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1 Boundary stabilization in finite time of one-dimensional
2 linear hyperbolic balance laws with coefficients depending on
3 time and space

4 Jean-Michel Coron*, Long Hu†, Guillaume Olive‡, Peipei Shang§

5 April 24, 2020

6 **Abstract**

7 In this article we are interested in the boundary stabilization in finite time of one-dimensional
8 linear hyperbolic balance laws with coefficients depending on time and space. We extend the so
9 called “backstepping method” by introducing appropriate time-dependent integral transforma-
10 tions in order to map our initial system to a new one which has desired stability properties. The
11 kernels of the integral transformations involved are solutions to non standard multi-dimensional
12 hyperbolic PDEs, where the time dependence introduces several new difficulties in the treat-
13 ment of their well-posedness. This work generalizes previous results of the literature, where only
14 time-independent systems were considered.

15 **Keywords:** Hyperbolic systems, Boundary stabilization, Non-autonomous systems, Backstep-
16 ping method.

17 **1 Introduction and main result**

18 In the present paper we are interested in the one-sided boundary stabilization in finite time of
19 one-dimensional linear hyperbolic balance laws when the coupling coefficients of the system depend
20 on both time and space variables. To investigate this stabilization property we use the by now so-
21 called “backstepping method”, a method that consists in transforming our initial system into another
22 system - called target system - for which the stabilization properties are simpler to study. In finite
23 dimension it relies on a recursive design procedure, which in the case of partial differential equations
24 leads to Volterra transformations of the second kind.

25 The idea of the possibility to transform a control system into another one in order to study
26 its controllability or stabilization properties already goes back to the development of the control
27 theory for linear finite-dimensional systems in the late 60’s, notably with the celebrated work [Bru70]
28 where the author introduced the so-called “control canonical form”. Concerning infinite-dimensional
29 systems, such as systems modeled by partial differential equations (PDEs), this approach is much
30 more complicated. The first attempt in this direction seems to be [Rus78], where the author was
31 interested in the spectral determination (i.e. pole placement) of a particular 2×2 first-order hyperbolic

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32 system. The difficult task in this approach is, in general, to find an invertible transformation that
33 allows to pass from one system to another and, to the best of our knowledge, there is no general
34 theory for infinite-dimensional systems so far (if possible). In [Rus78], the author proposed to use
35 a Volterra transformation of the second kind to pass from what he called the “control normal form”
36 to the control canonical form of his hyperbolic system and, in this way, easily solved his spectral
37 determination problem. In that paper, the use of such a transformation was justified by the analogy
38 with finite-dimensional systems when using transformations of the simple form $\text{Id} + K$ with K being
39 a triangular matrix (while for Volterra transformations of the second kind, K is an integral operator
40 whose kernel is supported in a triangular domain). The use of a Volterra transformation of the second
41 kind to transform a PDE into another one was also introduced at almost the same time in [Col77].
42 Therein, the author showed that a one-dimensional perturbed heat equation, with a time and space
43 dependent perturbation, can be transformed into the classical heat equation by means of a Volterra
44 transformation of the second kind whose kernel has to satisfy some PDE posed on a non-standard
45 domain which is triangular. The equation that the kernel has to satisfy is now commonly referred to
46 as the “kernel equation” and the method was then referred by the author of [Col77] to as the “method
47 of integral operators”. The result of [Col77] was notably applied in [Sei84] to deduce the boundary
48 null-controllability in one space dimension of the perturbed heat equation from that of the classical
49 heat equation.

50 In the 90’s a method with similar spirit appeared under the name of “backstepping method”. This
51 method was primarily designed to transform, thanks to a recursive procedure, finite-dimensional control
52 systems, which may be nonlinear, into control systems which can be stabilized by means of simple
53 feedback laws. This method was later on extended to linear PDEs. The first result in this direction
54 is in [CdN98] for a beam equation; see also [LK00] for a Burgers’ equation. However, the main break-
55 through for the PDEs case are in [BKL01, BK02, Liu03], which deal with 1-D heat equations and
56 where Volterra transformations of the second kind are introduced or used. In particular in [BK02]
57 the backstepping recursive procedure in finite dimension is applied to the semi-discretized finite dif-
58 ference approximation of these equations and it is proved that, as the spatial step size tends to 0, the
59 backstepping transformation at the finite dimensional level is converging to a Volterra transformation
60 of the second kind. The fact that the transformation which appears with this approach is a Volterra
61 transformation of the second kind comes from the recursive procedure of the backstepping method.
62 With this method the authors, directly inspired by the backstepping in finite dimension, independ-
63 ently arrived at the use of exactly the same transformation as in the two above mentioned pioneering
64 references [Rus78] and [Col77]. This is the reason of the use of the terminology “backstepping” for the
65 construction of stabilizing feedback laws relying on the use of Volterra transformations of the second
66 kind to transform a given control PDE to another control PDE (called the target system) which can
67 be easily stabilized (usually with the null feedback law).

68 The use of Volterra transformations of the second kind also matches very well with the boundary
69 stabilization of one-dimensional systems since this transformation somehow removes the undesirable
70 terms (or adds desirable ones) of the equation by “bringing” them to the part of the boundary where
71 the feedback is acting (through the kernel equations). This approach rapidly turned out to be very
72 successful in the study of the boundary stabilization of various important PDEs such as heat equa-
73 tions, wave equations, Schrödinger equations, Korteweg-de Vries equations, Kuramoto-Sivashinsky
74 equations, etc. and it eventually leads to the by now reference book [KS08] on this subject. This
75 method is nowadays systematically used as a standard tool to analyze the boundary stabilization for
76 (mainly one-dimensional) PDEs. This method has also received some recent developments. Notably,
77 the use of Volterra transformations of the second kind has started to show some serious limitation for
78 some problems and it has been replaced by more general integral transformations such as Fredholm
79 integral transformations (see e.g. [CL14, CL15, BAK15, CHO16, CHO17, CGM18]) or other kind of
80 integral transformations (see e.g. [SGK09]). In these cases the transformation on the state does not
81 have any special structure and the method is no longer related to the finite dimensional backstepping
82 approach. It is related to the older notion of feedback equivalence, as initiated in [Bru70]; see also

83 [Kal72], [Won85, Section 5.7], and [Son98, Section 5.2].

84 Concerning more specifically systems of hyperbolic equations and the finite-time stabilization
 85 property, which is the focus of this article, the first result was obtained in [CVKB13]. In this paper,
 86 the authors developed the original backstepping method to prove the boundary stabilization of a
 87 2×2 hyperbolic system in finite time, with the best time that can be achieved. The generalization
 88 of the result of [CVKB13] to $n \times n$ systems was a non-trivial task which was eventually solved in
 89 [HDMVK16, HVDMK19] using the ideas introduced previously in [HDM15] for 3×3 systems. The
 90 key point was to add additional constraints on the kernel to obtain a specific structure of the coupling
 91 parameter in the target system. The time of stabilization found in [HDMVK16, HVDMK19] was then
 92 improved in [ADM16, CHO17], using two different target systems.

93 The goal of the present article is to extend the results of the previously mentioned references to
 94 time-dependent systems. For the finite-time stabilization of non-autonomous hyperbolic systems, the
 95 only works that we are aware of are [DJK16] and [AA18] which concerned a single equation with
 96 constant speed. Therefore, the non-autonomous case for systems was still left without investigation.
 97 The introduction of the time variable in the coupling coefficients obviously complicates the whole
 98 situation. As in [Col77, DJK16, AA18] we need to introduce integral transformations with time-
 99 dependent kernels, resulting in much more complex kernel equations to solve. Finally, in addition to
 100 the previous references, we would also like to mention the work [Wan06] on time-dependent quasilinear
 101 hyperbolic systems concerning the related notion of controllability and the works [SK05, KD19], with
 102 the references therein, concerning the stabilization of time-dependent parabolic systems (where strong
 103 regularity conditions are required to make the backstepping method work, because of the result of
 104 [Kan90]).

105 The rest of this paper is organized as follows. In the remaining part of Section 1 we present in
 106 details the class of hyperbolic systems that we consider and we state our main result. In Section
 107 2 we perform several transformations to show that our initial system can be mapped to a target
 108 system which is finite-time stable with desired settling time. In Section 3 we prove the existence
 109 and regularity of the kernels of the integral transformations that were used in the previous section.
 110 Finally, we gathered in Appendices A, B and C some auxiliary results.

111 1.1 System description

112 In this article, we focus on the following general $n \times n$ linear hyperbolic systems, which appear
 113 for instance in the linearized Saint-Venant equations, plug flow chemical reactors equations, heat
 114 exchangers equations and many other physical models of balance laws (see e.g. [BC16, Chapter 1])
 115 around time-varying trajectories:

$$116 \quad \begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial y}{\partial x}(t, x) = M(t, x)y(t, x), \\ y_-(t, 1) = u(t), \quad y_+(t, 0) = Q(t)y_-(t, 0), \\ y(t^0, x) = y^0(x). \end{cases} \quad (1)$$

117 In (1), $t > t^0 \geq 0$ and $x \in (0, 1)$, $y(t, \cdot)$ is the state at time t , y^0 is the initial data at time t^0 and
 118 $u(t)$ is the control at time t . The matrix M couples the equations of the system inside the domain
 119 and the matrix Q couples the equations of the system on the boundary $x = 0$. We assume that the
 120 matrix Λ is diagonal:

$$121 \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n). \quad (2)$$

122 We denote by $m \in \{1, \dots, n-1\}$ the number of equations with negative speeds and by $p = n - m \in$
 123 $\{1, \dots, n-1\}$ the number of equations with positive speeds (all along this work we assume that
 124 $n \geq 2$, see Remark 1.11 below for the case $m = n \geq 1$). We assume that there exists some $\varepsilon > 0$ such

125 that, for every $t \geq 0$ and $x \in [0, 1]$, we have

$$126 \quad \lambda_1(t, x) < \cdots < \lambda_m(t, x) < -\varepsilon < 0 < \varepsilon < \lambda_{m+1}(t, x) < \cdots < \lambda_n(t, x), \quad (3)$$

127 and, for every $i \in \{1, \dots, n-1\}$,

$$128 \quad \lambda_{i+1}(t, x) - \lambda_i(t, x) > \varepsilon. \quad (4)$$

129 Assumptions (3) and (4) will be commented, respectively, in Remarks 1.9 and 1.10 below.

All along this paper, for a vector (or vector-valued function) $v \in \mathbb{R}^n$ and a matrix (or matrix-valued function) $A \in \mathbb{R}^{n \times n}$, we use the notation

$$v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix}, \quad A = \begin{pmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{pmatrix},$$

130 where $v_- \in \mathbb{R}^m$, $v_+ \in \mathbb{R}^p$ and $A_{--} \in \mathbb{R}^{m \times m}$, $A_{-+} \in \mathbb{R}^{m \times p}$, $A_{+-} \in \mathbb{R}^{p \times m}$, $A_{++} \in \mathbb{R}^{p \times p}$.

131 We will always assume the following regularities for the parameters involved in the system (1):

$$132 \quad \Lambda \in C^1([0, +\infty) \times [0, 1])^{n \times n}, \quad M \in C^0([0, +\infty) \times [0, 1])^{n \times n}, \quad Q \in C^0([0, +\infty))^{p \times m},$$

$$133 \quad \Lambda, \frac{\partial \Lambda}{\partial x}, M \in L^\infty((0, +\infty) \times (0, 1))^{n \times n}, \quad Q \in L^\infty(0, +\infty)^{p \times m}. \quad (5)$$

133 In this article, we use the notion of “solution along the characteristics” or “broad solution” for
 134 the system (1). The necessary background on this notion is given in Appendix A (see also [Bre00,
 135 Section 3.4] for more information). For the moment we only need to know that, for every $F \in$
 136 $L^\infty((0, +\infty) \times (0, 1))^{m \times n}$, $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$, there exists a unique (broad) solution $y \in$
 137 $C^0([t^0, +\infty); L^2(0, 1)^n)$ to the system (1) with

$$138 \quad u(t) = \int_0^1 F(t, \xi) y(t, \xi) d\xi. \quad (6)$$

139 The relation (6) will be called the feedback law and the function F will be called the state-feedback
 140 gain function.

141 Let us now give the notion of stability that we are interested in this article (see, for example,
 142 [BB98, Definition], [BR05, Section 3.2] and [Cor07, Definitions 11.11 and 11.27] for time-varying
 143 systems in finite dimension).

144 **Definition 1.1.** Let $T > 0$. We say that the system (1) with feedback law (6) is finite-time stable
 145 with settling time T if the following two properties hold:

146 (i) **Finite-time global attractor.** For every $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$,

$$147 \quad y(t^0 + T, \cdot) = 0. \quad (7)$$

148 (ii) **Uniform stability.** For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $t^0 \geq 0$ and
 149 $y^0 \in L^2(0, 1)^n$,

$$150 \quad (\|y^0\|_{L^2(0,1)^n} \leq \delta) \implies (\|y(t, \cdot)\|_{L^2(0,1)^n} \leq \varepsilon, \quad \forall t \geq t^0). \quad (8)$$

Remark 1.2. The property (8) guarantees that, inside any time interval of the form $[t^0, t^0 + T]$, the solution is controlled solely by its value at the initial time t^0 , even if this time t^0 is very large. For

our system (1) this property is in fact a consequence of the first property (7) and that the state-feedback gain function F is in $L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ (see Remark A.3). Such an implication is in general not true for time-dependent hyperbolic systems. A simple example is the following transport equation:

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \frac{\partial y}{\partial x}(t, x) = 0, \\ y(t, 1) = f(t) \int_0^1 y(t, \xi) d\xi, \\ y(t^0, x) = y^0(x), \end{cases}$$

where $f \in C^\infty([0, +\infty))$ is such that, for every $k \in \mathbb{N}$,

$$\begin{cases} f(t) = 0, & \forall t \in [2k, 2k + 1], \\ f(t) = t, & \forall t \in \left[2k + \frac{5}{4}, 2k + \frac{7}{4}\right], \end{cases}$$

151 (note that $f \notin L^\infty(0, +\infty)$). Then the finite-time global attractor property holds (with $T = 3$) but
 152 the uniform stability property does not hold (consider the sequences $y_\delta^0(x) = \delta$ for every $x \in (0, 1)$
 153 and $t_\delta^0 = 2 \lceil \frac{1}{\delta} \rceil + \frac{5}{4}$, where $\lceil \cdot \rceil$ denotes the ceiling function).

Remark 1.3. As we are trying to find a state-feedback gain function F so that (1) with feedback law (6) is finite-time stable, let us first point out that, in general, $F = 0$ does not work. A simple example is provided by the 2×2 system with constant coefficients ($t^0 = 0$ to simplify)

$$\begin{cases} \frac{\partial y_-}{\partial t}(t, x) - \frac{\partial y_-}{\partial x}(t, x) = -cy_+(t, x), \\ \frac{\partial y_+}{\partial t}(t, x) + \frac{\partial y_+}{\partial x}(t, x) = -cy_-(t, x), \\ y_-(t, 1) = 0, \quad y_+(t, 0) = y_-(t, 0), \\ y(0, x) = y^0(x), \end{cases}$$

154 which is exponentially unstable for $c > \pi$ (see e.g. [BC16, Proposition 5.12] with $y_-(t, x) = S_1(t, 1-x)$
 155 and $y_+(t, x) = S_2(t, 1-x)$), and thus not finite-time stable.

156 1.2 The characteristics

157 To state the main result of this paper we need to introduce the characteristic curves associated with
 158 system (1). To this end, it is convenient to first extend Λ to a function of \mathbb{R}^2 (still denoted by Λ).

Remark 1.4. This extension procedure can be done in such a way that the properties (2), (3), (4) and (5) remain valid on \mathbb{R}^2 . We can take for instance

$$\bar{\lambda}_i(t, x) = \begin{cases} \lambda_i(t, x) & \text{if } t \geq 0, \\ \lambda_i(0, x) + \delta \left(\lambda_i(0, x) - \lambda_i \left(1 - e^{t/\delta}, x \right) \right) & \text{if } t < 0, \end{cases}$$

159 where $\delta > 0$ is small enough so that $-\varepsilon + 4\delta \max_i \|\lambda_i\|_{L^\infty((0,1) \times (0,1))} < -\varepsilon/2$ to guarantee the
 160 properties (3) and (4) with $\varepsilon/2$ in place of ε . This extends the function to $\mathbb{R} \times [0, 1]$. We can use
 161 a similar procedure to then extend it to \mathbb{R}^2 . We can check that the results of this paper do not
 162 depend on such a choice of extension (all the important data are uniquely determined on the domain
 163 of interest $(0, +\infty) \times (0, 1)$).

164 **1.2.1 The flow**

165 For every $i \in \{1, \dots, n\}$, let χ_i be the flow associated with λ_i , i.e. for every $(t, x) \in \mathbb{R} \times \mathbb{R}$, the
 166 function $s \mapsto \chi_i(s; t, x)$ is the solution to the ODE

$$167 \quad \begin{cases} \frac{\partial \chi_i}{\partial s}(s; t, x) = \lambda_i(s, \chi_i(s; t, x)), & \forall s \in \mathbb{R}, \\ \chi_i(t; t, x) = x. \end{cases} \quad (9)$$

168 The existence and uniqueness of the solution to the ODE (9) follows from the (local) Cauchy-Lipschitz
 169 theorem and this solution is global since λ_i is bounded (by the finite time blow-up theorem, see e.g.
 170 [Har02, Theorem II.3.1]). The uniqueness of the solution to the ODE (9) also yields the group
 171 property

$$172 \quad \chi_i(\sigma; s, \chi_i(s; t, x)) = \chi_i(\sigma; t, x), \quad \forall \sigma \in \mathbb{R}. \quad (10)$$

173 By classical regularity results on ODEs (see e.g. [Har02, Theorem V.3.1]), χ_i has the regularity

$$174 \quad \chi_i \in C^1(\mathbb{R}^3), \quad (11)$$

175 and, for every $s, t, x \in \mathbb{R}$, we have

$$176 \quad \frac{\partial \chi_i}{\partial t}(s; t, x) = -\lambda_i(t, x) e^{\int_t^s \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta; t, x)) d\theta}, \quad \frac{\partial \chi_i}{\partial x}(s; t, x) = e^{\int_t^s \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta; t, x)) d\theta}. \quad (12)$$

177 Note in particular that

$$178 \quad \begin{cases} \frac{\partial \chi_i}{\partial t}(s; t, x) > 0 & \text{if } i \in \{1, \dots, m\}, \\ \frac{\partial \chi_i}{\partial t}(s; t, x) < 0 & \text{if } i \in \{m+1, \dots, n\}, \\ \frac{\partial \chi_i}{\partial x}(s; t, x) > 0. \end{cases} \quad (13)$$

179 **1.2.2 The entry and exit times**

180 For every $i \in \{1, \dots, n\}$, $t \in \mathbb{R}$ and $x \in [0, 1]$, let $s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x) \in \mathbb{R}$ be the entry and exit times
 181 of the flow $\chi_i(\cdot; t, x)$ inside the domain $[0, 1]$, i.e. the respective unique solutions to

$$182 \quad \begin{cases} \chi_i(s_i^{\text{in}}(t, x); t, x) = 1, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 0, & \text{if } i \in \{1, \dots, m\}, \\ \chi_i(s_i^{\text{in}}(t, x); t, x) = 0, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 1, & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (14)$$

183 The existence and uniqueness of $s_i^{\text{out}}(t, x)$ and $s_i^{\text{in}}(t, x)$ are guaranteed by the assumption (3). Note
 184 that we always have

$$185 \quad s_i^{\text{in}}(t, x) \leq t \leq s_i^{\text{out}}(t, x) \quad (15)$$

186 and the cases of equalities are given by

$$187 \quad \begin{cases} s_i^{\text{in}}(t, x) = t \iff x = 1, & s_i^{\text{out}}(t, x) = t \iff x = 0, & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{in}}(t, x) = t \iff x = 0, & s_i^{\text{out}}(t, x) = t \iff x = 1, & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (16)$$

188 It readily follows from (10) and the uniqueness of $s_i^{\text{in}}, s_i^{\text{out}}$ that, for every $s \in [s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x)]$,

$$189 \quad s_i^{\text{in}}(s, \chi_i(s; t, x)) = s_i^{\text{in}}(t, x), \quad s_i^{\text{out}}(s, \chi_i(s; t, x)) = s_i^{\text{out}}(t, x). \quad (17)$$

190 From (11) and by the implicit function theorem, we have

$$191 \quad s_i^{\text{in}}, s_i^{\text{out}} \in C^1(\mathbb{R} \times [0, 1]). \quad (18)$$

192 Moreover, integrating the ODE (9) and using the assumption (3), we have the following bounds, valid
193 for every $t \in \mathbb{R}$ and $x \in [0, 1]$,

$$194 \quad t - s_i^{\text{in}}(t, x) < \frac{1}{\varepsilon}, \quad s_i^{\text{out}}(t, x) - t < \frac{1}{\varepsilon}. \quad (19)$$

195 On the other hand, differentiating (14) and using (13) with (3), we see that, for every $t \in \mathbb{R}$ and
196 $x \in [0, 1]$, we have

$$197 \quad \begin{cases} \frac{\partial s_i^{\text{in}}}{\partial t}(t, x) > 0, & \frac{\partial s_i^{\text{out}}}{\partial t}(t, x) > 0, \\ \frac{\partial s_i^{\text{in}}}{\partial x}(t, x) > 0, & \frac{\partial s_i^{\text{out}}}{\partial x}(t, x) > 0 & \text{if } i \in \{1, \dots, m\}, \\ \frac{\partial s_i^{\text{in}}}{\partial x}(t, x) < 0, & \frac{\partial s_i^{\text{out}}}{\partial x}(t, x) < 0 & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (20)$$

198 Finally, from the assumption (3) and classical results on comparison for ODEs (see e.g. [Har02,
199 Corollary III.4.2]), we have, for every $t \in \mathbb{R}$ and $x \in [0, 1]$,

$$200 \quad \begin{cases} s_m^{\text{in}}(t, x) < \dots < s_1^{\text{in}}(t, x) & \text{if } x \neq 1, & s_1^{\text{out}}(t, x) < \dots < s_m^{\text{out}}(t, x) & \text{if } x \neq 0, \\ s_{m+1}^{\text{in}}(t, x) < \dots < s_n^{\text{in}}(t, x) & \text{if } x \neq 0, & s_n^{\text{out}}(t, x) < \dots < s_{m+1}^{\text{out}}(t, x) & \text{if } x \neq 1. \end{cases} \quad (21)$$

201 1.3 Main result and comments

202 We are now in position to state the main result of this paper:

203 **Theorem 1.5.** *Let Λ , M and Q satisfy (2), (3), (4) and (5). Then, there exists a state-feedback gain
204 function $F \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ such that the system (1) with feedback law (6) is finite-time
205 stable with settling time $T_{\text{unif}}(\Lambda)$ defined by*

$$206 \quad T_{\text{unif}}(\Lambda) = \sup_{t^0 \geq 0} s_{m+1}^{\text{out}}(s_m^{\text{out}}(t^0, 1), 0) - t^0. \quad (22)$$

207 Moreover, if for some $\tau > 0$, Λ , M and Q are τ -periodic with respect to time (that is $\Lambda(t + \tau, x) =$
208 $\Lambda(t, x)$ for every $t \geq 0$ and $x \in [0, 1]$, same for M and Q) then one can also impose to F to be
209 τ -periodic with respect to time (almost everywhere).

Let us remark that, thanks to (15), (16) and (19), we always have

$$0 < T_{\text{unif}}(\Lambda) < \frac{2}{\varepsilon}.$$

Note as well, thanks to (21) and the first line in (20), that we have

$$T_{\text{unif}}(\Lambda) = \max_{j \in \{m+1, \dots, n\}} \max_{i \in \{1, \dots, m\}} \sup_{t^0 \geq 0} s_j^{\text{out}}(s_i^{\text{out}}(t^0, 1), 0) - t^0.$$

210 **Example 1.6.** Theorem 1.5 applies for instance to the following coupled 2×2 system:

$$211 \quad \left\{ \begin{array}{l} \frac{\partial y_1}{\partial t}(t, x) - \frac{\partial y_1}{\partial x}(t, x) = m_{11}(t, x)y_1(t, x) + m_{12}(t, x)y_2(t, x), \\ \frac{\partial y_2}{\partial t}(t, x) + \left(1 + \frac{1}{1+t}\right) \frac{\partial y_2}{\partial x}(t, x) = m_{21}(t, x)y_1(t, x) + m_{22}(t, x)y_2(t, x), \\ y_1(t, 1) = u(t), \quad y_2(t, 0) = q(t)y_1(t, 0), \\ y(t^0, x) = y^0(x), \end{array} \right. \quad (23)$$

where $M = (m_{ij})_{1 \leq i, j \leq 2}$ and $Q = (q)$ are any parameters with the regularity (5). Let us show how to compute $T_{\text{unif}}(\Lambda)$ for this example. First of all, it is clear that $\chi_1(s; t, x) = -s + t + x$, so that $s_1^{\text{out}}(t^0, 1) = t^0 + 1$. On the other hand, we have $\chi_2(s; t, x) = s + \ln(1+s) - t - \ln(1+t) + x$. Therefore, $h(t^0) = s_2^{\text{out}}(t^0 + 1, 0) - t^0$ solves $\Psi(h(t^0), t^0) = 0$, where

$$\Psi(h, t^0) = h + \ln(1 + h + t^0) - 2 - \ln(2 + t^0).$$

Taking the derivative of the relation $\Psi(h(t^0), t^0) = 0$ and using the fact that $h \geq 1$ by (15), we see that $h'(t^0) \geq 0$, so that h is non-decreasing. Since $h \leq 2$ by (19), the function h is thus a bounded non-decreasing function and, consequently, $\lim_{t^0 \rightarrow +\infty} h(t^0)$ exists and is equal to $\sup_{t^0 \geq 0} h(t^0) = T_{\text{unif}}(\Lambda)$. Writing the relation $\Psi(h(t^0), t^0) = 0$ as follows for $t^0 > 0$

$$h(t^0) + \ln\left(\frac{1}{t^0} + \frac{h(t^0)}{t^0} + 1\right) - 2 - \ln\left(\frac{2}{t^0} + 1\right) = 0,$$

212 and letting $t^0 \rightarrow +\infty$ we obtain the value $T_{\text{unif}}(\Lambda) = 2$.

Remark 1.7. Observe that the time $T_{\text{unif}}(\Lambda)$ does not depend on the parameters M and Q . It depends only on Λ on $[0, +\infty) \times (0, 1)$. Moreover, this is the best time one can obtain, uniformly with respect to all the possible choices of M and Q (this explains our notation “ $T_{\text{unif}}(\Lambda)$ ”). More precisely,

$$T_{\text{unif}}(\Lambda) = \min E,$$

where E is the set of $T > 0$ such that, for every M and Q with the regularity (5), there exists a state-feedback gain function $F \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ so that the system (1) with feedback law (6) is finite-time stable with settling time T . Indeed, Theorem 1.5 establishes that $T_{\text{unif}}(\Lambda) \in E$, so that $E \neq \emptyset$. On the other hand, taking $M = 0$ and the constant matrix

$$Q = \left(\underbrace{\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}}_{m-1} \right\} 1 \quad \left. \vphantom{\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}} \right\} p-1 \quad .$$

213 we can check from the very definition of broad solution (see Definition A.1) that, if $T < T_{\text{unif}}(\Lambda)$,
214 then there exist $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$ such that the corresponding solution to (1) satisfies
215 $y(t^0 + T, \cdot) \neq 0$, whatever $u \in L^\infty(t^0, t^0 + T)^m$ is.

216 Of course, for particular choices of M and Q one may obtain a better settling time (a trivial
217 example being $M = 0$ and $Q = 0$). In the case of time-independent systems, the minimal time
218 in which one can achieve the stabilization and related controllability properties has been recently
219 discussed in [CN19] and [HO19] (see also the references therein).

220 *Remark 1.8.* If the speeds do not depend on time, i.e. $\lambda_\ell(t, x) = \lambda_\ell(x)$ for every $t \geq 0$, then we have
 221 a more explicit formula for the time $T_{\text{unif}}(\Lambda)$, namely:

$$222 \quad T_{\text{unif}}(\Lambda) = \int_0^1 \frac{1}{-\lambda_m(\xi)} d\xi + \int_0^1 \frac{1}{\lambda_{m+1}(\xi)} d\xi. \quad (24)$$

223 The value (24) is obtained by integrating over $\xi \in [0, 1]$ the differential equation satisfied by the
 224 inverse functions $\xi \mapsto \chi_m^{-1}(\xi; t, 1)$ and $\xi \mapsto \chi_{m+1}^{-1}(\xi; t, 0)$.

Remark 1.9. The assumption (3) that the negative (resp. positive) speeds are uniformly bounded from above (resp. below), despite not being necessary for the existence of a solution to (1), is to be expected for the system (1) to be finite-time stable. This is an issue that is not specific to systems and that already occurs for a single equation. Indeed, let us consider for instance the equation with speed $\lambda(t) = -e^{-t}$ (and $t^0 = 0$ to simplify):

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - e^{-t} \frac{\partial y}{\partial x}(t, x) = 0, \\ y(t, 1) = u(t), \\ y(0, x) = y^0(x). \end{cases}$$

Then, whatever $y^0 \in L^2(0, 1)$ and $u \in L^\infty(0, +\infty)$ are, if $y^0 \neq 0$ in a neighborhood of 1 we have

$$y(T, \cdot) \neq 0, \quad \forall T > 0.$$

This is easily seen thanks to the explicit representation of the solution (obtained by the characteristic method):

$$y(t, x) = \begin{cases} y^0(1 - (e^{-t} - x)) & \text{if } 0 < x < e^{-t}, \\ u\left(\ln\left(\frac{1}{1 + e^{-t} - x}\right)\right) & \text{if } e^{-t} < x < 1. \end{cases}$$

225 *Remark 1.10.* Contrary to (3), the assumption (4) is mainly technical. This assumption is needed
 226 because we will have to divide in the sequel by the quantities $\lambda_j - \lambda_i$ (see in particular (67) below)
 227 and we will need this inverse function to be bounded. However, this condition is clearly not necessary
 228 for some systems (1) to be finite-time stable. Indeed, consider for instance the following 3×3 system:

$$229 \quad \begin{cases} \frac{\partial y_1}{\partial t}(t, x) - \frac{\partial y_1}{\partial x}(t, x) = y_2(t, x), \\ \frac{\partial y_2}{\partial t}(t, x) - \left(1 - \frac{e^{-t}}{2}\right) \frac{\partial y_2}{\partial x}(t, x) = 0, \\ \frac{\partial y_3}{\partial t}(t, x) + \frac{\partial y_3}{\partial x}(t, x) = 0, \\ y_1(t, 1) = u_1(t), \quad y_2(t, 1) = u_2(t), \quad y_3(t, 0) = y_2(t, 0), \\ y(t^0, x) = y^0(x). \end{cases} \quad (25)$$

230 Then, using the characteristic method it is not difficult to see that the system (25) with $u_1 = u_2 = 0$
 231 is finite-time stable with settling time $T + 1$, where T is the unique positive solution to the equation
 232 $T + \frac{e^{-T}}{2} = \frac{3}{2}$.

233 *Remark 1.11.* The case $m = n \geq 1$ (no boundary conditions at $x = 0$) is easier and does not require
 234 the techniques presented in this paper. Indeed, it can be checked using for instance the constructive
 235 method of [LR03, Wan06] that in this case the system (1) with $u = 0$ is finite-time stable with settling
 236 time equal to $\sup_{t^0 \geq 0} s_m^{\text{out}}(t^0, 1) - t^0$.

237 **2 System transformations**

238 The goal of this section is to show that we can use several invertible transformations in order to
 239 remove or transform some coupling terms in the initial system (1) and to obtain in the end a system
 240 for which we can directly establish that it is finite-time stable with settling time $T_{\text{unif}}(\Lambda)$. The plan
 241 of this section is as follows:

- 242 1) In Section 2.1, we use a diagonal transformation to remove the diagonal terms in M .
- 243 2) Next, in Section 2.2, inspired by the seminal works [Col77, Rus78, BK02] for equations and
 244 [CVKB13, HDM15, HDMVK16, HVDMK19] for hyperbolic systems, we use a Volterra trans-
 245 formation of the second kind to transform the system obtained by the previous step into a new
 246 system in the so-called “control normal form” and with an additional triangular structure for
 247 the couplings.
- 248 3) Finally, in Section 2.3, inspired by the work [CHO17] for time-independent systems, we use an
 249 invertible Fredholm integral transformation to transform the system obtained by the previous
 250 step into a new system with a very simple coupling structure that allows us to readily see that
 251 it is finite-time stable with settling time $T_{\text{unif}}(\Lambda)$.

252 In Section 2 only the properties of the transformations and new systems are discussed. The
 253 existence of the transformations is the main technical point of this paper and will be proved in
 254 Section 3 below for the sake of the presentation.

255 Finally, because of the nature of the transformations that we will use in the sequel, we are led to
 256 consider a class of systems that is slightly more general than (1). All the systems of this paper will
 257 have the following form:

$$258 \quad \begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial y}{\partial x}(t, x) = M(t, x)y(t, x) + G(t, x)y(t, 0), \\ y_-(t, 1) = \int_0^1 F(t, \xi)y(t, \xi) d\xi, \quad y_+(t, 0) = Q(t)y_-(t, 0), \\ y(t^0, x) = y^0(x), \end{cases} \quad (26)$$

where M and Q will have at least the regularity (5), $F \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ and

$$G \in C^0([0, +\infty) \times [0, 1])^{n \times n} \cap L^\infty((0, +\infty) \times (0, 1))^{n \times n}.$$

Therefore, (26) is similar to (1) but has the extra term with G . In what follows, we will also refer to a system of the form (26) as

$$(M, G, F, Q).$$

259 Hyperbolic equations similar to $(0, G, F, Q)$ were called in “control normal form” in the pioneering
 260 work [Rus78, p. 212] for the similarity with the finite-dimensional setting (see also the earlier paper
 261 [Bru70]).

262 **2.1 Removal of the diagonal terms**

263 In this section we just perform a simple preliminary transformation in order to remove the diagonal
 264 terms in M . This is only a technical step, which is nevertheless necessary in view of the existence of
 265 the transformation that we will use in the next section, see Remark 2.7 below. This step is sometimes
 266 called “exponential pre-transformation” in the case of time-independent systems (see Remark 2.3
 267 below). More precisely, the goal of this section is to establish the following result:

268 **Proposition 2.1.** *There exists $M^1 = (m_{ij}^1)_{1 \leq i, j \leq n} \in C^0([0, +\infty) \times [0, 1])^{n \times n} \cap L^\infty((0, +\infty) \times$*
 269 *$(0, 1))^{n \times n}$ with diagonal terms equal to zero:*

$$270 \quad m_{ii}^1 = 0, \quad \forall i \in \{1, \dots, n\}, \quad (27)$$

and there exists $Q^1 \in C^0([0, +\infty))^{p \times m} \cap L^\infty(0, +\infty)^{p \times m}$ such that, for every $F^1 \in L^\infty((0, +\infty) \times$
 $(0, 1))^{m \times n}$, there exists $F \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ such that the following property holds for every
 $T > 0$:

$$\begin{aligned} (M^1, 0, F^1, Q^1) \text{ is finite-time stable with settling time } T \\ \implies (M, 0, F, Q) \text{ is finite-time stable with settling time } T. \end{aligned} \quad (28)$$

271 2.1.1 Formal computations

272 To prove Proposition 2.1, the idea is to show that, for every F^1 , there exists F such that we can
 273 transform a solution of $(M, 0, F, Q)$ into a solution of $(M^1, 0, F^1, Q^1)$. Let then y be the solution to
 274 the system $(M, 0, F, Q)$ with state-feedback gain function F to be determined below and initial data
 275 y^0 . Let $\Phi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^{n \times n}$ be a smooth matrix-valued function and set

$$276 \quad w(t, x) = \Phi(t, x)y(t, x). \quad (29)$$

Let us now perform some formal computations in order to see what w can solve. Using the equation
 satisfied by y , we have

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = \left(\frac{\partial \Phi}{\partial t} + \Phi M + \Lambda \frac{\partial \Phi}{\partial x} \right) y + (-\Phi \Lambda + \Lambda \Phi) \frac{\partial y}{\partial x}.$$

On the other hand, using the boundary condition satisfied by y at $x = 0$, we have

$$\begin{aligned} w_+(t, 0) - Q^1(t)w_-(t, 0) \\ = (\Phi_{+-}(t, 0) + \Phi_{++}(t, 0)Q(t) - Q^1(t)\Phi_{--}(t, 0) - Q^1(t)\Phi_{-+}(t, 0)Q(t)) y_-(t, 0). \end{aligned}$$

Finally, at $x = 1$, we have

$$w_-(t, 1) - \int_0^1 F^1(t, \xi)w(t, \xi) d\xi = \int_0^1 (\Phi_{--}(t, 1)F(t, \xi) - F^1(t, \xi)\Phi(t, \xi)) y(t, \xi) d\xi + \Phi_{-+}(t, 1)y_+(t, 1).$$

277 Thus, we see that w satisfies at $x = 1$ the boundary condition $w_-(t, 1) = \int_0^1 F^1(t, \xi)w(t, \xi) d\xi$ if
 278 $\Phi_{-+}(t, 1) = 0$ and

$$279 \quad F(t, \xi) = \Phi_{--}(t, 1)^{-1}F^1(t, \xi)\Phi(t, \xi), \quad (30)$$

280 provided that $\Phi_{--}(t, 1)$ is also invertible. Moreover, note that F belongs to $L^\infty((0, +\infty) \times (0, 1))^{m \times n}$
 281 provided that F^1 belongs to this space as well and

$$282 \quad \exists C > 0, \quad \|\Phi_{--}(\cdot, 1)^{-1}\|_{L^\infty(0, +\infty)^{m \times m}} \leq C. \quad (31)$$

283 In summary, w defined by (29) is the solution of $(M^1, 0, F^1, Q^1)$ with state-feedback gain function
 284 F^1 (which is assumed to be known) and initial data $w^0(\cdot) = \Phi(0, \cdot)y^0(\cdot)$ if we have the following four
 285 properties:

- 286 (i) $\Lambda(t, x)\Phi(t, x) = \Phi(t, x)\Lambda(t, x)$ for every $t \geq 0$ and $x \in [0, 1]$.
- 287 (ii) The matrices $\Phi(t, x)$ and $\Phi_{--}(t, 0) + \Phi_{-+}(t, 0)Q(t)$ are invertible for every $t \geq 0$ and $x \in [0, 1]$.
- 288 (iii) $\Phi_{-+}(t, 1) = 0$ for every $t \geq 0$ (it then follows with (ii) that $\Phi_{--}(t, 1)$ is invertible).

289 (iv) M^1 and Q^1 are defined by

$$290 \quad \begin{cases} M^1(t, x) = \left(\frac{\partial \Phi}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \Phi}{\partial x}(t, x) + \Phi(t, x)M(t, x) \right) \Phi(t, x)^{-1}, \\ Q^1(t) = (\Phi_{+-}(t, 0) + \Phi_{++}(t, 0)Q(t)) (\Phi_{--}(t, 0) + \Phi_{-+}(t, 0)Q(t))^{-1}. \end{cases} \quad (32)$$

291 Finally, it is not difficult to check that the stability property (28) is indeed satisfied since the
292 state-feedback gain function F is solely determined by the state-feedback gain function F^1 and, at
293 every fixed $t \geq 0$, the transformation (29) defines an injective (in fact, invertible) map of $L^2(0, 1)^n$.

294 2.1.2 Existence of the transformation

295 Let us now prove the existence of a function Φ with the properties listed above and which in addition
296 ensures that the condition (27) on M^1 holds.

297 **Proposition 2.2.** *There exists Φ with $\Phi, \frac{\partial \Phi}{\partial t} + \Lambda \frac{\partial \Phi}{\partial x} \in C^0([0, +\infty) \times [0, 1])^{n \times n} \cap L^\infty((0, +\infty) \times$
298 $(0, 1))^{n \times n}$ such that the properties (i), (ii), (iii) and (31) are satisfied and such that the matrix-
299 valued function M^1 defined in (32) satisfies (27).*

Proof. Let Φ be the diagonal matrix-valued function defined for every $t \geq 0$ and $x \in [0, 1]$ by

$$\Phi(t, x) = \text{diag}(\phi_1(t, x), \dots, \phi_n(t, x)),$$

300 where, for every $i \in \{1, \dots, n\}$,

$$301 \quad \phi_i(t, x) = e^{-\int_{s_i^{\text{in}}(t, x)}^t m_{ii}(\sigma, \chi_i(\sigma; t, x)) d\sigma}, \quad (33)$$

302 where m_{ii} is extended to negative times by an arbitrary function that keeps the regularity (5). Clearly,
303 $\phi_i \in C^0([0, +\infty) \times [0, 1])$ and it follows from (19) that $\phi_i \in L^\infty((0, +\infty) \times (0, 1))$.

It is clear that the first property (i) holds since Λ and Φ are both diagonal matrices. Since
 $\Phi_{-+} = 0$, the third property (iii) is automatically satisfied. It also follows that, to check the second
property (ii), we only need to show that $\Phi(t, x)$ is invertible, which readily follows from the explicit
expression of ϕ_i . The estimate (31) is obviously true since $\Phi_{--}(t, 1) = \text{Id}_{\mathbb{R}^m \times m}$ (recall (16)). Finally,
 M^1 defined in (32) satisfies (27) since ϕ_i satisfies the following linear hyperbolic equation:

$$\frac{\partial \phi_i}{\partial t}(t, x) + \lambda_i(t, x) \frac{\partial \phi_i}{\partial x}(t, x) + m_{ii}(t, x) \phi_i(t, x) = 0.$$

304 □

305 *Remark 2.3.* There are obviously other possible choices for ϕ_i , for instance in the time-independent
306 case we can take the slightly simpler function $\phi_i(t, x) = e^{-\int_0^x \frac{m_{ii}(\xi)}{\lambda_i(\xi)} d\xi}$ (which coincides with (33) only
307 for $i \in \{m+1, \dots, n\}$).

308 2.2 Volterra transformation

309 In this section we perform a second transformation to remove some coupling terms of the system.
310 The system will then have a triangular coupling structure, which is the key point to show later on
311 (Section 2.3 below) that this system is finite-time stable with settling time $T_{\text{unif}}(\Lambda)$. More precisely,
312 the goal of this section is to establish the following result:

313 **Proposition 2.4.** *There exists a strictly lower triangular matrix $G_{--}^2 = (g_{ij}^2)_{1 \leq i, j \leq m} \in C^0([0, +\infty) \times$*
 314 *$[0, 1])^{m \times m} \cap L^\infty((0, +\infty) \times (0, 1))^{m \times m}$:*

$$315 \quad g_{ij}^2 = 0, \quad \forall 1 \leq i \leq j \leq m, \quad (34)$$

and there exists $G_{+-}^2 \in C^0([0, +\infty) \times [0, 1])^{p \times m} \cap L^\infty((0, +\infty) \times (0, 1))^{p \times m}$ such that, for every $F^2 \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$, there exists $F^1 \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ such that the following property holds for every $T > 0$:

$$\begin{aligned} (0, G^2, F^2, Q^1) \text{ is finite-time stable with settling time } T \\ \implies (M^1, 0, F^1, Q^1) \text{ is finite-time stable with settling time } T, \end{aligned} \quad (35)$$

316 *where*

$$317 \quad G^2 = \begin{pmatrix} G_{--}^2 & 0 \\ G_{+-}^2 & 0 \end{pmatrix}. \quad (36)$$

Remark 2.5. Thanks to the triangular structure (34) and (36) of G^2 , we can check from the very definition of broad solution (see Definition A.1) that the system provided by Proposition 2.4 with state-feedback gain function equal to zero, i.e. $(0, G^2, 0, Q^1)$, is finite-time stable with settling time $T(0, G^2, 0, Q^1)$ defined by

$$T(0, G^2, 0, Q^1) = \sup_{t^0 \geq 0} s_{m+1}^{\text{out}}(T_m(t^0), 0) - t^0,$$

where

$$\begin{cases} T_1(t^0) = s_1^{\text{out}}(t^0, 1), \\ T_i(t^0) = s_i^{\text{out}}(T_{i-1}(t^0), 1), \quad \forall i \in \{2, \dots, m\}. \end{cases}$$

We do not detail this point here because it is not needed, and we refer to the arguments used in the proof of Proposition 2.12 below for an idea of the proof of this assertion. As a result, the combination of Proposition 2.4 with Proposition 2.1 already shows that our initial system $(M, 0, F, Q)$ is finite-time stable for some F , with settling time $T(0, G^2, 0, Q^1)$. However, this time $T(0, G^2, 0, Q^1)$ is always strictly larger than the time $T_{\text{unif}}(\Lambda)$ given in Theorem 1.5 (as long as $m > 1$). In the case of time-independent systems, the time $T(0, G^2, 0, Q^1)$ is the time obtained in [HDMVK16, HVDMK19] and it has the more explicit expression

$$T(0, G^2, 0, Q^1) = \sum_{i=1}^m \int_0^1 \frac{1}{-\lambda_i(\xi)} d\xi + \int_0^1 \frac{1}{\lambda_{m+1}(\xi)} d\xi.$$

318 2.2.1 Formal computations

319 Let us now show how to establish Proposition 2.4. As before, the goal is to show that, for every
 320 F^2 , there exists F^1 such that we can transform a solution of $(M^1, 0, F^1, Q^1)$ into a solution of
 321 $(0, G^2, F^2, Q^1)$. Let then w be the solution to the system $(M^1, 0, F^1, Q^1)$ with state-feedback gain
 322 function F^1 to be determined below and initial data w^0 . Inspired by the works mentioned at the
 323 beginning of Section 2, we use a Volterra transformation of the second kind as follows:

$$324 \quad \gamma(t, x) = w(t, x) - \int_0^x K(t, x, \xi) w(t, \xi) d\xi, \quad (37)$$

where we suppose for the moment that the kernel K is smooth on $\overline{\mathcal{T}}$, where \mathcal{T} is the infinite triangular prism defined by

$$\mathcal{T} = \{(t, x, \xi) \in (0, +\infty) \times (0, 1) \times (0, 1), \quad \xi < x\}.$$

Let us now perform some formal computations to see what γ can solve. We have

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= \frac{\partial w}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial w}{\partial x}(t, x) \\ &\quad - \int_0^x \frac{\partial K}{\partial t}(t, x, \xi) w(t, \xi) d\xi - \int_0^x K(t, x, \xi) \frac{\partial w}{\partial t}(t, \xi) d\xi \\ &\quad - \Lambda(t, x) K(t, x, x) w(t, x) - \Lambda(t, x) \int_0^x \frac{\partial K}{\partial x}(t, x, \xi) w(t, \xi) d\xi. \end{aligned}$$

Using the equation satisfied by w , we obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= M^1(t, x) w(t, x) \\ &\quad - \int_0^x \frac{\partial K}{\partial t}(t, x, \xi) w(t, \xi) d\xi - \int_0^x K(t, x, \xi) \left(-\Lambda(t, \xi) \frac{\partial w}{\partial \xi}(t, \xi) + M^1(t, \xi) w(t, \xi) \right) d\xi \\ &\quad - \Lambda(t, x) K(t, x, x) w(t, x) - \Lambda(t, x) \int_0^x \frac{\partial K}{\partial x}(t, x, \xi) w(t, \xi) d\xi. \end{aligned}$$

325 Integrating by parts the third term of the right hand side and using the boundary condition $w_+(t, 0) =$
326 $Q^1(t)w_-(t, 0)$, we finally obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= \int_0^x \left(-\frac{\partial K}{\partial t}(t, x, \xi) - \frac{\partial K}{\partial \xi}(t, x, \xi) \Lambda(t, \xi) - K(t, x, \xi) \frac{\partial \Lambda}{\partial \xi}(t, \xi) \right. \\ &\quad \left. - K(t, x, \xi) M^1(t, \xi) - \Lambda(t, x) \frac{\partial K}{\partial x}(t, x, \xi) \right) w(t, \xi) d\xi \\ &\quad + \left(M^1(t, x) + K(t, x, x) \Lambda(t, x) - \Lambda(t, x) K(t, x, x) \right) w(t, x) - K(t, x, 0) \Lambda(t, 0) \begin{pmatrix} \text{Id}_{\mathbb{R}^{m \times m}} \\ Q^1(t) \end{pmatrix} w_-(t, 0). \end{aligned}$$

On the other hand, since $\gamma(t, 0) = w(t, 0)$, γ satisfies the same boundary condition as w at $x = 0$:

$$\gamma_+(t, 0) - Q^1(t) \gamma_-(t, 0) = w_+(t, 0) - Q^1(t) w_-(t, 0) = 0.$$

Finally, at $x = 1$, we have

$$\begin{aligned} \gamma_-(t, 1) - \int_0^1 F^2(t, \xi) \gamma(t, \xi) d\xi &= \\ &= \int_0^1 \left(F^1(t, \xi) - K_-(t, 1, \xi) - F^2(t, \xi) + \int_\xi^1 F^2(t, \zeta) K(t, \zeta, \xi) d\zeta \right) w(t, \xi) d\xi, \end{aligned}$$

327 where K_- denotes the $m \times n$ sub-matrix of K formed by its first m rows. Thus, we see that γ satisfies
328 at $x = 1$ the boundary condition $\gamma_-(t, 1) = \int_0^1 F^2(t, \xi) \gamma(t, \xi) d\xi$ if we take

$$329 \quad F^1(t, \xi) = K_-(t, 1, \xi) + F^2(t, \xi) - \int_\xi^1 F^2(t, \zeta) K(t, \zeta, \xi) d\zeta. \quad (38)$$

Note that F^1 belongs to $L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ provided that F^2 belongs to this space as well and

$$K \in L^\infty(\mathcal{T})^{n \times n}.$$

330 In summary, γ defined by (37) is the solution to $(0, G^2, F^2, Q^1)$ with initial data $\gamma^0(x) = w^0(x) -$
 331 $\int_0^x K(t^0, x, \xi) w^0(\xi) d\xi$ if we have the following two properties:

332 (i) For every $(t, x, \xi) \in \mathcal{T}$,

$$\begin{cases} \frac{\partial K}{\partial t}(t, x, \xi) + \Lambda(t, x) \frac{\partial K}{\partial x}(t, x, \xi) + \frac{\partial K}{\partial \xi}(t, x, \xi) \Lambda(t, \xi) \\ \quad + K(t, x, \xi) \left(\frac{\partial \Lambda}{\partial \xi}(t, \xi) + M^1(t, \xi) \right) = 0, \\ K(t, x, x) \Lambda(t, x) - \Lambda(t, x) K(t, x, x) = -M^1(t, x). \end{cases} \quad (39)$$

333

(ii) G^2 is defined by

$$G^2 = \begin{pmatrix} G_{--}^2 & 0 \\ G_{+-}^2 & 0 \end{pmatrix},$$

334

with

$$\begin{cases} G_{--}^2(t, x) = -K_{--}(t, x, 0) \Lambda_{--}(t, 0) - K_{-+}(t, x, 0) \Lambda_{++}(t, 0) Q^1(t), \\ G_{+-}^2(t, x) = -K_{+-}(t, x, 0) \Lambda_{--}(t, 0) - K_{++}(t, x, 0) \Lambda_{++}(t, 0) Q^1(t). \end{cases} \quad (40)$$

335

336 Finally, the stability property (35) is clearly satisfied since, at every fixed $t \geq 0$, the Volterra
 337 transformation (37) defines an injective map of $L^2(0, 1)^n$ (see e.g. [Hoc73, Theorem 2.6]).

338 2.2.2 The kernel equations

339 We can prove that there exists $K \in C^0(\overline{\mathcal{T}})^{n \times n} \cap L^\infty(\mathcal{T})^{n \times n}$ that satisfies the so-called “kernel equa-
 340 tions” (39) in the sense of broad solutions. However, it is in general not enough to deduce sta-
 341 bility results for the initial system $(M, 0, F, Q)$ since the investigation of the stability properties of
 342 the system $(0, G^2, F^2, Q^1)$ is not an easier task without knowing any more information about it.
 343 The breakthrough idea of the conference paper [HDM15] in the time-independent case (see also
 344 [HDMVK16, HVDMK19]) was to construct a solution K to the kernel equations which, in addition,
 345 yields a simpler structure for the matrix G_{--}^2 defined in (40). This is the key point to prove stability
 346 results for the system $(0, G^2, F^2, Q^1)$ (see Remark 2.5 and Section 2.3 below). Such a construction
 347 is possible by adding some conditions for K_{--} at $(t, x, 0)$ (see (40)) but the price to pay is that it
 348 introduces discontinuities for K_{--} , so that K will not be globally C^0 anymore but only piecewise C^0
 349 in general. We will prove the following result:

350 **Theorem 2.6.** *There exists a $n \times n$ matrix-valued function $K = (k_{ij})_{1 \leq i, j \leq n}$ such that:*

351 (i) $K \in L^\infty(\mathcal{T})^{n \times n}$.

352 (ii) For every $i, j \in \{1, \dots, n\}$ with $j \notin \{i+1, \dots, m\}$, we have $k_{ij} \in C^0(\overline{\mathcal{T}})$.

(iii) For every $i, j \in \{1, \dots, m\}$ with $i < j$, we have $k_{ij} \in C^0(\overline{\mathcal{T}_{ij}^-}) \cap C^0(\overline{\mathcal{T}_{ij}^+})$, where (see Figure 1)

$$\begin{aligned} \mathcal{T}_{ij}^- &= \{(t, x, \xi) \in \mathcal{T}, \quad \xi < \psi_{ij}(t, x)\}, \\ \mathcal{T}_{ij}^+ &= \{(t, x, \xi) \in \mathcal{T}, \quad \xi > \psi_{ij}(t, x)\}, \end{aligned}$$

353 where $\psi_{ij} \in C^1([0, +\infty) \times [0, 1])$ satisfies the following semi-linear hyperbolic equation for every
 354 $t \geq 0$ and $x \in [0, 1]$:

$$355 \quad \begin{cases} \frac{\partial \psi_{ij}}{\partial t}(t, x) + \lambda_i(t, x) \frac{\partial \psi_{ij}}{\partial x}(t, x) - \lambda_j(t, \psi_{ij}(t, x)) = 0, \\ \psi_{ij}(t, 0) = 0. \end{cases} \quad (41)$$

356 (iv) K is a broad solution of (39) in \mathcal{T} (the exact meaning of this statement will be detailed during
 357 the proof of the theorem, in Section 3.2 below).

358 (v) For every $t \geq 0$ and $x \in [0, 1]$, the matrix $G_{--}^2(t, x)$ defined in (40) is strictly lower triangular,
 359 i.e. it satisfies (34) (it then follows from (ii) that $G_{--}^2 \in C^0([0, +\infty) \times [0, 1])^{m \times m}$).

360 The proof of Theorem 2.6 is one of the main technical difficulties of this article and it is postponed
 361 to Section 3.2 below for the sake of the presentation. We conclude this section with some important
 362 remarks.

363 *Remark 2.7.* Let us rewrite the second condition of (39) component-wise:

$$364 \quad (\lambda_j(t, x) - \lambda_i(t, x)) k_{ij}(t, x, x) = -m_{ij}^1(t, x). \quad (42)$$

365 Therefore, we see that for $i = j$ we shall necessarily have $m_{ii}^1 = 0$ and it explains why we had to
 366 perform a preliminary transformation in Section 2.1 to remove these terms (otherwise the equation
 367 (42), and thus the kernel equations (39), have no solution).

368 *Remark 2.8.* It is in general not possible to solve (39) with $G_{--}^2 = 0$, unless $m = 1$.

Remark 2.9. Observe that, with the regularity stated in Theorem 2.6, we have in particular that, for every $w \in C^0([t^0, +\infty); L^2(0, 1)^n)$, $t^0 \geq 0$,

$$(t, x) \mapsto \int_0^x K(t, x, \xi) w(t, \xi) d\xi \in C^0([t^0, +\infty) \times [0, 1])^n.$$

369 This follows from Lebesgue's dominated convergence theorem. This shows that γ defined by (37) has
 370 the good regularity to be a broad solution (see Definition A.1), if so has w .

Remark 2.10. Observe that the condition (iii) shows that the kernel has possible discontinuities on
 $\xi = \psi_{ij}(t, x)$ for $i < j \leq m$. Besides, these discontinuities also depend on the component of the kernel
 that we consider. The appearance of such discontinuities is explained by the requirement of the last
 condition (v) because we somehow force two boundary conditions at the points $(t, 0, 0)$, one by the
 condition already required in (39) (which concerns $i \neq j$, see Remark 2.7) and another one by (v)
 (which only concerns $i \leq j \leq m$). This results in discontinuities along the characteristics passing
 through these points. Note that this also complicates the justification of formal computations that we
 performed above since regularity problems will occur during the computation of the following term
 (when $i < j \leq m$):

$$\frac{\partial}{\partial t} \left(\int_0^x k_{ij}(t, x, \xi) w_j(t, \xi) d\xi \right) + \lambda_i(t, x) \frac{\partial}{\partial x} \left(\int_0^x k_{ij}(t, x, \xi) w_j(t, \xi) d\xi \right).$$

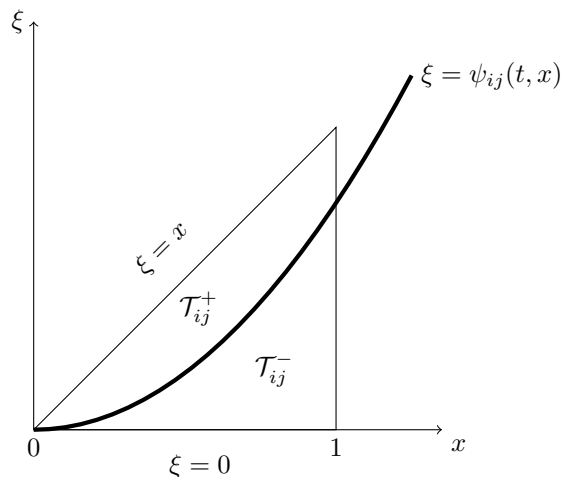
More precisely, writing $\int_0^x = \int_0^{\psi_{ij}(t, x)} + \int_{\psi_{ij}(t, x)}^x$ and using integration by parts, we see that the
 following jump terms notably appear:

$$\begin{aligned} & \left(\frac{\partial \psi_{ij}}{\partial t}(t, x) - \lambda_j(t, \psi_{ij}(t, x)) + \lambda_i(t, x) \frac{\partial \psi_{ij}}{\partial x}(t, x) \right) \\ & \quad \times (k_{ij}^-(t, x, \psi_{ij}(t, x)) - k_{ij}^+(t, x, \psi_{ij}(t, x))) w_j(t, \psi_{ij}(t, x)), \end{aligned}$$

where k_{ij}^- (resp. k_{ij}^+) denotes the trace of the restriction of k_{ij} to $\partial\mathcal{T}_{ij}^-$ (resp $\partial\mathcal{T}_{ij}^+$). This is why it is crucial to precise that ψ_{ij} solves the first equation in (41) so that such undesired terms vanish in the end. In the case of time-independent systems, we have in fact

$$\psi_{ij}(t, x) = \phi_j^{-1}(\phi_i(x)),$$

371 where we introduced $\phi_\ell(x) = \int_0^x \frac{1}{-\lambda_\ell(\xi)} d\xi$ for $\ell \in \{1, \dots, m\}$ (ψ_{ij} is well defined because $i < j$). This
 372 is the same function as in [HVDKM19, (A.1)].



373

374

Figure 1: 2D cross-section of the domain \mathcal{T} at a fixed t

375 2.3 Fredholm integral transformation

376 We recall that at the moment we already know that the system $(0, G^2, F^2, Q^1)$ of Proposition 2.4 is
 377 finite-time stable if we take $F^2 = 0$, but only with a settling time which is strictly larger than $T_{\text{unif}}(\Lambda)$
 378 (unless $m = 1$), see Remark 2.5. In this section, we perform a third and last transformation to remove
 379 the coupling term G_{+-}^2 in the system $(0, G^2, F^2, Q^1)$ and we show that the resulting system has the
 380 desired stability properties. More precisely, the goal of this section is to establish the two following
 381 results:

Proposition 2.11. *There exists $F^2 \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ such that the following property holds for every $T > 0$:*

$$\begin{aligned} (0, G^3, 0, Q^1) \text{ is finite-time stable with settling time } T \\ \implies (0, G^2, F^2, Q^1) \text{ is finite-time stable with settling time } T, \end{aligned} \quad (43)$$

382 where

383

$$G^3 = \begin{pmatrix} 0 & 0 \\ G_{+-}^2 & 0 \end{pmatrix}. \quad (44)$$

384 **Proposition 2.12.** *The system $(0, G^3, 0, Q^1)$ is finite-time stable with settling time $T_{\text{unif}}(\Lambda)$ defined
 385 by (22).*

386 Note that the proof of our main result – Theorem 1.5 – will then be complete (recall Propositions
 387 2.1 and 2.4), except for the τ -periodicity statement which will be studied later on in Section 3.3.

388 **2.3.1 Finite-time stability of the system** $(0, G^3, 0, Q^1)$

389 In this section we prove Proposition 2.12 in four steps.

390 1) Let $t^0 \geq 0$ be fixed. From the very definition of broad solution (see Definition A.1) and
 391 the simple structure of G^3 , we see that the first m components of the system vanish at time
 392 $t^0 + T_{\text{unif}}(\Lambda)$ if (recall that the feedback is equal to zero)

$$393 \quad s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > t^0, \quad \forall x \in [0, 1], \quad \forall i \in \{1, \dots, m\}, \quad (45)$$

394 and the remaining p components of the system vanish at time $t^0 + T_{\text{unif}}(\Lambda)$ if

$$\begin{cases} s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > t^0, & \forall x \in [0, 1], \quad \forall i \in \{m+1, \dots, n\} \\ s_j^{\text{in}}(s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x), 0) > t^0, & \forall x \in [0, 1), \quad \forall j \in \{1, \dots, m\}, \quad \forall i \in \{m+1, \dots, n\}. \end{cases} \quad (46)$$

396 2) First of all, observe that, from (20), (17) and (16) we have the following inverse formula for
 397 every $t, \bar{t} \in \mathbb{R}$:

$$398 \quad \begin{cases} s_i^{\text{in}}(t, 0) > \bar{t} \iff t > s_i^{\text{out}}(\bar{t}, 1), & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{in}}(t, 1) \geq \bar{t} \iff t \geq s_i^{\text{out}}(\bar{t}, 0), & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \quad (47)$$

3) Let us establish (45). Let then $i \in \{1, \dots, m\}$ be fixed. We have:

$$\begin{aligned} s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > t^0, \quad \forall x \in [0, 1] &\iff s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), 0) > t^0, \quad (\text{by (20)}), \\ &\iff t^0 + T_{\text{unif}}(\Lambda) > s_i^{\text{out}}(t^0, 1), \quad (\text{by (47)}), \end{aligned}$$

and this last statement holds true since, by definition of $T_{\text{unif}}(\Lambda)$ and (15)-(16), we have, for an arbitrary $j \in \{m+1, \dots, n\}$,

$$t^0 + T_{\text{unif}}(\Lambda) \geq s_j^{\text{out}}(s_i^{\text{out}}(t^0, 1), 0) > s_i^{\text{out}}(t^0, 1).$$

4) Let us now establish (46). We focus on the second inequality since the first one is obtained similarly to (45). Let then $i \in \{m+1, \dots, n\}$ and $j \in \{1, \dots, m\}$ be fixed. We have:

$$\begin{aligned} s_j^{\text{in}}(s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x), 0) > t^0, \quad \forall x \in [0, 1] \\ \iff s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > s_j^{\text{out}}(t^0, 1), \quad \forall x \in [0, 1), \quad (\text{by (47)}), \\ \iff s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), 1) \geq s_j^{\text{out}}(t^0, 1), \quad (\text{by (20)}), \\ \iff t^0 + T_{\text{unif}}(\Lambda) \geq s_i^{\text{out}}(s_j^{\text{out}}(t^0, 1), 0), \quad (\text{by (47)}), \end{aligned}$$

399 and this last statement holds true by definition of $T_{\text{unif}}(\Lambda)$.
 400 □

401 **2.3.2 Proof of Proposition 2.11**

402 We start the proof with some computations. We will show that we can transform a solution of
 403 $(0, G^3, 0, Q^1)$ into a solution of $(0, G^2, F^2, Q^1)$ (note the difference in the order of the transformation
 404 with respect to the previous sections and see Remark 2.15 below for the reason). Let then z be the
 405 solution to the system $(0, G^3, 0, Q^1)$ with initial data z^0 . Inspired by the work [CHO17] mentioned
 406 before (for time-independent systems), we propose to use a Fredholm integral transformation as
 407 follows:

$$408 \quad \gamma(t, x) = z(t, x) - \int_0^1 H(t, x, \xi) z(t, \xi) d\xi, \quad (48)$$

where we suppose for the moment that the kernel H is smooth on $\overline{\mathcal{R}}$, where \mathcal{R} is the infinite rectangular prism defined by

$$\mathcal{R} = (0, +\infty) \times (0, 1) \times (0, 1).$$

Let us now perform some formal computations and see what γ can solve. We have

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= \frac{\partial z}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial z}{\partial x}(t, x) \\ &\quad - \int_0^1 \frac{\partial H}{\partial t}(t, x, \xi) z(t, \xi) d\xi - \int_0^1 H(t, x, \xi) \frac{\partial z}{\partial t}(t, \xi) d\xi - \Lambda(t, x) \int_0^1 \frac{\partial H}{\partial x}(t, x, \xi) z(t, \xi) d\xi. \end{aligned}$$

Using the equation satisfied by z , we obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= G^3(t, x) z(t, 0) - \int_0^1 \frac{\partial H}{\partial t}(t, x, \xi) z(t, \xi) d\xi \\ &\quad - \int_0^1 H(t, x, \xi) \left(-\Lambda(t, \xi) \frac{\partial z}{\partial \xi}(t, \xi) + G^3(t, \xi) z(t, 0) \right) d\xi - \Lambda(t, x) \int_0^1 \frac{\partial H}{\partial x}(t, x, \xi) z(t, \xi) d\xi. \end{aligned}$$

409 Integrating by parts the third term of the right hand side, we obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= \int_0^1 \left(-\frac{\partial H}{\partial t}(t, x, \xi) - \frac{\partial H}{\partial \xi}(t, x, \xi) \Lambda(t, \xi) - H(t, x, \xi) \frac{\partial \Lambda}{\partial \xi}(t, \xi) \right. \\ &\quad \left. - \Lambda(t, x) \frac{\partial H}{\partial x}(t, x, \xi) \right) z(t, \xi) d\xi \\ &\quad + H(t, x, 1) \Lambda(t, 1) z(t, 1) + \left(G^3(t, x) - H(t, x, 0) \Lambda(t, 0) - \int_0^1 H(t, x, \xi) G^3(t, \xi) d\xi \right) z(t, 0). \end{aligned}$$

Using the formula (48) with $x = 0$ we finally obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) &= \int_0^1 \left(-\frac{\partial H}{\partial t}(t, x, \xi) - \frac{\partial H}{\partial \xi}(t, x, \xi) \Lambda(t, \xi) - H(t, x, \xi) \frac{\partial \Lambda}{\partial \xi}(t, \xi) \right. \\ &\quad \left. - \Lambda(t, x) \frac{\partial H}{\partial x}(t, x, \xi) + \left(G^3(t, x) - H(t, x, 0) \Lambda(t, 0) - \int_0^1 H(t, x, \zeta) G^3(t, \zeta) d\zeta \right) H(t, 0, \xi) \right) z(t, \xi) d\xi \\ &\quad + H(t, x, 1) \Lambda(t, 1) z(t, 1) + \left(G^3(t, x) - H(t, x, 0) \Lambda(t, 0) - \int_0^1 H(t, x, \xi) G^3(t, \xi) d\xi \right) \gamma(t, 0). \end{aligned}$$

410 Since

$$411 \quad z_-(t, 1) = 0, \quad (49)$$

the boundary term $H(t, x, 1)\Lambda(t, 1)z(t, 1)$ vanishes if we require that H satisfies

$$H_{-+}(t, x, 1) = H_{++}(t, x, 1) = 0.$$

On the other hand, γ and z satisfy the same boundary condition at $x = 0$ provided that

$$H(t, 0, \xi) = 0.$$

Finally, at $x = 1$, we have (recall (49))

$$\gamma_{-}(t, 1) - \int_0^1 F^2(t, \xi)\gamma(t, \xi) d\xi = \int_0^1 \left(-H_{-}(t, 1, \xi) - F^2(t, \xi) + \int_0^1 F^2(t, \zeta)H(t, \zeta, \xi) d\zeta \right) z(t, \xi) d\xi,$$

412 where H_{-} denotes again the $m \times n$ sub-matrix of H formed by its first m rows. Thus, we see that
 413 γ satisfies at $x = 1$ the boundary condition $\gamma_{-}(t, 1) = \int_0^1 F^2(t, \xi)\gamma(t, \xi) d\xi$ if $F^2(t, \cdot)$ satisfies the
 414 following Fredholm integral equation (at t fixed):

$$415 \quad F^2(t, \xi) - \int_0^1 F^2(t, \zeta)H(t, \zeta, \xi) d\zeta = -H_{-}(t, 1, \xi). \quad (50)$$

416 In summary, γ defined by (48) is the solution of $(0, G^2, F^2, Q^1)$ with state-feedback gain function
 417 F^2 satisfying (50) (whenever it exists) and initial data $\gamma^0(x) = z^0(x) - \int_0^1 H(t^0, x, \xi)z^0(\xi) d\xi$ if we
 418 have the following two properties:

419 (i) For every $(t, x, \xi) \in \mathcal{R}$,

$$420 \quad \begin{cases} \frac{\partial H}{\partial t}(t, x, \xi) + \Lambda(t, x)\frac{\partial H}{\partial x}(t, x, \xi) + \frac{\partial H}{\partial \xi}(t, x, \xi)\Lambda(t, \xi) + H(t, x, \xi)\frac{\partial \Lambda}{\partial \xi}(t, \xi) = 0, \\ H_{-+}(t, x, 1) = H_{++}(t, x, 1) = H(t, 0, \xi) = 0. \end{cases} \quad (51)$$

421 (ii) G^3 satisfies the Fredholm integral equation

$$422 \quad G^3(t, x) - \int_0^1 H(t, x, \xi)G^3(t, \xi) d\xi = G^2(t, x) + H(t, x, 0)\Lambda(t, 0). \quad (52)$$

423 Finally, the stability property (43) is clearly satisfied if, for every $t \geq 0$, the Fredholm transformation
 424 (48) defines a surjective map of $L^2(0, 1)^n$.

425 It remains to prove the existence of F^2 and H satisfying the above properties and so that the
 426 Fredholm transformation (48) is invertible (let us recall that, unlike Volterra transformations of the
 427 second kind, Fredholm transformations are not always invertible). Note that $H = 0$ is a solution of
 428 (51). Taking into account the very particular structure (36) of G^2 , this motivates our attempt to
 429 look for a kernel H with the following simple structure:

$$430 \quad H = \begin{pmatrix} H_{--} & 0 \\ 0 & 0 \end{pmatrix}. \quad (53)$$

This structure implies that the Fredholm equation (52) is equivalent to

$$\begin{cases} G^3_{--}(t, x) - \int_0^1 H_{--}(t, x, \xi)G^3_{--}(t, \xi) d\xi = G^2_{--}(t, x) + H_{--}(t, x, 0)\Lambda_{--}(t, 0), \\ G^3_{-+}(t, x) - \int_0^1 H_{--}(t, x, \xi)G^3_{-+}(t, \xi) d\xi = 0, \\ G^3_{+-}(t, x) = G^2_{+-}(t, x), \\ G^3_{++}(t, x) = 0. \end{cases}$$

456 *Remark 2.15.* If we prefer to use the inverse transformation

$$457 \quad z(t, x) = \gamma(t, x) - \int_0^1 L(t, x, \xi) \gamma(t, \xi) d\xi, \quad (55)$$

where L has the same structure as H (i.e. only L_{--} is not zero), then the corresponding kernel equations are

$$\left\{ \begin{array}{l} \frac{\partial L_{--}}{\partial t}(t, x, \xi) + \Lambda_{--}(t, x) \frac{\partial L_{--}}{\partial x}(t, x, \xi) + \frac{\partial L_{--}}{\partial \xi}(t, x, \xi) \Lambda_{--}(t, \xi) \\ \quad + L_{--}(t, x, \xi) \frac{\partial \Lambda_{--}}{\partial \xi}(t, \xi) - L_{--}(t, x, 1) \Lambda_{--}(t, 1) L_{--}(t, 1, \xi) = 0, \\ L_{--}(t, 0, \xi) = 0, \\ L_{--}(t, x, 0) = \left(G_{--}^2(t, x) - \int_0^1 L_{--}(t, x, \xi) G_{--}^2(t, \xi) d\xi \right) \Lambda_{--}(t, 0)^{-1}. \end{array} \right.$$

458 We see that these equations are slightly more complicated than (54) since there is a nonlinear
459 and nonlocal term. This explains why we had a preference for the transformation (48) over (55) but
460 there is no obstruction to work with (55).

461 3 Existence of a solution to the kernel equations

462 In this section we prove Theorem 2.6 and Theorem 2.13, which are the two key results for the present
463 article, and we describe in Section 3.3 how to obtain a time-periodic feedback. We propose to start
464 with the proof of Theorem 2.13 because it is far more simpler (in particular, no fixed-point argument
465 is needed).

466 3.1 Kernel for the Fredholm transformation

467 In this section we prove Theorem 2.13, that is we prove the existence of a suitably smooth matrix-
468 valued function $H_{--} = (h_{ij})_{1 \leq i, j \leq m}$ which is strictly lower triangular and satisfies (54) (in some
469 sense).

470 Writing (54) component-wise, this gives

$$471 \quad \left\{ \begin{array}{l} \frac{\partial h_{ij}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial h_{ij}}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial h_{ij}}{\partial \xi}(t, x, \xi) + \frac{\partial \lambda_j}{\partial \xi}(t, \xi) h_{ij}(t, x, \xi) = 0, \\ h_{ij}(t, 0, \xi) = 0, \\ h_{ij}(t, x, 0) = -\frac{g_{ij}^2(t, x)}{\lambda_j(t, 0)}. \end{array} \right. \quad (56)$$

Since we see that the equations are uncoupled, we can fix the indices i, j for the remainder of Section 3.1:

$$i, j \in \{1, \dots, m\} \text{ are fixed.}$$

472 3.1.1 The characteristics of (56)

For each $(t, x, \xi) \in \mathbb{R}^3$ fixed, we introduce the characteristic curve $\chi_{ij}(\cdot; t, x)$ associated with the hyperbolic equation (56) passing through the point (t, x, ξ) , i.e.

$$\chi_{ij}(s; t, x, \xi) = (s, \chi_i(s; t, x), \chi_j(s; t, \xi)), \quad \forall s \in \mathbb{R},$$

where we recall that χ_i and χ_j are defined in (9). For every $(t, x, \xi) \in \mathcal{R}$, we have

$$\chi_{ij}(s; t, x, \xi) \in \mathcal{R}, \quad \forall s \in (s_{ij}^{\text{in}}(t, x, \xi), s_{ij}^{\text{out}}(t, x, \xi)),$$

where we introduced

$$s_{ij}^{\text{in}}(t, x, \xi) = \max \{0, s_i^{\text{in}}(t, x), s_j^{\text{in}}(t, \xi)\} > 0, \quad s_{ij}^{\text{out}}(t, x, \xi) = \min \{s_i^{\text{out}}(t, x), s_j^{\text{out}}(t, \xi)\}.$$

Since the speeds λ_i, λ_j are negative ($i, j \leq m$), when s is increasing, $s \mapsto \chi_i(s; t, x), s \mapsto \chi_j(s; t, \xi)$ are decreasing. Therefore, the associated characteristic $\chi_{ij}(\cdot; t, x, \xi)$ will exit the domain \mathcal{R} through the planes $x = 0$ or $\xi = 0$. This is why we can impose boundary conditions at $(t, 0, \xi)$ and $(t, x, 0)$ (see (56)) and this is why it is enough to (uniquely) determine a solution on \mathcal{R} . To be more precise, we can split \mathcal{R} into three disjoint subsets:

$$\mathcal{R} = \mathcal{R}_{ij}^+ \cup \mathcal{R}_{ij}^- \cup \mathcal{D}_{ij},$$

where

$$\begin{aligned} \mathcal{R}_{ij}^+ &= \{(t, x, \xi) \in \mathcal{R}, \quad s_i^{\text{out}}(t, x) < s_j^{\text{out}}(t, \xi)\}, \\ \mathcal{R}_{ij}^- &= \{(t, x, \xi) \in \mathcal{R}, \quad s_i^{\text{out}}(t, x) > s_j^{\text{out}}(t, \xi)\}, \\ \mathcal{D}_{ij} &= \{(t, x, \xi) \in \mathcal{R}, \quad s_i^{\text{out}}(t, x) = s_j^{\text{out}}(t, \xi)\}. \end{aligned}$$

473 With these notations, the characteristic $\chi_{ij}(\cdot; t, x, \xi)$ will either exit the domain \mathcal{R} through the plane
474 $x = 0$ if $(t, x, \xi) \in \mathcal{R}_{ij}^+$ or through the plane $\xi = 0$ if $(t, x, \xi) \in \mathcal{R}_{ij}^-$:

475 **Proposition 3.1.**

476 (i) For every $(t, x, \xi) \in \mathcal{R}_{ij}^+$, we have $\chi_{ij}(s; t, x, \xi) \in \mathcal{R}_{ij}^+$ for every $s \in (t, s_i^{\text{out}}(t, x))$.

477 (ii) For every $(t, x, \xi) \in \mathcal{R}_{ij}^-$, we have $\chi_{ij}(s; t, x, \xi) \in \mathcal{R}_{ij}^-$ for every $s \in (t, s_j^{\text{out}}(t, \xi))$.

478 (iii) For every $(t, x, \xi) \in \mathcal{D}_{ij}$, we have $\chi_{ij}(s; t, x, \xi) \in \mathcal{D}_{ij}$ for every $s \in (t, s_i^{\text{out}}(t, x)) = (t, s_j^{\text{out}}(t, \xi))$.

479 These three points directly follow from (17).

480 3.1.2 Existence and regularity of a solution to (56)

481 Writing the solution of (56) along the characteristic curve $\chi_{ij}(s; t, x, \xi)$ for $s \in [s_{ij}^{\text{in}}(t, x, \xi), s_{ij}^{\text{out}}(t, x, \xi)]$
482 and using the boundary conditions, we obtain the following ODE:

$$483 \begin{cases} \frac{d}{ds} h_{ij}(\chi_{ij}(s; t, x, \xi)) = -\frac{\partial \lambda_j}{\partial \xi}(s, \chi_j(s; t, \xi)) h_{ij}(\chi_{ij}(s; t, x, \xi)), \\ h_{ij}(\chi_{ij}(s_{ij}^{\text{out}}(t, x, \xi); t, x, \xi)) = b_{ij}(t, x, \xi), \end{cases} \quad (57)$$

484 where

$$485 b_{ij}(t, x, \xi) = \begin{cases} 0 & \text{if } (t, x, \xi) \in \mathcal{R}_{ij}^+, \\ -\frac{g_{ij}^2(s_j^{\text{out}}(t, \xi), \chi_i(s_j^{\text{out}}(t, \xi); t, x))}{\lambda_j(s_j^{\text{out}}(t, \xi), 0)} & \text{if } (t, x, \xi) \in \mathcal{R}_{ij}^-. \end{cases} \quad (58)$$

486 Integrating this ODE over $[t, s_{ij}^{\text{out}}(t, x, \xi)]$ yields the integral equation

$$487 h_{ij}(t, x, \xi) = b_{ij}(t, x, \xi) + \int_t^{s_{ij}^{\text{out}}(t, x, \xi)} \frac{\partial \lambda_j}{\partial \xi}(s, \chi_j(s; t, \xi)) h_{ij}(\chi_{ij}(s; t, x, \xi)) ds. \quad (59)$$

488 In this case, this integral equation is very easily solved by taking (as it is in fact directly seen from
 489 the ODE (57)):

$$490 \quad h_{ij}(t, x, \xi) = b_{ij}(t, x, \xi) e^{\int_t^{s_{ij}^{\text{out}}(t, x, \xi)} \frac{\partial \lambda_j}{\partial \xi}(s, \chi_j(s; t, \xi)) ds}. \quad (60)$$

Clearly, $h_{ij} = 0$ for $i \leq j$ (i.e. H_{--} is indeed strictly lower triangular) since $g_{ij}^2 = 0$ for such indices
 (see (34)). Obviously, $h_{ij} \in C^0(\overline{\mathcal{R}_{ij}^+}) \cap L^\infty(\mathcal{R}_{ij}^+)$. On the other hand, thanks in particular to the
 regularities (11), (18), the bounds (19) and the assumption $\frac{\partial \Lambda}{\partial x} \in L^\infty((0, +\infty) \times (0, 1))^{n \times n}$, we can
 check that

$$h_{ij} \in C^0(\overline{\mathcal{R}_{ij}^-}) \cap L^\infty(\mathcal{R}_{ij}^-).$$

491 3.1.3 Characterization of \mathcal{R}_{ij}^\pm and \mathcal{D}_{ij}

492 Let us now show that

$$493 \quad \begin{aligned} \mathcal{R}_{ij}^- &= \{(t, x, \xi) \in \mathcal{R}, \quad \xi < \psi_{ij}(t, x)\}, \\ \mathcal{R}_{ij}^+ &= \{(t, x, \xi) \in \mathcal{R}, \quad \xi > \psi_{ij}(t, x)\}, \end{aligned} \quad (61)$$

494 where $\psi_{ij} \in C^1([0, +\infty) \times [0, 1])$ satisfies the semi-linear hyperbolic equation (41). First of all, it
 495 follows from (20) and the implicit function theorem that there exists a function $\psi_{ij} \in C^1([0, +\infty) \times$
 496 $[0, 1])$, $0 \leq \psi_{ij} \leq 1$, such that

$$497 \quad s_i^{\text{out}}(t, x) = s_j^{\text{out}}(t, \xi) \iff \xi = \psi_{ij}(t, x). \quad (62)$$

498 This shows that

$$499 \quad \mathcal{D}_{ij} = \{(t, x, \xi) \in \mathcal{R}, \quad \xi = \psi_{ij}(t, x)\}. \quad (63)$$

On the other hand, thanks to (20) and (62) we have

$$\xi > \psi_{ij}(t, x) \iff s_j^{\text{out}}(t, \xi) > s_j^{\text{out}}(t, \psi_{ij}(t, x)) = s_i^{\text{out}}(t, x).$$

500 This shows the equality (61) for \mathcal{R}_{ij}^+ . The equality for \mathcal{R}_{ij}^- can be proved similarly.

It remains to show that ψ_{ij} satisfies the semi-linear hyperbolic equation (41). This in fact follows
 from (63) and (iii) of Proposition 3.1. Indeed, thanks to these results, we have

$$\chi_j(s; t, \psi_{ij}(t, x)) = \psi_{ij}(s, \chi_i(s; t, x)), \quad \forall s \in (t, s_i^{\text{out}}(t, x)) = (t, s_j^{\text{out}}(t, \psi_{ij}(t, x))).$$

501 Taking the derivative of this identity at $s = t^+$, we immediately obtain the equation in (41). On
 502 the other hand, letting $s \rightarrow s_i^{\text{out}}(t, x)^- = s_j^{\text{out}}(t, \psi_{ij}(t, x))^-$ and then letting $x \rightarrow 0^+$, we obtain the
 503 second condition $\psi_{ij}(t, 0) = 0$.

504 3.2 Kernel for the Volterra transformation

505 In this section we prove Theorem 2.6, that is we prove the existence of a suitably smooth matrix-
 506 valued function $K = (k_{ij})_{1 \leq i, j \leq n}$ such that

$$507 \quad \begin{cases} \frac{\partial K}{\partial t}(t, x, \xi) + \Lambda(t, x) \frac{\partial K}{\partial x}(t, x, \xi) + \frac{\partial K}{\partial \xi}(t, x, \xi) \Lambda(t, \xi) + K(t, x, \xi) \widetilde{M}^1(t, \xi) = 0, \\ K(t, x, x) \Lambda(t, x) - \Lambda(t, x) K(t, x, x) = -M^1(t, x), \end{cases} \quad (64)$$

508 where we introduced the notation

$$509 \quad \widetilde{M}^1(t, \xi) = \frac{\partial \Lambda}{\partial \xi}(t, \xi) + M^1(t, \xi). \quad (65)$$

519 1) If $j \neq i$, then

$$\begin{cases} \frac{\partial k_{ij}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial k_{ij}}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial k_{ij}}{\partial \xi}(t, x, \xi) + \sum_{\ell=1}^n k_{i\ell}(t, x, \xi) \tilde{m}_{\ell j}^1(t, \xi) = 0, \\ k_{ij}(t, x, x) = r_{ij}(t, x). \end{cases} \quad (68)$$

521 2) If $j = i$, then

$$522 \quad \frac{\partial k_{ii}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial k_{ii}}{\partial x}(t, x, \xi) + \lambda_i(t, \xi) \frac{\partial k_{ii}}{\partial \xi}(t, x, \xi) + \sum_{\ell=1}^n k_{i\ell}(t, x, \xi) \tilde{m}_{\ell i}^1(t, \xi) = 0. \quad (69)$$

The geometric situation of the characteristics is more complicated than in Section 3.1, it is detailed in Section 3.2.2 below. For the moment, let us just point out that we will have to consider parameters $s < t$ (compare with Section 3.1) and, consequently, we should also add an artificial boundary condition at $t = 0$ (the value of k_{ij} at a point $(t, x, \xi) \in \mathcal{T}$ for sufficiently small t can not be obtained from its values on the planes $\xi = x$ or $x = 1$). To avoid imposing such a condition we can equivalently study (68)-(69) on the domain extended in time

$$\mathcal{P} = \{(t, x, \xi) \in \mathbb{R} \times (0, 1) \times (0, 1), \quad \xi < x\}.$$

Therefore, we need the values of $\tilde{m}_{\ell j}^1$ and r_{ij} for negative t . We also need the values of $q_{\ell j}^1$ for negative t since we want to consider the property (66). To this end we extend M to $\mathbb{R} \times [0, 1]$ (recall that its diagonal elements were already extended in the proof of Proposition 2.2) and we extend Q to \mathbb{R} in such a way that the property (5) is preserved. This extends $\tilde{m}_{\ell j}^1$ and r_{ij} to $\mathbb{R} \times [0, 1]$ and $q_{\ell j}^1$ to \mathbb{R} through the formula (65), (67) and (32), (33), with

$$\tilde{m}_{\ell j}^1, r_{ij} \in C^0(\mathbb{R} \times [0, 1]) \cap L^\infty(\mathbb{R} \times (0, 1)), \quad q_{\ell j}^1 \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

523 3.2.2 The characteristics of (68)-(69)

For each $(t, x, \xi) \in \mathbb{R}^3$ fixed, we still denote by $\chi_{ij}(\cdot; t, x, \xi)$ the characteristic curve associated with the hyperbolic system (68)-(69) passing through the point (t, x, ξ) , i.e.

$$\chi_{ij}(s; t, x, \xi) = (s, \chi_i(s; t, x), \chi_j(s; t, \xi)), \quad \forall s \in \mathbb{R}.$$

We now need to find for which parameters s the characteristic $\chi_{ij}(s; t, x, \xi)$ stays in the domain \mathcal{P} when $(t, x, \xi) \in \mathcal{P}$. To this end, we introduce the following sets for $j \in \{1, \dots, m\}$:

$$\begin{aligned} \mathcal{P}_{ij}^{\text{in},+} &= \{(t, x, \xi) \in \mathcal{P}, \quad s_i^{\text{in}}(t, x) < s_j^{\text{in}}(t, \xi)\}, \\ \mathcal{P}_{ij}^{\text{in},-} &= \{(t, x, \xi) \in \mathcal{P}, \quad s_i^{\text{in}}(t, x) > s_j^{\text{in}}(t, \xi)\}, \\ \mathcal{D}_{ij}^{\text{in}} &= \{(t, x, \xi) \in \mathcal{P}, \quad s_i^{\text{in}}(t, x) = s_j^{\text{in}}(t, \xi)\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_{ij}^{\text{out},+} &= \{(t, x, \xi) \in \mathcal{P}, \quad s_i^{\text{out}}(t, x) < s_j^{\text{out}}(t, \xi)\}, \\ \mathcal{P}_{ij}^{\text{out},-} &= \{(t, x, \xi) \in \mathcal{P}, \quad s_i^{\text{out}}(t, x) > s_j^{\text{out}}(t, \xi)\}, \\ \mathcal{D}_{ij}^{\text{out}} &= \{(t, x, \xi) \in \mathcal{P}, \quad s_i^{\text{out}}(t, x) = s_j^{\text{out}}(t, \xi)\}. \end{aligned}$$

524 As in Section 3.1.3 we can show that $\mathcal{P}_{ij}^{\text{out},+} \cap \mathcal{T} = \mathcal{T}_{ij}^+$ and $\mathcal{P}_{ij}^{\text{out},-} \cap \mathcal{T} = \mathcal{T}_{ij}^-$ (we recall that \mathcal{T}_{ij}^+ and
525 \mathcal{T}_{ij}^- are defined in the statement of Theorem 2.6).

526 The following proposition gives precise information about the exit of the characteristics from the
527 domain \mathcal{P} (the proof is postponed to Appendix B for the sake of the presentation; we refer to Figures
528 2, 3, 4 and 5 for a clarification of the geometric situation at a fixed t):

529 **Proposition 3.3.**

(i) For every $j \in \{1, \dots, i-1\}$, there exists a unique $s_{ij}^{\text{in}} \in C^0(\overline{\mathcal{P}})$ with $(t, x, \xi) \mapsto t - s_{ij}^{\text{in}}(t, x, \xi) \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 \leq \xi < x < 1$, we have $s_{ij}^{\text{in}}(t, x, \xi) < t$ (and $s_{ij}^{\text{in}}(t, x, \xi) = t$ otherwise) with

$$\chi_{ij}(s; t, x, \xi) \in \mathcal{P}, \quad \forall s \in (s_{ij}^{\text{in}}(t, x, \xi), t),$$

and

$$\begin{cases} \chi_j(s_{ij}^{\text{in}}(t, x, \xi); t, \xi) = \chi_i(s_{ij}^{\text{in}}(t, x, \xi); t, x) & \text{if } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{in},+}}, \\ \chi_i(s_{ij}^{\text{in}}(t, x, \xi); t, x) = 1 & \text{if } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{in},-}}. \end{cases}$$

(ii) For $j = i$, there exists a unique $s_{ii}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t, x, \xi) \mapsto s_{ii}^{\text{out}}(t, x, \xi) - t \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 < \xi \leq x \leq 1$, we have $s_{ii}^{\text{out}}(t, x, \xi) > t$ (and $s_{ii}^{\text{out}}(t, x, \xi) = t$ otherwise) and, if in addition $\xi < x$, then we have

$$\chi_{ii}(s; t, x, \xi) \in \mathcal{P}, \quad \forall s \in (t, s_{ii}^{\text{out}}(t, x, \xi)),$$

and

$$\chi_i(s_{ii}^{\text{out}}(t, x, \xi); t, \xi) = 0.$$

(iii) For every $j \in \{i+1, \dots, m\}$, there exists a unique $s_{ij}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t, x, \xi) \mapsto s_{ij}^{\text{out}}(t, x, \xi) - t \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 < \xi < x \leq 1$, we have $s_{ij}^{\text{out}}(t, x, \xi) > t$ (and $s_{ij}^{\text{out}}(t, x, \xi) = t$ otherwise) with

$$\chi_{ij}(s; t, x, \xi) \in \mathcal{P}, \quad \forall s \in (t, s_{ij}^{\text{out}}(t, x, \xi)),$$

and

$$\begin{cases} \chi_j(s_{ij}^{\text{out}}(t, x, \xi); t, \xi) = \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x) & \text{if } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out},+}}, \\ \chi_j(s_{ij}^{\text{out}}(t, x, \xi); t, \xi) = 0 & \text{if } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out},-}}. \end{cases}$$

(iv) For every $j \in \{m+1, \dots, n\}$, there exists a unique $s_{ij}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t, x, \xi) \mapsto s_{ij}^{\text{out}}(t, x, \xi) - t \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 \leq \xi < x \leq 1$, we have $s_{ij}^{\text{out}}(t, x, \xi) > t$ (and $s_{ij}^{\text{out}}(t, x, \xi) = t$ otherwise) with

$$\chi_{ij}(s; t, x, \xi) \in \mathcal{P}, \quad \forall s \in (t, s_{ij}^{\text{out}}(t, x, \xi)),$$

and

$$\chi_j(s_{ij}^{\text{out}}(t, x, \xi); t, \xi) = \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x).$$

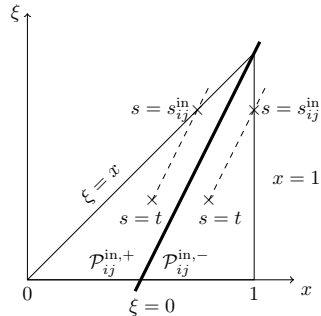


Figure 2: Definition of s_{ij}^{in}

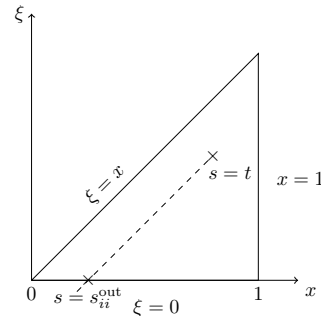
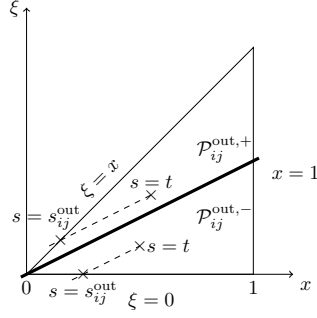
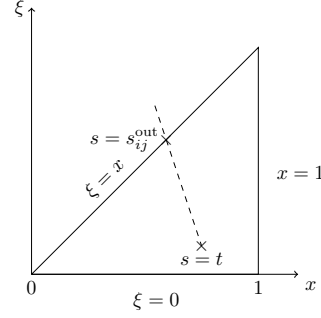


Figure 3: Definition of s_{ii}^{out}

531

Figure 4: Definition of s_{ij}^{out} when $i < j \leq m$ Figure 5: Definition of s_{ij}^{out} when $j > m$

532

533

In order to show that the system (68)-(69) is well-posed, we see from Proposition 3.3 that we need to add some conditions:

- 1) when $j \in \{1, \dots, i-1\}$, we will consider the following artificial boundary condition at $x = 1$:

$$k_{ij}(t, 1, \xi) = a_{ij}(t, \xi), \quad \forall j \in \{1, \dots, i-1\},$$

534

535

where $a_{ij} \in C^0(\mathbb{R} \times [0, 1]) \cap L^\infty(\mathbb{R} \times (0, 1))$ is any function that satisfies the corresponding C^0 -compatibility conditions at $(t, x, \xi) = (t, 1, 1)$, namely:

536

$$a_{ij}(t, 1) = r_{ij}(t, 1), \quad \forall t \in \mathbb{R}. \quad (70)$$

- 2) when $j \in \{i, \dots, m\}$, we have some freedom for the boundary condition. We choose to consider the following one in order to obtain (66):

$$k_{ij}(t, x, 0) = \sum_{\ell=1}^p k_{i, m+\ell}(t, x, 0) \tilde{q}_{\ell j}^1(t), \quad \forall j \in \{i, \dots, m\},$$

where we set

$$\tilde{q}_{\ell j}^1(t) = -\frac{1}{\lambda_j(t, 0)} \lambda_{m+\ell}(t, 0) q_{\ell j}^1(t).$$

537

Note that $\tilde{q}_{\ell j}^1 \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

538

In summary, we are going to solve the following coupled hyperbolic system:

539

- 1) If $j \in \{1, \dots, i-1\}$, then

$$\begin{cases} \frac{\partial k_{ij}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial k_{ij}}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial k_{ij}}{\partial \xi}(t, x, \xi) + \sum_{\ell=1}^n k_{i\ell}(t, x, \xi) \tilde{m}_{\ell j}^1(t, \xi) = 0, \\ k_{ij}(t, x, x) = r_{ij}(t, x), \\ k_{ij}(t, 1, \xi) = a_{ij}(t, \xi). \end{cases} \quad (71)$$

540

541

- 2) If $j = i$, then

$$\begin{cases} \frac{\partial k_{ii}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial k_{ii}}{\partial x}(t, x, \xi) + \lambda_i(t, \xi) \frac{\partial k_{ii}}{\partial \xi}(t, x, \xi) + \sum_{\ell=1}^n k_{i\ell}(t, x, \xi) \tilde{m}_{\ell i}^1(t, \xi) = 0, \\ k_{ii}(t, x, 0) = \sum_{\ell=1}^p k_{i, m+\ell}(t, x, 0) \tilde{q}_{\ell i}^1(t). \end{cases} \quad (72)$$

542

543 3) If $j \in \{i+1, \dots, m\}$, then

$$544 \quad \begin{cases} \frac{\partial k_{ij}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial k_{ij}}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial k_{ij}}{\partial \xi}(t, x, \xi) + \sum_{\ell=1}^n k_{i\ell}(t, x, \xi) \tilde{m}_{\ell j}^1(t, \xi) = 0, \\ k_{ij}(t, x, x) = r_{ij}(t, x), \\ k_{ij}(t, x, 0) = \sum_{\ell=1}^p k_{i, m+\ell}(t, x, 0) \tilde{q}_{\ell j}^1(t). \end{cases} \quad (73)$$

545 4) If $j \in \{m+1, \dots, n\}$, then

$$546 \quad \begin{cases} \frac{\partial k_{ij}}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial k_{ij}}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial k_{ij}}{\partial \xi}(t, x, \xi) + \sum_{\ell=1}^n k_{i\ell}(t, x, \xi) \tilde{m}_{\ell j}^1(t, \xi) = 0, \\ k_{ij}(t, x, x) = r_{ij}(t, x). \end{cases} \quad (74)$$

547 3.2.3 Transformation into integral equations

548 To prove the existence and uniqueness of the solution to the kernel equations (71)-(74) on \mathcal{P} , we use
 549 the classical strategy that consists in transforming these hyperbolic equations into integral equations.
 550 Then, in the next subsection, we will prove that this system of integral equations has a unique solution
 551 by using a fixed-point argument and appropriate estimates.

Let us introduce

$$\tilde{k}_{ij}^0(t, x, \xi) = \begin{cases} r_{ij}(s_{ij}^{\text{in}}(t, x, \xi), \chi_i(s_{ij}^{\text{in}}(t, x, \xi); t, x)) & \text{if } j \in \{1, \dots, i-1\} \text{ and } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{in},+}}, \\ a_{ij}(s_{ij}^{\text{in}}(t, x, \xi), \chi_j(s_{ij}^{\text{in}}(t, x, \xi); t, \xi)) & \text{if } j \in \{1, \dots, i-1\} \text{ and } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{in},-}}, \\ r_{ij}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x)) & \text{if } j \in \{i+1, \dots, m\} \text{ and } (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out},+}}, \\ r_{ij}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x)) & \text{if } j \in \{m+1, \dots, n\}. \end{cases}$$

552 Thanks to the C^0 -compatibility condition (70), note in particular that

$$553 \quad \tilde{k}_{ij}^0 \in C^0(\overline{\mathcal{P}}), \quad \forall j \in \{1, \dots, i-1\}. \quad (75)$$

554 Using now Proposition 3.3, we can obtain that

- 1) For $j \in \{1, \dots, i-1\}$, integrating (71) along the characteristic curve $\chi_{ij}(s; t, x, \xi)$ for $s \in (s_{ij}^{\text{in}}(t, x, \xi), t)$ yields the following integral equation:

$$k_{ij}(t, x, \xi) = \tilde{k}_{ij}^0(t, x, \xi) - \sum_{\ell=1}^n \int_{s_{ij}^{\text{in}}(t, x, \xi)}^t k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \tilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds.$$

- 2) For $j = i$, integrating (72) along the characteristic curve $\chi_{ii}(s; t, x, \xi)$ for $s \in (t, s_{ii}^{\text{out}}(t, x, \xi))$ yields the following integral equation:

$$k_{ii}(t, x, \xi) = \sum_{\ell=1}^p k_{i, m+\ell}(s_{ii}^{\text{out}}(t, x, \xi), \chi_i(s_{ii}^{\text{out}}(t, x, \xi); t, x), 0) \tilde{q}_{\ell i}^1(s_{ii}^{\text{out}}(t, x, \xi)) \\ + \sum_{\ell=1}^n \int_t^{s_{ii}^{\text{out}}(t, x, \xi)} k_{i\ell}(\chi_{ii}(s; t, x, \xi)) \tilde{m}_{\ell i}^1(s, \chi_i(s; t, \xi)) ds. \quad (76)$$

- 3) For $j \in \{i+1, \dots, m\}$, integrating (73) along the characteristic curve $\chi_{ij}(s; t, x, \xi)$ for $s \in (t, s_{ij}^{\text{out}}(t, x, \xi))$ yields the following integral equations:

$$k_{ij}(t, x, \xi) = \tilde{k}_{ij}^0(t, x, \xi) + \sum_{\ell=1}^n \int_t^{s_{ij}^{\text{out}}(t, x, \xi)} k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \tilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds, \quad (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out},+}},$$

and

$$k_{ij}(t, x, \xi) = \sum_{\ell=1}^p k_{i, m+\ell}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0) \tilde{q}_{\ell j}^1(s_{ij}^{\text{out}}(t, x, \xi)) + \sum_{\ell=1}^n \int_t^{s_{ij}^{\text{out}}(t, x, \xi)} k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \tilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds, \quad (t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out},-}}. \quad (77)$$

- 555 4) For $j \in \{m+1, \dots, n\}$, integrating (74) along the characteristic curve $\chi_{ij}(s; t, x, \xi)$ for $s \in$
556 $(t, s_{ij}^{\text{out}}(t, x, \xi))$ yields the following integral equation:

557
$$k_{ij}(t, x, \xi) = \tilde{k}_{ij}^0(t, x, \xi) + \sum_{\ell=1}^n \int_t^{s_{ij}^{\text{out}}(t, x, \xi)} k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \tilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds. \quad (78)$$

- 5) We now want to plug (78) into (76) and (77), respectively. From (78) we have

$$k_{i, m+\ell}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0) = \tilde{k}_{i, m+\ell}^0(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0) + \sum_{q=1}^n \int_{s_{ij}^{\text{out}}(t, x, \xi)}^{s_{i, m+\ell}^{\text{out}}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)} k_{iq}(\chi_{i, m+\ell}(s; s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)) \times \tilde{m}_{q, m+\ell}^1(s, \chi_{m+\ell}(s; s_{ij}^{\text{out}}(t, x, \xi), 0)) ds. \quad (79)$$

Plugging (79) into (76) and (77), we obtain, for every $j \in \{i, \dots, m\}$ and $(t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out},-}}$,

$$\begin{aligned} k_{ij}(t, x, \xi) &= \widehat{k}_{ij}^0(t, x, \xi) \\ &+ \sum_{\ell=1}^p \left(\sum_{q=1}^n \int_{s_{ij}^{\text{out}}(t, x, \xi)}^{s_{i, m+\ell}^{\text{out}}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)} k_{iq}(\chi_{i, m+\ell}(s; s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)) \right. \\ &\quad \left. \times \tilde{m}_{q, m+\ell}^1(s, \chi_{m+\ell}(s; s_{ij}^{\text{out}}(t, x, \xi), 0)) ds \right) \tilde{q}_{\ell j}^1(s_{ij}^{\text{out}}(t, x, \xi)) \\ &\quad + \sum_{\ell=1}^n \int_t^{s_{ij}^{\text{out}}(t, x, \xi)} k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \tilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds, \end{aligned}$$

where we introduced

$$\widehat{k}_{ij}^0(t, x, \xi) = \sum_{\ell=1}^p \tilde{k}_{i, m+\ell}^0(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0) \tilde{q}_{\ell j}^1(s_{ij}^{\text{out}}(t, x, \xi)).$$

558 Note that

559
$$\widehat{k}_{ij}^0 \in C^0(\overline{\mathcal{P}_{ij}^{\text{out},-}}) \cap L^\infty(\mathcal{P}_{ij}^{\text{out},-}), \quad \forall j \in \{i+1, \dots, m\}, \quad (80)$$

560 and, since $\mathcal{P}_{ii}^{\text{out},-} = \mathcal{P}$ (because of (20)),

561
$$\widehat{k}_{ii}^0 \in C^0(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P}). \quad (81)$$

Remark 3.4. Observe that, in general, for $j \in \{i+1, \dots, m\}$, we have

$$\widehat{k}_{ij}^0 \neq \widetilde{k}_{ij}^0 \text{ on } \mathcal{D}_{ij}^{\text{out}}.$$

562 This is the reason why we have to consider discontinuous kernels.

563 3.2.4 Solution to the integral equations

564 In this subsection we show that there exists a unique solution to the system of integral equations of
565 the previous section. This will conclude the proof of Theorem 2.6.

Fixed-point argument. As it is classical, we reformulate the existence of such a solution into the existence of a fixed-point of the mapping defined by the right-hand sides of these equations. Let us first introduce $K^0 = (k_{ij}^0)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ defined by

$$k_{ij}^0(t, x, \xi) = \begin{cases} \widetilde{k}_{ij}^0(t, x, \xi) & \text{if } j \in \{1, \dots, i-1\}, \\ \widehat{k}_{ii}^0(t, x, \xi) & \text{if } j = i, \\ \widetilde{k}_{ij}^0(t, x, \xi) & \text{if } j \in \{i+1, \dots, m\} \text{ and } (t, x, \xi) \in \mathcal{P}_{ij}^{\text{out},+}, \\ \widehat{k}_{ij}^0(t, x, \xi) & \text{if } j \in \{i+1, \dots, m\} \text{ and } (t, x, \xi) \in \mathcal{P}_{ij}^{\text{out},-}, \\ \widetilde{k}_{ij}^0(t, x, \xi) & \text{if } j \in \{m+1, \dots, n\}. \end{cases}$$

Thanks in particular to (75), (80) and (81), we see that

$$\begin{aligned} k_{ij}^0 &\in C^0(\overline{\mathcal{P}_{ij}^{\text{out},+}}) \cap C^0(\overline{\mathcal{P}_{ij}^{\text{out},-}}) \cap L^\infty(\mathcal{P}) \quad \text{if } j \in \{i+1, \dots, m\}, \\ k_{ij}^0 &\in C^0(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P}) \quad \text{otherwise.} \end{aligned}$$

566 It is this regularity that dictates the space in which we can work. More precisely, let us introduce
567 the vector space B defined by

$$568 \quad B = \left\{ K = (k_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}, \begin{cases} k_{ij} \in C^0(\overline{\mathcal{P}_{ij}^{\text{out},+}}) \cap C^0(\overline{\mathcal{P}_{ij}^{\text{out},-}}) \cap L^\infty(\mathcal{P}) \text{ if } j \in \{i+1, \dots, m\}, \\ k_{ij} \in C^0(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P}) \text{ otherwise.} \end{cases} \right\}. \quad (82)$$

We can check that B is a Banach space when equipped with the L^∞ norm. Let us now introduce the mapping

$$\Phi : B \longrightarrow B,$$

defined, for every $K \in B$, by

$$\Phi(K) = K^0 + \Phi_1(K) + \Phi_2(K),$$

where, for every $(t, x, \xi) \in \overline{\mathcal{P}}$,

$$(\Phi_1(K))_{ij}(t, x, \xi) = \begin{cases} - \sum_{\ell=1}^n \int_{s_{ij}^{\text{in}}(t, x, \xi)}^t k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \widetilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds, & \text{if } j \in \{1, \dots, i-1\}, \\ \sum_{\ell=1}^n \int_t^{s_{ij}^{\text{out}}(t, x, \xi)} k_{i\ell}(\chi_{ij}(s; t, x, \xi)) \widetilde{m}_{\ell j}^1(s, \chi_j(s; t, \xi)) ds, & \text{if } j \in \{i, \dots, n\}, \end{cases} \quad (83)$$

and

$$\begin{aligned}
& (\Phi_2(K))_{ij}(t, x, \xi) = \\
& \sum_{\ell=1}^p \left(\sum_{q=1}^n \int_{s_{ij}^{\text{out}}(t, x, \xi)}^{s_{i, m+\ell}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)} k_{iq} \left(\chi_{i, m+\ell}(s; s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0) \right) \right. \\
& \quad \left. \times \tilde{m}_{q, m+\ell}^1(s, \chi_{m+\ell}(s; s_{ij}^{\text{out}}(t, x, \xi), 0)) ds \right) \tilde{q}_{\ell j}^1(s_{ij}^{\text{out}}(t, x, \xi)), \quad (84)
\end{aligned}$$

569 if $j \in \{i, \dots, m\}$ and $(t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out}, -}}$, and $(\Phi_2(K))_{ij}(t, x, \xi) = 0$ otherwise (recall that $\mathcal{P}_{ii}^{\text{out}, -} = \mathcal{P}$).

570 **Regularity of the mapping.** First of all, we have to show that Φ is well defined, i.e. that for
571 every $K \in B$, we have indeed

$$572 \quad \Phi_1(K) \in B, \quad \Phi_2(K) \in B. \quad (85)$$

573 This is not obvious since the function $s \mapsto \chi_{ij}(s; t, x, \xi)$ may take values in the set $\mathcal{D}_{i\ell}^{\text{out}}$, where $k_{i\ell}$ is
574 discontinuous (even for $j \notin \{i+1, \dots, m\}$, where we expect $(\Phi_1(K))_{ij}$ to be continuous by definition
575 of B). The following result, close to Proposition 3.3, shows that this may happen only at one point:

576 **Proposition 3.5.** *Let $\ell \in \{i+1, \dots, m\}$ be fixed.*

577 (i) *For every $j \in \{1, \dots, i-1\}$, for every $(t, x, \xi) \in \overline{\mathcal{P}}$, there is at most one $s_{ij\ell}^{\text{disc}} \in (s_{ij}^{\text{in}}(t, x, \xi), t)$
578 such that $\chi_{ij}(s_{ij\ell}^{\text{disc}}; t, x, \xi) \in \mathcal{D}_{i\ell}^{\text{out}}$.*

579 (ii) *For every $j \in \{i, \dots, n\}$ with $j \neq \ell$, for every $(t, x, \xi) \in \overline{\mathcal{P}}$, there is at most one $s_{ij\ell}^{\text{disc}} \in$
580 $(t, s_{ij}^{\text{out}}(t, x, \xi))$ such that $\chi_{ij}(s_{ij\ell}^{\text{disc}}; t, x, \xi) \in \mathcal{D}_{i\ell}^{\text{out}}$.*

This result shows in fact a stronger regularity than (85), namely,

$$\Phi_1(K), \Phi_2(K) \in C^0(\overline{\mathcal{P}})^{m \times n} \cap L^\infty(\mathcal{P})^{m \times n}.$$

581 The proof of Proposition 3.5 is postponed to Appendix B for the sake of the presentation.

Contraction of the mapping. We will now prove that Φ^N is a contraction for $N \in \mathbb{N}^*$ large
enough. Therefore, the Banach fixed-point theorem can be applied, giving the existence (and unique-
ness) of $K \in B$ such that

$$K = \Phi(K).$$

582 This will conclude the proof of Theorem 2.6. Now, to show that Φ^N is a contraction when N is large,
583 it is sufficient to prove the following estimate:

584 **Proposition 3.6.** *There exists $C > 0$ such that, for every $N \in \mathbb{N}^*$ and $K, H \in B$,*

$$585 \quad \|\Phi^N(K) - \Phi^N(H)\|_{L^\infty(\mathcal{P})^{m \times n}} \leq \frac{C^N}{N!} \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}. \quad (86)$$

586 To establish (86) we will use the following key lemma:

587 **Lemma 3.7.** *For every $i \in \{1, \dots, m\}$, there exist a function $\Omega_i \in C^1(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P})$ and $\varepsilon_0 > 0$ such
588 that, for every $(t, x, \xi) \in \overline{\mathcal{P}}$, we have $\Omega_i(t, x, \xi) \geq 0$ with*

$$589 \quad \frac{\partial \Omega_i}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial \Omega_i}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial \Omega_i}{\partial \xi}(t, x, \xi) \geq \varepsilon_0, \quad \forall j \in \{1, \dots, i-1\}, \quad (87)$$

590 and

$$591 \quad \frac{\partial \Omega_i}{\partial t}(t, x, \xi) + \lambda_i(t, x) \frac{\partial \Omega_i}{\partial x}(t, x, \xi) + \lambda_j(t, \xi) \frac{\partial \Omega_i}{\partial \xi}(t, x, \xi) \leq -\varepsilon_0, \quad \forall j \in \{i, \dots, n\}. \quad (88)$$

592 The proof of Lemma 3.7 is postponed to Appendix C for the sake of the presentation.

593 *Remark 3.8.* In the time-independent case, we can take $\Omega_i(x, \xi) = \phi_i(x) - \nu\phi_i(\xi)$ (we recall that
 594 $\phi_i(x) = \int_0^x \frac{1}{-\lambda_i(y)} dy$) where $\nu \in [0, 1]$ is any number such that $\nu > \max_{1 \leq j < i \leq m} \max_{\xi \in [0, 1]} \lambda_i(\xi)/\lambda_j(\xi)$.
 595 This function appeared for instance in [HDMVK16, Lemma 6.2] for systems with constant coefficients
 596 and in [HVDMK19, (A.32)] for systems with time-independent coefficients (see also [CVKB13, Lemma
 597 A.4] for 2×2 systems, where it is enough to take $\nu = 0$ since (87) becomes void).

Remark 3.9. Observe that it follows from the estimate (88) that, for every $j \in \{i, \dots, n\}$,

$$s \mapsto \Omega_i(\chi_{ij}(s; t, x, \xi)) \text{ is strictly decreasing.}$$

598 This is the analogue to [HDMVK16, Remark 10].

599 We can now prove Proposition 3.6:

Proof of Proposition 3.6. Let us denote by

$$R = \max \left\{ \left\| \widetilde{M}^1 \right\|_{L^\infty(\mathbb{R} \times (0, 1))^{n \times n}}, \left\| \widetilde{Q}^1 \right\|_{L^\infty(\mathbb{R})^{p \times m}} \right\}.$$

1) We start with the estimate of $\|\Phi_1(K) - \Phi_1(H)\|_{L^\infty(\mathcal{P})^{m \times n}}$. Set

$$C_1 = \frac{n}{\varepsilon_0} R.$$

Let $j \in \{1, \dots, i-1\}$. From the definition (83) of Φ_1 we see that

$$\left| (\Phi_1(K) - \Phi_1(H))_{ij}(t, x, \xi) \right| \leq nR \left(\int_{s_{ij}^{\text{in}}(t, x, \xi)}^t 1 ds \right) \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}.$$

Thanks to the estimate (87) we can perform the change of variable $s \mapsto \theta(s) = \Omega_i(\chi_{ij}(s; t, x, \xi))$ and obtain

$$\varepsilon_0 \left(\int_{s_{ij}^{\text{in}}(t, x, \xi)}^t 1 ds \right) \leq \int_{s_{ij}^{\text{in}}(t, x, \xi)}^t \frac{d\theta}{ds}(s) ds = \theta(t) - \theta(s_{ij}^{\text{in}}(t, x, \xi)) \leq \theta(t) = \Omega_i(t, x, \xi).$$

This gives the estimate

$$\left| (\Phi_1(K) - \Phi_1(H))_{ij}(t, x, \xi) \right| \leq C_1 \Omega_i(t, x, \xi) \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}.$$

It is important to point out that the right-hand side does not depend on the second index j . Computing $\Phi_1^2(H) - \Phi_1^2(K) = \Phi_1(\Phi_1(H)) - \Phi_1(\Phi_1(K))$ and using the previous estimate, we obtain

$$\begin{aligned} & \left| (\Phi_1^2(K) - \Phi_1^2(H))_{ij}(t, x, \xi) \right| \\ & \leq nRC_1 \left(\int_{s_{ij}^{\text{in}}(t, x, \xi)}^t \Omega_i(\chi_{ij}(s; t, x, \xi)) ds \right) \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}. \end{aligned}$$

Using again the change of variable $s \mapsto \theta(s)$ and (87), we obtain

$$\begin{aligned} \varepsilon_0 \left(\int_{s_{ij}^{\text{in}}(t, x, \xi)}^t \Omega_i(\chi_{ij}(s; t, x, \xi)) ds \right) &= \varepsilon_0 \int_{s_{ij}^{\text{in}}(t, x, \xi)}^t \theta(s) ds \leq \int_{s_{ij}^{\text{in}}(t, x, \xi)}^t \theta(s) \frac{d\theta}{ds}(s) ds \\ &= \frac{\theta(t)^2}{2} - \frac{\theta(s_{ij}^{\text{in}}(t, x, \xi))^2}{2} \leq \frac{\theta(t)^2}{2} = \frac{\Omega_i(t, x, \xi)^2}{2}. \end{aligned}$$

This gives the estimate

$$\left| (\Phi_1^2(K) - \Phi_1^2(H))_{ij}(t, x, \xi) \right| \leq \frac{C_1^2 \Omega_i(t, x, \xi)^2}{2} \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}.$$

600 By induction, we easily obtain that, for every $N \in \mathbb{N}^*$,

$$601 \quad \left| (\Phi_1^N(K) - \Phi_1^N(H))_{ij}(t, x, \xi) \right| \leq \frac{C_1^N \Omega_i(t, x, \xi)^N}{N!} \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}. \quad (89)$$

Using the estimate (88) instead of (87), we can obtain exactly the same estimate as (89) for $j \in \{i, \dots, n\}$. Since Ω_i is bounded, it follows that

$$\|\Phi_1^N(K) - \Phi_1^N(H)\|_{L^\infty(\mathcal{P})^{m \times n}} \leq \frac{C^N}{N!} \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}},$$

602 for some C independent of N and K, H .

2) Let us now take care of $\Phi_2(K) - \Phi_2(H)$. The idea to estimate this term is essentially the same as before, with the extra use of the decreasing property stated in Remark 3.9. Set

$$C_2 = \frac{n}{\varepsilon_0} R^2 p.$$

From the definition (84) of Φ_2 we see that, for $j \in \{i, \dots, m\}$ and $(t, x, \xi) \in \overline{\mathcal{P}_{ij}^{\text{out}, -}}$,

$$\begin{aligned} & \left| (\Phi_2(K) - \Phi_2(H))_{ij}(t, x, \xi) \right| \\ & \leq nR^2 \sum_{\ell=1}^p \left(\int_{s_{ij}^{\text{out}}(t, x, \xi)}^{s_{i, m+\ell}^{\text{out}}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)} 1 ds \right) \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}. \end{aligned}$$

Thanks to the estimate (88) we can perform again the change of variable

$$s \mapsto \theta(s) = \Omega_i(\chi_{ij}(s; t, x, \xi)),$$

which is decreasing since $j \geq i$ (see Remark 3.9), and obtain

$$\begin{aligned} & \varepsilon_0 \left(\int_{s_{ij}^{\text{out}}(t, x, \xi)}^{s_{i, m+\ell}^{\text{out}}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)} 1 ds \right) \\ & \leq \int_{s_{ij}^{\text{out}}(t, x, \xi)}^{s_{i, m+\ell}^{\text{out}}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)} -\frac{d\theta}{ds}(s) ds \\ & = -\theta(s_{i, m+\ell}^{\text{out}}(s_{ij}^{\text{out}}(t, x, \xi), \chi_i(s_{ij}^{\text{out}}(t, x, \xi); t, x), 0)) + \theta(s_{ij}^{\text{out}}(t, x, \xi)) \leq \theta(t) = \Omega_i(t, x, \xi). \end{aligned}$$

This gives the estimate

$$\left| (\Phi_2(K) - \Phi_2(H))_{ij}(t, x, \xi) \right| \leq C_2 \Omega_i(t, x, \xi) \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}}.$$

Note that this estimate is also valid if $j \notin \{i, \dots, m\}$ or $(t, x, \xi) \notin \overline{\mathcal{P}_{ij}^{\text{out}, -}}$ since $(\Phi_2(\cdot))_{ij} = 0$ in this case. Reasoning by induction as before, it is now not difficult to obtain the estimate

$$\|\Phi_2^N(K) - \Phi_2^N(H)\|_{L^\infty(\mathcal{P})^{m \times n}} \leq \frac{C^N}{N!} \|K - H\|_{L^\infty(\mathcal{P})^{m \times n}},$$

603 for some C independent of N and K, H .

604

□

3.3 On the time-periodicity of F

In this section we assume that, for some $\tau > 0$, Λ , M and Q are τ -periodic with respect to time and show that the above construction of F leads, with minor modifications, to a F which is also τ -periodic with respect to time.

First of all, concerning the extension of Λ to a function of \mathbb{R}^2 (and of M and Q whenever needed), it is clear that one can extend Λ to $\mathbb{R} \times [0, 1]$ by just requiring the τ -periodicity with respect to time of this extension. This extension is still denoted by Λ . Then one extends Λ to \mathbb{R}^2 so that this extension, still denoted by Λ , is τ -periodic with respect to time and so that the properties (2), (3), (4) and (5) remain valid on \mathbb{R}^2 (see e.g. Remark 1.4).

From the construction of F (see (30), (38) and (50)) it is clear that F is τ -periodic with respect to time if so are all the matrix-valued functions involved in the several transformations of this article. Now, in order to obtain the τ -periodicity of these transformations, the minor modifications/comments are essentially the following ones.

- 1) Concerning the diagonal transformation to remove the diagonal terms in M (see Section 2.1), one simply observes that the function (33) is τ -periodic with respect to time if so is m_{ii} , thanks to the properties

$$\chi_i(s + \tau; t + \tau, \xi) = \chi_i(s; t, \xi), \quad s_i^{\text{in}}(t + \tau, \xi) = s_i^{\text{in}}(t, \xi) + \tau. \quad (90)$$

- 2) Concerning the kernel H of the Fredholm transformation (see Section 3.1) one easily checks that it is indeed τ -periodic with respect to time. This follows from the uniqueness of the solution to (59) and similar properties to (90).

- 3) Concerning the kernel K of the Volterra transformation of the second kind (see Section 3.2), to construct it in such a way that it is τ -periodic with respect to time, it suffices to observe that $\tilde{m}_{\ell j}^1$, r_{ij} and $q_{\ell j}^1$ become τ -periodic with respect to time once M and Q are, and to modify the definition of the space B given in (82) by adding the condition that the k_{ij} , $1 \leq i \leq m$, $1 \leq j \leq n$, are τ -periodic with respect to time (alternatively, one can keep (82) and deduce from the uniqueness of the fixed point of Φ that it has to be τ -periodic with respect to time).

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A Background on broad solutions

We recall that all the systems of this paper have the following form:

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial y}{\partial x}(t, x) = M(t, x)y(t, x) + G(t, x)y(t, 0), \\ y_-(t, 1) = \int_0^1 F(t, \xi)y(t, \xi) d\xi, \quad y_+(t, 0) = Q(t)y_-(t, 0), \\ y(t^0, x) = y^0(x), \end{cases} \quad (91)$$

where M and Q have at least the regularity (5), $F \in L^\infty((0, +\infty) \times (0, 1))^{m \times n}$ and

$$G \in C^0([0, +\infty) \times [0, 1])^{n \times n} \cap L^\infty((0, +\infty) \times (0, 1))^{n \times n}.$$

641 A.1 Definition of broad solution

Let us now introduce the notion of solution for such systems. To this end, we have to restrict our discussion to the domain where the system (91) evolves, i.e. on $(t^0, +\infty) \times (0, 1)$. For every $(t, x) \in (t^0, +\infty) \times (0, 1)$, we have

$$(s, \chi_i(s; t, x)) \in (t^0, +\infty) \times (0, 1), \quad \forall s \in (\bar{s}_i^{\text{in}}(t^0; t, x), s_i^{\text{out}}(t, x)),$$

where we introduced

$$\bar{s}_i^{\text{in}}(t^0; t, x) = \max \{t^0, s_i^{\text{in}}(t, x)\} < t.$$

642 Formally, writing the i -th equation of the system (91) along the characteristic $\chi_i(s; t, x)$ for $s \in$
643 $[\bar{s}_i^{\text{in}}(t^0; t, x), s_i^{\text{out}}(t, x)]$, and using the chain rules yields the ODE

$$\begin{cases} \frac{d}{ds} y_i(s, \chi_i(s; t, x)) = \sum_{j=1}^n m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) + \sum_{j=1}^n g_{ij}(s, \chi_i(s; t, x)) y_j(s, 0), \\ y_i(\bar{s}_i^{\text{in}}(t^0; t, x), \chi_i(\bar{s}_i^{\text{in}}(t^0; t, x); t, x)) = b_i(y)(t, x), \end{cases} \quad (92)$$

644 where the initial condition $b_i(y)(t, x)$ is given by the appropriate boundary or initial conditions of the system (91):

$$b_i(y)(t, x) = \begin{cases} \sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t, x), \xi) y_j(s_i^{\text{in}}(t, x), \xi) d\xi & \text{if } s_i^{\text{in}}(t, x) > t^0 \text{ and } i \in \{1, \dots, m\}, \\ \sum_{j=1}^m q_{i-m, j}(s_i^{\text{in}}(t, x)) y_j(s_i^{\text{in}}(t, x), 0) & \text{if } s_i^{\text{in}}(t, x) > t^0 \text{ and } i \in \{m+1, \dots, n\}, \\ y_i^0(\chi_i(t^0; t, x)) & \text{if } s_i^{\text{in}}(t, x) < t^0. \end{cases} \quad (93)$$

Integrating the ODE (92) over $s \in [\bar{s}_i^{\text{in}}(t^0; t, x), t]$, we obtain the following system of integral equations:

$$y_i(t, x) = b_i(y)(t, x) + \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds + \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t g_{ij}(s, \chi_i(s; t, x)) y_j(s, 0) ds. \quad (94)$$

645 This leads to the following notion of “solution along the characteristics” or “broad solution”:

646 **Definition A.1.** Let $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$ be fixed. We say that a function $y : (t^0, +\infty) \times$
647 $(0, 1) \rightarrow \mathbb{R}^n$ is a broad solution to the system (91) if

$$648 \quad y \in C^0([t^0, t^0 + T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(t^0, t^0 + T)^n), \quad \forall T > 0, \quad (95)$$

649 and if the integral equation (94) is satisfied for every $i \in \{1, \dots, n\}$, for a.e. $t > t^0$ and a.e. $x \in (0, 1)$.

650 A.2 Well-posedness

651 This section is devoted to the following well-posedness result regarding system (91):

652 **Theorem A.2.** *For every $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$, there exists a unique broad solution to (91).
653 Moreover, there exists $C > 0$ such that, for every $T > 0$, $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$, the corresponding
654 broad solution y satisfies*

$$655 \quad \|y\|_{C^0([t^0, t^0+T]; L^2(0, 1)^n)} + \|y\|_{C^0([0, 1]; L^2(t^0, t^0+T)^n)} \leq Ce^{CT} \|y^0\|_{L^2(0, 1)^n}. \quad (96)$$

656 *Remark A.3.* It follows from the uniformity of the constant Ce^{CT} with respect to the initial time
657 t^0 in the estimate (96) that, for systems of the form (91), the uniform stability property (8) is a
658 consequence of the finite-time global attractor property (7) (simply take $\delta > 0$ such that $Ce^{CT}\delta \leq \varepsilon$).

659 Let us first point out that this well-posedness result for our initial system (1) for the particular
660 F that we have constructed in Section 2 follows in fact from the well-posedness result for the final
661 target system of Proposition 2.11 (easier to establish), since we have shown that both systems are
662 equivalent by means of several invertible transformations. However, it is still important to have
663 such a well-posedness result for any F within the class studied, which is a result that also has its
664 own interest. We will provide a complete proof since, to the best of our knowledge, there are no
665 references that show the well-posedness for the initial-boundary value problem (91) with non-local
666 terms $G(t, x)y(t, 0)$, with weak regularity (95) and with uniform estimate (96).

667 *Proof of Theorem A.2.* We first remark that it is enough to prove the theorem for $\|Q\|_{L^\infty}$ small
668 enough, say

$$669 \quad \|Q\|_{L^\infty} \leq \alpha, \quad (97)$$

where $\alpha > 0$ does not depend on T, t^0, y^0 nor on M, G, F . This follows from the following change of variable:

$$y = D\tilde{y}, \quad D = \begin{pmatrix} \frac{\alpha}{\|Q\|_{L^\infty} + \alpha} \text{Id}_{\mathbb{R}^m} & 0 \\ 0 & \text{Id}_{\mathbb{R}^p} \end{pmatrix},$$

where \tilde{y} is the solution to the system $(\tilde{M}, \tilde{G}, \tilde{F}, \tilde{Q})$ with

$$\tilde{M} = D^{-1}MD, \quad \tilde{G} = D^{-1}GD, \quad \tilde{F} = \left(\frac{\alpha}{\|Q\|_{L^\infty} + \alpha} \right)^{-1} FD, \quad \tilde{Q} = \frac{\alpha}{\|Q\|_{L^\infty} + \alpha} Q.$$

Let us now show how to prove the theorem under the smallness condition (97) with the Banach fixed point theorem ($\alpha > 0$ will be fixed adequately below). Let $T > 0$, $t^0 \geq 0$ and $y^0 \in L^2(0, 1)^n$ be fixed for the remainder of the proof. It is clear that a function $y : (t^0, +\infty) \times (0, 1) \rightarrow \mathbb{R}^n$ satisfies the integral equation (94) if, and only if, it is a fixed point of the map $\mathcal{F} : B \rightarrow B$, where

$$B = C^0([t^0, t^0 + T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(t^0, t^0 + T)^n),$$

and $(\mathcal{F}(y))_i(t, x)$ is given by the expression on the right-hand side of (94). It can be checked that \mathcal{F} indeed maps B into itself (actually, by computations similar to the upcoming ones). Let us now make B a Banach space by equipping it with the following weighted norm:

$$\|y\|_B = \|y\|_{B_1} + \|y\|_{B_2},$$

where

$$\|y\|_{B_1} = \max_{t \in [t^0, t^0+T]} e^{-\frac{L_1}{2}(t-t^0)} \sqrt{\int_0^1 \sum_{i=1}^n |y_i(t, x)|^2 e^{-L_2 x} dx},$$

and

$$\|y\|_{B_2} = \max_{x \in [0,1]} e^{\frac{L_2}{2}(1-x)} \sqrt{\int_{t^0}^{t^0+T} \sum_{i=1}^n |y_i(t,x)|^2 e^{-L_1(t-t^0)} dt},$$

670 where $L_1, L_2 > 0$ are constants independent of T, t^0 and y^0 that will be fixed below. Our goal is to
671 show that, for $L_1, L_2 > 0$ large enough,

$$672 \quad \|\mathcal{F}(y^1) - \mathcal{F}(y^2)\|_B \leq \frac{1}{2} \|y^1 - y^2\|_B, \quad \forall y^1, y^2 \in B. \quad (98)$$

It is then not difficult to check that the fixed point of \mathcal{F} satisfies the estimate (96). Indeed, using (98), we easily see that the fixed point y of \mathcal{F} will satisfy

$$\frac{1}{2} \|y\|_B \leq \|\mathcal{F}(0)\|_B,$$

and some straightforward computations show that

$$\begin{aligned} \|y\|_{C^0([t^0, t^0+T]; L^2(0,1)^n)}^2 &\leq e^{L_2} e^{L_1 T} \|y\|_{B_1}^2, & \|y\|_{C^0([0,1]; L^2(t^0, t^0+T)^n)}^2 &\leq e^{L_1 T} \|y\|_{B_2}^2, \\ \|\mathcal{F}(0)\|_B^2 &\leq 2 \left(1 + \frac{e^{L_2}}{\varepsilon}\right) e^{\|\frac{\partial \Lambda}{\partial x}\|_{L^\infty} T} \|y^0\|_{L^2(0,1)^n}^2. \end{aligned}$$

Let us now establish (98). We introduce

$$y = y^1 - y^2,$$

so that $\mathcal{F}(y^1) - \mathcal{F}(y^2)$ is equal to the right-hand side of (94) with $y^0 = 0$. We have to estimate four types of terms in each $\|\cdot\|_{B_i}$ -norm ($i = 1, 2$). For convenience, we denote by

$$R_1 = \max \left\{ \|\Lambda\|_{L^\infty}, \left\| \frac{\partial \Lambda}{\partial x} \right\|_{L^\infty} \right\}, \quad R_2 = \max \{ \|M\|_{L^\infty}, \|G\|_{L^\infty}, \|F\|_{L^\infty} \}.$$

673 We recall that it is crucial that α does not depend on R_2 .

674 **Estimate of the $\|\cdot\|_{B_1}$ -norm.** Let $t \in [t^0, t^0 + T]$ be fixed. Let $I = \{x \in (0, 1), s_i^{\text{in}}(t, x) > t^0\}$.

1) Let $i \in \{1, \dots, m\}$. For a.e. $x \in I$, using Cauchy-Schwarz inequality, we have

$$\left| \sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t, x), \xi) y_j(s_i^{\text{in}}(t, x), \xi) d\xi \right|^2 \leq n R_2^2 e^{L_2} \|y\|_{B_1}^2 e^{L_1(s_i^{\text{in}}(t, x) - t^0)}.$$

675 Using a finer version of (15), namely,

$$676 \quad \frac{1-x}{R_1} \leq t - s_i^{\text{in}}(t, x), \quad (99)$$

(obtained similarly to (19)) we obtain the estimate

$$\int_I \left| \sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t, x), \xi) y_j(s_i^{\text{in}}(t, x), \xi) d\xi \right|^2 e^{-L_2 x} dx \leq \left(n R_2^2 \frac{1}{\frac{L_1}{R_1} - L_2} \right) e^{L_1(t-t^0)} \|y\|_{B_1}^2,$$

677 provided that

$$678 \quad \frac{L_1}{R_1} - L_2 > 0. \quad (100)$$

2) Let $i \in \{m+1, \dots, n\}$. Using (15), we have

$$\begin{aligned} & \int_I \left| \sum_{j=1}^m q_{i-m,j}(s_i^{\text{in}}(t,x)) y_j(s_i^{\text{in}}(t,x), 0) \right|^2 e^{-L_2 x} dx \\ & \leq m\alpha^2 e^{L_1(t-t^0)} \sum_{j=1}^m \int_I |y_j(s_i^{\text{in}}(t,x), 0)|^2 e^{-L_1(s_i^{\text{in}}(t,x)-t^0)} dx. \end{aligned}$$

Doing the change of variables $\sigma = s_i^{\text{in}}(t,x)$ and using the estimate (see (14), (12) and (19))

$$\frac{\partial s_i^{\text{in}}(t,x)}{\partial x} = \frac{-e^{-\int_{s_i^{\text{in}}(t,x)}^t \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta; t, x)) d\theta}}{\lambda_i(s_i^{\text{in}}(t,x), 0)} \leq -\frac{1}{R_1} e^{-\frac{R_1}{\varepsilon}},$$

we obtain

$$\int_I \left| \sum_{j=1}^m q_{i-m,j}(s_i^{\text{in}}(t,x)) y_j(s_i^{\text{in}}(t,x), 0) \right|^2 e^{-L_2 x} dx \leq \left(m\alpha^2 R_1 e^{\frac{R_1}{\varepsilon}} e^{-L_2} \right) e^{L_1(t-t^0)} \|y\|_{B_2}^2.$$

3) For the next term, we have

$$\begin{aligned} & \int_0^1 \sum_{i=1}^n \left| \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds \right|^2 e^{-L_2 x} dx \\ & \leq nR_2^2 \frac{1}{\varepsilon} \sum_{i=1}^n \int_0^1 \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t \sum_{j=1}^n |y_j(s, \chi_i(s; t, x))|^2 ds dx. \end{aligned}$$

Using the change of variable $(\sigma, \xi) = (s, \chi_i(s; t, x))$, whose Jacobian determinant is (see (12))

$$\det \begin{pmatrix} 1 & 0 \\ \lambda_i(s, \chi_i(s; t, x)) & \frac{\partial \chi_i}{\partial x}(s; t, x) \end{pmatrix} = e^{-\int_s^t \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta; t, x)) d\theta} \geq e^{-\frac{R_1}{\varepsilon}}, \quad \forall s \in (s_i^{\text{in}}(t, x), t),$$

we obtain

$$\begin{aligned} & \int_0^1 \sum_{i=1}^n \left| \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds \right|^2 e^{-L_2 x} dx \\ & \leq \left(n^2 R_2^2 \frac{1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} e^{L_2} \frac{1}{L_1} \right) e^{L_1(t-t^0)} \|y\|_{B_1}^2. \end{aligned}$$

4) Finally, the estimate of the remaining term is easy:

$$\begin{aligned} & \int_0^1 \sum_{i=1}^n \left| \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t g_{ij}(s, \chi_i(s; t, x)) y_j(s, 0) ds \right|^2 e^{-L_2 x} dx \\ & \leq \left(n^2 R_2^2 \frac{1}{\varepsilon} e^{-L_2} \right) e^{L_1(t-t^0)} \|y\|_{B_2}^2. \end{aligned}$$

In summary, we have established the following estimate (provided that (100) holds):

$$\begin{aligned} \|\mathcal{F}(y^1) - \mathcal{F}(y^2)\|_{B_1}^2 &\leq 3 \left(mnR_2^2 \frac{1}{\frac{L_1}{R_1} - L_2} + n^2 R_2^2 \frac{1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} e^{L_2} \frac{1}{L_1} \right) \|y\|_{B_1}^2 \\ &\quad + 3 \left((n-m)m\alpha^2 R_1 e^{\frac{R_1}{\varepsilon}} e^{-L_2} + n^2 R_2^2 \frac{1}{\varepsilon} e^{-L_2} \right) \|y\|_{B_2}^2. \end{aligned} \quad (101)$$

679 **Estimate of the $\|\cdot\|_{B_2}$ -norm.** Let $x \in [0, 1]$ be fixed. Let $J = \{t \in (t^0, t^0 + T), \quad s_i^{\text{in}}(t, x) > t^0\}$.

1) Let $i \in \{1, \dots, m\}$. We have

$$\begin{aligned} \int_J \left| \sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t, x), \xi) y_j(s_i^{\text{in}}(t, x), \xi) d\xi \right|^2 e^{-L_1(t-t^0)} dt \\ \leq nR_2^2 \int_0^1 \left(\int_J \sum_{j=1}^n |y_j(s_i^{\text{in}}(t, x), \xi)|^2 e^{-L_1(s_i^{\text{in}}(t, x)-t^0)} e^{L_1(s_i^{\text{in}}(t, x)-t)} dt \right) d\xi. \end{aligned}$$

680 Using once again (99), (100), performing the change of variable $\sigma = s_i^{\text{in}}(t, x)$, and using the
681 estimate (see (14), (12) and (19))

$$682 \frac{\partial s_i^{\text{in}}(t, x)}{\partial t} = \frac{\lambda_i(t, x) e^{-\int_{s_i^{\text{in}}(t, x)}^t \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta; t, x)) d\theta}}{\lambda_i(s_i^{\text{in}}(t, x), 1)} \geq \frac{\varepsilon}{R_1} e^{-\frac{R_1}{\varepsilon}}, \quad (102)$$

we obtain the estimate

$$\int_J \left| \sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t, x), \xi) y_j(s_i^{\text{in}}(t, x), \xi) d\xi \right|^2 e^{-L_1(t-t^0)} dt \leq \left(n \frac{R_1}{\varepsilon} R_2^2 e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2} \right) e^{-L_2(1-x)} \|y\|_{B_2}^2.$$

2) The next estimate is where we will need the smallness assumption on Q . Let $i \in \{m+1, \dots, n\}$. Using (15) and the change of variables $\sigma = s_i^{\text{in}}(t, x)$ (recall the estimate (102)), we obtain

$$\int_J \left| \sum_{j=1}^m q_{i-m, j}(s_i^{\text{in}}(t, x)) y_j(s_i^{\text{in}}(t, x), 0) \right|^2 e^{-L_1(t-t^0)} dt \leq \left(m\alpha^2 \frac{R_1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} \right) e^{-L_2(1-x)} \|y\|_{B_1}^2.$$

3) For $i \in \{1, \dots, m\}$, using the estimate

$$-R_1(t-s) \leq x - \chi_i(s; t, x),$$

we have

$$\begin{aligned} \left| \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds \right|^2 e^{-L_1(t-t^0)} \\ \leq nR_2^2 \frac{1}{\varepsilon} \int_{\bar{s}_i^{\text{in}}(t^0; t, x)}^t \sum_{j=1}^n |y_j(s, \chi_i(s; t, x))|^2 e^{-L_1(s-t^0)} e^{-\frac{L_1}{R_1}(\chi_i(s; t, x)-x)} ds. \end{aligned}$$

Integrating and using the change of variable $(\sigma, \xi) = (s, \chi_i(s; t, x))$, whose Jacobian determinant is uniformly estimated for $s \in (s_i^{\text{in}}(t, x), t)$ by (see (12) and (19))

$$\left| \det \begin{pmatrix} 1 & 0 \\ \lambda_i(s, \chi_i(s; t, x)) & \frac{\partial \chi_i}{\partial t}(s; t, x) \end{pmatrix} \right| = |\lambda_i(t, x)| e^{-\int_s^t \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta; t, x)) d\theta} \geq \varepsilon e^{-\frac{R_1}{\varepsilon}},$$

we obtain (using also that $x \leq \chi_i(s; t, x) \leq 1$)

$$\begin{aligned} & \int_{t^0}^{t^0+T} \sum_{i=1}^m \left| \sum_{j=1}^n \int_{s_i^{\text{in}}(t^0; t, x)}^t m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds \right|^2 e^{-L_1(t-t^0)} dt \\ & \leq mnR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \int_x^1 \left(\int_{t^0}^{t^0+T} \sum_{j=1}^n |y_j(\sigma, \xi)|^2 e^{-L_1(\sigma-t^0)} d\sigma \right) e^{-\frac{L_1}{R_1}(\xi-x)} d\xi \\ & \leq \left(mnR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{\frac{L_1}{R_1} - L_2} \right) e^{-L_2(1-x)} \|y\|_{B_2}^2, \end{aligned}$$

provided that (100) holds. A similar reasoning shows that

$$\begin{aligned} & \int_{t^0}^{t^0+T} \sum_{i=m+1}^n \left| \sum_{j=1}^n \int_{s_i^{\text{in}}(t^0; t, x)}^t m_{ij}(s, \chi_i(s; t, x)) y_j(s, \chi_i(s; t, x)) ds \right|^2 e^{-L_1(t-t^0)} dt \\ & \leq \left((n-m)nR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2 + \frac{L_1}{R_1}} \right) e^{-L_2(1-x)} \|y\|_{B_2}^2. \end{aligned}$$

4) For the remaining term, using a similar reasoning to the one used in the previous step, we obtain

$$\begin{aligned} & \int_{t^0}^{t^0+T} \sum_{i=1}^n \left| \sum_{j=1}^n \int_{s_i^{\text{in}}(t^0; t, x)}^t g_{ij}(s, \chi_i(s; t, x)) y_j(s, 0) ds \right|^2 e^{-L_1(t-t^0)} dt \\ & \leq \left(n^2 R_1 R_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_1} \right) e^{-L_2(1-x)} \|y\|_{B_2}^2. \end{aligned}$$

In summary, we have established the following estimate (provided that (100) holds):

$$\begin{aligned} \|\mathcal{F}(y^1) - \mathcal{F}(y^2)\|_{B_2}^2 & \leq 3 \left((n-m)m\alpha^2 \frac{R_1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} \right) \|y\|_{B_1}^2 \\ & + 3 \left(mn \frac{R_1}{\varepsilon} R_2^2 e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2} + mnR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{\frac{L_1}{R_1} - L_2} + (n-m)nR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2 + \frac{L_1}{R_1}} \right. \\ & \quad \left. + n^2 R_1 R_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_1} \right) \|y\|_{B_2}^2. \quad (103) \end{aligned}$$

683 Consequently, we see from (101) and (103) that \mathcal{F} indeed satisfies the contraction property (98)
684 if α is small enough (depending only on $n-m, m, R_1$ and ε) and if we fix $L_2 > 0$ and then $L_1 > 0$
685 large enough. This concludes the proof of Theorem A.2.

686 \square

Remark A.4. It can be shown that the broad solution is also the classical solution if the data of the system are smooth enough. It then follows by standard approximation arguments that the broad solution is also the so-called weak solution. We recall that the notion of weak solution for (91) is obtained by multiplying (91) by a smooth function and integrating by parts, that is, a function $y : (t^0, +\infty) \times (0, 1) \rightarrow \mathbb{R}^n$ is a weak solution to (91) if $y \in C^0([t^0, +\infty); L^2(0, 1)^n)$ and if it satisfies:

$$\begin{aligned} & \int_0^1 y(t^0 + T, x) \cdot \varphi(t^0 + T, x) dx - \int_0^1 y^0(x) \cdot \varphi(t^0, x) dx \\ & + \int_{t^0}^{t^0+T} \int_0^1 y(t, x) \cdot \left(-\frac{\partial \varphi}{\partial t}(t, x) - \Lambda(t, x) \frac{\partial \varphi}{\partial x}(t, x) - \left(\frac{\partial \Lambda}{\partial x}(t, x) + M(t, x)^{\text{Tr}} \right) \varphi(t, x) \right) dx dt \\ & + \int_{t^0}^{t^0+T} \int_0^1 y(t, \xi) \cdot F(t, \xi)^{\text{Tr}} \Lambda_{--}(t, 1) \varphi_-(t, 1) d\xi dt = 0, \quad (104) \end{aligned}$$

for every $T > 0$ and every $\varphi \in C^1([t^0, t^0 + T] \times [0, 1])^n$ such that, for every $t \in [t^0, t^0 + T]$,

$$\begin{aligned} & \varphi_+(t, 1) = 0, \\ & \varphi_-(t, 0) = -\Lambda_{--}(t, 0)^{-1} \left(Q(t)^{\text{Tr}} \Lambda_{++}(t, 0) \varphi_+(t, 0) + \left(\text{Id}_{\mathbb{R}^m} \quad Q(t)^{\text{Tr}} \right) \int_0^1 G(t, x)^{\text{Tr}} \varphi(t, x) dx \right). \end{aligned}$$

687 In (104), we denoted by A^{Tr} the transpose of a matrix A and $v_1 \cdot v_2$ denotes the canonical scalar
688 product between two vectors v_1, v_2 of \mathbb{R}^n .

689 A.3 Justification of the formal computations

690 In this section, we finally rigorously prove that the transformations that we used all along this paper
691 are preserving broad solutions. We show how it works only for the Fredholm transformation of Section
692 2.3 (because it is simpler to present) but the reasoning is general and can be used for the Volterra
693 transformation of Section 2.2 as well. More precisely, the goal of this section is to prove the following
694 result:

Proposition A.5. *Let $H_{--} = (h_{ij})_{1 \leq i, j \leq m}$, where h_{ij} is the solution to the differential equation (57). Let $t^0 \geq 0$ be fixed. Let $z^0 \in L^2(0, 1)^n$ and let z be the broad solution to*

$$\begin{cases} \frac{\partial z}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial z}{\partial x}(t, x) = G^3(t, x) z(t, 0), \\ z_-(t, 1) = 0, \quad z_+(t, 0) = Q^1(t) z_-(t, 0), \\ z(t^0, x) = z^0(x). \end{cases}$$

Then, the function γ defined by the Fredholm transformation (48) is the broad solution to

$$\begin{cases} \frac{\partial \gamma}{\partial t}(t, x) + \Lambda(t, x) \frac{\partial \gamma}{\partial x}(t, x) = G^2(t, x) \gamma(t, 0), \\ \gamma_-(t, 1) = \int_0^1 F^2(t, \xi) \gamma(t, \xi) d\xi, \quad \gamma_+(t, 0) = Q^1(t) \gamma_-(t, 0), \\ \gamma(t^0, x) = \gamma^0(x), \end{cases}$$

695 where $\gamma^0(x) = z^0(x) - \int_0^1 H(t^0, x, \xi) z^0(\xi) d\xi$.

696 We recall that H is given by (53), F^2 is the solution of (50), Q^1 is provided by Proposition 2.1,
 697 G^2 is provided by Proposition 2.4 and, finally, G^3 is given by (44).

698 *Remark A.6.* Obviously, we could use the explicit expression (60)-(58) of the solution H to simplify
 699 the forthcoming arguments but we choose not to do so and to only use the differential equation (57) in
 700 order to give a general procedure that can also be used to justify the formal computations of Section
 701 2.2 as well.

702 A similar result to Proposition A.5 can be found in [CN19, Proposition 3.5]. Here we propose a
 703 different and self-contained proof, based on the following characterization of broad solutions:

704 **Lemma A.7.** *A function $y : (t^0, +\infty) \times (0, 1) \rightarrow \mathbb{R}^n$ is the broad solution to (91) if, and only if, y
 705 has the regularity (95) and, for every $i \in \{1, \dots, n\}$, for a.e. $t > t^0$ and a.e. $x \in (0, 1)$, the function
 706 $s \mapsto y_i(s, \chi_i(s; t, x))$ belongs to $H^1(\bar{s}_i^{\text{in}}(t^0; t, x), s_i^{\text{out}}(t, x))$ and it satisfies the ODE (92).*

707 The proof of Lemma A.7 is not difficult, it simply relies on the properties (10) and (17).

708 *Proof of Proposition A.5.*

1) The required regularity

$$\gamma \in C^0([t^0, t^0 + T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(t^0, t^0 + T)^n), \quad \forall T > 0,$$

709 is clear since z also has this regularity and $(t, x) \mapsto \int_0^1 H_{--}(t, x, \xi) z_-(t, \xi) d\xi$ is continuous (see
 710 e.g. Remark 2.9).

2) The initial condition in the ODE formulation

$$\gamma_i(\bar{s}_i^{\text{in}}(t^0; t, x), \chi_i(\bar{s}_i^{\text{in}}(t^0; t, x); t, x)) = b_i(\gamma)(t, x)$$

711 is not difficult to check by using the boundary condition $z_-(t, 1) = 0$ with the definition (50) of
 712 F^2 and Fubini's theorem (case $s_i^{\text{in}}(t, x) > t^0$ and $i \in \{1, \dots, m\}$), the condition $H_{--}(t, 0, \xi) = 0$
 713 (case $s_i^{\text{in}}(t, x) > t^0$ and $i \in \{m+1, \dots, n\}$) and the definition of γ^0 (case $s_i^{\text{in}}(t, x) < t^0$).

714 3) It remains to check that, for every $i \in \{1, \dots, n\}$, for a.e. $t > t^0$ and $x \in (0, 1)$, the function
 715 $s \mapsto \gamma_i(s, \chi_i(s; t, x))$ belongs to $H^1(\bar{s}_i^{\text{in}}(t^0; t, x), s_i^{\text{out}}(t, x))$ with

$$716 \quad \frac{d}{ds} \gamma_i(s, \chi_i(s; t, x)) = \sum_{j=1}^m g_{ij}^2(s, \chi_i(s; t, x)) \gamma_j(s, 0). \quad (105)$$

By definition (48) of γ , we have

$$\gamma_i(s, \chi_i(s; t, x)) = z_i(s, \chi_i(s; t, x)) - \sum_{j=1}^m \int_0^1 h_{ij}(s, \chi_i(s; t, x), \xi) z_j(s, \xi) d\xi.$$

717 For $i \in \{m+1, \dots, n\}$, the identity (105) easily follows from the equation satisfied by z_i , the
 718 relation $z(\cdot, 0) = \gamma(\cdot, 0)$, and the fact that $h_{ij} = 0$ for such indices (recall (53)).

Let us now assume that $i \in \{1, \dots, m\}$. The equation satisfied by z_i then gives

$$\frac{d}{ds} z_i(s, \chi_i(s; t, x)) = 0, \quad \forall i \in \{1, \dots, m\}.$$

On the other hand, since we know some information of h_{ij} along the characteristic curve $s \mapsto$
 $\chi_{ij}(s; t, x, \theta) = (s, \chi_i(s; t, x), \chi_j(s; t, \theta))$, we would like to perform the change of variable

$$\xi = \chi_j(s; t, \theta).$$

719
720

Thanks to (13) and the implicit function theorem there exists $\theta_j \in C^1(\mathbb{R}^3)$ such that, for every $(s, t, \xi) \in \mathbb{R}^3$, we have

$$721 \quad \xi = \chi_j(s; t, \theta_j(s; t, \xi)), \quad \frac{\partial \theta_j}{\partial \xi}(s; t, \xi) > 0. \quad (106)$$

Using this change of variable, we have

$$\gamma_i(s, \chi_i(s; t, x)) = z_i(s, \chi_i(s; t, x)) - \sum_{j=1}^m \int_{a_j(s)}^{b_j(s)} \eta_{ij}(s, \theta) d\theta,$$

where

$$\begin{aligned} a_j(s) &= \theta_j(s; t, 0), & b_j(s) &= \theta_j(s; t, 1), \\ \eta_{ij}(s, \theta) &= h_{ij}(\chi_{ij}(s; t, x, \theta)) z_j(s, \chi_j(s; t, \theta)) \frac{\partial \chi_j}{\partial x}(s; t, \theta). \end{aligned}$$

We would like to use the formula

$$\frac{d}{ds} \left(\int_{a_j(s)}^{b_j(s)} \eta_{ij}(s, \theta) d\theta \right) = b'_j(s) \eta_{ij}(s, b_j(s)) - a'_j(s) \eta_{ij}(s, a_j(s)) + \int_{a_j(s)}^{b_j(s)} \frac{\partial \eta_{ij}}{\partial s}(s, \theta) d\theta.$$

Clearly, $a_j, b_j \in C^1(\mathbb{R})$. Differentiating the relation $\xi = \chi_j(s; t, \theta_j(s; t, \xi))$ with respect to s we obtain

$$a'_j(s) = \frac{-\lambda_j(s, 0)}{\frac{\partial \chi_j}{\partial x}(s; t, \theta_j(s; t, 0))}.$$

On the other hand, using (57) with $\xi = 0$, (106) and the boundary condition $z_-(\cdot, 1) = 0$, we have

$$\eta_{ij}(s, a_j(s)) = -\frac{g_{ij}^2(s, \chi_i(s; t, x))}{\lambda_j(s, 0)} z_j(s, 0) \frac{\partial \chi_j}{\partial x}(s; t, \theta_j(s; t, 0)), \quad \eta_{ij}(s, b_j(s)) = 0.$$

Using the ODEs satisfied along the characteristics by h_{ij} (see (57)) and z_j , and using the relation (see (12))

$$\frac{\partial^2 \chi_j}{\partial s \partial x}(s; t, \theta) = \frac{\partial \lambda_j}{\partial x}(s, \chi_j(s; t, \theta)) \frac{\partial \chi_j}{\partial x}(s; t, \theta),$$

we can check that η_{ij} has weak derivative with respect to s which is equal to zero:

$$\frac{\partial \eta_{ij}}{\partial s}(s, \theta) = 0.$$

It follows from all the previous computations and the relation $z(\cdot, 0) = \gamma(\cdot, 0)$ that

$$\frac{d}{ds} \gamma_i(s, \chi_i(s; t, x)) = \sum_{j=1}^m a'_j(s) \eta_{ij}(s, a_j(s)) = \sum_{j=1}^m g_{ij}^2(s, \chi_i(s; t, x)) \gamma_j(s, 0).$$

722

□

723 B Constructions of $s_{ij}^{\text{in}}, s_{ij}^{\text{out}}$ and $s_{ij\ell}^{\text{disc}}$

724 In this appendix, we give a proof of the existence $s_{ij}^{\text{in}}, s_{ij}^{\text{out}}$ and $s_{ij\ell}^{\text{disc}}$ satisfying the properties stated
725 in Proposition 3.3 and Proposition 3.5. We will make use of the following simple lemma:

726 **Lemma B.1.** *Let $f \in C^1([a, b])$ ($a < b$) satisfy the following property:*

$$727 \quad \forall s \in [a, b], \quad f(s) = 0 \implies f'(s) < 0. \quad (107)$$

Then, there exists a unique $c \in [a, b]$ such that

$$f(s) > 0, \quad \forall s \in (a, c), \quad f(s) < 0, \quad \forall s \in (c, b).$$

728 *Moreover, c has the properties listed in Table 1 (an \emptyset means that such a situation can not occur).*

	$f(b) > 0$	$f(b) = 0$	$f(b) < 0$
$f(a) > 0$	$c = b$	$c = b$	$f(c) = 0$
$f(a) = 0$	\emptyset	\emptyset	$c = a$
$f(a) < 0$	\emptyset	\emptyset	$c = a$

Table 1: Properties of c

729 *Proof of Proposition 3.3.* We recall that $i \in \{1, \dots, m\}$ and we refer to Figures 2, 3, 4 and 5 for a
730 clarification of the geometric situation (at a fixed t). We only focus on the existence part since the
731 uniqueness readily follows from the properties that have to be satisfied.

- 1) Assume that $j \in \{1, \dots, i-1\}$. For every $(t, x, \xi) \in \overline{\mathcal{P}}$ such that $x < 1$, we introduce the C^1 function

$$f : s \in [\max\{s_i^{\text{in}}(t, x), s_j^{\text{in}}(t, \xi)\}, t] \mapsto \chi_j(s; t, \xi) - \chi_i(s; t, x).$$

Note that the interval has a non empty interior since $x < 1$ and $\xi \leq x < 1$ (see (15)-(16)). This function clearly satisfies the property (107) thanks to the ODE (9) and the assumption (3) since $j < i$. Consequently, Lemma B.1 applies and gives the existence of $s_{ij}^{\text{in}}(t, x, \xi)$ with

$$\max\{s_i^{\text{in}}(t, x), s_j^{\text{in}}(t, \xi)\} \leq s_{ij}^{\text{in}}(t, x, \xi) \leq t,$$

and such that

$$\chi_j(s; t, \xi) < \chi_i(s; t, x), \quad \forall s \in (s_{ij}^{\text{in}}(t, x, \xi), t).$$

732 Clearly, $(t, x, \xi) \mapsto t - s_{ij}^{\text{in}}(t, x, \xi) \in L^\infty(\mathcal{P})$ thanks to (19). Moreover, it follows from Table 1
733 that

$$734 \quad \begin{cases} s_{ij}^{\text{in}}(t, x, \xi) = t & \text{if } s_i^{\text{in}}(t, x) < s_j^{\text{in}}(t, \xi) \text{ and } \xi = x, \\ f(s_{ij}^{\text{in}}(t, x, \xi)) = 0 & \text{if } s_i^{\text{in}}(t, x) < s_j^{\text{in}}(t, \xi) \text{ and } \xi < x, \\ s_{ij}^{\text{in}}(t, x, \xi) = s_i^{\text{in}}(t, x) & \text{if } s_i^{\text{in}}(t, x) = s_j^{\text{in}}(t, \xi) \text{ and } \xi < x, \\ s_{ij}^{\text{in}}(t, x, \xi) = s_i^{\text{in}}(t, x) & \text{if } s_i^{\text{in}}(t, x) > s_j^{\text{in}}(t, \xi) \text{ and } \xi < x. \end{cases} \quad (108)$$

735 Let us now complete the definition of s_{ij}^{in} on the remaining parts of $\overline{\mathcal{P}}$. The missing case in
736 (108) is when $s_i^{\text{in}}(t, x) \geq s_j^{\text{in}}(t, \xi)$ and $\xi = x$. However, unless $x = 1$, these conditions are not
737 compatible since $s_i^{\text{in}}(t, x) < s_j^{\text{in}}(t, x)$ for $j < i \leq m$ (see (21)). Consequently, it only remains to
738 define s_{ij}^{in} in the part where $x = 1$, which we do now by setting

$$739 \quad s_{ij}^{\text{in}}(t, 1, \xi) = t. \quad (109)$$

740 We can check that s_{ij}^{in} defined by (108)-(109) belongs to $C^0(\overline{\mathcal{P}})$ (for the second case in (108)
741 this follows from the implicit function theorem). Therefore, such a s_{ij}^{in} clearly satisfies all the
742 properties claimed in the statement of item (i) of Proposition 3.3.

2) Assume that $j = i$. We will show that, in this case, we can simply take

$$s_{ii}^{\text{out}}(t, x, \xi) = s_i^{\text{out}}(t, \xi).$$

Clearly, $s_{ii}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t, x, \xi) \mapsto s_{ii}^{\text{out}}(t, x, \xi) - t \in L^\infty(\mathcal{P})$ thanks to (19) and $s_{ii}^{\text{out}}(t, x, \xi) > t$
as long as $\xi > 0$ (see (15)-(16)). Let us now observe that, for $\xi < x$, we have from (13):

$$\chi_i(s; t, \xi) < \chi_i(s; t, x), \quad \forall s \in \mathbb{R},$$

743 and $\chi_i(s; t, \xi) > 0$ for $s \in (t, s_{ii}^{\text{out}}(t, x, \xi))$ since $s_{ii}^{\text{out}}(t, \xi) < s_i^{\text{out}}(t, x)$ by (20) (recall that $j = i \in$
744 $\{1, \dots, m\}$).

745 3) The proof for the case $j \in \{i + 1, \dots, m\}$ is similar to the proof of part 1) by considering, for
746 each $(t, x, \xi) \in \overline{\mathcal{P}}$ such that $\xi, x > 0$, the function

$$747 \quad f : s \in [t, \min \{s_i^{\text{out}}(t, x), s_j^{\text{out}}(t, \xi)\}] \mapsto \chi_i(s; t, x) - \chi_j(s; t, \xi). \quad (110)$$

748 4) The proof for the case $j \in \{m + 1, \dots, n\}$ is also similar to the proof of part 1) by considering,
749 for each $(t, x, \xi) \in \overline{\mathcal{P}}$ such that $0 \leq \xi < x \leq 1$, the function f defined again by (110).

750 □

751 *Proof of Proposition 3.5.* The difference with the proof of Proposition 3.3 is that we do not need to
752 neither track the regularity of the point where the function f vanishes nor its sign on the left and
753 right of this zero. It is a straightforward consequence of Lemma B.1 applied to the following functions
754 (it is enough to consider non empty intervals):

1) For $j \in \{1, \dots, i - 1\}$, we use

$$f : s \in [s_{ij}^{\text{in}}(t, x, \xi), t] \mapsto \chi_j(s; t, \xi) - \psi_{i\ell}(s, \chi_i(s; t, x)).$$

Using the ODE (9) satisfied by χ_j and using the equation (41) satisfied by $\psi_{i\ell}$, we have

$$f'(s) = \lambda_j(s, \chi_j(s; t, \xi)) - \lambda_\ell(s, \psi_{i\ell}(s, \chi_i(s; t, x))).$$

755 Since $j < \ell$, this shows that such a f satisfies the property (107) of Lemma B.1.

756 2) For $j \in \{i, \dots, \ell - 1\}$, we use

$$757 \quad f : s \in [t, s_{ij}^{\text{out}}(t, x, \xi)] \mapsto \chi_j(s; t, \xi) - \psi_{i\ell}(s, \chi_i(s; t, x)). \quad (111)$$

758 3) For $j \in \{\ell + 1, \dots, m\}$, we use the function $-f$, where f is given by (111).

759 4) For $j \in \{m + 1, \dots, n\}$, we use the same function f given by (111) (in fact, the result then
760 directly follows from the intermediate value theorem).

761 □

762 C Construction of Ω_i

763 This appendix is devoted to the proof of Lemma 3.7, that is to the existence of the key change of
764 variable needed in the proof of Proposition 3.6. We recall that $i \in \{1, \dots, m\}$.

1) Inspired by the time-independent case (see Remark 3.6), we look for Ω_i in the following form:

$$\Omega_i(t, x, \xi) = \omega_i^1(t, x) - \omega_i^\nu(t, \xi),$$

765 where, at each fixed $\nu \in (0, 1]$, $\omega_i^\nu(\cdot, \cdot)$ is the solution to the following linear hyperbolic equation:

$$766 \quad \begin{cases} \frac{\partial \omega_i^\nu}{\partial t}(t, x) + \frac{\lambda_i(t, x)}{\nu} \frac{\partial \omega_i^\nu}{\partial x}(t, x) = 0, \\ \omega_i^\nu(t, 0) = t, \end{cases} \quad t \in \mathbb{R}, x \in [0, 1]. \quad (112)$$

767 The solution of (112) is explicit:

$$768 \quad \omega_i^\nu(t, x) = \omega_i^\nu(s_i^{\text{out}, \nu}(t, x), 0) = s_i^{\text{out}, \nu}(t, x), \quad (113)$$

769 where $s_i^{\text{out}, \nu}(t, x) \geq t$ (with $s_i^{\text{out}, \nu}(t, x) = t \iff x = 0$) is the unique number such that

$$770 \quad \chi_i^\nu(s_i^{\text{out}, \nu}(t, x); t, x) = 0, \quad (114)$$

771 where $s \mapsto \chi_i^\nu(s; t, x)$ is the solution to the ODE

$$772 \quad \begin{cases} \frac{\partial \chi_i^\nu}{\partial s}(s; t, x) = \frac{1}{\nu} \lambda_i(s, \chi_i^\nu(s; t, x)), \quad \forall s \in \mathbb{R}, \\ \chi_i^\nu(t; t, x) = x. \end{cases} \quad (115)$$

773 We can check that the map $(t, x, \nu) \mapsto \omega_i^\nu(t, x)$ belongs to $C^1(\mathbb{R} \times [0, 1] \times (0, 1])$.

774 2) We now prove that there exists $\delta > 0$ such that, for every $t \in \mathbb{R}$, $x \in [0, 1]$ and $\nu \in (0, 1]$,

$$775 \quad \frac{\partial \omega_i^\nu}{\partial t}(t, x) \geq \varepsilon \delta, \quad \frac{\partial \omega_i^\nu}{\partial x}(t, x) \geq \nu \delta, \quad \frac{\partial \omega_i^\nu}{\partial \nu}(t, x) \geq 0. \quad (116)$$

Using the equation (112) and the assumption (3), it is clear that the estimate for $\partial \omega_i^\nu / \partial t$ follows from the estimate of $\partial \omega_i^\nu / \partial x$. Note from (113) that $\partial \omega_i^\nu / \partial x = \partial s_i^{\text{out}, \nu} / \partial x$. Taking the derivative of (114) with respect to x , we obtain

$$\frac{1}{\nu} \lambda_i(s_i^{\text{out}, \nu}(t, x), \chi_i^\nu(s_i^{\text{out}, \nu}(t, x); t, x)) \frac{\partial s_i^{\text{out}, \nu}}{\partial x}(t, x) + \frac{\partial \chi_i^\nu}{\partial x}(s_i^{\text{out}, \nu}(t, x); t, x) = 0.$$

Since $\lambda_i \in L^\infty(\mathbb{R} \times (0, 1))$, we have to bound $\frac{\partial \chi_i^\nu}{\partial x}(s_i^{\text{out}, \nu}(t, x); t, x)$ from below by a positive constant that does not depend on t, x and ν . From (115) we can show that

$$\frac{\partial \chi_i^\nu}{\partial x}(s; t, x) = e^{\frac{1}{\nu} \int_t^s \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i^\nu(\theta; t, x)) d\theta},$$

so that

$$\frac{\partial \chi_i^\nu}{\partial x}(s_i^{\text{out}, \nu}(t, x); t, x) \geq e^{\frac{1}{\nu} (s_i^{\text{out}, \nu}(t, x) - t) \inf_{\mathbb{R} \times [0, 1]} \frac{\partial \lambda_i}{\partial x}}.$$

776 This establishes the desired lower bound since $\frac{\partial \lambda_i}{\partial x} \in L^\infty(\mathbb{R} \times (0, 1))$ and $0 \leq s_i^{\text{out}, \nu}(t, x) - t \leq \frac{x}{\varepsilon} \nu$
777 (the proof is similar to the one of (19)). Note that it follows as well from this estimate that
778 $\Omega_i \in L^\infty(\mathbb{R} \times (0, 1) \times (0, 1))$.

To prove the remaining estimate in (116), we denote by $\gamma^\nu = \partial\omega_i^\nu/\partial\nu$ and observe that it satisfies

$$\begin{cases} \frac{\partial\gamma^\nu}{\partial t}(t, x) + \frac{\lambda_i(t, x)}{\nu} \frac{\partial\gamma^\nu}{\partial x}(t, x) = \frac{\lambda_i(t, x)}{\nu^2} \frac{\partial\omega_i^\nu}{\partial x}(t, x) \leq 0, \\ \gamma^\nu(t, 0) = 0, \end{cases} \quad t \in \mathbb{R}, x \in [0, 1].$$

779 It immediately follows that $\gamma^\nu \geq 0$.

3) Let us now check the estimates (87) and (88). We have

$$\begin{aligned} & \frac{\partial\Omega_i}{\partial t}(t, x, \nu) + \lambda_i(t, x) \frac{\partial\Omega_i}{\partial x}(t, x, \nu) + \lambda_j(t, \xi) \frac{\partial\Omega_i}{\partial \xi}(t, x, \nu) \\ &= \frac{\partial\omega_i^1}{\partial t}(t, x) + \lambda_i(t, x) \frac{\partial\omega_i^1}{\partial x}(t, x) - \frac{\partial\omega_i^\nu}{\partial t}(t, \xi) - \lambda_j(t, \xi) \frac{\partial\omega_i^\nu}{\partial x}(t, \xi) \\ &= -\frac{\partial\omega_i^\nu}{\partial t}(t, \xi) \left(1 - \nu \frac{\lambda_j(t, \xi)}{\lambda_i(t, \xi)} \right). \end{aligned}$$

780 Since $\lambda_j/\lambda_i \leq 1$ for $i \leq j$ and $i \leq m$, we see that the estimate (88) is obtained by simply taking

$$781 \quad 0 < \varepsilon_0 \leq \varepsilon\delta(1 - \nu), \quad 0 < \nu < 1. \quad (117)$$

On the other hand, let us introduce

$$r = \max_{1 \leq j < i} \sup_{\substack{t \in \mathbb{R} \\ \xi \in [0, 1]}} \frac{\lambda_i(t, \xi)}{\lambda_j(t, \xi)}.$$

Clearly, $0 < r \leq 1$. In fact, $r < 1$ since from (4) we have, for $j < i \leq m$,

$$\sup_{\substack{t \in \mathbb{R} \\ \xi \in [0, 1]}} \frac{\lambda_i(t, \xi)}{\lambda_j(t, \xi)} \leq 1 - \frac{\varepsilon}{\|\lambda_j\|_{L^\infty}}.$$

782 The estimate (87) now follows from (116) by taking

$$783 \quad 0 < \varepsilon_0 \leq \varepsilon\delta \left(\frac{\nu}{r} - 1 \right), \quad r < \nu \leq 1. \quad (118)$$

784 Note that the conditions (117) and (118) are compatible by taking ν close enough to 1.

4) It remains to check that $\Omega_i \geq 0$ on $\bar{\mathcal{P}}$. Since both functions $\nu \mapsto \omega_i^\nu(t, x)$ and $x \mapsto \omega_i^\nu(t, x)$ are nondecreasing by (116) and $\xi \leq x$, we have

$$\omega_i^1(t, x) \geq \omega_i^\nu(t, x) \geq \omega_i^\nu(t, \xi).$$

785 □

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