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Proof of the Twin Prime Conjecture (Together with the proof of Polignac's Conjecture for Cousin Primes)

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Abstract In this paper proof of the twin prime conjecture is going to be presented. Originally very difficult problem (in observational space) has been transformed into a sampler one (in generative space) that can be solved. It will be shown that twin primes could be obtained through two stage sieve process, and that will be used to obtain a reasonable estimation of the number of twin primes. The same approach is used to prove the Polignac's conjecture for cousin primes.

1 Introduction

In number theory, Polignac's conjecture states: For any positive even number *n*, there are infinitely many prime gaps of size *n*. In other words: there are infinitely many cases of two consecutive prime numbers with the difference *n* [1]. For n = 2 it is known as twin prime conjecture.

Conditioned on the truth of the generalized Elliot-Halberstam Conjecture [2], in [3] it has been shown that there are infinitely many primes' gaps that have value of at least 6. In this paper gaps 2 and 4 are analyzed. The problem is addressed in generative space, which means that prime numbers are not going to be analyzed directly, but rather their representatives that can be used to produce them. It will be shown that twin primes could be generated by two stage process (sieve), and that will be used to generate formula for the number of twin primes. It will be shown that exist lower bound for the number of twin primes smaller than some natural number $n, n \in N$, and that will be used to show that overall number of twin primes is infinite.

In the last part of the paper it will be shown that the number of cousin primes is infinite, too.

<u>Remark 1</u>: In this paper any infinite series in the form $c_1 \cdot l \pm c_2$ is going to be called a thread defined by number c_1 (in literature these forms are known as linear factors – however, it seems that the term thread is probably better choice in this context). Here c_1 and c_2 are numbers that belong to the set of natural numbers (c_2 can also be zero and usually is smaller than c_1) and l represents an infinite series of consecutive natural numbers in the form (1, 2, 3, ...).

2 Proof of the twin prime conjecture

It is well known that every two consecutive odd numbers (ps_k, pl_k) between two consecutive odd numbers divisible by 3 (e.g. 9 11 13 15, or 39 41 43 45), can be expressed as

$$ps_k = 6k - 1$$
$$pl_k = 6k + 1, k \in N.$$

Twin prime numbers are obtained in the case when both ps_k and pl_k are prime numbers. If any of the ps_k or pl_k (or both) is a composite number, then we cannot have twin primes. In the text that follows we will call numbers ps_k - numbers in *mps* form and numbers pl_k - numbers in *mpl* form. Here we are going to present a two stage process that can be used for generation of twin primes. In the first stage we are going to produce prime numbers by removing all composite numbers from the set of natural numbers. In the second stage, we are going to remove all prime numbers that have an bigger odd neighbor that is composite number. At the end, only the prime numbers in the *mps* form, that represent the smaller number of a twin prime pair, are going to stay. Their number is equal to the number of twin primes. It is going to be shown that that number is infinite. It is easy to check that all numbers in *mpl* form are going to be removed from the set, since their neighbors are composite numbers divisible by 3.

STAGE 1

Prime numbers can be obtained in the following way:

First, we remove all even numbers (except 2) from the set of natural numbers. Then, it is necessary

to remove the composite odd numbers from the rest of the numbers. In order to do that, the formula for the composite odd numbers is going to be analyzed. It is well known that odd numbers bigger than 1, here denoted by a, can be represented by the following formula

$$a = 2n + 1$$

where $n \in N$. It is not difficult to prove that all composite odd numbers a_c can be represented by the following formula

$$a_{c} = 2(2ij+i+j) + 1 = 2((2j+1)i+j) + 1.$$
(1)

where $i, j \in N$. It is simple to conclude that all composite numbers could be represented by product (i + 1)(j + 1), where $i, j \in N$. If it is checked how that formula looks like for the odd numbers, after simple calculation, equation (1) is obtained. This calculation is presented here. The form $2m + 1, m \in N$ will represent odd numbers that are composite. Then the following equation holds

$$2m+1=(i_1+1)(j_1+1)$$

where $i_1, j_1 \in N$. Now, it is easy to see that the following equation holds

$$m = \frac{i_1 j_1 + i_1 + j_1}{2}$$

In order to have $m \in N$, it is easy to check that i_1 and j_1 have to be in the forms

$$i_1 = 2i$$
 and $j_1 = 2j$,

where $i, j \in N$. From that, it follows that *m* must be in the form

$$m = 2ij + i + j = (2i + 1)j + i.$$
 (2)

When all numbers represented by m are removed from the set of odd natural numbers bigger than 1, only the numbers that represent odd prime numbers are going to stay. In other words, only odd numbers that cannot be represented by (1) will stay. This process is equivalent to the sieve of Sundaram [4].

Let us denote the numbers used for the generation of odd prime numbers with m2 (here we ignore number 2). Those are the numbers that are left after the implementation of Sundaram sieve. The number of those numbers that are smaller than some natural number n, is equivalent to the number of prime numbers smaller than n. If we denote with $\pi(n)$ number of primes smaller than n, the following equation holds

$$\pi(n)\approx \frac{n}{\ln(n)}.$$

STAGE 2

What was left after the first stage are prime numbers. With the exception of number 2, all other prime numbers are odd numbers. All odd primes can be expressed in the form 2n + 1, $n \in N$. It is simple to understand that their bigger odd neighbor must be in the form 2n + 3, $n \in N$. Now, we should implement a second step in which we are going to remove number 2 (since 2 cannot make twin pair) and all odd numbers in the form 2m + 3, $m \in N$ and that are composite. If we make the same analysis again, it is simple to understand that m must be in the form

$$m = 2ij + i + j - 1 = (2i + 1)j + i - 1.$$
(3)

All numbers (in observational space) that are going to stay must be numbers in *mps* form and they represent a smaller primes of the twin pairs (it is simple to understand that prime numbers in *mpl* form have neighbors that are composite odd numbers divisible by 3).

Since the methods that are applied in the first and the second stage are similar, it can be intuitively concluded that the number of numbers left after the second "Sundaram" sieve, should be comparable to tplb(n), $n \in N$, defined by the following equation

$$tplb(n) = \frac{\pi(n)}{\ln(\pi(n))} \quad . \tag{4}$$

The tblp(n) would be obtained in the case when second stage sieve wold produce the same amount of numbers removed with each thread, like the original Sundaram sieve. However, the result is not

correct and it requires some compensation terms since the second "Sundaram" sieve is applied on an incomplete set, that is depleted by previously implemented Sundaram sieve. Actually, tplb(n)represents a lower bound for the number of twin primes that are smaller than some number *n*. In order to understand why it is so, we are going to analyze (2) and (3) in more detail.

It is not difficult to be seen that m in (2) and (3) is represented by the threads that are defined by odd prime numbers. For details see any of the previous version of this paper.

Now we are going to compare stages 1 and 2 step by step. Removal of number 2 in second stage is ignored.

Step	Stage 1	Step	Stage 2
1	Remove even numbers (except 2) amount of numbers left $-1/2$	1	Remove numbers defined by thread defined by 3 (obtained for $i = 1$) amount of numbers left 1/2
2	Remove numbers defined by thread defined by 3 (obtained for $i = 1$) amount of numbers left 2/3	2	Remove numbers defined by thread defined by 5 (obtained for $i = 2$) amount of numbers left 3/4
3	Remove numbers defined by thread defined by 5 (obtained for $i = 2$) amount of numbers left 4/5	3	Remove numbers defined by thread defined by 7 (obtained for $i = 3$) amount of numbers left 5/6
4	Remove numbers defined by thread defined by 7 (obtained for $i = 3$) amount of numbers left 6/7	4	Remove numbers defined by thread defined by 11 (obtained for $i = 5$) amount of numbers left 9/10
5	Remove numbers defined by thread defined by 11 (obtained for $i = 5$) amount of numbers left 10/11	5	Remove numbers defined by thread defined by 13 (obtained for $i = 6$) amount of numbers left 11/12
6	Remove numbers defined by thread defined by 13 (obtained for $i = 6$) amount of numbers left 12/13	6	Remove numbers defined by thread defined by 17 (obtained for $i = 8$) amount of numbers left 15/16

Table 1 Comparison of the stages 1 and 2

What can be seen is that in every step, except step 1, threads in the second stage will leave bigger percentage of numbers than the corresponding threads in the first stage. It can be noticed that threads defined by the same number in first and second stage will not remove the same percentage of numbers. The reason is obvious – consider for instance the thread defined by 3: in the first stage

it will remove 1/3 of the numbers left, but in the second stage it will remove ½ of the numbers left, since the thread defined by 3 in stage 1 has already removed one third of the numbers (odd numbers divisible by 3 in observation space). So, only odd numbers (in observational space) that give residual 1 and -1 when they are divided by 3 are left, and there are approximately same number of numbers that give residual -1 and numbers that give residual 1, when the number is divided by 3. Same way of reasoning can be applied for all other threads defined by same prime in different stages. So, from Table we can see that bigger number of numbers is left in every step of stage 2 then in the stage 1 (except 1st step). From that, we can conclude that after every step bigger than 1, part of the numbers that is left in stage 2 is bigger than number of numbers left in the stage 1 (that is even more noticeable if we consider amount of numbers left after removal of all numbers generated by threads that are defined by all prime numbers smaller than some natural number). Let us mark the number of twin primes smaller than some natural number *n* with $\pi_{g2}(n)$. From previous analysis we can safely conclude that the following equation holds for every n > 4

$$\pi_{g^2}(n) > \frac{1}{3} tplb(n) \quad ,$$

or for *n* > 100

$$\pi_{g2}(n) > tplb(n)$$

(Reason for introduction of constant 1/3 or postponing the relevance of the relationship for some big enough n (which is here 100), comes from the fact that fractures that were presented in Table 1 are not correct in the case when small number of numbers are analyzed. For small numbers their values can fluctuate significantly).

Since it it easy to show that following holds

$$\lim_{n \to \infty} tplb(n) = \lim_{n \to \infty} \frac{\pi(n)}{\ln(\pi(n))} = \infty \quad ,$$

we can safely conclude that the number of twin primes is infinite. That concludes the proof.

Without going into details, here we will state the following conjecture: for n big enough, number of twin primes is given by the following equation

$$\pi_{g_2}(n) \sim \frac{\pi^2}{8} \frac{\pi(n)}{\ln(\pi(n))} = \frac{3}{4} \zeta(2) \frac{\pi(n)}{\ln(\pi(n))},$$

where ζ represents Riemann zeta function. If we mark the number of primes smaller than some natural number *n* with $\pi_{g^2}(n) = f(n)$, where function f(n) gives good estimation of the number of primes smaller than *n*, than $\pi_{g^2}(n)$, for *n* big enough, is given by the following equation

$$\pi_{g^2}(n) \sim \frac{3}{4} \zeta(2) \cdot f(f(n)).$$

If particular case f(n) = Li(n), the following equation holds

$$\pi_{g2}(n) \sim \frac{3}{4}\zeta(2) \cdot \int_{2}^{n} \left(\frac{dx}{\ln\left(\int_{2}^{x} \left(\frac{dt}{\ln(t)} \right) \right)} \right).$$

3. Proof that the number of cousin primes is infinite

The cousin primes are successive prime numbers with gap 4. It is clear that cousin primes represent pairs of odd numbers that surround odd number divisible by 3 (e.g. (7 9 11), or (13 15 17)). A pair can only represent a cousin primes if both those numbers are primes. So, if we denote a pair of odd numbers that surround an odd number divisible by 3 as $pl_k = 6k + 1$ and $ps_k = 6(k + 1) - 1$, $k \in N$, these numbers can represent cousin primes only in the case when both pl_k and ps_k are prime numbers. If any of the ps_k or pl_k (or both) is a composite number, then we cannot have cousin primes.

Here, similar to the case of twin primes we are going to create a two stage process for generation of cousin primes.

STAGE 1

Using the same methodology as previously, generate all prime numbers. In order to do that, from the set of all natural numbers bigger than 1, remove all even numbers (except 2) and all odd numbers generated by equation (2).

STAGE 2

What was left after first stage are prime numbers. With the exception of number 2, all other prime numbers are odd numbers. All odd primes can be expressed in the form 2m + 1, $m \in N$. It is simple to understand that their bigger odd cousin must be in the form 2m + 5, $m \in N$. Now, we should implement a second step in which we are going to remove number 2 (since 2 cannot make cousin pair) and all odd numbers in the form 2m + 5, $m \in N$ and that are composite. If we make the same analysis like in the case of twin primes, it is simple to understand that m must be in the form

$$m = 2ij + i + j - 2 = (2i + 1)j + i - 2.$$
(5)

All numbers (in observational space) that are going to stay must be numbers in *mpl* form and they represent a smaller primes of the cousin pairs (it is simple to understand that prime numbers in *mps* form have cousins that are composite odd numbers divisible by 3).

Now, using the same method like in the case of the twin prime conjecture, it can be proved that exists infinitely many cousin primes.

Let us mark the number of cousin primes smaller than some natural number *n* with $\pi_{g^4}(n)$. Without going into details, here we will state the following conjecture: for *n* big enough, number of cousin primes is given by the following equation

$$\pi_{g^{4}}(n) \sim \frac{\pi^{2}}{8} \frac{\pi(n)}{\ln(\pi(n))} = \frac{3}{4} \zeta(2) \frac{\pi(n)}{\ln(\pi(n))},$$

where ζ represents Riemann zeta function. If we mark the number of primes smaller than some natural number *n* with $\pi_{\alpha}(n) = f(n)$, where function f(n) gives good estimation of the number of

primes smaller than *n*, than $\pi_{g4}(n)$, for *n* big enough, is given by the following equation

$$\pi_{g4}(n) \sim \frac{3}{4} \zeta(2) \cdot f(f(n)).$$

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