# Asymptotic behaviour for a time-inhomogeneous Kolmogorov type diffusion 

Mihai Gradinaru and Emeline Luirard<br>Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France<br>\{Mihai.Gradinaru, Emeline.Luirard\}@univ-rennes1.fr


#### Abstract

We study a kinetic stochastic model with a non-linear time-inhomogeneous drag force and a Brownian-type random force. More precisely, the Kolmogorov type diffusion ( $V, X$ ) is considered: here $X$ is the position of the particle and $V$ is its velocity and is solution of a stochastic differential equation driven by a one-dimensional Brownian motion, with the drift of the form $t^{-\beta} F(v)$. The function $F$ satisfies some homogeneity condition and $\beta$ is positive. The behaviour of the process $(V, X)$ in large time is proved by using stochastic analysis tools.


Keywords: kinetic stochastic equation; time-inhomogeneous diffusions; explosion times; scaling transformations; asymptotic distributions; ergodicity.
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## 1 Introduction

In several domains as fluids dynamics, statistical mechanics, biology, a number of models are based on the Fokker-Planck and Langevin equations driven by Brownian motion or could be non-linear or driven by other random noises. For example, in CCM10 the persistent turning walker model was introduced, inspired by the modelling of fish motion. An associated two-component Kolmogorov type diffusion solves a kinetic Fokker-Planck equation based on an Ornstein-Uhlenbeck Gaussian process and the authors studied the large time behaviour of this model by using appropriate tools from stochastic analysis. One of the natural questions is the behaviour in large time of the solution to the corresponding stochastic differential equation (SDE). Although the tools of partial differential equations allow us to ask of this kind of questions, since these models are probabilistic, tools based on stochastic processes could be more natural to use.

In the last decade the asymptotic study of solutions of non-linear Langevin's type was the subject of an important number of papers, see CNP19, [EG15, [T21. For instance, in [FT21] the following system is studied

$$
V_{t}=v_{0}+B_{t}-\frac{\rho}{2} \int_{0}^{t} F\left(V_{s}\right) \mathrm{d} s \quad \text { and } \quad X_{t}=x_{0}+\int_{0}^{t} V_{s} \mathrm{~d} s
$$

In other words one considers a particle moving such that its velocity is a diffusion with an invariant measure behaving like $\left(1+|v|^{2}\right)^{-\rho / 2}$, as $|v| \rightarrow+\infty$. The authors prove that for large time, after a suitable rescaling, the position process behaves as a Brownian motion or other stable processes, following the values of $\rho$. Results have been extended to additive functional of
$V$ in Bét21. It should be noticed that these cited papers use the standard tools associated with time-homogeneous equations: invariant measure, scale function and speed measure. Several of these tools will not be available when the drag force is depending explicitly on time. In GO13, a non-linear SDE driven by a Brownian motion but having time-inhomogeneous drift coefficient was studied and its large time behaviour was described. Moreover, sharp rates of convergence are proved for the 1-dimensional marginal of the solution. In the present paper, we consider the velocity process as satisfying the same kind of SDE.

Let us describe our framework: consider a one-dimensional time-inhomogeneous stochastic kinetic model driven by a Brownian motion. We denote by $\left(X_{t}\right)_{t \geq 0}$ the one-dimensional process describing the position of a particle at time $t$ having the velocity $V_{t}$. The velocity process $\left(V_{t}\right)_{t \geq 0}$ is supposed to follow a Brownian dynamic in a potential $U(t, v)$, varying in time:

$$
\mathrm{d} V_{t}=\mathrm{d} B_{t}-\frac{1}{2} \partial_{v} U\left(t, V_{t}\right) \mathrm{d} t \quad \text { and } \quad X_{t}=X_{0}+\int_{0}^{t} V_{s} \mathrm{~d} s
$$

This system can be viewed as a perturbation of the classical two-component Kolmogorov diffusion

$$
\mathrm{d} V_{t}=\mathrm{d} B_{t} \quad \text { and } \quad X_{t}=X_{0}+\int_{0}^{t} V_{s} \mathrm{~d} s
$$

In the present paper the potential is supposed to grow slowly to infinity, and it will be supposed to be of the form $t^{-\beta} \int_{0}^{v} F(u) \mathrm{d} u$, with $\beta>0$ and $F$ satisfying some homogeneity condition. It describes a one dimensional particle evolving in a force field $F t^{-\beta}$ and undergoing many small random shocks. A natural question is to understand the behaviour of the process $(V, X)$ in large time. More precisely we look for the limit in distribution of $v(\varepsilon)\left(V_{t / \varepsilon}, \varepsilon X_{t / \varepsilon}\right)_{t}$, as $\varepsilon \rightarrow 0$, where $v(\varepsilon)$ is some rate of convergence. Our results are proved on the product of path spaces and consequently contain those of GO13.
If $F=0$, it is not difficult to see that the rescaled position process $\left(\varepsilon^{1 / 2} V_{t / \varepsilon}, \varepsilon^{3 / 2} X_{t / \varepsilon}\right)_{t}$ converges in distribution towards the Kolmogorov diffusion $\left(B_{t}, \int_{0}^{t} B_{s} \mathrm{~d} s\right)_{t}$. We prove that this kinetic behaviour still holds for sufficiently "small at infinity" potential. The strategy to tackle this problem is based on estimates of moments of the velocity process. The main result can then be extended for the case when the potential is equally weighted in some sense as the random noise. The potential either offsets the random noise (critical regime) or swings with it (sub-critical regime).
As suggested at the beginning of the introduction, other random noises can be considered. In GL21, the case of a Lévy random noise is analysed. The case of a stochastic system in a harmonic potential is the purpose of a future work (see Lui22]).

The organisation of our paper is as follows: in the next section we introduce notations, and we state our main results. Results about existence and non-explosion of solutions are stated in Section 3. Estimates of the moments of the velocity process are given in Section 4 while the proofs of our main results are presented in Section 5.

## 2 Notations and main results

Let $\left(B_{t}\right)_{t \geq 0}$ be a standard Brownian motion, $\beta$ a real number and $F$ a continuous function which is supposed to satisfy either

$$
\begin{equation*}
\text { for some } \gamma \in \mathbb{R}, \forall v \in \mathbb{R}, \lambda>0, F(\lambda v)=\lambda^{\gamma} F(v) \text {, } \tag{1}
\end{equation*}
$$

$$
|F| \leq G \text { where } G \text { is a positive function satisfying } H_{1}^{\gamma} \text {. }
$$

Each assumption implies that there exist a positive constant $K$ such that, for all $v \in \mathbb{R}$, $|F(v)| \leq K|v|^{\gamma}$. Obviously $\left.H_{2}^{\gamma}\right\rangle$ is a generalisation of $\left.H_{1}^{\gamma}\right\rangle$. In the following, sgn is the sign function with convention $\operatorname{sgn}(0)=0$. As an example of function satisfying $H_{1}^{\gamma}$ one can keep in mind $F: v \mapsto \operatorname{sgn}(v)|v|^{\gamma}$ (see also GO13), and as an example of function satisfying $H_{2}^{\gamma}$ ) (with $\gamma=0) F: v \mapsto v /\left(1+v^{2}\right)$ (see also FT21).
Remark 2.1. If a function $\pi$ satisfies $\left.H_{1}^{\gamma}\right)$, then for all $x \in \mathbb{R}, \pi(x)=\pi(\operatorname{sgn}(x))|x|^{\gamma}$.
We consider the following one-dimensional stochastic kinetic model, for $t \geq t_{0}>0$,

$$
\begin{equation*}
\mathrm{d} V_{t}=\mathrm{d} B_{t}-t^{-\beta} F\left(V_{t}\right) \mathrm{d} t, V_{t_{0}}=v_{0}>0, \quad \text { and } \quad \mathrm{d} X_{t}=V_{t} \mathrm{~d} t, X_{t_{0}}=x_{0} \in \mathbb{R} \tag{SKE}
\end{equation*}
$$

Most of the convergences take place in the space of continuous functions $\mathcal{C}((0,+\infty), \mathbb{R})$ endowed by the uniform topology

$$
\mathrm{d}_{u}: f, g \in \mathcal{C}((0,+\infty), \mathbb{R}) \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^{n}} \min \left(1, \sup _{\left[\frac{1}{n}, n\right]}|f-g|\right) .
$$

For a family $\left(\left(Z_{t}^{(\varepsilon)}\right)_{t>0}\right)_{\varepsilon>0}$ of continuous processes, we write

$$
\left(Z_{t}^{(\varepsilon)}\right)_{t>0} \quad \Longrightarrow \quad\left(Z_{t}\right)_{t>0}
$$

if $\left(Z_{t}^{(\varepsilon)}\right)_{t>0}$ converges in distribution to $\left(Z_{t}\right)_{t>0}$ in $\mathcal{C}((0,+\infty), \mathbb{R})$, as $\varepsilon \rightarrow 0$.
We write

$$
\left(Z_{t}^{(\varepsilon)}\right)_{t>0} \xrightarrow{\text { f.d.d. }}\left(Z_{t}\right)_{t>0},
$$

if for all finite subsets $S \subset(0,+\infty)$, the vector $\left(Z_{t}^{(\varepsilon)}\right)_{t \in S}$ converges in distribution to $\left(Z_{t}\right)_{t \in S}$ in $\mathbb{R}^{S}$, as $\varepsilon \rightarrow 0$.
Let us state our main results. Set $q:=\frac{\beta}{\gamma+1}$.
Theorem 2.2. Consider $\gamma \geq 0$, and $q>\frac{1}{2}$. Assume that either $H_{1}^{\gamma}$ or $H_{2}^{\gamma}$ is satisfied. Let $\left(V_{t}, X_{t}\right)_{t \geq t_{0}}$ be the solution to (SKE) and $\left(\mathcal{B}_{t}\right)_{t \geq 0}$ be a standard Brownian motion. Furthermore, if $\gamma \geq 1$, we suppose that for all $v \in \mathbb{R}, v F(v) \geq 0$.
Then, as $\varepsilon \rightarrow 0$,

$$
\left(\sqrt{\varepsilon} V_{t / \varepsilon}, \varepsilon^{3 / 2} X_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}} \Longrightarrow\left(\mathcal{B}_{t}, \int_{0}^{t} \mathcal{B}_{s} \mathrm{~d} s\right)_{t \geq 0}
$$

Theorem 2.3. Consider $\gamma \geq 0$ and $q=\frac{1}{2}$. Assume that $H_{1}^{\gamma}$ is satisfied. Let $\left(V_{t}, X_{t}\right)_{t \geq t_{0}}$ be the solution to (SKE). If $\gamma \geq 1$, we suppose furthermore that for all $v \in \mathbb{R}, v F(v) \geq 0$.
Call $\widetilde{H}$ the eternal ergodic process, solution to the homogeneous SDE

$$
\mathrm{d} H_{s}=\mathrm{d} W_{s}-\frac{H_{s}}{2} \mathrm{~d} s-F\left(H_{s}\right) \mathrm{d} s
$$

such that the law of $H_{-\infty}$ is the invariant measure, where $\left(W_{t}\right)_{t \geq 0}$ is again a standard Brownian motion. Setting $\Lambda_{F, t_{1}, \cdots, t_{d}}$ for the f.d.d. of $\widetilde{H}$, we call $\left(\mathcal{V}_{t}\right)_{t \geq 0}$ the process whose finite dimensional
 the linear map $T\left(u_{1}, \cdots, u_{d}\right):=\left(\sqrt{t_{1}} u_{1}, \cdots, \sqrt{t_{d}} u_{d}\right)$, that is $\left(\mathcal{V}_{t}\right)_{t \geq 0}=\left(\sqrt{t} \widetilde{H}_{\log (t)}\right)_{t \geq 0}$.
Then, as $\varepsilon \rightarrow 0$,

$$
\left(\sqrt{\varepsilon} V_{t / \varepsilon}, \varepsilon^{3 / 2} X_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}} \Longrightarrow\left(\mathcal{V}_{t}, \int_{0}^{t} \mathcal{V}_{s} \mathrm{~d} s\right)_{t \geq 0}
$$

Remark 2.4. The one-dimensional distribution of $\left(\mathcal{V}_{t}\right)_{t \geq 0}$ has already been explicitly computed (see Theorem 4.1 in [GO13]).
Theorem 2.5. Consider $\gamma \geq 1$ and $q<\frac{1}{2}$. Assume that $F: v \mapsto \rho \operatorname{sgn}(v)|v|^{\gamma}$ with $\rho>0$. Let $\left(V_{t}, X_{t}\right)_{t \geq t_{0}}$ be the solution to (SKE). Call $\widehat{H}$ the ergodic process, solution to the homogeneous SDE

$$
\mathrm{d} H_{s}=\mathrm{d} W_{s}-F\left(H_{s}\right) \mathrm{d} s
$$

where $\left(W_{t}\right)_{t \geq 0}$ is a standard Brownian motion. Call $\Pi_{F}$ its invariant measure. We call $\left(\mathscr{V}_{t}\right)_{t \geq 0}$ the process whose f.d.d. are $T *\left(\otimes^{d} \Pi_{F}\right)$, the pushforward measure of $\otimes^{d} \Pi_{F}$ by the linear map $T\left(u_{1}, \cdots, u_{d}\right):=\left(t_{1}{ }^{q} u_{1}, \cdots, t_{d}{ }^{q} u_{d}\right)$.
Then, as $\varepsilon \rightarrow 0$,

$$
\left(\varepsilon^{q} V_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}} \stackrel{f . d . d .}{\Longrightarrow}\left(\mathscr{V}_{t}\right)_{t \geq 0} .
$$

Moreover, in the linear case (i.e. $\gamma=1$ ) and if $\beta>-\frac{1}{2}$, we define $\left(\mathscr{X}_{t}\right)_{t \geq 0}$ the centered Gaussian process with covariance function $K(s, t):=\left(\rho^{2}(1+2 \beta)\right)^{-1}(s \wedge t)^{1+2 \beta}$.
Then, as $\varepsilon \rightarrow 0$,

$$
\begin{equation*}
\left(\varepsilon^{\beta+\frac{1}{2}} X_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}} \stackrel{\text { f.d.d. }}{\Longrightarrow}\left(\mathscr{X}_{t}\right)_{t \geq 0} . \tag{1}
\end{equation*}
$$

Remark 2.6. If $\beta=0$, one can prove using the martingale method, that $\left(\sqrt{\varepsilon} X_{t / \varepsilon}\right)_{t \geq 0}$ converges towards a Brownian motion. Assume, by way of contradiction, that the process $\left(\varepsilon^{q} V_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}}$ would converge (i.e. were tight), then by the continuous mapping theorem, the process $\left(\varepsilon X_{t / \varepsilon}\right)_{t \geq 0}$ should converge. This is a contradiction with (1). Here is why we deal only with finite-dimensional convergence for the velocity process.

## 3 Changed-of-time processes

In the following, we suppose that $\gamma>-1$ and set $\Omega=\overline{\mathcal{C}}\left(\left[t_{0},+\infty\right)\right)$ the set of continuous functions, that equal $+\infty$ after their (possibly infinite) explosion time. Following the idea used in GO13, we first perform a change of time in (SKE) in order to produce at least one time-homogeneous coefficient in the transformed equation. For every $\mathcal{C}^{2}$-diffeomorphism $\varphi:\left[0, t_{1}\right) \rightarrow\left[t_{0},+\infty\right)$, let introduce the scaling transformation $\Phi_{\varphi}$ defined, for $\omega \in \Omega$, by

$$
\Phi_{\varphi}(\omega)(s):=\frac{\omega(\varphi(s))}{\sqrt{\varphi^{\prime}(s)}}, \text { with } s \in\left[0, t_{1}\right)
$$

The result containing the change of time transformation can be found in GO13, Proposition 2.1, p. 187.

Let $V$ be solution to the equation (SKE). Thanks to Lévy's characterization theorem of the Brownian motion, $\left(W_{t}\right)_{t \geq 0}:=\left(\int_{0}^{t} \frac{\mathrm{~d} B_{\varphi(s)}}{\sqrt{\varphi^{\prime}(s)}}\right)_{t \geq 0}$ is a standard Brownian motion. Then, by a change of variable $t=\varphi(s)$, one gets

$$
V_{\varphi(t)}-V_{\varphi(0)}=\int_{0}^{t} \sqrt{\varphi^{\prime}(s)} \mathrm{d} W_{s}-\int_{0}^{t} \frac{F\left(V_{\varphi(s)}\right)}{\varphi(s)^{\beta}} \varphi^{\prime}(s) \mathrm{d} s
$$

The integration by parts formula yields

$$
\mathrm{d}\left(\frac{V_{\varphi(s)}}{\sqrt{\varphi^{\prime}(s)}}\right)=\mathrm{d} W_{s}-\frac{\sqrt{\varphi^{\prime}(s)}}{\varphi(s)^{\beta}} F\left(V_{\varphi(s)}\right) \mathrm{d} s-\frac{\varphi^{\prime \prime}(s)}{2 \varphi^{\prime}(s)} \frac{V_{\varphi(s)}}{\sqrt{\varphi^{\prime}(s)}} \mathrm{d} s
$$

As a consequence, we can state the following result in our context.

Proposition 3.1. If $V$ is a solution to the equation (SKE), then $V^{(\varphi)}:=\Phi_{\varphi}(V)$ is a solution to

$$
\begin{equation*}
\mathrm{d} V_{s}^{(\varphi)}=\mathrm{d} W_{s}-\frac{\sqrt{\varphi^{\prime}(s)}}{\varphi(s)^{\beta}} F\left(\sqrt{\varphi^{\prime}(s)} V_{s}^{(\varphi)}\right) \mathrm{d} s-\frac{\varphi^{\prime \prime}(s)}{\varphi^{\prime}(s)} \frac{V_{s}^{(\varphi)}}{2} \mathrm{~d} s, V_{0}^{(\varphi)}=\frac{V_{\varphi(0)}}{\sqrt{\varphi^{\prime}(0)}} \tag{2}
\end{equation*}
$$

where $W_{t}:=\int_{0}^{t} \frac{\mathrm{~d} B_{\varphi(s)}}{\sqrt{\varphi^{\prime}(s)}}$.
If $V^{(\varphi)}$ is a solution to (2), then $\Phi_{\varphi}^{-1}\left(V^{(\varphi)}\right)$ is a solution to the equation (SKE), where $B_{t}-B_{t_{0}}:=$ $\int_{t_{0}}^{t} \sqrt{\left(\varphi^{\prime} \circ \varphi^{-1}\right)(s)} \mathrm{d} W_{\varphi^{-1}(s)}$.
Furthermore, uniqueness in law, pathwise uniqueness or strong existence hold for the equation (SKE) if and only if they hold for the equation (2).

In the following, we will use two particular changes of time, depending on which term of (2) should become time-homogeneous.

- The exponential change of time: denoting $\varphi_{e}: t \mapsto t_{0} e^{t}$, the exponential scaling transformation is defined by $\Phi_{e}(\omega): s \in \mathbb{R}^{+} \mapsto \frac{{\stackrel{\omega}{t_{0}}} e^{s}}{\sqrt{t_{0}} e^{s / 2}}$, for $\omega \in \Omega$. Thanks to Proposition 3.1 the process $V^{(e)}:=\Phi_{e}(V)$ satisfies the equation

$$
\mathrm{d} V_{s}^{(e)}=\mathrm{d} W_{s}-\frac{V_{s}^{(e)}}{2} \mathrm{~d} s-t_{0}^{1 / 2-\beta} e^{(1 / 2-\beta) s} F\left(\sqrt{t_{0}} e^{s / 2} V_{s}^{(e)}\right) \mathrm{d} s
$$

where $\left(W_{t}\right)_{t \geq 0}$ is a standard Brownian motion.

- The power change of time: for $q=\frac{\beta}{\gamma+1} \neq \frac{1}{2}$, consider $\varphi_{q} \in \mathcal{C}^{2}\left(\left[0, t_{1}\right)\right)$ the solution to the Cauchy problem

$$
\varphi_{q}^{\prime}=\varphi_{q}^{2 q}, \varphi_{q}(0)=t_{0}
$$

Clearly, $\varphi_{q}(t)=\left(t_{0}^{1-2 q}+(1-2 q) t\right)^{1 /(1-2 q)}$, when $2 q \neq 1$, and $\varphi_{q}=\varphi_{e}$, when $2 q=1$.
The time $t_{1}$ satisfies $t_{1}=+\infty$, when $2 q \leq 1$, and $t_{1}=t_{0}^{1-2 q}(2 q-1)^{-1}$, when $2 q>1$. The power scaling transformation is defined by $\Phi_{q}(\omega): s \in \mathbb{R}^{+} \mapsto \frac{\omega\left(\varphi_{q}(s)\right)}{\varphi_{q}(s)^{q}}$. The process $V^{(q)}:=V^{\left(\varphi_{q}\right)}$ satisfies the equation

$$
\begin{equation*}
\mathrm{d} V_{s}^{(q)}=\mathrm{d} W_{s}-\varphi_{q}^{-\gamma q}(s) F\left(\sqrt{\varphi_{q}^{\prime}(s)} V_{s}^{(q)}\right) \mathrm{d} s-q \varphi_{q}^{2 q-1}(s) V_{s}^{(q)} \mathrm{d} s \tag{3}
\end{equation*}
$$

where $\left(W_{t}\right)_{t \geq 0}$ is a standard Brownian motion.
Adapting the proof of Propositions 3.2, 3.6 and 3.7 p. 188, in GO13, one can prove the following proposition.
Proposition 3.2. For $\gamma \geq 0$, there exists a pathwise unique strong solution to (SKE), defined up to the explosion time $\tau_{\infty}$ of $V$.

- When $\gamma \leq 1$ or for all $v \in \mathbb{R}, v F(v) \geq 0$, then $\tau_{\infty}$ is a.s. infinite.
- When $2 q>1$, then $\mathbb{P}\left(\tau_{\infty}=+\infty\right)>0$.
- Under $\left.H_{1}^{\gamma}\right]$, if $\gamma>1$ and $\left.(F(-1), F(1)) \in((0,+\infty)) \times[0,+\infty)\right) \cup(\mathbb{R} \times(-\infty, 0))$, then $\mathbb{P}\left(\tau_{\infty}=+\infty\right)<1$.

Remark 3.3. Assume that $\overline{H_{1}^{\gamma}}$ is satisfied. In the linear case $(\gamma=1)$, the drift and the diffusion terms are Lipschitz and satisfy locally linear growth condition. The existence and nonexplosion of $V$ follow from Theorem 2.9, p. 289, in [KS98].

For more details, we refer to Lui22].

## 4 Moment estimates of the velocity process

In this section, we give estimates for the moment of the velocity process. It will be useful to control some stochastic terms appearing later.
Proposition 4.1. Assume that $\gamma \geq 0$ and $\beta \in \mathbb{R}$. The inequality

$$
\forall t \geq t_{0}, \mathbb{E}\left[\left|V_{t}\right|^{\kappa}\right] \leq C_{\gamma, \kappa, \beta, t_{0}} t^{\frac{\kappa}{2}}
$$

holds for

- $\kappa \in[0,1]$, when $\gamma<1$ and $\beta \geq \frac{\gamma+1}{2}$,
- $\kappa \geq 0$, when for all $v \in \mathbb{R}, v F(v) \geq 0$.

If $\kappa \in[0,1], \gamma<1$ and $\beta<\frac{\gamma+1}{2}$, then

$$
\forall t \geq t_{0}, \mathbb{E}\left[\left|V_{t}\right|^{\kappa}\right] \leq C_{\gamma, \kappa, \beta, t_{0}} t^{\kappa \frac{1-\beta}{1-\gamma}}
$$

Remark 4.2. When $-1<\gamma<0$, it can be proved that for all $t \geq t_{0}, \mathbb{E}\left[\left|V_{t}\right|\right] \leq C_{\gamma, \beta, t_{0}} \sqrt{t}$, without hypothesis of the positivity of the function $v \mapsto v F(v)$.
Proof. STEP 1. Assume that $\gamma \geq 1$ and that for all $v \in \mathbb{R}, v F(v) \geq 0$.
Define, for all $n \geq 0$, the stopping times $T_{n}:=\inf \left\{t \geq t_{0},\left|V_{t}\right| \geq n\right\}$. By Itô's formula, for all $t \geq t_{0}$, we have

$$
\begin{aligned}
V_{t \wedge T_{n}}^{2} & =v_{0}^{2}+\int_{t_{0}}^{t \wedge T_{n}} 2 V_{s} \mathrm{~d} B_{s}-\int_{t_{0}}^{t \wedge T_{n}} 2 s^{-\beta} V_{s} F\left(V_{s}\right) \mathrm{d} s+\left(t \wedge T_{n}-t_{0}\right) \\
& =v_{0}^{2}+\int_{t_{0}}^{t} \mathbb{1}_{s \leq T_{n}} 2 V_{s} \mathrm{~d} B_{s}-\int_{t_{0}}^{t \wedge T_{n}} 2 s^{-\beta} V_{s} F\left(V_{s}\right) \mathrm{d} s+\left(t \wedge T_{n}-t_{0}\right) \\
& \leq v_{0}^{2}+\int_{t_{0}}^{t} \mathbb{1}_{s \leq T_{n}} 2 V_{s} \mathrm{~d} B_{s}+\left(t-t_{0}\right) .
\end{aligned}
$$

Since $\int_{t_{0}}^{t} 4 \mathbb{1}_{s \leq T_{n}} V_{s}^{2} \mathrm{~d} s \leq 4 n^{2}\left(t-t_{0}\right)<+\infty$, taking expectation yields

$$
\mathbb{E}\left[V_{t \wedge T_{n}}^{2}\right] \leq v_{0}^{2}+\left(t-t_{0}\right) \leq C_{t_{0}} t .
$$

Set $\kappa \in[0,2]$, we obtain by Jensen's inequality that

$$
\begin{equation*}
\mathbb{E}\left[\left|V_{t}\right|^{\kappa}\right] \leq \mathbb{E}\left[\left|V_{t}\right|^{2}\right]^{\frac{\kappa}{2}} \leq\left(\liminf _{n \rightarrow+\infty} \mathbb{E}\left[V_{t \wedge T_{n}}^{2}\right]\right)^{\frac{\kappa}{2}} \leq C_{\kappa, t_{0}} t^{\frac{\kappa}{2}} \tag{4}
\end{equation*}
$$

When $\kappa>2$, the function $v \mapsto|v|^{\kappa}$ is $\mathcal{C}^{2}$, so by Itô's formula, we can write for all $t \geq t_{0}$,

$$
\begin{aligned}
&\left|V_{t \wedge T_{n}}\right|^{\kappa}=\left|v_{0}\right|^{\kappa}+\int_{t_{0}}^{t \wedge T_{n}} \kappa \operatorname{sgn}\left(V_{s}\right)\left|V_{s}\right|^{\kappa-1} \mathrm{~d} B_{s}-\int_{t_{0}}^{t \wedge T_{n}} \kappa s^{-\beta}\left|V_{s}\right|^{\kappa-1} \operatorname{sgn}\left(V_{s}\right) F\left(V_{s}\right) \mathrm{d} s \\
&+\int_{t_{0}}^{t \wedge T_{n}} \frac{\kappa(\kappa-1)}{2}\left|V_{s}\right|^{\kappa-2} \mathrm{~d} s
\end{aligned}
$$

In addition, using the hypothesis on the sign of $F$, we have

$$
\begin{equation*}
\left|V_{t \wedge T_{n}}\right|^{\kappa} \leq\left|v_{0}\right|^{\kappa}+\int_{t_{0}}^{t} \mathbb{1}_{s \leq T_{n}} \kappa \operatorname{sgn}\left(V_{s}\right)\left|V_{s}\right|^{\kappa-1} \mathrm{~d} B_{s}+\int_{t_{0}}^{t \wedge T_{n}} \frac{\kappa(\kappa-1)}{2}\left|V_{s}\right|^{\kappa-2} \mathrm{~d} s \tag{5}
\end{equation*}
$$

We observe that $\int_{t_{0}}^{t} \kappa^{2} V_{s}^{2 \kappa-2} \mathbb{1}_{s \leq T_{n}} \mathrm{~d} s \leq \kappa^{2} n^{2 \kappa-2}\left(t-t_{0}\right)<+\infty$. Taking expectation in (5), we obtain

$$
\mathbb{E}\left[\left|V_{t}\right|^{\kappa}\right] \leq \liminf _{n \rightarrow+\infty} \mathbb{E}\left[\left|V_{t \wedge T_{n}}\right|^{\kappa}\right] \leq\left|v_{0}\right|^{\kappa}+\int_{t_{0}}^{t} \frac{\kappa(\kappa-1)}{2} \mathbb{E}\left[\left|V_{s}\right|^{\kappa-2}\right] \mathrm{d} s
$$

When $0 \leq \kappa-2 \leq 2$, we can upper bound $\mathbb{E}\left[\left|V_{s}\right|^{\kappa-2}\right]$ by injecting (4) and get

$$
\mathbb{E}\left[\left|V_{t}\right|^{\kappa}\right] \leq\left|v_{0}\right|^{\kappa}+\int_{t_{0}}^{t} \frac{\kappa(\kappa-1)}{2} C_{\kappa, t_{0}} s^{\frac{\kappa-2}{2}} \mathrm{~d} s \leq C_{\kappa, t_{0}} s^{\frac{\kappa}{2}}
$$

The same method is then applied inductively to prove the inequality for all $\kappa>2$.
Step 2. Assume now that $\gamma \in[0,1[$. Fix $\kappa \in[0,1]$. Then Jensen's inequality yields, for all $t \geq t_{0}, \mathbb{E}\left[\left|V_{t}\right|^{\kappa}\right] \leq \mathbb{E}\left[\left|V_{t}\right|\right]^{\kappa}$, hence it suffices to verify the inequality only for $\kappa=1$.
Define, for all $n \geq 0$, the stopping times $T_{n}:=\inf \left\{t \geq t_{0},\left|V_{t}\right| \geq n\right\}$ and let us recall that under both hypotheses $\left(H_{1}^{\gamma}\right)$ or $\left\langle H_{2}^{\gamma}\right\rangle$, there exists a positive constant $K$, such that $|F(v)| \leq K|v|^{\gamma}$. We can write, for $t \geq t_{0}$ and $n \geq 0$,

$$
\begin{aligned}
\left|V_{t \wedge T_{n}}\right| & \leq\left|v_{0}-B_{t_{0}}\right|+\left|B_{t \wedge T_{n}}\right|+\int_{t_{0}}^{t \wedge T_{n}} s^{-\beta}\left|F\left(V_{s \wedge T_{n}}\right)\right| \mathrm{d} s \\
& \leq\left|v_{0}-B_{t_{0}}\right|+\left|B_{t \wedge T_{n}}\right|+\int_{t_{0}}^{t \wedge T_{n}} s^{-\beta} K\left|V_{s \wedge T_{n}}\right|^{\gamma} \mathrm{d} s
\end{aligned}
$$

By noting that $\gamma \in\left[0,1\left[\right.\right.$ and $\left(B_{t}^{2}-t\right)_{t \geq 0}$ is a martingale, taking expectation we get

$$
\begin{aligned}
\mathbb{E}\left[\left|V_{t \wedge T_{n}}\right|\right] & \leq \mathbb{E}\left[\left|v_{0}-B_{t_{0}}\right|\right]+\mathbb{E}\left[\left|B_{t \wedge T_{n}}\right|\right]+\int_{t_{0}}^{t} s^{-\beta} K \mathbb{E}\left[\left|V_{s \wedge T_{n}}\right|^{\gamma}\right] \mathrm{d} s \\
& \leq \mathbb{E}\left[\left|v_{0}-B_{t_{0}}\right|\right]+\sqrt{\mathbb{E}\left[B_{t \wedge T_{n}}^{2}\right]}+\int_{t_{0}}^{t} s^{-\beta} K \mathbb{E}\left[\left|V_{s \wedge T_{n}}\right|\right]^{\gamma} \mathrm{d} s \\
& \leq \mathbb{E}\left[\left|v_{0}-B_{t_{0}}\right|\right]+\sqrt{\mathbb{E}\left[t \wedge T_{n}\right]}+\int_{t_{0}}^{t} s^{-\beta} K \mathbb{E}\left[\left|V_{s \wedge T_{n}}\right|\right]^{\gamma} \mathrm{d} s \\
& \leq C_{t_{0}} \sqrt{t}+\int_{t_{0}}^{t} s^{-\beta} K \mathbb{E}\left[\left|V_{s \wedge T_{n}}\right|\right]^{\gamma} \mathrm{d} s .
\end{aligned}
$$

The function $g_{n}: t \mapsto \mathbb{E}\left[\left|V_{t \wedge T_{n}}\right|\right]$ is bounded by $n$. Applying a Gronwall-type lemma, stated below (Lemma 4.3) and Fatou's lemma, for $\beta \neq 1$ and for all $t \geq t_{0}$, we end up with

$$
\begin{aligned}
\mathbb{E}\left[\left|V_{t}\right|\right] \leq \liminf _{n \rightarrow+\infty} \mathbb{E}\left[\left|V_{t \wedge T_{n}}\right|\right] & \leq C_{\gamma}\left[C_{t_{0}} \sqrt{t}+\left(\frac{1-\gamma}{1-\beta} K\left(t^{1-\beta}-t_{0}^{1-\beta}\right)\right)^{\frac{1}{1-\gamma}}\right] \\
& \leq C_{\gamma, \beta, t_{0}} \begin{cases}\sqrt{t} & \text { if } \beta \geq \frac{\gamma+1}{2} \\
t^{\frac{1-\beta}{1-\gamma}} & \text { else. }\end{cases}
\end{aligned}
$$

The case $\beta=1$ can be treated similarly.

Lemma 4.3 (Gronwall-type lemma). Fix $r \in[0,1)$ and $t_{0} \in \mathbb{R}$. Assume that $g$ is a nonnegative real-valued function, $b$ is a positive function and $a$ is a differentiable real-valued function. Moreover, suppose that the function $b g^{r}$ is continuous. If

$$
\begin{equation*}
\forall t \geq t_{0}, g(t) \leq a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s \tag{6}
\end{equation*}
$$

then,

$$
\forall t \geq t_{0}, g(t) \leq 2^{\frac{1}{1-r}}\left[a(t)+\left((1-r) \int_{t_{0}}^{t} b(s) \mathrm{d} s\right)^{\frac{1}{1-r}}\right]
$$

Proof. For $t \geq t_{0}$, since $r \geq 0$,

$$
g(t)^{r} \leq\left(a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s\right)^{r}
$$

then, multiplying by $b(t)>0$,

$$
b(t) g(t)^{r} \leq b(t)\left(a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s\right)^{r}
$$

Now, let us make appear the derivative of $H$

$$
a^{\prime}(t)+b(t) g(t)^{r} \leq a^{\prime}(t)+b(t)\left(a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s\right)^{r}
$$

that is

$$
\frac{a^{\prime}(t)+b(t) g(t)^{r}}{\left(a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s\right)^{r}} \leq b(t)+\frac{a^{\prime}(t)}{\left(a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s\right)^{r}} \leq b(t)+\frac{a^{\prime}(t)}{a(t)^{r}} .
$$

Integrating, since $r \neq 1$, we obtain
$(1-r)^{-1}\left[\left(a(t)+\int_{t_{0}}^{t} b(s) g(s)^{r} \mathrm{~d} s\right)^{1-r}-a\left(t_{0}\right)^{1-r}\right] \leq(1-r)^{-1}\left[a(t)^{1-r}-a\left(t_{0}\right)^{1-r}\right]+\int_{t_{0}}^{t} b(s) \mathrm{d} s$ or equivalently, setting $H$ for the right-hand side of (6) and using that $r<1$, we get

$$
H(t)^{1-r} \leq a(t)^{1-r}+(1-r) \int_{t_{0}}^{t} b(s) \mathrm{d} s
$$

Since $\frac{1}{1-r}>0$ and using (6)

$$
g(t) \leq\left(a(t)^{1-r}+(1-r) \int_{t_{0}}^{t} b(s) \mathrm{d} s\right)^{\frac{1}{1-r}} \leq C_{r}\left[a(t)+\left((1-r) \int_{t_{0}}^{t} b(s) \mathrm{d} s\right)^{\frac{1}{1-r}}\right]
$$

This concludes the proof of the lemma.
Remark 4.4. Call $H(t)$ the right-hand side of (6). If $g$ is not continuous, note that the function $H$ is continuous and satisfies (6) (since $b$ is positive and $g \leq H$ ). Therefore, one can apply the lemma to $H$ and then use the inequality $g \leq H$.

## 5 Proof of the asymptotic behaviour of the solution

This section is devoted to the proofs of our main results.

### 5.1 Asymptotic behaviour in the super-critical regime under both assumptions

In this section, we assume that $\gamma \geq 0$ and $q>\frac{1}{2}$.
Proof of Theorem 2.2. We split the proof into three steps.
Step 1. We note that it is enough to prove that the process

$$
\left(V_{t}^{(\varepsilon)}\right)_{t \geq 0}:=\left(\sqrt{\varepsilon} V_{t / \varepsilon}\right)_{t \geq 0}
$$

converges in distribution to a Brownian motion in the space of continuous functions $\mathcal{C}([0,+\infty))$ endowed by the uniform topology. In order to see $V^{(\varepsilon)}$ as a process of $\mathcal{C}([0,+\infty))$, let us state for all $s \in\left[0, \varepsilon t_{0}\right], V_{s}^{(\varepsilon)}:=V_{\varepsilon t_{0}}^{(\varepsilon)}=\sqrt{\varepsilon} v_{0}$.
For every $\varepsilon \in(0,1]$ and $t \geq \varepsilon t_{0}$, we can write

$$
\varepsilon^{3 / 2} X_{t / \varepsilon}=\varepsilon^{3 / 2} x_{0}+\int_{\varepsilon t_{0}}^{t} V_{s}^{(\varepsilon)} \mathrm{d} s
$$

Clearly, the theorem will be proved once we show that $g_{\varepsilon}\left(V_{\bullet}^{(\varepsilon)}\right):=\left(V_{\bullet}^{(\varepsilon)}, \int_{\varepsilon t_{0}}^{\bullet} V_{s}^{(\varepsilon)} \mathrm{d} s\right)$ converges weakly in $\mathcal{C}\left([0,+\infty)\right.$ ) endowed by the uniform topology. Here the mapping $g_{\varepsilon}: v \mapsto$ $\left(v_{t}, \int_{\varepsilon t_{0}}^{t} v_{s} \mathrm{~d} s\right)_{t \geq 0}$ is defined and valued on $\mathcal{C}((0,+\infty))$. This mapping is converging, as $\varepsilon \rightarrow 0$, to the continuous mapping $g: v \mapsto\left(v_{t}, \int_{0}^{t} v_{s} \mathrm{~d} s\right)_{t \geq 0}$.
We have, for every $\varepsilon \in(0,1]$ and $t \geq \varepsilon t_{0}$,

$$
\begin{aligned}
V_{t}^{(\varepsilon)}=\sqrt{\varepsilon} V_{t / \varepsilon} & =\sqrt{\varepsilon}\left(v_{0}-B_{t_{0}}\right)+\sqrt{\varepsilon} B_{t / \varepsilon}-\sqrt{\varepsilon} \int_{t_{0}}^{t / \varepsilon} F\left(V_{s}\right) s^{-\beta} \mathrm{d} s \\
& =\sqrt{\varepsilon}\left(v_{0}-B_{t_{0}}\right)+B_{t}^{(\varepsilon)}-\varepsilon^{\beta-1 / 2} \int_{\varepsilon t_{0}}^{t} F\left(V_{u / \varepsilon}\right) u^{-\beta} \mathrm{d} u
\end{aligned}
$$

By self-similarity, $B^{(\varepsilon)}:=\left(\sqrt{\varepsilon} B_{t / \varepsilon}\right)_{t \geq 0}$ has the same distribution as a standard Brownian motion. Assume that the convergence of the rescaled velocity process is proved in the strong way, that is

$$
\begin{equation*}
\forall T \geq t_{0}, \sup _{\varepsilon t_{0} \leq t \leq T}\left|V_{t}^{(\varepsilon)}-B_{t}^{(\varepsilon)}\right| \xrightarrow{\mathbb{P}} 0, \quad \text { as } \varepsilon \rightarrow 0 \tag{7}
\end{equation*}
$$

Then it suffices to prove that $g_{\varepsilon}\left(B^{(\varepsilon)}\right) \Longrightarrow g(\mathcal{B})$ and $\mathrm{d}_{u}\left(g_{\varepsilon}\left(V^{(\varepsilon)}\right), g_{\varepsilon}\left(B^{(\varepsilon)}\right)\right) \xrightarrow{\mathbb{P}} 0$, as $\varepsilon \rightarrow 0$ (see Theorem 3.1, p. 27, in Bil99].
On the one hand, the process $B^{(\varepsilon)}$ being a Brownian motion and $|\cdot|_{\mathbb{R}^{2}}$ denoting a norm on $\mathbb{R}^{2}$, the first convergence follows from

$$
\begin{equation*}
\forall T \geq t_{0}, \sup _{\varepsilon t_{0} \leq t \leq T}\left|g_{\varepsilon}\left(\mathcal{B}_{t}\right)-g\left(\mathcal{B}_{t}\right)\right|_{\mathbb{R}^{2}} \xrightarrow{\mathbb{P}} 0, \quad \text { as } \varepsilon \rightarrow 0 . \tag{8}
\end{equation*}
$$

Let us prove (8). For every $\varepsilon t_{0} \leq t \leq T$, we get

$$
\begin{aligned}
\left|g_{\varepsilon}\left(\mathcal{B}_{t}\right)-g\left(\mathcal{B}_{t}\right)\right|_{\mathbb{R}^{2}} & =\left|\int_{0}^{\varepsilon t_{0}} \mathcal{B}_{s} \mathrm{~d} s\right| \\
& \leq \int_{0}^{\varepsilon t_{0}}\left|\mathcal{B}_{s}\right| \mathrm{d} s
\end{aligned}
$$

Hence,

$$
\mathbb{E}\left[\sup _{\varepsilon t_{0} \leq t \leq T}\left|g_{\varepsilon}\left(\mathcal{B}_{t}\right)-g\left(\mathcal{B}_{t}\right)\right|_{\mathbb{R}^{2}}\right] \leq \int_{0}^{\varepsilon t_{0}} \mathbb{E}\left|\mathcal{B}_{s}\right| \mathrm{d} s \leq C \int_{0}^{\varepsilon t_{0}} \sqrt{s} \mathrm{~d} s \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0
$$

On the other hand, we prove that

$$
\begin{equation*}
\forall T \geq t_{0}, \sup _{\varepsilon t_{0} \leq t \leq T}\left|g_{\varepsilon}\left(V_{t}^{(\varepsilon)}\right)-g_{\varepsilon}\left(B_{t}^{(\varepsilon)}\right)\right|_{\mathbb{R}^{2}} \xrightarrow{\mathbb{P}} 0, \quad \text { as } \varepsilon \rightarrow 0 \tag{9}
\end{equation*}
$$

For every $\varepsilon t_{0} \leq t \leq T$, using (7)

$$
\begin{aligned}
\left|g_{\varepsilon}\left(V_{t}^{(\varepsilon)}\right)-g_{\varepsilon}\left(B_{t}^{(\varepsilon)}\right)\right|_{\mathbb{R}^{2}} & =\left|V_{t}^{(\varepsilon)}-B_{t}^{(\varepsilon)}\right|+\left|\int_{\varepsilon t_{0}}^{t} V_{s}^{(\varepsilon)}-B_{s}^{(\varepsilon)} \mathrm{d} s\right| \\
& \leq\left(1+T-\varepsilon t_{0}\right) \sup _{\varepsilon t_{0} \leq t \leq T}\left|V_{t}^{(\varepsilon)}-B_{t}^{(\varepsilon)}\right| \xrightarrow{\mathbb{P}} 0 .
\end{aligned}
$$

Step 2. Let us prove now (7). Recall that under both hypothesis $\left.H_{1}^{\gamma}\right\rangle$ and $\left(\overrightarrow{H_{2}^{\gamma}}\right)$, there exists a positive constant $K$, such that $(\sqrt{\varepsilon})^{\gamma}\left|F\left(\frac{V_{u}^{(\varepsilon)}}{\sqrt{\varepsilon}}\right)\right| \leq K\left|V_{u}^{(\varepsilon)}\right|^{\gamma}$. Modifying the factor in front of the integral part, we get

$$
V_{t}^{(\varepsilon)}=\sqrt{\varepsilon}\left(v_{0}-B_{t_{0}}\right)+\sqrt{\varepsilon} B_{t / \varepsilon}-\varepsilon^{\beta-(\gamma+1) / 2} \int_{\varepsilon t_{0}}^{t}(\sqrt{\varepsilon})^{\gamma} F\left(\frac{V_{u}^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta} \mathrm{d} u
$$

It follows that, for all $t_{0} \leq T$,

$$
\begin{aligned}
\sup _{\varepsilon t_{0} \leq t \leq T}\left|V_{t}^{(\varepsilon)}-B_{t}^{(\varepsilon)}\right| & \leq \sqrt{\varepsilon}\left|v_{0}-B_{t_{0}}\right|+\varepsilon^{\beta-(\gamma+1) / 2} \sup _{\varepsilon t_{0} \leq t \leq T}\left|\int_{\varepsilon t_{0}}^{t}(\sqrt{\varepsilon})^{\gamma} F\left(\frac{V_{u}^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta} \mathrm{d} u\right| \\
& \leq \sqrt{\varepsilon}\left|v_{0}-B_{t_{0}}\right|+\varepsilon^{\beta-(\gamma+1) / 2} \int_{\varepsilon t_{0}}^{T} K\left|V_{u}^{(\varepsilon)}\right|^{\gamma} u^{-\beta} \mathrm{d} u
\end{aligned}
$$

Taking the expectation and using moment estimates (Proposition4.1), we obtain, when $\beta \neq \frac{\gamma}{2}+1$ and since $\beta>\frac{\gamma+1}{2}$,

$$
\begin{aligned}
\varepsilon^{\beta-(\gamma+1) / 2} \mathbb{E}\left[\int_{\varepsilon t_{0}}^{T} K\left|V_{u}^{(\varepsilon)}\right|^{\gamma} u^{-\beta} \mathrm{d} u\right] & =\varepsilon^{\beta-(\gamma+1) / 2} \int_{\varepsilon t_{0}}^{T} K \mathbb{E}\left[\left|V_{u}^{(\varepsilon)}\right|^{\gamma}\right] u^{-\beta} \mathrm{d} u \\
& \leq \varepsilon^{\beta-(\gamma+1) / 2} \int_{\varepsilon t_{0}}^{T} K C_{\gamma, \beta, t_{0}} u^{\frac{\gamma}{2}-\beta} \mathrm{d} u \\
& \leq C\left(\varepsilon^{\beta-(\gamma+1) / 2} T^{\frac{\gamma}{2}-\beta+1}-t_{0}^{\frac{\gamma}{2}-\beta+1} \sqrt{\varepsilon}\right) \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0 .
\end{aligned}
$$

Hence, setting $r=\min \left(\frac{1}{2}, \beta-(\gamma+1) / 2\right)>0$

$$
\mathbb{E}\left[\sup _{\varepsilon t_{0} \leq t \leq T}\left|V_{t}^{(\varepsilon)}-B_{t}^{(\varepsilon)}\right|\right]=O\left(\varepsilon^{r}\right)
$$

The case $\beta=\frac{\gamma}{2}+1$ can be treated similarly to get

$$
\mathbb{E}\left[\sup _{\varepsilon t_{0} \leq t \leq T}\left|V_{t}^{(\varepsilon)}-B_{t}^{(\varepsilon)}\right|\right]=O(\sqrt{\varepsilon} \ln (\varepsilon))
$$

This concludes the proof.

Remark 5.1. One can observe that the only moment in this proof, when we need the condition " $\gamma<1$ or for all $v \in \mathbb{R}, v F(v)$ " is when we are proving the moment estimates.

### 5.2 Asymptotic behaviour in the critical regime under $H_{1}^{\gamma}$

Assume in this section that $\beta=\frac{\gamma+1}{2}$ and $H_{1}^{\gamma}$ is satisfied.
Proof of Theorem 2.3. Step 1. As in the first step of the previous section, it suffices to prove the convergence of the rescaled velocity process $\left(\sqrt{\varepsilon} V_{t / \varepsilon}\right)_{t}$. Keeping same notations, we prove that $g_{\varepsilon}\left(V^{(\varepsilon)}\right)$ converges in distribution in $\mathcal{C}([0,+\infty))$ to $g(\mathcal{V})$. In order to see $V^{(\varepsilon)}$ as a process of $\mathcal{C}\left([0,+\infty)\right.$ ), let us set for all $s \in\left[0, \varepsilon t_{0}\right], V_{s}^{(\varepsilon)}:=V_{\varepsilon t_{0}}^{(\varepsilon)}=\sqrt{\varepsilon} v_{0}$. Call $P_{\varepsilon}, P$ the distribution of $V^{(\varepsilon)}, \mathcal{V}$ respectively. Then, using Pormanteau theorem (see Theorem 2.1 p. 16 in Bil99), it suffices to prove that for all function $h: \mathcal{C}([0,+\infty)) \times \mathcal{C}([0,+\infty)) \rightarrow \mathbb{R}$ bounded and uniformly continuous,

$$
\int_{\mathcal{C}([0,+\infty))^{2}} h\left(g_{\varepsilon}(\omega)\right) \mathrm{d} P_{\varepsilon}(\mathrm{d} \omega) \underset{\varepsilon \rightarrow 0}{\longrightarrow} \int_{\mathcal{C}([0,+\infty))^{2}} h(g(\omega)) \mathrm{d} P(\mathrm{~d} \omega) .
$$

Take a bounded and uniformly continuous function $h$. By assumption, one knows that $P_{\varepsilon} \Longrightarrow$ $P$, hence, by Problem 4.12 p .64 , in KS98, it suffices to prove that the uniformly bounded sequence $\left(h \circ g_{\varepsilon}\right)$ of continuous functions on $\mathcal{C}([0,+\infty))$ converges uniformly on compact subsets of $\mathcal{C}([0,+\infty))$ to the continuous function $h \circ g$. Let $K$ be a compact set of $\mathcal{C}([0,+\infty))$. Then, for all $\omega \in K, \max _{\left[0, \varepsilon t_{0}\right]}|\omega|$ is uniformly bounded by a constant, called $M$.
Fix $\eta>0$. By the uniform continuity of $h$, there exists $\delta>0$ such that for all $\omega \in K$,

$$
\mathrm{d}_{u}\left(g_{\varepsilon}(\omega), g(\omega)\right) \leq \delta \Rightarrow\left|h \circ g_{\varepsilon}(\omega), h \circ g(\omega)\right| \leq \eta
$$

However, there exists $\varepsilon_{1}>0$ small enough, such that for all $\varepsilon \leq \varepsilon_{1}$, for all $\omega \in K$,

$$
\mathrm{d}_{u}\left(g_{\varepsilon}(\omega), g(\omega)\right) \leq C\left|\int_{0}^{\varepsilon t_{0}} \omega(s) \mathrm{d} s\right| \leq C \varepsilon t_{0} M \leq \delta
$$

STEP 2. We first prove the f.d.d. convergence. The exponential scaling process $V^{(e)}$ satisfies the time-homogeneous equation

$$
\begin{equation*}
\mathrm{d} V_{s}^{(e)}=\mathrm{d} W_{s}-\frac{V_{s}^{(e)}}{2} \mathrm{~d} s-F\left(V_{s}^{(e)}\right) \mathrm{d} s \tag{10}
\end{equation*}
$$

where $\left(W_{t}\right)_{t \geq 0}$ is a standard Brownian motion.
Using the bijection induced by the exponential change of time (Proposition 3.1), we get

$$
\left(\frac{V_{t_{0} e^{t}}}{\sqrt{t_{0}} e^{t / 2}}\right)_{t \geq 0}=\left(H_{t}\right)_{t \geq 0}
$$

as solutions of the same SDE , starting at the same point. This can also be written as

$$
\left(\frac{V_{t}}{\sqrt{t}}\right)_{t \geq t_{0}}=\left(H_{\log \left(t / t_{0}\right)}\right)_{t \geq t_{0}}
$$

So, we have, for all $\varepsilon>0$, and $\left(t_{1}, \cdots, t_{d}\right) \in\left[\varepsilon t_{0},+\infty\right)^{d}$,

$$
\begin{equation*}
\left(\frac{V_{\varepsilon^{-1} t_{1}}}{\sqrt{\varepsilon^{-1} t_{1}}}, \cdots, \frac{V_{\varepsilon^{-1} t_{d}}}{\sqrt{\varepsilon^{-1} t_{d}}}\right)=\left(H_{\log \left(t_{1}\right)+\log \left(\left(\varepsilon t_{0}\right)^{-1}\right)}, \cdots, H_{\log \left(t_{d}\right)+\log \left(\left(\varepsilon t_{0}\right)^{-1}\right)}\right) . \tag{11}
\end{equation*}
$$

As in [GO13], the scale function and the speed measure of $H$ are respectively

$$
\mathrm{p}(x):=\int_{0}^{x} \exp \left(\frac{y^{2}}{2}+\frac{2}{\gamma+1} \operatorname{sgn}(y) F(\operatorname{sgn}(y))|y|^{\gamma+1}\right) \mathrm{d} y
$$

and

$$
\nu_{F}(\mathrm{~d} x):=\exp \left(-\frac{x^{2}}{2}-\frac{2}{\gamma+1} \operatorname{sgn}(x) F(\operatorname{sgn}(x))|x|^{\gamma+1}\right) \mathrm{d} x
$$

By the ergodic theorem (Theorem 23.15 p. 465 in (Kal02), $H$ is $\Lambda_{F}$-ergodic, where $\Lambda_{F}$ is the probability measure associated to $\nu_{\widetilde{F}}$. Call $\widetilde{H}$ the solution of the time homogeneous equation (10) such that the initial condition $\widetilde{H}_{-\infty}$ has the distribution $\Lambda_{F}$.

For $t_{1}, \cdots, t_{d} \in\left[\varepsilon t_{0},+\infty\right)^{d}$, let $\Lambda_{F, t_{1}, \cdots, t_{d}}:=\mathcal{L}\left(\widetilde{H}_{t_{1}}, \cdots, \widetilde{H}_{t_{d}}\right)$ be the distribution of $\left(\widetilde{H}_{t_{1}}, \cdots, \widetilde{H}_{t_{d}}\right)$. Then, for all $s \geq 0, \Lambda_{F, t_{1}, \cdots, t_{d}}=\Lambda_{F, t_{1}+s, \cdots, t_{d}+s}$. Indeed, thanks to the invariance property of $\Lambda_{F},\left(\widetilde{H}_{t}\right)_{t \in \mathbb{R}}$ and $\left(\widetilde{H}_{t+s}\right)_{t \in \mathbb{R}}$ satisfy the same SDE, starting at the same distribution. As a consequence, for all $\varepsilon>0$,

$$
\begin{equation*}
\mathcal{L}\left(\widetilde{H}_{\log \left(t_{1}\right)+\log \left(\left(\varepsilon t_{0}\right)^{-1}\right)}, \cdots, \widetilde{H}_{\log \left(t_{d}\right)+\log \left(\left(\varepsilon t_{0}\right)^{-1}\right)}\right)=\Lambda_{F, \log \left(t_{1}\right), \cdots, \log \left(t_{d}\right)} . \tag{12}
\end{equation*}
$$

Moreover, by exponential ergodicity, for every $\psi: \mathbb{R}^{d} \rightarrow \mathbb{R}$ continuous and bounded function, we can prove that

$$
\begin{equation*}
\left|\mathbb{E}\left[\psi\left(H_{\log \left(t_{1} /\left(t_{0} \varepsilon\right)\right)}, \cdots, H_{\log \left(t_{d} /\left(t_{0} \varepsilon\right)\right)}\right)\right]-\mathbb{E}\left[\psi\left(\widetilde{H}_{\log \left(t_{1} /\left(t_{0} \varepsilon\right)\right)}, \cdots, \widetilde{H}_{\log \left(t_{d} /\left(t_{0} \varepsilon\right)\right)}\right)\right]\right| \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0 \tag{13}
\end{equation*}
$$

We postpone the proof of this convergence in Step 3.
To conclude this step, gather (11), (12) and (13) to get

$$
\left(\frac{V_{\varepsilon^{-1} t_{1}}}{\sqrt{\varepsilon^{-1} t_{1}}}, \cdots, \frac{V_{\varepsilon^{-1} t_{d}}}{\sqrt{\varepsilon^{-1} t_{d}}}\right) \underset{\varepsilon \rightarrow 0}{\Longrightarrow} \Lambda_{F, \log \left(t_{1}\right), \cdots, \log \left(t_{d}\right)} .
$$

This can be written as

$$
\left(\sqrt{\varepsilon} V_{t_{1} / \varepsilon}, \cdots, \sqrt{\varepsilon} V_{t_{d} / \varepsilon}\right) \underset{\varepsilon \rightarrow 0}{\Longrightarrow} T * \Lambda_{F, \log \left(t_{1}\right), \cdots, \log \left(t_{d}\right)}
$$

where $T * \Lambda_{F, \log \left(t_{1}\right), \cdots, \log \left(t_{d}\right)}$ is the pushforward of the measure $\Lambda_{F, \log \left(t_{1}\right), \cdots, \log \left(t_{d}\right)}$ by the linear $\operatorname{map} T\left(u_{1}, \cdots, u_{d}\right):=\left(\sqrt{t_{1}} u_{1}, \cdots, \sqrt{t_{d}} u_{d}\right)$.

Step 3. Let us now prove (13). Pick $\varepsilon t_{0} \leq s \leq t$. Set $h_{0}=v_{0}{\sqrt{t_{0}}}^{-1}$. Actually we prove a more general result, which will also be useful in the last regime. The convergence (13) will be a direct consequence of this lemma.

Lemma 5.2. Let $H$ be an exponential ergodic process with invariant measure $\nu$, solution to a SDE driven by a Brownian motion. Pick a continuous function $\phi:\left[t_{0},+\infty\right) \rightarrow \mathbb{R}$ satisfying $\lim _{s \rightarrow+\infty} \phi(s)=+\infty$.
Then, for all integer $d \geq 1$, every continuous and bounded function $\psi: \mathbb{R}^{d} \rightarrow \mathbb{R}$, all $h_{0} \in \mathbb{R}$ and all $t_{1}, \cdots, t_{d} \in\left[\varepsilon t_{0},+\infty\right)^{d}$,

$$
\left|\mathbb{E}\left[\psi\left(H_{\phi\left(\varepsilon^{-1} t_{1}\right)}, \cdots, H_{\phi\left(\varepsilon^{-1} t_{d}\right)}\right) \mid H_{0}=h_{0}\right]-\mathbb{E}\left[\psi\left(H_{\phi\left(\varepsilon^{-1} t_{1}\right)}, \cdots, H_{\phi\left(\varepsilon^{-1} t_{d}\right)}\right) \mid H_{0} \sim \nu\right]\right| \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0 .
$$

Proof. For the sake of clarity, let us give a proof for $d=2$. The general case $d \geq 2$ is similar. Let $\psi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous and bounded function.
We set $\mu_{\varepsilon}:=\mathcal{L}\left(H_{\phi\left(\varepsilon^{-1} s\right)} \mid H_{0}=h_{0}\right)$. We use the generalized Markov property of solution to SDE driven by Brownian motion process (see Theorem 21.11 p. 421 in [Kal02]). This leads to

$$
\mathbb{E}\left[\psi\left(H_{\phi\left(\varepsilon^{-1} s\right)}, H_{\phi\left(\varepsilon^{-1} t\right)}\right) \mid H_{0}=h_{0}\right]=\mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0} \sim \mu_{\varepsilon}\right]
$$

and, since $\Lambda_{F}$ is invariant,

$$
\mathbb{E}\left[\psi\left(H_{\phi\left(\varepsilon^{-1} s\right)}, H_{\phi\left(\varepsilon^{-1} t\right)}\right) \mid H_{0} \sim \nu\right]=\mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0} \sim \nu\right] .
$$

Then, we are reduced to prove

$$
\left|\mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0} \sim \mu_{\varepsilon}\right]-\mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0} \sim \nu\right]\right| \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0
$$

Hence, setting $p(t, x, \mathrm{~d} y):=\mathbb{P}_{x}\left(H_{t} \in \mathrm{~d} y\right)$ and $\|\cdot\|_{T V}$ for the total variation norm, we get

$$
\begin{gathered}
\left|\mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0} \sim \mu_{\varepsilon}\right]-\mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0} \sim \nu\right]\right| \\
\leq\left|\int_{\mathbb{R}} \mathbb{E}\left[\psi\left(H_{0}, H_{\phi\left(\varepsilon^{-1} t\right)-\phi\left(\varepsilon^{-1} s\right)}\right) \mid H_{0}=y\right]\left(\mu_{\varepsilon}(\mathrm{d} y)-\nu(\mathrm{d} y)\right)\right| \\
\leq\|\psi\|_{\infty} \int_{\mathbb{R}}\left|p\left(\phi\left(\varepsilon^{-1} s\right), h_{0}, \mathrm{~d} y\right)-\nu(\mathrm{d} y)\right| \\
\leq\|\psi\|_{\infty}\left\|p\left(\phi\left(\varepsilon^{-1} s\right), h_{0}, \cdot\right)-\nu\right\|_{T V}
\end{gathered}
$$

We let $\varepsilon \rightarrow 0$, using the exponential ergodicity of $H$.
STEP 4. Let us prove now the tightness of the family of laws of continuous process $\left(V^{(\varepsilon)}\right)_{t \geq \varepsilon t_{0}}=$ $\left(\sqrt{\varepsilon} V_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}}$ on every compact interval $[m, M], 0<m \leq M$. We prove the Kolmogorov criterion stated in Problem 4.11 p. 64, in KS98.
Take $\varepsilon_{0}$ small enough such that for all $\varepsilon \leq \varepsilon_{0}, \varepsilon t_{0} \leq m$. Fix $m \leq s \leq t \leq M$ and $\alpha>2$. Recalling that $B^{(\varepsilon)}$ is a Brownian motion, using Jensen's inequality, moment estimates (Proposition 4.1)
and the relation $\beta=\frac{\gamma+1}{2}$, we can write

$$
\begin{aligned}
\mathbb{E}\left[\left|V_{t}^{(\varepsilon)}-V_{s}^{(\varepsilon)}\right|^{\alpha}\right] & \leq C_{\alpha} \mathbb{E}\left[\left|B_{t}^{(\varepsilon)}-B_{s}^{\varepsilon}\right|^{\alpha}\right]+C_{\alpha} \mathbb{E}\left[\left|\sqrt{\varepsilon} \int_{s / \varepsilon}^{t / \varepsilon} F\left(V_{u}\right) u^{-\beta} \mathrm{d} u\right|^{\alpha}\right] \\
& \leq C_{\alpha} \mathbb{E}\left[\left|B_{t}-B_{s}\right|^{\alpha}\right]+C_{\alpha} \varepsilon^{1-\frac{\alpha}{2}}(t-s)^{\alpha-1} \mathbb{E}\left[\int_{s / \varepsilon}^{t / \varepsilon}\left|F\left(V_{u}\right)\right|^{\alpha} u^{-\beta \alpha} \mathrm{d} u\right] \\
& \leq C_{\alpha} \mathbb{E}\left[\left|B_{t-s}\right|^{\alpha}\right]+C_{\alpha} \varepsilon^{1-\frac{\alpha}{2}}(t-s)^{\alpha-1} \int_{s / \varepsilon}^{t / \varepsilon} u^{\frac{\gamma \alpha}{2}-\beta \alpha} \mathrm{d} u \\
& \leq C_{\alpha}(t-s)^{\frac{\alpha}{2}}+C_{\alpha} \varepsilon^{1-\frac{\alpha}{2}}(t-s)^{\alpha-1} \int_{s / \varepsilon}^{t / \varepsilon} u^{-\frac{\alpha}{2}} \mathrm{~d} u \\
& \leq C_{\alpha}(t-s)^{\frac{\alpha}{2}}+C_{\alpha}(t-s)^{\alpha-1}\left(t^{1-\frac{\alpha}{2}}-s^{1-\frac{\alpha}{2}}\right) \\
& \leq C_{\alpha}(t-s)^{\frac{\alpha}{2}}+C_{\alpha, m, M}(t-s)^{\alpha-1} \\
& \leq C_{\alpha, m, M}(t-s)^{\frac{\alpha}{2}} .
\end{aligned}
$$

Since $\alpha>2$, then $\frac{\alpha}{2}>1$ and the upper bound does not depend on $\varepsilon$. Furthermore, by moment estimates (Proposition 4.1),

$$
\sup _{\varepsilon \leq \varepsilon_{0}} \mathbb{E}\left[\left|V_{m}^{(\varepsilon)}\right|\right] \leq \sqrt{m}<+\infty
$$

Conclusion. The previous steps yields weak convergence on every compact set (Theorem 13.1 p. 139, in Bil99]). The conclusion follows from Theorem 16.7 p. 174, in [Bil99], since all processes considered are continuous.

Example 5.1. We will see that the limiting process $\mathcal{V}$ is more explicit in the linear case $(\gamma=1)$. Choose $F(1)=1, F(-1)=-1$, the process $\widetilde{H}$ solution of (10) is in fact an Ornstein Uhlenbeck process with invariant measure $\Lambda_{F}(d x):=e^{-\frac{3 x^{2}}{2}} \mathrm{~d} x$. It is a centered Gaussian process, hence for all $s_{1}, \cdots, s_{d}$, its f.d.d. $\Lambda_{F, s_{1}, \cdots, s_{d}}$ are Gaussian. As a consequence, knowing the covariance function $K$ is enough to provide the law of the process. Since $\widetilde{H}$ is a stationary OrnsteinUhlenbeck process, one has $K: s, t \mapsto \frac{1}{3} e^{-\frac{3}{2}|t-s|}$. Hence, the limiting process $\mathcal{V}$ having f.d.d $T * \Lambda_{F, \log \left(t_{1}\right), \cdots, \log \left(t_{d}\right)}$ is a centered Gaussian process with covariance function $s, t \mapsto \frac{1}{3} \frac{(s \wedge t)^{2}}{s \vee t}$.

### 5.3 Asymptotic behaviour in the subcritical regime under $H_{1}^{\gamma}$

Assume in this section that $\beta<\frac{\gamma+1}{2}$ and $F: v \mapsto \rho \operatorname{sgn}(v)|v|^{\gamma}$ with $\gamma \geq 1$. For simplicity, we shall write $\varphi$ instead of $\varphi_{q}$.
Proof of Theorem 2.5. Step 1. We first prove the f.d.d. convergence of the velocity process $\left(V_{t}^{(\varepsilon)}\right)_{t \geq \varepsilon t_{0}}:=\left(\varepsilon^{q} V_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}}$. Again we give a proof only for $d=2$, since the general case $d \geq 2$ is similar.
The power scaling process $V^{(q)}$, solution to (3) satisfies

$$
\mathrm{d} V_{s}^{(q)}=\mathrm{d} W_{s}-F\left(V_{s}^{(q)}\right) \mathrm{d} s-q \varphi^{2 q-1}(s) V_{s}^{(q)} \mathrm{d} s
$$

We call $H$ the ergodic process solution to the SDE

$$
\begin{equation*}
\mathrm{d} H_{s}=\mathrm{d} W_{s}-F\left(H_{s}\right) \mathrm{d} s, \text { with } H_{0}=h_{0}:=v_{0} t_{0}^{-q} \tag{14}
\end{equation*}
$$

We denote by $\Pi_{F}(\mathrm{~d} x):=e^{-\frac{2 \rho}{\gamma+1}|x|^{\gamma+1}} \mathrm{~d} x$ its invariant measure. Using the bijection induced by the power change of time (Proposition 3.1), as solutions of the same SDE starting at the same point, we have, for all $\varepsilon>0$, and $s, t \in\left[\varepsilon t_{0},+\infty\right)^{2}$,

$$
\left(\varepsilon^{q} \frac{V_{\varepsilon^{-1} s}}{s^{q}}, \varepsilon^{q} \frac{V_{\varepsilon^{-1} t}}{t^{q}}\right)=\left(V_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}^{(q)}, V_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}^{(q)}\right)
$$

Using Theorem 3.1 p. 27, in [Bil99], it suffices to prove that for all $s, t \in\left[\varepsilon t_{0},+\infty\right)^{2}$,

$$
\begin{aligned}
& \text { - }\left|\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}, H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right)-\left(V_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}^{(q)}, V_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}^{(q)}\right)\right|_{\mathbb{R}^{2}} \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0 . \\
& \text { - }\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}, H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right) \underset{\varepsilon \rightarrow 0}{\Longrightarrow} \Pi_{F} \otimes \Pi_{F} .
\end{aligned}
$$

STEP 2. We prove that for all $t \geq \varepsilon t_{0}, \mathbb{E}\left[\left(H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}-V_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}^{(q)}\right)^{2}\right] \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0$.
Pick $t \geq \varepsilon t_{0}$. For simplicity of notation, we write $H_{t}^{(\varphi, \varepsilon)}:=H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}$ and $V_{t}^{(\varphi, \varepsilon)}:=V_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}^{(q)}$. We have

$$
\mathrm{d}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)=-\varepsilon^{2 q-1}\left(F\left(H_{t}^{(\varphi, \varepsilon)}\right)-F\left(V_{t}^{(\varphi, \varepsilon)}\right)\right) t^{-2 q} \mathrm{~d} t+q t^{-1} V_{t}^{(\varphi, \varepsilon)} \mathrm{d} t
$$

Pick $\delta>0$. By straightforward differentiation, we can write

$$
\begin{aligned}
& \mathrm{d}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)^{2} \leq-\frac{2 \varepsilon^{2 q-1}}{t^{2 q}}\left(F\left(H_{t}^{(\varphi, \varepsilon)}\right)-F\left(V_{t}^{(\varphi, \varepsilon)}\right)\right)\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right) \mathbb{1}_{\left.\right|_{t} ^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)} \mid>\delta} \mathrm{d} t \\
&+2 t^{-1} q V_{t}^{(\varphi, \varepsilon)}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right) \mathrm{d} t
\end{aligned}
$$

Since $\gamma \geq 1$, the function $F^{-1}$ is $\frac{1}{\gamma}$-Hölder, therefore there exists $C_{\gamma}>0$ such that,

$$
\begin{align*}
& \mathrm{d}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)^{2} \leq-\frac{2 \varepsilon^{2 q-1}}{t^{2 q}} C_{\gamma} \delta^{\gamma-1}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)^{2} \mathbb{1}_{\left|H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right|>\delta} \mathrm{d} t \\
&+2 t^{-1} q V_{t}^{(\varphi, \varepsilon)}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right) \mathrm{d} t \tag{15}
\end{align*}
$$

We set $g_{\varepsilon}(t)=\mathbb{E}\left[\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)^{2}\right]$ and $\widetilde{g}_{\varepsilon}(t)=\mathbb{E}\left[\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)^{2} \mathbb{1}_{H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)} \mid>\delta}\right]$.
Taking expectation in (15), we get

$$
\begin{equation*}
\widetilde{g}_{\varepsilon}^{\prime}(t) \leq-\frac{2 \varepsilon^{2 q-1}}{t^{2 q}} C_{\gamma} \delta^{\gamma-1} \widetilde{g}_{\varepsilon}(t)+b_{\varepsilon}(t), \text { with } \widetilde{g}_{\varepsilon}\left(\varepsilon t_{0}\right)=0 \tag{16}
\end{equation*}
$$

where

$$
b_{\varepsilon}(t):=2 t^{-1} q \mathbb{E}\left[V_{t}^{(\varphi, \varepsilon)}\left(H_{t}^{(\varphi, \varepsilon)}-V_{t}^{(\varphi, \varepsilon)}\right)\right] .
$$

Using Cauchy-Schwarz inequality and moment estimates (Proposition 4.1), we have

$$
\left|b_{\varepsilon}(t)\right| \leq 2 t^{-1}|q| \sqrt{\mathbb{E}\left[\left(V_{t}^{(\varphi, \varepsilon)}\right)^{2}\right]} \sqrt{g_{\varepsilon}(t)} \leq C 2 t^{-1}|q| \sqrt{\varepsilon^{2 q-1} t^{1-2 q} g_{\varepsilon}(t)}
$$

Set $h(t):=\frac{1}{1-2 q} C_{\gamma} \delta^{\gamma-1} t^{1-2 q}$. We use the comparison theorem for ordinary differential equation on (16) to get

$$
\widetilde{g}_{\varepsilon}(t) \leq \int_{\varepsilon t_{0}}^{t} b_{\varepsilon}(s) \exp \left(-2 \varepsilon^{2 q-1}(h(t)-h(s))\right) \mathrm{d} s
$$

As a consequence, we deduce that

$$
\begin{aligned}
g_{\varepsilon}(t) & \leq \delta^{2}+\widetilde{g}_{\varepsilon}(t) \\
& \leq \delta^{2}+\exp \left(-2 \varepsilon^{2 q-1} h(t)\right) C \int_{\varepsilon t_{0}}^{t} 2 s^{-1} \sqrt{\varepsilon^{2 q-1} s^{1-2 q}} \sqrt{g_{\varepsilon}(s) \exp \left(2 \varepsilon^{2 q-1} h(s)\right)} \exp \left(\varepsilon^{2 q-1} h(s)\right) \mathrm{d} s .
\end{aligned}
$$

Applying a Gronwall-type lemma (Lemma 4.3) to the function $g_{\varepsilon} \exp \left(2 \varepsilon^{2 q-1} h\right)$, we obtain

$$
g_{\varepsilon}(t) \leq C \delta^{2}+C\left(\int_{\varepsilon t_{0}}^{t} s^{-1} \sqrt{\varepsilon^{2 q-1} s^{1-2 q}} \exp \left(-\varepsilon^{2 q-1}(h(t)-h(s))\right) \mathrm{d} s\right)^{2} .
$$

We conclude, using the dominated convergence theorem, since $1-2 q>0$, that for all $\delta>0$

$$
\begin{equation*}
0 \leq \limsup _{\varepsilon \rightarrow 0} g_{\varepsilon}(t) \leq \delta^{2} \tag{17}
\end{equation*}
$$

To prove the domination hypothesis, notice that by optimization of the function $x \mapsto \sqrt{x} \exp (-A x)$,

$$
\mathbb{1}_{\varepsilon t_{0} \leq s \leq t} s^{-1} \sqrt{\varepsilon^{2 q-1} s^{1-2 q}} \exp \left(-\varepsilon^{2 q-1}(h(t)-h(s))\right) \leq \mathbb{1}_{0 \leq s \leq t} s^{-\frac{1}{2}-q} \frac{1}{\sqrt{h(t)-h(s)}}
$$

This function is integrable, since $1-2 q>0$.
We let $\delta \rightarrow 0$ in (17) to conclude that for all $t>0, \lim _{\varepsilon \rightarrow 0} g_{\varepsilon}(t)=0$.
Step 3. Pick $s, t \in\left[\varepsilon t_{0},+\infty\right)^{2}$. We prove that the solution $H$ to (14) satisfies

$$
\begin{equation*}
\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}, H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right) \underset{\varepsilon \rightarrow 0}{\Longrightarrow} \Pi_{F} \otimes \Pi_{F} \tag{18}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)=\frac{t^{1-2 q}-s^{1-2 q}}{\varepsilon^{1-2 q}} \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0 \tag{19}
\end{equation*}
$$

By Lemma 5.2, for every continuous and bounded function $\varphi$, we can write

$$
\left|\mathbb{E}\left[\psi\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}, H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right) \mid H_{0}=h_{0}\right]-\mathbb{E}\left[\psi\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}, H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right) \mid H_{0} \sim \Pi_{F}\right]\right| \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0
$$

Hence, it suffices to prove that for every bounded continuous functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$, the following convergence holds

$$
\lim _{\varepsilon \rightarrow 0} \mathbb{E}\left[f\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}\right) g\left(H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right) \mid H_{0} \sim \Pi_{F}\right]=\Pi_{F}(f) \Pi_{F}(g)
$$

The following reasoning is inspired from the proof of Lemma 3.2 p. 7-8 in CCM10. Since $H_{0}$ is starting from the invariant measure, up to considering $f-\Pi_{F}(f)$ and $g-\Pi_{F}(g)$, we can assume that $f$ and $g$ have zero $\Pi_{F}$-mean. We call $\left(P_{t}\right)_{t \geq 0}$ the semigroup of $H$, then we get, by invariance property of $\Pi_{F}$,

$$
\begin{aligned}
\mathbb{E}\left[f\left(H_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}\right) g\left(H_{\varphi^{-1}\left(\varepsilon^{-1} t\right)}\right) \mid H_{0} \sim \Pi_{F}\right] & =\int P_{\varphi^{-1}\left(\varepsilon^{-1} s\right)}\left(f P_{\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)} g\right) \mathrm{d} \Pi_{F} \\
& =\int f P_{\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)} g \mathrm{~d} \Pi_{F}
\end{aligned}
$$

Note that $U: v \mapsto \frac{|v|^{1+\gamma}}{1+\gamma}$ is a convex function, thus a $\lambda$-Poincaré inequality holds for the process $H$ (see Bob99 p. 1904). This implies the exponential decay of the variance (see Theorem 4.2.5 p. 183, in [BGL14]), i.e. there exists a constant $C>0$ such that, since $\Pi_{F}$ is a probability measure,

$$
\begin{aligned}
\left|\int f P_{\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)} g \mathrm{~d} \Pi_{F}\right| & \leq\left\|f P_{\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)} g\right\|_{2} \\
& \leq\|f\|_{\infty}\left\|P_{\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)} g\right\|_{2} \\
& \leq C\|f\|_{\infty}\|g\|_{\infty} e^{-\lambda\left(\varphi^{-1}\left(\varepsilon^{-1} t\right)-\varphi^{-1}\left(\varepsilon^{-1} s\right)\right)}
\end{aligned}
$$

We deduce (18) from (19).
Step 4. We prove the f.d.d. convergence of the position process $\left(X_{t}^{(\varepsilon)}\right)_{t \geq \varepsilon t_{0}}:=\left(\varepsilon^{\beta+\frac{1}{2}} X_{t / \varepsilon}\right)_{t \geq \varepsilon t_{0}}$. Take $\gamma=1$ and $\beta \in\left(-\frac{1}{2}, 1\right)$. Pick $t \geq \varepsilon t_{0}$. By Itô's formula applied to $t^{\beta} V_{t}$, we get

$$
\rho X_{t}^{(\varepsilon)}=\varepsilon^{\beta+\frac{1}{2}}\left(t_{0}^{\beta} v_{0}+x_{0}\right)-\varepsilon^{\frac{1-\beta}{2}} t^{\beta} V_{t}^{(\varepsilon)}+\varepsilon^{\beta+\frac{1}{2}} \int_{t_{0}}^{t / \varepsilon} s^{\beta} \mathrm{d} B_{s}+\varepsilon^{\beta+\frac{1}{2}} \int_{t_{0}}^{t / \varepsilon} \beta s^{\beta-1} V_{s} \mathrm{~d} s
$$

Since $\beta>-\frac{1}{2}$, the first term converges to 0 in probability as $\varepsilon \rightarrow 0$. Moreover, by Itô's formula, for all $t \geq t_{0}$,

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbb{E}\left[V_{t}^{2}\right]=-2 \rho s^{-\beta} \mathbb{E}\left[V_{s}^{2}\right]+1
$$

Hence, by comparison theorem for ordinary differential equation,

$$
\mathbb{E}\left[V_{t}^{2}\right] \leq \exp \left(-2 \rho \frac{t^{1-\beta}}{1-\beta}\right)\left(v_{0}^{2}+\int_{t_{0}}^{t} \exp \left(2 \rho \frac{s^{1-\beta}}{1-\beta}\right) \mathrm{d} s\right)
$$

Using an integration by parts, we deduce that there exists a positive constant $C$ such that, for all $t \geq t_{0}$,

$$
\mathbb{E}\left[V_{t}^{2}\right] \leq C t^{\beta}
$$

As a consequence, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\left|-\varepsilon^{\frac{1-\beta}{2}} t^{\beta} V_{t}^{(\varepsilon)}+\varepsilon^{\beta+\frac{1}{2}} \int_{t_{0}}^{t / \varepsilon} \beta s^{\beta-1} V_{s} \mathrm{~d} s\right|\right] & \leq \varepsilon^{\frac{1-\beta}{2}} t^{\beta} \mathbb{E}\left[\left|V_{t}^{(\varepsilon)}\right|\right]+\varepsilon^{\beta+\frac{1}{2}} \int_{t_{0}}^{t / \varepsilon} \beta s^{\beta-1} \mathbb{E}\left[\left|V_{s}\right|\right] \mathrm{d} s \\
& \leq C \varepsilon^{\frac{1}{2}} t^{\frac{3 \beta}{2}}+C \varepsilon^{\frac{1-\beta}{2}} t^{\frac{3 \beta}{2}}-C \varepsilon^{\beta+\frac{1}{2}} t_{0}^{\frac{3 \beta}{2}} \underset{\varepsilon \rightarrow 0}{\longrightarrow} 0
\end{aligned}
$$

It remains to study the centered Gaussian process $M_{t}^{(\varepsilon)}:=\varepsilon^{\beta+\frac{1}{2}} \int_{t_{0}}^{t / \varepsilon} s^{\beta} \mathrm{d} B_{s}$. By Itô's isometry and since $\beta>-\frac{1}{2}$, for all $\varepsilon t_{0} \leq s \leq t$, we can write

$$
\operatorname{Cov}\left(M_{s}^{(\varepsilon)}, M_{t}^{(\varepsilon)}\right)=\varepsilon^{2 \beta+1} \int_{t_{0}}^{s / \varepsilon} u^{2 \beta} \mathrm{~d} s \underset{\varepsilon \rightarrow 0}{\sim} \frac{s^{1+2 \beta}}{1+2 \beta}
$$

Since the convergence of centered Gaussian processes is characterized by the convergence of their covariance functions, the conclusion follows from Theorem 3.1, p. 27, in Bil99.

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