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# Diffusive behaviour for some time-inhomogeneous stochastic kinetic models

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**Abstract:** In this paper, we consider a kinetic stochastic model with a non-linear time-inhomogeneous drag force and a Brownian random force. More precisely, we study the position  $X_t$  of a particle having the velocity as solution of a stochastic differential equation driven by a one-dimensional Brownian motion, with the drift of the form  $t^{-\beta}F(v)$ ,  $F$  satisfying some homogeneity condition and  $\beta > 0$ . The diffusive behaviour of the position in large time is proven and the precise rate of convergence is pointed out by using stochastic analysis tools.

**Key words:** kinetic stochastic equation; time-inhomogeneous diffusions; explosion times; scaling transformations; asymptotic distributions.

**MSC2010 Subject Classification:** Primary 60J60; Secondary 60H10; 60J65; 60F05.

## 1 Introduction

It is classical that the kinetic Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle submitted to some drag and random forces. Moreover the Feynman-Kac formula allows to make the link between the kinetic Fokker-Planck equation and stochastic differential equation driven by a Brownian motion called Langevin equation. In the simple linear case the solution of the Langevin equation is a well-known Gaussian process, the Ornstein-Uhlenbeck process.

Some models in several domains as fluids dynamics, statistical mechanics, biology, are based on the Fokker-Planck and Langevin equations in their classical form or on generalisations, for instance non-linear or driven by other random noises than Brownian motion. The behaviour in large time of the solution to the corresponding stochastic differential equation is one of the usual questions when studying these models. Although the tools of partial differential equations allowed to ask of this kind of questions, since these models are probabilistic, tools based on stochastic processes could be used. For instance, in [2] the persistent turning walker model was introduced, inspired from the modelling of fish motion. An associated diffusion solves a kinetic Fokker-Planck equation based on an Ornstein-Uhlenbeck Gaussian process and the authors studied the large time diffusive behaviour of this model by using appropriate tools from stochastic analysis.

In the last decade the asymptotic study of solutions of non-linear Langevin's type was the subject of an important number of papers, see for instance [3], [5], [6]. For instance in [6] the

following system is studied

$$V_t = v_0 + B_t - \frac{\rho}{2} \int_0^t F(V_s) ds \quad \text{and} \quad X_t = x_0 + \int_0^t V_s ds.$$

In other words one considers a particle moving such that its velocity is a diffusion with an invariant measure behaving like  $(1 + |v|^2)^{-\rho/2}$ , as  $|v| \rightarrow \infty$ . The authors prove that for large time, after a suitable rescaling, the position process behaves as a Brownian motion or other stable processes, following the values of  $\rho$ . It should be noted that in these cited papers the standard tools associated to time-homogeneous equations are used: invariant measure, scale function, speed measure and so on. Several of these tools will not be available when the drag force is depending explicitly on time.

Let us first describe our problem: consider a one-dimensional time-inhomogeneous stochastic kinetic model driven by a Brownian motion. We denote by  $(X_t)_{t \geq 0}$  the one-dimensional process describing the position of a particle at time  $t$  having the velocity  $V_t$ . The velocity process  $(V_t)_{t \geq 0}$  is supposed to follow a Brownian dynamics in a potential  $U(t, v)$ , varying in time :

$$dV_t = dB_t - \frac{1}{2} \partial_v U(t, V_t) dt \quad \text{and} \quad X_t = X_0 + \int_0^t V_s ds. \quad (1)$$

In the present paper the potential is of the form  $t^{-\beta} \int_0^v F(u) du$ , with  $\beta > 0$  and  $F$  satisfying some homogeneity condition. A natural question is to understand the behaviour of the position process in large time. More precisely we look for the limit in distribution of  $v(\varepsilon)X_{t/\varepsilon}$ , as  $\varepsilon \rightarrow 0$ , where  $v(\varepsilon)$  is some rate of convergence.

We will see that the position process has a diffusive behaviour and we give the precise rate and variance of the limit, by studying a Brownian martingale. The strategy to tackle this problem is based on the following idea: use a scaling transformation in order to get a stochastic differential equation having a negligible term, as  $t$  is large, in order to compare with a simpler stochastic differential equation (see also [7] for a similar reasoning).

Our paper is organised as follows: in the next section we introduce notations and we state our main result. Existence and non-explosion of solutions are studied in Section 3 and the proof of our main result is given in Section 4. We collect in the Appendix our technical results.

## 2 Notations and main result

Assume first that the system velocity - position is given, for  $t \geq t_0 > 0$ , by

$$dV_t = dB_t - t^{-\beta} F(V_t) dt, \quad V_{t_0} = v_0 > 0, \quad \text{and} \quad dX_t = V_t dt, \quad X_{t_0} = x_0 \in \mathbb{R}. \quad (\text{SKE})$$

where  $\beta > 0$  and  $(B_t)_{t \geq 0}$  is a standard Brownian motion.  $F$  is supposed to satisfy either  $(H_1\alpha)$  which is a homogeneity condition or  $(H_2\alpha)$  when it is dominated by a positive function satisfying the homogeneity condition. That is,  $F$  satisfies either

$$\text{for some } \alpha \in \mathbb{R}, \quad \forall x \in \mathbb{R}, \quad \lambda > 0, \quad F(\lambda x) = \lambda^\alpha F(x), \quad (H_1\alpha)$$

or

$$\begin{aligned} & \forall x \in \mathbb{R}, \quad \forall \lambda > 0, \quad \text{sgn}(F(\lambda x)) = \text{sgn}(F(x)) \\ & \text{and } F \text{ is dominated by a positive function } G \text{ satisfying } (H_1\alpha). \end{aligned} \quad (H_2\alpha)$$

Obviously  $(H_2\alpha)$  is a generalization of  $(H_1\alpha)$ . Nevertheless, we keep both assumptions since some proofs are simpler written under  $(H_1\alpha)$  and are similar under  $(H_2\alpha)$ . As an example of function satisfying  $(H_1\alpha)$  one can keep in mind  $F : v \mapsto \text{sgn}(v) |v|^\alpha$  (see also [7]), and as an example of function satisfying  $(H_2\alpha)$  (with  $\alpha = 0$ )  $F : v \mapsto v/(1+v^2)$  (see also [6]).

**Remark 2.1.** *If a function  $\pi$  satisfies  $(H_1\alpha)$ , then*

$$\pi(x) = \begin{cases} \pi(1)x^\alpha & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ \pi(-1)|x|^\alpha & \text{if } x < 0. \end{cases} \quad (2)$$

Let us state our main result which holds under both assumptions.

**Theorem 2.2.** *Consider  $\alpha \geq 0$  and  $\beta > \frac{\alpha+1}{2}$ . Let  $(V_t, X_t)_{t \geq t_0}$  be a solution to (SKE). If  $\alpha < 1$  or  $(F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+$ . Then, as  $\varepsilon \rightarrow 0$ ,*

$$(\varepsilon^{3/2} X_{t/\varepsilon})_{t \geq t_0} \Longrightarrow (\mathcal{B}_{t^3/3})_{t \geq t_0}, \quad (3)$$

*in the space of continuous functions  $\mathcal{C}([t_0, \infty))$  endowed by the uniform topology, where  $(\mathcal{B}_t)_{t \geq 0}$  is a standard Brownian motion.*

**Remark 2.3.**

1. *When  $\alpha = 1$  and  $(F(-1), F(1)) \notin \mathbb{R}^- \times \mathbb{R}^+$  the result is still true by assuming some technical conditions on  $t_0$ ,  $F(1)$  and  $F(-1)$  (see Remark A.5).*
2. *By trying to adapt naively the proof of Theorem 1a) p. 2 in [6], we are led to find a solution to the Poisson equation  $\frac{1}{2}\partial_{xx}^2 g(s, x) + \partial_s g(s, x) - F(x)s^{-\beta}\partial_x g(s, x) = -x$ . This PDE does not admit an evident solution and seems to be ill-posed. Thus, due to the time-dependence of the stochastic differential equation satisfied by the velocity process, one has to proceed quite differently.*

### 3 Existence and non-explosion of solution

In the following, suppose  $\alpha \geq 0$  and  $\beta > \frac{\alpha+1}{2}$  and set  $\Omega = \overline{\mathcal{C}}([t_0, \infty))$ , the set of continuous functions  $\omega : [t_0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$  which equals  $\infty$  after their explosion time (possibly infinite). Following the idea used in [7], we first perform a change of time in (SKE) in order to produce at least one time-homogeneous coefficient in the transformed equation. For every  $\mathcal{C}^2$ -diffeomorphism  $\phi : [0, t_1) \rightarrow [t_0, \infty)$ , let introduce the scaling transformation  $\Phi_\phi$  given, for  $\omega \in \Omega$ , by

$$\Phi_\phi(\omega)(s) := \frac{\omega(\phi(s))}{\sqrt{\phi'(s)}}, \text{ with } s \in [0, t_1).$$

The result containing the change of time transformation is given in [7], Proposition 2.1, p. 187. For the sake of completeness we state and sketch the proof in our context.

**Proposition 3.1.** *If  $V$  is a solution to equation (SKE), then  $V^{(\phi)}$  is a solution to*

$$dV_s^{(\phi)} = dW_s - \frac{\sqrt{\phi'(s)}}{\phi(s)^\beta} F(\sqrt{\phi'(s)} V_s^{(\phi)}) ds - \frac{\phi''(s)}{\phi'(s)} \frac{V_s^{(\phi)}}{2} ds, \quad V_0^{(\phi)} = \frac{V_{\phi(0)}}{\sqrt{\phi'(0)}}, \quad (4)$$

where  $V^{(\phi)} = \Phi_\phi(V)$  and  $W_t := \int_0^t \frac{dB_{\phi(s)}}{\sqrt{\phi'(s)}}$ .

If  $V^{(\phi)}$  is a solution to (4), then  $V$  is a solution to equation (SKE), where  $V = \Phi_\phi^{-1}(V^{(\phi)})$

and  $B_t - B_{t_0} := \int_{t_0}^t \sqrt{(\phi' \circ \phi^{-1})(s)} dW_{\phi^{-1}(s)}$ .

Furthermore uniqueness in law, pathwise uniqueness or strong existence hold for equation (SKE) if and only if they hold for equation (4).

*Proof.* Let  $V$  be a solution to equation (SKE). Thanks to Lévy's characterization theorem of the Brownian motion,  $W$  is a standard Brownian motion. Then, by a change of variable  $t = \phi(s)$ , one gets

$$V_{\phi(t)} - V_{\phi(0)} = \int_0^t \sqrt{\phi'(s)} dW_s - \int_0^t \frac{F(V_{\phi(s)})}{\phi(s)^\beta} \phi'(s) ds.$$

The integration by parts formula yields

$$d\left(\frac{V_{\phi(s)}}{\sqrt{\phi'(s)}}\right) = dW_s - \frac{\sqrt{\phi'(s)}}{\phi(s)^\beta} F(V_{\phi(s)}) ds - \frac{\phi''(s)}{2\phi'(s)} \frac{V_{\phi(s)}}{\sqrt{\phi'(s)}} ds.$$

From which follows (4). The proof of the second part is similar.  $\square$

In the following, we will use two particular changes of time, depending on which term of (4) should become time-homogeneous:

- *exponential change of time:* denoting  $\phi_e : t \mapsto t_0 e^t$ , the exponential scaling transformation is given by  $\Phi_e(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega_{t_0 e^s}}{\sqrt{t_0 e^{s/2}}}$ , for  $\omega \in \Omega$ . Set  $V^{(e)} := \Phi_e(V)$ . Thanks to Proposition 3.1, the process  $(V^{(e)})_{t \geq 0}$  satisfies the equation

$$dV_s^{(e)} = dW_s - \frac{V_s^{(e)}}{2} ds - t_0^{1/2-\beta} e^{(1/2-\beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds, \quad (5)$$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion.

- *power change of time:* setting  $\gamma := \frac{2\beta}{\alpha+1} > 1$ , consider  $\phi_\gamma \in \mathcal{C}^2([0, t_1])$  the solution to the Cauchy problem

$$\phi'_\gamma = \phi_\gamma^\gamma, \quad \phi_\gamma(0) = t_0.$$

Clearly  $\phi_\gamma(t) = (t_0^{1-\gamma} + (1-\gamma)t)^{1/(1-\gamma)}$ : the maximal time  $t_1$  satisfies  $(\gamma-1)t_1 = t_0^{1-\gamma}$  and the power scaling transformation is given by  $\Phi_\gamma(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega(\phi_\gamma(s))}{\phi_\gamma(s)^{\gamma/2}}$ . The process  $V^{(\gamma)} := V^{(\phi_\gamma)}$  satisfies the equation

$$dV_s^{(\gamma)} = dW_s - \rho \phi_\gamma^{-\alpha\beta/(\alpha+1)}(s) F\left(\sqrt{\phi'_\gamma(s)} V_s^{(\gamma)}\right) ds - \gamma \phi_\gamma^{\gamma-1}(s) \frac{V_s^{(\gamma)}}{2} ds, \quad (6)$$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion.

In the following we will study the existence and the behaviour of the solution to (SKE), first under the homogeneity assumption  $(H_1\alpha)$  and then under the domination assumption  $(H_2\alpha)$ .

### 3.1 Study under $(H_1\alpha)$

In the following we assume  $(H_1\alpha)$ . Then the process  $V^{(\gamma)}$  satisfies the equation

$$dV_s^{(\gamma)} = dW_s - F(V_s^{(\gamma)}) ds - \gamma \phi_\gamma^{\gamma-1}(s) \frac{V_s^{(\gamma)}}{2} ds, \quad s \in [0, t_1), \quad (7)$$

which can be written, by using the expression of  $\phi_\gamma$ , as

$$dV_s^{(\gamma)} = dW_s - F(V_s^{(\gamma)}) ds - \delta \frac{V_s^{(\gamma)}}{t_1 - s} ds, \quad s \in [0, t_1), \quad (8)$$

where  $\delta = \frac{\gamma}{2(\gamma-1)}$ . Proposition 3.2, p. 188, in [7] can be stated in the present situation:

**Proposition 3.2.** *For  $\alpha > -1$ , there exists a pathwise unique strong solution to (SKE), defined up to the explosion time.*

*Proof.* We sketch the proof in our context. Note that, since  $\alpha > -1$ ,  $x \mapsto |x|^\alpha$  is locally integrable. Leaving out the third term on the right-hand side of (7), one gets

$$dH_s = dW_s - F(H_s) ds, \quad s \in [0, t_1). \quad (9)$$

By using Proposition 2.2, p. 28, in [4], there exists a unique weak solution  $H$  to the time-homogeneous equation (9) defined up to the explosion time. Moreover, the Girsanov transformation induces a linear bijection between weak solutions defined up to the explosion time to equations (7) and (9). It follows that there exists a unique weak solution  $V^{(\gamma)}$  to equation (7). Therefore, by using Proposition 3.1, there exists a unique weak solution  $V$  to equation (SKE). Besides, by using Corollary 3.4 and Proposition 3.2, pp. 389-390, in [11], pathwise uniqueness holds for the equation (SKE). The conclusion follows from Theorem 1.7, p. 368, in [11].  $\square$

**Remark 3.3.** *When  $\alpha = 1$ , drift and diffusion are Lipschitz and satisfy locally linear growth. The existence and non-explosion of  $V$  follow from Theorem 2.9, p. 289, in [9].*

**Proposition 3.4.**

- When  $\alpha \leq 1$  or  $(F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+$ , the explosion time of  $V$  is a.s. infinite.
- Else, i.e. if  $\alpha > 1$  and  $(F(-1), F(1)) \in (\mathbb{R}_+^* \times \mathbb{R}_+) \cup (\mathbb{R} \times \mathbb{R}_-)$ ,  $\mathbb{P}(\tau_\infty = \infty) \in (0, 1)$ , where  $\tau_\infty$  denotes the explosion time of  $V$ .

*Proof.* We split the proof in several steps.

STEP 1. Assume first that  $\alpha \leq 1$  or  $(F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+$ . We will use Theorem 10.2.1, p. 254, in [12]. Call  $\mathcal{L}_t$  the time-inhomogeneous infinitesimal generator of  $V$ , given by

$$\mathcal{L}_t := \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{F(x)}{t^\beta} \frac{\partial}{\partial x}. \quad (10)$$

Let  $\varphi$  be a twice continuous differentiable positive function such that  $\varphi(x) = 1 + x^2$  for all  $|x| \geq 1$ ,  $\varphi(x) = 1$  for all  $|x| \leq \frac{1}{2}$  and  $\varphi \geq 1$ . Note that  $\varphi$  does not depend on time. Hence  $(\partial_t + \mathcal{L}_t)\varphi = \mathcal{L}_t\varphi$ .

Let  $T \geq t_0$  and call  $c_T$  the supremum of  $\mathcal{L}_t\varphi$  on  $[t_0, T] \times [-1, 1]$ . Then, for all  $|x| \leq 1$  and  $t \in [t_0, T]$ ,

$$\mathcal{L}_t\varphi(x) \leq c_T \leq c_T\varphi(x).$$

Moreover, for all  $|x| > 1$  and  $t \in [t_0, T]$ , for  $C$  a positive constant,

$$\mathcal{L}_t\varphi(x) = -2x \frac{F(x)}{t^\beta} + 1 \leq \begin{cases} 1 \leq \varphi(x), & \text{if } (F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+, \\ 2 \max(|F(1)|, |F(-1)|)x^2 + 1 \leq C\varphi(x), & \text{if } \alpha \leq 1. \end{cases}$$

So, by using Theorem 10.2.1, p. 254, in [12], we deduce that  $\tau_\infty$  is infinite a.s.

STEP 2. Assume now the contrary, that is  $\alpha > 1$  and  $(F(-1), F(1)) \in (\mathbb{R}_+^* \times \mathbb{R}_+) \cup (\mathbb{R} \times \mathbb{R}_-)$ . We follow the ideas of the proof of Proposition 3.7, pp. 191-192, in [7]. We first show that  $\mathbb{P}(\tau_\infty = \infty) > 0$ . Let  $V^{(\gamma)}$  be the pathwise unique strong solution to equation (8). Also denote by  $b$ , the  $\delta$ -Brownian bridge, the pathwise unique strong solution to equation

$$db_s = dW_s - \delta \frac{b_s}{t_1 - s} ds, \quad b_0 = x_0, \quad s \in [0, t_1]. \quad (11)$$

Note that the equation (11) is obtained from (8) by omitting the second term on the right-hand side. Denote by  $\tau_\infty^{(\gamma)}$  the explosion time of  $V^{(\gamma)}$ , clearly,  $\tau_\infty^{(\gamma)} \in [0, t_1] \cup \{\infty\}$  a.s. and  $\{\tau_\infty^{(\gamma)} \geq t_1\} = \{\tau_\infty = \infty\}$ . Note that  $b$  becomes continuous on  $[0, t_1]$ , with  $b_{t_1} = 0$  a.s.

Fix  $n \geq 1$ , for all  $s \in [0, t_1]$ , define

$$T_n := \inf \left\{ s \in [0, t_1], \left| V_s^{(\gamma)} \right| \geq n \right\}, \quad \sigma_n := \inf \{ s \in [0, t_1], |b_s| \geq n \},$$

and

$$\mathcal{E}(s) := \exp \left( \int_0^s -F(b_u) dW_u - \frac{1}{2} \int_0^s F(b_u)^2 du \right).$$

Then, one has, since  $\alpha \geq 0$ ,

$$\begin{aligned} \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^{s \wedge \sigma_n} F(b_u)^2 du \right) \right] &\leq \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^{s \wedge \sigma_n} n^{2\alpha} \max(F(1)^2, F(-1)^2) du \right) \right] \\ &\leq \exp \left( \frac{t_1}{2} n^{2\alpha} \max(F(1)^2, F(-1)^2) \right), \end{aligned}$$

so Novikov's condition applies to  $(\mathcal{E}_{s \wedge \sigma_n})_{s \geq 0}$ . By using the Girsanov transformation between  $b$  and  $V^{(\gamma)}$ , we can write for every integer  $n \geq 1$ ,  $s \in [0, t_1]$  and  $A \in \mathcal{F}_s$ ,

$$\mathbb{E} \left[ \mathbb{1}_A \left( V_{\bullet \wedge T_n}^{(\gamma)} \right) \mathbb{1}_{T_n > s} \right] = \mathbb{E} \left[ \mathbb{1}_A (b_{\bullet \wedge \sigma_n}) \mathcal{E}(s \wedge \sigma_n) \mathbb{1}_{\sigma_n > s} \right].$$

Letting  $n \rightarrow \infty$ , we obtain by the dominated convergence theorem and Fatou's lemma,

$$\mathbb{E} \left[ \mathbb{1}_A \left( V^{(\gamma)} \right) \mathbb{1}_{\tau_\infty^{(\gamma)} > s} \right] \geq \mathbb{E} \left[ \mathbb{1}_A (b) \mathcal{E}(s) \right].$$

Hence,  $\mathbb{P}(\tau_\infty = \infty) = \mathbb{P}(\tau_\infty^{(\gamma)} \geq t_1) \geq \mathbb{E}[\mathcal{E}(t_1)] > 0$ .

STEP 3. We will show that  $\mathbb{P}(\tau_\infty = \infty) < 1$  when  $F(1) > 0$  and  $F(-1) > 0$ . Our strategy is to apply Theorem 10.2.1, p. 254, in [12]. Let  $T > t_0$  and choose  $a \in (1, \alpha)$ . Also, one can choose  $k \geq 1$  such that  $a(a-1)^{-1} < k(T-t_0)$ . Introduce the continuous differentiable negative

function  $f_1 : x \mapsto \frac{-1/2}{1+|x|^a}$ , and, for  $\mu > 0$ , the bounded twice continuous differentiable function

$$G_{1,\mu}(x) = \exp\left(\mu \int_{-\infty}^x f_1(y) dy\right), \quad x \in \mathbb{R}.$$

For all  $t \in [t_0, T]$  and  $x \in \mathbb{R}$ ,

$$\begin{aligned} (\partial_t + \mathcal{L}_t)G_{1,\mu}(x) &= \mathcal{L}_t G_{1,\mu}(x) = \mu G_{1,\mu}(x) \left[ F(x)t^{-\beta} |f_1(x)| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x) \right] \\ &\geq \mu G_{1,\mu}(x) \left[ F(x)T^{-\beta} |f_1(x)| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x) \right]. \end{aligned}$$

Since  $|f_1(x)| \underset{|x| \rightarrow \infty}{\sim} \frac{1}{2}|x|^{-a}$ ,  $\lim_{|x| \rightarrow \infty} F(x)|f_1(x)| = +\infty$ , and using that  $\lim_{|x| \rightarrow \infty} f_1'(x) = 0$ , there exists  $r \geq 1$  such that, for all  $\mu > 0$ ,

$$(\partial_t + \mathcal{L}_t)G_{1,\mu}(x) \geq \mu G_{1,\mu}(x) \left[ F(x)T^{-\beta} |f_1(x)| + \frac{1}{2}f_1'(x) \right] \geq k\mu G_{1,\mu}(x) \text{ on } [t_0, T] \times [-r, r]^c.$$

Moreover, since  $f_1^2$  is bounded away from zero, while  $|f_1'|$  is bounded on  $[-r, r]$ , there exists  $\mu_0$ , such that, since  $F$  is non-negative,

$$(\partial_t + \mathcal{L}_t)G_{1,\mu_0}(x) \geq \mu_0 G_{1,\mu_0}(x) \left[ \frac{1}{2}f_1'(x) + \frac{\mu_0}{2}f_1^2(x) \right] \geq k\mu_0 G_{1,\mu_0}(x) \text{ on } [t_0, T] \times [-r, r].$$

Hence, for all  $t \in [t_0, T]$  and  $x \in \mathbb{R}$ ,  $(\partial_t + \mathcal{L}_t)G_{1,\mu_0}(x) \geq k\mu_0 G_{1,\mu_0}(x)$ . Besides, since  $|f_1(x)| \leq 1 \wedge |x|^{-a}$ ,

$$\int_{-\infty}^{x_0} (-f_1(x)) dx \leq \int_{\mathbb{R}} (1 \wedge |x|^{-a}) dx = a(a-1)^{-1} < k(T-t_0).$$

Thus,  $G_{1,\mu_0}(x_0) > e^{-k\mu_0(T-t_0)} \geq e^{-k\mu_0(T-t_0)} \sup_{x \in \mathbb{R}} G_{1,\mu_0}(x)$ . Therefore, Theorem 10.2.1, p. 254, in [12] applies and  $V$  explodes in finite time with positive probability.

When  $F(-1) < 0$  and  $F(1) < 0$ , one can work in the same way, using instead  $G_{1,\mu}$  the function  $x \mapsto \exp\left(\mu \int_x^{+\infty} f_1(y) dy\right)$ , in order to get that  $\mathbb{P}(\tau_\infty = \infty) < 1$ .

STEP 4. It remains to show that  $\mathbb{P}(\tau_\infty = \infty) < 1$  when  $F(1) < 0$  and  $F(-1) > 0$ . As in the previous step, we choose  $a \in (1, \alpha)$  and for any  $T > t_0$ , one can choose again  $k \geq 1$  such that  $a(a-1)^{-1} < k(T-t_0)$ . Moreover, one can see that there exists a continuous differentiable odd function  $f_2$ , defined on  $\mathbb{R}$ , vanishing only at  $x = 0$ , such that  $|f_2(x)| \leq 1 \wedge |x|^{-a}$ , and

$$f_2(x) := kx, \quad x \in \left[-\frac{1}{2k}, \frac{1}{2k}\right], \quad \lim_{|x| \rightarrow \infty} |x|^\alpha |f_2(x)| = \infty \text{ and } \lim_{|x| \rightarrow \infty} f_2'(x) = 0.$$

For  $\mu > 0$ , we introduce the bounded twice continuous differentiable function

$$G_{2,\mu}(x) := \exp\left(\mu \int_0^x f_2(y) dy\right), \quad x \in \mathbb{R}.$$

Note that for all  $x \in \mathbb{R}$  and  $t \in [t_0, T]$ ,

$$\begin{aligned} (\partial_t + \mathcal{L}_t)G_{2,\mu}(x) &= \mathcal{L}_t G_{2,\mu}(x) = \mu G_{2,\mu}(x) \left[ \frac{|F(x)f_2(x)|}{t^\beta} + \frac{1}{2}f_2'(x) + \frac{\mu}{2}f_2^2(x) \right] \\ &\geq \mu G_{2,\mu}(x) \left[ \rho \frac{|x|^\alpha |f_2(x)|}{t^\beta} + \frac{1}{2}f_2'(x) + \frac{\mu}{2}f_2^2(x) \right], \end{aligned}$$



where  $\rho = \min \{ |F(1)|, |F(-1)| \} > 0$ . One can conclude, using the same argument as in the proof of Proposition 3.7, p. 13, in [7].  $\square$

### 3.2 Study under $(H_2\alpha)$

We assume now  $(H_2\alpha)$ . Since, the equation (6) doesn't have any time-homogeneous term, the previous method cannot be used to conclude to the existence up to explosion. Instead, one will use the exponential change of time process to get

**Proposition 3.5.** *If  $\alpha \geq 0$ , there exists a pathwise unique strong solution to (SKE), defined up to the explosion time.*

*Proof.* The proof is identical to that of Proposition 3.2, by considering  $V^{(e)}$  instead of  $V^{(\gamma)}$ .  $\square$

**Proposition 3.6.**

- When  $\alpha \leq 1$  or  $(F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+$ , the explosion time of  $V$  is a.s. infinite.
- Else, i.e.  $\alpha > 1$  and  $(F(-1), F(1)) \in (\mathbb{R}_+^* \times \mathbb{R}_+) \cup (\mathbb{R} \times \mathbb{R}_-)$ ,  $\mathbb{P}(\tau_\infty = \infty) > 0$ , where  $\tau_\infty$  denotes the explosion time of  $V$ .

*Proof.* The proof is identical to that of Proposition 3.4 by considering  $G$  instead of  $|F|$ .  $\square$

## 4 Asymptotic behaviour of the solution

Let observe the SDE (5) satisfied by  $V^{(e)}$ : leaving out the last term, it yields the equation of the Ornstein-Uhlenbeck process:

$$dU_s = dW_s - \frac{U_s}{2} ds.$$

*Proof of Theorem 2.2.* STEP 1. Firstly, for all  $t \geq 0$ ,

$$\begin{aligned} X_t^{(e)} &:= \int_0^t V_s^{(e)} ds = \int_{t_0}^{t_0 e^t} \frac{V_u}{u^{3/2}} du \stackrel{IBP}{=} \frac{X_{t_0 e^t}}{t_0^{3/2} e^{3t/2}} - \frac{X_{t_0}}{t_0^{3/2}} + \frac{3}{2} \int_{t_0}^{t_0 e^t} \frac{X_s}{s^{5/2}} ds \\ &= X_{t_0 e^t} t_0^{-3/2} e^{-3t/2} - X_{t_0} t_0^{-3/2} + \frac{3}{2} \int_0^t X_{t_0 e^u} t_0^{-3/2} e^{-3u/2} du. \end{aligned} \quad (12)$$

The behaviour of the third term on the right-hand side is unknown. However, by setting  $G : t \mapsto \int_0^t X_{t_0 e^u} t_0^{-3/2} e^{-3u/2} du$ , (12) may be written as

$$G'(t) + \frac{3}{2}G(t) = X_t^{(e)} + X_{t_0} t_0^{-3/2}, \quad G(0) = 0.$$

This ODE can be solved :

$$G : t \mapsto e^{-3t/2} \int_0^t \left( X_s^{(e)} + X_{t_0} t_0^{-3/2} \right) e^{3s/2} ds = e^{-3t/2} \int_0^t X_s^{(e)} e^{3s/2} ds + \frac{2}{3} X_{t_0} t_0^{-3/2} (1 - e^{-3t/2}).$$

Hence, using the two equalities of  $G'$ , one obtains that, for all  $t \geq 0$ ,

$$X_{t_0 e^t} t_0^{-3/2} e^{-3t/2} = X_t^{(e)} - \frac{3}{2} e^{-3t/2} \int_0^t X_s^{(e)} e^{3s/2} ds + X_{t_0} t_0^{-3/2} e^{-3t/2}.$$

STEP 2. We now study the second term on the right-hand side. Since for  $u \geq 0$ ,  $X_u^{(e)} = \int_0^u V_s^{(e)} ds$ , by Itô's formula one gets,

$$\frac{3}{2} e^{-3t/2} \int_0^t X_u^{(e)} e^{3u/2} du = X_t^{(e)} - e^{-3t/2} \int_0^t V_s^{(e)} e^{3s/2} ds.$$

Replacing in the preceding equality,

$$X_{t_0 e^t} t_0^{-3/2} e^{-3t/2} = X_{t_0} t_0^{-3/2} e^{-3t/2} + e^{-3t/2} \int_0^t V_s^{(e)} e^{3s/2} ds. \quad (13)$$

Moreover, applying again Itô's formula,

$$\begin{aligned} V_t^{(e)} e^{3t/2} &= V_0^{(e)} + \frac{3}{2} \int_0^t V_s^{(e)} e^{3s/2} ds + \int_0^t e^{3s/2} dV_s^{(e)} \\ &= V_0^{(e)} + \int_0^t V_s^{(e)} e^{3s/2} ds + \int_0^t e^{3s/2} dW_s - \int_0^t e^{3s/2} t_0^{1/2-\beta} e^{(1/2-\beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds. \end{aligned}$$

Hence,

$$\begin{aligned} X_{t_0 e^t} t_0^{-3/2} e^{-3t/2} &= e^{-3t/2} (X_{t_0} t_0^{-3/2} - V_0^{(e)}) + V_t^{(e)} - \int_0^t e^{-3(t-s)/2} dW_s \\ &\quad + t_0^{1/2-\beta} \int_0^t e^{-3(t-s)/2} e^{(1/2-\beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds. \end{aligned}$$

One can express  $V^{(e)}$  with the help of Itô's formula again:

$$V_t^{(e)} = V_0^{(e)} e^{-t/2} + \int_0^t e^{-(t-s)/2} dW_s - \int_0^t e^{-(t-s)/2} t_0^{1/2-\beta} e^{(1/2-\beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds. \quad (14)$$

Hence, it becomes

$$\begin{aligned} X_{t_0 e^t} t_0^{-3/2} e^{-3t/2} &= e^{-3t/2} (X_{t_0} t_0^{-3/2} - V_0^{(e)}) + V_0^{(e)} e^{-t/2} + \int_0^t \left[ e^{-\frac{(t-s)}{2}} - e^{-\frac{3(t-s)}{2}} \right] dW_s \\ &\quad - t_0^{1/2-\beta} \int_0^t \left[ e^{-\frac{(t-s)}{2}} - e^{-\frac{3(t-s)}{2}} \right] e^{(1/2-\beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds. \end{aligned}$$

It follows that, for all  $u \geq t_0$ , and  $\varepsilon > 0$ ,

$$\begin{aligned} \varepsilon^{3/2} X_{u/\varepsilon} &= \varepsilon^{3/2} (X_{t_0} - V_0^{(e)} t_0^{3/2}) + V_0^{(e)} \sqrt{t_0} \varepsilon u + u^{3/2} \int_0^{\ln(u/t_0 \varepsilon)} \left[ e^{-\frac{(\ln(u/t_0 \varepsilon) - s)}{2}} - e^{-\frac{3(\ln(u/t_0 \varepsilon) - s)}{2}} \right] dW_s \\ &\quad - u^{3/2} t_0^{1/2-\beta} \int_0^{\ln(u/t_0 \varepsilon)} \left[ e^{-\frac{(\ln(u/t_0 \varepsilon) - s)}{2}} - e^{-\frac{3(\ln(u/t_0 \varepsilon) - s)}{2}} \right] e^{(1/2-\beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds. \end{aligned} \quad (15)$$

Set, for  $t \geq 0$ , the continuous local martingale, vanishing at 0,

$$M_t := \int_0^t \left( e^{3t/2} e^{-(t-s)/2} - e^{3s/2} \right) dW_s.$$

The quadratic variation of  $(M_t)_{t \geq 0}$  is  $\langle M, M \rangle_t = (e^t - 1)^3/3$ . Hence  $\langle M, M \rangle_\infty = \infty$ , so by the Dambis-Dubins-Schwarz theorem (Theorem 1.6, p. 181, in [11]), there exists a standard Brownian motion  $(\mathcal{B}_t)_{t \geq 0}$  such that  $M_t = \mathcal{B}_{(e^t - 1)^3/3}$ . Then one can write

$$\begin{aligned} \varepsilon^{3/2} X_{u/\varepsilon} &= \varepsilon^{3/2} (X_{t_0} - V_0^{(e)} t_0^{3/2}) + V_0^{(e)} \sqrt{t_0 \varepsilon} u + (t_0 \varepsilon)^{3/2} M_{\ln(u/t_0 \varepsilon)} \\ &\quad - u^{3/2} t_0^{1/2 - \beta} \int_0^{\ln(u/t_0 \varepsilon)} \left[ e^{-\frac{(\ln(u/t_0 \varepsilon) - s)}{2}} - e^{-\frac{3(\ln(u/t_0 \varepsilon) - s)}{2}} \right] e^{(1/2 - \beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) ds. \end{aligned} \quad (16)$$

STEP 3. Letting  $\varepsilon \rightarrow 0$  in (16), the first two terms on the right-hand side converge to 0 a.s. and uniformly on compact sets, while the third term converges to 0 a.s. uniformly on compact sets, by the dominated convergence theorem. We fix  $T \geq t_0$ . For all  $s \geq 0$ ,

$$\begin{aligned} \sup_{u \in [t_0, T]} &\left| \mathbb{1}_{[0, \ln(u/t_0 \varepsilon)]} u^{3/2} t_0^{1/2 - \beta} \left[ e^{-\frac{(\ln(u/t_0 \varepsilon) - s)}{2}} - e^{-\frac{3(\ln(u/t_0 \varepsilon) - s)}{2}} \right] e^{(1/2 - \beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) \right| \\ &\leq \mathbb{1}_{\mathbb{R}^+}(s) t_0^{1/2 - \beta} \left[ T \sqrt{\varepsilon t_0} e^{s/2} + (t_0 \varepsilon)^{3/2} e^{3s/2} \right] e^{(1/2 - \beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ a.s.} \end{aligned}$$

For all  $\varepsilon > 0$  and  $s \geq 0$ ,

$$\begin{aligned} \sup_{u \in [t_0, T]} &\left| \mathbb{1}_{[0, \ln(u/t_0 \varepsilon)]} u^{3/2} t_0^{1/2 - \beta} \left[ e^{-\frac{(\ln(u/t_0 \varepsilon) - s)}{2}} - e^{-\frac{3(\ln(u/t_0 \varepsilon) - s)}{2}} \right] e^{(1/2 - \beta)s} F(\sqrt{t_0} e^{s/2} V_s^{(e)}) \right| \\ &\leq \begin{cases} 2T^{3/2} t_0^{(\alpha+1)/2 - \beta} e^{(\frac{\alpha+1}{2} - \beta)s} (2 \ln(s))^{\alpha/2} \left| F\left(\frac{V_s^{(e)}}{\sqrt{2 \ln(s)}}\right) \right| \mathbb{1}_{\mathbb{R}^+}(s) & \text{under } (H_1 \alpha), \\ 2T^{3/2} t_0^{(\alpha+1)/2 - \beta} e^{(\frac{\alpha+1}{2} - \beta)s} (2 \ln(s))^{\alpha/2} G\left(\frac{V_s^{(e)}}{\sqrt{2 \ln(s)}}\right) \mathbb{1}_{\mathbb{R}^+}(s) & \text{under } (H_2 \alpha). \end{cases} \end{aligned}$$

Recall that  $\alpha \geq 0$ , then  $F$  is a continuous function on  $\mathbb{R}$ . Thus, thanks to Remark A.3,  $\left| F\left(\frac{V_s^{(e)}}{\sqrt{2 \ln(s)}}\right) \right|$  is bounded for  $s$  large enough. Using the continuity of  $G$ , the same argument applies under  $(H_2 \alpha)$ .

Finally, we obtain

$$\varepsilon^{3/2} X_{u/\varepsilon} = Y_u^{\varepsilon, t_0} + (t_0 \varepsilon)^{3/2} \mathcal{B}_{(\frac{u}{t_0 \varepsilon} - 1)^3/3} \stackrel{d}{=} Y_u^{\varepsilon, t_0} + \mathcal{B}_{(u - t_0 \varepsilon)^3/3}.$$

where  $\sup_{u \in [t_0, T]} Y_u^{\varepsilon, t_0} \xrightarrow{\varepsilon \rightarrow 0} 0$  almost surely, for all  $T \geq t_0$ . Thus, as  $\varepsilon \rightarrow 0$ ,

$$d\left(\varepsilon^{3/2} X_{u/\varepsilon}, (t_0 \varepsilon)^{3/2} \mathcal{B}_{(\frac{u}{t_0 \varepsilon} - 1)^3/3}\right) \xrightarrow{\mathbb{P}} 0,$$

where

$$d : f, g \in C([t_0, +\infty)) \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min\left(1, \sup_{[t_0, n]} |f(t) - g(t)|\right) \text{ is a metric on } C([t_0, +\infty)).$$

Indeed, fix  $a > 0$  and choose  $N > 0$  such that  $\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \frac{a}{2}$ , then,

$$d\left(\varepsilon^{3/2} X_{u/\varepsilon}, (t_0 \varepsilon)^{3/2} \mathcal{B}_{(\frac{u}{t_0 \varepsilon} - 1)^{3/3}}\right) \leq \frac{a}{2} + \sum_{n=1}^N \frac{1}{2^n} \sup_{[t_0, n]} |Y_u^{\varepsilon, t_0}|.$$

It follows that

$$\mathbb{P}\left(d\left(\varepsilon^{3/2} X_{u/\varepsilon}, (t_0 \varepsilon)^{3/2} \mathcal{B}_{(\frac{u}{t_0 \varepsilon} - 1)^{3/3}}\right) > a\right) \leq \sum_{n=1}^N \mathbb{P}\left(\sup_{[t_0, n]} |Y_u^{\varepsilon, t_0}| > a'\right) \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where  $a' = a(\sum_{n \geq N}^{+\infty} 1/2^n)^{-1}$ . The proof will be complete by applying Theorem 3.1, p. 27, in [1]. □

## Appendix

We collect in this section some technical results. First, we discuss results of existence, explosion and behaviour of some time-homogeneous processes  $V^+$  and  $V^-$ . For  $\alpha > -1$ , let  $\pi$  be a non-negative function satisfying

$$\forall x \in \mathbb{R}, \lambda > 0, \pi(\lambda x) = \lambda^\alpha \pi(x).$$

Recall that  $\pi$  satisfies (2). Under  $(H_1\alpha)$  we take  $\pi = |F|$  and under  $(H_2\alpha)$ , we take  $\pi = G$ .

Define the pathwise unique strong solution (up to the explosion time) to the time-homogeneous equation

$$dV_s^\pm = dW_s - \frac{V_s^\pm}{2} ds \pm t_0^{\frac{\alpha+1}{2}-\beta} \pi(V_s^\pm) \mathbb{1}_{\{\pm F(V_s^\pm) < 0\}} ds. \quad (17)$$

**Lemma A.1.** *Set  $\tau_\infty^\pm$ ,  $\tau_\infty$ , respectively the explosion time of  $V^\pm$  and  $V$ .*

- (i) *If  $\alpha \leq 1$  or  $F(1) \geq 0$ , then  $\tau_\infty^+ = \infty$  a.s.*
- (ii) *If  $\alpha \leq 1$  or  $F(-1) \leq 0$ , then  $\tau_\infty^- = \infty$  a.s.*
- (iii) *If  $\alpha > 1$  and  $F(1) < 0$ , then  $\mathbb{P}(\tau_\infty^+ = \infty) = 0$ .*
- (iv) *If  $\alpha > 1$  and  $F(-1) > 0$ , then  $\mathbb{P}(\tau_\infty^- = \infty) = 0$ .*

*Proof.* STEP 1. Firstly let us prove that  $V^- \leq V^{(e)} \leq V^+$  almost surely. Indeed, if we denote

$$\mathbf{b}(t, x) = -\frac{x}{2} - t_0^{1/2-\beta} e^{(1/2-\beta)t} F(\sqrt{t_0} e^{t/2} x) \quad \text{and} \quad \mathbf{b}^+(x) = -\frac{x}{2} + t_0^{(\alpha+1)/2-\beta} \pi(x) \mathbb{1}_{\{F(x) \leq 0\}},$$

we can write, for all  $t \geq 0$  and all  $x \in \mathbb{R}$ ,

$$\mathbf{b}(t, x) \leq \mathbf{b}^+(x) \iff -e^{(1/2-\beta)t} F(\sqrt{t_0} e^{t/2} x) \leq t_0^{\frac{\alpha}{2}} \pi(x) \mathbb{1}_{\{F(x) \leq 0\}}.$$

This inequality holds true by the choice of  $\pi$  and the assumption  $(\alpha+1)/2 - \beta < 0$ . By using a comparison theorem (see Theorem 1.1, Chap. VI p.437, in [8]) one gets,  $V^{(e)} \leq V^+$ , almost surely. The other inequality can be obtained in the same way.

STEP 2. Call  $\tau_\infty^{(e)}$  the explosion time of  $V^{(e)}$ , then  $\{\tau_\infty^{(e)} = \infty\} = \{\tau_\infty = \infty\}$ . So,  $\{\tau_\infty^- = \infty\} \cap \{\tau_\infty^+ = \infty\} \subset \{\tau_\infty = \infty\}$ .

We give the detailed proof for (i) and (iii), the other parts could be obtained by changing "+" and "-" in the reasoning. First, we prove (i). The scale function of  $V^+$  is given, for  $x \in \mathbb{R}$ , by

$$\mathbf{p}^+(x) := \int_0^x \exp\left(\frac{y^2}{2} - 2t_0^{\frac{(\alpha+1)}{2}-\beta} \int_0^y \pi(z) \mathbb{1}_{\{F(z) \leq 0\}} dz\right) dy.$$

Note that, if  $x < 0$ ,  $-\mathbf{p}^+(x) \geq \int_x^0 e^{y^2/2} dy$ . Thus  $\mathbf{p}^+(-\infty) = -\infty$ . Suppose that  $F(1) \geq 0$ , then for  $x \geq 0$ ,  $\mathbf{p}^+(x) = \int_0^x e^{y^2/2} dy$ , so  $\mathbf{p}^+(\infty) = \infty$ . By Proposition 5.22, p. 345, in [9], the conclusion follows. Assume now that  $\alpha < 1$  and  $F(1) < 0$ . Then, for  $x \geq 0$ ,  $\mathbf{p}^+(x) = \int_0^x \exp\left(\frac{y^2}{2} - 2t_0^{\frac{\alpha+1}{2}-\beta} \pi(1) \frac{y^{\alpha+1}}{\alpha+1}\right) dy$ , so  $\mathbf{p}^+(\infty) = \infty$ . Using the same result in [9], the conclusion follows. If  $\alpha = 1$ , the drift has linear growth and the conclusion is clear.

STEP 3. We proceed with the proof of (iii). Assume  $\alpha > 1$  and  $F(1) < 0$ . As previously,  $\mathbf{p}^+(-\infty) = -\infty$ . Besides,  $\mathbf{p}^+(\infty) < \infty$ . Denote  $\mathbf{m}^+ : y \mapsto 2/(\mathbf{p}^+)'(y)$  the speed measure of  $V^+$ . Fix  $y > 0$ , then, setting  $\mathbf{c} = 2t_0^{\frac{\alpha+1}{2}-\beta} \pi(1) > 0$ , one can apply integration by parts to get:

$$\begin{aligned} (\mathbf{p}^+(\infty) - \mathbf{p}^+(y)) \mathbf{m}^+(y) &= 2 \exp\left(-\frac{y^2}{2} + \mathbf{c} \frac{y^{\alpha+1}}{\alpha+1}\right) \int_y^{+\infty} \exp\left(\frac{z^2}{2} - \mathbf{c} \frac{z^{\alpha+1}}{\alpha+1}\right) dz \\ &= \frac{2}{\mathbf{c}y^\alpha - y} + 2 \exp\left(-\frac{y^2}{2} + \mathbf{c} \frac{y^{\alpha+1}}{\alpha+1}\right) \int_y^\infty e^{\frac{z^2}{2} - \mathbf{c} \frac{z^{\alpha+1}}{\alpha+1}} \frac{1 - \mathbf{c}\alpha z^{\alpha-1}}{(z - \mathbf{c}z^\alpha)^2} dz. \end{aligned}$$

One can deduce, by integrating small  $o$ , that

$$(\mathbf{p}^+(\infty) - \mathbf{p}^+(y)) \mathbf{m}^+(y) \underset{y \rightarrow \infty}{\sim} \frac{2}{\mathbf{c}y^\alpha - y}$$

which is an integrable function at  $\infty$ . The conclusion follows from Theorem 5.29, p. 348, in [9].  $\square$

We turn now to the study of the growth rate of the velocity process  $V^{(e)}$ .

**Lemma A.2.** *When  $\alpha < 1$  or  $(F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+$ ,*

$$\limsup_{t \rightarrow \infty} \frac{V_t^+}{\sqrt{2 \ln(t)}} \leq 1 \text{ a.s.}, \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{-V_t^-}{\sqrt{2 \ln(t)}} \leq 1 \text{ a.s.} \quad (18)$$

Moreover,

$$\limsup_{t \rightarrow \infty} \frac{|V_t^{(e)}|}{\sqrt{2 \ln(t)}} \leq 1 \text{ a.s.} \quad (19)$$

**Remark A.3.** *As a consequence of (19), when  $\alpha < 1$  or  $(F(-1), F(1)) \in \mathbb{R}^- \times \mathbb{R}^+$ , for  $t$  large enough,  $|V_t^{(e)}| \leq C\sqrt{2 \ln(t)}$  for some positive constant  $C$ .*

*Proof.* Assuming (18), one gets (19) writing

$$\limsup_{t \rightarrow \infty} \frac{V_t^{(e)}}{\sqrt{2 \ln(t)}} \leq \limsup_{t \rightarrow \infty} \frac{V_t^+}{\sqrt{2 \ln(t)}} \leq 1 \text{ a.s. and } \limsup_{t \rightarrow \infty} \frac{-V_t^{(e)}}{\sqrt{2 \ln(t)}} \leq \limsup_{t \rightarrow \infty} \frac{-V_t^-}{\sqrt{2 \ln(t)}} \leq 1 \text{ a.s.}$$

Before, proving (18) let us state Motoo's theorem which proof is given in [10].

**Theorem A.4** (Motoo). *Let  $X$  be a regular continuous strong Markov process in  $(a, \infty)$ ,  $a \in [-\infty, \infty)$ . Assume also that  $X$  is time-homogeneous, with scale function  $\mathbf{s}$ . For every real positive increasing function  $h$ ,*

$$\mathbb{P} \left( \limsup_{t \rightarrow \infty} \frac{X_t}{h(t)} \geq 1 \right) = 0 \text{ or } 1 \text{ according to whether } \int^\infty \frac{dt}{\mathbf{s}(h(t))} < \infty \text{ or } = \infty.$$

Motoo's theorem yields for all  $\varepsilon > 0$ ,

$$\mathbb{P} \left( \limsup_{t \rightarrow \infty} \frac{V_t^+}{\sqrt{2 \ln(t)}} \geq 1 + \varepsilon \right) = 0 \text{ and } \mathbb{P} \left( \limsup_{t \rightarrow \infty} \frac{-V_t^-}{\sqrt{2 \ln(t)}} \geq 1 + \varepsilon \right) = 0.$$

Indeed, define  $\tilde{V}^- := -V^-$ , then

$$d\tilde{V}_s^- = -dW_s - \frac{\tilde{V}_s^-}{2} ds + t_0^{(\alpha+1)/2-\beta} \pi(-\tilde{V}_s^-) \mathbb{1}_{\{F(-\tilde{V}_s^-) > 0\}} ds.$$

Fix  $y_0 > 0$ . The scale function of  $V^+$  and  $\tilde{V}^-$  is given, for  $y \geq y_0$ , by

$$\mathbf{s}^\pm(y) = \kappa \int_{y_0}^y \exp \left( \frac{z^2}{2} - 2C^\pm \frac{z^{\alpha+1}}{\alpha+1} \right) dz.$$

Here and elsewhere  $\kappa$  denotes positive constants which can change of value from line to line, and

$$C^+ := t_0^{\frac{\alpha+1}{2}-\beta} \pi(1) \mathbb{1}_{\{F(1) < 0\}} \quad \text{for } V^+ \quad \text{and} \quad C^- := t_0^{\frac{\alpha+1}{2}-\beta} \pi(-1) \mathbb{1}_{\{F(-1) > 0\}} \quad \text{for } \tilde{V}^-.$$

Let  $\epsilon > 0$ . Define the positive increasing function  $h : t \mapsto (1 + \epsilon) \sqrt{2 \ln(t)}$ . We will show that  $1/\mathbf{s}(h)$  is integrable at infinity. Firstly, remark that

$$\int_{y_0}^{+\infty} \frac{1}{\mathbf{s}(h(t))} dt = \int_{h(y_0)}^{+\infty} \frac{1}{\mathbf{s}(y)} \frac{dy}{h'(h^{-1}(y))} = \int_{h(y_0)}^{+\infty} \frac{1}{\mathbf{s}(y)} \frac{y \exp(y^2/(2(1+\epsilon)^2))}{(1+\epsilon)^2} dy.$$

It remains to find an equivalent of  $\mathbf{s}$  at infinity. In the following " $\asymp$ " means equality up to a multiplicative positive constant. Fix  $y > y_0$ , integrating by parts, one gets,

$$\begin{aligned} \mathbf{s}(y) &\asymp \int_{y_0}^y \exp \left( \frac{z^2}{2} - 2C^\pm \frac{z^{\alpha+1}}{\alpha+1} \right) (z - 2C^\pm z^\alpha) \cdot \frac{1}{z - 2C^\pm z^\alpha} dz \\ &\asymp \frac{\exp \left( \frac{y^2}{2} - 2C^\pm \frac{y^{\alpha+1}}{\alpha+1} \right)}{y - 2C^\pm y^\alpha} - \kappa + \int_{y_0}^y \frac{1 - 2\alpha C^\pm z^{\alpha-1}}{(z - 2C^\pm z^\alpha)^2} \exp \left( \frac{z^2}{2} - 2C^\pm \frac{z^{\alpha+1}}{\alpha+1} \right) dz. \end{aligned}$$

Since  $\alpha > -1$ ,  $\lim_{y \rightarrow \infty} \frac{1 - 2\alpha C^\pm y^{\alpha-1}}{(y - 2C^\pm y^\alpha)^2} = 0$ , except when  $\alpha = 1$  and  $C^\pm = \frac{1}{2}$ . Moreover the function  $y \mapsto \exp \left( \frac{y^2}{2} - 2C^\pm \frac{y^{\alpha+1}}{\alpha+1} \right)$  is not integrable at infinity, when  $C^\pm = 0$ , or  $\alpha < 1$ , or  $\alpha = 1$  and  $C^\pm < \frac{1}{2}$ . In these cases one gets, by integration,

$$\mathbf{s}(y) \underset{y \rightarrow \infty}{\sim} \kappa \frac{\exp \left( \frac{y^2}{2} - 2C^\pm \frac{y^{\alpha+1}}{\alpha+1} \right)}{y - 2C^\pm y^\alpha}.$$

Hence,

$$\frac{1}{s(y)} \frac{y \exp\left(\frac{y^2}{2(1+\varepsilon)^2}\right)}{(1+\varepsilon)^2} \underset{y \rightarrow \infty}{\sim} \kappa(y^2 - 2C^\pm y^{\alpha+1}) \exp\left(-\frac{y^2}{2}\left(1 - \frac{1}{(1+\varepsilon)^2}\right) + 2C^\pm \frac{y^{\alpha+1}}{\alpha+1}\right).$$

which is integrable if  $C^\pm = 0$  or  $\alpha < 1$ , or  $\alpha = 1$  and  $C^\pm < \frac{1}{2}$ . One can conclude using Motoo's theorem.  $\square$

**Remark A.5.** *The preceding proof shows that the result of Lemma A.2 is still true for  $\alpha = 1$  and some condition depending on  $\pi(1)$  and  $\pi(-1)$  which ensure that  $C^\pm < \frac{1}{2}$ . Consequently, our main result is also true under the same condition.*

## References

- [1] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley-Blackwell, New York, 1999.
- [2] P. Cattiaux, D. Chafai, and S. Motsch, *Asymptotic analysis and diffusion limit of the Persistent Turning Walker Model*, Asymptotic Analysis **67** (2010), no. 1–2, 17–31.
- [3] P. Cattiaux, E. Nasreddine, and M. Puel, *Diffusion limit for kinetic Fokker-Planck equation with heavy tails equilibria: the critical case*, Kinet. Relat. Models **12** (2019), no. 4, 727–748.
- [4] A. S. Cherny and H. J. Engelbert, *Singular stochastic differential equations*, Lecture notes in mathematics, no. 1858, Springer, Berlin, 2005.
- [5] R. Eon and M. Gradinaru, *Gaussian asymptotics for a non-linear Langevin type equation driven by an  $\alpha$ -stable Lévy noise*, Electronic Journal of Probability **20** (2015).
- [6] N. Fournier and C. Tardif, *One dimensional critical kinetic Fokker-Planck equations, Bessel and stable processes*, May 2018.
- [7] M. Gradinaru and Y. Offret, *Existence and asymptotic behaviour of some time-inhomogeneous diffusions*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **49** (2013), no. 1, 182–207.
- [8] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North Holland, Amsterdam ; New York : Tokyo : New York, 1981.
- [9] I. Karatzas and E. S. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Graduate Texts in Mathematics, Springer-Verlag, New York, 1998.
- [10] M. Motoo, *Proof of the law of iterated logarithm through diffusion equation*, Annals of the Institute of Statistical Mathematics **10** (1959), no. 1, 21–28.
- [11] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 3rd ed., Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin Heidelberg, 1999.
- [12] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Classics in Mathematics, Springer-Verlag, Berlin Heidelberg, 2006.