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The Medieval Moon in a Matrix: Double Argument Tables for Lunar Motion*

José Chabás (Universitat Pompeu Fabra, Barcelona) and
Bernard R. Goldstein (University of Pittsburgh)

Abstract

Astronomers have always considered the motion of the Moon as highly complicated, and this motion is decisive in determining the circumstances of such critical celestial phenomena as eclipses. Table-makers devoted much ingenuity in trying to find ways to present it in tabular form. In the late Middle Ages, double argument tables provided a smart and compact solution to address this problem satisfactorily, and many tables of this kind were compiled by both Christian and Jewish astronomers. This paper presents multiple examples of the diversity of approaches adopted by compilers of tables who used this powerful tool, and brings to light intellectual interactions among them that are otherwise hidden from view.

Keywords

Ibn al-Kammād, Jacob ben Makhir, Parisian Alfonsine Tables, John Vimond, Bonfils

Beginning at the end of the thirteenth century, double argument tables—both in Latin and Hebrew—became a powerful tool for presenting astronomical information to determine the positions of the Moon and the planets. They were put in the format of what is now called a matrix with its columns and rows, very convenient for dealing simultaneously with two quantities, or variables as they are presently known. In this respect, it seems that European astronomers followed the lead of astronomers in the Muslim world, and already in the late tenth century Ibn Yūnus (Egypt, d. 1009) had compiled an extensive double argument table for the lunar equation (King 1974; Saliba 1976; North 1977, p. 279; King, Samsó, and Goldstein 2001, pp. 79, 85). This paper deals with the application of double argument tables to lunar motion, for which we have identified seven categories: (i) tables for the lunar equation, or the equivalent; (ii) tables for the true lunar position; (iii) tables for lunar velocity at any time; (iv) tables for the time from mean to true syzygy; (v) tables for the distance in longitude from mean to true syzygy; (vi) tables to determine the position of the Moon between syzygies; and (vii) table for solar eclipses. Since double argument tables in (iv) have previously been addressed systematically by the authors of the present paper (Chabás and Goldstein 1997), here we focus on the other categories. Double argument tables were not only used to describe the lunar motions; they were also applied to the planets, a subject that will be addressed in a subsequent paper.

I. Double argument tables for the lunar equation

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In Book V of the *Almagest* Ptolemy described his second lunar model which is rather complicated (see Figure 1). He gave an example for finding the lunar equation directly from the geometric model, which requires solving several triangles by means of trigonometry (*Almagest* V.6). But, then, in *Almagest* V.8, Ptolemy displayed a table for the lunar equation in 6 columns, plus a column for the lunar latitude. The total number of entries in the columns for determining the lunar equation is 270 (= 45 · 6) and, for computing the lunar equation, only addition, subtraction, multiplication, and interpolation are required. If c_i refers to an entry in the i -th column, then the general equation for the second lunar model is:

$$c = c_4(\alpha) + c_5(\alpha) \cdot c_6(2\eta\text{-bar}) \quad [1]$$

where

$$\alpha = \alpha\text{-bar} + c_3(2\eta\text{-bar}). \quad [2]$$

The lunar equation is c , and c_3 , c_4 , c_5 , and c_6 represent the equation of center, the equation of anomaly, the increment, and the minutes of proportion (for purposes of interpolation), respectively.¹

The argument is presented in 2 columns, c_1 , and c_2 . In Ptolemy's table the argument for some columns is α (true anomaly, in contrast to $\alpha\text{-bar}$, mean anomaly) and for others it is $2\eta\text{-bar}$ (mean double elongation, or just double elongation).² This tabular presentation of Ptolemy's second lunar model was generally accepted in the Middle Ages without alteration; even in the case of the Alfonsine Tables where the parameter for the lunar eccentricity was changed, the presentation remained essentially the same although the order of the columns is different. In other words, Ptolemy achieved great economy in his table for the lunar equations. An important consideration in limiting the number of entries was to ease the burden on the copyist.

<<Insert Figure 1 and its caption about here.>>

<<Caption>> Figure 1: Ptolemy's second lunar model. O is the observer, C is the center of the epicycle, S-bar is the direction to the mean Sun, V is the direction to Aries 0°, D is the center of the deferent whose radius is DC, L is the true position of the Moon, A_e on line DC extended is the true apogee of the epicycle, $\eta\text{-bar}$ is the mean elongation, that is the distance in longitude of the mean Moon from the mean Sun, $\alpha\text{-bar}$ is the mean argument of anomaly, and λ is the true longitude of the Moon. OD = OD' = 10;19,

¹ The order of the columns in *Almagest* V.8 was not followed in the tables for the lunar equation compiled by subsequent astronomers. Moreover, al-Battānī added a column for the solar equation. For the correspondence of the various columns in this table in the *Almagest*, the *zij* of al-Battānī, the Toledan Tables, and the Parisian Alfonsine Tables, see Chabás and Goldstein 2012b, p. 71.

² Elongation is the difference between the mean position of the Moon and that of the Sun, and double elongation is twice this difference. Strictly speaking, this angular distance should be called "mean elongation" in contrast to "true elongation", understood as the difference between the true positions of the Moon and the Sun. Unless otherwise specified, in this paper "elongation" refers to "mean elongation". In section I.4, below, we refer to an unusual expression found in a medieval text, namely, "true" elongation, which does not fit either of the two definitions above, for it represents the difference between the mean Sun and the true Moon.

CL = 5;15, and DC = 49;41, such that OD + DC = 60. Angle q_1 is a function of the mean double elongation ($c_3(2\eta\text{-bar})$ in eq. [2]), whereas angle q_2 is a function of both the mean double elongation and the lunar anomaly (c in eq. [1]).

In the late Middle Ages some astronomers focused on two ways to modify the tabular presentation of Ptolemy's lunar model, without changing the underlying model. One way was to eliminate subtractions by displacing the entries so that they are all positive, thus avoiding the cumbersome rules which were necessary before the introduction of negative numbers.³ The other way was to present a double argument table, or a matrix with its rows and columns, which simplifies interpolation. This second way involved more work for the compiler of the table (and the copyist) and less work for the user: in this sense these tables are "user-friendly". In this section we address double argument tables for the lunar equation which include some examples of displacement.

We denote an entry in a double argument table $c(\alpha, 2\eta\text{-bar})$, that is, the two arguments are, in principle, α and $2\eta\text{-bar}$, and this is what we find in eq. [1], but other combinations are possible. For the first argument, the choice is quite limited. Some table-makers, e.g., John of Lignères, preferred $\alpha\text{-bar}$, mean anomaly, rather than α , true anomaly. This approach reduces the work required of the user of the table, for there is no need to compute α from $\alpha\text{-bar}$ by means of eq. [2]. For the second argument, there are two choices for replacing the mean double elongation: the minutes of proportion, $c_6(2\eta\text{-bar})$, and mean elongation, $\eta\text{-bar}$. In some cases, such as the tables of Judah ben Verga, mean elongation is replaced by an integer number of days since syzygy. As the mean motion in elongation is about $12^\circ/\text{d}$, using successive days as the argument is more or less equivalent to using values of mean elongation separated by intervals of 10° or 15° . Regardless of the way the second argument is represented, when $2\eta\text{-bar} = 0^\circ$, it follows that $c_3(2\eta\text{-bar}) = 0^\circ$ and $c_6(2\eta\text{-bar}) = 0$, and eq. [1] reduces to $c = c_4(\alpha)$. Thus, the first (or last) column in the double argument table for the lunar equation reproduces the equation of anomaly. By locating its maximum value, it is then easy to verify the tradition to which a particular table belongs: al-Battānī (5;1°) or Alfonsine astronomy (4;56°).

We have identified 10 authors who used double argument tables for the lunar equation: five in the tradition of al-Battānī (and the Toledan Tables) and five in the Alfonsine tradition. Among those in the first group are Jacob ben Makhir; the compiler of a table in Paris, uniquely preserved in Bibliothèque nationale de France [BnF], MS lat. 7411; Joseph Ibn Waqār; the anonymous author of the set in Vatican, Biblioteca Apostolica Vaticana [BAV], MS Heb. 384; and Judah ben Verga (see section I.7, below). Those in the Alfonsine framework are John of Lignères, William Batecombe (Oxford Tables), John Mülhus, Giovanni Bianchini, and Farissol Botarel. The number of astronomers writing in Hebrew and those in Latin is also well balanced.

1. The first case in the West of a double argument table for the complete lunar equation is found in the *Almanac* of Jacob ben Makhir (c. 1236–c. 1305), also known by his Provençal name, Profeit Tibbon, later rendered in Latin as Profatius or Profatius Judeus (Steinschneider 1964, pp. 111–113). The *Almanac* was composed in Hebrew c. 1300 and translated into Latin shortly thereafter (Chabás and Goldstein 2019). The two arguments are the minutes of proportion and the true anomaly. The use of true anomaly

³ See Kennedy 1977 and Samsó 2003 on displaced tables in Islamic astronomy and Chabás and Goldstein 2013 for those in Latin, and the references therein.

requires the compilation of an additional table to transform mean anomaly into true anomaly according to eq. [2]. This is indeed the case in the *Almanac*, where a large and unprecedented table for this purpose is found (see Figure 2). The use of the minutes of proportion as a second argument takes advantage of one option. In this case, the argument for the minutes of proportion is displayed at intervals of 5 min from 0 min to 60 min. We note that the same two arguments were later used by Joseph Ibn Waqār of Seville (ca. 1357; Castells 1996) and Moses Farissol Botarel (Avignon, late fifteenth century), although with different intervals. The number of entries in Jacob's table is 4680, given in zodiacal signs, degrees, and minutes. An outstanding feature of Jacob's double argument table is the vertical displacement of $7;40^\circ$ that affects all entries to facilitate further computation, as mentioned above. In this case the extreme values of the entries in the column for 0 minutes of proportion are $2;39^\circ$ (at arguments 92° – 98°) and $12;41^\circ$ (at arguments 261° – 269°), such that their sum, $15;20^\circ$, is twice the amplitude of the displacement. On the other hand, $7;40^\circ - 2;39^\circ = 5;1^\circ$ and $12;41^\circ - 7;40^\circ = 5;1^\circ$ show these values of the equation of anomaly are in the tradition of al-Battānī and the Toledan Tables. For a detailed recomputation of the entries, see Chabás and Goldstein 2019, section 10).

<<Insert Figure 2 and its caption about here>>

<<Caption>> Figure 2: Excerpt of Jacob ben Makhir's double argument table for the lunar equation in his *Almanac* (Madrid, Biblioteca Nacional, MS 9288, 75r)

2. Another example of a double argument table for the lunar equation in the same tradition is uniquely preserved in a late thirteenth-century manuscript: Paris, BnF, MS lat. 7411. It contains canons to the Toledan Tables, the Tables of Novara by Campanus, and a large set of tables associated with the Toledan Tables (for descriptions of this manuscript, see Boudet 1994, pp. 93–98; and Pedersen 2002, p. 160). The double argument table for the Moon occupies folios 17v–23r, where the argument for the columns is double elongation, here called *longitudo duplex*, displayed at intervals of 30° from 0s 0° to 11s 0° , and the argument for the rows is true anomaly, here called *portio lune*, at intervals of 1° from 0s 1° to 12s 0° . Note that the argument for the columns differs from that in the *Almanac* of Jacob ben Makhir. The entries, of which there are 4320, are also given in degrees and minutes and show no vertical displacement. In the column for $2\eta\text{-bar} = 0^\circ$, the maximum entry is $5;1^\circ$ and it occurs for values of true anomaly of 91° – 97° and 262° – 270° , a clear sign of dependence on the Toledan Tables.

It is noteworthy that in both tables described above, true anomaly is given to degrees and therefore both have 360 rows, whereas there are only 13 columns in the *Almanac* of Jacob ben Makhir and 12 columns in Paris, BnF, MS lat. 7411. The intervals displayed differ considerably for the two arguments (1° vs. 5 minutes of proportion or 30° in double elongation), thus making linear interpolation much more problematic, hence less reliable, in one of the arguments. It was just a matter of time—in fact, very little time—before someone compiled a more homogeneous table, with similar intervals for both arguments.

3. In his *Tabule magne*, dated 1325, John of Lignères, one of the contributors to the standard Parisian Alfonsine Tables, made extensive use of double argument tables, for he applied them to the equations of the Moon and the planets, as well as for finding the true position of the Moon between syzygies (see section IV, below). In his table for the lunar equation, the argument for the rows, at the left of the table, is mean lunar anomaly,

α -bar, a possibility mentioned above, given at intervals of 6° , from $0s\ 6^\circ$ to $6s\ 0^\circ$. As far as we know, this is the first double argument table of this type using this argument. At the head of the table, the argument is double elongation, 2η -bar, also at intervals of 6° , from $0s\ 6^\circ$ to $12s\ 0^\circ$. Note the use of signs of 30° (in contrast to the standard Parisian Alfonsine Tables which use signs of 60°). The entries are given in degrees and minutes, and their maximum value when 2η -bar = 0° is $4;56^\circ$ for α -bar = $3s\ 6^\circ / 8s\ 24^\circ$. The maximum value for the equation of anomaly, $4;56^\circ$, is a characteristic feature of Alfonsine astronomy. The number of entries, 1800, is much lower than in the two preceding cases but, in some manuscripts such as London, British Library, MS add. 24070, 18r–20v, for each entry we are also given the differences, both horizontal and vertical, between successive entries (see Figure 3). This increases considerably the amount of data and facilitates the task of interpolation. As this is the earliest known double argument table for the general lunar equation using Alfonsine parameters, it is plausible that John of Lignères took the layout from previous tables based on parameters in the tradition of al-Battānī and the Toledan Tables (e.g., the tables in Paris, BnF, MS lat. 7411, discussed above), harmonized the intervals of the two arguments, and adapted the parameters to Alfonsine astronomy.

<<Insert Figure 3 and its caption about here.>>

<<Caption>> Figure 3: John of Lignères’s table in the *Tabule magne*; excerpt (London, British Library, MS 24070, 18r)

4. The Oxford Tables of 1348, attributed to William Batecombe and called *Tabule anglicane* at the time, provide an example of another double argument table applied to the Moon. The Oxford Tables also include double argument tables for the longitudes and the latitudes of the planets and show a close relationship with John of Lignères’s *Tabule magne*. They too were computed with Alfonsine parameters (North 1977; Chabás and Goldstein 2016). In the case of the Moon, the two arguments involved are the mean lunar anomaly for the columns at intervals of 6° from $0s\ 6^\circ$ to $12s\ 0^\circ$, and the mean elongation for the rows at intervals of 3° from $0s\ 3^\circ$ to $12s\ 0^\circ$. Note that anomaly is the argument for the columns, in contrast to all the other double argument tables examined here. However, the most relevant difference in this table is that the entries do not represent the lunar equation (as in all other cases), but “true” elongation, the angular distance between the mean Sun and the true Moon, as indicated in its title: *Tabula (...) continens veram elongationem lune a medio motu solis (...)*. Indeed, when mean elongation is 0° (= $12s\ 0^\circ$), “true” elongation reduces to the equation of anomaly, whose maximum value is $4;56^\circ$, as expected. Again, the purpose of this new Alfonsine table is to facilitate computation, for the total number of entries, 3600, is double that in John of Lignères’s table.

It is worth emphasizing that the entries in the lunar table in the Oxford Tables are “true” elongations: this is most unusual, although not unprecedented. Cambridge, Gonville and Caius College, MS 110/179, 90v–99r, has a similar double argument table, with the same arguments, mean elongation and mean lunar anomaly, but their positions in the table are switched with respect the Oxford Tables. The relevant difference, however, is that the maximum value for the equation of anomaly of the Moon is $5;1^\circ$, which is the usual parameter found in the Toledan Tables. It would thus seem that the author of the Oxford Tables, presumably William Batecombe, proceeded in the same way as John of Lignères: he borrowed a previously existing layout and adapted its contents to the new astronomy, Alfonsine in both cases. Astronomical tables always

convey scientific knowledge and sometimes they also inform us about the practices of table-makers.

As we recently pointed out, the Oxford Tables were adapted to other latitudes, such as Paris and Louvain, and they were also translated into Hebrew by Mordecai Finzi (*fl.* 1440–1475) who adapted them to Mantua (Chabás and Goldstein 2018), thus enhancing the dissemination of double argument tables within the astronomical community.

5. Joseph Ibn Waqār’s double argument table for the lunar equation is found in Munich, Bayerische Staatsbibliothek, MS Heb. 230, 37a–38b (see Table 1). The heading is in Arabic written in Hebrew characters. As mentioned above, he used the same two arguments as those in Jacob ben Makhir’s *Almanac*, minutes of proportion and true anomaly, with two important changes: the entries have no vertical displacement and the interval for the minutes of proportion is 10 min, twice as much as in the *Almanac*. This implies that the number of entries in the table is half that of Jacob’s. This table is in the tradition of al-Battānī, for the maximum value of c_4 is $5;1^\circ$ for true anomalies $3s\ 3^\circ - 3s\ 6^\circ$ and $0'$ of proportion. Comparison with the entries in the *Almanac* shows agreement, that is, when adding the corresponding entries in both tables, the result is $7;40^\circ$.

Table 1: Excerpts of the subtables for 0s and 5s from Ibn Waqār’s double argument table for the lunar equation (Munich, Bayerische Staatsbibliothek, MS Heb. 230, 37a–38b)

0s: to be subtracted							
	0'	10'	20'	...	50'	60'	
1	0; 5°	0; 6°	0; 6°		0; 7°	0; 8°	29
2	0; 9	0;10	0;11		0;13	0;14	28
3	0;14	0;15	0;16		0;20	0;21	27
...							
29	2;15	2;26	2;38		3;11	3;23	1
30	2;20	2;31	2;43		3;18	3;30	0
11s: to be added							

5s: to be subtracted							
	0'	10'	20'	...	50'	60'	
1	2;37°	2;52°	3; 1°		3;53°	4; 9°	29
2	2;33	2;48	3; 3		3;47	4; 2	28
3	2;28	2;43	2;57		3;40	3;54	27
...							
29	0; 6	0; 7	0; 7		0;10	0;10	1
30	0; 0	0; 0	0; 0		0; 0	0; 0	0
6s: to be added							

6. An anonymous set of tables for radix 1400 (Vatican, BAV, MS Heb. 384, 263a–278b; Goldstein 2003) includes a double argument table for the lunar equation on f. 266b. The argument for the columns is mean elongation at intervals of 15° from $0s\ 0^\circ$ to $3s\ 0^\circ$ (with the same set of entries to be applied to symmetrical mean elongations from $12s\ 0^\circ$ to $9s\ 0^\circ$ above the table, and $6s\ 0^\circ$ to $3s\ 0^\circ$, and $6s\ 0^\circ$ to $9s\ 0^\circ$ below the table), and the argument for the rows is true anomaly at intervals of 10° from 0° to 350° , for a total of 252 entries. The maximum value of c_4 is $5;0^\circ$ at true anomaly 90° and 100° in the tradition of al-Battānī; and the maximum entry, $7;39^\circ$, occurs at mean elongation 90° and true anomaly 100° , which is consistent with Ptolemy's second lunar model.

7. Judah ben Verga, active in Lisbon from 1455 to 1475, composed a set of tables which survives in two copies: Paris, BnF, MS Heb. 1085, 86b–98a; and Oxford, Bodleian Library, MS Poc. 368, 22b–236a (Goldstein 2001). The double argument table for the lunar equation is in the tradition of al-Battānī, for the maximum value of c_4 is $5;1^\circ$ (0d of mean elongation and 7d of mean anomaly). The argument for the columns is the number of days since syzygy from 0d to 14d (with values for the corresponding motion in longitude, motion in elongation, equation of center, and minutes of proportion displayed as well), and the argument for the rows is the number of days since syzygy from 0d to 27d together with the corresponding mean anomaly. There is a total of 420 (= $15 \cdot 28$) entries for the lunar equation (Goldstein 2001, pp. 247, 269).

8. Moses Farissol Botarel (Avignon, late fifteenth century), a student of Moses ben Abraham of Nîmes (*fl.* 1460) who translated the Parisian Alfonsine Tables into Hebrew, composed astronomical tables, uniquely preserved in Munich, Bayerische Staatsbibliothek, MS Heb. 343, 92a–103b, that are partly in the tradition of Alfonsine astronomy and partly in the Hebrew tradition of Levi ben Gerson and others (Goldstein and Chabás 2017). Of interest here is his double argument table for the lunar equation (f. 102a–b), where the argument for the rows is the lunar anomaly for Ari 1° and Lib 1° and then at intervals of 5° from Ari 5° to Psc 30° , and the argument for the columns is the minutes of proportion (representing the double elongation) at intervals of 5 min from 0 min to 60 min, for a total of 962 (= $74 \cdot 13$) entries. The entries are all displaced by $7;40^\circ$, as was the case for Jacob ben Makhir. As noted above, Farissol Botarel used Alfonsine parameters (e.g., the maximum value of c_4 , the equation of anomaly, is $4;56^\circ$) but otherwise his table is very similar to that of Jacob ben Makhir (Goldstein and Chabás 2016–2017, pp. 57–60).

9. In all double argument tables considered so far, the extent of the intervals for both arguments were adjusted to facilitate interpolation. In Vatican, BAV, MS Pal. lat. 1374, 87r–102r, we have found another table of the same kind that goes far beyond all previous ones, for it displays 10,800 (= $180 \cdot 60$) entries for the complete lunar equation. The manuscript includes references to Prague and 1407, and contains an incomplete copy of the Parisian Alfonsine Tables, as well as tabular materials by John of Lignères and John of Genoa. For a description of this manuscript, see Schuba 1992, 86–88. Although computed with Alfonsine parameters, the double argument table seems isolated from the items mentioned above. As indicated in a short text at the end of the table, it was compiled to determine the true position of the Moon by an otherwise unknown Nicolaus Mülhus, a wool merchant and layman in Zittau (*lanificem et laicum morantem in Zittavia*). The minutes of proportion are used as argument for the columns, from 0 to 60, at intervals of 1 min, and the true anomaly is the argument for the rows, from 0° to 180° , at intervals of 1° . Each column is presented as a separate subtable, with

the argument divided in three parts (1° – 60° , 61° – 120° , and 121° – 180°), and below it there is a smaller table where we are given the values of the equation of center for that particular value of the minutes of proportion (see Figure 4). Note also that an extra column displays the difference between two successive entries. It is easy to verify that the subtable for 0 minutes of proportion agrees with the Alfonsine equation of anomaly, rounded to minutes, with a maximum of $4;56^{\circ}$, and that all entries in the associated small tables correspond to excerpts of the equation of center used by Alfonsine astronomers, with a maximum of $13;9^{\circ}$. All entries follow by applying equations 1 and 2, above. It is worth noting that the compiler of this vast table at about the very beginning of the fifteenth century was not a clergyman or a member of a university faculty; rather, he was a wool merchant, as indicated in the text: an example of a new kind of actor in mathematical astronomy.

<<Insert Figure 4 and its caption about here.>>

<<Caption>> Figure 4: Lunar equation when the minutes of proportion are 0 (left) and 1 (right). The small tables below give the Alfonsine equation of center for these two values as a function of double elongation (Vatican, BAV, MS Pal. lat. 1374, 87r)

II. Double argument tables for the true lunar position

The tenth table for the lunar equation is included as a component in a remarkable table composed by the astronomer Giovanni Bianchini (d. after 1469). He worked as an administrator for the Ferrarese d'Este family, and had no known relation to a university. He produced various sets of tables but the set for the planets was the only one to be printed (first edition Venice, 1495). Among this set are double argument tables for the Sun, the Moon, and each of the planets. They are exceptional in many respects and differ from all other double argument tables previously compiled. The table for the lunar motions has the double elongation at the time when the mean lunar anomaly is 0° as argument for the columns, given at intervals of 10° from 0° to 350° , thus displaying 36 columns (see Figure 5 for an excerpt; another excerpt is reproduced in Chabás and Goldstein 2009, pp. 52–53). The argument for the rows is absolutely original, for it is the time elapsed since mean lunar anomaly was 0° , and it is expressed in integer days from 0d to 27d. Hence, the resulting matrix has 1008 ($= 36 \cdot 28$) cells. This arrangement requires an unusual set of mean motion tables, where the emphasis is on the times when the mean lunar anomaly is 0° and the other mean motions (including the double elongation) are given for those times. An outstanding feature here is that each cell contains four entries, whereas in all other previous such tables there was only one. In this sense, Bianchini compiled what could be characterized as multivalued double argument tables. In particular, for each day and each value of the mean lunar anomaly, we are given entries for the increment in true lunar longitude; a quantity closely related to the lunar equation, and called *equatio* (actually, one tenth of successive horizontal differences of the increment in longitude); a quantity similar to the hourly velocity (actually, one twenty-fourth of successive vertical differences of the increment in longitude); and the true argument of lunar latitude. Therefore, the total number of entries in this table is 4032 ($= 4 \cdot 1008$). All tabular entries were computed with the standard Alfonsine parameters (for details, recomputation, and excerpts of the table, see Chabás and Goldstein 2009, pp. 45–57).

With Bianchini, double argument tables for the Moon reached their highest level of sophistication since their appearance in Europe more than a century and a half earlier. His approach was certainly clever and innovative, but the table he produced was far from being a paragon of user-friendliness.

<<Insert Figure 5 and its caption about here.>>

<<Caption>> Figure 5: Bianchini's table for the lunar position fills 18 pages in the printed edition (Venice 1495), of which only an excerpt is shown here (c1r)

III. Double argument tables for lunar velocity at any time

We are aware of only one table in this category: Vatican, BAV, MS Heb. 384, 375a–384b, uniquely preserves a double argument table for lunar velocity (see Goldstein 1998, pp. 180–181). According to Richler et al. (2008, pp. 327–328), this manuscript was copied in the fifteenth century and contains a collection of astronomical and other texts. It has been assumed that Judah ben Asher II (d. 1391) was the author of this table, but his relevant astronomical work, uniquely preserved in this manuscript on ff. 284a–341b, has not been studied. There is no heading for the table, and its entries are the lunar velocity at any time, that is, it is not restricted to syzygy in contrast to the other known medieval tables for lunar velocity (Goldstein 1996). The argument for the rows is true anomaly, and the argument for the columns is double elongation. Both arguments use signs of 60° , as in the standard version of the Parisian Alfonsine Tables. The argument for the rows ranges from 2° to $6,0^\circ (= 360^\circ)$, at intervals of 2° . The argument for the columns ranges from 0° to $5,46^\circ (= 346^\circ)$ at irregular intervals. There are 180 rows and 34 columns, for a total of 6120 entries, and each entry is the lunar velocity given in degrees, minutes, and seconds per day. This table is most unusual and it is associated in this manuscript with similar unusual tables for velocities of each of the planets (364a–374a). The only example of such a table that we have found outside this manuscript is in the Almanac of Abraham Zacut, where there is a table for the velocity of Mercury, given to minutes, which is similar in structure to the table for the velocity of Mercury in this manuscript, given to seconds (see Chabás and Goldstein 2000, pp. 145–146). We have not succeeded in determining the underlying algorithm for computing the entries in the lunar velocity table (or for the entries in the planetary velocity tables) and numerous copyist's errors make the task difficult. The initial entry for $\alpha = 0^\circ$ and $2\eta\text{-bar} = 0^\circ$ is $11;49,58^\circ/\text{d}$, equivalent to about $0;29,35^\circ/\text{h}$, which is the minimum lunar velocity in Levi ben Gerson's table for lunar velocity at syzygy (Goldstein 1974, p. 182) and close to the minimum value in John of Genoa's table for lunar velocity at syzygy, $0;29,37,13^\circ/\text{h}$ (Goldstein 1992, pp. 12–13). Note that Levi ben Gerson constructed tables for lunar motion, but not for planetary motion. John of Genoa was associated with the compilers of the Parisian Alfonsine Tables in the 1330s (Goldstein 1992, p. 4 n. 8). The significance of the agreement for the initial entry is not clear, but this entry indicates that the standard lunar velocity table that goes back to al-Battānī, where the minimum lunar velocity is $0;30,18^\circ/\text{h}$, does not underlie this table (Nallino 1903–1907, 2:88).

IV. Double argument tables for the time from mean to true syzygy

The determination of Δt , the time from mean to true syzygy is a major and unavoidable problem when computing the circumstances of an eclipse. As this was the main goal of most astronomers since antiquity, various strategies were devised for that purpose (Chabás and Goldstein 1997, 2015b, pp. 40–56).

Ptolemy had no table for determining Δt , but in *Almagest* VI.4 he presented an algorithm for an approximate solution: Δt can be obtained as the product of 13/12 and the result of dividing the true elongation at mean syzygy by the velocity of the Moon at the time of mean syzygy. Once Δt has been added to (or subtracted from) the time at mean syzygy, the process can be repeated, until the solar and lunar longitudes are found to be equal, to the precision of minutes of arc. Fortunately, this iterative procedure converges rapidly.

1. The earliest solution to this problem we have found in the form of a double argument table was provided by the Andalusian astronomer Ibn al-Kammād (c. 1116) in his *zij al-Muqtabis*, which survives in Latin and Hebrew versions but not in the original Arabic (for the Latin and Hebrew versions, see Chabás and Goldstein 2015a, pp. 624–625; on the author of the Hebrew version, Solomon Franco, fl. 1375, see Goldstein 2013). The argument for the rows is the true elongation, η , displayed at intervals of $0;30^\circ$ from $0;30^\circ$ to 12° , and that for the columns is the difference between the hourly velocities of the Moon and the Sun, $v_m(t) - v_s(t)$, from $0;27,30^\circ/\text{h}$ to $0;33,30^\circ/\text{h}$, at intervals of $0;0,30^\circ/\text{h}$. The entries give directly Δt , and are simply obtained dividing the true elongation by the difference in velocities, according to the formula:

$$\Delta t = -\eta / [v_m(t) - v_s(t)]. \quad [3]$$

Although simple to use, this approach does not yield very accurate results, for the relative velocity is taken to be constant during the interval Δt , which can amount to 12h or even more.

Nonetheless, this procedure was followed, and analogous tables were compiled with the same arguments, by various astronomers, including Solomon Franco whose own set reproduces Ibn al-Kammād's table. Following the same pattern, other astronomers increased the number of entries in the table: Juan Gil of Burgos (c. 1350) reduced the interval for the elongation to $0;6^\circ$, and in the Tables of Barcelona (c. 1381), probably compiled by Jacob Corsuno, it is reduced to $0;10^\circ$ (Chabás and Goldstein 2015a, p. 625; Chabás 1996, pp. 507–508).

2. A similar approach was used by Joseph Ibn Waqār, who has already been mentioned (see section I.5, above). As was the case with his predecessors, true elongation is taken as one argument, whereas the other is lunar velocity, not the difference between lunar and solar velocities. The entries, Δt , are derived in the same way as those in Ibn al-Kammād's table, but for the fact that the solar velocity, $v_s(t)$, in eq. [3] is taken to be $1^\circ/\text{d}$ in computing all the entries (Chabás and Goldstein 2015b, pp. 44–45).

3. Around 1320, John Vimond seems to have played a decisive role in the introduction of new parameters and methods in the Parisian community of astronomers. His two double argument tables to compute Δt differ in presentation from those mentioned above. Both have the true elongation as the argument for the rows at the left of the table and the difference between the lunar and solar velocities as the argument for the columns at the head of the table. However, the velocity in elongation, $v_m(t) - v_s(t)$, is no longer displayed in $^\circ/\text{h}$ but in $^\circ/\text{d}$, and the four values considered are 11, 12, 13, and

14°/d, equivalent to 0;27,30°/h to 0;35,0°/h. The other argument, true elongation, is given in integer degrees, from 1 to 7, in the first table, and in minutes of arc, from 1 to 60, in the second. For a detailed explanation, see Chabás and Goldstein 2004, pp. 226–228.

4. In various manuscripts⁴ containing the tables for 1322 compiled by John of Lignères we find a double argument table for the same purpose: see Figure 6. The two arguments are the same as in previous cases, but again a few modifications were made. The argument for the columns, the velocity in elongation, is displayed here in integer numbers from 27′/h or 28′/h to 33′/h, and the elongation is given in integer numbers from 1 to 33 minutes of arc. The entries, in minutes and seconds, are computed by dividing the true elongation by the velocity in elongation according to eq. [3], and therefore entries in common with the table of Ibn al-Kammād agree, but for scribal errors.

<<Insert Figure 6 and its caption about here.>>

<<Caption>> Figure 6: Double argument table for the time from mean to true syzygy associated with John of Lignères (Bernkastel-Kues, Cusanusstiftsbibliothek, MS 212, 91r, the table on the right side of the figure)

A similar table is found in a mid-fourteenth century manuscript containing several texts by John of Lignères: Erfurt, Universitäts- und Forschungsbibliothek, MS Amplon. Q 366, 55r. In this case, the range of the velocity in elongation is greater, from 27′/h to 34′/h, and the entries are also displayed in minutes and seconds. The table includes columns for the differences between successive entries in a row. This very same table appears in the printed version of the Parisian Alfonsine Tables published in Venice by Johannes Lucilius Santritter in 1492, h7r–v, although it is not in the *editio princeps* of 1483. In h6v, Santritter included another such table, with the same format and columns for differences, which complements the first one. Here the true elongation is given in degrees, from 1° to 8°, and the entries are thus displayed in hours, minutes, and seconds. Both tables were later reproduced by P. Duhamel in his edition of the Parisian Alfonsine published in Paris in 1545, pp. 155–157.

5. Levi ben Gerson (d. 1344) was active in Orange, Southern France. He did not belong to the Parisian milieu and there is no trace of Alfonsine astronomy in his tables. Levi produced two sets of four tables for the time from mean to true syzygy: the first three in each set are single argument tables, where the argument is either the day of the year or the lunar anomaly. The fourth in each set is a double argument table (Goldstein 1974, pp. 137–144 and 235–241; Mancha 1998, pp. 321–322). The entries in this particular table, given in hours and minutes, are to be added to the entries in the other three previous tables to obtain $\Delta t + 24h + E$, where E is the equation of time, and $24h$ is added to avoid negative entries. The arguments also differ from those in the tables computed by other astronomers for the same purpose. In this case, the argument for the rows is the day of the year at intervals of about 6 days, and that for the columns is lunar

⁴ Among them are Erfurt, Universitäts- und Forschungsbibliothek, MS Amplon. F 377, 46r; Bernkastel-Kues, Cusanusstiftsbibliothek, MS 212, 91r; Paris, BnF, MS lat. 7286C, 53v; Paris, BnF, MS lat. 7295A, 162r. For the Tables of John of Lignères for 1322, see Chabás 2019, forthcoming.

anomaly at intervals of 15° . A distinctive feature of Levi ben Gerson's tables for determining Δt is that they include the equation of time.

6. A different type of double argument table for determining the time from mean to true syzygy was developed in the fourteenth century. John of Murs, in collaboration with Firmin de Beauval, authored an unprecedented table for that purpose, called *Tabulae permanentes* (Porres and Chabás 2001).⁵ Two different arguments are used here: mean solar anomaly for the columns at the head of the table, and mean lunar anomaly for the rows at the left of the table, both given in integer degrees at intervals of 6° . The entries are displayed in hours and minutes, and their maximum value is 14;0h. Most copies of this table also show the differences between consecutive entries, both vertically and horizontally. The same table is found in some manuscripts containing sets of tables by John of Gmunden (Vienna, c. 1384–1442), and it comes as no surprise that many fifteenth-century astronomers appreciated that table, which was probably the most accurate and compact solution ever provided to the problem of finding the time from mean to true syzygy.

7. The same presentation is found in another work on eclipses, known as *The book of six wings*, originally written in Hebrew by Immanuel ben Jacob Bonfils of Tarascon (Southern France) in the mid-fourteenth century (Solon 1968 and 1970; Goldstein and Chabás 2017). The table is called *Wing 2* and its two arguments are mean solar anomaly and true lunar anomaly, both in signs of 30° and degrees, at intervals of 6° (see Table 2). This table differs from previous ones in that the entries are $T = \Delta t + E + 24h$, where the initial value is 24;16h (when both anomalies are zero); E is the equation of time and 24h are added to avoid calculation with negative terms. We have checked a few entries and they mostly agree with computations based on the *zij* of al-Battānī. But an analysis of this table suggests inconsistencies in the underlying algorithms used to compute the entries. It is noteworthy that Bonfils included the equation of time in the entries of his tables, as did Levi ben Gerson. However, some entries agree with those in Levi's tables, whereas others do not (cf. Goldstein 1974, p. 142).

Table 2: Excerpt of Bonfils's table, with entries for T (in hours) and L (in degrees)*

(s) ($^\circ$)	0s 0°		0s 6°		...	3s 0°		...	6s 0°		...	11s 24°	
	[T]	[L]	[T]	[L]		[T]	[L]		[T]	[L]		[T]	[L]
0 0	24;16	3; 0	25;18	3; 2		34; 7	3;25		24;16	3; 0		23;14	2;58
0 6	23;49	2;47	24;51	2;50		33;43	3;13		23;54	2;47		22;46	2;46
0 12	23;22	2;34	24;23	2;37		33;18	2;59		23;30	2;34		22;20	2;32
...													
3 0	20; 6	0;51	21; 8	0;53		30;20	1;16		20;50	<u>0;51</u>		19; 4	0;49
...													
5 24	23;51	2;46	24;54	2;48		33;48	3;11		23;56	2;46		22;51	2;44
6 0	24;16	3; 0	25;19	3; 2		34;10	3;25		24;16	3; 0		23;13	2;58
6 6	24;41	3;14	25;44	3;16		34;31	3;39		24;36	3;14		23;38	3;12

⁵ We have been informed that R. L. Kremer is about to publish a paper on the mathematical aspects of this table.

...	9	0	28;26	5; 9	<u>29;27</u>	<u>5;10</u>	37;54	<u>5;33</u>	27;42	5; 9	27;25	5; 7
...	11	18	25;10	3;26	26;13	3;28	34;56	3;51	25; 2	3;26	24; 5	3;23
11	24	24;44	3;13	25;46	3;14	34;31	3;37	24;39	3;13	23;42	3;10	
0	0	24;16	3; 0	25;18	3; 2	34; 7	<u>3;35</u>	24;16	3; 0	23;14	2;58	

* The entries of *Wing 2* have been taken from Solon 1968, pp. 299–328, and checked against those in Munich, Bayerische Staatsbibliothek, MS Heb. 386, 43a–55a (Mn1) and Munich, Bayerische Staatsbibliothek, MS Heb. 343, 6r–17b (Mn 2), with the following variants:

3s0°/6s0°L: 0;50 Mn1 & Mn2.
 9s0°/0s6°T: 29;28 Mn1 & Mn2.
 9s0°/0s6°L: 5;11 Mn1 & Mn2.
 9s0°/3s0°L: 5;34 Mn1 & Mn2.
 0s0°/3s0°L: 3;25 Mn1 & Mn2 (as in the first row!).

Moses Farissol Botarel has two single argument tables for this purpose based on Bonfils's table: in one table Moses only reproduced the first column and in the other table he only reproduced the first row of Bonfils's table, in both cases subtracting 24h from the entries in Bonfils's table so that some entries are positive and some negative (Goldstein and Chabás 2016–2017, pp. 44–46). Evidently, Moses did not properly understand Bonfils's double argument table.

8. About a century later, the table in the *Tabulae permanentes* was considerably enlarged, to reach 34,560 entries, by the Viennese astronomer, Georg Peurbach (1423–1461). This impressive double argument table covers 48 pages in print of his *Tabulae eclipsum*, published in 1514 in Vienna by Georg Transtetter. In this case, the solar anomaly is given at intervals of 2° and the lunar anomaly at intervals of 1°, and the entries are in hours and minutes. Undoubtedly, this type of table provided more accurate values than those found in the tables by the Ibn al-Kammād and his followers.

Table A summarizes some features of the double argument tables for determining the time from mean to true syzygy and shows the variety of approaches developed in compiling them.

Table A: Characteristics of the double argument tables to determine Δt

	Argument for the rows	Interval	Argument for the columns	Interval
Ibn al-Kammād	elongation	0;30°	velocity in elongation	0;0,30°/h
Juan Gil of Burgos	elongation	0;6°	velocity in elongation	0;0,30°/h
Tables of Barcelona	elongation	0;10°	velocity in elongation	0;0,30°/h

Joseph Ibn Waqār	lunar velocity	0;10°/d	elongation	0;10° and 1°
John Vimond	elongation	1° and 0;1°	velocity in elongation	1°/d and 0;1°/d
John of Lignères?	elongation	1° and 0;1°	velocity in elongation	0;0,30°/h
Levi ben Gerson	day in a year	6 days	lunar anomaly	15°
<i>Tabulae permanentes</i>	lunar anomaly	6°	solar anomaly	6°
Immanuel ben Jacob Bonfils	lunar anomaly	6°	solar anomaly	6°
John of Gmunden	lunar anomaly	6°	solar anomaly	6°
Georg Peurbach	lunar anomaly	1°	solar anomaly	2°

V. Double argument tables for the distance in longitude from mean to true syzygy

1. In Bonfils's *Wing 2* (see Table 2), in addition to the columns for $\Delta t + 24;16h$ there are also columns for the difference in longitude between true and mean syzygy, $\Delta\lambda$. Actually, the tabulated entries in these columns are the longitude, $L = \Delta\lambda + 3^\circ$, and its extreme values are $0;26^\circ$ and $5;34^\circ$. The addition of 3° in longitude is to make sure that all entries are positive, indicating that Bonfils, as had previously been done by Levi ben Gerson, introduced vertical displacements to produce more user-friendly tables.

2. Ibn Waqār has a separate double argument table for determining the position of the Sun at true syzygy. In this case, the argument for the rows is the velocity of the Sun, ranging from $0;57^\circ/d$ to $1;2^\circ/d$ and the argument for the columns is Δt , already found in the corresponding double argument table (see above). The entries represent the longitudinal difference from mean and true syzygy (Chabás and Goldstein 2015b, pp. 44–45). These two tables complement each other, for Ibn Waqār noticed that the position of true syzygy depends mostly on the Sun (which moves slowly) whereas the time of true syzygy depends mostly on the Moon (which moves quickly). So, for computing the time of true syzygy he considered the solar motion to be constant at $1^\circ/d$ (rather than varying from $0;57^\circ/d$ to $1;2^\circ/d$) and let the lunar motion vary, and for the position at true syzygy he let the solar velocity vary, and used the time that was already determined. In this way, the two tables are linked together.

VI. Double argument tables for true lunar positions between syzygies

The computation of true lunar positions for each day between syzygies generated another type of double argument table and provides a relevant example of the interaction between astronomers in the fourteenth century, especially in the Parisian milieu. In all we have identified seven such tables, where the two arguments are the number of complete days after a mean syzygy and the mean lunar anomaly. The number of days varies from one author to another, and the intervals for the anomaly vary depending on the table.

1. John Vimond also compiled an unprecedented double argument table to correct the lunar position for each day between syzygies, where mean lunar anomaly, at steps of 12° , is the argument for the rows at the left of the table, and the number of days from syzygy, from 1 day to 14 days, is the argument at the head of the columns (Chabás and Goldstein 2004, pp. 228–230). For each day and each value of the mean lunar anomaly, we are given the increment in lunar longitude (*motus completus*) to be added to the mean longitude at the preceding mean syzygy, as well as one sixtieth of the differences between successive entries in the same row (*motus ad minutum diei*): see Figure 7. The entries for the *motus completus*—420 in total—are given in degrees and minutes, and those for the *motus ad minutum diei*—also 420 in total—in minutes of arc. We note that the two intervals chosen (12° of anomaly and 1 day) are mutually consistent, for 12° is close to the mean motion in lunar anomaly in one day. This is indeed a very convenient table requiring interpolation, and it would come as no surprise if later astronomers compiled similar tables with a greater number of entries.

<<Insert Figure 7 and its caption about here.>>

<<Caption>> Figure 7: John Vimond’s table for true lunar positions between syzygies; excerpt (Paris, BnF, MS lat. 7286C, 3v)

2. Vimond’s Parisian contemporary, John of Murs, composed similar tables with the same layout, but he switched the arguments and modified the intervals. As was the case with other tables he compiled, he made several versions of this double argument table. They are found in his *Kalendarium*, the *Tables of 1321*, and the *Patefit* (Chabás and Goldstein 2012a). In the *Kalendarium*, the lunar anomaly is given at intervals of 1 zodiacal sign of 30° , whereas in his *Tables of 1321* it is displayed for each 6° , and for each 3° in the *Patefit*. John of Murs gave progressively more and more entries by shortening the interval for the values of lunar anomaly (see Table B). In the version of this table in the *Tables of 1321* there is a surprising feature: signs of 60° are used, in contrast to other works by John of Murs where he used zodiacal signs of 30° . Another oddity is that the range of the other argument is from 1 to 30 days, which is exceptionally large, since the starting point is a syzygy, whether conjunction or opposition, and the time to the next syzygy is only half of a synodic month or about $14\frac{1}{2}$ days. In his third table, found in the *Patefit*, John of Murs made a more substantial change, for the common entries in it and in the *Tables of 1321* do not fully agree. Moreover, since they systematically differ in the minutes, it would seem that John of Murs computed anew all 1920 entries in this table. These double argument tables for syzygies show us John of Murs’s table-making practices, oftenly recasting and updating tables he had compiled previously, by modifying their entries or increasing their number, or both.

3. As it happened, John of Lignères used John of Murs’s new values in the *Patefit* to compile his own double argument table for syzygies. John of Lignères’s table is found among his *Tabule magne*, where the arguments are the same, and presented in the same way, as in John of Murs’s table. A novelty in John of Lignères’s *Tabule magne* is that for each entry we are also given the corresponding hourly lunar velocity, simply obtained by dividing the difference between two successive entries by 24. Here we see Alfonsine Parisians at work.

4. A step further was taken by Levi ben Gerson who compiled a double argument table for syzygies with a range of 14 successive days and with mean anomaly displayed at steps of 10° . An outstanding feature of Levi's table is that its entries have a higher precision, to seconds, than all those compiled by his predecessors. Moreover, Levi constructed his own lunar model together with a lunar equation table based on it, and then used his own tables for computing the entries in this double argument table (Goldstein 1974, pp. 148–149, 246–254; see also Mancha 1998, p. 333). See Table B for a survey of the double argument tables for syzygies.

Table B: Characteristics of the double argument tables to correct the lunar position for each day between syzygies

	Number of days between syzygies	Mean lunar anomaly (interval)	Entries (precision)	Number of entries
John Vimond	1–14 days	12°	minutes	420
John of Murs, <i>Kalendarium</i>	1–15 days	30°	minutes	180
John of Murs, <i>Tables of 1321</i>	1–30 days	6°	minutes	1800
John of Murs, <i>Patefit</i>	1–16 days	3°	minutes	1920
John of Lignères, <i>Tabule magne</i>	1–15 days	6°	minutes	900
Levi ben Gerson	1–14 days	10°	seconds	504
Anonymous tables for 1400	1–14 days	1 day	minutes	392

5. In an anonymous set of tables in Hebrew for radix 1400 there is a double argument table for correcting the lunar position for each day between syzygies (Goldstein 2003, p. 166; Vatican, BAV, MS Heb. 384, 267a). The argument for the columns has three components: the minutes of proportion, the number of days since syzygy (from 1 to 14), and the mean elongation, where the value for 1 day is $12;11^\circ$ and the value for 14 days is $170;40^\circ$ (see Table 3). The argument for the rows is the mean anomaly from day 0 to day 27, where the value for 1 day is $13;4^\circ$ and the value for 27 days is $352;47^\circ$. An entry is to be added to the mean lunar longitude at the preceding syzygy to yield the true lunar longitude. The entries are given to minutes of arc. But for the fact of displaying three components in the headings, this table is similar in structure to all those compiled previously (see Table B). We also note that all three components are mutually consistent and represent a single variable, for the values assigned to the mean elongation are multiples of the number of days since the previous syzygy and the minutes of proportion are those appropriate to twice the mean elongation. Moreover, the format of this table is similar to that of Ben Verga for the lunar equation (see section I.7, above), and the entries in this table agree closely with those that follow from the corresponding entries in Ben Verga's table (cf. Goldstein 2001, p. 247).

Table 3: "Table for correcting the position of the Moon from day to day" (Vatican, BAV, MS Heb. 384, 267a), excerpt

Minutes of prop.	2	8	18	...	1
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	Days	1	2	3	...	14
	Elongation	0s 12;11°	0s 24;23°	1s 6;34°		5s 20;40°
Days	Anomaly					
0	0s 0; 0°	0s 12;53°	0s 25;45	1s 8;35		6s 4;41
1	0s 13; 4	0s 11;52	0s 24;39	1s 7;29		6s 3;38
2	0s 26; 8	0s 10;51	0s 23;37	1s 6;20		6s 2;36
3	1s 9;12	0s 10; 1	0s 22;44	1s 5;23		6s 1;38
...						
27	11s 22;47	0s 13;29	0s 26;22	1s 9;18		6s 5;16

John Vimond, John of Murs, and John of Lignères, all worked in Paris at about the same time in the same domain. Nevertheless, nothing in the texts they wrote provides any insight into their possible interactions as astronomers. In contrast, the tables they compiled speak for themselves on that issue, and show their mutual dependence. In particular, in the case of the double argument table for syzygies, the picture that emerges is that John Vimond was responsible for the first version, with a presentation unprecedented in Latin. This table was then borrowed and enlarged by John of Murs for his *Tables of 1321*, who also adapted it for his *Kalendarium*. At a later stage, John of Murs modified slightly the entries in the table, enlarging it anew, to be included in his *Patefit*. Finally, John of Lignères incorporated this last version to his *Tabule magne*, with slight variations. In all their interventions, John of Murs and John of Lignères did not limit themselves to mere copying. Both added value to the initial table, by inserting additional information, such as successive differences of the entries, or the like, in a general process of increasing the user-friendliness of the table and simplifying the task of the user. These practices in table-making reveal a much more intense interaction than previously recognized between the astronomers that were at that time shaping the Parisian Alfonsine Tables.

VII. Tables for solar eclipses

To complete a comprehensive account of double argument tables for lunar motion, one must also refer to another category of such tables, dealing in this case with eclipses, which was much less common among medieval astronomers.

In addition to the double argument table for the time from mean to true syzygy (see section IV.1, above), Ibn al-Kammād has another double argument table to determine the half duration of a solar eclipse (Chabás and Goldstein 1994, p. 23). The argument for the rows is presented in two related columns: one displays the adjusted latitude at conjunction, from 0;34,13° to 0°, and the other the corresponding eclipse magnitude, in digits of the apparent solar diameter, from 0;15d to 11;40d, at regular intervals. The argument for the columns is the same as in Ibn al-Kammād's previous table: the hourly velocity in elongation, $v_m(t) - v_s(t)$, from 0;27,30°/h to 0;33,30°/h, at intervals of 0;0,30°/h. The entries, of which there are 252 (= 36 · 7), are given in hours and minutes and represent the half duration of the eclipse.

The same table, but for variant readings, is found in the *zij* of Yah<.>yā b. Abī Mans<.>ūr (Bagdad, c. 830). According to Kennedy and Faris (1970, pp. 27–30), the relation between the entries in the first two columns is given by the modern expression

$$\beta = r_s + r_m - d \cdot r_s/6,$$

where β is the lunar latitude, r_s and r_m are the radii of the Sun and the Moon, respectively, and d is the eclipse magnitude, in digits. The entries for the half duration, t , of the solar eclipse can be computed, although with only rough agreement, by means of the modern expression

$$t = [(r_s + r_m)^2 - \beta^2]^{1/2} / (v_m - v_s).$$

Ibn al-Kammād's table is later found in other sets compiled in the second half of the fourteenth century by Solomon Franco (Vatican, BAV, MS Heb. 498, 61a), Isaac Ibn al-Hadib (Chabás and Goldstein 2015b, p. 357), and in the Tables of Barcelona (Chabás 1996, pp. 511–512).

Conclusion

Beginning in 1300 double argument tables were progressively generated by European table-makers, as had been the case in the Islamic world, and they proliferated among astronomers throughout Europe. In the case of the Moon, both for its equation and for syzygies, the aim was to present single and compact tables that reflected simultaneously the related effects of the Sun and the Moon. The use of a single table to solve complex problems that had previously required several tables indicates substantial progress in facilitating the task of practitioners of astronomy at the time and represents a qualitative change in medieval table-making: cf. North 1977, p. 290. To judge from the number of such tables that were compiled, this new practice seems to have been well accepted among astronomers. Moreover, a number of table-makers put much effort into modifying the quantities that served as arguments, shortening the intervals used to present them, and adding columns and rows for successive differences in order to make their tables even more useful. That is, innovation was mostly restricted to presentation without changing either parameters or models.

It is also noteworthy that, except for Levi ben Gerson, there were no changes introduced by these astronomers either in the solar and lunar models or in the underlying parameters. Rather, they were engaged in seeking solutions to a standard astronomical problem and found various ways to tabulate what would later be considered a function of two variables.

Compliance with ethical standards

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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