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Normal form near orbit segments of convex Hamiltonian systems.

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In the study of Hamiltonian systems on cotangent bundles, it is natural to perturb Hamiltonians by adding potentials (functions depending only on the base point). This led to the definition of Mañé genericity: a property is generic if, given a Hamiltonian H , the set of potentials u such that $H + u$ satisfies the property is generic. This notion is mostly used in the context of Hamiltonians which are convex in p , in the sense that $\partial_{pp}^2 H$ is positive definite at each points. We will also restrict our study to this situation.

There is a close relation between perturbations of Hamiltonians by a small additive potential and perturbations by a positive factor close to one. Indeed, the Hamiltonians $H + u$ and $H/(1 - u)$ have the same level one energy surface, hence their dynamics on this energy surface are reparametrisation of each other, this is the Maupertuis principle. This remark is particularly relevant when H is homogeneous in the fibers (which corresponds to Finsler metrics) or even fiberwise quadratic (which corresponds to Riemannian metrics). In these cases, perturbations by potentials of the Hamiltonian correspond, up to parametrisation, to conformal perturbations of the metric.

One of the widely studied aspects is to understand to what extent the return map associated to a periodic orbit can be perturbed by adding a small potential. This kind of question depend strongly on the context in which they are posed. Some of the most studied contexts are, in increasing order of difficulty, perturbations of general vector fields, perturbations of Hamiltonian systems inside the class of Hamiltonian systems, perturbations of Riemannian metrics inside the class of Riemannian metrics, Mañé perturbations of convex Hamiltonians. It is for example well-known that each vector field can be perturbed to a vector field with only hyperbolic periodic orbits, this is part of the Kupka-Smale theorem, see [5, 13]. There is no such result in the context of Hamiltonian vector fields, but it remains true that each Hamiltonian can be perturbed to a Hamiltonian with only non-degenerate periodic orbits (including the iterated ones), see [11, 12]. The same result is true in the context of Riemannian metrics: every Riemannian metric can be perturbed to a Riemannian metric with only non-degenerate closed geodesics, this is the bumpy metric theorem, see [4, 2, 1]. The question was investigated only much more recently in the context of Mañé perturbations of convex Hamiltonians, see [9, 10]. It is proved in [10] that the same result holds : If H is a convex Hamiltonian and a is a regular value of H , then there exist arbitrarily small potentials u such that all periodic orbits (including iterated ones) of $H + u$ at energy a are non-degenerate. The proof given in [10] is actually rather similar to the ones given in papers on the perturbations of Riemannian metrics. In all these proofs, it is very useful to work

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in appropriate coordinates around an orbit segment. In the riemannian case, one can use the so-called Fermi coordinates. In the Hamiltonian case, appropriate coordinates are considered in [10, Lemma 3.1] itself taken from [3, Lemma C.1]. However, as we shall detail below, the proof of this Lemma in [3], Appendix C, is incomplete, and the statement itself is actually wrong. Our goal in the present paper is to state and prove a corrected version of this normal form Lemma. Our proof is different from the one outlined in [3], Appendix C. In particular, it is purely Hamiltonian and does not rest on the results of [7] on Finsler metrics, as [3] did. Although our normal form is weaker than the one claimed in [10], it is actually sufficient to prove the main results of [10, 6], and probably also of the other works using [3, Lemma C.1].

1 Introduction.

When studying Mañé generic properties of convex Hamiltonians, a natural group of changes of coordinates to consider is the group of symplectic diffeomorphisms preserving the vertical fibration, *i. e.* of the form $\psi(q, p) = (\varphi(q), G(q, p))$. We call such transformations fibered. It is well known that such a diffeomorphism is symplectic if and only if the second coordinate $G(q, p)$ is of the form

$$G(q, p) = \omega_q + p \circ (d\varphi_q)^{-1}$$

for some closed form ω . We will say that the fibered symplectic diffeomorphism ψ is homogeneous if it preserves the zero section, which implies that there exists a diffeomorphism φ of the base such that

$$\psi(q, p) = (\varphi(q), p \circ (d\varphi_q)^{-1}).$$

We will say that the above diffeomorphism is vertical if it is of the form $(q, p) \mapsto (q, p + \omega_q)$, that is if it preserves the first coordinate. Each fibered symplectic diffeomorphism is thus the composition of a vertical and of a homogeneous symplectic diffeomorphisms (in any order). Note that fibered symplectic diffeomorphisms preserve convexity, since their restrictions to fibers is affine.

If ψ is a fibered symplectic diffeomorphism and $g(q)$ is a potentiel, then $(H \circ \psi) + g = (H + g \circ \varphi^{-1}) \circ \psi$. So if a property is invariant under symplectic diffeomorphisms, for example having only non-degenerate periodic orbits on a given energy level, then this property is satisfied by $H + g$ for arbitrarily small g if and only if it is satisfied by $H \circ \psi + g$ for arbitrarily small g .

These considerations allow us to enlarge the group of transformations considered in [3, Lemma C.1], where only homogeneous diffeomorphisms are considered (restricting to homogeneous transformations seems natural only for Homogeneous Hamiltonians). This is, as we shall see, a necessary (but not sufficient) modification.

Since all considerations are local, we will always work on the manifold \mathbb{R}^{1+d} and on its cotangent bundle $\mathbb{R}^{1+d} \times (\mathbb{R}^{1+d})^*$. We will use the notations $q = (q_0, q_*) \in \mathbb{R} \times \mathbb{R}^d$ and similarly $p = (p_0, p_*) \in \mathbb{R}^* \times \mathbb{R}^{d*}$. We denote by e_0, e_1, \dots, e_d both the standard base of \mathbb{R}^{1+d} and $\mathbb{R}^{(1+d)*}$.

If (Q, P) is an orbit segment such that $\dot{Q}(0) \neq 0$, then there exists a local diffeomorphism φ of the base which sends the orbit segment $Q(t), t \in [-\delta, \delta]$ to the straight line segment $te_0, t \in [-\delta, \delta]$. Once this reduction has been performed, we consider fiber-preserving symplectic diffeomorphisms the horizontal component of which is the identity on the segment $[-\delta, \delta]e_0$. We call such diffeomorphisms *admissible*. This implies that

$$d\varphi_{te_0} \cdot e_0 = e_0 \quad , \quad (p \circ (d\varphi_{te_0})^{-1})_0 = p_0, \tag{1}$$

where the second equality is obtained by applying the linear form $p \circ d\varphi_{te_0}^{-1}$ to the first equality, and where p_0 is the first coordinate $p_0 = p \cdot e_0$.

Theorem 1. Let \underline{H} be a Hamiltonian convex in p (meaning that $\partial_{pp}^2 H$ is positive definite at each point), and $(\underline{Q}(t), \underline{P}(t))$ be an orbit such that $\dot{\underline{Q}}(0) \neq 0$. Then there exists a local fibered symplectic diffeomorphism ψ and $\delta > 0$ such that the new Hamiltonian $H = \underline{H} \circ \psi$ and the new orbit $(Q(t), P(t)) = \psi^{-1}(\underline{Q}(t), \underline{P}(t))$ satisfy, for all $t \in [0, \delta]$,

$$P(t) = 0, \quad (2)$$

$$Q(t) = te_0, \quad (3)$$

$$\partial_{p_0 p_*}^2 H(te_0, P(t)) = 0, \quad (4)$$

$$\partial_{q p_*}^2 H(te_0, P(t)) = 0, \quad (5)$$

$$\partial_{p_* p_*}^2 H(te_0, P(t)) = Id. \quad (6)$$

In the case where H is homogeneous, the equalities (3), (4), (5) can still be obtained with the additional restriction that the diffeomorphism ψ is homogeneous. The additional equalities $\partial_{qp}^2 H(te_0, P(t)) = 0$ and $P(t) = (P_0(0), 0)$ then automatically hold.

If $\partial_{q_* p_0}^2 H(te_0, 0)$ does not identically vanish, then this quantity can not be reduced to zero by applying an admissible change of coordinates preserving the other equalities.

The part of the statement concerning homogeneous Hamiltonians is equivalent to [7, Lemma 3.1]. We will not use this reference and provide a direct Hamiltonian proof. Note that equality (2) can obviously not be obtained using only homogeneous changes of coordinates, since they preserve the zero section.

Moreover, it follows from (1) that the first coordinate $P_0(t)$ is invariant under the action of homogeneous admissible diffeomorphisms. As a consequence, it is not true that orbits of general convex Hamiltonians can be reduced to $(Q(t), P(t)) = (te_0, e_0)$ by such diffeomorphisms, as is claimed in [3, Lemma C.1]. Once vertical changes of coordinates are allowed, this becomes possible, as we shall verify below, and (2) could obviously be replaced by $P(t) = e_0$ (but this is less natural and less convenient).

Finally, contrarily to [3, Lemma C.1], the last claim of the theorem implies that (5) can't be strengthened to $\partial_{qp}^2 H(te_0, 0) = 0$, even if vertical symplectic diffeomorphisms, in addition to homogeneous ones, are permitted.

Yet, if H satisfies all the conclusions of the theorem, it is of the form

$$H(q, p) = f(q) + v(q)p_0 + \frac{1}{2}a(q_0)p_0^2 + \frac{1}{2}\langle p_*, p_* \rangle + O_3(q_*, p),$$

with $f(q_0, 0) = f(0, 0)$, $w(q_0, 0) = 1$. We thus have

$$\tilde{H}(q, p) := (H(q, p) - H(0))/v(q) = \tilde{f}(q) + p_0 + \frac{1}{2}\tilde{a}(q_0)p_0^2 + \frac{1}{2}\langle p_*, p_* \rangle + O_3(q_*, p),$$

with $\tilde{f}(q_0, 0) = 0$. This means that the additional conclusion $\partial_{qp}^2 H(te_0, 0) = 0$ can be achieved provided we translate H so that our orbit has energy 0 (which does not change anything to the dynamics), and then multiply H by a function of q (which is a reparametrization of the dynamics on the energy surface $\tilde{H}^{-1}(0) = H^{-1}(H(0))$). This fact is sufficient to derive most of the applications of [3, Lemma C.1] existing in the literature.

Let us illustrate for example how the results of [10] can be obtained. We denote by E_t the space $\{q_0 = t\} \cap \{p_0 = 0\}$, which projects isomorphically to $\mathbb{R}^d \times \mathbb{R}^{d*}$. Let $L : E_0 \rightarrow E_\delta$ be the differential at zero of the transition maps between the sections $\{q_0 = 0\} \cap \{\tilde{H} = 0\}$ and $\{q_0 = \delta\} \cap \{\tilde{H} = 0\}$, seen as a symplectic $2d \times 2d$ matrix. Since the dynamics of H and \tilde{H} in restriction to $\tilde{H}^{-1}(0)$ are reparametrizations of each other, they have the same transition map hence $L_H = L_{\tilde{H}}$. The main statement of [10] is that, if X is a dense set of symplectic $2d \times 2d$ matrices, there exist arbitrarily small potentials $g(q)$ such that $L_{H+g} \in X$. However,

this statement is proved in [10] only under the additional assumption that $v(q) = 1$, which, as we saw, can not be obtained in general by an admissible change of coordinates, contrarily to what is stated in [10]. In other words, the proof of [10] applies to \tilde{H} , but not to H . Applying the statement to \tilde{H} gives a small potential $\tilde{g}(q)$ such that $L_{\tilde{H}+\tilde{g}} \in X$. We now observe that $\tilde{H} + \tilde{g} = (H - H(0) + v\tilde{g})/v$, hence

$$L_{H+v\tilde{g}} = L_{\tilde{H}+\tilde{g}} \in X,$$

and $v\tilde{g}$ is an arbitrarily small potential (note that v is a function of q only, not of (q, p)).

2 Proof of the Normal Form.

We will always work in coordinates such that (3) holds, and consider only admissible changes of coordinates (i. e. changes of coordinates whose horizontal component fixes the axis $\mathbb{R}e_0$). We shall lead simultaneously the proofs in the homogeneous case (which is presented here just for completeness since the original proof in [7] is perfectly fine) and in the general case (which is the case of interest in the present paper). The proof will consist of applying several admissible diffeomorphisms. In each step, we will denote by \underline{H} the initial Hamiltonian and by $H = \underline{H} \circ \psi$ the transformed Hamiltonian. In the matrix computations below, we most of the time consider the momenta p as line matrices hence denote by pM what might also be denoted by $M^t p$.

PROOF OF (4). We consider a base diffeomorphism of the form $\varphi(q_0, q_*) = (q_0 + l(q_0) \cdot q_*, q_*)$, where $q_0 \mapsto l(q_0)$ is a smooth map with values in \mathbb{R}^{d^*} . The corresponding homogeneous diffeomorphism is

$$\psi : (q_0, q_*, p_0, p_*) \mapsto (q_0 + l(q_0) \cdot q_*, q_*, p_0, p_* + p_0 l(q_0)).$$

We then have $\partial_{p_*}(H \circ \psi)_{(q,p)} = \partial_{p_*} H_{\psi(q,p)}$ hence

$$\partial_{p_0 p_*}^2 (H \circ \psi)_{(q,p)} = \partial_{p_0 p_*}^2 H_{\psi(q,p)} + \partial_{p_* p_*}^2 H_{\psi(q,p)} \cdot l(q_0).$$

Denoting by $(te_0, P(t))$ the orbits in the new coordinates, we obtain (4) by choosing

$$l(q_0) := -(\partial_{p_* p_*}^2 H_{\psi(q_0 e_0, P(q_0))})^{-1} \cdot \partial_{p_0 p_*}^2 H_{\psi(q_0 e_0, P(q_0))}.$$

That $\partial_{p_* p_*}^2 H_{\psi(q_0 e_0, P(q_0))}$ is invertible is a consequence of the hypothesis that $\partial_{pp}^2 H$ is positive definite at each point. \square

The next step is to prescribe the vertical component of the orbit.

PROOF OF (2). Let $\underline{P}_0(t)$ be the first component of $\underline{P}(t)$ (the orbit before the change of coordinates). We consider a function $v(t) : \mathbb{R} \rightarrow \mathbb{R}$ such that $v' = \underline{P}_0$ and the function $u(q_0, q_*) := v(q_0) + \underline{P}_*(q_0) \cdot q_*$. We have $du_{te_0} = \underline{P}(t)$, hence applying the vertical diffeomorphism $\psi(q, p) = (q, p + du_q)$, the new orbit $(Q(t), P(t)) = \psi^{-1}(\underline{Q}(t), \underline{P}(t))$ satisfies $P(t) = 0$. \square

In the homogeneous case, we do not need to apply any change of coordinates, since the equality $P(t) = (P_0(0), 0)$ follows automatically from (4) : The map $p \mapsto \partial_{p_*} H(q, p)$ being also homogeneous, the relation $\partial_{p_*} H(te_0, P(t)) = 0$ implies that

$$0 = \partial_{p_* p_*}^2 H(te_0, P(t)) \cdot P(t) = \partial_{p_0 p_*}^2 H(te_0, P(t)) \cdot P_0(t) + \partial_{p_* p_*}^2 H(te_0, P(t)) \cdot P_*(t).$$

If $\partial_{p_0 p_*}^2 H(te_0, P(t)) = 0$, this implies that $P_*(t) = 0$. Then the Euler relation

$$P_0(t) = \partial_p H(te_0, P(t)) \cdot P(t) = \alpha H(te_0, P(t)),$$

where α is the degree of homogeneity, implies that

$$P_0(t) = \alpha H(te_0, P(t)) = \alpha H(0, P(0)) = P_0(0),$$

where we have used the preservation of H along the orbit. We have proved, in the Homogeneous case, that the relation $P(t) = (P_0(0), 0)$ holds automatically (without the necessity of changing coordinates).

PROOF OF (5). This equality can be obtained by a further homogeneous change of coordinates preserving (3) and (4). In order to treat simultaneously the homogeneous and the general case, we make the assumption that $P(t) = P(0) = (P_0, 0)$ for some constant P_0 . We consider the vector field

$$\underline{V}(q) := \partial_p \underline{H}(q, P)$$

on \mathbb{R}^{d+1} . We will apply a variant of the Flow Box Theorem to the vector field \underline{V} . More precisely, we consider the diffeomorphism $\varphi(q) = (q_0, \phi(q_0, q_*))$, defined in a neighborhood of $[-\delta, \delta] \times \{0\}$ in such a way that $q_* \mapsto \phi(t, q_*)$ is the transition map along the orbits of \underline{V} between the sections $\{q_0 = 0\}$ and $\{q_0 = t\}$. In other words, $\varphi(q_0, q_*) = \Phi^{q_0}(0, q_*)$, where Φ^t is the flow of the reparametrized vector field $\underline{V}(q)/\underline{V}_0(q)$ (\underline{V}_0 is the first coordinate of \underline{V}). It is a smooth diffeomorphism near $[-\delta, \delta] \times \{0\}$, and $\underline{V} = \underline{V}_0 \varphi_{\sharp} e_0$, where $\varphi_{\sharp} e_0$ is the forward image of the constant vector field e_0 .

Since φ is fixing the axis $\mathbb{R}e_0$ and preserving q_0 , the associated homogeneous diffeomorphism ψ preserves (3) and (4). Moreover, $\psi(q, P(0)) = (\varphi(q), P(0))$. Denoting as usual $H := \underline{H} \circ \psi$, and $V(q) = \partial_p H(q, P(0))$, we have $\varphi_{\sharp} V = \underline{V} = \underline{V}_0 \varphi_{\sharp} e_0$, hence $V = (\underline{V}_0 \circ \varphi^{-1})e_0$, and $V_* = 0$, or in other words $\partial_{p_*} H(q, P(0)) = 0$ for all q . Differentiating with respect to q gives (5). \square

We have proved that the equalities (3), (4), (5), as well as $P(t) = (P_0(0), 0)$ can be obtained by a change of coordinates, which can be assumed homogeneous if H is homogeneous. We now finish the proof of the homogeneous case by observing that the equality $\partial_{qp_0} H(te_0, P(t)) = 0$ (hence $\partial_{qp} H(te_0, P(t)) = 0$) automatically hold in this case. Indeed, since $P(t) = P(0)$ is constant, we have $\partial_q H(te_0, P(0)) = 0$. Since the map $p \mapsto \partial_q H(te_0, P(0))$ is homogeneous, we deduce by the Euler relation that $P(0)\partial_{p_0q} H(te_0, P(0)) = \partial_{qp} H(te_0, P(0)) \cdot (P_0(0), 0) = 0$. Since $P(0) = 0$ (otherwise $\dot{Q}(0) = 0$), we obtain $\partial_{p_0q} H(te_0, P(0)) = 0$. This ends the proof in the homogeneous case.

PROOF OF (6). We assume that the equations (2) to (5) initially hold. We will obtain (6) by an admissible (usually not homogeneous) transformation preserving all these equalities. This transformation will be decomposed into first a homogeneous transformation and second a vertical transformation none of which preserve (5).

The first step consists of applying the homogeneous change of coordinates ψ associated to a diffeomorphism of the form

$$\varphi(q_0, q_*) = (q_0, M(q_0) \cdot q_*),$$

where $M(t)$ is a $d \times d$ invertible matrix depending smoothly on t . The matrix of the differential of φ is

$$J(q) = \begin{bmatrix} 1 & 0 \\ M'(q_0)q_* & M(q_0) \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} 1 & 0 \\ -M^{-1}(q_0)M'(q_0)q_* & M^{-1}(q_0) \end{bmatrix},$$

where $M'(q_0)$ is the derivative. We thus have

$$\psi(q, p) = (q_0, M(q_0)q_*, p_0 - p_* M^{-1}(q_0)M'(q_0)q_*, p_* M^{-1}(q_0)).$$

The Hamiltonian in original coordinates is of the form

$$\underline{H}(q, p) = \underline{H}(q, 0) + \underline{v}(q)p_0 + \frac{1}{2}\underline{a}(q_0)p_0^2 + \frac{1}{2}\langle p_* \underline{A}(q_0), p_* \rangle + O_3(q_*, p),$$

with $\underline{v}(q) = \partial_{p_0} \underline{H}(q, 0)$, $\underline{a}(q_0) = \partial_{p_0 p_0}^2 \underline{H}(q_0 e_0, 0)$, $\underline{A}(q_0) = \partial_{p_* p_*}^2 \underline{H}(q_0 e_0, 0)$. We get

$$\begin{aligned} H(q, p) &= \underline{H} \circ \psi(q, p) = H(q, 0) + v(q)p_0 - p_* M^{-1}(q_0) M'(q_0) q_* + \frac{1}{2} a(q_0) p_0^2 \\ &\quad + \frac{1}{2} \langle p_* M^{-1}(q_0) \underline{A}(q_0), p_* M^{-1}(q_0) \rangle + O_3(q_*, p), \end{aligned}$$

We get (6) provided $M(q_0) M^t(q_0) = \underline{A}(q_0)$ for each q_0 . We could for example take $M(q_0) = \underline{A}^{1/2}(q_0)$ for each q_0 (remember that $\underline{A}(q_0)$ is positive definite for each q_0). However, the unavoidable apparition term $p_* M^{-1}(q_0) M'(q_0) q_*$ means that (5) have been destroyed. In order to be able to restore it by a vertical change of coordinates, we need a better choice of $M(q_0)$:

Lemma 2.1. *We can choose $M(q_0)$ in such a way that*

$$H(q, p) = H(q, 0) + v(q)p_0 - p_* B(q_0) q_* + \frac{1}{2} a(q_0) p_0^2 + \frac{1}{2} \langle p_*, p_* \rangle + O_3(q_*, p),$$

where $B(q_0)$ is symmetric for each q_0 .

PROOF. We need the matrix $M(t)$ to satisfy the two conditions that $M(t) M^t(t) = \underline{A}(t)$ and $B(t) := M'(t) M^{-1}(t)$ is symmetric. Derivating the first condition, we get $M' M^t + M (M')^t = \underline{A}'$. Expressing this relation in terms of B , we obtain the equation $B M M^t + M M^t B = \underline{A}'$, which can be rewritten

$$B \underline{A} + \underline{A} B = \underline{A}'.$$

We claim that the linear map $L_P : X \mapsto PX + XP$ from $\mathcal{S}_d(\mathbb{R})$ (the set of symmetric matrices) to itself is an isomorphism if P is a positive definite symmetric matrix. Indeed, if P is diagonal, then $(PX + XP)_{ij} = (P_{ii} + P_{jj}) X_{ij}$, with $P_{ii} + P_{jj} > 0$. In general, $P = O D O^t$ for some orthogonal matrix O , and then $L_P(X) = O L_D(O^t X O) O^t$, hence L_P is also an isomorphism. The map $P \mapsto L_P$ is smooth, hence so is the map $P \mapsto (L_P)^{-1} =: L_P^{-1}$. We deduce that the equation $R(t) \underline{A}(t) + \underline{A}(t) R(t) = \underline{A}'(t)$ has a unique symmetric solution

$$B(t) = L_{\underline{A}(t)}^{-1}(\underline{A}'(t)),$$

this solution depends smoothly on t . With this $B(t)$, we consider the solution $M(t)$ of the differential equation

$$M'(t) = B(t) M(t)$$

with initial condition $M(0) = \underline{A}^{1/2}(0)$. By definition of B , we have

$$\underline{A}'(t) = M'(t) M^{-1}(t) \underline{A}(t) + \underline{A}(t) M'(t) M^{-1}(t).$$

Seing this as a differential equation in A , we just need to check that $M(t) M^t(t)$ solves this equation to deduce that $A(t) = M(t) M^t(t)$. This follows from the simple computation

$$(M M^t)' = M' M^t + M (M^t)' = M' M^{-1} (M M^t) + (M M^t) M' M^{-1},$$

where we have used that $(M^t)' = M^t M' M^{-1}$, which holds because $M' M^{-1}$ is symmetric. \square

The second step consists of applying the vertical change of coordinates

$$\Psi : (q, p) \mapsto (q, p + du_q)$$

with $u(q) = \langle B(q_0) q_*, q_* \rangle / 2$, so that $du_q = (*, B(q_0) q_*)$. It is then a direct computation that

$$H \circ \Psi(q, p) = f(q) + w(q)p_0 + \frac{1}{2} a(q_0) p_0^2 + \frac{1}{2} \langle p_*, p_* \rangle + O_3(q_*, p),$$

for some smooth functions f such that $f(q_0, 0) = H(0, 0)$, and w such that $w(q_0, 0) = 1$. \square

We now prove the last statement of the theorem, about the impossibility of achieving the additional condition that $\partial_{qp_0}^2 H = 0$. We shall only consider admissible diffeomorphisms which preserve (2), (3) and (4). Every fibration preserving symplectic diffeomorphism preserving (2) and (3) is the composition of a homogeneous and a vertical diffeomorphisms each of which preserve (2) and (3).

Let us first observe that $\partial_{q_*p_0}^2 H(te_0, 0)$ can't be changed by applying a vertical diffeomorphism preserving (2). Such a diffeomorphism is of the form $\psi(q, p) = (q, p + du_q)$ for some smooth function u satisfying $du_{te_0} = 0$, hence in particular u is constant on $\mathbb{R}e_0$. Then,

$$\partial_{p_0}(H \circ \psi)(q, p) = \partial_{p_0}H(q, p + \partial_q u(q))$$

and

$$\begin{aligned} \partial_{q_*p_0}^2 (H \circ \psi)(te_0, 0) &= \partial_{q_*p_0}^2 H(te_0, 0) + \sum_{i=0}^d \partial_{p_i p_0}^2 H(te_0, 0) \partial_{q_*q_i}^2 u(te_0) \\ &= \partial_{q_*p_0}^2 H(te_0, 0) + \partial_{p_0 p_0}^2 H(te_0, 0) \partial_{q_*q_0}^2 u(te_0) = \partial_{q_*p_0}^2 H(te_0, 0). \end{aligned}$$

In this computation, we have used first that $\partial_{p_*p_0}^2 H(te_0, 0) = 0$, and then that $\partial_{q_*q_0}^2 u(te_0) = \partial_{q_0 q_*}^2 u(te_0) = 0$, which holds since $\partial_{q_*} u(te_0)$ is identically zero.

We now consider the action of homogeneous admissible transformations.

Lemma 2.2. *The homogeneous symplectic diffeomorphism ψ associated to φ preserves (3) and (4) if and only if the matrix of the differential of φ has the following $1 + d$ block form for each $t \in [0, \delta]$:*

$$J\varphi(te_0, P(t)) = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}.$$

PROOF. Since φ is admissible, the matrix $J\varphi$ along the orbit has the triangular block form

$$J(t) = J\varphi(te_0, P(t)) = \begin{bmatrix} 1 & b(t) \\ 0 & B(t) \end{bmatrix}, \quad J^{-1}(t) = \begin{bmatrix} 1 & -b(t)B^{-1}(t) \\ 0 & B^{-1}(t) \end{bmatrix},$$

hence

$$\psi(q, p) = (\varphi(q), p_0, p_* B^{-1}(t) - p_0 b(t) B^{-1}(t)).$$

We have

$$\partial_{pp}^2 (H \circ \psi)(te_0, P(t)) = J^{-1}(t) \partial_{pp}^2 H_{\psi(te_0, P(t))} J^{-1t}(t).$$

In matrix form,

$$\begin{aligned} \partial_{pp}^2 (H \circ \psi) &= \begin{bmatrix} 1 & -b(t)B^{-1}(t) \\ 0 & B^{-1}(t) \end{bmatrix} \begin{bmatrix} \partial_{p_0 p_0}^2 H & 0 \\ 0 & \partial_{p_* p_*}^2 H \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B^{-1t}(t)b^t(t) & B^{-1t}(t) \end{bmatrix} \\ &= \begin{bmatrix} * & -b(t)B^{-1}(t)A(t)B^{-1t} \\ B^{-1}(t)A(t)B^{-1t}(t)b^t & B^{-1}(t)A(t)B^{-1t} \end{bmatrix}. \end{aligned}$$

This matrix is block diagonal (which is equivalent to $\partial_{p_0 p_*}^2 (H \circ \psi) = 0$) if and only if $b(t) = 0$. \square

Let now H be a Hamiltonian satisfying (2) to (4), and ψ be the homogeneous transformation associated to the diffeomorphism φ . We assume that $H \circ \psi$ still satisfies (2) to (4), and moreover that $\partial_{qp_0}^2 (H \circ \psi)(te_0, 0) = 0$ for each $t \in [-\delta, \delta]$. We denote by $\alpha(q)$ the first coordinate of φ . In view of the previous Lemma, we have the block diagonal form for $t \in [\delta, \delta]$,

$$J\varphi_{te_0, 0} = \begin{bmatrix} 1 & 0 \\ 0 & Z(t) \end{bmatrix},$$

in particular $\partial_q \alpha(q_0, 0) = e_0$, $\partial_{q_*} \alpha(q_0, 0) = 0$ for each $q_0 \in [-\delta, \delta]$ thus $\partial_{q_0 q_*}^2 \alpha(q_0, 0) = 0$. It is convenient for the following computations to denote $V(q) := \partial_q H(q, 0)$, $W(q) = \partial_p(H \circ \psi)(q, 0)$, so that

$$V(\varphi(q)) = d\varphi_q \cdot W(q)$$

and, focusing on the first coordinates, $V_0(\varphi(q)) = \partial_q \alpha(q) \cdot W(q)$. Differentiating with respect to q_* at the point te_0 , $t \in [-\delta, \delta]$ and using our hypothesis that $\partial_{q_*} W_0(te_0) = 0$, we obtain :

$$\begin{aligned} \partial_{q_*} V_0(te_0) \cdot Z(t) &= \partial_{q_* q}^2 \alpha(te_0) \cdot W(te_0) + \partial_q \alpha(te_0) \cdot \partial_{q_*} W(te_0) \\ &= \partial_{q_* q}^2 \alpha(te_0) \cdot e_0 + \partial_{q_*} W_0(te_0) \\ &= \partial_{q_* q_0}^2 \alpha(te_0) = 0. \end{aligned}$$

Since $Z(t)$ is invertible for each t , we have proved that $\partial_{q_* p_0}^2 H(te_0, 0) = \partial_{q_*} V_0(te_0) = 0$. Si if H did not satisfy this condition from the start, then neither does $H \circ \psi$. \square

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