \rightarrow \infty$  of the right-hand side is a strictly positive quantity, in view of property (ii), which does not depend on  $\delta$ . On the other hand

$$\lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; (\tilde{x} - \delta, \tilde{x} + \delta)) = \mathcal{F}(u, 1; (\tilde{x} - \delta, \tilde{x} + \delta)) = \int_{\tilde{x} - \delta}^{\tilde{x} + \delta} h(|u'|) dx,$$

and therefore we conclude that  $\int_{\tilde{x} - \delta}^{\tilde{x} + \delta} h(|u'|) dx \geq \frac{1}{2} \liminf_{k \rightarrow \infty} (1 - m_k)^2 > 0$ . This is a contradiction since  $\delta$  can be chosen arbitrarily small.

(iv): By arguing as in the previous step and using the fact that  $x_k \in I_\eta$  for  $k$  large by (iii) we have the inequality

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; I_\eta) &\geq \int_{\tilde{x} - \eta}^{\tilde{x} + \eta} (1 - v_k) |v'_k| dx \geq \frac{1}{2} \int_{\tilde{x} - \eta}^{x_k} [(1 - v_k)^2]' dx - \frac{1}{2} \int_{x_k}^{\tilde{x} + \eta} [(1 - v_k)^2]' dx \\ &\geq (1 - m_k)^2 - \frac{1}{2} (1 - v_k(\tilde{x} - \eta))^2 - \frac{1}{2} (1 - v_k(\tilde{x} + \eta))^2. \end{aligned}$$

By letting  $k \rightarrow \infty$ , and using that  $v_k(\tilde{x} \pm \eta) \rightarrow 1$ , we have

$$\limsup_{k \rightarrow \infty} (1 - m_k)^2 \leq \int_{I_\eta} h(|u'|) dx + g(|[u](\tilde{x})|),$$

and the right-hand side is strictly smaller than one provided  $\eta$  is small enough.  $\square$

In the following we fix  $\eta > 0$  such that the conclusions of Lemma 7.2 hold, and we define  $x_k$  and  $m_k$  as in (7.5). Since these quantities depend on the choice of  $\eta$ , in the arguments below we will always let  $k \rightarrow \infty$  and  $\eta \rightarrow 0$  in this order. The following formula, obtained by a simple change of variables, will be often useful: for every fixed  $T > 0$  and for  $k \geq k_T$ , for some  $k_T$  large enough,

$$\begin{aligned} \mathcal{F}_k(u_k, v_k; (x_k - \varepsilon_k T, x_k + \varepsilon_k T)) &= \int_{x_k - \varepsilon_k T}^{x_k + \varepsilon_k T} \left( f_k^2(v_k) |u'_k|^2 + \frac{(1 - v_k)^2}{4\varepsilon_k} + \varepsilon_k |v'_k|^2 \right) dx \\ &= \int_{-T}^T \left( \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 + \frac{(1 - w_k)^2}{4} + |w'_k|^2 \right) dx, \end{aligned} \quad (7.7)$$

where  $w_k$  and  $z_k$  are the functions defined in (7.2) and (7.3) respectively (for  $\varepsilon = \varepsilon_k$ ).

**Lemma 7.3** (Compactness of  $w_k$ ). *Up to subsequences,  $w_k \rightharpoonup w$  weakly in  $H^1_{\text{loc}}(\mathbb{R})$  and therefore locally uniformly for some function  $w$  such that  $1 - w \in H^1(\mathbb{R})$ . In particular  $w$  satisfies*

$$\lim_{|x| \rightarrow \infty} w(x) = 1, \quad w(0) = \lim_{k \rightarrow \infty} v_k(x_k) = \lim_{k \rightarrow \infty} m_k < 1.$$

*Proof.* The result is an immediate consequence of the uniform bound on  $\|1 - w_k\|_{H^1(-T, T)}$  for every  $T > 0$ , which follows from (7.7) and the fact that  $(u_k, v_k)$  is a recovery sequence and has therefore equibounded energy. The fact that  $w(0) < 1$  follows from Lemma 7.2(ii).  $\square$

In order to proceed with the proof of Theorem 7.1, it is convenient to distinguish between the cases  $w(0) = 0$  and  $w(0) > 0$ . Indeed, we will see that these conditions correspond to  $g(|[u](\tilde{x})|) = 1$  or  $g(|[u](\tilde{x})|) < 1$  respectively; according to Theorem 3.4 the behaviour of minimizing sequences for the minimum problem defining  $g$  is different in the two cases.

We first consider the (easier) case  $w(0) = 0$ .

**Lemma 7.4.** *If  $w(0) = 0$ , then Theorem 7.1(i) holds.*

*Proof.* By (iv) of Lemma 7.2 it has to be  $g(|[u](\bar{x})|) = 1$ . Therefore for  $\eta > 0$  and  $T > 0$  we have by (7.7)

$$\begin{aligned} \mathcal{F}(u, 1; I_\eta) &= \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) \\ &\geq \liminf_{k \rightarrow \infty} \int_{-T}^T \left( \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx \geq \int_{-T}^T \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx, \end{aligned}$$

where the last inequality follows by lower semicontinuity with respect to the weak convergence of  $w_k$  to  $w$  in  $H^1(-T, T)$ , proved in Lemma 7.3. By letting first  $T \rightarrow \infty$  and then  $\eta \rightarrow 0$  we obtain

$$1 = g(|[u](\bar{x})|) = \lim_{\eta \rightarrow 0} \mathcal{F}(u, 1; I_\eta) \geq \int_{-\infty}^{\infty} \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx.$$

As  $w(0) = 0$ , by (3.8) we conclude that  $w(x) = 1 - e^{-\frac{|x|}{2}}$  is the optimal profile for  $g(|[u](\bar{x})|)$ .

It remains to prove the strong convergence in  $H_{\text{loc}}^1(\mathbb{R})$ . To this aim, it is sufficient to show convergence of the energies: suppose by contradiction that for some  $M > 0$  and  $\sigma > 0$

$$\limsup_{k \rightarrow \infty} \int_{-M}^M \left( \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx = \int_{-M}^M \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx + \sigma. \quad (7.8)$$

Then for every  $\eta > 0$  and  $T > M$  one has

$$\begin{aligned} \mathcal{F}(u, 1; I_\eta) &= \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) \geq \limsup_{k \rightarrow \infty} \int_{-T}^T \left( \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx \\ &\geq \limsup_{k \rightarrow \infty} \int_{-M}^M \left( \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx + \liminf_{k \rightarrow \infty} \int_{[-T, T] \setminus [-M, M]} \left( \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx \\ &\geq \int_{-T}^T \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx + \sigma, \end{aligned}$$

so that by letting first  $T \rightarrow \infty$  and then  $\eta \rightarrow 0$  we end up with

$$g(|[u](\bar{x})|) \geq \int_{-\infty}^{\infty} \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx + \sigma,$$

which is a contradiction. This completes the proof.  $\square$

**Lemma 7.5.** *If  $w(0) > 0$ , there is a subsequence such that  $z_k$  converges to some  $z \in H_{\text{loc}}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  strongly in  $H_{\text{loc}}^1(\mathbb{R})$ , and  $w_k$  converges to  $w \in H_{\text{loc}}^1(\mathbb{R})$  strongly in  $H_{\text{loc}}^1(\mathbb{R})$ . The pair  $(\bar{s}^{-1}z, w)$ , where  $\bar{s} := [u](\bar{x})$ , is a minimizer of the functional  $\mathcal{G}_{|\bar{s}|}$ , introduced in (3.5), in the class  $\mathcal{U}_1$ .*

*Proof.* Preliminaries. We first prove the weak compactness of the sequence  $z_k$ . For every fixed  $T > 0$  and for  $k \geq k_T$  one has

$$\min_{[-T, T]} w_k = \min_{[x_k - \varepsilon_k T, x_k + \varepsilon_k T]} v_k = v_k(x_k) = w_k(0) \rightarrow w(0) > 0,$$

so that

$$\inf_{k \geq k_T} \min_{[-T, T]} \frac{1}{\varepsilon_k} f_k^2(w_k) = \inf_{k \geq k_T} \frac{1}{\varepsilon_k} \wedge f^2(w_k(0)) > 0.$$

By (7.7) it follows that for every  $T > 0$

$$\sup_{k \geq k_T} \int_{-T}^T |z'_k|^2 dx < \infty. \quad (7.9)$$



Since by assumption the sequence  $z_k$  is also uniformly bounded in  $L^\infty$ , we can extract a further subsequence such that

$$z_k \rightharpoonup z \quad \text{weakly in } H_{\text{loc}}^1(\mathbb{R}) \quad (7.10)$$

for some function  $z \in H_{\text{loc}}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Possibly after extracting a subsequence, by (7.9) we can assume that  $|z'_k|^2 \mathcal{L}^1$  converges weakly in measures to a Radon measure  $\mu$ . By lower semicontinuity, this implies

$$|z'|^2 \mathcal{L}^1 \leq \mu \quad \text{as measures,} \quad (7.11)$$

with  $\mu(-T, T) = \int_{-T}^T |z'|^2 dx$  if and only if  $z_k \rightarrow z$  strongly in  $H^1(-T, T)$ .

*Step 1.* We prove that  $\mu(\{w = 1\}) = 0$  and

$$g(|[u](\bar{x})|) \geq \int_{\mathbb{R}} f^2(w) d\mu + \int_{-\infty}^{\infty} \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx.$$

To see this, we first observe that for almost any  $T > 0$  we have  $\mu(\{-T, T\}) = 0$  and, by (7.7),

$$\mathcal{F}(u, 1; I_\eta) = \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) \geq \limsup_{k \rightarrow \infty} \int_{-T}^T \left( \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 + \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx.$$

For  $\delta > 0$ , we let  $A_\delta := \{x \in (-T, T) : w(x) < 1 - \delta\}$ . For almost every  $\delta$  we have  $\mu(\partial A_\delta) = 0$ . By uniform convergence (Lemma 7.3), for sufficiently large  $k$  we have  $w_k(x) < 1 - \frac{1}{2}\delta$  on  $A_\delta$ , therefore  $f(w_k) \rightarrow f(w)$  uniformly on  $A_\delta$ . Since  $\mu(\partial A_\delta) = 0$ ,  $|z'_k|^2 \mathcal{L}^1 \rightharpoonup \mu$ , and  $f(w) \in C^0(\bar{A}_\delta)$ ,

$$\int_{A_\delta} f^2(w) d\mu = \lim_{k \rightarrow \infty} \int_{A_\delta} f^2(w) |z'_k|^2 dx = \lim_{k \rightarrow \infty} \int_{A_\delta} f^2(w_k) |z'_k|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{-T}^T \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 dx.$$

By monotone convergence, this implies

$$\int_{(-T, T) \cap \{w < 1\}} f^2(w) d\mu = \lim_{\delta \rightarrow 0} \int_{A_\delta} f^2(w) d\mu \leq \liminf_{k \rightarrow \infty} \int_{-T}^T \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 dx. \quad (7.12)$$

Also by uniform convergence of  $w_k$  to  $w$ , for  $k$  sufficiently large we have  $1 - 2\delta \leq w_k(x)$  on  $B_\delta := (-T, T) \setminus A_\delta$ . Recalling (2.3), for large  $k$  we have  $f^2(1 - 2\delta) \leq \varepsilon_k^{-1} \wedge f^2(w_k) = \varepsilon_k^{-1} f_k^2(w_k)$  on  $B_\delta$ . Since  $\mu(\partial B_\delta) = 0$ ,

$$f^2(1 - 2\delta) \mu(B_\delta) = f^2(1 - 2\delta) \lim_{k \rightarrow \infty} \int_{B_\delta} |z'_k|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{B_\delta} \frac{1}{\varepsilon_k} f_k^2(w_k) |z'_k|^2 dx \leq C.$$

By monotone convergence,

$$\mu((-T, T) \cap \{w = 1\}) = \lim_{\delta \rightarrow 0} \mu(B_\delta) \leq \lim_{\delta \rightarrow 0} \frac{C}{f^2(1 - 2\delta)} = 0.$$

Since  $w_k$  converges to  $w$  weakly in  $H^1(-T, T)$  (Lemma 7.3), (7.12) yields

$$\int_{(-T, T)} f^2(w) d\mu + \int_{(-T, T)} \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx \leq \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) = \mathcal{F}(u, 1; I_\eta)$$

for all  $T$  and all  $\eta$ . Sending first  $T \rightarrow \infty$  and then  $\eta \rightarrow 0$ ,

$$\int_{\mathbb{R}} f^2(w) d\mu + \int_{\mathbb{R}} \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx \leq g(|[u](\bar{x})|). \quad (7.13)$$

*Step 2.* By Lemma 7.3 and (7.10), for every  $j \in \mathbb{N}$  we can pick  $T_j > j$  such that  $|w(\pm T_j) - 1| \leq 2^{-j}$  and  $z_k(\pm T_j) \rightarrow z(\pm T_j)$  as  $k \rightarrow \infty$ . Since  $u_k$  and  $z_k$  are bounded in  $L^\infty$ , possibly passing to a subsequence we have

$$z_+ := \lim_{j \rightarrow \infty} z(T_j), \quad z_- := \lim_{j \rightarrow \infty} z(-T_j). \quad (7.14)$$

We intend to show that  $z_+ - z_- = [u](\bar{x})$ .

To do so, we separate the part of the profile in the inner interval  $(a, b) := (a_k^j, b_k^j) := (x_k - \varepsilon_k T_j, x_k + \varepsilon_k T_j)$  from the part outside (for sufficiently large  $k$  we can assume  $[a - \varepsilon_k, b + \varepsilon_k] \subset I_\eta$ ). Specifically, we set

$$u_{k,j}^{\text{in}}(x) := \begin{cases} u_k(a) & \text{if } x \leq a, \\ u_k(x) & \text{if } a < x \leq b, \\ u_k(b) & \text{if } x > b, \end{cases} \quad u_{k,j}^{\text{out}}(x) := \begin{cases} u_k(x) - u_k(a) & \text{if } x \leq a, \\ 0 & \text{if } a \leq x \leq b, \\ u_k(x) - u_k(b) & \text{if } x > b \end{cases}$$

which implies, since  $u_k(a) = z_k(-T_j) \rightarrow z(-T_j)$  and  $u_k(b) = z_k(T_j) \rightarrow z(T_j)$ , that

$$u_{k,j}^{\text{in}}(x) \rightarrow u_j^{\text{in}}(x) := \begin{cases} z(-T_j) & \text{if } x \leq \bar{x}, \\ z(T_j) & \text{if } x > \bar{x}, \end{cases} \quad u_{k,j}^{\text{out}} \rightarrow u_j^{\text{out}} := u - u_j^{\text{in}}$$

in  $L^1(I_\eta)$ . Correspondingly, we define

$$v_{k,j}^{\text{in}}(x) := \begin{cases} 1 & \text{if } x \leq a - \varepsilon_k, \\ v_k(x) & \text{if } a \leq x \leq b, \\ 1 & \text{if } x \geq b + \varepsilon_k, \\ \text{affine interp.} & \text{in between,} \end{cases} \quad v_{k,j}^{\text{out}}(x) := \begin{cases} v_k(x) & \text{if } x \leq a, \\ 1 & \text{if } a + \varepsilon_k \leq x \leq b - \varepsilon_k, \\ v_k(x) & \text{if } x \geq b, \\ \text{affine interp.} & \text{in between.} \end{cases}$$

We observe that  $v_{k,j}^{\text{in}} \rightarrow 1$  with  $\min v_{k,j}^{\text{in}} = v_{k,j}^{\text{in}}(x_k) = m_k \rightarrow w(0) < 1$ ;  $v_{k,j}^{\text{out}} \rightarrow 1$ . The energy obeys

$$\mathcal{F}_k(u_{k,j}^{\text{in}}, v_{k,j}^{\text{in}}; I_\eta) \leq \mathcal{F}_k(u_k, v_k; (a, b)) + c2^{-j},$$

$$\mathcal{F}_k(u_{k,j}^{\text{out}}, v_{k,j}^{\text{out}}; I_\eta) \leq \mathcal{F}_k(u_k, v_k; I_\eta \setminus (a, b)) + c2^{-j},$$

so that

$$\mathcal{F}_k(u_{k,j}^{\text{in}}, v_{k,j}^{\text{in}}; I_\eta) + \mathcal{F}_k(u_{k,j}^{\text{out}}, v_{k,j}^{\text{out}}; I_\eta) \leq \mathcal{F}_k(u_k, v_k; I_\eta) + c2^{-j}.$$

Passing to the limit and using (7.1),

$$\mathcal{F}(u_j^{\text{in}}, 1; I_\eta) + \mathcal{F}(u_j^{\text{out}}, 1; I_\eta) \leq \liminf_{k \rightarrow \infty} \mathcal{F}_k(u_k, v_k; I_\eta) + c2^{-j} = \mathcal{F}(u, 1; I_\eta) + c2^{-j}$$

for all  $j$  and  $\eta$ . Using the explicit form of  $u_j^{\text{in}}$ ,  $u_j^{\text{out}}$  and  $\mathcal{F}$ , we obtain

$$g(|[u_j^{\text{in}}]|) + g(|[u_j^{\text{out}}]|) \leq g(|[u]|) + c2^{-j}, \quad (7.15)$$

where all jumps are evaluated at  $\bar{x}$ . Recalling  $\min v_{k,j}^{\text{in}} = v_{k,j}^{\text{in}}(x_k) = m_k \rightarrow w(0) < 1$  and  $v_{k,j}^{\text{in}} = 1$  on  $\partial I_\eta$ ,

$$(1 - m_k)^2 \leq \int_{I_\eta} |1 - v_{k,j}^{\text{in}}| |(v_{k,j}^{\text{in}})'| dx \leq \int_{I_\eta} \left( \frac{(1 - v_{k,j}^{\text{in}})^2}{4\varepsilon_k} + \varepsilon_k |(v_{k,j}^{\text{in}})'|^2 \right) dx \leq \mathcal{F}_k(u_{k,j}^{\text{in}}, v_{k,j}^{\text{in}}; I_\eta)$$

which gives, by the same argument, first

$$(1 - w(0))^2 + \mathcal{F}(u_j^{\text{out}}, 1; I_\eta) \leq \mathcal{F}(u, 1; I_\eta) + c2^{-j}$$

and then

$$(1 - w(0))^2 + g(|[u_j^{\text{out}}]|) \leq g(|[u]|) + c2^{-j}. \quad (7.16)$$

Taking  $j \rightarrow \infty$  in (7.15) and (7.16) we obtain, with (7.14),

$$g(|z^+ - z^-|) + g(|[u] - (z^+ - z^-)|) \leq g(|[u]|) \quad \text{and} \quad (1 - w(0))^2 + g(|[u] - (z^+ - z^-)|) \leq g(|[u]|).$$

Recalling that  $g$  is strictly subadditive on  $(0, \infty)$ , we see that one of the two terms in the first inequality must vanish. The second inequality excludes the case  $z^+ = z^-$ , therefore  $z^+ - z^- = [u]$ .

In order to show that  $z' \in L^1(\mathbb{R})$ , we prove that  $z$  is monotone. Assume for definiteness that  $z^- \leq z^+$ . We define

$$\bar{z} := z_- \vee (z_+ \wedge z) \quad \text{and} \quad \hat{z}(x) := \begin{cases} \min \bar{z}([x, 0]) & \text{if } x \leq 0, \\ \max \bar{z}([0, x]) & \text{if } x > 0. \end{cases}$$

Then  $z_- \leq \hat{z} \leq \bar{z}$  on  $(-\infty, 0)$  and  $\bar{z} \leq \hat{z} \leq z_+$  on  $(0, \infty)$ , hence  $\hat{z}(\pm t) \rightarrow z_{\pm}$  for  $t \rightarrow \infty$ . Obviously  $\hat{z}$  is monotone. Further,  $\hat{z} \in H_{\text{loc}}^1(\mathbb{R})$ ,  $0 \leq \hat{z}' \leq |z'|$ ,  $\int_{\mathbb{R}} \hat{z}' dx = z^+ - z^-$ , and

$$\int_{-\infty}^{\infty} f^2(w) |\hat{z}'|^2 dx \leq \int_{-\infty}^{\infty} f^2(w) |z'|^2 dx,$$

with equality if and only if  $\hat{z} = z$  almost everywhere, which would imply that  $z$  is also monotone. In particular,  $(\bar{s}^{-1} \hat{z}, w) \in \mathcal{U}_1$  for  $\bar{s} = [u](\bar{x})$ . Using Proposition 3.2(i),  $|\hat{z}'| \leq |z'|$ , (7.11) and (7.13) we obtain

$$\begin{aligned} g(|[u](\bar{x})|) &\leq \int_{-\infty}^{\infty} \left( f^2(w) |\hat{z}'|^2 + \frac{(1-w)^2}{4} + |w'|^2 \right) dx \\ &\leq \int_{-\infty}^{\infty} \left( f^2(w) |z'|^2 + \frac{(1-w)^2}{4} + |w'|^2 \right) dx \\ &\leq \int_{\mathbb{R}} f^2(w) d\mu + \int_{-\infty}^{\infty} \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx \leq g(|[u](\bar{x})|). \end{aligned}$$

Therefore equality holds throughout. This implies in particular  $\hat{z} = z$ , so that  $z$  is monotone,  $(\bar{s}^{-1} z, w) \in \mathcal{U}_1$ , and  $\mu = |z'|^2 \mathcal{L}^1$ , which gives strong convergence of  $z_k$  to  $z$  in  $H^1(-T, T)$  for all  $T$ .

*Step 3.* In order to complete the proof it only remains to show the strong convergence of  $w_k$  to  $w$ . We have for all  $T > 0$

$$\liminf_{k \rightarrow \infty} \int_{-T}^T \left( \frac{(1-w_k)^2}{4} + |w'_k|^2 \right) dx \geq \int_{-T}^T \left( \frac{(1-w)^2}{4} + |w'|^2 \right) dx \quad (7.17)$$

and

$$\liminf_{k \rightarrow \infty} \int_{-T}^T \frac{1}{\varepsilon_k} f_{\varepsilon_k}^2(w_k) |z'_k|^2 dx \geq \int_{-T}^T f^2(w) |z'|^2 dx.$$

Then arguing as in the case  $w(0) = 0$  and assuming by contradiction that (7.8) holds, it is easily seen that one gets the contradiction

$$g(|[u](\bar{x})|) \geq \int_{-\infty}^{\infty} \left( f^2(w) |z'|^2 + \frac{(1-w)^2}{4} + |w'|^2 \right) dx + \sigma,$$

proving that (7.17) actually holds with an equality and a limit on the left-hand side. This gives the strong convergence of  $w_k$  to  $w$ .  $\square$

**Remark 7.6.** *Using the uniqueness result of Proposition 3.3(v), convergence of the entire sequence (after suitable translations) can be obtained. Hence Theorem 7.1(i) is proved also in the case  $w(0) > 0$ .*

We finally turn to the second assertion of Theorem 7.1.

**Lemma 7.7.** *Theorem 7.1(ii) holds.*

*Proof.* We fix  $\eta > 0$  and we let  $A := (0, 1) \setminus \bigcup_{x \in J_u} [x - \eta, x + \eta]$ . We first remark that

$$v_{\varepsilon} \rightarrow 1 \quad \text{uniformly on } A.$$

Indeed, this follows by the same argument used to prove Lemma 7.2(iii). Therefore by assumption (2.5) we can write

$$(1 - v_\varepsilon)f(v_\varepsilon) \geq (1 - \omega(\varepsilon))\ell \quad \text{on } A,$$

for some modulus of continuity  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . We introduce the set  $A_\varepsilon := \{x \in A : f_\varepsilon(v_\varepsilon)(x) < 1\}$  and we observe that, since  $f_\varepsilon(v_\varepsilon) = \sqrt{\varepsilon}f(v_\varepsilon)$  on  $A_\varepsilon$ ,

$$\begin{aligned} \int_A \left( f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx &\geq \int_{A_\varepsilon} \left( \varepsilon f^2(v_\varepsilon)|u'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx + \int_{A_\varepsilon^c} f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 dx \\ &\geq \int_{A_\varepsilon} (1 - v_\varepsilon)f(v_\varepsilon)|u'_\varepsilon| dx + \int_{A_\varepsilon^c} |u'_\varepsilon|^2 dx \\ &\geq \ell(1 - \omega(\varepsilon)) \int_{A_\varepsilon} |u'_\varepsilon| dx + \int_{A_\varepsilon^c} |u'_\varepsilon|^2 dx. \end{aligned} \quad (7.18)$$

The sequence  $u'_\varepsilon \chi_{A_\varepsilon^c}$  is bounded in  $L^2(A)$ , and therefore, possibly passing to a subsequence, has a weak limit  $p \in L^2(A)$ . The sequence  $u'_\varepsilon \chi_{A_\varepsilon}$  is bounded in  $L^1(A)$ , and analogously we can assume that  $u'_\varepsilon \chi_{A_\varepsilon} \rightharpoonup \mu$  and  $|u'_\varepsilon| \chi_{A_\varepsilon} \rightharpoonup \nu$  weakly in measures, with

$$|\mu|(A) \leq \nu(A) \leq \liminf_{\varepsilon \rightarrow 0} \int_A |u'_\varepsilon| \chi_{A_\varepsilon} dx.$$

Since  $u'_\varepsilon = u'_\varepsilon \chi_{A_\varepsilon} + u'_\varepsilon \chi_{A_\varepsilon^c}$  converges distributionally to  $u' \in L^\infty(A)$ , we have  $\mu + p \mathcal{L}^1 = u' \mathcal{L}^1$ , which implies that  $\mu = (u' - p) \mathcal{L}^1$  and  $|\mu|(A) = \int_A |u' - p| dx$ . Thus (7.18) gives

$$\liminf_{\varepsilon \rightarrow 0} \int_A \left( f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx \geq \int_A (\ell|u' - p| + p^2) dx \geq \int_A (|u'|^2 + |u' - p|^2) dx,$$

where in the last step we used that for any  $t \in [-\frac{\ell}{2}, \frac{\ell}{2}]$  and any  $y \in \mathbb{R}$  one has

$$\ell|t - y| + y^2 \geq t^2 + (t - y)^2.$$

Indeed, this inequality is the same as  $\ell|t - y| \geq 2t(t - y)$ , which is true since  $\ell \geq 2|t|$ .

Since we know that

$$\limsup_{\varepsilon \rightarrow 0} \int_A \left( f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx \leq \mathcal{F}(u, 1; A) = \int_A |u'|^2 dx,$$

we conclude that  $p = u'$ . In particular

$$\begin{aligned} \int_A |u'|^2 dx &\leq \liminf_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int_A f_\varepsilon^2(v_\varepsilon)|u'_\varepsilon|^2 dx \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) = \mathcal{F}(u, 1; A) = \int_A |u'|^2 dx, \end{aligned}$$

hence equality holds throughout and (7.4) is proven. This completes the proof.  $\square$

## A A relaxation result

In order to construct a time-discrete evolution in the cohesive setting including our notion of irreversibility, we made use in Section 4 of a relaxation result in the spirit of [11, 12]. The main difference is that the surface part of our energy contains also information on points representing a preexisting crack, and therefore is not considered in the existing literature; however, the proof is a small variant of the standard theory, and we present it in this section. A similar problem was discussed by Giacomini in [35].

Let  $b : \{0, 1\} \rightarrow \mathbb{R}$  be a given boundary datum. Let also  $(\Gamma, s)$  be a given pair with  $\Gamma \subset [0, 1]$  countable and  $s : \Gamma \rightarrow (0, \infty)$ . Let  $\Phi_0 : \text{BV}(0, 1) \rightarrow [0, \infty]$  be the functional defined by

$$\Phi_0(u) := \begin{cases} \int_0^1 |u'|^2 dx + \sum_{x \in \Gamma} \bar{g}([u](x), s(x)) + \sum_{x \in J_u^b \setminus \Gamma} g([u](x)) & \text{if } u \in \text{SBV}(0, 1), \\ \infty & \text{otherwise,} \end{cases} \quad (\text{A.1})$$

where  $J_u^b$  and the jump  $[u](x)$  at the boundary points  $x \in \{0, 1\}$  are defined in (4.5) and (4.6) respectively. The surface energy densities  $g, \bar{g}$  are defined in (2.10) and (3.33) respectively.

**Theorem A.1.** *The relaxation of the functional  $\Phi_0$  with respect to the weak\*-topology of  $\text{BV}(0, 1)$  is given by  $\Phi : \text{BV}(0, 1) \rightarrow [0, \infty]$ ,*

$$\Phi(u) := \int_0^1 h(|u'|) dx + \sum_{x \in \Gamma} \bar{g}([u](x), s(x)) + \sum_{x \in J_u^b \setminus \Gamma} g([u](x)) + \ell |D^c u|(0, 1), \quad (\text{A.2})$$

where the elastic energy density  $h$  is defined in (2.9).

*Proof.* In order to handle the boundary conditions, we work in a larger open interval  $I$  containing  $[0, 1]$ , for instance  $I := (-1, 2)$ , and we extend the boundary values as  $b(x) = b(0)$  for  $x \leq 0$ ,  $b(x) = b(1)$  for  $x \geq 1$ . We consider the functional

$$\Psi_0(u) := \begin{cases} \int_0^1 |u'|^2 dx + \sum_{x \in \Gamma} \bar{g}([u](x), s(x)) + \sum_{x \in J_u \setminus \Gamma} g([u](x)) & \text{if } u \in \text{SBV}(I), \\ \infty & \text{otherwise in } \text{BV}(I). \end{cases} \quad \begin{matrix} u = b \text{ on } I \setminus (0, 1), \\ \end{matrix}$$

The statement is equivalent to proving that the relaxation of the functional  $\Psi_0$  with respect to the weak\*-topology of  $\text{BV}(I)$  is given by

$$\Psi(u) := \int_0^1 h(|u'|) dx + \sum_{x \in \Gamma} \bar{g}([u](x), s(x)) + \sum_{x \in J_u \setminus \Gamma} g([u](x)) + \ell |D^c u|(0, 1) \quad (\text{A.3})$$

if  $u \in \text{BV}(I)$  with  $u = b$  on  $I \setminus (0, 1)$ ,  $\Psi(u) = \infty$  otherwise in  $\text{BV}(I)$ .

*Step 1: lower semicontinuity of  $\Psi$ .* Let  $u_n \in \text{BV}(I)$  be a sequence converging weakly\* in  $\text{BV}$  to some function  $u$ ; in particular,  $u_n \rightarrow u$  in  $L^1(I)$  and  $Du_n \xrightarrow{*} Du$  in the sense of measures. We can also assume without loss of generality that  $u_n = b$  on  $I \setminus (0, 1)$  for every  $n$ , and that  $\Psi(u_n)$  has a limit as  $n \rightarrow \infty$ . Since  $\Psi$  can be obtained as the supremum of functionals corresponding to a finite set  $\Gamma$ , we may also assume that  $\Gamma = \{x_1, \dots, x_m\}$ , with  $s_i = s(x_i)$ .

We now isolate the possible jumps of  $u_n$  at the points  $x_i \in \Gamma$ : namely, we consider the atomic measures

$$\mu_n := \sum_{i=1}^m [u_n](x_i) \delta_{x_i},$$

and we can assume that  $[u_n](x_i) \rightarrow a_i$  for suitable values  $a_i \in \mathbb{R}$ , or equivalently

$$\mu_n \xrightarrow{*} \mu = \sum_{i=1}^m a_i \delta_{x_i}. \quad (\text{A.4})$$

We then consider the function  $v_n \in \text{BV}(I)$  defined by  $v_n(x) = b(0) + (Du_n - \mu_n)((-1, x))$ . Notice in particular that  $v_n$  is continuous at the points  $x_i \in \Gamma$ ,  $i = 1, \dots, m$ . Moreover  $v_n \rightarrow v$  weakly\* in  $\text{BV}$ , with  $v \in \text{BV}(I)$  satisfying  $Du = Dv + \mu$ . With this decomposition we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Psi(u_n) &\geq \liminf_{n \rightarrow \infty} \left[ \int_0^1 h(|v_n'|) dx + \sum_{x \in J_{v_n}} g([v_n](x)) + \ell |D^c v_n|(0, 1) \right] \\ &\quad + \liminf_{n \rightarrow \infty} \sum_{i=1}^m \bar{g}([u_n](x_i), s_i). \end{aligned}$$

By standard results the first term on the right-hand side is lower semicontinuous with respect to weak\*-convergence in BV, see for instance [6, Theorem 5.2]; therefore, using also (A.4) and the continuity of  $\bar{g}$ ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Psi(u_n) &\geq \left[ \int_0^1 h(|v'|) dx + \sum_{x \in J_v} g(|[v](x)|) + \ell |D^c v|(0, 1) \right] + \sum_{i=1}^m \bar{g}(|a_i|, s_i) \\ &= \int_0^1 h(|u'|) dx + \sum_{x \in J_u \setminus \Gamma} g(|[u](x)|) + \ell |D^c u|(0, 1) \\ &\quad + \sum_{i=1}^m \left( g(|[v](x_i)|) + \bar{g}(|a_i|, s_i) \right). \end{aligned}$$

Finally, as  $[u](x_i) = [v](x_i) + a_i$ , using the subadditivity property of  $\bar{g}$  proved in (3.38) we can bound from below the last sum by  $\sum_{i=1}^m \bar{g}(|[u](x_i)|, s_i)$ , and in turn we recover the desired inequality  $\liminf_{n \rightarrow \infty} \Psi(u_n) \geq \Psi(u)$ .

*Step 2: relaxation.* For all open sets  $A \subset I$  we consider the localized versions  $\Psi_0(\cdot; A)$  and  $\Psi(\cdot; A)$  of  $\Psi_0$  and  $\Psi$  respectively. We denote by  $\bar{\Psi}_0(\cdot; A)$  the relaxation of  $\Psi_0(\cdot; A)$  with respect to the weak\*-topology of BV. Notice that this coincides with the lower semicontinuous envelope of  $\Psi_0(\cdot; A)$  with respect to the strong topology of  $L^1$ , that is

$$\bar{\Psi}_0(u; A) = \inf \left\{ \liminf_{n \rightarrow \infty} \Psi_0(u_n; A) : u_n \rightarrow u \text{ in } L^1(A) \right\}$$

(see Remark A.3 below). The thesis amounts to show that  $\bar{\Psi}_0 = \Psi$ .

We further remark that since  $\bar{g}$  is nondecreasing in the first argument,  $\sum_{x \in \Gamma} \bar{g}(0, s(x)) \leq \Psi(u) \leq \Psi_0(u)$  for all  $u$ . In particular, we can assume that  $\sum_{x \in \Gamma} \bar{g}(0, s(x)) < \infty$  for the rest of the proof.

By the lower semicontinuity of  $\Psi$  proved in the previous step, the inequality  $\Psi \leq \bar{\Psi}_0$  is immediate. We therefore show the opposite inequality. By the same argument as in [12, Proposition 3.3] we have that for every fixed  $u \in \text{BV}(I)$  the set function  $\bar{\Psi}_0(u; \cdot)$  is the restriction to the family of open subsets of  $I$  of a regular Borel measure.

Notice that for every  $u \in \text{SBV}(I)$  with  $u = b$  on  $I \setminus (0, 1)$  and every open set  $A \subset I$  we have, using (3.38),

$$\Psi_0(u; A) \leq \int_A |u'|^2 dx + \sum_{x \in J_u \cap A} g(|[u](x)|) + \sum_{x \in \Gamma \cap A} \bar{g}(0, s(x)).$$

As a consequence of [12, Theorem 3.1] we obtain for every  $u \in \text{BV}(I)$  with  $u = b$  on  $I \setminus (0, 1)$  and every open set  $A \subset I$

$$\bar{\Psi}_0(u; A) \leq \int_A h(|u'|) dx + \ell |D^c u|(A) + \sum_{x \in J_u \cap A} g(|[u](x)|) + \sum_{x \in \Gamma \cap A} \bar{g}(0, s(x)).$$

In particular, from the previous inequality it follows that

$$\bar{\Psi}_0(u; \cdot) \llcorner (I \setminus \Gamma) \leq h(|u'|) \mathcal{L}^1 + \ell |D^c u| + \sum_{x \in J_u \setminus \Gamma} g(|[u](x)|) \delta_x. \quad (\text{A.5})$$

It remains to evaluate  $\bar{\Psi}_0(u; \cdot) \llcorner \Gamma$ . Without loss of generality we may assume that  $u \in \text{BV}(I)$  with  $u = b$  on  $I \setminus (0, 1)$ . Let  $K \subset \Gamma$  be any finite subset of  $\Gamma$ . For  $\varepsilon > 0$  we take an open set  $A_\varepsilon \subset I$  with  $K \subset A_\varepsilon$  and

$$|Du|(A_\varepsilon \setminus K) < \varepsilon, \quad \sum_{x \in A_\varepsilon \cap \Gamma \setminus K} \bar{g}(0, s(x)) < \varepsilon.$$

We can construct a sequence of functions  $u_h \in \text{SBV}(I)$ , with  $u_h = b$  on  $I \setminus (0, 1)$ , such that  $u_h$  is piecewise constant,  $|Du_h|(A_\varepsilon \setminus K) \leq |Du|(A_\varepsilon \setminus K) < \varepsilon$ , and  $u_h \rightarrow u$  strongly in  $L^\infty(I)$ . Then

$$\begin{aligned} \bar{\Psi}_0(u; A_\varepsilon) &\leq \liminf_{h \rightarrow \infty} \Psi_0(u_h; A_\varepsilon) = \liminf_{h \rightarrow \infty} \left( \sum_{x \in A_\varepsilon \cap \Gamma} \bar{g}(|[u_h](x)|, s(x)) + \sum_{x \in (J_{u_h} \setminus \Gamma) \cap A_\varepsilon} g(|[u_h](x)|) \right) \\ &\leq \liminf_{h \rightarrow \infty} \left( \sum_{x \in K} \bar{g}(|[u_h](x)|, s(x)) + \sum_{x \in A_\varepsilon \cap \Gamma \setminus K} \bar{g}(0, s(x)) + \sum_{x \in (J_{u_h} \setminus K) \cap A_\varepsilon} g(|[u_h](x)|) \right) \\ &\leq \sum_{x \in K} \bar{g}(|[u](x)|, s(x)) + \sum_{x \in A_\varepsilon \cap \Gamma \setminus K} \bar{g}(0, s(x)) + \ell \limsup_{h \rightarrow 0} |Du_h|(A_\varepsilon \setminus K) \\ &\leq \sum_{x \in K} \bar{g}(|[u](x)|, s(x)) + (1 + \ell)\varepsilon, \end{aligned}$$

where we used (3.38) in the second line, and the inequality  $g(s) \leq \ell s$  in the third line. By letting  $\varepsilon \rightarrow 0$  we obtain

$$\bar{\Psi}_0(u; K) \leq \sum_{x \in K} \bar{g}(|[u](x)|, s(x)),$$

and since  $K$  is an arbitrary finite subset of  $\Gamma$  we conclude that

$$\bar{\Psi}_0(u; \cdot) \llcorner \Gamma \leq \sum_{x \in \Gamma} \bar{g}(|[u](x)|, s(x)) \delta_x. \quad (\text{A.6})$$

The combination of (A.5) and (A.6) yields the inequality  $\bar{\Psi}_0 \leq \Psi$  and concludes the proof of the theorem.  $\square$

**Remark A.2.** Observe that the functional  $\Phi$  is also the lower semicontinuous envelope of

$$\tilde{\Phi}_0(u) := \begin{cases} \int_0^1 |u'|^2 dx + \sum_{x \in \Gamma} \bar{g}(|[u](x)|, s(x)) + \sum_{x \in J_u^b \setminus \Gamma} g(|[u](x)|) & \text{if } u \in \text{SBV}(0, 1) \\ \infty & \text{with } \mathcal{H}^0(J_u) < \infty, \\ & \text{otherwise,} \end{cases}$$

which is finite only on functions with a finite number of jumps. Indeed, the functions of the recovery sequence constructed in the second step of the proof of Theorem A.1 are piecewise constant and therefore they satisfy the additional constraint.

**Remark A.3.** Notice that the relaxation  $\bar{\Phi}$  of  $\Phi_0$  with respect to the weak\*-topology of BV coincides with the relaxation of  $\Phi_0$  in the  $L^1$ -topology, which is given by

$$\Phi^1(u) := \inf \left\{ \liminf_{n \rightarrow \infty} \Phi_0(u_n) : u_n \rightarrow u \text{ in } L^1(0, 1) \right\}. \quad (\text{A.7})$$

Indeed, the inequality  $\Phi^1 \leq \bar{\Phi}$  is obvious. For the converse, first notice that the same estimate as in (4.13) gives that for every sequence  $(u_n)_n$  with  $\sup_n \|u_n\|_\infty < \infty$  and  $\sup_n \Phi_0(u_n) < \infty$  one also has  $\sup_n |Du_n|(0, 1) < \infty$ . Then, given any sequence  $u_n \rightarrow u$  in  $L^1(0, 1)$  with  $\sup_n \Phi_0(u_n) < \infty$ , consider the truncation  $u_n^M := (-M) \vee (u_n \wedge M)$ , for  $M > 0$  sufficiently large. By the previous observation we have  $u_n^M \rightarrow u^M$  in the weak\*-topology of BV, and therefore

$$\bar{\Phi}(u^M) \leq \liminf_{n \rightarrow \infty} \Phi_0(u_n^M) \leq \liminf_{n \rightarrow \infty} \Phi_0(u_n).$$

By passing to the limit as  $M \rightarrow \infty$ , the inequality  $\bar{\Phi} \leq \Phi^1$  follows.

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