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Boundary Observer Design for Cascaded ODE–Hyperbolic PDE Systems: A Matrix Inequalities Approach (Extended Version)

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Abstract

Boundary observer design for a system of ODEs in cascade with hyperbolic PDEs is studied. An infinite dimensional observer is used to solve the state estimation problem. The interconnection of the observer and the system is written in estimation error coordinates and analyzed as an abstract dynamical system. The design of the observer is performed to achieve global exponential stability of the estimation error with respect to a suitable norm and with a tunable convergence rate. Sufficient conditions in the form matrix inequalities are given for the design of the observer. The effectiveness of the approach is shown in a numerical example.

Key words: Distributed-parameter systems, Observers, Convex optimization, Lyapunov methods, Computer-aided design

1 Introduction

1.1 Background

Systems of balance laws are often studied when considering control applications. Indeed, such a class of infinite-dimensional systems can be adopted to model many physical phenomena as hydraulic networks [15], multi-phase flow [13], transmission networks [17], road traffic networks [18] or gas flow in pipelines [14]; see [4] for an introduction on this class of hyperbolic Partial Differential Equations (PDEs) and physical motivations. For such distributed parameter systems, sensors are often placed pointwise at the boundary of the domain, and similarly for (potentially non-located) actuators. This gives rise to challenging boundary control design problems, investigated using various techniques, e.g., in

[9,23,25]; just to mention a few. In what pertains to boundary state estimation problems, due to the practical need of state observers, the analysis of observability properties and observer design problems for infinite-dimensional systems have seen an increasing interest in the community. Boundary observability has been formally analyzed in [29] using abstract semigroup theory. The design of boundary observers has been investigated via Lyapunov methods in [1,7]. The use of backstepping techniques has been explored in [28] for the design of boundary observers for parabolic PDEs and in [2,19] for PDE–ODE cascade systems. Recently, the design of high-gain observers for hyperbolic systems of balance laws, yet with distributed in-domain measurements, has been addressed in [21].

1.2 Contribution

In this paper, we consider the problem of exponentially estimating the state of a system of hyperbolic PDEs with linear time-invariant dynamic boundary conditions via boundary measurements. Systems modeled as the interconnection of PDEs and ODEs can be found in numerous applications; see, e.g., [4, Chapter 1], [11,26]. To address this problem, we consider an infinite-dimensional

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observer that is a copy of the system with a linear boundary injection term to be designed. Our contributions in the solution to this problem are as follows. Partly inspired by [7], where boundary observer design for conservation laws with static and asymptotically stable dynamic boundary conditions is addressed, we present results for the design of the observer based on matrix inequalities. The approach we pursue relies on Lyapunov theory for infinite-dimensional systems and leads to a set of sufficient conditions to ensure exponential convergence of the estimation error with guaranteed convergence speed. Connections between the feasibility of those conditions and structural properties of the system under estimation are illustrated. As a second step, our conditions guaranteeing exponential convergence of the estimation error are exploited to derive an observer design algorithm based on the solution to some linear matrix inequalities (LMI), which can be efficiently solved via available software [5].

The unique feature of our paper with respect to [7] is the use of a nondiagonal Lyapunov functional for the design of the observer. This enables us to relax the assumption on the boundary conditions in [7], which in our work do not need to be asymptotically stable. As a side effect, the use of such a nondiagonal functional renders the construction of a numerically affordable algorithm for the design of the observer more challenging than in [7]. This is a relevant contribution of our work.

A preliminary version of this work has been presented in the conference paper [16]. While in [16] the design of the observer is based on the solution to some bilinear matrix inequalities, the present paper proposes a design algorithm based on the solution to some LMIs coupled to a two-dimensional line search. In addition, this paper contains full proofs of the main results.

The remainder of the paper is organized as follows. Section 2 presents the system under consideration, illustrates the problem we solve, and provides some preliminary results. Section 3 provides sufficient conditions in the form of matrix inequalities for stability analysis of the estimation error dynamics. Section 4 is dedicated to the design of the proposed observer. Section 5 showcases the application of our methodology in a numerical example. Some auxiliary results are included in Appendix A.

Notation

We denote by \mathbb{N} the set of nonnegative integers, \mathbb{R} the set of real numbers, $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers, and \mathbb{C} the set of complex numbers. Given $z \in \mathbb{C}$, $\Re(z)$ stands for the real part of z . The symbols \mathbb{S}^n (\mathbb{S}_+^n) and \mathbb{D}_+^n denote, respectively, the set of real $n \times n$ symmetric (symmetric positive definite) matrices and the set of diagonal positive definite matrices. For a matrix $A \in \mathbb{R}^{n \times m}$, A^\top denotes the transpose of A , $\text{spec}(A)$ its

spectrum, and, when $n = m$, $\text{He}(A) = A + A^\top$. Given two matrices A and B , $A \oplus B$ denotes the block diagonal matrix with matrices A and B on its diagonal. For a symmetric matrix A , positive definiteness (negative definiteness) and positive semidefiniteness (negative semidefiniteness) are denoted, respectively, by $A \succ 0$ ($A \prec 0$) and $A \succeq 0$ ($A \preceq 0$). Given $A, B \in \mathbb{S}^n$, we say that $A \preceq B$ ($A \succeq B$) if $A - B \preceq 0$ ($A - B \succeq 0$). Given $A \in \mathbb{S}^n$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stand, respectively, for the largest and the smallest eigenvalue of A . In partitioned symmetric matrices, the symbol \bullet stands for symmetric blocks. The symbol \mathbf{I} denotes the identity matrix or the identity operator, depending on the context. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given $x, y \in \mathbb{R}^n$, we denote by $\langle x, y \rangle_{\mathbb{R}^n}$ the standard Euclidean inner product. Let X and Y be linear normed spaces, the symbol $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y . Let $U \subset \mathbb{R}$, $V \subset \mathbb{R}^n$, and $f: U \rightarrow V$, we denote by $\|f\|_{\mathcal{L}^2} = (\int_U |f(x)|^2 dx)^{\frac{1}{2}}$ the \mathcal{L}^2 norm of f . In particular, we say that $f \in \mathcal{L}^2(U; V)$ if $\|f\|_{\mathcal{L}^2}$ is finite. Given $f, g \in \mathcal{L}^2(U; V)$, $\langle f, g \rangle_{\mathcal{L}^2} := \int_U \langle f(x), g(x) \rangle_{\mathbb{R}^n} dx$. Let $U \subset \mathbb{R}$ be open and V be a linear normed space, we denote

$$\mathcal{H}^1(U; V) := \left\{ f \in \mathcal{L}^2(U; V) : f \text{ is absolutely continuous on } U, \frac{d}{dz} f \in \mathcal{L}^2(U; V) \right\}$$

where $\frac{d}{dz}$ stands for the weak derivative of f . The symbols $\mathcal{C}^k(U; V)$ and $\mathcal{C}^\infty(U; V)$ denote, respectively, the set of class k functions $f: U \rightarrow V$ and set of smooth compactly supported functions $f: U \rightarrow V$. Let $I \subset \mathbb{R}$, $\phi: I \rightarrow \mathcal{H}^1(U; V)$, $t \in I$, and $z^* \in U$. We denote by $(\phi(t))(z^*) \in V$ the value of $\phi(t)$ at $z = z^*$. Given a linear operator \mathcal{A} , we denote by \mathcal{A}^* its adjoint. Let X and Y be linear normed spaces, U be an open subset of X , $f: U \rightarrow Y$, and $x \in U$, we denote by $Df(x)$ the Fréchet derivative of f at x .

2 Problem Statement and Outline of the Solution

2.1 System Description

Let $\Omega := (0, 1)$, we consider a system of n_x linear 1-D hyperbolic PDEs with dynamic boundary conditions formally written as:

$$\begin{cases} \partial_t x(t, z) + \Lambda \partial_z x(t, z) + Fx(t, z) = 0 \\ x(t, 0) = C\chi(t) \\ \dot{\chi}(t) = A\chi(t) \\ y(t) = Mx(t, 1) \end{cases} \quad \forall (t, z) \in \mathbb{R}_{>0} \times \Omega \quad (1)$$

where $\partial_t x$ and $\partial_z x$ denote, respectively, the derivative of x with respect to “time” and the “spatial” variable

$z, (z \mapsto x(t, z), \chi(t)) \in \mathcal{Z} := \mathcal{L}^2(0, 1; \mathbb{R}^{n_x}) \times \mathbb{R}^{n_\chi}$ is the system state, and $y \in \mathbb{R}^{n_y}$ is a measured output. We assume that the matrices $\Lambda \in \mathbb{D}_+^{n_x}, F \in \mathbb{R}^{n_x \times n_x}, C \in \mathbb{R}^{n_x \times n_x}, A \in \mathbb{R}^{n_\chi \times n_\chi}$, and $M \in \mathbb{R}^{n_y \times n_x}$ are given. Our goal is to design an observer providing an exponentially converging estimate $(\hat{x}(t, z), \hat{\chi}(t))$ of the system state $(x(t, z), \chi(t))$ from any initial condition in \mathcal{Z} .

Remark 1 System (1) is an LTI ODE plant in cascade with a hyperbolic PDE in the sensing path; [22]. In this setting, one could be interested in getting an estimate of the finite-dimensional state χ only. However, due to the presence of infinite-dimensional dynamics in the sensing path, an infinite dimensional observer providing an estimate of the full state (x, χ) is needed.

2.2 Outline of the proposed Observer

Inspired by [7], we consider the following observer with state in \mathcal{Z}

$$\begin{cases} \partial_t \hat{x}(t, z) + \Lambda \partial_z \hat{x}(t, z) + F \hat{x}(t, z) = 0 \\ \hat{x}(t, 0) = C \hat{\chi}(t) \\ \dot{\hat{\chi}}(t) = A \hat{\chi}(t) + L(y(t) - M \hat{x}(t, 1)) \end{cases} \quad \forall (t, z) \in \mathbb{R}_{>0} \times \Omega \quad (2)$$

where $L \in \mathbb{R}^{n_\chi \times n_y}$ is the observer gain to be designed; see Fig. 1 for a schematic representation of the interconnection of system (1) and observer (2).

Remark 2 It is worth mentioning that one can potentially add an injection term at the boundary of the \hat{x} -dynamics. However, the design of this additional term is hard to handle from a numerical standpoint. This issue is left for future research.

Remark 3 Observer design for ODE–PDE cascades of has been addressed in [2, 19, 22] via the use of backstepping techniques. A wholly similar approach, yet for control design is proposed in [12]. The application of such an approach requires the use of plant parameters-dependent state transformations, which makes it prone to a lack of robustness in the presence of parametric uncertainties. As opposed, our approach does not rely on any transformation, hence we believe that it is very well suited to account for parametric uncertainties in an output feedback control scheme based on the proposed observer. Another interesting aspect of our methodology is that it allows to tune the decay rate directly in the design and to quantify the overshoot on the estimation error.

At this stage, define the following two estimation errors $\varepsilon := x - \hat{x}$ and $\eta := \chi - \hat{\chi}$. The dynamics of these errors

are as follows:

$$\begin{cases} \partial_t \varepsilon(t, z) + \Lambda \partial_z \varepsilon(t, z) + F \varepsilon(t, z) = 0 \\ \varepsilon(t, 0) = C \eta(t) \\ \dot{\eta}(t) = A \eta - LM \varepsilon(t, 1) \end{cases} \quad \forall (t, z) \in \mathbb{R}_{>0} \times \Omega \quad (3)$$

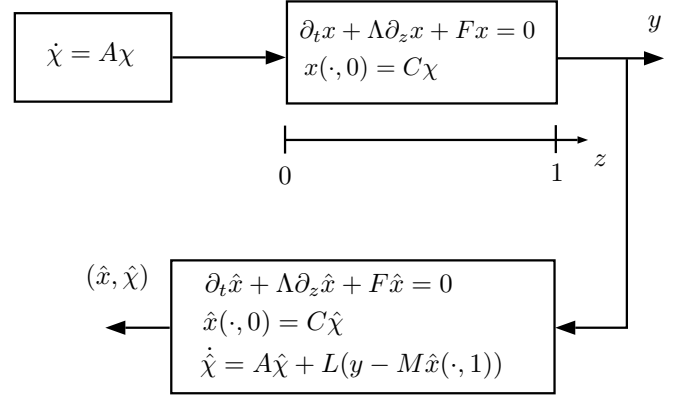


Fig. 1. Schematic representation of the considered state estimation setting.

Remark 4 In this paper, we assume that (1) is not subject to any external inputs. Nevertheless, notice that, since (1) is linear, our methodology can be trivially extended to deal with known exogenous inputs as, e.g., control inputs.

2.3 Abstract Formulation of the Problem

Similarly as in [4], in this paper, we focus on mild solutions to (3). To this end, as in [10, 3, 20], we reformulate the error dynamics as an abstract differential equation on the Hilbert space \mathcal{Z} endowed with the following inner product:

$$\langle (f_1, f_2), (g_1, g_2) \rangle_{\mathcal{Z}} := \langle f_1, g_1 \rangle_{\mathcal{L}^2} + \langle f_2, g_2 \rangle_{\mathbb{R}^{n_\chi}} \quad (4)$$

In particular, let $\mathcal{X} := \mathcal{H}^1(0, 1; \mathbb{R}^{n_x}) \times \mathbb{R}^{n_\chi} \subset \mathcal{Z}$. Define $\mathcal{D} := \{(\varepsilon, \eta) \in \mathcal{X} : \varepsilon(0) = C \eta\}$ and consider the following unbounded operator:

$$\begin{aligned} \mathcal{A} : \text{dom } \mathcal{A} &\rightarrow \mathcal{Z} \\ (\varepsilon, \eta) &\mapsto \begin{pmatrix} -\Lambda \frac{d}{dz} - F & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ -LM \varepsilon(1) \end{pmatrix} \end{aligned} \quad (5)$$

where $\text{dom } \mathcal{A} := \mathcal{D}$. Then, the error dynamics can be formally written as the following abstract differential

equation on the Hilbert space \mathcal{Z}

$$\begin{pmatrix} \dot{\varepsilon} \\ \dot{\eta} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \quad (6)$$

In particular, following the lines of [3], we consider the following notion of (mild) solution for (6):

Definition 2.1 *Let $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ be an interval containing zero. A function $\psi \in C^0(\mathcal{I}, \mathcal{L}^2(0, 1; \mathbb{R}^{n_x}) \times \mathbb{R}^{n_x})$ is a solution to (6) if for all $t \in \mathcal{I}$*

$$\int_0^t \psi(s) ds \in \text{dom } \mathcal{A}, \quad \psi(t) = \psi(0) + \mathcal{A} \int_0^t \psi(s) ds$$

Moreover, we say that ψ is maximal if its domain cannot be extended and it is complete if $\sup \mathcal{I} = \infty$. \circ

2.4 Well-posedness of the Error Dynamics

In this subsection, we illustrate existence and uniqueness of the solutions to (6). In principle existence and uniqueness issues should be addressed for the interconnection of plant (1) and observer (2). On the other hand, for the sake of brevity, in this paper we limit such an analysis only to the error dynamics (6). Nonetheless, it is worth mentioning that the extension of the analysis we propose next for (6) to the interconnection of (1) and (2) is straightforward.

The result given next shows existence and uniqueness of solutions to (6). The proof of this result is largely inspired by the techniques used in¹ [4, Theorem A.1, page 244].

Proposition 1 (Well-posedness) *Let $(\varepsilon_0, \eta_0) \in \mathcal{Z}$. Then, there exists a unique maximal solution $\psi = (\psi_\varepsilon, \psi_\eta)$ to (6) such that $\psi(0) = (\varepsilon_0, \eta_0)$. Moreover ψ is complete.*

PROOF. From [3, Proposition 3.1.11, Proposition 3.1.12, page 115] existence and uniqueness of mild solutions to (6) hold if and only if \mathcal{A} in (5) generates a C_0 -semigroup $t \mapsto \mathcal{T}(t)$ on the Hilbert space \mathcal{Z} . In particular, in this case, for any

¹ Although the result given in [4, Theorem A.1, page 244] applies only to system of hyperbolic PDEs, the authors provide a similar result for systems of hyperbolic PDEs with linear differential boundary conditions, i.e., [4, Theorem A.6, page 254]. However, the proof of such a result is therein omitted. In this paper we propose a complete proof for the case of interest.

$\psi_0 = (\varepsilon_0, \eta_0) \in \mathcal{L}^2(0, 1; \mathbb{R}^{n_x}) \times \mathbb{R}^{n_x}$, the unique maximal solution ψ to (6) with $\psi(0) = (\varepsilon_0, \eta_0)$ reads as

$$\psi(t) = \mathcal{T}(t)\psi_0 \quad \forall t \geq 0$$

Therefore, to complete the proof, we show that (5) generates a C_0 -semigroup on the Hilbert space \mathcal{Z} . To this end, we make use of [10, Corollary 2.2.3] and rely upon Lemma A.1, Lemma A.2, and Lemma A.4 in the Appendix. In particular, from Lemma A.1 and Lemma A.2, it follows that \mathcal{A} is densely defined and closed. From Lemma A.4, there exists $\omega \in \mathbb{R}$ such that

$$\begin{aligned} \langle \mathcal{A}\theta, \theta \rangle_{\mathcal{Z}} &\leq \omega \langle \theta, \theta \rangle_{\mathcal{Z}} \quad \forall \theta \in \text{dom } \mathcal{A} \\ \langle \mathcal{A}^*\theta, \theta \rangle_{\mathcal{Z}} &\leq \omega \langle \theta, \theta \rangle_{\mathcal{Z}} \quad \forall \theta \in \text{dom } \mathcal{A}^* \end{aligned}$$

Thus, by invoking [10, Corollary 2.2.3] the result is established. \blacksquare

Now we are in a position to formally state the problem we solve in this paper.

Problem 1 *Given $A \in \mathbb{R}^{n_x \times n_x}$, $C \in \mathbb{R}^{n_x \times n_x}$, $M \in \mathbb{R}^{n_y \times n_x}$, $\Lambda \in \mathbb{D}_+^{n_x}$, and $\lambda > 0$. Design $L \in \mathbb{R}^{n_x \times n_y}$ such that for each solution ψ to (6) one has, for all $t \in \text{dom } \psi$*

$$\|\psi(t)\|_{\mathcal{Z}} \leq \kappa e^{-\lambda t} \|\psi(0)\|_{\mathcal{Z}} \quad (7)$$

for some $\kappa \in \mathbb{R}_{>0}$. \circ

3 Stability Analysis of the Error Dynamics

In this section we propose sufficient conditions for the solution to Problem 1. To this end, consider the following preliminary result, whose role will be clarified later in the proof of Theorem 1.

Proposition 2 *Let $P_1 \in \mathbb{D}_+^{n_x}$, $P_2 \in \mathbb{R}^{n_x \times n_x}$, $P_3 \in \mathbb{S}_+^{n_x}$, and $\mu \in \mathbb{R}$. Define*

$$\begin{aligned} \Pi: [0, 1] &\rightarrow \mathbb{S}_+^{n_x + n_x} \\ z &\mapsto \begin{bmatrix} e^{-\mu z} P_1 & P_2^\top \\ \bullet & P_3 \end{bmatrix} \end{aligned} \quad (8)$$

Then, for all $(\varepsilon, \eta) \in \text{dom } \mathcal{A}$ one has

$$\begin{aligned} 2 \int_0^1 \left\langle \Pi(z) \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix}, \mathcal{A} \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x + n_x}} dz = \\ \int_0^1 \left\langle \begin{bmatrix} \varepsilon(z) \\ \varepsilon(1) \\ \eta \end{bmatrix}, \Phi(z) \begin{bmatrix} \varepsilon(z) \\ \varepsilon(1) \\ \eta \end{bmatrix} \right\rangle_{\mathbb{R}^{2n_x + n_x}} dz \end{aligned} \quad (9)$$

where Φ is defined in (10) (at the top of the next page).

$$\Phi(z) = \begin{bmatrix} e^{-\mu z}(-\mu\Lambda P_1 - \text{He}(F^\top P_1)) & -P_2^\top LM & P_2^\top A - F^\top P_2^\top \\ \bullet & -\Lambda P_1 e^{-\mu z} & -\Lambda P_2^\top - M^\top L^\top P_3 \\ \bullet & \bullet & \text{He}(P_3 A + P_2 \Lambda C) + C^\top \Lambda P_1 C \end{bmatrix} \quad (10)$$

PROOF. Pick any $(\varepsilon, \eta) \in \mathcal{Z}$. Then, one has

$$2 \int_0^1 \left\langle \Pi(z) \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix}, \mathcal{A} \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x+n_x}} = \Theta_1 + \Theta_2 + \Theta_3$$

where:

$$\begin{aligned} \Theta_1 &:= -2 \int_0^1 \left\langle \frac{d}{dz} \varepsilon(z), \Lambda P_1 \varepsilon(z) \right\rangle_{\mathbb{R}^{n_x}} e^{-\mu z} dz \\ &\quad - 2 \int_0^1 \langle \varepsilon(z), P_1 F \varepsilon(z) \rangle_{\mathbb{R}^{n_x}} e^{-\mu z} dz \\ \Theta_2 &:= 2 \int_0^1 \langle P_2^\top A \eta, \varepsilon(z) \rangle_{\mathbb{R}^{n_x}} dz \\ &\quad - 2 \int_0^1 \langle P_2^\top LM \varepsilon(1), \varepsilon(z) \rangle_{\mathbb{R}^{n_x}} dz \\ &\quad - 2 \int_0^1 \langle \Lambda P_2^\top \eta, \frac{d}{dz} \varepsilon(z) \rangle_{\mathbb{R}^{n_x}} dz \\ &\quad - 2 \int_0^1 \langle F^\top P_2^\top \eta, \varepsilon(z) \rangle_{\mathbb{R}^{n_x}} dz \\ \Theta_3 &:= 2 \langle \eta, P_3 A \eta \rangle_{\mathbb{R}^{n_x}} - 2 \langle \eta, P_3 LM \varepsilon(1) \rangle_{\mathbb{R}^{n_x}} \end{aligned}$$

To conclude the proof, next we develop Θ_1 , Θ_2 , and Θ_3 .

By integrating by parts the first term in Θ_1 and using the fact that $(\varepsilon, \eta) \in \text{dom } \mathcal{A}$, one gets

$$\begin{aligned} \Theta_1 &= -\varepsilon(1)^\top \Lambda P_1 \varepsilon(1) e^{-\mu} + \eta^\top C^\top \Lambda P_1 C \eta \\ &\quad - \mu \int_0^1 \varepsilon(z)^\top \Lambda P_1 \varepsilon(z) e^{-\mu z} dz \\ &\quad - 2 \int_0^1 \varepsilon(z)^\top P_1 F \varepsilon(z) e^{-\mu z} dz \end{aligned}$$

Concerning Θ_2 , using the elementary identity

$$\int_0^1 \frac{d}{dz} \varepsilon(z) dz = \varepsilon(1) - \varepsilon(0)$$

and using the fact that $(\varepsilon, \eta) \in \text{dom } \mathcal{A}$, it turns out that

$$\begin{aligned} \Theta_2 &= 2 \int_0^1 \eta^\top A^\top P_2 \varepsilon(z) dz \\ &\quad - 2 \int_0^1 \varepsilon(1)^\top M^\top L^\top P_2 \varepsilon(z) dz \\ &\quad - 2 \eta^\top P_2 \Lambda \varepsilon(1) + 2 \eta^\top P_2 \Lambda C \eta \\ &\quad - 2 \int_0^1 \eta^\top P_2 F \varepsilon(z) dz \end{aligned}$$

Finally, notice that $\Theta_3 = \eta^\top \text{He}(P_3 A) \eta - 2 \eta^\top P_3 LM \varepsilon(1)$. Combining the expressions of Θ_1 , Θ_2 , and Θ_3 , straightforward manipulations enable to rewrite $\Theta_1 + \Theta_2 + \Theta_3$ as in (9). Thus the result is established. \blacksquare

The result given next provides sufficient conditions in the form of matrix inequalities for the solution to Problem 1.

Theorem 1 *Let $\lambda > 0$ be given. If there exist $P_1 \in \mathbb{D}_+^{n_x}$, $P_2 \in \mathbb{R}^{n_x \times n_x}$, $P_3 \in \mathbb{S}_+^{n_x}$, $L \in \mathbb{R}^{n_x \times n_u}$, and $\mu \in \mathbb{R}_{>0}$ such that:*

$$\begin{bmatrix} P_1 e^{-\mu} & P_2^\top \\ \bullet & P_3 \end{bmatrix} \succ 0 \quad (11a)$$

$$\Phi(1) + 2\lambda \begin{bmatrix} P_1 e^{-\mu} & 0 & P_2^\top \\ \bullet & 0 & 0 \\ \bullet & \bullet & P_3 \end{bmatrix} \preceq 0 \quad (11b)$$

Then, L solves Problem 1. In particular, any maximal solution to (6) satisfies (7) with $\kappa = \sqrt{\frac{\alpha_2}{\alpha_1}}$ where

$$\alpha_1 := \lambda_{\min} \left(\begin{bmatrix} P_1 e^{-\mu} & P_2^\top \\ \bullet & P_3 \end{bmatrix} \right), \quad \alpha_2 := \lambda_{\max} \left(\begin{bmatrix} P_1 & P_2^\top \\ \bullet & P_3 \end{bmatrix} \right) \quad (12)$$

PROOF. Let $z \mapsto \Pi(z)$ be defined as in (8). Consider the following Lyapunov functional candidate on \mathcal{Z} :

$$V(\varepsilon, \eta) := \int_0^1 \left\langle \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix}, \Pi(z) \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x+n_x}} dz \quad (13)$$

In particular, observe that for all $(\varepsilon, \eta) \in \mathcal{Z}$ one has

$$\alpha_1 \|(\varepsilon, \eta)\|_{\mathcal{Z}}^2 \leq V(\varepsilon, \eta) \leq \alpha_2 \|(\varepsilon, \eta)\|_{\mathcal{Z}}^2 \quad (14)$$

where α_1 and α_2 are strictly positive thanks to (11a). As a first step, we show that exponential convergence of the estimation error holds true for any initial condition in $\text{dom } \mathcal{A}$. Pick a solution $\mathcal{I} \ni t \mapsto \psi(t) := (\psi_\varepsilon(t), \psi_\eta(t))$ to (6) and assume that $\psi(0) \in \text{dom } \mathcal{A}$. Then, as already discussed in the proof of Proposition 1, \mathcal{A} generates a \mathcal{C}_0 -semigroup on the Hilbert space \mathcal{Z} , from [3, Proposition 3.1.9, item (h), page 112], one has that ψ is

a classical solution to (6). More precisely, one has that $\psi \in C^1(\text{int } \mathcal{I}, \mathcal{Z})$, $\psi(t) \in \text{dom } \mathcal{A}$ for all $t \in \mathcal{I}$, and

$$\dot{\psi}(t) = \mathcal{A}\psi(t) \quad \forall t \in \text{int } \mathcal{I} \quad (15)$$

For all $t \in \text{int } \mathcal{I}$, ψ being differentiable, one has

$$\dot{V}(t) := \frac{d}{dt}V(\psi(t)) = DV(\psi(t))\dot{\psi}(t)$$

where the above identity follows from the chain rule for the Fréchet derivative. Using Lemma A.5 in the appendix, one gets

$$\dot{V}(t) = 2 \int_0^1 \left\langle \Pi(z) \begin{bmatrix} (\psi_\varepsilon(t))(z) \\ \psi_\eta(t) \end{bmatrix}, \mathcal{A} \begin{bmatrix} (\psi_\varepsilon(t))(z) \\ \psi_\eta(t) \end{bmatrix} \right\rangle_{\mathbb{R}^{2n_x+n_x}} dz$$

By invoking Proposition 2, it follows that for all $t \in \text{int } \mathcal{I}$

$$\dot{V}(t) = \int_0^1 \left\langle \begin{bmatrix} (\psi_\varepsilon(t))(z) \\ (\psi_\varepsilon(t))(1) \\ \psi_\eta(t) \end{bmatrix}, \Phi(z) \begin{bmatrix} (\psi_\varepsilon(t))(z) \\ (\psi_\varepsilon(t))(1) \\ \psi_\eta(t) \end{bmatrix} \right\rangle_{\mathbb{R}^{2n_x+n_x}} dz \quad (16)$$

For all $z \in [0, 1]$, define

$$\Xi(z) := \Phi(z) + 2\lambda \begin{bmatrix} P_1 e^{-\mu z} & 0 & P_2^\top \\ \bullet & 0 & 0 \\ \bullet & \bullet & P_3 \end{bmatrix} \quad (17)$$

and observe that the matrix on the lefthand side of (11b) corresponds to $\Xi(1)$. Using (16), for all $t \in \text{int } \mathcal{I}$, one has:

$$\dot{V}(t) + 2\lambda V(\psi(t)) = \int_0^1 \left\langle \begin{bmatrix} (\psi_\varepsilon(t))(z) \\ (\psi_\varepsilon(t))(1) \\ \psi_\eta(t) \end{bmatrix}, \Xi(z) \begin{bmatrix} (\psi_\varepsilon(t))(z) \\ (\psi_\varepsilon(t))(1) \\ \psi_\eta(t) \end{bmatrix} \right\rangle_{\mathbb{R}^{2n_x+n_x}} dz \quad (18)$$

At this stage, notice that from (11b), it follows that for $z \in [0, 1]$, $\Xi(z) \preceq \Xi(1) \preceq 0$. This implies, thanks to (18), that for all $t \in \text{int } \mathcal{I}$, $\dot{V}(t) + 2\lambda V(\psi(t)) \leq 0$. The latter, due to (14), by standard manipulations, yields for all $t \in \mathcal{I}$

$$\|\psi(t)\|_{\mathcal{Z}} \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\lambda t} \|\psi(0)\|_{\mathcal{Z}} \quad (19)$$

which reads as (7) with $\kappa = \sqrt{\frac{\alpha_2}{\alpha_1}}$.

Now we conclude the proof by showing that the above result holds true also for “mild” solutions to (6). Let $\psi \in C_0(\mathcal{I}, \mathcal{Z})$ be a solution to (6). Then, since, as stated in

Lemma A.1, $\text{dom } \mathcal{A}$ is dense in \mathcal{Z} , there exists a sequence $\{\psi_k^0\} \subset \text{dom } \mathcal{A}$ such that

$$\|\psi_k^0 - \psi(0)\|_{\mathcal{Z}} \xrightarrow{k \rightarrow \infty} 0 \quad (20)$$

Let $\mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{T}(t)$ be the C_0 -semigroup generated by \mathcal{A} on \mathcal{Z} . For each $k \in \mathbb{N}$, define

$$t \mapsto \psi_k(t) := \mathcal{T}(t)\psi_k^0 \quad t \in \mathcal{I}$$

Then, from [10, Theorem 2.1.10, items (a), (b), page 21] for all $k \in \mathbb{N}$

$$\begin{aligned} \psi_k &\in C^1(\text{int } \mathcal{I}, \mathcal{Z}) \\ \psi_k(t) &\in \text{dom } \mathcal{A} \quad \forall t \in \mathcal{I} \\ \dot{\psi}_k(t) &= \mathcal{A}\psi_k(t) \quad \forall t \in \text{int } \mathcal{I} \end{aligned}$$

This shows that for each $k \in \mathbb{N}$, ψ_k is a classical solution to (6). As such, for all $k \in \mathbb{N}$, (19) holds for ψ_k . In particular, for all $k \in \mathbb{N}$, $t \in \mathcal{I}$, one has

$$\|\psi_k(t)\|_{\mathcal{Z}} \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\lambda t} \|\psi_k(0)\|_{\mathcal{Z}} \quad (21)$$

Moreover, it can be easily shown that (20) implies that for all $t \in \mathcal{I}$

$$\|\psi_k(t) - \psi(t)\|_{\mathcal{Z}} \xrightarrow{k \rightarrow \infty} 0 \quad (22)$$

Therefore, from (21), taking the limit for $k \rightarrow \infty$ and by using (20)-(22) one gets (19). This shows that (19) holds for all solutions to (6). Hence, the result is established. ■

The above result establishes sufficient conditions for the solution to Problem 1. A natural question is whether the satisfaction of (11b) relies on some detectability assumptions on the system data. A positive answer is given by the following result:

Proposition 3 *Let $\lambda > 0$ be given. If there exist $P_1 \in D_+^{n_x}$, $P_2 \in \mathbb{R}^{n_x \times n_x}$, $P_3 \in S_+^{n_x}$, $L \in \mathbb{R}^{n_x \times n_y}$, and $\mu \in \mathbb{R}_{>0}$ such that (11b) holds. Then, the pair (A, MC) is λ -detectable, i.e., there exists L such that $\text{spec}(A - LMC) \subset \{z \in \mathbb{C} : \Re(z) \leq \lambda\}$.*

² This property follows directly from the fact that any solution ψ to (6) (nonnecessarily differentiable) satisfies

$$\psi(t) = \mathcal{T}(t)\psi(0) \quad \forall t \in \text{dom } \psi$$

where \mathcal{T} is the C_0 semigroup generated by \mathcal{A} on \mathcal{Z} ; see [3, Proposition 3.1.11, page 115].

PROOF. Assume that (11b) holds. Define

$$S := \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & -\mathbf{I} & C \\ 0 & 0 & \mathbf{I} \end{bmatrix}$$

Hence, using (11b), it follows that:

$$0 \succeq S^T \left(\Phi(1) + 2\lambda \begin{bmatrix} P_1 e^{-\mu} & 0 & P_2^T \\ \bullet & 0 & 0 \\ \bullet & \bullet & P_3 \end{bmatrix} \right) S =$$

$$\begin{bmatrix} -e^{-\mu}(\mu\Lambda P_1 + \text{He}(F^T P_1)) & P_2^T L M & P_2^T A - F^T P_2^T - P_2^T L M C \\ \bullet & -\Lambda P_1 e^{-\mu} & \Lambda P_1 e^{-\mu} C + \Lambda P_2^T + M^T L^T P_3 \\ \bullet & \bullet & (1 - e^{-\mu})C^T \Lambda P_1 C + \text{He}(P_3(A - L M C)) \end{bmatrix}$$

$$+ 2\lambda \begin{bmatrix} P_1 e^{-\mu} & 0 & P_2^T \\ \bullet & 0 & 0 \\ \bullet & \bullet & P_3 \end{bmatrix}$$

By inspecting the righthand side of the above inequality, it turns out that:

$$(1 - e^{-\mu})C^T \Lambda P_1 C + \text{He}(P_3(A - L M C)) + 2\lambda P_3 \preceq 0$$

Now, observe that since $\Lambda P_1 \succ 0$ and $\mu > 0$, the above inequality implies that:

$$\text{He}(P_3(A - L M C)) + 2\lambda P_3 \preceq 0$$

This shows that the eigenvalues of $A - L M C$ are contained in the set

$$\{z \in \mathbb{C} : \Re(z) \leq \lambda\}$$

that is, $(A, M C)$ is λ -detectable, thereby concluding the proof. \blacksquare

Remark 5 Let us notice that, as long as the matrix F is diagonal and positive definite, and the matrix A is Hurwitz, inequality (11a) is trivially satisfied with $L = 0$, $P_2 = 0$, and a sufficiently large $\mu > 0$. This scenario is considered, e.g., in [7].

Remark 6 As opposed to [7], the feasibility of (11b) does not require A to be Hurwitz. This aspect is connected to the selection of the non-block diagonal structure of the Lyapunov functional (13). More specifically, by taking a diagonal functional, i.e., by enforcing $P_2 = 0$, the (3, 3) block of (11b) reads as $\text{He}(P_3 A) + C^T \Lambda P_1 C + \lambda P_3$. This clearly points out that, when $P_2 = 0$, the feasibility of (11b) implies that A is Hurwitz.

4 Observer Design

Condition (11b) is a bilinear matrix inequality (BMI). As such, numerical solution to (11b) can be challenging when the size of the unknown matrices gets large; see [30]. This aspect is connected to the selection of the

non-block diagonal structure of the Lyapunov functional (13). To overcome this drawback, next we propose sufficient conditions for the solution to Problem 1 involving a set of LMIs coupled with a line search on two scalar parameters. The main advantage of this approach is that LMIs can be efficiently solved via interior point methods in polynomial time; see [5] for more details on the use of LMIs in systems and control.

Proposition 4 Let $\lambda \in \mathbb{R}_{>0}$ be given. If there exist $P_1 \in \mathbb{D}_+^{n_x}$, $P_2 \in \mathbb{R}^{n_x \times n_x}$, $P_3 \in \mathbb{S}_+^{n_x}$, $J \in \mathbb{R}^{n_x \times n_y}$, $\vartheta \in \mathbb{R}_{>0}$, and $\mu \in \mathbb{R}_{>0}$ such that:

$$\Sigma := \begin{bmatrix} \Upsilon & \Gamma & Q^T J^T \\ \star & -\vartheta P_3 & 0 \\ \star & \star & -\frac{1}{\vartheta} P_3 \end{bmatrix} \preceq 0 \quad (23a)$$

where

$$\Upsilon := \begin{bmatrix} -e^{-\mu}(\mu\Lambda P_1 + \text{He}(F^T P_1)) & 0 & P_2^T A - F^T P_2^T \\ \bullet & -\Lambda P_1 e^{-\mu} & -\Lambda P_2^T - M^T J^T \\ \bullet & \bullet & \text{He}(P_3 A + P_2 \Lambda C) + C^T \Lambda P_1 C \end{bmatrix}$$

$$+ 2\lambda \begin{bmatrix} e^{-\mu} P_1 & 0 & P_2^T \\ \bullet & 0 & 0 \\ \bullet & \bullet & P_3 \end{bmatrix}, \Gamma := \begin{bmatrix} -P_2^T \\ 0 \\ 0 \end{bmatrix}, Q := \begin{bmatrix} 0 & M & 0 \end{bmatrix} \quad (23b)$$

Then, $P_1, P_2, P_3, L = P_3^{-1} J$ and μ satisfy (11b).

PROOF. The proof hinges upon the statement of Theorem 1, by showing that the hypotheses of the above result imply that $\Xi(1) \prec 0$, where Ξ is defined in (17).

As a first step we show that (23a) implies

$$\Psi := \Upsilon + \frac{1}{\vartheta} \Gamma P_3^{-1} \Gamma^T + \vartheta Q^T L^T P_3 L Q \preceq 0 \quad (24)$$

To this end, observe that, by defining $R := \mathbf{I} \oplus \mathbf{I} \oplus P_3^{-1}$, it follows that $\Sigma = R \widehat{\Sigma} R$, where:

$$\widehat{\Sigma} := \begin{bmatrix} \Upsilon & \Gamma & Q^T L^T \\ \star & -\vartheta P_3 & 0 \\ \star & \star & -\frac{1}{\vartheta} P_3^{-1} \end{bmatrix}$$

This shows that (23a) is equivalent to $\widehat{\Sigma} \preceq 0$, which in turn, by Schur complement, implies that (24) holds. To conclude the proof, notice that $\Xi(1) = \Upsilon + \text{He}(\Gamma L Q)$ and recall that (cf. [24, Claim 1]) for any $T \in \mathbb{S}_+^{n_x}$

$$\text{He}(\Gamma L Q) \preceq \Gamma T \Gamma^T + Q^T L^T T^{-1} L Q$$

Picking $T = \frac{1}{\vartheta} P_3^{-1}$, the above relationship yields $\Xi(1) \preceq \Psi$, which thanks to (24) gives $\Xi(1) \preceq 0$. This concludes the proof. \blacksquare

5 Numerical Example

Consider a system of the form (1) defined by the following data

$$\Lambda = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} -0.5 & 0.1 \\ 0.5 & 0.1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

It is worth mentioning that, since A is not Hurwitz,

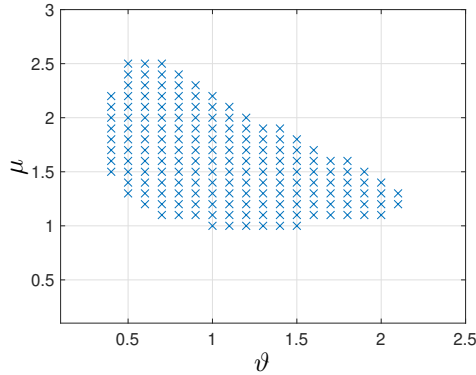


Fig. 2. Feasible set projected onto the plan (ϑ, μ) . Crosses stands for feasible points.

the results in [7] cannot be used to design a boundary observer in this case; see Remark 6. Now we show how Proposition 4 enables to overcome this limitation. For design purposes, we select $\lambda = \frac{1}{2} \times 0.03$ and solve (11b)–(23a) by performing a line search on the parameters μ and ϑ on a grid of 900 points uniformly taken over the box $[0.1, 3] \times [0.1, 3]$. In Fig. 2 we report the set of feasible points for the selected grid. With the objective of improving the transient response of the observer, a possible approach consists of selecting among all feasible solutions, the one characterized by the smallest value of the overshoot $\kappa = \sqrt{\frac{\alpha_2}{\alpha_1}}$ in (7) where α_1 and α_2 are defined in (12). Indeed, as pointed out by Fig. 3, the value of κ depends on the value of μ and ϑ . In particular, the smallest value of κ can be achieved by selecting $\mu = 1.3$ and $\vartheta = 1$. For those values, the solution to the (11b)–

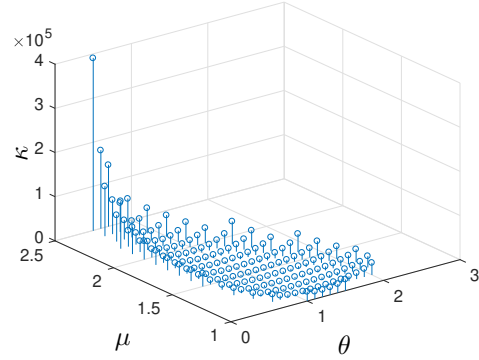


Fig. 3. Value of the overshoot κ as a function of (θ, μ) .

(23a) reads as:

$$\begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} 0.0040 & 0 & -0.0026 & -0.0227 & 0.18 \\ 0 & 0.0192 & -0.0029 & -0.0296 & 0.1654 \\ -0.0026 & -0.0029 & 0.0376 & -0.1892 & -3.332 \\ -0.0227 & -0.0296 & -0.1892 & 6.815 & -55.61 \\ 0.18 & 0.1654 & -3.332 & -55.61 & 2755 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 0.2287 & 0.0156 & 0.0005 \end{bmatrix}$$

In Fig. 4, we report the evolution of the Lyapunov functional (13) along the solution to (6) from the following initial condition³:

$$\begin{aligned} x_1(0, z) &= 0.1(\cos(2\pi z) - 1) \quad \forall z \in [0, 1] \\ x_2(0, z) &= -0.1(\cos(4\pi z) - 1) \quad \forall z \in [0, 1] \\ \chi(0) &= (0.1, -0.1, 0.2) \\ \hat{x}(0, z) &= (0, 0) \quad \forall z \in [0, 1] \\ \hat{\chi}(0) &= (0, 0, 0) \end{aligned} \quad (25)$$

As expected, V exponentially converges to zero, hence ensuring that the estimation error (ε, η) converges to zero exponentially. Moreover, the picture points out that in this simulation the exponential upper bound on V is very tight. Exponential state reconstruction is confirmed by Fig. 5 and Fig. 6 where the evolution of ε and η , respectively, are reported. For the sake of comparison, in Fig. 7 we portrait the evolution of $\|(\varepsilon, \eta)\|_{\mathcal{Z}}$ for the initial condition in (25) obtained by considering different selections of the variables θ and μ . The picture clearly emphasizes that minimizing the overshoot κ leads to an improved response. In particular, among the three considered pairs, the best pair of parameters seems to be

³ Numerical integration of hyperbolic PDEs is performed via the use of the Lax-Friedrichs (Shampine's two-step variant) scheme implemented in Matlab[®] by Shampine [27]. Code available at <https://github.com/f-ferrante/AUT190observerHypODE>

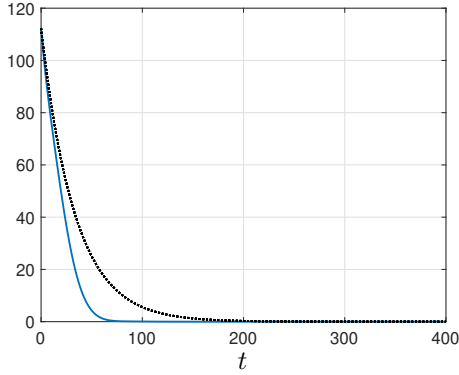


Fig. 4. Evolution of the Lyapunov functional (13) along the solution to (6) from the initial condition in (25) (solid line) the upper bound $e^{-2\lambda t}V(\varepsilon(0), \eta(0))$ (dotted line).

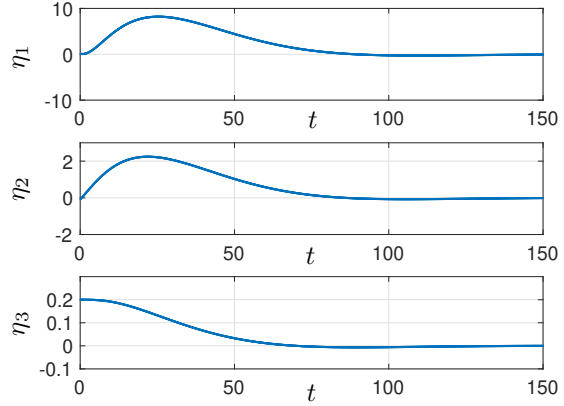


Fig. 6. Evolution of the error η for the initial condition in (25).

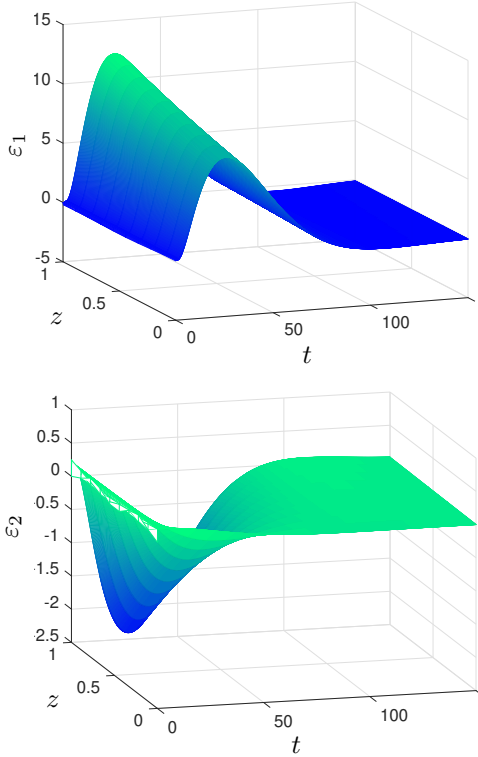


Fig. 5. Evolution of the error ε for the initial condition in (25).

$(\vartheta, \mu) = (1, 1.3)$, which is consistent with Fig. 3. In this case, the estimation error converges close to zero in about 90 seconds. We believe that this large convergence time is mostly due to the estimation error overshoot caused by the unstable boundary dynamics. This overshoot is likely to be generated by the time lag induced by the convective behavior of the ε -dynamics. A possible approach to mitigate this unwanted behavior consists of adding an additional injection term in the domain of the observer. However, this extension is beyond the scope of our paper.

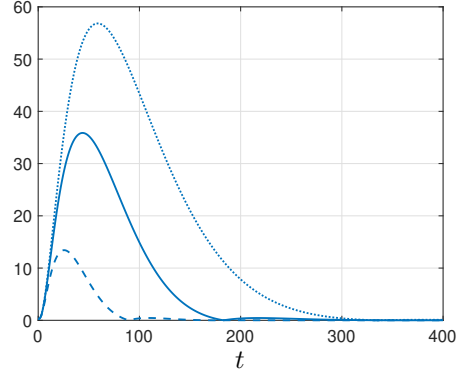


Fig. 7. Evolution of $\|(\varepsilon, \eta)\|_{\mathcal{Z}}$ for the initial condition in (25) obtained for different designs: $(\vartheta, \mu) = (1, 1.3)$ (dashed line), $(\vartheta, \mu) = (0.5, 2.2)$ (solid line), and $(\vartheta, \mu) = (1, 2.2)$ (dotted line).

6 Conclusion

In this paper, we considered the problem of designing an observer to estimate the state of a coupled ODEs–hyperbolic PDEs system. The error dynamics are analyzed via abstract differential equations tools. The observer is designed to induce global exponential stability of the error dynamics with respect to a specific norm and with a prescribed convergence rate. By pursuing a Lyapunov approach, the observer design problem is recast into the feasibility problem of some bilinear matrix inequalities. Then, such conditions are exploited to devise an observer design algorithm based on the solution to some LMIs coupled to a line search on two scalars. Numerical simulations are used to illustrate the effectiveness of the proposed design strategy in an example. Future research directions include the use of an in-domain injection to improve the transient behavior of the observer.

A Technical Results

A.1 Technical Results for Section 2.3

In this subsection, we provide some auxiliary results that are used to build the proof of Proposition 1.

Lemma A.1 *The set*

$$\mathcal{D} := \{(\varepsilon, \eta) \in \mathcal{X} : \varepsilon(0) = C\eta\} \quad (\text{A.1})$$

is dense in \mathcal{Z} .

PROOF. Let $f = (f_\varepsilon, f_\eta) \in \mathcal{Z}$. Then, since $\mathcal{C}_c^\infty(0, 1; \mathbb{R}^{n_x})$ is dense in $\mathcal{L}^2(0, 1; \mathbb{R}^{n_x})$, there exists a sequence $\{\varepsilon_k\} \subset \mathcal{C}_c^\infty(0, 1; \mathbb{R}^{n_x})$ such that

$$\|\varepsilon_k - f_\varepsilon\|_{\mathcal{L}^2} \rightarrow 0$$

Define the following sequence in $\mathcal{H}^1(0, 1; \mathbb{R}^{n_x})$

$$\eta_k(z) := \begin{cases} (1 - kz)Cf_\eta & \text{if } z \in (0, \frac{1}{k}) \\ 0 & \text{if } z \in [\frac{1}{k}, 1] \end{cases}$$

and let $\omega_k := \eta_k + \varepsilon_k$. Obviously, since for all $k \in \mathbb{N}$, $\varepsilon_k(0) = \varepsilon_k(1) = 0$ and $\eta_k, \varepsilon_k \in \mathcal{H}^1(0, 1; \mathbb{R}^{n_x})$, it follows that $\{(\omega_k, f_\eta)\} \subset \mathcal{D}$. Moreover one has for each $k \in \mathbb{N}$

$$\|(\omega_k, f_\eta) - f\|_{\mathcal{Z}} = \|\omega_k - f_\varepsilon\|_{\mathcal{L}^2(0, 1; \mathbb{R}^{n_x})}$$

hence, since by construction

$$\|\omega_k - f_\varepsilon\|_{\mathcal{L}^2} \rightarrow 0$$

one has

$$\|(\omega_k, f_\eta) - f\|_{\mathcal{Z}} \rightarrow 0$$

Thereby concluding the proof. \blacksquare

Lemma A.2 *Let \mathcal{A} be defined as in (5). Then, \mathcal{A} is closed in \mathcal{Z} .*

PROOF. The proof builds on the general ideas contained in the proof of [4, Theorem A.1, page 244]. Pick $\{(\varepsilon_k, \eta_k)\} \subset \text{dom } \mathcal{A}$ and assume that

$$(\varepsilon_k, \eta_k) \xrightarrow[k \rightarrow \infty]{\mathcal{Z}} (\varepsilon, \eta) \quad (\text{A.2})$$

$$\mathcal{A} \begin{pmatrix} \varepsilon_k \\ \eta_k \end{pmatrix} \xrightarrow[k \rightarrow \infty]{\mathcal{Z}} \begin{pmatrix} y_\varepsilon \\ y_\eta \end{pmatrix} \quad (\text{A.3})$$

we show that

$$(\varepsilon, \eta) \in \text{dom } \mathcal{A} \\ \begin{pmatrix} y_\varepsilon \\ y_\eta \end{pmatrix} = \mathcal{A} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}$$

First observe that for all $k \in \mathbb{N}$

$$\mathcal{A} \begin{pmatrix} \varepsilon_k \\ \eta_k \end{pmatrix} = \begin{pmatrix} -\Lambda \frac{d}{dz} \varepsilon_k(z) - F\varepsilon_k(z) \\ A\eta_k \end{pmatrix} + \begin{pmatrix} 0 \\ -LM\varepsilon_k(1) \end{pmatrix}$$

hence, from (A.2)-(A.3) it follows that

$$\Lambda \frac{d}{dz} \varepsilon_k \xrightarrow[k \rightarrow \infty]{\mathcal{L}^2(0, 1; \mathbb{R}^{n_x})} F\varepsilon - y_\varepsilon \quad (\text{A.4})$$

In particular, the above identity shows that $\frac{d}{dz} \varepsilon_k$ converges in $\mathcal{L}^2(0, 1; \mathbb{R}^{n_x})$. Thus, since $\{\varepsilon_k\} \subset \mathcal{H}^1(0, 1; \mathbb{R}^{n_x})$, and $\frac{d}{dz} \varepsilon_k$ and ε_k converge in $\mathcal{L}^2(0, 1; \mathbb{R}^{n_x})$, it follows that

$$\frac{d}{dz} \varepsilon_k \xrightarrow[k \rightarrow \infty]{\mathcal{L}^2(0, 1; \mathbb{R}^{n_x})} \frac{d}{dz} \varepsilon$$

thereby implying that

$$\varepsilon_k \xrightarrow[k \rightarrow \infty]{\mathcal{H}^1(0, 1; \mathbb{R}^{n_x})} \varepsilon \quad (\text{A.5})$$

Now observe that since, for each $k \in \mathbb{N}$, $(\varepsilon_k, \eta_k) \in \text{dom } \mathcal{A}$, one has that

$$\varepsilon_k(0) = C\eta_k \quad (\text{A.6})$$

which gives

$$\lim_{k \rightarrow \infty} \varepsilon_k(0) = C\eta$$

Moreover, from [6, Theorem 8.8, page 212] one has that there exists $\varsigma \in \mathbb{R}$ such that for all k

$$\|\varepsilon_k - \varepsilon\|_\infty \leq \varsigma \|\varepsilon_k - \varepsilon\|_{\mathcal{H}^1(0, 1; \mathbb{R}^{n_x})}$$

This implies

$$\lim_{k \rightarrow \infty} \|\varepsilon_k - \varepsilon\|_\infty = 0 \quad (\text{A.7})$$

and the latter, due to $\varepsilon, \varepsilon_k \in \mathcal{H}^1(0, 1; \mathbb{R}^{n_x})$, gives

$$\lim_{k \rightarrow \infty} \varepsilon_k(0) = \varepsilon(0)$$

which, using (A.6), yields $\varepsilon(0) = C\eta$. This shows that $(\varepsilon, \eta) \in \text{dom } \mathcal{A}$. To conclude the proof, we show that

$$\begin{pmatrix} y_\varepsilon \\ y_\eta \end{pmatrix} = \mathcal{A} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}$$

Consider the following partition

$$\mathcal{A} \begin{pmatrix} \varepsilon_k \\ \eta_k \end{pmatrix} =: \begin{pmatrix} y_{\varepsilon_k} \\ y_{\eta_k} \end{pmatrix}$$

Obviously, since for all k

$$y_{\eta_k} = A\eta_k - LM\varepsilon_k(1)$$

$\eta_k \rightarrow \eta$, and from (A.7) $\varepsilon_k(1) \rightarrow \varepsilon(1)$, it follows that

$$y_\eta = A\eta - LM\varepsilon(1)$$

Moreover, from (A.4)

$$0 = \lim_{k \rightarrow \infty} \|y_{\varepsilon_k} - y_\varepsilon\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})} \quad (\text{A.8})$$

At this stage, for all k define

$$\delta_k := F(\varepsilon - \varepsilon_k) + \Lambda \frac{d}{dz}(\varepsilon - \varepsilon_k)$$

and observe that from (A.5)

$$\delta_k \xrightarrow[k \rightarrow \infty]{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})} 0 \quad (\text{A.9})$$

Consider the following rewriting of (A.8):

$$0 = \lim_{k \rightarrow \infty} \left\| \delta_k - \Lambda \frac{d}{dz}\varepsilon - F\varepsilon + y_\varepsilon \right\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})}$$

by using standard arguments, the relation above yields

$$0 = \lim_{k \rightarrow \infty} \left\| \delta_k \right\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})} - \left\| \Lambda \frac{d}{dz}\varepsilon + F\varepsilon - y_\varepsilon \right\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})}$$

using (A.9) it turns out that

$$0 = \left\| \Lambda \frac{d}{dz}\varepsilon + F\varepsilon - y_\varepsilon \right\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})}$$

Namely

$$-\Lambda \frac{d}{dz}\varepsilon - F\varepsilon \Big|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_x})} = y_\varepsilon$$

and this concludes the proof. \blacksquare

In the result given next, we determine the adjoint of the operator \mathcal{A} . Such an adjoint is used in Lemma A.4

Lemma A.3 *Let*

$$\mathcal{D}^* := \{(\varepsilon^*, \eta^*) \in \mathcal{X} : \varepsilon^*(1) = -\Lambda^{-1}M^\top L^\top \eta^*\} \quad (\text{A.10})$$

For the operator \mathcal{A} defined in (5), one has

$$\mathcal{A}^* : \text{dom } \mathcal{A}^* \rightarrow \mathcal{Z} \\ (\varepsilon^*, \eta^*) \mapsto \begin{pmatrix} \Lambda \frac{d}{dz}\varepsilon^* - F^\top \varepsilon^* \\ A^\top \eta^* + C^\top \Lambda \varepsilon^*(0) \end{pmatrix} \quad (\text{A.11})$$

with $\text{dom } \mathcal{A}^* = \mathcal{D}^*$.

PROOF. As a first step, define

$$\widehat{\mathcal{A}} : \mathcal{D} \rightarrow \mathcal{Z} \\ (\varepsilon, \eta) \mapsto \begin{pmatrix} -\Lambda \frac{d}{dz}\varepsilon \\ -LM\varepsilon(1) \end{pmatrix} \\ \mathcal{T} : \mathcal{Z} \rightarrow \mathcal{Z} \\ (\varepsilon, \eta) \mapsto \begin{pmatrix} -F\varepsilon \\ A\eta \end{pmatrix}$$

where the set \mathcal{D} is defined in (A.1) and observe that $\mathcal{A} = \widehat{\mathcal{A}} + \mathcal{T}$. Therefore, since \mathcal{T} is bounded and $\text{dom } \mathcal{T} = \mathcal{Z}$, from [10, Lemma A.3.65, page 603] it follows that $\mathcal{A}^* = \widehat{\mathcal{A}}^* + \mathcal{T}^*$ with $\text{dom}((\mathcal{A} + \mathcal{T})^*) = \text{dom } \widehat{\mathcal{A}}^*$. To determine \mathcal{T}^* , pick $\theta := (\varepsilon, \eta), \theta^* := (\varepsilon^*, \eta^*) \in \mathcal{Z}$ and notice that

$$\langle \mathcal{T}\theta, \theta^* \rangle_{\mathcal{Z}} = \int_0^1 \langle -F\varepsilon(z), \varepsilon^*(z) \rangle_{\mathbb{R}^{n_x}} dz + \langle A\eta, \eta^* \rangle_{\mathbb{R}^{n_x}}$$

that is

$$\langle \mathcal{T}\theta, \theta^* \rangle_{\mathcal{Z}} = \int_0^1 \langle \varepsilon(z), -F^\top \varepsilon^*(z) \rangle_{\mathbb{R}^{n_x}} dz + \langle \eta, A^\top \eta^* \rangle_{\mathbb{R}^{n_x}}$$

which allows one to conclude that

$$\mathcal{T}^* : \mathcal{Z} \rightarrow \mathcal{Z} \\ (\varepsilon^*, \eta^*) \mapsto \begin{pmatrix} -F^\top \varepsilon^* \\ A^\top \eta^* \end{pmatrix}$$

To conclude, pick $\theta = (\varepsilon, \eta) \in \text{dom } \widehat{\mathcal{A}}$ and observe that

$$\langle \widehat{\mathcal{A}}\theta, \theta^* \rangle_{\mathcal{Z}} = -\int_0^1 \langle \Lambda \frac{d}{dz}\varepsilon(z), \varepsilon^*(z) \rangle_{\mathbb{R}^{n_x}} dz - \langle LM\varepsilon(1), \eta^* \rangle_{\mathbb{R}^{n_x}}$$

Since $(\varepsilon, \eta) \in \text{dom } \widehat{\mathcal{A}}$, by integrating by parts one gets

$$\langle \widehat{\mathcal{A}}\theta, \theta^* \rangle_{\mathcal{Z}} = \langle \eta, C^\top \Lambda \varepsilon^*(0) \rangle_{\mathbb{R}^{n_x}} - \langle \Lambda \varepsilon(1), \varepsilon^*(1) \rangle_{\mathbb{R}^{n_x}} - \langle \varepsilon(1), M^\top L^\top \eta^* \rangle_{\mathbb{R}^{n_x}} + \int_0^1 \langle \varepsilon(z), \Lambda \frac{d}{dz}\varepsilon^*(z) \rangle_{\mathbb{R}^{n_x}} dz$$

The latter allows one to conclude that $\widehat{\mathcal{A}}^*$ corresponds to (A.11), thereby concluding the proof. \blacksquare

The result given next establishes *quasi-dissipativity* for

the operator \mathcal{A} and its adjoint⁴; see [4, Theorem A.1, page 244].

Lemma A.4 *There exists $\omega \in \mathbb{R}$ such that*

$$\begin{aligned} \langle \mathcal{A}\theta, \theta \rangle_{\mathcal{Z}} &\leq \omega \langle \theta, \theta \rangle_{\mathcal{Z}} \quad \forall \theta \in \text{dom } \mathcal{A} \\ \langle \mathcal{A}^*\theta, \theta \rangle_{\mathcal{Z}} &\leq \omega \langle \theta, \theta \rangle_{\mathcal{Z}} \quad \forall \theta \in \text{dom } \mathcal{A}^* \end{aligned} \quad (\text{A.12})$$

PROOF. Let

$$\Pi := \begin{bmatrix} -\frac{1}{2} \text{He}(F) & 0 & 0 \\ \bullet & -\frac{1}{2} \Lambda & \frac{M^T L^T}{2} \\ \bullet & \bullet & \frac{\text{He}(A)}{2} + C^T \Lambda C \end{bmatrix} \quad (\text{A.13})$$

Then, it can be easily shown that for all $\theta = (\varepsilon, \eta) \in \text{dom } \mathcal{A}$

$$\langle \mathcal{A}\theta, \theta \rangle_{\mathcal{Z}} = \int_0^1 \begin{bmatrix} \varepsilon(z) \\ \varepsilon(1) \\ \eta \end{bmatrix}^T \Pi \begin{bmatrix} \varepsilon(z) \\ \varepsilon(1) \\ \eta \end{bmatrix} dz$$

From Lemma A.6, there exists $H \in \mathbb{S}^{n_x}$ such that

$$\Pi \preceq \begin{bmatrix} -\frac{1}{2} \text{He}(F) & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & H \end{bmatrix}$$

Therefore, one has that

$$\langle \mathcal{A}\theta, \theta \rangle_{\mathcal{Z}} \leq \int_0^1 -\langle \varepsilon(z), F\varepsilon(z) \rangle_{\mathbb{R}^{n_x}} dz + \langle \eta, H\eta \rangle_{\mathbb{R}^{n_x}}$$

which implies

$$\langle \mathcal{A}\theta, \theta \rangle_{\mathcal{Z}} \leq \underbrace{\max\{\lambda_{\max}\left(-\frac{1}{2} \text{He}(F)\right), \lambda_{\max}(H)\}}_{\omega_1} \langle \theta, \theta \rangle_{\mathcal{Z}} \quad (\text{A.14})$$

Analogously, it can be shown that that for all $\theta = (\varepsilon, \eta) \in \text{dom } \mathcal{A}^*$

$$\langle \mathcal{A}^*\theta, \theta \rangle_{\mathcal{Z}} = \int_0^1 \begin{bmatrix} \varepsilon(z) \\ \varepsilon(0) \\ \eta \end{bmatrix}^T \Pi_* \begin{bmatrix} \varepsilon(z) \\ \varepsilon(0) \\ \eta \end{bmatrix} dz$$

with

$$\Pi_* := \begin{bmatrix} -\frac{1}{2} \text{He}(F) & 0 & 0 \\ \bullet & -\frac{1}{2} \Lambda & \Lambda C \\ \bullet & \bullet & \frac{\text{He}(A)}{2} + LM\Lambda^{-1}M^T L^T \end{bmatrix}$$

⁴ In the proof of [4, Theorem A.1, page 244], a specific inner product is used to establish quasi-dissipativity properties. In the proof of Lemma A.4, we consider a more standard inner product. This renders the corresponding proof easier to work out.

Thus, by invoking again Lemma A.6, one can conclude that there exists $S_* \in \mathbb{S}^{n_x}$ such that for all $\theta \in \text{dom } \mathcal{A}^*$

$$\langle \mathcal{A}^*\theta, \theta \rangle_{\mathcal{Z}} \leq \underbrace{\max\{\lambda_{\max}\left(-\frac{1}{2} \text{He}(F)\right), \lambda_{\max}(S_*)\}}_{\omega_2} \langle \theta, \theta \rangle_{\mathcal{Z}} \quad (\text{A.15})$$

Finally, by taking $\omega = \max\{\omega_1, \omega_2\}$, using (A.14) and (A.15), one gets (A.12). This concludes the proof. ■

A.2 Definitions and auxiliary results

Definition A.1 ([10]) *Let \mathcal{Z} be a Hilbert space. The function $\mathcal{T}: \mathbb{R}_{\geq 0} \rightarrow \mathcal{L}(\mathcal{Z}, \mathcal{Z})$ is a \mathcal{C}_0 -semigroup on \mathcal{Z} if it satisfies the following properties:*

- (a) For all $t, s \in \mathbb{R}_{\geq 0}$, $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$
- (b) $\mathcal{T}(0) = \mathbf{I}$
- (c) For all $z_0 \in \mathcal{Z}$, $\lim_{t \rightarrow 0^+} \|\mathcal{T}(t)z_0 - z_0\| = 0$

○

Definition A.2 [10] *Let \mathcal{Z} be a Hilbert space and $\mathcal{A}: \text{dom } \mathcal{A} \rightarrow \mathcal{Z}$. We say that \mathcal{A} generates a \mathcal{C}_0 -semigroup \mathcal{T} on \mathcal{Z} if for all $z \in \text{dom } \mathcal{A}$*

$$\mathcal{A}z = \lim_{t \rightarrow 0^+} \frac{1}{t} (\mathcal{T}(t) - \mathbf{I})z$$

○

Definition A.3 ([8]) *Let X and Y be linear normed spaces, U be an open subset of X , $f: U \rightarrow Y$, and $x \in U$. We say that f is Fréchet differentiable at x if there exists $L \in \mathcal{L}(X, Y)$ such that*

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Lh\|_Y}{\|h\|_X} = 0$$

In particular L is the Fréchet derivative of f at x and is denoted by $Df(x)$. When $X = \mathbb{R}$, we denote

$$\dot{f}(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

○

The following lemma is instrumental to compute the Fréchet derivative of Lyapunov functionals.

Lemma A.5 *Let $\Pi \in \mathcal{C}^0([0, 1]; \mathbb{S}^{n_x+n_x})$ and \mathcal{Z} be endowed with the inner product defined in (4). Consider the following functional*

$$V: \mathcal{Z} \rightarrow \mathbb{R}$$

$$(\varepsilon, \eta) \mapsto V(\varepsilon, \eta) := \int_0^1 \left\langle \Pi(z) \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix}, \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x+n_x}} dz$$

Then, V is Fréchet differentiable on \mathcal{Z} . In particular, for each $X := (\varepsilon, \eta)$, $h := (h_\varepsilon, h_\eta) \in \mathcal{Z}$

$$DV(X)h = 2 \int_0^1 \left\langle \Pi(z) \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix}, \begin{bmatrix} h_\varepsilon(z) \\ h_\eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x+n_x}} dz$$

PROOF. Let $X, h \in \mathcal{Z}$. For the sake of convenience, define

$$\mathcal{K}(X, h) := 2 \int_0^1 \left\langle \Pi(z) \begin{bmatrix} \varepsilon(z) \\ \eta \end{bmatrix}, \begin{bmatrix} h_\varepsilon(z) \\ h_\eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x+n_x}}$$

For any $X, h \in \mathcal{Z}$, one has

$$\begin{aligned} V(X+h) - V(X) &= \mathcal{K}(X, h) + \\ &\int_0^1 \left\langle \Pi(z) \begin{bmatrix} h_\varepsilon(z) \\ h_\eta \end{bmatrix}, \begin{bmatrix} h_\varepsilon(z) \\ h_\eta \end{bmatrix} \right\rangle_{\mathbb{R}^{n_x+n_x}} dz \\ &\leq \max_{z \in [0,1]} \lambda_{\max}(\Pi(z)) \|h\|_{\mathcal{Z}} \\ &+ \mathcal{K}(X, h) \end{aligned}$$

Thus, it follows that

$$\lim_{\|h\|_{\mathcal{Z}} \rightarrow 0} \frac{|V(X+h) - V(X) - \mathcal{K}(X, h)|}{\|h\|_{\mathcal{Z}}} = 0$$

This concludes the proof. \blacksquare

Lemma A.6 Let $A \in \mathbb{S}_+^n$, $C \in \mathbb{S}^m$, and $B \in \mathbb{R}^{n \times m}$. Then, there exists $\Gamma \in \mathbb{S}^m$ such that

$$\begin{bmatrix} -A & B \\ \bullet & C \end{bmatrix} \preceq \begin{bmatrix} 0 & 0 \\ \bullet & \Gamma \end{bmatrix} \quad (\text{A.16})$$

PROOF. Pick $\Gamma = C + B^T A^{-1} B$. Then (A.16) holds if and only if

$$\begin{bmatrix} -A & B \\ \bullet & -B^T A^{-1} B \end{bmatrix} \preceq 0 \quad (\text{A.17})$$

In particular, the Schur complement of the lefthand side of (A.17) is 0. Thus, since $A \in \mathbb{S}_+^n$, it follows that (A.17) holds. This concludes the proof. \blacksquare

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