The End of an Error: Bianchini, Regiomontanus, and the Tabulation of Stellar Coordinates

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Abstract

Giovanni Bianchini's 15th-century *Tabulae primi mobilis* is a collection of 50 pages of canons and 100 pages of tables of spherical astronomy and mathematical astrology, beginning with a treatment of the conversion of stellar coordinates from ecliptic to equatorial. His new method corrects a long-standing error made by a number of his antecedents, and with his tables the computations are much more efficient than in Ptolemy's *Almagest*. The completely novel structure of Bianchini's tables, here and in his *Tabulae magistrales*, was taken over by Regiomontanus in the latter's *Tabulae directionum*. One of the tables Regiomontanus imported from Bianchini contains the first appearance of the tangent function in Latin Europe, which both used as an auxiliary quantity for the calculation of stellar coordinates.

Introduction

Mathematical astronomy in medieval Europe was inspired by two eastern Arabic sources: al-Khwārizmī's *Zīj* influenced by India and Persia, and al-Battānī's *Ṣābi' Zīj* in the style of Ptolemy's *Almagest*. These texts traveled to al-Andalus in the 10th century, helping to establish a distinct astronomical tradition in the Islamic west. The two most important works in this genre, the Toledan[1] and Alfonsine Tables[2] (mid-11th and mid-13th centuries), were used and adapted in various European centres; the latter became the authority guiding astronomers' work through the 14th and into the 15th century. Although its tables were frequently restructured to make them more user-friendly,[3] the underlying astronomy and mathematics remained mostly stable.

This was about to change. The 15th-century Viennese astronomers John of Gmunden (c. 1380-1442), Georg Peurbach (1423-1461), and Regiomontanus (1436-1476) were among those who revived interest in studying and altering the mathematics underlying Alfonsine astronomy. Regiomontanus became the most recognized European astronomer of his time, due partly to the publication of a number of his treatises after the invention of printing. However, some of Regiomontanus's work was directly inspired by his lesser-known older colleague, Giovanni Bianchini.

Bianchini (ca. 1410-1469), chief fiscal officer to the ruling d'Este family in Ferrara, composed several collections of astronomical tables, all related to aspects of his *magnum opus* of mathematical astronomy, the *Flores Almagesti*.[4] One of these, the *Tabulae primi mobilis*, is a set of tables and

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[1] On the Toledan Tables, see Toomer 1968 and especially F. Pedersen 2002.

[2] There is a wealth of literature on the Alfonsine Tables and related works; begin with Chabás/Goldstein 2003.

[3] For a survey of astronomical tables in the late Middle Ages, see Chabás/Goldstein 2012. This work, and many of their research papers, explore the notion of "user-friendliness" in the tables of this time period; see especially Chabás/Goldstein 2013. The ALFA project (see the acknowledgments), within whose auspices this paper was written, aims to refine and complement this perspective on Alfonsine astronomy.

[4] The set of tables dealing with the motions of the planets is the subject of Chabás/Goldstein 2009.

instructions for solving problems in spherical astronomy and astrology. A smaller set of tables, the *Tabulae magistrales*, is a companion to the *Tabulae primi mobilis* and has been the subject of a recent study.[5] Here we shall concentrate on the tables in both works that solve the problem: given a star's longitude λ and latitude β with respect to the ecliptic, determine the star's right ascension α and declination δ with respect to the equator.[6] At the end of this paper we shall consider the influence of Bianchini's work especially on Regiomontanus's most popular set of tables, the *Tabulae directionum.*

Stellar coordinates in Ptolemy's Almagest

The end of Book I of the *Almagest* concerns the fundamentals of spherical astronomy. The positions of celestial bodies are measured with the ecliptic as the reference circle: *longitudes* λ are measured along the ecliptic from the vernal equinox ♈; and *latitudes* β are measured perpendicularly to the ecliptic (Figure 1). The angle between the ecliptic and the equator, the *obliquity of the ecliptic*, is $\varepsilon \approx 23½°$. Since the Sun is always in the ecliptic and the planets do not diverge from the ecliptic by more than about 6º, the ecliptic is a natural reference circle from which to locate the planets. However, since the inclination of the ecliptic to the horizon varies constantly and stars can be any distance from the ecliptic, it is better to use the equator as the reference circle for coordinates of stars, since it maintains a constant inclination to the horizon and meridian.

<Insert Figure 1 here>

Ptolemy does not consider conversion of celestial coordinates in Book I; in fact, he does not think of declination and right ascension as coordinates at all.[7] Instead a star's right ascension is a measure of the time between the crossing of the meridian of the vernal equinox and the star; while the declination is the distance on the meridian from the equator to the star's culmination. He does show how to calculate the solar declination $\delta_\odot$ and right ascension $\alpha_\odot$ of points on the ecliptic, quantities needed for solar and lunar theory. For these he finds equivalents to

$$\sin\delta_\odot = \sin\lambda \sin\varepsilon \tag{1}$$

and

$$\sin\alpha_\odot = \tan\delta_\odot \cot\varepsilon,^{8} \tag{2}$$

and composes a table of $\delta_\odot(\lambda)$ in I.15. Of course, Ptolemy did not have modern trigonometric functions and theorems; instead he used a table of chords in a circle of radius $R = 60$, and applied the chord equivalent of what is today called Menelaus's Theorem: in Figure 2,

$$\frac{\sin(a+b)}{a} = \frac{\sin(g+h)}{\sin g} \cdot \frac{\sin f}{\sin(e+f)} \tag{3}$$

and

[5] Chabás 2016; see also Rosińska 1981a.
[6] A collection of letters between Bianchini and Regiomontanus on astronomical topics also discusses this problem, among others. This correspondence was first published in Curtze 1902, and was studied thoroughly in Gerl 1989. A translation of part of the most important letter (Regiomontanus's last) appears in Swerdlow 1990, 170-174.
[7] See O. Pedersen 2011, 99-101.
[8] *Almagest* I.14-16; see Toomer 1984, 69-74.

$$\frac{\sin a}{\sin b} = \frac{\sin(c+d)}{\sin d} \cdot \frac{\sin g}{\sin h}. \tag{4}$$

<Insert Figure 2 here>

Later when Ptolemy considers the fixed stars (VIII.5), he does address the problem of finding the declination of any star, but his method is not complete. In Figure 3, *N* is the pole of the equator, *P* is the pole of the ecliptic, and *V* is the intersection of the meridian with the equator. He applies Menelaus's Theorem (3) to the configuration *PNVZY*★ and arrives at an equivalent of

$$\cos\varepsilon = \frac{\cos XZ}{\sin(XZ+\beta)} \cdot \sin\delta. \tag{5}$$

This formula is correct, but what precisely is *XZ*? It is not the declination of point *X*, since it is perpendicular to the ecliptic rather than to the equator. *Almagest* VIII.5 does not explain how to find *XZ*. One method considers triangle ΥXZ. If we reverse the roles of the ecliptic and equator, then $\lambda = \Upsilon X$ is a "right ascension" corresponding to "longitude" ΥZ. The *Almagest*'s table of oblique ascensions contains a table of solar right ascensions $\alpha_\odot$ (the columns corresponding to terrestrial latitude 0°), so one may enter this table inversely with argument λ to obtain ΥZ. Finally, we enter the solar declination table with argument ΥZ to find *XZ*.[9]

<Insert Figure 3 here>

Between Ptolemy and Bianchini

Ptolemy's work on stellar coordinates was transmitted to his successors along with the rest of the *Almagest*. One of the earliest to comment on it was al-Battānī, who describes the process in the *Ṣābi' Zīj*.[10] However, he does not explain how to find *XZ* either. He was taken to task for this by 10th-century Egyptian astronomer Ibn Yūnus in his *Ḥākimī Zīj*, who notes that *XZ* may be found using right ascension tables.[11] The quantity *XZ* was soon named the *second declination* (we shall refer to it as $\delta_2(\lambda)$); it became standard to include a table for it in eastern Islamic *zījes*. Tables of the second declination may be found from the late 10th century onward, by Kūshyār ibn Labbān, al-Bīrūnī, al-Khāzinī, Naṣīr al-Dīn al-Ṭūsī, Jamshīd al-Kāshī, and Ulugh Beg, among others.[12]

However, while al-Battānī's *Zīj* continued to influence medieval European astronomy through the Toledan and the Alfonsine Tables, most later eastern Islamic developments, including the second declination, were unknown in Europe. Thus, stellar coordinate conversions continued to omit the computation of *XZ*. When the problem was considered (which was not frequent), astronomers simply substituted the value of $\delta_\odot(\lambda)$ for *XZ*. Richard of Wallingford (1292-1336), although he was aware of the

[9] The method may be found in *Almagest* VIII.5 (Toomer 1984, 410-413), where Toomer makes the same error that Bianchini is about to correct (note 201, p. 411); the table of oblique ascensions is *Almagest* II.8 (Toomer 1984, 100-103). See O. Pedersen 2011, 97-99.
[10] See Nallino 1899-1907: the Latin translation, vol. 1, 31-32; commentary by Nallino, vol. 1, 192-193; and Arabic text, vol. 3, 46-48.
[11] See King 1972, 293-295.
[12] See Kennedy 1956.

correct solution, did this when performing a sample calculation in his *Quadripartitum*.[13] Around the same time, John of Lignères made the same substitution.[14] A similar instruction is found in the canons to the astronomical tables of John of Gmunden (1380-1442), although in one of the manuscripts a marginal note by Regiomontanus indicates that the method is "false".[15]

Bianchini's works

Giovanni Bianchini's collection of astronomical treatises appears to have been written between 1440 and 1460. The *Flores Almagesti*, intended to be a detailed exposition and occasional extension of the *Almagest* with all required supplementary mathematical techniques, reached only the end of Book VI.[16] It seems to have grown over time; the extant manuscripts contain between eight and ten chapters. Although it explicitly follows the *Almagest*, its theoretical treatment contains a number of innovations. Most obvious is the addition of two chapters at the beginning on arithmetic and algebra. The arithmetic was necessary, since Bianchini was introducing decimal numeration presumably inspired by his training at an Italian abacus school. Of relevance to us are Chapters III through V on trigonometry and spherical astronomy.

The *Flores Almagesti* provided mathematical demonstrations for its procedures, but Bianchini's other astronomical works were intended for practical use. They contain sets of tables for planetary motions, eclipses, and spherical astronomy; when justifications of new procedures are called for, Bianchini refers back to the *Flores Almagesti*.[17] We shall focus on the part of the *Tabulae primi mobilis* on spherical astronomy. This may have been Bianchini's most popular work, judging by the 15 available manuscripts (compared to seven for the *Flores Almagesti*). They are as follows:

- Cracow BJ 556, 9r–23v (canons), 55r–104v (tables)
- Munich, Bayerische Staatsbibliothek, MS Clm 14111, 109v–115v (tables, incomplete)
- Paris, Bibliothèque Nationale de France, MS 7270, 143r–146v (canons, truncated), 147r–167r (canons, complete), 183r–236r (tables, incomplete)
- Paris, Bibliothèque Nationale de France, MS 7271, 147r–161r (canons), 181r, 190r–216v, 242v-245v (tables, incomplete)
- Paris, Bibliothèque Nationale de France, MS 7286, 65r–81v (canons), 98v–131v, 139r-148v (tables, incomplete)
- Paris, Bibliothèque Nationale de France, MS 10265, 88v–92v, 153v–221v (tables, incomplete)
- Paris, Bibliothèque Nationale de France, MS 10267, 45r–80v (canons)
- Vatican, BAV, Vat. Lat. 2228, 52r–78r (canons)

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[13] See North 1976: Latin edition and translation, vol. 1, 126-134 (especially pp. 130-131) and associated commentary, vol. 2, 71. See also the edition and translation, vol. 1, 158-159 and associated commentary, vol. 2, 79-80; and the edition/translation, vol. 1, 438-439 and associated commentary, vol. 2, 306.
[14] See Saby 1987, vol. 1, 57.
[15] Porres de Mateo 2003, vol. 2, 464.
[16] On the chapter on arithmetic in the *Flores Almagesti*, see Rosińska 1995 (law of signs for negative numbers). On the chapter on algebra, see Rosińska 1997-98. On Bianchini's concept of number in the chapters on arithmetic and algebra, see Rosińska 1998. On the *Flores Almagesti* as an inspiration for university teaching especially in Cracow, see Rosińska 2006.
[17] The planetary tables were studied in detail in Chabás/Goldstein 2009.

- Bologna, Biblioteca Comunale, MS 1601, 1r–17r (canons), 33r, 34r–66r, 67r–v, 74v–85r, 94v–99r (tables, incomplete)
- Florence, Biblioteca Medicea Laurenziana, MS 142, 1–22
- Florence, Biblioteca Medicea Laurenziana, MS Ashb. 216[18]
- Florence, Biblioteca Medicea Laurenziana, MS Plut. 29.33, 122r–134v (canons), 143r–149v (tables, incomplete)
- Nuremberg, Stadtbibliothek, MS Cent. V 58, 133r–154r (canons)
- Oxford, Bodleian Library, MS Can. Misc. 517, 82r–99r (canons), 119v–153r, 160r–171r (tables)
- Würzburg, Universitätsbibliothek, MS M. ch. f. 254, 111r–126r

We have consulted the first eight of these manuscripts.

Associated with the *Tabulae primi mobilis* is a set of eight tables called the *Tabulae magistrales*, available in at least eight manuscripts, in seven of which, they appear alongside or even intermingled with the *Tabulae primi mobilis*. The functions tabulated in the *Tabulae magistrales* are clearly intended for spherical astronomy; we shall demonstrate that two of them were designed to convert stellar coordinates – including the fourth table, the earliest tangent table in Europe.[19] (While the *Tabulae magistrales* includes tables of both the tangent and the cosecant, we are hesitant to call these tables trigonometric. For Bianchini and his immediate successors they fell within the category of "auxiliary tables": not as basic as tables of primitive trigonometric functions, yet not representing quantities with immediate astronomical interpretations.)

The table of second declinations

The tables in the *Tabulae primi mobilis* are in different orders in different manuscripts; sometimes they even apply different ranges of arguments. Occasionally they are intermingled with tables from other treatises. This was not uncommon; tables were collected with less care for order and coherence than canons were. Figure 4 gives the manuscript folio numbers of the tables we examine in this study.

<Insert Figure 4 here>

Collections of tables in spherical astronomy often begin with the most important fundamental quantity, the solar declination $\delta_\odot$ as a function of ecliptic longitude λ. We find such a table in the *Tabulae primi mobilis*, with a slightly unusual title: *Tabula novissima declinationis ecliptice per arcum secundum Johannem Blanchinum*.[20] This table poses an immediate problem. Although it clearly uses Bianchini's parameter for the obliquity of the ecliptic ($\varepsilon = 23;33;30°$), its entries vary systematically and significantly from the correct values of $\delta_\odot(\lambda)$, the difference gradually increasing with λ up to 3/4° in the middle of the table before decreasing back to zero (Figure 5). The entries also disagree with the solar declination values in the *Tabulae primi mobilis* canons; for instance, in Chapter 10, Bianchini states that $\delta_\odot(38°) = 14;15°$ and $\delta_\odot(39°) = 14;34°$, while the table reads 15;1° and 15;20° respectively.[21]

[18] A table of contents of our treatise in this manuscript is given in Boffito 1908, 457-460.
[19] On the *Tabulae magistrales*, see Rosińska 1981a and Chabás 2016.
[20] Here and elsewhere, for tables and passages from the *Tabulae primi mobilis* we shall refer to folio numbers in the manuscript Cracow BJ 556. This table is on f. 59r.
[21] Cracow BJ 556, f. 12v.

<Insert Figure 5 here>

However, in Chapters 14 and 15 of the canons Bianchini calculates the declination of a star by adding the (solar) "declination" to the star's latitude,[22] suggesting the possibility that our table may be of the second declination. Recomputations according to the modern formula

$$\tan \delta_2 = \tan \varepsilon \sin \lambda \tag{6}$$

produce nearly a perfect fit to the table, with 44 differences of ±0;1° and one of +0;2° in 180 entries. Thus Bianchini has rediscovered and tabulated the second declination.[23]

Now that Bianchini has a second declination table, he can compute the stellar declination $\delta(\lambda, \beta)$ using the method of *Almagest* VIII.5:

$$\cos \varepsilon = \frac{\cos \delta_2(\lambda)}{\sin\big(\delta_2(\lambda) + \beta(\lambda)\big)} \cdot \sin \delta(\lambda, \beta). \tag{7}$$

He describes how to do this in Chapter 15 of the canons to the *Tabulae primi mobilis*, but refers back to Book V of the *Flores Almagesti* for the geometric proof. Indeed, Book V refers to the matter several times, and in its introduction Bianchini draws attention to a distinction between the two declinations:

> *Et arcus* [★*X*] *designat quantitatem latitudinis stelle et* [*XZ*] *declinationem non equatam secundum Joannem Blanchinum.*[24]
>
> And arc ★*X* marks the quantity of the latitude of the star, and the declination is not made equal to *XZ*, according to Giovanni Bianchini.

In V.6 we find a complete calculation of the stellar declination for $\lambda = 60°, \beta = 23°$, as follows.[25] In Figure 3, consider the bold Menelaus configuration and apply (3):[26]

$$\frac{\sin VP}{\sin VN} = \frac{\sin PZ}{\sin \bigstar Z} \cdot \frac{\sin \bigstar Y}{\sin NY}. \tag{8}$$

Now, $VP = 90° + \varepsilon = 113;33{,}30°$; $VN = 90°$; $PZ = 90° + \delta_2(60°) = 90° + 20;41° = 110;41°$; $\bigstar Z = \beta + \delta_2(60°) = 23° + 20;41° = 43;41°$; and $NY = 90°$. Substituting sine values into (8) using Bianchini's sine table of radius 60,000, we have

$$\frac{55{,}000}{60{,}000} = \frac{56{,}133}{41{,}440} \cdot \frac{\sin \bigstar Y}{60{,}000}. \tag{9}$$

From this we find $\delta(\lambda, \beta) = \bigstar Y = 42;35°$.

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[22] Cracow BJ 556, f. 13r.
[23] The table is given to seconds, but in all but one manuscript the seconds place is 0. In Cracow BJ 556, the seconds place is 30 in five entries.
[24] Here and elsewhere, for tables and passages from the *Flores Almagesti* we shall refer to folio numbers in the manuscript Vatican, BAV, Vat. Lat. 2228. This passage is on f. 44v. Bianchini adheres to the unusual practice of referring to himself by name when he believes he has contributed something original.
[25] Vatican, BAV, Vat. Lat. 2228, ff. 48v-49r.
[26] On Bianchini's application of Menelaus's Theorem in general, see Zepeda 2013, 323-329.

Bianchini concludes Chapter 15 of the canons with an admission that he himself had made the error of confusing the two solar declinations when constructing his own tables, but his reading of the *Almagest* set him straight:

> *…quia multi processores et compositores tabularum in hoc manifeste et sensibiliter erraverunt. Et egomet in simili alias deceptus sum. Et antequam cognovissem preciosum Almagesti librum cum falsa doctrina ab aliis tam famosis accepta, certas composui tabulas ad clima nostrum (gradus 45) correspondentes que aliquantum a veritate discrepabant. Non debet homo verecundari si error suus ab aliis corrigatur; nec ego verecundor errorem meum propositum corrigere. Composui ergo tabulas ad hoc necessarias. Prout circa finem huius opusculi dicam quae verissime et pro veris operandae sunt, ut supra potentem demonstravi.*[27]
>
> …because many users and makers of tables had erred clearly and perceptibly in this matter. And even I was deceived in something similar at another time. And before I had understood the valuable book of the *Almagest*, when false doctrine was accepted by other quite famous [people], I composed certain tables corresponding to our climate (45º), and they diverged from the truth by a fair amount. No man must be ashamed if his own error is corrected by others; nor am I ashamed to correct my own erroneous proposition. Therefore I have composed tables necessary for this [purpose]. Accordingly, around the end of this little work let me state that these things are performed most truly and according to truths, as I have demonstrated its power above.

We are not aware whether these erroneous tables are still extant.

The table of the radicum ascensionum

Recall from our discussion of Ptolemy's solution to the problem of stellar coordinates that $XZ = \delta_2(\lambda)$ is one of two quantities needed after applying Menelaus's Theorem. The other, ΥZ, is close to the solar right ascension $\alpha_\odot$. But it is not quite $\alpha_\odot$, since *XZ* is perpendicular to the ecliptic rather than to the equator. One might refer to ΥZ as the "second right ascension" $\alpha_2(\lambda)$ in analogy to the second declination, although this name is not historically attested. Bianchini calls it the ***radicum ascensionum***.

Flores Almagesti V.3 shows how to calculate $\alpha_2(\lambda)$, giving an example for $\lambda = 60°$.[28] In the bold Menelaus configuration in Figure 3, apply (4):

$$\frac{\sin \Upsilon Y}{\sin VY} = \frac{\sin \Upsilon X}{\sin XU} \cdot \frac{\sin PU}{\sin PV}. \tag{10}$$

In our case $\Upsilon X = 60°$, $XU = 90° - \lambda = 30°$, $PU = 90°$, and $PV = 90° + \varepsilon = 113;33{,}30°$. Then

$$\frac{\sin \Upsilon Y}{\sin VY} = \frac{51{,}962}{30{,}000} \cdot \frac{60{,}000}{55{,}000} = \frac{3{,}117{,}720{,}000}{1{,}650{,}000{,}000} = \frac{25{,}981}{13{,}750}. \tag{11}$$

But $VY = 90° - \Upsilon Y = 90° - \alpha_2(60°)$. So to complete the solution, Bianchini must find the equivalent of an arc tangent. He goes through an extraordinarily long procedure to do this, eventually finding $\Upsilon Y = \alpha_2(60°) = 62;7°$. In modern terms,

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[27] Cracow, BJ 556, f. 13r.
[28] Vatican, BAV, Vat. Lat. 2228, ff. 46r-47r. See also Gerl 1988, 236-240.

$$\tan \alpha_2 = \frac{\tan \lambda}{\cos \varepsilon}. \tag{12}$$

Bianchini concludes the section with the following passage, in which he suggests that the obscurity of the ancient astronomers' exposition of these matters caused modern astronomers to commit the error of assuming that the arcs of the solar declination and of the latitude lie on the same great circle:

> *Et hoc ferretum credo quae antiqui sapientes in istis demonstrationibus occultaverunt, qua propter multi moderni in compositionibus tabularum et canonibus suis erraverunt. Quia, cum tabulis communibus de ascensionibus signorum in circulo directo operabantur; et docebant operari, quod falsum est, quia ponebant declinationes et latitudines esse super unum circulum.*[29]
>
> And I believe that the ancient sages obscured [this matter] in their demonstrations, and because of this many moderns erred in their compositions of tables and their canons. Wherefore, they were working with the common tables of the ascensions of signs [α rather than α_2]; and they were teaching [these things] to be done, which is false, because they put the declinations and latitudes on a single circle.

In Chapter 12 of the *Tabulae primi mobilis* Bianchini describes his demonstration in the *Flores Almagesti* as "most beautiful" (*pulcherrima*), noting that others have erred in this matter. He then describes how to use his table of the *radicum ascensionum.*[30] Due to the symmetries of the function, he needed to tabulate it only for arguments up to 90º, but he gives values for every degree up to 360º. Minor differences between entries suggest that he did not employ the symmetries when computing the table. In the first 90º of the table at intervals of 1º, there are 38 differences of ±0;1° and four of +0;2° from recomputation with (12). As with the recomputation of the table of $\delta_2(\lambda)$, the fit is almost perfect.

We may now see how Bianchini computed his values of the second declination $\delta_2(\lambda)$, which he describes in *Flores Almagesti* V.5.[31] Relying once again on Menelaus's Theorem, he demonstrates an equivalent of

$$\sin \delta_2 = \sin \alpha_2 \sin \varepsilon. \tag{13}$$

So he computes the second declination from the *radicum ascensionum*.

Tabulating stellar right ascensions and the Tabulae magistrales

The problem remained to tabulate $\alpha(\lambda, \beta)$. Since α is a function of two arguments, it requires a double argument table. However, before attempting such an arduous task, Bianchini invented a method that requires much less work. He constructed two single-argument tables (one with argument λ, the other with argument β), from which α may be found by combining the entries in the two tables in a simple way. This approach has a number of ancient precedents; the *Almagest* contains several tables of this type.

[29] Vatican, BAV, Vat. Lat. 2228, f. 47r.
[30] Cracow, BJ 556, ff. 12v-13r; the table itself is on f. 59v.
[31] Vatican, BAV, Vat. Lat. 2228, f. 48r-v.

The process is described in Chapter 16 of the canons to the *Tabulae primi mobilis*[32] and demonstrated mathematically in *Flores Almagesti* V.8.[33] We begin by using the table of the second declination to determine $\bigstar Y = \delta(\lambda, \beta)$, as we described earlier (see Figure 3). Then spherical trigonometry gives us

$$\sin(\alpha_2 - \alpha) = \tan\delta \cdot \cot \measuredangle XZY, \tag{14}$$

where $\alpha_2 = \Upsilon Z$ is the *radicum ascensionum* and $\alpha = \Upsilon Y$ is the desired right ascension. Next Bianchini turns to his *Tabula ad inveniendum gradus ascensionum cum quibus stella quelibet oritus atque mediat caelum in omni regionem*.[34] Here we find two tables. The first, column T_3 in Figure 6, is the first table of tangents in the Latin West; Bianchini instructs us to use it to look up $\tan\delta$. The second, column T_4, gives the value of $\cot \measuredangle XZY$, which may be found using

$$\cot \measuredangle XZY = \tan\varepsilon \cos\alpha_2. \tag{15}$$

Look up the appropriate entries in T_3 and T_4, multiply them, find the arc sine of the product, and subtract the result from the *radicum ascensionum* α_2. The result is the desired stellar right ascension:

$$\alpha(\lambda, \beta) = \alpha_2 - \sin^{-1}(\tan\delta \tan\varepsilon \cos\alpha_2). \tag{16}$$

The two tables brought together in the *Tabula ad inveniendum gradus ascensionum* are identical to the fourth and third tables respectively of the *Tabulae magistrales*; Bianchini says as much in his canons.[35] So, we have here both the astronomical motivation and the mathematical definitions of these two tables.

<Insert Figure 6 here>

Now that Bianchini has the apparatus he needs to compute stellar declinations $\delta(\lambda, \beta)$ and right ascensions $\alpha(\lambda, \beta)$, he goes on to construct large double-argument tables of both functions. Presumably these tables were generated from the tables described above. We expect to subject them to further study in the near future.

Regiomontanus

From the beginning of Bianchini's correspondence with Regiomontanus, he poses problems relating to his interests. In the last of his extant letters dated Feb 5, 1464, one of the problems he sets for Regiomontanus is the trisection of a 60º arc, which he himself had used while describing how to construct a sine table in *Flores Almagesti* Book III.[36] Elsewhere in the same letter, he refers explicitly to his method for determining stellar declinations. He does not go into detail, but reiterates his claim that he is the first

[32] Cracow, BJ 556, f. 13v.
[33] Vatican, BAV, Vat. Lat. 2228, f. 49r-v.
[34] This table is found under this title only in the manuscript Cracow BJ 556, ff. 55r-56v. However, the two tables from the *Tabulae magistrales* from which it is assembled are present in every manuscript we have consulted that includes tables.
[35] See Chabás 2016 for an analysis of the *Tabulae magistrales*.
[36] Curtze 1902, 238. Regiomontanus's solution, which differs from the one used by Bianchini, is on pp. 258-259. In *Flores Almagesti* MS Cracow BJ 558 ff. 22v, 23r in the section where Bianchini describes his trisection and its use in constructing a sine table, there are marginal notes by Regiomontanus indicating that he believes that Bianchini's method is better than Ptolemy's, and that the trisection is due to the Banū Mūsā.

to solve and tabulate the problem correctly.[37] In Regiomontanus's response (in which he may have terminated the correspondence inadvertently by displaying his mathematical superiority to Bianchini), he says that he looks forward to seeing the *Flores Almagesti*. So, throughout the correspondence Regiomontanus may have been unaware both of Bianchini's tables and of the mathematics behind them.

At the time, Regiomontanus was completing his own *Tabulae primi mobilis*. However, our interest is in his 1467 *Tabulae directionum*, a set of tables composed within a couple of years of the Bianchini correspondence. It was extremely popular, printed in various forms into the early 17th century. In it we find a striking number of similarities to Bianchini's *Tabulae primi mobilis*, so many that one must conclude that Bianchini was at the very least the inspiration for the *Tabulae directionum*.[38] Indeed, the 16th century mathematician Gerolamo Cardano went so far as to accuse Regiomontanus of plagiarism.[39]

We shall describe which of Bianchini's tables may be found in some form in the *Tabulae directionum*. Regiomontanus did not lift any tables directly from Bianchini; this is easily seen by the fact that Bianchini used the parameter $\varepsilon = 23;33{,}30°$ from the *Toledan Tables*, while Regiomontanus used his more modern $\varepsilon = 23;30°$. Nevertheless, the tabular structures for stellar coordinate conversion used by the two texts are essentially identical. Below, we refer to folio numbers from the 1504 edition of the *Tabulae directionum*.

Stellar declinations: Bianchini's table of the second declination $\delta_2(\lambda)$ corresponds to the first function in Regiomontanus's *Tabula declinationum generalis* (f. 7r-7v, Figure 7).[40] Regiomontanus's second function is his only elaboration of Bianchini's tabular structure. From (7) we have

$$\sin\delta(\lambda,\beta) = \sin\big(\delta_2(\lambda) + \beta(\lambda)\big)\cdot\frac{\cos\varepsilon}{\cos\delta_2(\lambda)}. \tag{17}$$

Bianchini had instructed the user how to compute δ after looking up δ_2 in his table. Regiomontanus goes a step further, tabulating the auxiliary function (the "*numerus multiplicandus*")

$$\frac{\cos\varepsilon}{\cos\delta_2(\lambda)}. \tag{18}$$

This reduces the user's effort substantially.

<Insert Figure 7 here>

Stellar right ascensions: Recall that to find $\alpha(\lambda,\beta)$, Bianchini had employed his table of the *radicum ascensionum* $\alpha_2(\lambda)$ and two tables from his *Tabulae magistrales*: one for the function $\tan\varepsilon\cos\alpha_2(\lambda)$, and the other for $\tan\delta$. In Regiomontanus's work, we find the first two of these tables in his *Tabula generalis celi mediationum* (ff. 20v-22r). He calls the table of $\alpha_2(\lambda)$ the *radix ascensionum*

[37] See a Latin edition in Curtze 1902, 239-241, an account of the correspondence in Byrne 2007, 176-198, and another in Zinner 1990, 60-69. Swerdlow 1990 is an analysis especially of Regiomontanus's final letter to Bianchini.
[38] This was noticed first in Rosińska 1981b, especially 571-573 and 577; see also Rosińska 1997-98, 140, 142. See Swerdlow (forthcoming) for a detailed discussion of the astrological aspects of the *Tabulae directionum.*
[39] Swerdlow 1990, 168, 193.
[40] This table is also found along with Regiomontanus's tangent table in Paris BNF 10265, f. 222v, a manuscript otherwise filled with Bianchini's works. (A marginal note stating Regiomontanus's name is found about 20 folios earlier in the same manuscript.)

(after Bianchini), and the table for $\tan\varepsilon\cos\alpha_2(\lambda)$ the *numerus multiplicandus*. Regiomontanus's tangent table, which he calls the *tabula fecunda*, appears on f. 8r.

Double-argument tables of stellar coordinates: Both works contain double-argument tables for $\delta(\lambda,\beta)$ (ff. 1v-7r in Regiomontanus) and $\alpha(\lambda,\beta)$ (ff. 8v-20r in Regiomontanus), taking up many pages. In both collections the latitude β varies in 1º increments from 0º to 8º both northward and southward, although in Bianchini's table of $\delta(\lambda,\beta)$, β also takes on some other values beyond 8º.

Thus, the first 40 pages of the *Tabulae directionum* are a unified set of tables employing the same structure as Bianchini's *Tabulae primi mobilis*, with one minor enhancement. Although they were recomputed for a new value of ε, they follow Bianchini's methods almost to the letter.

Concluding remarks

Bianchini's approach to stellar coordinate conversions was innovative; it provided a complete and novel solution to a problem that had been studied inadequately by his predecessors. His tabular structures include a rediscovery of the second declination, a function that had been part of the astronomical corpus in eastern Islam. But his collection of tables went well beyond the second declination, providing a universal treatment of stellar coordinates. Among these tables was one that would evolve eventually into our modern tangent function.

Clearly Bianchini was extremely proud of his accomplishment. Nevertheless he frequently gives credit and effusive praise to Ptolemy, both in the *Flores Almagesti* and the *Tabulae primi mobilis*. This is no mere gesture. Although he is fully aware of the improvements he has made over the *Almagest* and is not afraid to draw attention to them, his methods follow in Ptolemy's footsteps. Here and elsewhere, the *Flores Almagesti* reveals itself to be both an encomium of its namesake and an extension of Ptolemy's tradition into new territory – a path that, in the topic of stellar coordinates, Regiomontanus quickly followed.

Both Bianchini's *Tabulae primi mobilis* and Regiomontanus's *Tabulae directionum* contain much more than we have described here, particularly for mathematical astrology. We are translating and analyzing the remaining tables, and hope that a fuller understanding of the two works and the connections between them will result from this study.

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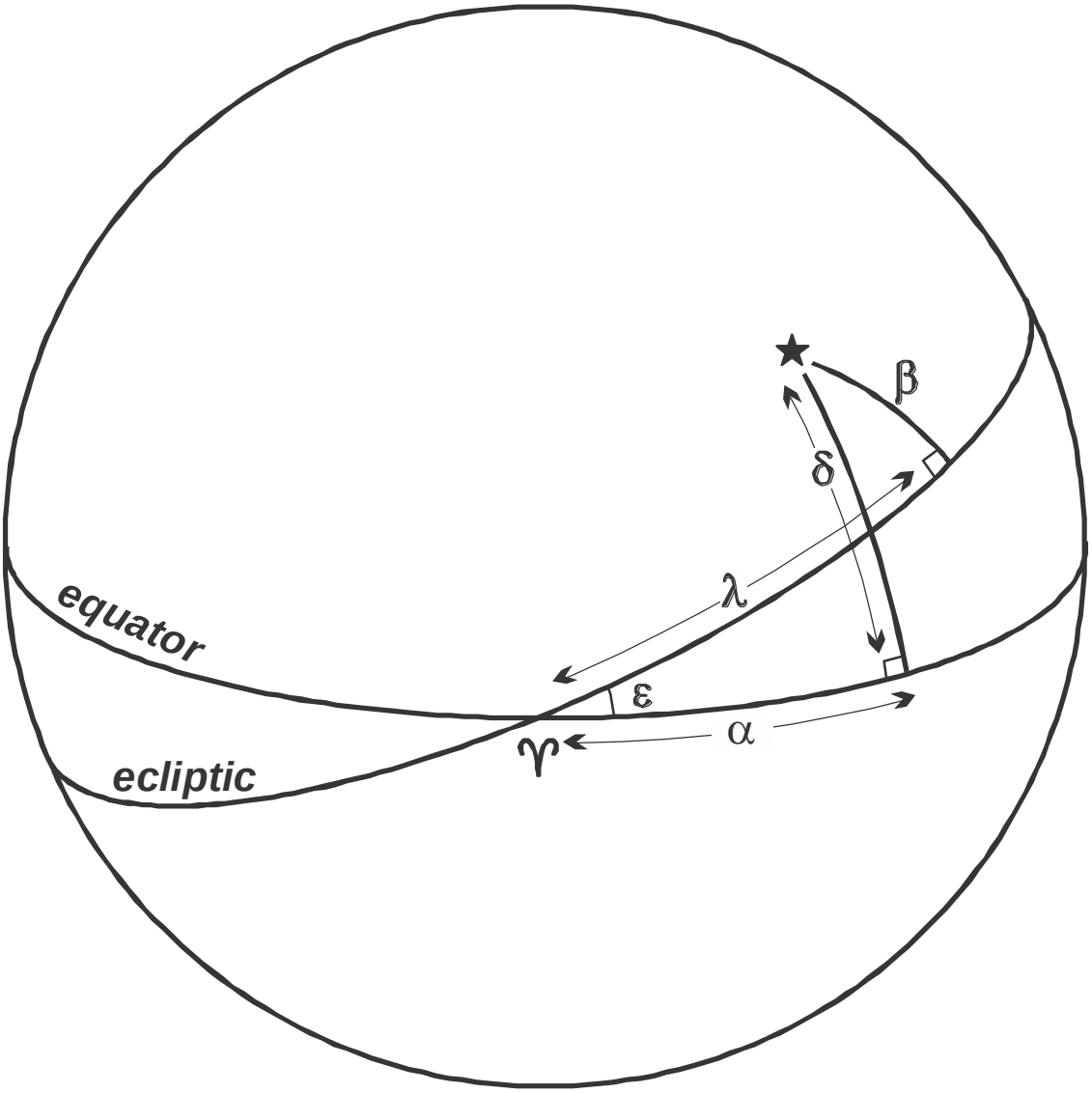


Figure 1: Coordinates on the celestial sphere

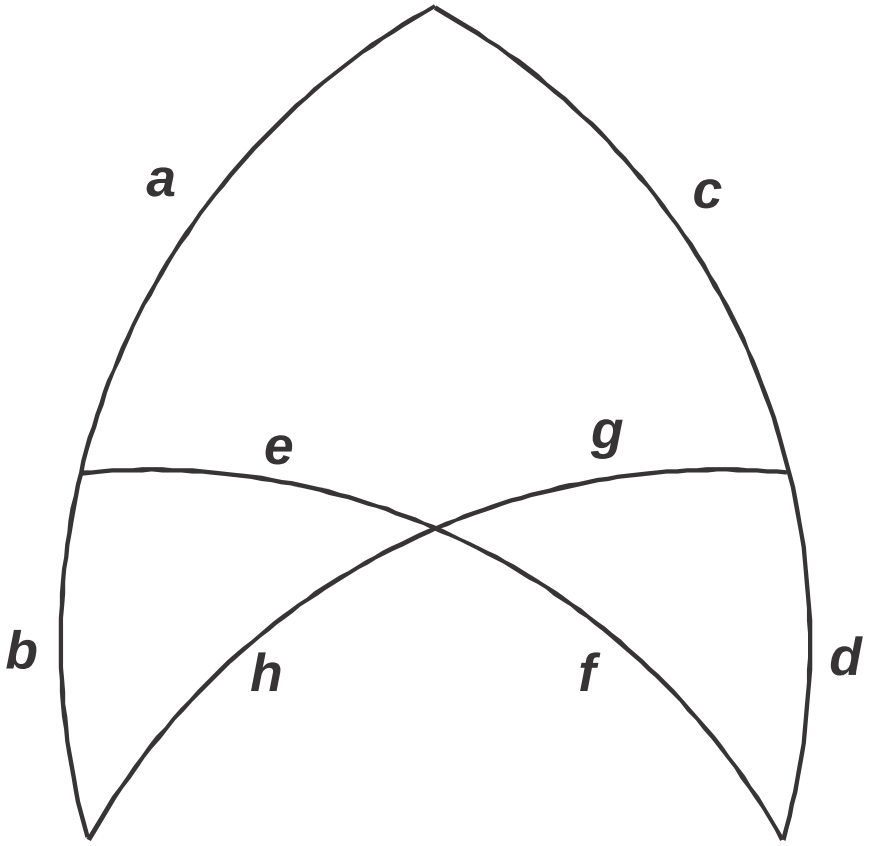


Figure 2: Menelaus's Theorem

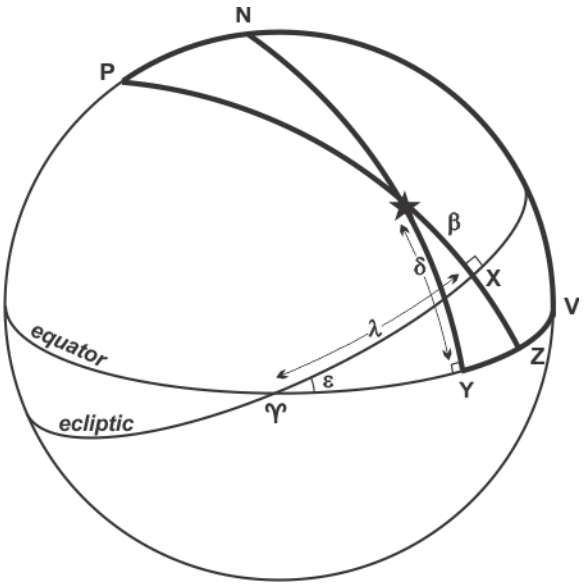


Figure 3: Ptolemy's solution to the problem of stellar coordinates

	Second Declination	*Radicum ascensionum*	*Tabula ad inveniendum gradus ascensionum*	Double-argument stellar declinations	Double-argument stellar right ascensions
Cracow BJ 556	59r	59v	55r-56v; 65v	60r-65r	66r-92v
Munich Clm 14111	–	–	–	111v-115r[41]	109v-111r
BNF 7270	236r	183r	185v-186r; 228r-228v	187r-v	191v-218r
BNF 7271	242v	181r	183v-184r; 185r-185v	244r-245r	190r-216v
BNF 7286	98v	99r	151r-152v	99v-104v	105r-131v
BNF 10265	92v	221v	226v-229r	88v-90r	153v-180r

Figure 4: The locations of the tables in the manuscripts. (Vat. Lat. 2228 and BNF 10267 contain no tables, only canons. Munich Clm 14111 contains only short fragments.) For the *Tabula ad inveniendum gradus ascensionum*, in all manuscripts other than Cracow BJ 556 the folio numbers refer to the tables in the *Tabulae magistrales.*

[41] A table with the same structure, but different entries.

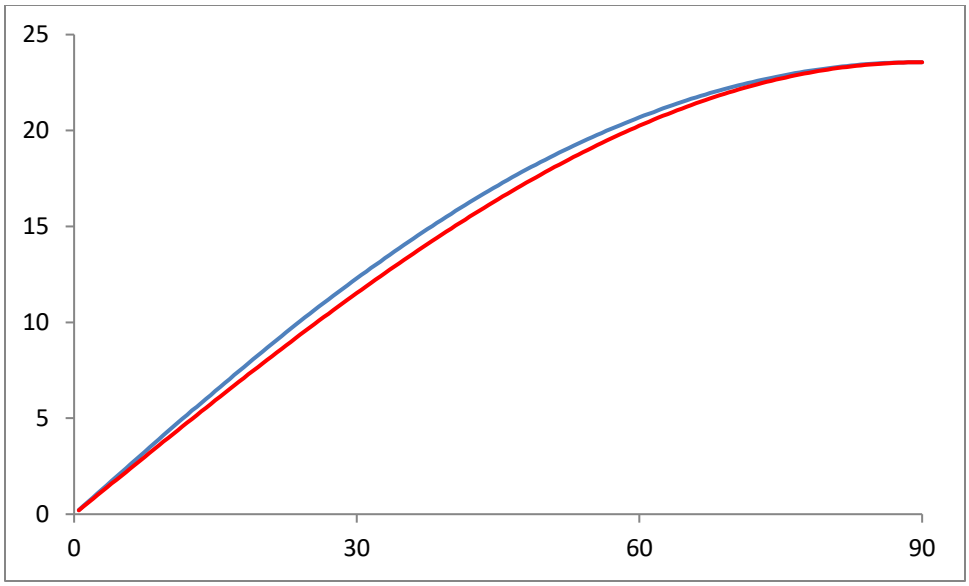
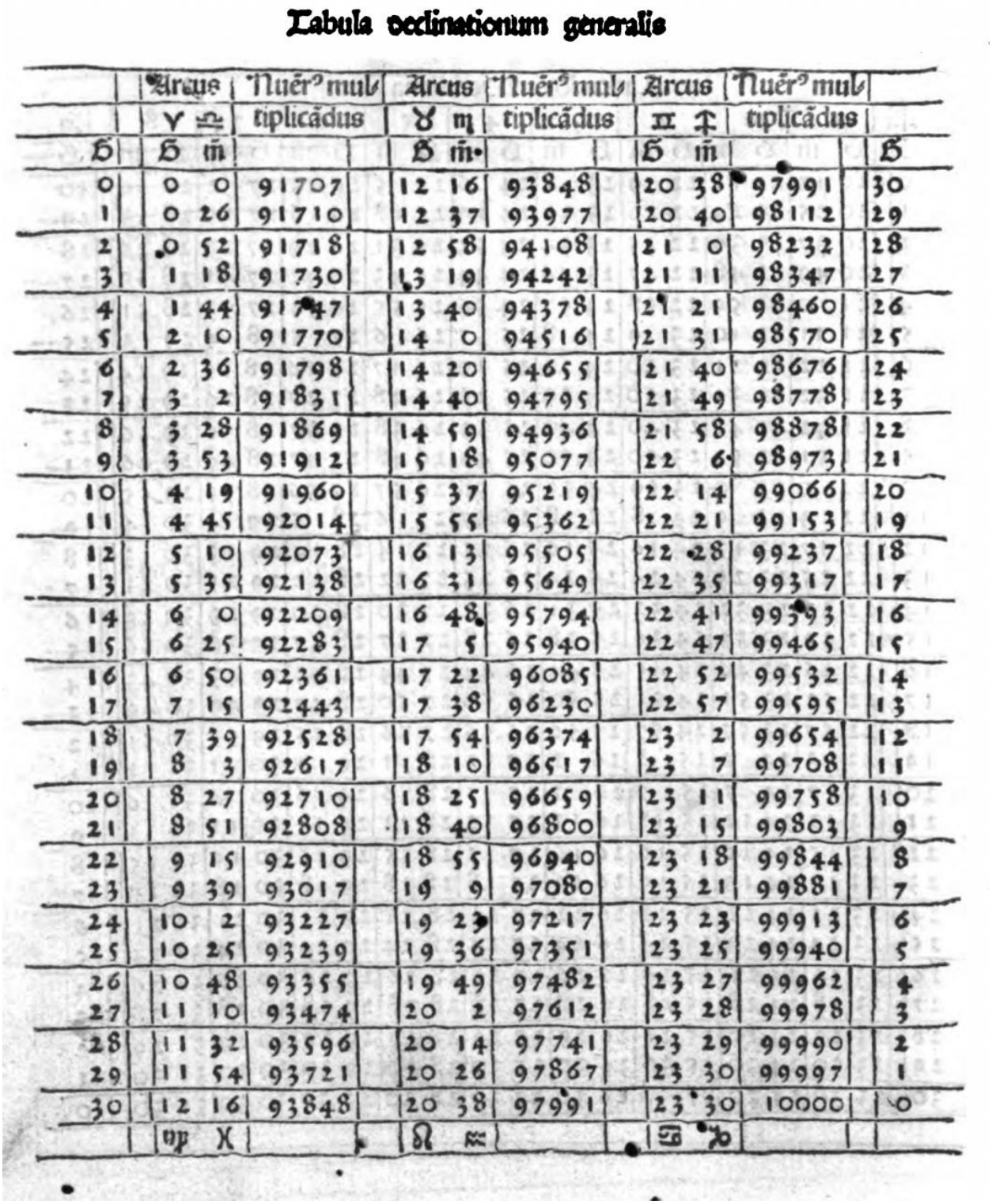


Figure 5: Bianchini's declination table. The values in his table are the higher curve; the lower curve is the correct declination values for $\varepsilon = 23;33{,}30°$.

λ	$T_4(\lambda)$	$T_3(\lambda)$	λ	$T_4(\lambda)$	$T_3(\lambda)$
0;0	26160	0	⋮	⋮	⋮
0;10	26158	29	⋮	⋮	⋮
0;20	26156	58	50	15939	11918
0;30	26154	87	⋮	⋮	⋮
0;40	26152	116	55	14123	14282
0;50	26149	145	⋮	⋮	⋮
1	26146	175	60	12234	17321
⋮	⋮	⋮	⋮	⋮	⋮
2	26132	347	65	10280	21445
⋮	⋮	⋮	⋮	⋮	⋮
3	26117	524	70	8278	27474
⋮	⋮	⋮	⋮	⋮	⋮
4	26075	699	75	6227	37321
⋮	⋮	⋮	⋮	⋮	⋮
5	26032	875	80	4161	56712
⋮	⋮	⋮	⋮	⋮	⋮
10	25680	1763	85	2079	114309
⋮	⋮	⋮	⋮	⋮	⋮
15	25188	2679	86	1663	143023
⋮	⋮	⋮	⋮	⋮	⋮
20	24305	3640	87	1247	190822
⋮	⋮	⋮	⋮	⋮	⋮
25	23308	4663	88	831	286342
⋮	⋮	⋮	⋮	⋮	⋮
30	22136	5773	89	415	572980
⋮	⋮	⋮	89;10	346	687216
35	20784	7002	89;20	277	859542
⋮	⋮	⋮	89;30	208	1145000
40	19292	8391	89;40	139	1719170
⋮	⋮	⋮	89;50	70	3420851
45	17663	10000	90	0	

Figure 6: Excerpts from the *tabula ad inveniendum gradus ascensionum*. The two sub-tables are equal to T_3 and T_4 of the *Tabulae magistrales*. Note that both are given in decimal notation. (Difference columns for use in interpolation are omitted here.)



Tabula declinationum generalis

	Arcus ♈ ♎	Nuēr⁹ multiplicādus	Arcus ♉ ♍	Nuēr⁹ multiplicādus	Arcus ♊ ♐	Nuēr⁹ multiplicādus	
♄	♄ m		♄ m		♄ m		♄
0	0 0	91707	12 16	93848	20 38	97991	30
1	0 26	91710	12 37	93977	20 40	98112	29
2	0 52	91718	12 58	94108	21 0	98232	28
3	1 18	91730	13 19	94242	21 11	98347	27
4	1 44	91747	13 40	94378	21 21	98460	26
5	2 10	91770	14 0	94516	21 31	98570	25
6	2 36	91798	14 20	94655	21 40	98676	24
7	3 2	91831	14 40	94795	21 49	98778	23
8	3 28	91869	14 59	94936	21 58	98878	22
9	3 53	91912	15 18	95077	22 6	98973	21
10	4 19	91960	15 37	95219	22 14	99066	20
11	4 45	92014	15 55	95362	22 21	99153	19
12	5 10	92073	16 13	95505	22 28	99237	18
13	5 35	92138	16 31	95649	22 35	99317	17
14	6 0	92209	16 48	95794	22 41	99393	16
15	6 25	92283	17 5	95940	22 47	99465	15
16	6 50	92361	17 22	96085	22 52	99532	14
17	7 15	92443	17 38	96230	22 57	99595	13
18	7 39	92528	17 54	96374	23 2	99654	12
19	8 3	92617	18 10	96517	23 7	99708	11
20	8 27	92710	18 25	96659	23 11	99758	10
21	8 51	92808	18 40	96800	23 15	99803	9
22	9 15	92910	18 55	96940	23 18	99844	8
23	9 39	93017	19 9	97080	23 21	99881	7
24	10 2	93227	19 23	97217	23 23	99913	6
25	10 25	93239	19 36	97351	23 25	99940	5
26	10 48	93355	19 49	97482	23 27	99962	4
27	11 10	93474	20 2	97612	23 28	99978	3
28	11 32	93596	20 14	97741	23 29	99990	2
29	11 54	93721	20 26	97867	23 30	99997	1
30	12 16	93848	20 38	97991	23 30	10000	0
	♈ ♎		♉ ♍		♊ ♐		

Figure 7: Regiomontanus's *Tabula declinationum generalis*. The first column is the argument; the second is the second declination; and the third is a new auxiliary function facilitating the calculation of the stellar declination.