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BETTI NUMBERS OF SHIMURA CURVES AND ARITHMETIC THREE–ORBIFOLDS

MIKOŁAJ FRĄCZYK AND JEAN RAIMBAULT

Abstract. We show that asymptotically the first Betti number b_1 of a Shimura curve satisfies the Gauss–Bonnet equality $2\pi(b_1 - 2) = \mathrm{vol}$ where vol is hyperbolic volume; equivalently $2g-2 = (1+o(1))\,\mathrm{vol}$ where g is the arithmetic genus. We also show that the first Betti number of a congruence hyperbolic 3–orbifold asymptotically vanishes relatively to hyperbolic volume, that is $b_1/\mathrm{vol} \to 0$. This generalises previous results obtained by the first author, on which we rely, and uses the same main tool, namely Benjamini–Schramm convergence.

1. Introduction

1.1. **Benjamini–Schramm convergence.** Let G be a semisimple Lie group, $K \subset G$ a maximal compact subgroup and $X = G/K$ the associated symmetric space. Benjamini–Schramm convergence of locally symmetric orbifolds $\Gamma\backslash X$ of finite volume was introduced in [1]. The Benjamini–Schramm convergence of a sequence of finite volume locally symmetric spaces $(\Gamma_i\backslash X)_{i\in\mathbb{N}}$ to the symmetric space X is equivalent to the following simple geometric condition:

$$\forall R > 0,\ \lim_{i\to\infty} \frac{\mathrm{vol}((\Gamma_i\backslash X)_{<R})}{\mathrm{vol}(\Gamma_i\backslash X)} = 0, \tag{1.1}$$

where $M_{<R}$ denotes the R-thin part of a Riemannian orbifold M (which we take to include the full singular set, see (3.1) below).

In addition to X there are other possible limits in the Benjamini-Schramm topology. In order to describe them it is convenient to pass to the language of invariant random subgroups (IRS) of the group G. These are the Borel probability measures on the Chabauty space Sub_G of closed subgroups which are invariant under conjugation by elements of G. For every lattice Γ of G there is a unique G-invariant probability measure on G/Γ and its pushforward by the map $g\Gamma \mapsto g\Gamma g^{-1}$ gives an IRS denoted μ_Γ. It was

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observed in [1] that $(\Gamma_i \backslash X)$ converges to X if and only if μ_{Γ_i} converge weakly-* to the trivial IRS $\delta_{\{1\}}$. In general a sequence $(\Gamma_i \backslash X)$ converges Benjamini-Schramm if and only if μ_Γ converges weakly-* to some IRS ν. The limit IRS ν is always supported on discrete subgroups and the Benjamini-Schramm limit is the random locally symmetric space X/Λ where Λ is a ν-random subgroup of G.

It was proven in [1], as a consequence of the Nevo–Stück–Zimmer theorem, that if G is semisimple of higher rank, with all factors having property (T) then any sequence of irreducible locally symmetric spaces converges in the Benjamini–Schramm sense to X. This was extended to all nontrivial products in [17] (see also [22] for more precise results in a very specific case).

This statement is known to be false when $G = \mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$, because in those cases there are lattices $\Gamma \subset G$ such that $H^1(\Gamma, \mathbb{R}) \neq 0$ (see [23], [18], [15]). On the other hand restricting attention to the family of *arithmetic congruence lattices* in G (see 1.4 below for a short description) the first author proved in [11] that for $G = \mathrm{SO}(2,1), \mathrm{SO}(3,1)$ the symmetric space $X = \mathbb{H}^2, \mathbb{H}^3$ is the only possible limit in the Benjamini-Schramm topology for a sequence of torsion-free congruence lattices. Previously the second author [25] had proven a similar result for the family of non-uniform, not necessarily torsion-free lattices (nonuniformity makes them much easier to deal with algebraically). In this paper we remove the torsion-free hypothesis in general.

Theorem A. If $G = \mathrm{PGL}_2(\mathbb{R})$ or $\mathrm{PGL}_2(\mathbb{C})$ and Γ_n is a sequence of irreducible arithmetic lattices in G, which are either all congruence and pairwise distinct, or pairwise non-commensurable, then the sequence of locally symmetric spaces $\Gamma_n \backslash X$ converges in the Benjamini–Schramm sense to X.

In [11] the torsion free assumption was necessary because the methods only allowed to control the volume of the subset of thin part consiting of the collars of short geodesics. For a sequence of general arithmetic congruence orbifolds $(\Gamma_n \backslash X)_{n \in \mathbf{N}}$ it could *a priori* happen that the vast majority of the thin part comes from the cusps or the conical singularities so the sequence does not converge to X. Theorem A excludes this possibility. For the proof we use the estimates developped in [11] to show that any weak-* limit of the sequence μ_{Γ_n} is supported on elementary subgroups. By [24] the only IRS supported on this set is the trivial IRS, hence the theorem. We carry out the second step of this scheme of proof in detail in Proposition A.4, which is valid for all sequences of lattices in proper Gromov-hyperbolic spaces.

We note that because we are using a soft method our approach does not indicate the rate of decay of $\mathrm{vol}((\Gamma_n \backslash X)_{<R}) / \mathrm{vol}(\Gamma_n \backslash X)$ as opposed to [11].

1.2. **Genus of Shimura curves.** One application of Theorem A is to determine the asymptotic genus of congruence surfaces of large volume. For compact surfaces without singularities the genus and volume are essentially linearly related by the Gauss-Bonnet formula. However for 2-orbifolds terms coming

from cone points and cusps appear in the formula, and it is easy to see that there exists sequences of hyperbolic orbifolds with underlying space a sphere and volume going to infinity. This also has an algebraic interpretation: if S is isomorphic as a Riemann surface to the $\mathbb{C}$-points of an algebraic variety defined over a number field, which is the case for orbifolds obtained from congruence groups (so-called Shimura curves, see [26]), then its arithmetic genus is given by the Riemann–Hurwitz formula and essentially proportional to the volume while its geometric genus equals the topological genus of the underlying surface and can be arbitrarily smaller than the former.

It is known that this phenomenon cannot occur for congruence orbifolds: using the uniform spectral gap for congruence quotients (see [7] for a more general result) and a theorem of P. Zograf [29] it follows that there is a lower bound of the form $g \geq c\,\mathrm{vol}$ for congruence subgroups (see also [19]). As a consequence of Theorem A we obtain the following asymptotically more precise result (we note that it was known for congruence covers of the modular surface by a result of J. G. Thompson [28]).

Theorem B. Let Γ_n be a sequence of congruence lattices in $\mathrm{PSL}_2(\mathbb{R})$, and let g_n be the topological genus of the orbifold $O_n = \Gamma_n \backslash \mathbb{H}^2$. Then, assuming the Γ_n are not pairwise conjugated, we have

$$\lim_{n\to+\infty} \frac{g_n}{\mathrm{vol}\, O_n} = \frac{1}{4\pi}.$$

1.3. **Betti numbers of 3–orbifolds.** Theorem B is equivalent to the statement that $b_1(\Gamma_n)/\,\mathrm{vol}(\Gamma_n\backslash\mathbb{H}^2)$ converges to $1/2\pi$ for a sequence of congruence lattices. Indeed, the rank of abelianisation is essentially equal to twice the genus in a BS-convergent sequence. This can be proven more directly by analytical means, as $1/2\pi$ is the first L^2-Betti number of the hyperbolic plane. While more complicated, the analytic approach generalizes to the dimension 3 and where obtain the following result.

Theorem C. Let Γ_n be a sequence of congruence lattices in $\mathrm{PSL}_2(\mathbb{C})$. Then

$$\lim_{n\to+\infty} \frac{b_1(\Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash\mathbb{H}^3)} = 0.$$

This was proven in [25] for non-uniform lattices, and in [11] in the case of all torsion-free lattices. Our proof is very similar to the proof for hyperbolic 3–manifolds appearing in [1].

1.4. **Congruence lattices.** For completeness we give an explicit description of the congruence arithmetic latices in $G = \mathrm{PGL}(2,\mathbb{R}), \mathrm{PGL}(2,\mathbb{C})$, though we will not directly use this structure theory in the rest of the paper. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We start by choosing a number field k with Archimedean places $\nu_1, \ldots, \nu_d$ such that $k_{\nu_1} \simeq \mathbb{K}$ and $k_{\nu_i} \simeq \mathbb{R}$ for $i \geq 2$. In what follows $\mathbb{A}, \mathbb{A}_f$ stand for the ring of adèles, respectively finite adèles of k. We will write $k \ni x \mapsto (x)_\nu \in k_\nu$ for the embedding of k in its completion k_ν. Let

$a, b \in k^\times$ be such that $(a)_{\nu_i}, (b)_{\nu_i}$ are positive for $i \geq 2$ and $(a)_{\nu_1}$ or $(b)_{\nu_1}$ is negative if $\mathbb{K} \simeq \mathbb{R}$. We define the quaternion algebra A as

$$A = k + \mathbf{i}k + \mathbf{j}k + \mathbf{ij}k,$$

subject to the relations $\mathbf{i}^2 = -a, \mathbf{j}^2 = -b, \mathbf{ij} = -\mathbf{ji}$. By our choice of a, b we have $A \otimes_k k_{\nu_1} \simeq M(2, \mathbb{K})$ and for $i \geq 2$ the algebra $A \otimes_k k_{\nu_i}$ is isomorphic to the Hamilton's quaternions. We form an algebraic group $\mathrm{PA}^\times = A^\times / k^\times$. It is an adjoint simple group of type A_1 defined over k. Note that $\mathrm{PA}^\times(\mathbb{A}) = \mathrm{PA}^\times(k \otimes_{\mathbb{Q}} \mathbb{R}) \times \mathrm{PA}^\times(\mathbb{A}_f)$ and

$$\mathrm{PA}^\times(k \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{i=1}^{d} \mathrm{PA}^\times(k_{\nu_i}) \simeq \mathrm{PGL}(2, \mathbb{K}) \times \mathrm{PO}(3)^{d-1}.$$

Choose an open compact subgroup U of $\mathrm{PA}^\times(\mathbb{A}_f)$. Let $\Gamma_U = \mathrm{PA}^\times(k) \cap (\mathrm{PA}^\times(k \otimes_{\mathbb{Q}} \mathbb{R}) \times \mathrm{PA}^\times(\mathbb{A}_f))$. By a classical result of Borel-Harish-Chandra [4] the group Γ_U is a lattice in $\mathrm{PA}^\times(k \otimes_{\mathbb{Q}} \mathbb{R}) \times \mathrm{PA}^\times(\mathbb{A}_f) \simeq \mathrm{PGL}(2, \mathbb{K}) \times \mathrm{PO}(3)^{d-1} \times U$. The projection of Γ_U to the factor $\mathrm{PGL}(2, \mathbb{K})$ is a *congruence arithmetic lattice* in $\mathrm{PGL}(2, \mathbb{K})$. Every congruence arithmetic lattice of $\mathrm{PGL}(2, \mathbb{K})$ arises in this way.

1.5. **Outline of the paper.** In Section 2 we apply a "soft" criterion for Benjamini–Schramm convergence, together with the estimates from [11], to deduce Theorem A. The criterion is proven, in a general form including lattices in the isometry group of any proper Gromov-hyperbolic space, in Appendix A. Next, in section 3 we give a precise metric description of the singular locus of hyperbolic 2- and 3-orbifolds, and (in the 3-dimensional case) a way to smooth the boundary of the thick part while keeping control of the geometry (the technical details of which are left to a second Appendix B). We use this description of singularities and Theorem A to deduce Theorem B in section 4. In section 5 we use heat kernel methods (for which we need the precise description of the smoothed thick part) to deduce Theorem C from Theorem A.

2. Benjamini–Schramm convergence of quotients of hyperbolic spaces

2.1. **A criterion for convergence.** Let G be a semisimple Lie group and γ a semisimple element of G. Let G_γ be the centraliser in G of Γ, then for any sufficiently decreasing (for example compactly supported) continuous function on G the following integral makes sense.

$$\mathcal{O}_f(\gamma) = \int_{G/G_\gamma} f(\gamma^{-1} x \gamma) dx \tag{2.1}$$

The following proposition is a generalisation of [25, Proposition 2.2]. We provide a self-contained proof (along the same lines as that of loc. cit.) of a much more general result valid for all Gromov-hyperbolic spaces in Proposition A.4 below.

Proposition 2.1. *Let Γ_n be a sequence of lattices in either* $\mathrm{PGL}_2(\mathbb{R})$ *or* $\mathrm{PGL}_2(\mathbb{C})$ *and $d = 2, 3$ accordingly. Let U be the subset of loxodromic elements in G. If for every smooth compactly supported function f on G the limit*

$$\lim_{n\to+\infty} \frac{\sum_{[\gamma]_{\Gamma_n}\subset U} \mathrm{vol}((\Gamma_n)_\gamma\backslash G_\gamma)\mathcal{O}_f(\gamma)}{\mathrm{vol}(\Gamma_n\backslash G)} = 0 \tag{2.2}$$

holds, then $\Gamma_n\backslash\mathbb{H}^d$ is BS-convergent to $\mathbb{H}^d$.

This is essentially tautological if the Γ_n are torsion-free; the nontrivial part is that it allows us to avoid studying the elliptic conjugacy classes (and the parabolic classes if the Γ_n are noncompact) in order to establish BS-convergence of a sequence of orbifolds.

2.2. **Proof of Theorem A.** If X is a rank-one irreducible symmetric space such as $\mathbb{H}^2$ or $\mathbb{H}^3$ and $G = \mathrm{Isom}(X)$ then G is a simple Lie group of non-compact type and its elliptic radical is trivial. Theorem A thus follows immediately from Proposition 2.1 and the following result extracted from [11].

Theorem 2.2. *Let $G = \mathrm{PGL}_2(\mathbb{R})$ or $\mathrm{PGL}_2(\mathbb{C})$ and let U be the set of hyperbolic elements of G. Let Γ_n a sequence of arithmetic congruence lattices in G, such that $\mathrm{vol}(\Gamma_n\backslash G) \to +\infty$ or any sequence of pairwise non-commensurable arithmetic lattices. Then for any $f \in C_0^\infty(G)$ we have :*

$$\frac{1}{\mathrm{vol}(\Gamma_n\backslash G)} \left| \sum_{[\gamma]\subset U} \mathrm{vol}((\Gamma_n)_\gamma\backslash G_\gamma)\mathcal{O}_f(\gamma) \right| \xrightarrow{n\to+\infty} 0. \tag{2.3}$$

Proof. If Γ is an arithmetic lattice in $\mathrm{PGL}_2(\mathbb{R})$ or $\mathrm{PGL}_2(\mathbb{C})$ then an element $\gamma \in \Gamma$ is hyperbolic if and only if it is semisimple and of infinite order. In the proof of [11, Theorem 1.8], starting form the lines (10.7-10.9) the author bounds the sum

$$\sum_{\substack{[\gamma]_\Gamma \\ \text{non torsion}}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f) \tag{2.4}$$

for congruence arithmetic lattices. The line (10.7) of [11, p. 67] is the adèlic version of the last sum where we group together the classes conjugate over $\mathrm{PA}^\times(k)$, where $\mathrm{PA}^\times$ is the group used to construct the lattice Γ as explained in Section 1.4. The passage between the adèlic and classical trace formula is explained in [11, Theorem 4.21]. Proceeding as in [11, p. 67-69] we obtain the bound

$$\sum_{\substack{[\gamma]_\Gamma \\ \text{non torsion}}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f) \ll \mathrm{vol}(\Gamma\backslash G)^{0.986}.$$

Any hyperbolic conjugacy class $[\gamma]_\Gamma$ is non-torsion so we can deduce the that the sum (2.3) converges to 0 as $\mathrm{vol}(\Gamma\backslash X) \to \infty$ and Γ is a congruence arithmetic lattice. In order to establish the convergence for sequences of

pairwise non-commensurable arithmetic lattices $(\Gamma_n)_{n\in\mathbb{N}}$ we choose for each n a maximal arithmetic lattice Λ_n containing Γ_n. It is always a congruence arithmetic lattice. We have

$$\frac{1}{\mathrm{vol}(\Gamma_n\backslash X)}\left|\sum_{[\gamma]_{\Gamma_n}\in U}\mathrm{vol}((\Gamma_n)_\gamma\backslash G_\gamma)\mathcal{O}_f(\gamma)\right| \leq$$
$$\frac{1}{\mathrm{vol}(\Gamma_n\backslash X)}\sum_{[\gamma]_{\Gamma_n}\in U}\mathrm{vol}((\Gamma_n)_\gamma\backslash G_\gamma)\mathcal{O}_{|f|}(\gamma) \leq$$
$$\frac{1}{\mathrm{vol}(\Lambda_n\backslash X)}\sum_{[\gamma]_{\Lambda_n}\in U}\mathrm{vol}((\Lambda_n)_\gamma\backslash G_\gamma)\mathcal{O}_{|f|}(\gamma) = o(1).$$

□

3. Structure of the singular locus of closed hyperbolic orbifolds

To be able to deduce from the sole Benjamini–Schramm convergence of a sequence of orbifolds further asymptotic results on topological invariants we need a fine metric description of the singular locus. The results in this section provide it; they are not really original but precise statements such as we need are not easily found in the litterature. As usual our main tool is the Margulis lemma.

Theorem 3.1. *For every $n \geq 2$ there exists $\varepsilon = \varepsilon(n) > 0$ such that the following holds. Let Γ be a discrete subgroup of isometries of $\mathbb{H}^n$, then for any $x \in \mathbb{H}^n$ the subgroup*

$$\Gamma_\varepsilon := \langle \gamma \in \Gamma : d(x, \gamma x) \leq \varepsilon\rangle$$

is virtually abelian.

In the sequel we will only work in 2 or 3-dimensional hyperbolic space, and we let ε denote a Margulis constant which is valid for both cases. Recall that $O_{\leq\varepsilon}$ stands for the ε-thin part of an orbifold O, for which we use the following definition: if $O = \Gamma\backslash X$ where X is the orbifold universal cover and we assume X to be CAT(0) then

$$O_{\leq\varepsilon} = \Gamma\backslash\{\tilde{x} \in X : \exists\gamma \in \Gamma \setminus \{\mathrm{Id}\},\, d(\tilde{x}, \gamma\tilde{x}) \leq \varepsilon\} \tag{3.1}$$

which includes the singular locus of O—note that in the litterature, e.g. in [3], a different convention is often used where only points with large stabilisers are included. The closure of the complement of $O_{\leq\varepsilon}$ (the ε-thick part) will be denoted by $O_{\geq\varepsilon}$.

In fact we need to tweak a bit the definition of the thin part around that part of the singular locus where the cone angle is π: around these vertices or geodesics we put a collar whose width is $\varepsilon/6$ (instead of $\varepsilon/2$).

3.1. **2-dimensional orbifolds.** In $\mathrm{PGL}_2(\mathbb{R})^+$ all the virtually abelian discrete subgroups are given by the following list:

(1) An infinite cyclic group generated by an hyperbolic or parabolic isometry;
(2) A finite cyclic group generated by an elliptic isometry;
(3) An infinite dihedral group generated by two elliptic isometries of order 2.

As a first consequence we see that the singular locus of an orientable hyperbolic 2-orbifold consists only of *cone points*, that is all non-manifold points have a neighbourhood which is isometric to the quotient of a disc by a finite cyclic group.

In addition we can deduce from this classification a metric description of the singular locus. We need the following notation: given an elliptic isometry γ with fixed point x and rotation angle θ, let $\ell(\theta, \varepsilon)$ be the smallest ℓ such that $d(y, \gamma y) \geq \varepsilon$ for $d(x, y) = \ell$. Similarly, given a hyperbolic isometry γ of minimal displacement ℓ we define $r(\ell, \varepsilon)$ to be the minimal distance from its axis at which an hyperbolic isometry translates of at least ε.

Lemma 3.2. *Let $O = \Gamma\backslash\mathbb{H}^2$ be an orientable hyperbolic 2-orbifold and x a point in its singular locus. Then x is an isolated cone point and one of the following possibilities hold:*

(1) *If its angle is $2\pi/m$ with $m \geq 3$ then there is no other singular point in the ball $B_O(x, \ell)$ where $\ell = \ell(2\pi/m, \varepsilon)$.*
(2) *If the angle is equal to π then either there is no other singular point within distance $\ell(\pi, \varepsilon)$, or there is one (and its cone angle is also π) at distance $\ell_x < \ell(\pi, \varepsilon)$ but no other within distance $r(\ell_x, \varepsilon)$ of x.*

Proof. Let $\Gamma x \in O$ be as in the statement, with $x \in \mathbb{H}^2$. Then x is a fixed point of a non-trivial element of Γ, and it follows that the subgroup

$$\Gamma_x^\varepsilon = \{\gamma \in \Gamma : \ d(x, \gamma x) \leq \varepsilon\}$$

must be one of those described in (2) or (3) at the beginning of this section; let γ_0 be a generator (with minimal rotation angle) of the cyclic subgroup fixing x and $m > 1$ its order.

In any case x lies above a conical point in O. Assume now that $m \geq 3$; then $\Gamma_x = \langle \gamma_0 \rangle$ and by the Margulis lemma there is no other fixed point of a non-trivial element in Γ within the set

$$C = \{y \in \mathbb{H}^2 : \ d(y, \gamma_0 y) \leq \varepsilon)\}.$$

By definition the ball $B_{\mathbb{H}^2}(x, \ell(2\pi/m, \varepsilon))$ is contained in C, so it contains no other singular point.

If $m = 2$ and there is another elliptic fixed point $x' \in \mathbb{H}^2$ with $d(x, x') \leq \ell(\pi, \varepsilon)$ then we might assume that x' is the closest such point. By the previous paragraph any nontrivial $\gamma_0' \in \Gamma$ fixing x' must be of order 2. Let $\eta = \gamma_0\gamma_0'$. It is a hyperbolic isometry with axis containing the geodesic α joining x to x' and translation distance $2d(x, x')$. Write Γ_α for the setwise

stabilizer of α in Γ. For every $\gamma \in \Gamma_\alpha$ not fixing x we will have $d(x, \gamma x) \geq 2d(x, x')$ as otherwise $\gamma_0\gamma$ would have a fixed point closer to x than x'. We deduce that $\Gamma_\alpha = \langle \gamma_0, \gamma_0' \rangle$. The former is a maximal virtually abelian subgroup of Γ (it is an intersection of Γ with the normaliser of a split torus). The Margulis lemma now implies that within the ball $B_{\mathbb{H}^2}(x, \ell(\pi, \varepsilon))$ (resp. $B_{\mathbb{H}^2}(x, r(\ell_x, \varepsilon))$) any other elliptic fixed point must be a translate of either x or x' by a power of η, as any such point is moved by at most ε by γ_0 (resp. η) and hence its stabiliser in Γ must belong to Γ_α. □

3.2. **3-dimensional orbifolds.**

3.2.1. *Description of the singular locus.* The list of discrete virtually abelian subgroups of $\mathrm{PGL}_2(\mathbb{C})$ is long enough to make us avoid giving a complete description. Rather, we will assume that Γ is a cocompact lattice in $\mathrm{PGL}_2(\mathbb{C})$ and Λ a maximal virtually abelian subgroup of Γ which contains torsion elements (which is all we need to prove Theorem C). If Λ contains a hyperbolic element γ then it must normalise $\langle \gamma \rangle$, so it is contained in the normalizer of a maximal torus. Any such normalizer is isomorphic to $\mathbb{C}^\times \rtimes \mathbb{Z}/2$. Otherwise Λ contains only elements of finite order and so by Burnside's theorem it must be a finite subgroup of the maximal compact $\mathrm{PU}(2)$. It follows that Λ is one of the following groups:

(1) $\langle \gamma, \eta \rangle \cong \mathbb{Z} \times \mathbb{Z}/m$ where γ, η are respectively hyperbolic and elliptic isometries sharing the same axis;
(2) $\langle \gamma, \eta, \rho \rangle \cong (\mathbb{Z} \times \mathbb{Z}/m) \rtimes \mathbb{Z}/2$ where η, γ are as above (with η possibly trivial) and ρ is an elliptic of order 2 with axis orthogonal to that of γ or η;
(3) One of the finitely many non-dihedral finite subgroups of $\mathrm{PU}(2)$.

We see from this description that the singular locus of an hyperbolic 3–orbifold consists of closed geodesics (which we will call *singular geodesics*), which can intersect each other. A singular point not on the intersection of two singular geodesics has a neighbourhood isometric to the quotient of a ball by a rotation; the angle of the latter we will call the cone angle of the singular geodesic. We will call a vertex which is at the intersection of two or more singular geodesics a *vertex* of the singular locus.

Together with the Margulis lemma the list above allows us to give the following metric description of the singular locus (see also [3, Corollary 6.3] for a more geometric description, and loc. cit., Fig. 5 on p. 33 for illustrations). This description is analogous to the situation from Lemma 3.2; we recall that ℓ and r were defined there.

Lemma 3.3. *Let O be a compact orientable 3-dimensional hyperbolic orbifold and Σ its singular locus. Let $x \in \Sigma$ be a vertex. Then one of the two following possibilities hold.*

(1) *The $\varepsilon/2$-neighbourhood of x is isometric to one of a finite list of orbifolds, whose singular locus has only one vertex and all singular geodesics go through x.*

(2) *There is at most one other singular vertex x' within distance $\varepsilon/2$ of x; x and x' are joined by a singular geodesic c of length ℓ and cone angle $2\pi/m$, there are two singular geodesics with cone angle π and orthogonal to c each going through one of x or x'. There are no further components of the singular locus within distance* $\max(\ell(2\pi/m,\varepsilon), r(\ell,\varepsilon))$ *of x and x'.*

Moreover if two non-intersecting singular geodesics of O are within distance $\varepsilon/2$ of each other then both have angle π.

Proof. Let $O = \Gamma\backslash\mathbb{H}^3$ a closed hyperbolic 3–orbifold. Let x be a vertex in the singular locus of O and Π the subgroup of Γ fixing a lift $\tilde{x}$ of x to $\mathbb{H}^3$. Then Π is either a dihedral group $\mathbb{Z}/m \rtimes \mathbb{Z}/2$ or one of finitely many finite non-dihedral subgroups of $\mathrm{PU}(2)$, according to the list of virtually abelian subgroups of Γ above.

If the vertex is as in (1) and $\eta \in \Gamma$, $\eta \notin \Pi$ is an elliptic isometry of order m then as (by the Margulis Lemma) Π contains all isometries moving $\tilde{x}$ by at most ε any fixed point of η must be at distance at least $\ell(2\pi/m,\varepsilon) \geq \ell(\pi,\varepsilon) = \varepsilon/2$ of $\tilde{x}$. Similarly any hyperbolic isometry in Γ must move $\tilde{x}$ by at least ε. Hence the quotient $\Pi\backslash B(\tilde{x},\varepsilon/2)$ embeds into O.

If the vertex has a dihedral stabiliser as in (2) let η be a generator of the $\mathbb{Z}/m$-subgroup and γ a generator of the $\mathbb{Z}$-subgroup commuting with η. We might assume that either $\ell < \varepsilon/2$ or $m > 5$ (otherwise we can add its neighbourhood to the finite list in (1)). Then any elliptic element of Γ which does not normalise $\langle\eta\rangle$ cannot fix a point in $B(\tilde{x},\varepsilon)$ (otherwise it and η would generate a subgroup moving a point by less than ε but not in the list given above, which is not possible by the Margulis Lemma). Similarly it cannot fix a point within $r(\ell,\varepsilon)$ of the axis of γ. □

3.2.2. *Smoothing the thick part.* Let $C = (C_0, C_1, \ldots) \in [0,+\infty[^{\mathbb{N}}$. As (a slight variation of) the definition in [20] we say that a Riemannian manifold has *C-bounded geometry* if its injectivity radius is at least C_0, the normal geodesic flow up to C_0 gives coordinates for a collar neighbourhood of the boundary, and the kth derivatives of the metric tensor and its inverse (in normal coordinates) are bounded in sup norm by C_k. In this section we prove the following lemma.

Lemma 3.4. *There exists C such that for any hyperbolic 3–orbifold O there exists a smooth submanifold O' such that:*

- *$O_{\geq\varepsilon} \subset O'$ and this is an homotopy equivalence;*
- *O' is of C-bounded geometry.*

We will deduce the lemma from the description of the singular locus and the following general proposition, the proof of which we give in appendix B.

Proposition 3.5. *Let X be a Riemannian d-manifold and H_1, H_2 two open subsets whose closures have smooth boundary. Assume the following holds:*

- *they intersect transversally in a compact subset; let α_0 such that the dihedral angles at the intersection stay within the interval $]\alpha_0, \pi-\alpha_0[$.*
- *Both manifolds $X \setminus H_i$ are of bounded geometry.*

Then for any $\delta > 0$ there exists an open subset H of X such that:

(1) *$H \supset H_1 \cup H_2$ and they are equal outside of the δ-neighbourhood of $H_1 \cap H_2$;*
(2) *the closure of H has a smooth boundary;*
(3) *$X \setminus H$ is of bounded geometry; the bounds depend only on δ, on the bounds on the geometry of X and $X \setminus H_i$ and on α_0.*

Proof of Lemma 3.4. Observe first that the boundary of the thin part is smooth away from the geodesics with cone angle π and the vertices of the singular locus, as follows from the third part of Lemma 3.3. Thus the non-smooth part of $\partial O_{\geq\varepsilon}$ comes from intersecting tubular neighbourhoods of singular geodesics and short geodesics. There are finitely many possible configurations where the geodesics are not orthogonal to each other (corresponding to case (1) of Lemma 3.3); we do not need to deal in detail with these, so the only problem left to deal with is the following: at all points in the intersection of the tubular neighbourhood N_1 (with varying radius) of one geodesic, and the $\varepsilon/6$-tubular neigbourhood N_2 of another geodesic orthogonal to the first, the dihedral angle between ∂N_1 and ∂N_2 stays bounded away from 0 and from π[1].

To prove this note that the maximum and minimum values for these angles both are continuous functions of the radius $0 \leq r < +\infty$ of N_1. It can be continuously extended to $r = +\infty$, the values then being those of the angle (in a conformal model of $\mathbb{H}^3$) between ∂N_1 and the boundary at infinity of $\mathbb{H}^3$. As N_1 and N_2 are never tangent to each other we see by compactness that the maximal and minimal values stay bounded away from 0 and π. □

4. The genus of congruence orbifolds

In this section we prove Theorem B. Let O be an hyperbolic orbifold of dimension 2, which is a quotient of the hyperbolic plane $\mathbb{H}^2$ by a lattice of $\mathrm{PSL}_2(\mathbb{R})$. Then the underlying topological space $|O|$ is a surface of finite type, that is it is homeomorphic to a compact surface S with a finite number of points removed. The *genus* of O is defined to be the genus of S.

Suppose that O has genus g, that it has k punctures and r conical singularities with angles $2\pi/m_1, \ldots, 2\pi/m_r$ (the tuple $(g, k, m_1, \ldots, m_r)$ is then called the *signature* of O). Then, computing the volume of a well-chosen fundamental polygon we get the following equality (see [2, Theorem 10.4.2]):

$$\operatorname{vol} O = 2\pi \left(2g - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{4.1}$$

[1]Note that the neighbourhoods corresponding to two geodesics orthogonal to a third one cannot intersect each other, because we took their radius to be $\varepsilon/3$ and the distance between the geodesics outside the ε-thin part is at least $\varepsilon/2$

From this equation we obtain the bound:

$$\left| g - \frac{\mathrm{vol}(O)}{4\pi} \right| \leq \frac{k+r+2}{4\pi}.$$

We now see that Theorem B follows from Theorem A together with the following proposition.

Proposition 4.1. *Let O_n be a sequence of hyperbolic 2–orbifolds which is Benjamini–Schramm convergent to $\mathbb{H}^2$. Let k_n, r_n be the numbers of cusps and conical points of O_n, respectively. Then $k_n + r_n = o(\mathrm{vol}\, O_n)$.*

Proof. To prove that $r_n = o(\mathrm{vol}\, O_n)$ we associate to each conical point x with angle θ the region

$$\Omega_x = B(x, \ell(\theta, \varepsilon))$$

if there is no other singular point within distance $\ell(\theta, \varepsilon)$. Otherwise let ℓ_x be the distance to the nearest singular point and put

$$\Omega_x = B(x, r(\ell_x, \varepsilon)).$$

We will check below the following facts:

(1) there exists $c > 0$ such that $\mathrm{vol}\, \Omega_x > c$ for all n and $x \in O_n$;
(2) Any point $p \in O_n$ is covered by at most two distinct sets Ω_x;
(3) for all conical points $x \in O_n$ we have $\Omega_x \subset (O_n)_{\leq \varepsilon}$.

It follows from these that:

$$r_n \leq \frac{1}{c} \sum_{x \in \Sigma_{O_n}} \mathrm{vol}\, \Omega_x \leq \frac{2}{c} \mathrm{vol} \left(\bigcup_{x \in \Sigma_{O_n}} \Omega_x \right) \leq \frac{2}{c} \mathrm{vol}(O_n)_{\leq \varepsilon}$$

and as the right-hand side is $o(\mathrm{vol}\, O_n)$ in a BS-convergent sequence we get that $r_n = o(\mathrm{vol}\, O_n)$.

That (3) holds follows immediately from the definitions of $\ell(\theta, \varepsilon)$ and $r(\ell, \varepsilon)$. Point (2) follows from Margulis lemma combined with Lemma 3.2.

It remains to prove 1. Let $x \in O_n$ be a singularity with cone angle $2\pi/m$ with $m > 2$, let $\tilde{x}$ be a lift of x to $\mathbb{H}^2$ and $\ell = \ell(2\pi/m, \varepsilon)$. Then we have

$$\mathrm{vol}(B_{O_n}(x, \ell)) = \frac{1}{m} B_{\mathbb{H}^2}(\tilde{x}, \ell) \gg \frac{e^\ell}{m}$$

so we need to prove that $e^\ell \gg m$. This follows easily from distance computations in the disk model: by definition of $\ell(\theta, \varepsilon)$ we have that $\ell(\theta, \varepsilon) = \log((1 + r)/(1 - r))$ where $0 < r < 1$ is such that $d(r, re^{i\theta}) = \varepsilon$. It follows that

$$\cosh(\varepsilon) = 1 + \frac{2r^2 |1 - e^{i\theta}|^2}{(1 - r^2)^2}$$

and by standard computations we get that

$$r = 1 - \frac{\theta}{\sqrt{2} \sinh(\varepsilon)} + O(\theta^2)$$

whence it follows that

$$\ell(\theta,\varepsilon) = -\log(\theta) - c + O(\theta)$$

for some constant c depending on ε. We finally get that $\ell \gg e^{\log(m/2\pi)} \gg m$.

Assume now that $m = 2$ and that there is another singular point x' within $\ell(2,\varepsilon)$ of x. In this case the volume of Ω_x is half that of a collar around a closed geodesic of length $r(\ell_x,\varepsilon) \ll \varepsilon$; as the latter is bounded from below (see [14]) so is that of Ω_x.

The proof that $k_n = o(\operatorname{vol} O_n)$ is similar: by the Margulis lemma the regions of the ε-thin part where a given conjugacy class of parabolic isometries realises the injectivity radius are pairwise disjoint, and an easy hyperbolic area computation shows that the volume of such a region is bounded below. □

5. Betti numbers of arithmetic 3–orbifolds

Recall that ε is the Margulis constant for $\mathbb{H}^3$. Let O be a 3–orbifold, then we will write O' for the manifold with boundary obtained by Lemma 3.4. We write Δ^1_{abs} for the maximal self-adjoint extension of the Hodge–Laplace operator on O' with absolute boundary condition. The goal of this section is to prove the following proposition, which we do by extending the analysis at the end of section 7 in [1] to the orbifold case.

Proposition 5.1. *Let O_n be a sequence of closed hyperbolic 3–orbifolds which BS-converge to $\mathbb{H}^3$, and let O'_n be the smoothings described in Lemma 3.4. Then for all $t > 0$ we have that*

$$\limsup_{t\to+\infty} \lim_{n\to+\infty} \frac{\operatorname{Tr}(e^{-t\Delta^1_{\mathrm{abs}}[O'_n]})}{\operatorname{vol} O_n} = 0.$$

Before giving the proof we explain how this implies Theorem C: let $O_n = \Gamma_n\backslash\mathbb{H}^3$. By Hodge theory we have $b_1(O'_n) \leq \operatorname{Tr}(e^{-t\Delta^1_{\mathrm{abs}}[O'_n]})$ for all t, and so Proposition 5.1 implies that

$$\lim_{n\to+\infty} \frac{b_1(O'_n)}{\operatorname{vol} O_n} = 0.$$

On the other hand we have that the orbifold fundamental group Γ_n is a quotient of $\pi_1(O'_n)$. Indeed, the universal cover of $(O_n)_{\geq\varepsilon}$ is a cover of the connected subset $(\widehat{O}_n)_{\geq\varepsilon}$ of those $x \in \mathbb{H}^3$ which are not displaced by less than ε by some non-trivial element of Γ_n, and $(O_n)_{\geq\varepsilon}$ is homotopy equivalent to O'_n. Moreover $H_1(O'_n)$ is the abelianisation of $\pi_1(O'_n)$. From these two facts it follows that $b_1(\Gamma_n) \leq b_1(O'_n)$, so that $b_1(\Gamma_n) = o(\operatorname{vol} O_n)$ as well.

The proof of Proposition 5.1 is done in four steps: first we prove an analogue of Proposition 4.1 and then deduce the convergence of the part of the trace formula for O_n coming from the ε-thick part: see (5.1). The two next steps together imply that the trace of the heat kernel on O'_n is asymptotically the same as that computed in (5.1): first we analyse the

integral of the difference on the R-thick part and show that it limit superior is $o(R)$ (see (5.6), then we prove that the integral on the R-thin part of O'_n asymptotically vanishes (see (5.7)). Altogether these three steps imply that

$$\lim_{n\to+\infty} \frac{\mathrm{Tr}(e^{-t\Delta^1_{\mathrm{abs}}[O'_n]})}{\mathrm{vol}\, O_n} = \mathrm{tr}\, e^{-t\Delta^1[\mathbb{H}^3]}$$

where we denoted $\mathrm{tr}\, e^{-t\Delta^1[\mathbb{H}^3]} = \mathrm{tr}\, e^{-t\Delta^1[\mathbb{H}^3]}(\tilde{x},\tilde{x})$ for any $\tilde{x} \in \mathbb{H}^3$. The proposition now follows from the vanishing of the first L^2-Betti number of $\mathbb{H}^3$, which means that $\lim_{t\to+\infty} \mathrm{tr}\, e^{-t\Delta^1[\mathbb{H}^3]} = 0$ (see [21]).

5.1. **Upper bound on the total length of singular geodesics.** Let Σ_n be the set of singular geodesics of O_n. To prove Proposition 5.1 we will need to control the total length $\sum_{c\in\Sigma_n} \ell_n$ in terms of the volume of the thin part of O_n. This is problematic for 3-orbifolds because of an issue with singular geodesics corresponding to order-2 elements. For these geodesics we will need to replace the lengths in the sum by another quantity. To make it precise let us introduce some notations.

Let O by a finite volume hyperbolic 3-orbifold and let Σ be the set of singular geodesics on O. For $c \in \Sigma$ we will write $\tilde{c}$ for a lift of c to $\mathbb{H}^3$; in our arguments below we will clarify the choice of $\tilde{c}$ whenever it matters. Let Γ be the orbifold fundamental group of O. Let $c \in \Sigma$ and write $\Gamma_{\tilde{c}}$ for the pointwise stabilizer of its lift $\tilde{c}$. Then $\Gamma_{\tilde{c}}$ is a lattice inside a maximal torus of $\mathrm{PGL}(2,\mathbb{C})$, so is it is of the form $\mathbb{Z}\times\mathbb{Z}/m_c$ for an integer $m_c \geq 2$. We write ℓ_c for the length of c.

Let M be the maximal order of a finite non-dihedral subgroup of $\mathrm{PU}(2)$. The relevance of M to the arguments below comes from the fact that finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$ either stabilize a geodesic in $\mathbb{H}^3$ or are conjugate to a non-dihedral subgroup of $\mathrm{PU}(2)$. Accordingly, we divide Σ into three sets $\Sigma^1, \Sigma^2, \Sigma^3$ defined as follows :

$$\begin{aligned}
\Sigma^1 &= \{c \in \Sigma | m_c = 2\},\\
\Sigma^2 &= \{c \in \Sigma | 2 < m_c \leq M\},\\
\Sigma^3 &= \{c \in \Sigma | M < m_c\}.
\end{aligned}$$

The sets do not depend on the choice of $\tilde{c}$. Let $c \in \Sigma^1$. A point $p \in c$ will be called a **type I vertex** if there exists a closed geodesic $a \notin \Sigma^3$ on O (not necessarily singular) such that $p \in c \cap a$ and $\ell_a \leq \varepsilon$. A point $p \in c$ is a **type II vertex** if there exists $b \in \Sigma^3$ such that $p \in c \cap b$. Write $T^I(c), T^{II}(c)$ for the sets of type I and type II vertices. For $p \in T^I(c), T^{II}(c)$ we let $r_p := \max\{r(\ell_a,\varepsilon), \ell(2\pi/m_a,\varepsilon)\}, \max\{r(\ell_b,\varepsilon), \ell(2\pi/m_b,\varepsilon)\}$ respectively. Define $\ell'_c := \ell_c - \sum_{p\in T^{II}(c)} 2r_p$.

Proposition 5.2. *For any hyperbolic 3-orbifold we have*

$$\sum_{c\in\Sigma\setminus\Sigma^1} \ell_c + \sum_{c\in\Sigma^1} \left(\ell'_c + |T^{II}(c)|\right) \ll \mathrm{vol}(O_{\leq\varepsilon}).$$

Proof. As in the proof of Proposition 4.1 we will construct sets $\Omega_c, \Omega_p^{II} \subset O$ attached to each singular geodesic $c \in \Sigma$ and to $p \in T^{II}(c)$ for $c \in \Sigma^1$ satisfying the following properties

(1) for $c \in \Sigma \setminus \Sigma^1$ we have $\mathrm{vol}(\Omega_c) \gg \ell_c$; for $c \in \Sigma^1$ we have $\mathrm{vol}(\Omega_c) \gg \ell'_c$ and for $p \in T^{II}(c)$ $\mathrm{vol}(\Omega_p^{II}) \gg 1$.
(2) any point $x \in O$ is covered by at most M distinct sets Ω_c, Ω_p^{II}.
(3) $\Omega_c, \Omega_p^{II} \subset O_{\leq \varepsilon}$.

If $A \subset \mathbb{H}^3$ write $[A]$ for the image of A in O under the covering map. The subset Σ^1 is the most problematic so let us first define the sets Ω_c for $c \in \Sigma^2, \Sigma^3$.

- For $c \in \Sigma^3$ let $\Omega_c := [B_{\mathbb{H}^3}(\tilde{c}, \ell(2\pi/m_c, \varepsilon))]$.
- For $c \in \Sigma^2$ let $\Omega_c := [B_{\mathbb{H}^3}(\tilde{c}, \varepsilon/2)]$.

Now let $c \in \Sigma^1$. We construct sets $\Omega_p^I, \Omega_p^{II}$ for $p \in T^I(c), T^{II}(c)$ respectively.

- $\Omega_p^I = [B_{\mathbb{H}^3}(\tilde{a}, r(\ell_a, \varepsilon))]$.
- $\Omega_p^{II} = [B_{\mathbb{H}^3}(\tilde{b}, r_p)]$ (recall that $\max\{r(\ell_b, \varepsilon), \ell(2\pi/m_b, \varepsilon)\}$).

Margulis lemma and the description of nilpotent subgroups from Section 3.2.1 imply that $\Omega_p^I, \Omega_q^{II}$ are pairwise disjoint if $p \in T^I(c), q \in T^{II}(c)$. We define

$$\Omega_c := [B_{\mathbb{H}^3}(\tilde{c}, \varepsilon/2)] \cup \bigcup_{p \in T^I(c)} \Omega_p^I \setminus \bigcup_{p \in T^{II}(c)} \Omega_p^{II}.$$

5.1.1. *Step 1.* We verify the condition (1). Recall that in the proof of Proposition 4.1 we showed that $e^{\ell(2\pi/m, \varepsilon)} \gg m$. For $c \in \Sigma^3$ the formula for integration in cylindrical coordinates [10, p. 205] yields

$$\mathrm{vol}(\Gamma_{\tilde{c}} \backslash B_{\mathbb{H}^3}(\tilde{c}, \ell(2\pi/m_c, \varepsilon)) \gg e^{2\ell(2\pi/m_c, \varepsilon)} \ell_c m_c^{-1} \gg \ell_c.$$

Using Margulis lemma and the description of nilpotent subgroups from Section 3.2.1 we can show that the map

$$\Gamma_{\tilde{c}} \backslash B_{\mathbb{H}^3}(\tilde{c}, \ell(2\pi/m_c, \varepsilon)) \to [\Gamma_{\tilde{c}} \backslash B_{\mathbb{H}^3}(\tilde{c}, \ell(2\pi/m_c, \varepsilon))] = \Omega_c$$

is at most 2-to-1, so $\mathrm{vol}(\Omega_c) \gg \ell_c$.

For $c \in \Sigma^2$ we similarly get

$$\mathrm{vol}(\Gamma_{\tilde{c}} \backslash B_{\mathbb{H}^3}(\tilde{c}, \ell(2\pi/m_c, \varepsilon)) \gg \ell_c.$$

By Lemma 3.3 and Margulis lemma the map

$$\Gamma_{\tilde{c}} \backslash B_{\mathbb{H}^3}(\tilde{c}, \varepsilon/2) \to [\Gamma_{\tilde{c}} \backslash B_{\mathbb{H}^3}(\tilde{c}, \varepsilon/2)]$$

is at most M-to-one. Hence $\mathrm{vol}(\Omega_c) \gg \ell_c$.

Now let $c \in \Sigma^1$. Since the sets $\Omega_p^I, \Omega_q^{II}$ for $p \in T^I(c), q \in T^{II}(c)$ are pairwise disjoint we can write

$$\ell_c = \ell'_c + \sum_{p \in T^{II}(c)} 2r_p \text{ and } \ell'_c = \ell''_c + \sum_{p \in T^I(c)} 2r_p$$

where $\ell''_c \geq 0$. Let $p \in T^I(c)$. Let γ be an element of $\Gamma_{\tilde{a}}$ translating $\tilde{a}$ by ℓ_a. Integration in cylindrical coordinates yields

$$\mathrm{vol}(\langle\gamma\rangle\backslash B_{\mathbb{H}^3}(\tilde{a}, r_p) \gg \ell_a^{-1} \gg r_p.$$

Note that we implicitly used here the fact that m_a is bounded. The Margulis lemma implies that the quotient map from the last set to $[B_{\mathbb{H}^3}(\tilde{a}, r(\ell_a, \varepsilon))]$ is at most M-to-1 so we deduce $\mathrm{vol}(\Omega_p^I) \gg r_p$. Reasoning as in the previous cases we get $\mathrm{vol}(\Omega_c) \gg \ell''_c + \sum_{p\in T^I(c)} r_p \gg \ell'_c$.

Finally let $p \in T^{II}(c)$. Integrating in cylindrical coordinates we get

$$\mathrm{vol}(\Gamma_b\backslash B_{\mathbb{H}^3}(\tilde{b}, \max\{r(\ell_b, \varepsilon), \ell(2\pi/m_b, \varepsilon)) \gg \frac{\ell_b}{m_b}\max\{m_b^2, \ell_b^{-2}\} \gg 1.$$

As before we deduce $\mathrm{vol}(\Omega_p^{II}) \gg 1$. This concludes the first step.

5.1.2. *Step 2.* We verify the condition (2). For $c \in \Sigma^3$ the sets Ω_c are pairwise disjoint. Indeed let $c_1, c_2 \in \Sigma^3$ and assume $\Omega_{c_1} \cap \Omega_{c_2} \neq \emptyset$. By Margulis lemma, for some lifts $\tilde{c}_1, \tilde{c}_2$ the torsion parts of the stabilizers $\Gamma_{\tilde{c}_1}, \Gamma_{\tilde{c}_2}$ generate a nilpotent subgroup. By discussion in Section 3.2.1 it is either contained in a normalizer of geodesic or in a finite non-dihedral subgroup of PU(2), and the definition of Σ^3 excludes the second option so $\Gamma_{\tilde{c}_1}, \Gamma_{\tilde{c}_2}$ both normalize the same geodesic. This can happen only if $\tilde{c}_1 = \tilde{c}_2$.

A similar argument shows that for $c_1 \in \Sigma^2, c_2 \in \Sigma^3$ the sets $\Omega_{c_1}, \Omega_{c_2}$ are disjoint.

By Lemma 3.3 and Margulis lemma the sets Ω_p^{II} are pairwise disjoint or equal. It is not hard to verify that we can have at most two different $p \in T^{II}(c), p' \in T^{II}(c')$ such that $\Omega_p^{II} = \Omega_{p'}^{II}$. By construction Ω_p^{II} contains exactly one set of form Ω_c with $c \in \Sigma^3$. By Lemma 3.3 together with Margulis lemma Ω_p^{II} are disjoint from Ω_c if $c \in \Sigma^1, \Sigma^2$. Again by Margulis lemma and Lemma 3.3, every point $x \in O$ can be covered by at most M sets Ω_c with $c \in \Sigma^1, \Sigma^2$. We conclude that any point is covered by at most M distinct sets of form $\Omega_c, c \in \Sigma$ and $\Omega_p^{II}, p \in T^{II}(c), c \in \Sigma^3$.

5.1.3. *Last Step.* Property (3) holds by construction. We get

$$\sum_{c\in\Sigma}\mathrm{vol}(\Omega_c) + \sum_{c\in\Sigma^3}\sum_{p\in T^{II}(c)}\mathrm{vol}(\Omega_p^{II}) \ll M\,\mathrm{vol}(O_{\leq\varepsilon}).$$

By the first step we conclude that

$$\sum_{c\in\Sigma\backslash\Sigma^1}\ell_c + \sum_{c\in\Sigma^1}\left(\ell'_c + |T^{II}(c)|\right) \ll \mathrm{vol}(O_{\leq\varepsilon}).$$

□

5.2. **Trace formula on the thick part.** Let O_n be a sequence as in Proposition 5.1. We prove here that

$$\int_{(O_n)_{\geq\varepsilon}} \operatorname{tr} e^{-t\Delta^1[O_n]}(x,x)dx - \operatorname{tr} e^{-t\Delta^1[\mathbb{H}^3]} \cdot \operatorname{vol} O_n = o(\operatorname{vol} O_n). \tag{5.1}$$

Let $\mathcal{C}_{n,e}$ and $\mathcal{C}_{n,h}$ be the sets of conjugacy classes of respectively elliptic and hyperbolic elements in Γ_n. For $\gamma \in \Gamma$ let $\mathcal{F}_\gamma$ be a fundamental domain for the centraliser Γ_γ of γ in Γ and $\mathcal{F}_\gamma^{\geq\varepsilon}$ the part of it on which the non-trivial elements of Γ displace by at least ε. The proof of the Selberg trace formula then gives that

$$\begin{aligned}\int_{(O_n)_{\geq\varepsilon}} \operatorname{tr} e^{-t\Delta^1[O_n]}(x,x)dx &= \operatorname{vol}(O_n)_{\geq\varepsilon} \operatorname{tr} e^{-t\Delta^1[\mathbb{H}^3]} \\ &\quad + \sum_{[\gamma]\in\mathcal{C}_{n,e}\cup\mathcal{C}_{n,h}} \int_{\mathcal{F}_\gamma^{\geq\varepsilon}} \operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx.\end{aligned} \tag{5.2}$$

Because of Benjamini–Schramm convergence we have $\operatorname{vol} O_n - \operatorname{vol}(O_n)_{\geq\varepsilon} = o(\operatorname{vol} O_n)$. Then (5.1) will follow from (5.2) together with the following limit:

$$\sum_{[\gamma]\in\mathcal{C}_{n,e}\cup\mathcal{C}_{n,h}} \int_{\mathcal{F}_\gamma^{\geq\varepsilon}} \operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx = o(\operatorname{vol} O_n). \tag{5.3}$$

We proceed to prove (5.3). The proof for the hyperbolic part is exactly the same as in [1, Section 7].

We deal now with the elliptic part; similar computations are done in [9, pp. 193 and following]. To simplify the computations we integrate over a subset $\mathcal{E}_\gamma^{\geq\varepsilon}$ of $\mathcal{F}_\gamma$ which is slightly larger than $\mathcal{F}_\gamma^{\geq\varepsilon}$.

If $[\gamma]$ is an elliptic conjugacy class let c be the singular geodesic on O_n corresponding to γ and $\tilde{c}$ the lift of c to $\mathbb{H}^3$ which is fixed by γ. Let ℓ_c be the length of c and m_c the order of the torsion subgroup of Γ_γ. If $m_c > 2$ we put

$$\mathcal{E}_\gamma^{\geq\varepsilon} = \mathcal{F}_\gamma \setminus B_{\mathbb{H}^3}(\tilde{c}, \max\{r(\ell_c,\varepsilon), \ell(2\pi/m_c,\varepsilon)\}).$$

The definition for γ with $m_c = 2$ is bit more involved. Recall from Proposition 5.2 that we call a point $p \in c$ a type II vertex if there exists a singular geodesic b in O_n such that $p \in c \cap b$ and the torsion part of $\Gamma_{\tilde{b}}$ is of order at least M (a constant defined there). Write $T^{II}(c)$ for the set of type II vertices on c. For each point $p \in T^{II}(c)$ the geodesic b is unique so the values ℓ_b, m_b are well defined. To shorten notation we will write $r_c := \max\{r(\ell_c,\varepsilon), \ell(2\pi/m_c)\}$ and $r_p := \max\{r(\ell_b,\varepsilon), \ell(2\pi/m_b)\}$. Let $T^{II}(\tilde{c}) \subset \tilde{c}$ be the set of lifts of $p \in T^{II}(c)$. Set $T^{II}(\tilde{c})$ is Γ_γ invariant. Define

$$\mathcal{E}_\gamma^{\geq\varepsilon} := \mathcal{F}_\gamma \setminus \left(B_{\mathbb{H}^3}(\tilde{c}, r_c) \cup \bigcup_{\tilde{p}\in T^{II}(\tilde{c})} B_{\mathbb{H}}(\tilde{p}, r_p) \right).$$

We are ready to bound the integrals in (5.3) corresponding to the elliptic elements. For γ with $m_c > 2$ we have

$$e \cdot \int_{\mathcal{E}_\gamma^{\geq\varepsilon}} \operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx = \frac{2\pi}{m_c}\ell_c \int_{\max(\ell(2\pi/m_c,\varepsilon),r(\ell_\gamma,\varepsilon))}^{+\infty} f_\theta(r)dr$$

where $f_\theta(r) = \sinh(r)\cosh(r)\operatorname{tr}(\gamma^* e^{-t\Delta[\mathbb{H}^3]}(x,\gamma x))$ for a point x at distance r from the axis, and $e = 1$ or $1/2$ according to whether $\Gamma_\gamma \cong \mathbb{Z}\times\mathbb{Z}/m$ or $(\mathbb{Z}\times\mathbb{Z}/m)\rtimes\mathbb{Z}/2$ (see 3.2.1 for the geometric significance of this). This is a consequence of desintegration of hyperbolic volume in cylindrical coordinates [10, p. 205]. By the Gaussian estimate of the heat kernel of $\mathbb{H}^3$ (which can be seen from its explicit expression; see [27, Proposition 2.2 on P. 425] for a more general statement) we have that

$$f_{2\pi/m_c}(r) \ll C(t)e^{-c(t)r^2}$$

uniformly for $r \geq \ell(2\pi/m_c,\varepsilon)$. We get

$$\int_{\mathcal{E}_\gamma^{\geq\varepsilon}} \operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx \ll \frac{\ell_c}{m_c}. \tag{5.4}$$

Now let γ be an elliptic element of order 2. The singular geodesic c can be identified with its lift to $\mathcal{F}_\gamma$. Let $\operatorname{pr}\colon \mathcal{F}_\gamma \to c$ be the "closest point projection" to c. By triangle inequality, for every point $y \in \mathcal{E}_\gamma^{\geq\varepsilon}$ we have $d(y,\operatorname{pr}(y)) \geq \max\{r_c, r_p - d(\operatorname{pr}(y),p) | p \in T^{II}(c)\}$. Let ℓ_c' be as in Proposition 5.2 and let $r_p := \max\{\ell(2\pi/m_b,\varepsilon), r(\ell_b,\varepsilon)\}$ where b is the singular geodesic of O such that $p \in c\cap b$ (see the definition of type II vertices). Write $c_0 = c \setminus \bigcup_{p\in T^{II}(c)} B(p,r_p)$ and $c_1 := c\setminus c_0$. Note that ℓ_c' is the length of c_0. We will split the integral over $\mathcal{E}_\gamma^{\geq\varepsilon}$ according to whether $\operatorname{pr}(y)$ falls into c_0 or c_1:

$$\begin{aligned}
&e \cdot \int_{\mathcal{E}_\gamma^{\geq\varepsilon}} \operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx \\
=&\int_{\operatorname{pr}^{-1}(c_0)\cap\mathcal{E}_\gamma^{\geq\varepsilon}} f_\pi(d(y,\operatorname{pr}(y)))dy + \int_{\operatorname{pr}^{-1}(c_1)\cap\mathcal{E}_\gamma^{\geq\varepsilon}} f_\pi(d(y,\operatorname{pr}(y)))dy \\
\leq&\pi\ell_c' \int_{\max(\ell(2\pi/m_c,\varepsilon),r(\ell_c,\varepsilon))}^{+\infty} f_\pi(r)dr + \pi \sum_{p\in T^{II}(c)} 2\int_0^{r_p}\int_s^{+\infty} f_\pi(r)dr\,ds.
\end{aligned}$$

Using the estimate for the heat kernel we get

$$\int_{\mathcal{E}_\gamma^{\geq\varepsilon}} \operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx \ll \ell_c' + |T^{II}(c)|. \tag{5.5}$$

Let Σ_n be the set of singular geodesics in O_n (so each is the image of an axis of an elliptic conjugacy class in Γ_n) with subsets $\Sigma_n^1, \Sigma_n^2, \Sigma_n^3$ defined as in Section 5.1. If γ is an elliptic isometry of order m, primitive in Γ, there are $m-1$ elliptic elements in Γ_γ sharing the same axis. We have $\mathcal{F}_\gamma^{\geq\varepsilon} \subset \mathcal{E}_\gamma^{\geq\varepsilon}$

so by (5.4) and (5.5) we get that

$$\sum_{[\gamma]\in\mathcal{C}_{n,e}}\int_{\mathcal{F}_\gamma^{\geq\varepsilon}}\operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx \ll \sum_{c\in\Sigma_n^2,\Sigma_n^3}\ell_c\frac{m_c-1}{m_c}+\sum_{c\in\Sigma_n^1}(\ell_c'+|T^{II}(c)|).$$

It follows that

$$\sum_{[\gamma]\in\mathcal{C}_{n,e}}\int_{\mathcal{F}_\gamma^{\geq\varepsilon}}\operatorname{tr}(\gamma^* e^{-t\Delta^1[\mathbb{H}^3]}(x,\gamma x))dx \ll \sum_{c\in\Sigma_n\setminus\Sigma_n^1}\ell_c+\sum_{c\in\Sigma_n^1}\left(\ell_c'+|T^{II}(c)|\right).$$

By Proposition 5.2 the right hand side is of order $O(\operatorname{vol}((O_n)_{\leq\varepsilon}))$. The sequence converges Benjamini-Schramm to $\mathbb{H}^3$ so $\operatorname{vol}((O_n)_{\leq\varepsilon})=o(\operatorname{vol}(O_n))$. Estimate (5.3) follows.

5.3. **Comparison between heat kernels.** We prove here that

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\frac{1}{\operatorname{vol}O_n}\int_{(O_n)_{\geq R}}\operatorname{tr}(e^{-t\Delta^1[O_n]}-e^{-t\Delta^1_{\mathrm{abs}}[O_n']})(x,x)dx=0. \tag{5.6}$$

To do this we let U_n be the subset of $\mathbb{H}^3$ covering O_n' and choose a fundamental domain D_n for Γ acting in the subset of U_n covering $(O_n)_{\geq R}$ (we assume R is large enough so that $(O_n)_{\geq R}\subset O_n'$). Then we can write

$$\begin{aligned}\int_{(O_n)_{\geq R}}\operatorname{tr}(e^{-t\Delta^1[O_n]}-e^{-t\Delta^1_{\mathrm{abs}}[O_n']})(x,x)dx &= \int_{D_n}\sum_{\gamma\in\Gamma}\operatorname{tr}\gamma^*(e^{-t\Delta^1[\mathbb{H}^3]}-e^{-t\Delta_{\mathrm{abs}}[U_n]})(x,\gamma x)dx\\ &\ll e^{-\frac{R^2}{Ct}}\int_{D_n}\sum_{\gamma\in\Gamma}e^{-\frac{d(x,\gamma x)^2}{Ct}}dx\end{aligned}$$

where $\Delta_{\mathrm{abs}}[U_n]$ is the Laplacian with absolute boundary conditions on the complete manifold U_n, and the second line follows from [20, Theorem 2.26]. By the same arguments as used above to demonstrate (5.1) the integral is $O(\operatorname{vol}O_n)$ (with a constant independent of R as the domain of integration shrinks when we take R to infinity). In the end we get that

$$\limsup_{n\to+\infty}\frac{1}{\operatorname{vol}O_n}\int_{(O_n)_{\geq R}}\operatorname{tr}(e^{-t\Delta^1_{\mathrm{abs}}[O_n]}-e^{-t\Delta^1[O_n']})(x,x)dx\ll e^{-\frac{R^2}{Ct}}$$

from which (5.6) follows immediately.

5.4. **Heat kernel near the boundary.** Here we prove the final ingredient for the proof of Proposition 5.1: for all $R>0$ we have

$$\int_{O_n'\setminus(O_n)_{\geq R}}\operatorname{tr}e^{-t\Delta^1_{\mathrm{abs}}[O_n']}(x,x)dx=o(\operatorname{vol}O_n). \tag{5.7}$$

By Benjamini–Schramm convergence we have that $\operatorname{vol}(O_n'\setminus(O_n)_{\geq R})=o(\operatorname{vol}O_n)$. So to prove (5.6) it suffices to see that $\operatorname{tr}e^{-t\Delta^1_{\mathrm{abs}}[O_n']}(x,x)=O_t(1)$ for $x\in O_n'$. As in [1, (7.19.4)] this follows from [20, Theorem 2.35]; the latter is applicable with a uniform constant in our context by Lemma 3.4.

Appendix A. Benjamini–Schramm convergence in Gromov-hyperbolic spaces

A.1. **Orbital integrals on hyperbolic spaces.** Let X be a proper Gromov-hyperbolic space and $G = \mathrm{Isom}(X)$. With the compact-open topology G is a locally compact second countable topological group. For $\gamma \in G$ we denote by G_γ its centraliser. The following lemma is a slight generalisation of [6, Corollary 3.10(2) on p. 463]—the latter dealing only with discrete groups. It might be possible to straightforwrdly adapt the arguments in loc. cit. to our case, but we give a different, mostly self-contained proof.

Lemma A.1. *Let $\gamma \in G$ be an hyperbolic isometry. Then $G_\gamma/\langle\gamma\rangle$ is compact.*

For the proof we use the following lemma, which should be standard but we could not find in the literature. The proof is a bit long and technical so we put it at the end of this appendix (see A.3).

Lemma A.2. *Let γ be an hyperbolic isometry of X. For any $x \in X$ there exists constants $C = C(x, \gamma, \delta)$ and $A = A(x, \gamma, \delta)$ such that for any $y \in X$ and any k sufficiently large (depending on γ, x, δ) we have*

$$d(y, \gamma^k y) \geq Ck + 2d(y, \langle\gamma\rangle x) - A.$$

Proof of lemma A.1. Let $\tau = d(\gamma) := \inf\{d(y, \gamma y) | y \in X\}$ be the minimal displacement of γ. Fix $x \in X$, let k, A, C as given by Lemma A.2 and define:

$$D = \{y \in X | d(y, \gamma^k y) \leq k\tau + 1\}.$$

It is a non-empty (by definition of τ) closed G_γ-invariant subset of X. Given that the action of G_γ on D is proper, the Lemma will follow once we prove that $\langle\gamma\rangle\backslash D$ is compact. The previous Lemma implies that

$$D \subset \{y \in X : \ d(y, \langle\gamma\rangle x) \leq (\tau - C)k + A + 1\}$$

so that $D \subset \gamma^{\mathbb{Z}} B(x, R)$ for some sufficiently large R, and as X is proper this in turn implies that $\langle\gamma\rangle\backslash D$ is compact. □

Let dg be a fixed Haar measure on G. According to the lemma above the subgroup G_γ admits a lattice so it is unimodular and we have a decomposition $dg = dxdh$ where dx is a G-invariant measure on G/G_γ and dh a Haar measure on G_γ, both depending only on the original choice of dg. For a function $f \in C_0(G)$ we can then define the *orbital integral* associated to γ by:

$$\mathcal{O}_f(\gamma) = \int_{G/G_\gamma} f(\gamma^{-1} x \gamma) dx \tag{A.1}$$

which depends only on the G-conjugacy class $[\gamma]_G$.

A.2. **General criterion for Benjamini–Schramm convergence.** Here again X is always a proper Gromov-hyperbolic space and $G = \mathrm{Isom}(X)$. We assume that the action of G on X is non-elementary. The *elliptic radical* of G can then be defined as its unique maximal normal compact subgroup (see [24, Proposition 3.4]; in our context, by properness of X bounded elements are the same as compact ones). The following lemma is a special case of [24, Theorem 1.5].

Lemma A.3. *Let μ be an invariant random subgroup of G. Then either μ is supported on the elliptic radical or it has full limit set.*

Recall from [12, Section 3] that there is a "Benjamini–Schramm topology" on the set of Borel probability measures on the Gromov–Hausdorff space of pointed proper metric spaces (up to isometry). The set of measures supported on spaces locally isometric to X is precompact in this topology. Moreover, if X is a locally symmetric space then (1.1) is equivalent to $\Gamma_i\backslash X$ converging in the Benjamini–Schramm topology to X.

There is a continuous injective map from the space of invariant random subgroups of G to the Benjamini–Schramm space. If Γ_i are lattices in G then the sequence of uniformly pointed spaces $\Gamma_i\backslash X$ converges to X if and only if the IRSs μ_{Γ_i} converge to the trivial IRS. We will use this to prove the following criterion for convergence, which is a more general version of Proposition 2.1.

Proposition A.4. *Let U the set of hyperbolic isometries in G. Assume that the elliptic radical of G is trivial. If Γ_n is a sequence of lattices in G which satisfies:*

$$\lim_{n\to+\infty} \frac{\sum_{[\gamma]_{\Gamma_n}\subset U} \mathrm{vol}((\Gamma_n)_\gamma\backslash G_\gamma)\mathcal{O}_f(\gamma)}{\mathrm{vol}(\Gamma_n\backslash G)} = 0 \tag{A.2}$$

then the sequence of metric spaces $\Gamma_n\backslash X$ converges to X in the Benjamini–Schramm topology.

Proof. Let μ_n be the invariant random subgroup of G supported on the conjugacy class of Γ_n. We want to prove that any weak limit μ of a subsequence of (μ_n) is equal to the trivial IRS δ_e. By Lemma A.3, and the fact that a subgroup of G containing no hyperbolic isometries has at most one limit point (cf. [13, Section 8.2]) it suffices to prove that any such μ contains no hyperbolic isometries.

To prove this choose a covering $U = \bigcup_{C\in\mathcal{C}} C$ of U where $\mathcal{C}$ is countable and every $C \in \mathcal{C}$ is compact. We can do this since Sub_G is metrizable [8, Proposition 2]. Let $W_C = \Lambda : \Lambda\cap C \neq \emptyset$; this is a Chabauty-closed subset of Sub_G. If ν is a nontrivial IRS then by Lemma A.3 and previous paragraph it almost surely contains a hyperbolic element. Hence, there is $C \in \mathcal{C}$ such that $\nu(W_C) > 0$. We need to prove the opposite for μ, which amounts to the following : for every C there exists a non-negative Borel function F on Sub_G which is positive on W_C and such that $\int_{\mathrm{Sub}_G} F(\Lambda)d\mu(\Lambda) = 0$.

Let us fix $C \in \mathcal{C}$ and prove this. There exists an open relatively compact subset V with $C \subset V$ and $\overline{V} \subset U$. Choose any $f \in C^\infty(G)$ such that $f > 0$ on C and $f = 0$ on $G \setminus V$ and define :

$$F(\Lambda) = \begin{cases} \sum_{\lambda \in \Lambda} f(\lambda) & \text{if } \Lambda \text{ is discrete;} \\ 1 & \text{if } \Lambda \text{ is not discrete and intersects } C; \\ 0 & \text{otherwise.} \end{cases}$$

Then F is lower semicontinuous on Sub_G, non-negative and positive on W_C. On the other hand we have :

$$\begin{aligned} \int_{\mathrm{Sub}_G} F(\Lambda) d\mu_n(\Lambda) &= \frac{1}{\mathrm{vol}(\Gamma_n \backslash G)} \int_{G/\Gamma_n} \sum_{\gamma \in g\Gamma_n g^{-1}} f(\gamma) dg \\ &= \frac{1}{\mathrm{vol}(\Gamma_n \backslash G)} \sum_{[\gamma]_{\Gamma_n} \subset U} \mathrm{vol}((\Gamma_n)_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f). \end{aligned}$$

By the so-called "Portemanteau theorem" [16, Theorem 13.16] the limit inferior of the left-hand side is larger or equal to $\int_{\mathrm{Sub}_G} F(\Lambda) d\mu(\Lambda)$. By (A.2) we have that the right-hand side converges to 0. It follows that

$$\int_{\mathrm{Sub}_G} F(\Lambda) d\mu(\Lambda) = 0$$

which finishes the proof. □

A.3. **Proof of Lemma A.2.** Let us recall the statement. We have a proper hyperbolic geodesic space X and an hyperbolic isometry γ of X. We fix $x \in X$ and we want to show that there exists constants $C = C(x, \gamma, \delta)$ and $A = A(x, \gamma, \delta)$ such that for any $y \in X$ and any k sufficiently large (depending on γ, x, δ) we have

$$d(y, \gamma^k y) \geq Ck + 2d(y, \langle \gamma \rangle x) - A. \tag{A.3}$$

Let $x, y \in X$. As γ is hyperbolic there exists a, c such that $L = \langle \gamma \rangle x$ is a (c, a)-quasi-geodesic. Regarding the conclusion of the proposition it does not change anything if we assume that x is the approximate projection of y on L, meaning that any point x' of L within distance $d(y, L)$ of y, satisfies $d(x', x) \leq K$ (where K depends only on the hyperbolicity constant δ).

Let $\ell = d(x, \gamma x)$. Note first that if k is large enough so that

$$k > 100 c \ell^{-1} K \log(k) + ac \tag{A.4}$$

holds, and y is close enough to L so that

$$d(y, x) > c^2 \ell^{-1} \log(k) + cK(2 + \log(2 + k)) + ca \tag{A.5}$$

does not then we see immediately that (A.3) holds, by the triangle inequality. Thus from now on we will assume that both inequalities above hold for y and k.

Let $x_i = \gamma^i x$, $y_i = \gamma^i y$ for $0 \leq i \leq k$. Let F be the finite set

$$F = \{x_0, x_1, \ldots, x_k\} \cup \{y_0, y_k\};$$

by [5, Proposition 7.3.1] there exists a choice of a "spanning tree" on F (that is, a tree whose edges are a subset of all pairs of geodesics segment between points of F) such that

$$\forall p, q \in F : \ d(p,q) \geq d_{T_F}(p,q) - (1 + \log(2+k))K \tag{A.6}$$

where K depends only on δ (so we take it equal to the K introduced above to simplify notation). One of y_0, y_k must be connected to one of the x_i in T_F; we may assume that $[y_0, x_i]$ is an edge in T_F for some i. We claim that this i must be unique, and we must have

$$i < c\ell^{-1}\left((\log(k+2)+2)K + a\right). \tag{A.7}$$

Indeed, let i be the smallest integer such that $[x_i, y_0] \subset T_F$. Then, because

$$d_{T_F}(x_0, y_0) \leq d(x_0, y_0) + (\log(k+2)+1)K$$

and

$$d_{T_F}(x_0, y_0) \geq d(x_0, x_i) + d(x_i, y_0) \geq \frac{i\ell}{c} - a + d(x_0, y_0) - K$$

we see that i must verify (A.7). Now assume that there is a $j > i$ such that $[x_j, y_0] \subset T_F$, and take it to be the smallest such; we want to reach a contradiction. Consider $i \leq l < j$ to be maximal such that the path in T_F from x_l to x_i does not go through y_0. Then the path in T_F from x_l to x_{l+1} must go through y_0 (otherwise we would have a path from x_{l+1} to x_i via x_l avoiding y_0). We have thus $d_{T_F}(x_l, x_{l+1}) \geq d(x_0, y_0) - K$ which together with (A.5) and (A.6) contradicts the fact that $d(x_l, x_{l+1}) = \ell$.

We now want to prove that $[y_0, y_k]$ is not an edge in T_F. To do so we must consider two possibilities. Assume first that $[y_k, x_j] \subset T_F$ for some j. Then reasoning as above we see that j is the only such index, and $j > k - c\ell^{-1}\left((\log(k+2)+2)K + a\right) > i$. In this case we reach a contradiction in the same way as in the previous paragraph: considering a maximal $i \leq l < j$ such that the path from x_l to x_i does not go through y_0 we see that $d_{T_F}(x_l, x_{l+1})$ is too large.

If there is no edge $[y_k, x_j]$ in T_F then the path from x_k to y_k must go first to x_i, then to y_0 and finally to y_k. But as $d(x_k, x_i) > (\log(k+2)+1)K$ by (A.7) and (A.4) we see that this contradicts $d(x_0, y_0) = d(x_k, y_k)$.

So we get that there must be a unique edge $[y_k, x_j]$ in T_F, and the path in T_F from y_0 to y_k must go through x_j and x_i. As before we must have

$$j > k - c\ell^{-1}\left((\log(k+2)+2)K + a\right)$$

and we finally get using first (A.6), then the fact that $(x_0, \ldots, x_k)$ is a quasi-geodesic, and finally the above together with (A.7) that:

$$\begin{aligned} d(y_0, y_k) &\geq d(y_0, x_i) + d(x_i, x_j) + d(x_j, y_k) - K - K\log(2+k) \\ &\geq 2d(x_0, y_0) + c^{-1}(j-i)\ell - a - 3K - K\log(2+k) \\ &\geq 2d(x_0, y_0) + c^{-1}\ell k - B - b\log(k) \end{aligned}$$

where B, b depend only on x, γ, δ. From the last inequality and (A.4) we can conclude that (A.3) holds.

Appendix B. Smoothing corners

In this appendix we prove Proposition 3.5; as the argument is technical but has no subtleties we will be quite sketchy in presenting it.

Recall that we have the following situation: X is a manifold with bounded geometry, $H_1, H_2 \subset X$ such that $X \setminus H_i$ both have bounded geometry, meet transversally and the dihedral angle between them is bounded away from 0 and π. We remark that constructing a smoothing of $Y = X \setminus (H_1 \cup H_2)$ satisfying the conclusions of Proposition 3.5 is immediate in the case where the intersection $I = H_1 \cap H_2$ has a neighbourhood in Y which is isometric to the product $[0, \delta[^2 \times I$. In general we will prove the following statement: there exists a diffeomorphism φ from $[0, \delta[^2 \times I$ to a neighbourhood of I in Y such that φ and φ^{-1} have all their derivatives uniformly bounded. In view of the preceding remark this proves the proposition.

To define φ we need some more auxiliary notation: for a vector field V and $t \geq 0$ we let Φ^t_V be its flow at time t; if $H \subset Z$ is open with smooth boundary we denote by N^Z_H the normal field of H in Z. We put:

$$\varphi_1(x,t,s) = \Phi^t_{N^X_{H_1}}(\Phi^s_{N^{H_1}_I}(x))$$

and

$$\varphi_2(x,t,s) = \Phi^s_{N^X_{H_2}}(\Phi^t_{N^{H_2}_I}(x))$$

We fix a smooth non-decreasing function $h : \mathbb{R} \to [0,1[$ such that h is zero on negative numbers, and at infinity it tends to 1 and all its derivatives vanish at all orders. Let $0 < a < 1$ such that the convex hull of all $\varphi_1(x,t,s)$ and $\varphi_2(x,t,s)$ for $as \leq t \leq a^{-1}s$ is contained in Y. For $x, y \in X$ and $u \in [0,1]$ let $ux + (1-u)y$ denote the barycenter of x, y on the geodesic segment between them[2]. With this notation we define:

$$\varphi(x,t,s) = h\left(\frac{at-s}{as-t}\right)\varphi_1(x,t,s) + \left(1 - h\left(\frac{at-s}{as-t}\right)\right)\varphi_2(x,t,s)$$

and we claim that φ has the desired properties. It is smooth as a composition of smooth maps. To deduce the remaining properties we will use the following lemma.

Lemma B.1. *For $i = 1, 2$ there is c depending only on the bounds on the geometry of H_i such that the following properties hold.*

(1) *Let $z \in \partial H_i$ and $0 \leq t \leq \delta$. The linear map $D_z\Phi^t_{N^X_{H_i}}$ is c-Lipschitz on angles. The same holds for $x \in I$ and $D_x\Phi^t_{N^{H_i}_I}$.*

(2) *For all $x \in I$ and all $0 \leq s, t < \delta$, let $y = \Phi^t_{N^X_{H_i}}(\Phi^s_{N^{H_i}_I}(x))$. Let γ be the geodesic (in X) from x to y, u_i the parallel transport along γ of the*

[2]This is well-defined for those pairs of points in X that we consider, as long as we take $\delta \ll \mathrm{inj}(X)$

outward normal vector to H_i at x and $v_i = \frac{\partial}{\partial\tau}\big|_{\tau=t} \Phi^{\tau}_{N^X_{H_i}}(\Phi^s_{N^{H_i}_I}(x))$. Then the angle between u_i and v_i is at most $c\delta$.

Proof. (1) follows from the boundedness of coefficients of the metric tensor and its inverse in normal exponential coordinates (in both $I \subset H_i$ and $\partial H_i \subset X$). (2) follows from (1), together with the fact that parallel transport along a closed curve stays close to the identity within the δ-neighbourhood. □

Let V_i be the vector fields given by the vectors v_i defined in the lemma. As for any $x \in I$ we have that the angle between $V_1(x)$ and $V_2(x)$ lies in $[\alpha_0, \pi - \alpha_0]$ it follows from (2) that if we choose $\delta < c^{-1}\alpha_0/2$ we have that the angle between V_1 and V_2 at any point x in the δ-neighbourhood of I lies in $[\alpha_0/2, \pi - \alpha_0/2]$. In particular V_1, V_2 define a plane field, and we define J to be its orthogonal.

Let π_J be orthogonal projection on J. The block decomposition of $D\varphi$ according to $TX = J \oplus (V_1 + V_2)$ is:

$$D_{(x,t,s)}\varphi = \begin{pmatrix} \pi_J D_x\varphi & C \\ (1-\pi_J)D_x\varphi & B \end{pmatrix}.$$

We need to prove that:

(1) $D_x\varphi, B$ and C have bounded coefficients (in terms of the bounds on the geometry);
(2) $\pi_J D_x\varphi$ and B are everywhere invertible and their inverses are bounded;
(3) $\|(1-\pi_J)D_x\varphi \ll \delta$.

Indeed, this shows that the map φ has a derivative which everywhere invertible. In particular, it is a local diffeomorphism and as it is the identity on I it is also a global diffeomorphism. This also implies that its derivative is uniformly bounded in terms of the geometry of H_i and α_0, and so is its inverse.

We deal first with $D_x\varphi$. We note that

$$(D_x\varphi)_{(x,t,s)} = h\left(\frac{at-s}{as-t}\right) D_x\varphi_1(x,t,s) + \left(1 - h\left(\frac{at-s}{as-t}\right)\right) D_x\varphi_2(x,t,s) + O(\delta)$$

because of bounded geometry and the fact that to obtain φ we move φ_1 and φ_2 by at most δ. It follows that $D_x\varphi$ is bounded. By point (1) of the Lemma we have that at all points the angle between the image of $D_x\varphi$ and V_i is at most $c\delta$; it follows that $\|(1-\pi_J)D_x\varphi\| \ll \delta$. Moreover $D_x\varphi$ is everywhere invertible with bounded inverse, because both $A_1 = D_x\varphi_1$ and $A_2 = D_x\varphi_2$ are, and for $w \in T_xI$ the vectors $A_1(w), A_2(w)$ have an angle $\leq c\delta$ between them by (1).

We also have

$$D_t\varphi = h\left(\frac{at-s}{as-t}\right) D_t\varphi_1(x,t,s) + \left(1 - h\left(\frac{at-s}{as-t}\right)\right) D_t\varphi_2(x,t,s) + O(\delta)$$

and similarly for $D_s\varphi$, so the coefficients of B, C are bounded.

It remains to prove that B is invertible and $\det(B)$ is bounded away from zero. At a point $x \in I$ we have $D_t\varphi$ and $D_s\varphi$ belong to two disjoint

open convex cones in T_xX/J_x; by (2) and (1) this remains true in the δ-neighbourhood and the angle between the cones remains bounded away from zero, hence the matrix B is invertible with uniformly bounded inverse.

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