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J. Mourad, D.A. Steer

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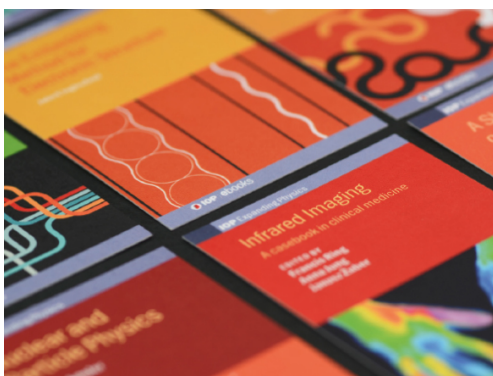
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Translation invariant time-dependent solutions to massive gravity II

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Abstract. This paper is a sequel to [JCAP 12 \(2013\) 004](#) and is also devoted to translation-invariant solutions of ghost-free massive gravity in its moving frame formulation. Here we consider a mass term which is linear in the vielbein (corresponding to a β_3 term in the 4D metric formulation) in addition to the cosmological constant. We determine explicitly the constraints, and from the initial value formulation show that the time-dependent solutions can have singularities at a finite time. Although the constraints give, as in the β_1 case, the correct number of degrees of freedom for a massive spin two field, we show that the lapse function can change sign at a finite time causing a singular time evolution. This is very different to the β_1 case where time evolution is always well defined. We conclude that the β_3 mass term can be pathological and should be treated with care.

Keywords: modified gravity, gravity

ArXiv ePrint: [1405.1862](https://arxiv.org/abs/1405.1862)



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1 Introduction

This paper is devoted to general translation invariant time-dependent solutions of ghost-free massive gravity (for reviews of massive gravity, see [1–3]), and is a follow up paper to [4] which will hereafter be referred to as [I]. The general framework of massive gravity and the motivations were presented in detail in [I] where one particular mass term — which is cubic in the vielbien in $D = 4$ dimensions, and corresponds to the β_1 mass term in the metric formulation (see e.g. [3]) — was considered. This case was singled out in [5] as allowing a simple and covariant way of deducing an extra scalar constraint. It is also singled out in the Hamiltonian analysis, being the only one allowing the constraints to be determined explicitly [6–8].

The general mass term in D dimensions depends on $D - 1$ constants β_i ($i = 1, \dots, D - 1$) in addition to the cosmological constant term. Here we shall consider the one linear in the vielbein, namely β_{D-1} . In that case, the covariant analysis of [5] shows that a symmetry condition is imposed on the moving frame vielbein components, but does it not lead to an extra scalar constraint. Within the simplified framework of space-independent solutions, we shall show the origin of the necessary extra scalar constraint, and thus determine explicitly all the constraints and the equations of motion. We show that the time evolution is well posed provided the lapse function N does not vanish. In fact, the lapse is obtained from the extra scalar constraint and is fully determined by the other fields and their first derivatives. As opposed to the β_1 case (namely the mass term which depends on $D - 1$ factors of the vielbein) in which the lapse function $N(t)$ is strictly positive, here we show that the sign of $N(t)$ can change. This leads to singularities in the time evolution which occur at a finite time. This is the crucial difference with respect to the β_1 case where the sign of N remains constant. In [I] we showed that there is a sector in the β_1 -theory which is stable (for related pathologies of massive gravity theories see also [9–12]): for β_3 -theory this is no longer the case, and singularities are generic for $D > 3$.

The paper is organised as follows. In section 2 we give a brief summary of the moving frame formulation of massive gravity. Section 3 is devoted to the general analysis of translation-invariant fields. We use a convenient ADM-like decomposition with a lapse function, a shift vector and a symmetric $(D - 1) \times (D - 1)$ matrix. We show that the Bianchi identities¹ together with the constraints arising from the $0i$ -components of the equations of motion lead to the vanishing of the shift vectors. This is similar to the β_1 case and we expect it to be a general property of massive gravity. The Bianchi identities leave one further scalar constraint which once used in the equations of motion — and in particular after putting these in a form showing their well-posedness — leads to a new scalar constraint. This extra scalar constraint, which was missing in the analysis of [5], provides the expression of the lapse function in terms of the symmetric matrix and its first derivative. The equations of motion are well posed provided the lapse function does not change sign. Contrary to the β_1 case, this condition is not manifestly true and a case by case study is necessary to prove its validity. This is what we do in section 4 where we consider some particular cases. In section 4.1, we solve analytically the three dimensional case and show that the lapse function is constant. Section 4.2 is devoted to the diagonal solutions in any $D \geq 4$ where all the eigenvalues are equal except one. We show that in that case, for initial conditions in a certain region, N can vanish. This seems to be the most important difference with respect to the β_1 case and in that respect the latter mass term does not have this pathology. In the appendix, we show how using the translation-invariant solutions of this paper and [I] it is possible, by performing a Lorentz transformation, to obtain “plane wave” solutions, which can also be seen as the generalisation of the pp -waves of general relativity.

2 Action, equations of motion, and constraints in non-linear massive gravity

We consider non-linear massive gravity in D -dimensions in the vierbein formulation, see [2, 3, 5, 13]. This is described by a dynamical metric $g_{\mu\nu}$ as well as a non-dynamical one $f_{\mu\nu}$, with the corresponding families of 1-forms given by θ^A and f^A where

$$\eta_{AB}\theta^A{}_{\mu}\theta^B{}_{\nu} = g_{\mu\nu}, \quad (2.1)$$

$$\eta_{AB}f^A{}_{\mu}f^B{}_{\nu} = f_{\mu\nu}. \quad (2.2)$$

Here the Lorentz indices $A, B = 0, \dots, D-1$ are raised and lowered with the Minkowski metric η_{AB} , and the dual vectors e_A to the 1-forms θ^A satisfy

$$\theta^A(e_B) = \theta^A{}_{\mu}e_B{}^{\mu} = \delta^A_B. \quad (2.3)$$

As has been discussed extensively in the literature [14–16], if the symmetry property

$$e^{C\mu}f^B{}_{\mu} = e^{B\mu}f^C{}_{\mu} \quad (2.4)$$

holds, then the matrix $g^{-1}f$ has a real square-root, and hence the potential in the metric formulation of massive gravity is well defined. As a result, the action in terms of vierbeins reads [5, 13, 17]

$$S = \frac{1}{2} \int \Omega^{AB} \wedge \theta^*_{AB} + \sum_{n=0}^{D-1} \beta_n \int f^{A_1} \wedge \dots \wedge f^{A_n} \wedge \theta^*_{A_1 \dots A_n}, \quad (2.5)$$

¹The Bianchi identities correspond, in the Hamiltonian language, to the secondary constraints.

where the β_n are arbitrary parameters, and [18]

$$\theta_{A_1 \dots A_n}^* \equiv \frac{1}{(D-n)!} \epsilon_{A_1 \dots A_n} \theta^{A_{n+1}} \wedge \dots \wedge \theta^{A_D} \quad (2.6)$$

is a $(D-n)$ -form. The curvature 2-form Ω^{AB} is defined by

$$\Omega^{AB} \equiv d\omega^{AB} + \omega^A_C \wedge \omega^{CB}, \quad (2.7)$$

where the spin-connection ω^{AB} results from the torsion-free condition $\mathcal{D}\theta^A \equiv d\theta^A + \omega^A_B \wedge \theta^B = 0$ and the antisymmetry in the indices A and B . From the definition (2.7) the curvature 2-form satisfies the Bianchi identity

$$\mathcal{D}\Omega^{AB} \equiv d\Omega^{AB} + \omega^A_C \wedge \Omega^{CB} + \omega^B_C \wedge \Omega^{AC} = 0. \quad (2.8)$$

We now take (2.5) as our starting point (that is, the condition (2.4) on the vielbeins is *not* imposed). As shown in [5], for some β_n this condition is obtained dynamically, though this is not always necessarily the case (for an example where it does not hold see [20]). In fact, the class of theories of massive gravity described by (2.5) is larger than that of the metric formulation, and it potentially has a larger space of solutions. Furthermore the moving frame formulation does not rely on matrix square roots and is technically much easier to deal with, particularly concerning the Bianchi identities which will be extensively used in the following. In fact the expression of the derivative of a matrix square root in terms of the derivative of the matrix is complicated and involves time ordering.

Here we focus on $\beta_0 \neq 0$ and $\beta_{D-1} \neq 0$ in which case (2.4) is imposed dynamically (see below and [5]). The action (2.5) breaks both diffeomorphism and local Lorentz invariance if the non-dynamical vielbein f^A is fixed. In the following we choose $f^A \equiv dx^A$, which is always possible when $f_{\mu\nu} = \eta_{\mu\nu}$. The isometry group $\text{SO}(1, D-1)$ of the background metric $f_{\mu\nu}$ is a global symmetry group of the theory. Thus $f^A{}_\mu = \delta^A{}_\mu$, and we can identify Lorentz and spacetime indices. Now we define the Einstein tensor as the $(D-1)$ -form

$$G_A \equiv -\frac{1}{2} \Omega^{BC} \wedge \theta_{ABC}^* \equiv G_A{}^B \theta_B^*, \quad (2.9)$$

where $G_{AB} = R_{AB} - \eta_{AB}R/2$ with $R_{AB} = \Omega_{ACB}{}^C$. Thus

$$\begin{aligned} G_{AB} = & e_C{}^\mu \partial_\mu \omega^C{}_{AB} - e_B{}^\mu \partial_\mu \omega^C{}_{AC} - \omega^C{}_{AD} \omega^D{}_{BC} + \omega^D{}_{AB} \omega^C{}_{DC} \\ & + \frac{\eta_{AB}}{2} [-\omega^C{}_{DE} \omega^{DE}{}_C + 2e_I{}^\mu \partial_\mu \omega^{CI}{}_C + \omega^D{}_{ED} \omega^{FE}{}_F] \end{aligned} \quad (2.10)$$

where

$$\omega^A{}_B = \omega^A{}_{BC} \theta^C.$$

Then the field equations following from (2.5) read

$$G_A = t_A, \quad (2.11)$$

or equivalently $G_{AB} = t_{AB}$, with

$$t_A \equiv \sum_{n=0}^{D-1} \beta_n f^{A_1} \wedge \dots \wedge f^{A_n} \wedge \theta_{AA_1 \dots A_n}^* \equiv t_A{}^B \theta_B^* \quad (2.12)$$

so that

$$t_A \wedge \theta_B = (-1)^{D-1} t_{AB} \varepsilon \quad (2.13)$$

with ε the volume element $\varepsilon \equiv \theta^0 \wedge \theta^1 \wedge \dots \wedge \theta^{D-1}$. For $\beta_0 \neq 0$ and $\beta_{D-1} \neq 0$, it follows that

$$t_{AB} = \beta_0 \eta_{AB} + \beta_{D-1} (D-1)! \frac{\theta_{BA}}{\det \theta}. \quad (2.14)$$

As mentioned before, Lorentz and space-time indices are identified so that $\theta_{AB} = \theta^C{}_{\mu} \eta_{CA} \delta_B^{\mu}$.

Notice that as a result of diffeomorphism invariance of the Einstein-Hilbert term, the Bianchi identity

$$\mathcal{D}G_A = 0 = \mathcal{D}t_A \quad (2.15)$$

must hold, whilst Lorentz invariance imposes

$$G_{[AB]} = 0 = t_{[AB]}. \quad (2.16)$$

Thus from (2.14), the symmetry of $t_{AB} = t_{BA}$ implies the symmetry of the matrix θ_{AB} . In order to guarantee that Minkowski space is a vacuum solution, and that the equations of motion reduce to those of Fierz-Pauli [21] in the linearized limit, we choose β_0 and β_{D-1} to be related to the mass m of the spin 2 field by $\beta_0 = m^2 = -\beta_{D-1} (D-1)!$. Thus the equations of motion (2.11) become

$$G_{AB} = m^2 \left[\eta_{AB} - \frac{\theta_{AB}}{\det \theta} \right]. \quad (2.17)$$

Finally [5] the Bianchi identity (2.15) leads to the D constraints

$$\omega^B{}_{AC} \theta^C{}_B = 0 \quad (2.18)$$

or equivalently

$$\partial_A \theta^B{}_B - \partial_B \theta^B{}_A = 0. \quad (2.19)$$

As discussed in [5] the various traces of the equation of motion do not lead to a further scalar constraint, contrary to the cases in which only β_1 or β_2 are non-zero. In that respect, the situation we consider here of $\beta_{D-1} \neq 0$ is singled out from those considered before.

In the following, we will use the simplified framework of translation invariant fields to analyse in detail the equations of motion and find the origin of this extra scalar constraint needed to have the correct number of degrees of freedom.

3 Equations of motion and constraints

We now consider solutions invariant under spatial translations and split the symmetric moving frame components e_{AB} and θ^{AB} into the ADM-type form

$$e_{00} = -N, \quad e_{0i} = -N n_i, \quad e_{ij} = \pi_{ij} - N n_i n_j, \quad (3.1)$$

and so from (2.3)

$$\theta^{00} = - \left(\frac{1}{N} - \zeta^{ij} n_i n_j \right), \quad \theta^{0i} = -\zeta^{ij} n_j, \quad \theta^{ij} = \zeta^{ij}. \quad (3.2)$$

Here ζ_{ij} is the inverse of π_{ij} , and the $D(D+1)/2$ variables all depend on $t = x^0$. Notice that $\det \theta = \frac{\det(\zeta)}{N}$.

In the following, we first write down the constraints arising from the Bianchi identities (2.19), and then those coming from the equations of motion (2.11). These latter constraints will enable us to show that shift vector n_i vanishes (subsection 3.2). As a result the remaining equations of motion take a simplified form, and these are discussed in section 3.3. Finally, we show how the equations of motion and Bianchi identities lead to further scalar constraint which determines the lapse function N in terms of $(\zeta, \partial_t \zeta)$.

3.1 Bianchi identities

The time component of the vector constraint (2.19) yields a first constraint (denoted by C_1 for future reference)

$$C_1 \equiv \partial_t(\text{tr}\zeta) = 0 \tag{3.3}$$

or equivalently $\text{tr}\zeta = c$ for a constant c , while the spatial components give

$$\partial_t(\zeta^{ij}n_j) = 0. \tag{3.4}$$

We now use the constraints coming from the equations of motion (2.11) to show that $n_i = 0$, so that (3.4) is trivially satisfied.

3.2 Vanishing shift, $n_i = 0$

The most succinct way of finding the D constraints included in the equations of motion is to proceed as follows.

Let $G_A^{(s)}$ denote the spatial components of the $D-1$ form G_A , that is

$$G_A^{(s)} \equiv \frac{1}{(D-1)!} G_{Ai_1i_2\dots i_{D-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{D-1}} \tag{3.5}$$

so that

$$G_A = G_A^{(s)} + G_A^{(t)}$$

where the remaining time components, $G_A^{(t)}$, contains one dt . Since we consider time-dependent metrics only, $dG_A = dG_A^{(s)}$ and the Bianchi identity (2.15) reads $dG_A^{(s)} + \omega_A^B G_B = 0$. Thus, at most, $G_A^{(s)}$ is first order in time derivatives meaning that the D equations

$$G_A^{(s)} = t_A^{(s)} \tag{3.6}$$

are constraint equations. The second order equations of motion are contained in

$$G_A^{(t)} = t_A^{(t)}. \tag{3.7}$$

From (2.9) it follows that only the components $\Omega^{AB}{}_{ij}$ of the curvature tensor are needed to determine the constraints, and furthermore from (2.7) we have

$$\Omega^{AB}{}_{ij} = \omega^A{}_{C[i}\omega^{CB}{}_{j]}. \tag{3.8}$$

The relevant components, $\omega^A{}_{Ci}$, of the spin connexion² can be determined from

$$\omega_{AB\mu} = \omega_{ABC}\theta^C{}_\mu = \frac{1}{2}(C_{ABC} + C_{ACB} + C_{CBA})\theta^C{}_\mu.$$

²Note that the first term of (2.7) will only contribute time-derivatives, and hence will appear in $\Omega^{AB}{}_{0i}$.

The only non-vanishing components of the structure functions

$$C_{AB}{}^C = e_{[A}{}^\mu \partial_\mu e_{B]}{}^\nu \theta^C{}_\nu = -e_A{}^\mu e_B{}^\nu \partial_{[\mu} \theta_{\nu]}{}^C$$

are $C_{0i}{}^0 = \partial_t(\zeta n)_i$ and $C_{0i}{}^k = -\partial_t \zeta_i{}^k$. Thus

$$\omega_{0ki} = -\frac{N}{2} \Sigma_{ki}, \quad \omega_{jki} = \frac{N}{2} (\Sigma^T)_{i[j} n_{k]}$$

where

$$\Sigma \equiv \{\partial_t \zeta, \pi\} \zeta.$$

Finally, substituting into (3.8) gives

$$\Omega^{qp}{}_{ij} = \frac{N^2}{4} \left\{ (1 - n^2) \Sigma^q{}_{[i} \Sigma^p{}_{j]} + \Sigma^q{}_{[i} (\Sigma^T n)_{j]} n^p + (\Sigma^T n)_{[i} \Sigma^p{}_{j]} n^q \right\}, \quad (3.9)$$

$$\Omega^{0q}{}_{ij} = \frac{N^2}{4} \left\{ (\Sigma^T n)_{[j} \Sigma^q{}_{i]} \right\}. \quad (3.10)$$

From (2.9) as well as (3.9) and (3.10), we then obtain

$$G_p^{(s)} = G_0^{(s)} n_p$$

with

$$G_0^{(s)} = \frac{N^2}{8} (\det \zeta) \left[\text{tr}(\mathbb{S}^2) - \text{tr}(\mathbb{S})^2 - n^T (\mathbb{S}^2 - \text{Str}(\mathbb{S})) n \right] dV$$

where $dV = dx^1 \wedge \dots \wedge dx^{D-1}$ is the spatial volume element, and the symmetric matrix \mathbb{S} is defined by

$$\mathbb{S} \equiv \{\partial_t \zeta, \pi\} \quad (3.11)$$

so that $\Sigma = \mathbb{S} \zeta$. The r.h.s. of (3.6), is determined directly from (2.12) and we find

$$t_p^{(s)} = m^2 n_p (\det \zeta) dV$$

$$t_0^{(s)} = m^2 [(\det \zeta) - 1] dV.$$

Hence, the we finally arrive at the D constraints contained in $G_A^{(s)} = t_A^{(s)}$:

$$\frac{N^2}{8} \left[\text{tr}(\mathbb{S}^2) - \text{tr}(\mathbb{S})^2 - n^T (\mathbb{S}^2 - \text{Str}(\mathbb{S})) n \right] n_p = m^2 n_p \quad (3.12)$$

$$\frac{N^2}{8} \left[\text{tr}(\mathbb{S}^2) - \text{tr}(\mathbb{S})^2 - n^T (\mathbb{S}^2 - \text{Str}(\mathbb{S})) n \right] = m^2 \left[1 - \frac{1}{(\det \zeta)} \right]. \quad (3.13)$$

On substituting (3.13) into (3.12) it follows that

$$n_i = 0 \quad (3.14)$$

whilst the remaining scalar constraint reads

$$\frac{N^2}{8} \left[\text{tr}(\mathbb{S}^2) - \text{tr}(\mathbb{S})^2 \right] = m^2 \left[1 - \frac{1}{(\det \zeta)} \right]. \quad (3.15)$$

3.3 Equations of motion

The constraint $n_i = 0$ which we have found above enormously simplifies the analysis. In this subsection we write down the remaining dynamical Einstein equations, namely (3.7).

To do so, it is convenient to change variables. Notice that the dynamical metric now takes the form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\frac{1}{N^2} dt^2 + (\pi^{-2})_{ij} dx^i dx^j, \\ &\equiv -dT^2 + (B^{-2})_{ij} dx^i dx^j \end{aligned} \quad (3.16)$$

where $B \equiv \pi^2$ and the new time coordinate T is unambiguously determined from (3.16) only if N does not vanish at some finite $t = t_*$. We will return to this important point and its interpretation below.³ In the following we denote by a dot a derivative with respect to T

$$\cdot \equiv \partial_T = N \partial_t. \quad (3.17)$$

On using

$$NS = -\zeta \dot{B} B^{-1} \pi,$$

(which follows from the definition of \mathbb{S} in (3.11)), the second constraint C_2 in (3.15) reads

$$C_2 \equiv \left(\text{tr}(\dot{B} B^{-1}) \right)^2 - \text{tr} \left((\dot{B} B^{-1})^2 \right) + 8m^2 (1 - \det \pi) = 0. \quad (3.18)$$

The equations of motion for $B(T)$, are given by the (ij) -component of the Einstein equation. From (2.10) and (2.17) they read

$$\begin{aligned} -\partial_T(\dot{B} B^{-1}) + \partial_T \text{tr}(\dot{B} B^{-1}) \mathbb{1} + \frac{1}{2} (\dot{B} B^{-1}) \text{tr}(\dot{B} B^{-1}) \\ - \frac{1}{4} \left[\left(\text{tr}(\dot{B} B^{-1}) \right)^2 + \text{tr} \left((\dot{B} B^{-1})^2 \right) \right] = -2m^2 [\zeta (N \det \pi) - \mathbb{1}], \end{aligned} \quad (3.19)$$

and we will refer to them as E (standing for ‘equations of motion’) in the following. Notice that even when they are written in terms of the new time coordinate T , these equations depend explicitly on the lapse function $N(T)$ which is still undetermined. As we will discuss in the next subsection, E and C_1 together in fact generate a third constraint C_3 which will determine $N(T)$.

Before doing so, let us discuss the main properties of the equations E. First notice that the l.h.s. of (3.19) can be simplified by using (3.18) to eliminate the second term in the square bracket. Then on defining

$$\epsilon e^f \equiv \det \pi, \quad \epsilon = \frac{\det \pi}{|\det \pi|} = \pm 1, \quad (3.20)$$

the equations E become

$$\frac{1}{2} \partial_T (e^{-f} \dot{B} B^{-1}) = \partial_T (e^{-f} \dot{f}) \mathbb{1} + m^2 \left[(\epsilon - 2e^{-f}) \mathbb{1} + N \zeta \epsilon \right]. \quad (3.21)$$

³Recall, [1], that for $\beta_1 \neq 0$ the constraints always imply that $N > 1$; in the case of β_{D-1} we will see that N can vanish for $D > 3$.

Second, note that the antisymmetric part of this equation yields the *conserved* matrix

$$\gamma \equiv \frac{1}{2}e^{-f}[\dot{B}, B^{-1}], \quad (3.22)$$

with $\dot{\gamma} = 0$. Thus the system contains $(D-1)(D-2)/2$ conserved quantities, which correspond to the expected conservation of angular momentum since the system is invariant under spatial rotations. These will be used extensively below when we construct exact solutions.

Third, on taking the trace of (3.21) we find

$$N\beta\epsilon(\text{tr}\zeta) = (D-2)\partial_T(e^{-f}\dot{f}) + m^2(D-1)(\epsilon - 2e^{-f}) \quad (3.23)$$

which, when substituted back into (3.21) can be used to eliminate N . Indeed we then find a unique traceless 2nd order equation for B which is independent of $N(T)$, namely

$$\frac{1}{2}\partial_T(e^{-f}\dot{B}B^{-1}) - \partial_T(e^{-f}\dot{f}) \left[\mathbb{1} - (D-2)\frac{\zeta}{\text{tr}(\zeta)} \right] = m^2(\epsilon - 2e^{-f}) \left[\mathbb{1} - (D-1)\frac{\zeta}{\text{tr}(\zeta)} \right]. \quad (3.24)$$

If this equation is well posed — something which is not manifest and will be discussed below — it follows that it can be used to find $\ddot{\zeta}$ in terms of lower derivatives of ζ . As a result, the first term on the right hand side of (3.23) can be expressed in terms of $(\dot{\zeta}, \zeta)$ thus eliminating all second derivatives from (3.23) which then becomes a constraint. This will be done in detail in the next subsection. Notice that in the β_1 case, see [I], the analogue of (3.23) is already a constraint since in that case the analogue of C_1 is $\dot{f} = 0$ so that N is determined directly in terms of ζ .

Before writing out this third constraint C_3 explicitly, it is useful to introduce a final set of variables which simplify the equations and constraints further. Motivated from (3.21) define

$$\partial_u \equiv e^{-f}\partial_T, \quad C \equiv e^{-2f}B, \quad (3.25)$$

with $' = \partial_u$. Then the conserved quantity γ and the equation of motion E become

$$\gamma = \frac{1}{2}[C', C^{-1}], \quad (3.26)$$

$$\partial_u[C^{-1}C'] = 2m^2e^{-f}[N\zeta\epsilon - (2e^{-f} - \epsilon)\mathbb{1}], \quad (3.27)$$

whilst the constraints C_1 and C_2 become

$$\tilde{C}_1 \equiv (\text{tr}\zeta)' = 0, \quad (3.28)$$

$$\tilde{C}_2 \equiv \text{tr}((C'C^{-1})^2) - 4(D-2)f'^2 - 8m^2(e^{-2f} - \epsilon e^{-f}) = 0. \quad (3.29)$$

eqs. (3.23) and (3.24) in turn read

$$-m^2(\text{tr}\zeta)N\epsilon = (D-2)f''e^f - m^2(D-1)(2e^{-f} - \epsilon), \quad (3.30)$$

$$\partial_u(C^{-1}C') + 2\frac{\zeta}{\text{tr}\zeta}(D-2)f'' = 2m^2e^{-f}(\epsilon - 2e^{-f}) \left[\mathbb{1} - (D-1)\frac{\zeta}{\text{tr}\zeta} \right]. \quad (3.31)$$

Fourth, in these new variables it is straightforward to verify the compatibility of the constraints \tilde{C}_1 and \tilde{C}_2 with the equation of motion. To do so, we first multiply (3.31) by $C^{-1}C'$ and then take the trace. This gives

$$\begin{aligned} & \frac{1}{2}\partial_u [\text{tr}(C^{-1}C')^2] - \frac{4(D-2)}{\text{tr}\zeta}f'' [(\text{tr}\zeta)' + f'\text{tr}\zeta] \\ &= 2m^2e^{-f}(\epsilon - 2e^{-f}) \left\{ 2f' + \frac{2(D-1)(\text{tr}\zeta)'}{\text{tr}\zeta} \right\} \end{aligned} \quad (3.32)$$

which can be rewritten as

$$\partial_u(\tilde{C}_2) = -8m^2(\text{tr}\zeta)'e^{-f}N\epsilon. \quad (3.33)$$

Thus we deduce that if \tilde{C}_2 is valid for all u and the equations of motion hold, then \tilde{C}_1 follows provides $N \neq 0$. In the Hamiltonian language, this is expressed as \tilde{C}_1 being a secondary constraint, with \tilde{C}_2 a primary constant. An alternative way of interpreting (3.33) is that if \tilde{C}_1 holds for all u and \tilde{C}_2 is true at an initial u , then must be true for all u .

Finally let us verify that Fierz-Pauli theory is recovered in the linearized limit, $\pi = \mathbb{1} + h$ and $B = \mathbb{1} + 2h$. In that case, the first constraint \tilde{C}_1 in equation (3.28) gives $\text{tr}h' = 0$ whilst from \tilde{C}_2 in (3.29) it follows that $\text{tr}h = 0$. Thus $\det \pi = 1$ so that from (3.20) that $f = 0 = f' = f''$. Also $\text{tr}(\zeta) = (D - 1)$, so that from (3.30), $N = 1$. Thus $u = T = t$ and finally the equation of motion (3.31) reduces to

$$\frac{d^2 h_{ij}}{dt^2} + m^2 h_{ij} = 0 \quad (3.34)$$

as required.

3.4 The third constraint

In the previous subsection we obtained N in terms of the second derivative of f (or equivalently ζ), see (3.30). We also have the equation of motion (3.31). If this equation is well posed it follows that it can be used to find ζ'' in terms of lower derivatives of ζ . As a result, the first term on the right hand side of (3.30) can be expressed in terms of (ζ', ζ) thus eliminating all second derivatives from (3.30) which then becomes a constraint.

Alternatively, we can determine N in terms of (ζ, ζ') directly from the observation that the equations of motion (3.31) combined with the first constraint \tilde{C}_1 yield a 3rd scalar constraint C_3 . Schematically the reason is the following: in terms of ζ , eq. (3.31) can be rewritten in the form $\zeta'' = g(\zeta, \dot{\zeta}, N)$ for some (matrix) function g which will be determined below but which is clearly linear in N . On taking the trace of this equation of motion, it follows that the left hand side must vanish by \tilde{C}_1 . Thus we are left with a third constraint C_3 namely $\text{tr}(g(\zeta, \zeta', N)) = 0$, which a priori determines N giving $N = N(\zeta, \zeta')$. (As mentioned above, in the $\beta_1 \neq 0$ case [I], the analogue of (3.31) contains no terms in first and second derivatives of ζ and directly determines $N = N(\zeta)$.)

The crucial step is to show the well posedness of the the equations of motion (3.31) — that is to put them in a form in which it is manifest that ζ'' is determined in terms of lower derivatives. To carry out this procedure, we write the real symmetric matrix $\zeta(u)$ in the form

$$\zeta(u) = \sum_{i=1}^{D-1} e^{-\Delta_i(u)} |v_i(u)\rangle \langle v_i(u)|, \quad (3.35)$$

where the $|v_i(u)\rangle$ are its orthonormal eigenvectors with eigenvalues⁴ $e^{-\Delta_i}$. Thus from (3.20) we have $f = \sum_i \Delta_i$, and from (3.25) C is given by $C = \pi^2 e^{-2f}$. The constraint \tilde{C}_1 in (3.28) then implies that

$$(\text{tr}\zeta)'' = 0 \quad \iff \quad \sum_i \Delta_i'' e^{-\Delta_i} = \sum_i \Delta_i'^2 e^{-\Delta_i}, \quad (3.36)$$

⁴From now on we only consider positive eigenvalues so that $\epsilon = +1$

whilst the conserved anti-symmetric matrix γ defined in (3.26) is given by

$$\gamma = \frac{1}{2}[C', C^{-1}] = 2 \sum_{i,j} \sinh^2(\Delta_i - \Delta_j) \langle v_i(u) | v'_j(u) \rangle | v_j(u) \rangle \langle v_i(u) | \quad (3.37)$$

so that

$$\langle v_i(u) | v'_j(u) \rangle = \frac{\langle v_j(u) | \gamma | v_i(u) \rangle}{2 \sinh^2(\Delta_i - \Delta_j)} \quad (i \neq j). \quad (3.38)$$

The matrix elements of γ in a time independent basis give $(D-1)(D-2)/2$ constants of motion. If we choose that basis to be the eigenvectors $|v_a(0)\rangle$ at the initial time $u=0$, these are given by

$$\begin{aligned} \gamma_{ab} &= 2 \sum_{i,j} \sinh^2(\Delta_i - \Delta_j) \langle v_i(u) | v'_j(u) \rangle \langle v_a(0) | v_j(u) \rangle \langle v_i(u) | v_b(0) \rangle \\ &= 2 \sinh^2(\Delta_a - \Delta_b) \langle v_b(0) | v'_a(0) \rangle. \end{aligned} \quad (3.39)$$

We can now proceed to find the third constraint C_3 . The (ii) components of equations of motion (3.27) reduce to

$$2(\Delta_i'' - f'') - A_i = 2m^2 e^{-f} \left[N e^{-\Delta_i} + (1 - 2e^{-f}) \right] \quad (3.40)$$

where

$$A_i \equiv \frac{1}{2} \sum_{j \neq i} \sinh 2(\Delta_i - \Delta_j) \left(\frac{\langle v_j(u) | \gamma | v_i(u) \rangle}{\sinh^2(\Delta_i - \Delta_j)} \right)^2, \quad (3.41)$$

so that $\sum_i A_i = 0$. The (ij) components of the equations of motion are a consequence of the constraints (as in the β_1 case [I]). On summing (3.40) over i , we find

$$f'' = -\frac{m^2 e^{-f}}{(D-2)} \left[N \text{tr} \zeta + (D-1)(1 - 2e^{-f}) \right]$$

which is of course identical to (3.30). Then substituting into (3.40) gives

$$\Delta_i'' = \frac{A_i}{2} + \frac{m^2 e^{-f}}{(D-2)} \left[N ((D-2)e^{-\Delta_i} - \text{tr} \zeta) - (1 - 2e^{-f}) \right]. \quad (3.42)$$

Finally, the constraint C_3 comes from combining (3.42) with (3.36):

$$C_3 \equiv m^2 e^{-f} N = \frac{\sum_i \Delta_i'^2 e^{-\Delta_i} + \frac{m^2 \text{tr} \zeta}{D-2} e^{-f} (1 - 2e^{-f}) - \frac{1}{2} \sum_i A_i e^{-\Delta_i}}{\text{tr}(\zeta^2) - \frac{(\text{tr} \zeta)^2}{D-2}}. \quad (3.43)$$

Notice that in the limit $\Delta \rightarrow 0$ then $\text{tr} \zeta \rightarrow D-1$, and $N \rightarrow 1$ as it should from the linear analysis above.⁵ For completeness, in terms of the variables Δ_i and $|v_i\rangle$, the first two constraints read

$$\tilde{C}_1 \equiv -\sum_i \Delta_i' e^{-\Delta_i} = 0 \quad (3.44)$$

$$\frac{1}{4} \tilde{C}_2 \equiv \sum_i \Delta_i'^2 - \left(\sum_i \Delta_i' \right)^2 + \frac{1}{4} \sum_{i \neq j} \left(\frac{\langle v_j(u) | \gamma | v_i(u) \rangle}{\sinh^2(\Delta_i - \Delta_j)} \right)^2 + 2m^2 e^{-f} (1 - e^{-f}) = 0 \quad (3.45)$$

⁵In the Hamiltonian language, this constraint C_3 is expressed as the determination of the Lagrange multiplier N .

Now we are in a position to analyse the initial value formulation. A combination of (3.43) and (3.42) shows that the problem is well posed provided N does not diverge. Indeed, suppose that at $u = 0$, we are given $\Delta_i(0)$, $\Delta'_i(0)$, $|v_i(0)\rangle$ and $|v'_i(0)\rangle$ that satisfy the two constraints (3.44) and (3.45) with γ given by (3.39). From these we obtain $\Delta_i(\delta u)$, $|v_i(\delta u)\rangle$. From (3.38) we determine $|v'_i(\delta u)\rangle$ and from eqs. (3.42) and (3.43) we get $\Delta'_i(\delta u)$ provided N does not diverge. Thus, as long as N remains finite, our system of equations are sufficient to solve the system completely once a correct set of initial values satisfying the constraints is given. It is possible that N diverges at a finite u , since the denominator in (3.43) can vanish consistently with the constraints \tilde{C}_1 and \tilde{C}_2 .

To finish the problem, we of course need to transform back to the original time variable t , defined from (3.17) and (3.25) by

$$\frac{dt}{du} = e^{-f(u)} N(u) \tag{3.46}$$

with $N(u)$ determined from (3.43). In order to be able to determine $u(t)$, $N(u)$ should not change sign. Indeed, if $N(u)$ were to change sign then t would not be a monotonic function of u and so we could not invert to find $t(u)$. From (3.43) notice that while the first term in the numerator of $N(u)$ is definitely positive, the remaining two terms do not have a definite sign. Hence generically nothing appears to guarantee that N can never vanish. Below we give an example where, indeed, $N(u)$ changes sign.

4 Examples

To understand whether N can change sign, study instabilities etc, it is instructive to search for exact solutions of the equations of motion E and constraints C_1, C_2, C_3 . We will consider two simple situations: $D = 3$ dimensions, and D -dimensional space-time but vanishing γ .

Before doing so, notice that due to the constraints, phase space (π_{ij}, π'_{ij}) contains the $D(D - 1) - 2$ degrees of freedom necessary to describe a massive spin 2 field, and of those, $(D - 1)(D - 2)/2 + 1$ are constants of motion.

4.1 Solutions in $D = 3$ dimensions

In $D = 3$ the system is integrable. It is simplest to proceed by writing $\pi(u) = \pi_0(u) + \pi_1(u)\sigma_1 + \pi_3(u)\sigma_3$, where σ_i are the Pauli matrices, so that

$$\det \pi = (\pi_0^2 - \pi_1^2 - \pi_3^2). \tag{4.1}$$

Then from \tilde{C}_1 ,

$$\text{tr} \zeta = c = 2 \frac{\pi_0}{\det \pi} \tag{4.2}$$

where c is a constant, and from the conserved matrix γ in (3.26)

$$\gamma_{12} \equiv L = c^2 [\pi'_1 \pi_3 - \pi'_3 \pi_1] \tag{4.3}$$

with $dL/du = 0$. The constraint equation \tilde{C}_2 in (3.29) reads

$$(\det \pi)(\pi_1'^2 + \pi_3'^2 - \pi_0'^2) + \left(\frac{L}{c^2}\right)^2 = m^2(1 - \det \pi). \tag{4.4}$$

One can check that the equation of motion (3.31) is the derivative of this equation.

In order to solve (4.4) we change variables to

$$\pi_0 = \rho \cosh \xi, \quad \pi_1 = \rho \sinh \xi \sin \theta, \quad \pi_3 = \rho \sinh \xi \cos \theta, \quad (4.5)$$

so that $\det \pi = \rho^2 = \left(\frac{2}{c} \cosh \xi\right)^2$ and $L = \theta' \sinh^2(2\xi)$. Then (4.4) reads

$$\xi'^2 + V_{\text{eff}}(\xi) = 0, \quad (4.6)$$

with

$$V_{\text{eff}} = \frac{1}{4} \left[\frac{L^2}{4 \sinh^2 \xi} - m^2 c^2 \left(\frac{c^2}{4 \cosh^2 \xi} - 1 \right) \right]. \quad (4.7)$$

This is the equation of motion for a particle moving in 1 dimension with an effective potential (4.7). Notice that as $\xi \rightarrow \pm\infty$, $V \rightarrow m^2 c^2/4$, and hence the field dynamics is bounded in a region of finite ξ . When $L = 0$, the potential is negative at $\xi = 0$ only for $c \geq 2$. When $L \neq 0$ there is a potential barrier at $\xi = 0$ meaning that ξ cannot change sign during the evolution. In fact it is straightforward to show that $V_{\text{eff}} < 0$ only if

$$L < mc(c - 2). \quad (4.8)$$

In fact the solution can be obtained exactly, and is given by

$$\xi(u) = \text{arcsinh} \left(\left[\frac{E^2}{2B^2} + \left(\frac{E^4}{4B^2} - F^2 \right)^{1/2} \frac{1}{B} \sin(2B(u - u_0)) \right] \right)^{1/2} \quad (4.9)$$

where

$$F = \frac{L}{4}, \quad B = \frac{mc}{2}, \quad E = \sqrt{\frac{m^2 c^4}{16} - F^2 - B^2}, \quad (4.10)$$

and u_0 determines the initial value of ξ . As expected the motion is periodic in terms of the variable u .

However, we must check whether or not $N(u)$ changes sign during the evolution. To do so, we use the third constraint C_3 given in (3.43). From (4.5), the two eigen-values and vectors of ζ (see (3.35)) are given by

$$e^{-\Delta_1} = \frac{c}{2} \frac{e^{-\xi}}{\cosh \xi}, \quad e^{-\Delta_2} = \frac{c}{2} \frac{e^{\xi}}{\cosh \xi}, \quad |v_1\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}$$

so that from its definition in (3.41) it follows that

$$A_1 = L^2 \frac{\cosh 2\xi}{\sinh^3(2\xi)} = -A_2. \quad (4.11)$$

On using (4.7) we finally find from (3.43) that N is given by

$$N = \frac{c}{2} \left(1 - \frac{L^2}{m^2 c^4} \right). \quad (4.12)$$

Thus for L 's satisfying the requirement (4.8), N is positive and constant for all ξ . Finally, in terms of the original time coordinate t ,

$$t = \int \frac{N(u)}{\det \pi} du = \frac{Nc^2}{4} \int \frac{1}{\cosh^2(\xi(u))} du \quad (4.13)$$

where $\xi(u)$ is given in (4.9). This can be integrated exactly, but the answer is not particularly illuminating. Crucially t is well defined and monotonic, and ξ is also periodic in t .

As we now show, however, the situation is very different in $D > 3$ dimensions.

4.2 The diagonal case, $\gamma = 0$, in D dimensions

We now study the diagonal case

$$\zeta = \text{diag}(e^{-\Delta_1}, \dots, e^{-\Delta_{D-2}}, e^{-\Delta}) \quad (4.14)$$

where $\Delta = \Delta_{D-1}$. Thus $\gamma = 0$, and the equations of motion and constraints C_1, C_2, C_3 given in (3.42)–(3.45) can be solved numerically. It is useful to get some analytic understanding of the dynamics by focusing on the simple case in which all but one of the Δ_i are identical,⁶ thus

$$\Delta_a(u) = \delta(u) \quad \text{for} \quad a = 1 \dots D - 2.$$

We have checked that the generic solutions show similar properties to the ones we discuss below.

In that case, the equations of motion and constraints directly determine δ since one can use the constraint \tilde{C}_1 to eliminate Δ :

$$\text{tr}\zeta = (D - 2)e^{-\delta} + e^{-\Delta} = c, \quad (4.15)$$

where c is a constant. Before doing so, notice that the previous condition implies that $e^{-\delta}$ is bound in the range

$$0 \leq e^{-\delta} \leq \frac{c}{D - 2}. \quad (4.16)$$

We then find that the constraint (3.45) reduces to⁷

$$\delta'^2 + V(\delta) = 0 \quad (4.17)$$

where

$$V = -2m^2 e^{-(D-2)\delta} \left(\frac{(c - (D-2)e^{-\delta})^2}{D-2} \right) \frac{[1 - c e^{-(D-2)\delta} + (D-2)e^{-(D-1)\delta}]}{\{c(D-3) - (D-2)(D-1)e^{-\delta}\}}. \quad (4.18)$$

(In $D = 3$ dimensions, this potential agrees with the one of the previous section when we set $L = 0$ and on correctly identifying δ and ξ .) In order to have a perturbative solution for $\delta \ll 1$ we require $V(0) < 0$, which leads to

$$\frac{(D-1)(D-2)}{D-3} > c > D-1. \quad (4.19)$$

Notice also that

$$V(\delta \rightarrow \infty) \rightarrow 0, \quad V(\delta \rightarrow -\infty) \rightarrow \infty \quad (4.20)$$

and furthermore that for $D > 3$, even though the system has no generalised angular momentum (that is $\gamma = 0$ here), the potential diverges at $\delta = \delta_*$ where

$$e^{-\delta_*} = \frac{c(D-3)}{(D-2)(D-1)} < \frac{c}{(D-2)}. \quad (4.21)$$

⁶The easiest particular case is the one in which all the Δ_i for $i = 1, \dots, D-1$ are equal, so that π is proportional to the identity. However, it is straightforward to see that in that case π must be the identity itself. Homogenous and diagonal numerical solutions to bigravity theories were considered in [19].

⁷One can check that the equation of motion (3.31) is the derivative of equation (4.17).

As a result of this potential barrier, the evolution of δ is divided into disconnected sectors depending on the initial conditions. Furthermore, in each sector we need to determine N which, from (3.43), is given by

$$N = \frac{c e^{-(D-2)\delta} [(D-3)e^{(D-2)\delta} - 2c(D-3) + 2(D-2)^2 e^{-\delta}]}{(e^{-\delta} - e^{-\delta_*})^2 (D-2)^2 (D-1)^2}. \quad (4.22)$$

(When $D = 3$ this reduces to $N = c/2$ as it should from (4.12) with $L = 0$.) Notice that N also diverges at δ_* . Whether or not N can change sign is determined by the square brackets in the numerator which we denote by $f(e^{-\delta})$. This function f has a minimum at

$$e^{-\delta_0} = \left(\frac{D-3}{2(D-2)} \right)^{\frac{1}{D-1}} \quad (4.23)$$

where it takes the value

$$f_{\min} = -2(D-3)[c - c_{\text{crit}}], \quad (4.24)$$

with

$$c_{\text{crit}} = \frac{D-1}{2^{1/(D-1)}} \left(\frac{D-2}{D-3} \right)^{\frac{D-2}{D-1}} \quad (4.25)$$

which lies in the range given by (4.19). (For instance, in $D = 4$ dimensions $6 \leq c \leq 3$ with $c_{\text{crit}} = (54)^{1/3} \simeq 3.78$.) Furthermore,

$$f(e^{-\delta_*}) = -2c \frac{D-3}{D-1} \left[1 - \left(\frac{c_{\text{crit}}}{c} \right)^{D-1} \right]. \quad (4.26)$$

Thus for $c < c_{\text{crit}}$ N is always positive, while for $c > c_{\text{crit}}$ N is negative in a range of values of δ around δ_0 which includes δ_* . Figure 1 shows this generic behaviour as well as that of V for $D = 4$.

To conclude, for $c > c_{\text{crit}}$, N necessarily vanishes at a finite time u_* so that the system is badly defined. For $c < c_{\text{crit}}$ there is well defined evolution if δ is initially in the minimum of V near $\delta = 0$. However, if initially $\delta > \delta_*$ then depending on the sign of δ' , either $\delta \rightarrow \delta_*$ in a finite u (and correspondingly infinite time t), or $\delta \rightarrow \infty$ as $u \rightarrow \infty$.

A numerical study of the generic system in $D = 4$ in which all the eigenvalues of ζ are initially different (with $\gamma = 0$, see (4.14)) shows that the behaviour of the system is qualitatively similar. Crucially there are initial conditions for which N changes sign in a finite time.

5 Conclusion

In this paper we have studied massive gravity in its vielbein formulation, considering the mass terms parametrised by the parameters β_0 and β_{D-1} only. In this case, the covariant analysis of [5] showed that a symmetry condition is imposed on the moving frame vielbein components, but does it not lead to an extra scalar constraint. By focusing on time-dependent and spatially translational invariant metrics — which in the context of Fierz-Pauli would correspond to studying plane waves, and in the context of General Relativity to Bianchi I and Kasner solutions — we were able to determine the origin of this extra scalar constraint which is required for the theory to have the correct $D(D-1) - 2$ degrees of freedom needed to describe a massive spin 2 particle.

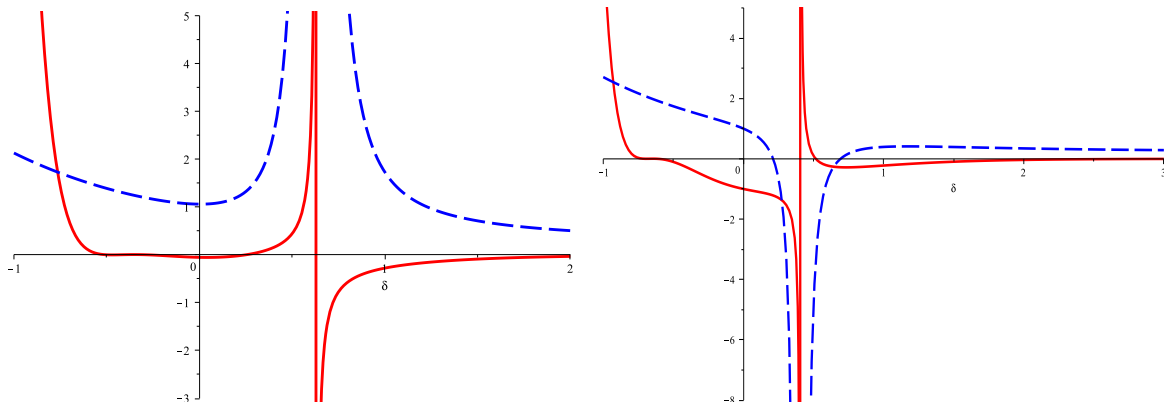


Figure 1. The potential $V(\delta)$ (red, solid lines) and $N(\delta)$ (blue dashed lines) in $D = 4$ dimensions for which $c_{\text{crit}} \simeq 3.78$. LH panel: $c = 3.3 < c_{\text{crit}}$; RH panel $c = 4 > c_{\text{crit}}$.

We carried out the analysis using a convenient ADM-like decomposition of the veilbein, with a lapse function N , a shift vector n_i and a symmetric $(D - 1) \times (D - 1)$ matrix π_{ij} (or equivalently its inverse ζ_{ij}). In section 3.2 we showed that as a result of the Bianchi identities and the constraints coming from the $0i$ -components of the equations of motion, $n_i = 0$. In terms of the variable u , which is linked to the original time coordinate t through (3.46), we also showed how the Bianchi identities give rise to the scalar constraint (which was denoted by C_1). This, once used in the equations of motion, was shown to lead to the new scalar constraint, C_3 , which was missing in the analysis of [5]. As discussed in section 3.4, a crucial step in the determination of C_3 was to put the equations of motion in a form showing their well-posedness, that is with ζ'' explicitly expressed in terms of its lower derivatives. In passing we also showed that of the $D(D - 1) - 2$ degrees of freedom, $(D - 1)(D - 2)/2 + 1$ are constants of motion (coming from the conserved anti-symmetric matrix γ as well as $\text{tr}\zeta$).

A notable difference with the β_1 case studied in [I] is that the equations of motion are well posed provided the lapse function does change sign (nor diverge). While this was guaranteed in the β_1 case (where $N(t) \geq 1 \forall t$), it is not longer manifestly true for the β_{D-1} mass term considered here. A case by case study was needed to prove its validity: in section 4.1, we solved analytically the theory in $D = 3$ and showed that the lapse function is constant and hence the theory is well defined. However, as shown in section 4.2 this no longer holds in $D = 4$ where for initial conditions in a certain region, N can change sign at a finite time t , leading to singular time evolution. In this respect, an important conclusion is that in $D = 4$ dimensions the β_3 mass term can be pathological and should be treated with care.

Finally, in the appendix, we have shown how the translation-invariant solutions presented here and in [I] can be generalised by performing a Lorentz transformation. As a result we obtain “plane wave” solutions, which can also be seen as the generalisation of the pp -waves of general relativity.

In the future it would be interesting to use the intuition gained here to understand how to obtain the constraint C_3 for a general space-time metric, and hence complete the covariant Lagrangian approach of [5] to this β_3 -case.

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A Plane waves

In this appendix we show how, from the time-dependent solutions presented above, we can obtain space- and time-dependent solutions. These are the generalisation of the plane waves of Fierz-Pauli theory, and can also be seen as the generalisation of the pp -waves of general relativity.

As we have taken $f^A = dx^A$ from the start, the theory is globally Lorenz invariant and hence it is sufficient to perform a boost on the time-dependent solutions, thus mapping the rest-frame D -vector $p_0^\mu = (m, \vec{0})$ to a general momentum D -vector $p^\mu = \Lambda^\mu{}_\nu p_0^\nu$ with

$$p^2 = -m^2. \quad (\text{A.1})$$

Recall that for a general D -vector v , we define the longitudinal and transverse components by

$$v_L = -\frac{v \cdot p}{m^2} p, \quad v_T = v - v_L \quad (\text{A.2})$$

with $v_T \cdot p = 0$, and similarly for a general D -tensor $t^{\mu\nu}$, its transverse and longitudinal parts are given by

$$t_{LL}^{\mu\nu} = \frac{p^\mu p^\nu}{m^4} p_\alpha p_\beta t^{\alpha\beta}, \quad t_{LT}^{\mu\nu} = -\frac{p^\mu p_\alpha}{m^2} \Delta^\nu{}_\beta t^{\alpha\beta}, \quad t_{TT}^{\mu\nu} = \Delta^\mu{}_\alpha \Delta^\nu{}_\beta t^{\alpha\beta}, \quad (\text{A.3})$$

where

$$\Delta_{\mu\nu} = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \quad (\text{A.4})$$

and $t = t_{TT} + t_{TL} + t_{LT} + t_{LL}$.

Define the D polarisation vectors $\epsilon^{(A)}(p)_\mu$, $A = 0, \dots, D-1$ by

$$\epsilon^{(A)}(\Lambda p_0) = \Lambda \epsilon^{(A)}(p_0) \quad (\text{A.5})$$

where

$$\epsilon^{(\nu)}(p_0)_\mu = \delta^\nu{}_\mu. \quad (\text{A.6})$$

These form a basis with

$$\epsilon^{(A)} \cdot \epsilon^{(B)} = \eta^{AB}, \quad \epsilon^{(0)} = \frac{p}{m}. \quad (\text{A.7})$$

The plane wave solution for the moving frame $\theta^A = \theta^A{}_\mu dx^\mu$ is then explicitly given in terms of the translation-invariant solutions by

$$\theta_{LL}^{\mu\nu}(x) = N(-x \cdot \epsilon^{(0)}) \epsilon_{(0)}^\mu \epsilon_{(0)}^\nu, \quad \theta_{TL}^{\mu\nu} = 0, \quad \theta_{TT}^{\mu\nu} = \pi_{ij}(-x \cdot \epsilon^{(0)}) \epsilon^{(i)\mu} \epsilon^{(j)\nu}. \quad (\text{A.8})$$

In particular the full solution for the metric $g_{\mu\nu}$ is given by $g_{\mu\nu}(x) = \eta_{AB} \theta^A{}_\mu(x) \theta^B{}_\nu(x)$ where $\theta^{AB} = \theta_{LL}^{AB} + \theta_{TT}^{AB}$. Thus, to conclude

$$g_{\mu\nu}(x) = -N^2(-x \cdot \epsilon^{(0)}) \frac{p_\mu p_\nu}{m^2} + (\pi^2)_{ij} \epsilon_\mu^{(i)} \epsilon_\nu^{(j)} \quad (\text{A.9})$$

where N and π_{ij} can be replaced by any of the time-dependent solutions determined above.

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