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**Abstract**

We consider a two-phase Darcy flow in a fractured porous medium consisting in a matrix flow coupled with a tangential flow in the fractures, described as a network of planar surfaces. This flow model is also coupled with the mechanical deformation of the matrix assuming that the fractures are open and filled by the fluids, as well as small deformations and a linear elastic constitutive law. The model is discretized using the gradient discretization method [20], which covers a large class of conforming and non conforming schemes. This framework allows for a generic convergence analysis of the coupled model using a combination of discrete functional tools. Here, we describe the model together with its numerical discretization, and we prove a convergence result. This is, to our knowledge, the first convergence result for this type of models taking into account two-phase flows and the non-linear poromechanical coupling. Previous related works consider a linear approximation obtained for a single phase flow by freezing the fracture conductivity [29, 30]. Numerical tests employing the Two-Point Flux Approximation (TPFA) finite volume scheme for the flows and $\mathbb{P}_2$ finite elements for the mechanical deformation are also provided to illustrate the behavior of the solution to the model.

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## 1 Introduction

Many real-life applications in geosciences involve processes like multi-phase flow and hydromechanical coupling in heterogeneous porous media. Such mathematical models are coupled systems of partial differential equations, including non-linear and degenerate parabolic ones. Besides the inherent difficulties posed by such equations, further complexities stem from the heterogeneity of the medium and the presence of discontinuities like fractures. This has a strong impact on the complexity of the models, challenging their mathematical and numerical analysis and the development of efficient simulation tools.

This work focuses on the so called hybrid-dimensional matrix fracture models obtained by averaging both the unknowns and the equations in the fracture width and by imposing appropriate transmission conditions at both sides of the matrix fracture interfaces. Given the high geometrical complexity of real-life fracture networks, the main advantages of these hybrid-dimensional compared with full-dimensional models are to facilitate the mesh generation and the discretization of

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the model, and to reduce the computational cost of the resulting schemes. This type of hybrid-dimensional models has been the object of intensive researches over the last 15 years due to the ubiquity of fractures in geology and their considerable impact on the flow and transport of mass and energy in porous media, and on the mechanical behavior of the rocks. For the derivation and analysis of such models, let us refer to [4, 25, 36, 39, 6, 11, 13, 43] for single-phase Darcy flows, [10, 45, 42, 32, 12, 21, 14, 2] for two-phase Darcy flows, and [41, 37, 34, 29, 30, 26, 35, 27, 48] for poroelastic models.

In this work we consider the two-phase Darcy flow in a network of fractures represented as $(d-1)$-dimensional planar surfaces coupled with the surrounding $d$-dimensional matrix. The fractures are assumed to be open and fully filled by the phases. Both phase pressures are assumed continuous at matrix fracture interfaces. This is a classical assumption for open fractures given the low pressure drop in the width of the fractures [10, 45, 42, 12]. For single-phase flows, the Poiseuille law is classically used to model the tangential velocity along the fractures, leading to a conductivity equal to $\frac{d_f^3}{12}$, where $d_f$ is the fracture aperture [29, 30]. Following [41], the extension to two-phase flow is based on the generalized Darcy laws using relative permeabilities combined with a capillary pressure law relating both phase pressures and accounting for surface tension effects in the fractures. This hybrid-dimensional two-phase Darcy flow model is coupled with the matrix mechanical deformation assuming a linear poroelastic behavior [41, 37, 34]. The extension of the single-phase poromechanical coupling [29, 30, 26, 35, 48] to two-phase Darcy flows is based on the so-called equivalent pressure used both in the matrix for the effective stress and at both sides of the fractures as boundary condition for the mechanics. It is defined as a convex combination of the phase pressures with many different choices proposed in the litterature [44]. Our choice of the equivalent pressure follows the pioneer monograph by Coussy [15] using the capillary energy which, as already noticed in [37, 34], plays a key role to obtain energy estimates for the coupled system. From the open fracture assumption, the fracture mechanical behavior reduces to the continuity of the normal stresses at both sides of the fracture matching with the fracture equivalent pressure times the unit normal vector.

In this work, the hybrid-dimensional coupled model is discretized using the gradient discretization method [20]. This framework is based on abstract vector spaces of discrete unknowns combined with reconstruction operators. The gradient scheme is then obtained by substitution of the continuous operators by their discrete counterparts in the weak formulation of the coupled model. The main asset of this framework is to allow a generic convergence analysis based on general properties of the reconstruction operators that hold for a large class of conforming and non conforming discretizations. The two main ingredients to discretize the coupled model are the discretizations of the hybrid-dimensional two-phase Darcy flow and the discretization of the mechanics. Let us briefly mention, in both cases, a few families of discretizations typically satisfying the gradient discretization properties. For the discretization of the Darcy flow, the gradient discretization framework typically covers the case of cell-centered finite volume schemes with Two-Point Flux Approximation on strongly admissible meshes [36, 6, 2], or some symmetric Multi-Point Flux Approximations [47, 46, 3] on tetrahedral or hexahedral meshes. It also accounts for the families of Mixed Hybrid Mimetic and Mixed or Mixed Hybrid Finite Element discretizations such as in [4, 39, 11, 13, 7]. The case of vertex-based discretizations such as Control Volume Finite Element approaches (i.e. conforming finite element with mass lumping) [10, 45, 42] or the Vertex Approximate Gradient scheme [11, 13, 12, 21, 14] is also accounted for. For the discretization of the elastic mechanical model, the gradient discretization framework covers conforming finite element methods such as in [29], the Crouzeix-Raviart discretization [31, 18], the Hybrid High Order discretization [17], and the Virtual Element Method [8].

The main objective of this work is to introduce the gradient discretization for the hybrid-dimensional coupled model and to prove the convergence of the discrete solution to a weak solution of the model. Without taking account of the poromechanical coupling, such a convergence result has been obtained in [6, 4, 39, 11, 13] for hybrid-dimensional single-phase Darcy flow models, and in

[12, 21] for hybrid-dimensional two-phase Darcy flow models. The well-posedness and convergence analysis of single-phase poromechanical models is studied in [29, 30]. Nevertheless those analyses consider a linear approximation of the coupled model obtained by freezing the fracture conductivity $\frac{d_f^3}{12}$, hence not taking into account the non-linear coupling between the fracture aperture and the Darcy flow. In this work, we are able to prove the convergence to a weak solution for the non-linear coupled model and two-phase flows based on the following main assumptions. It is assumed that the matrix porosity remains bounded from below and above by strictly positive constants, that the fracture aperture remains larger than a fixed aperture vanishing only at the tips, and that the mobility functions are bounded from below by strictly positive constants. The assumptions on the porosity and fracture aperture cannot be avoided since the continuous model does not ensure these properties, which are needed to ensure its well-posedness. The assumption on the mobilities are classical to carry out the stability and convergence analysis of two-phase Darcy flows with heterogeneous rock types (see [24, 12, 21]). This is, to our knowledge, the first convergence result for this type of hybrid-dimensional model taking into account the non-linear poromechanical coupling.

The rest of the article is organized as follows. Section 2 introduces the continuous hybrid-dimensional coupled model. Section 3 describes the gradient discretization method for the coupled model including the definition of the reconstruction operators, the discrete variational formulation and the properties of the gradient discretization needed for the subsequent convergence analysis. Section 4 proceeds with the convergence analysis. The a priori estimates are established in Subsection 4.1, the compactness properties in Subsection 4.2 and the convergence to a weak solution is proved in Subsection 4.3. In Section 5, numerical experiments based on the Two-Point Flux Approximation finite volume scheme for the flows and second-order finite elements for the mechanical deformation are carried out for a cross-shaped fracture network in a two dimensional porous medium, and illustrate the numerical convergence of the solution. Appendices A.1 and A.2 state some technical results used in the convergence analysis.

## 2 Continuous model

We consider a bounded polytopal domain $\Omega$ of $\mathbb{R}^d$, $d \in \{2, 3\}$, partitioned into a fracture domain $\Gamma$ and a matrix domain $\Omega \backslash \overline{\Gamma}$. The network of fractures is defined by

$$\overline{\Gamma} = \bigcup_{i \in I} \overline{\Gamma}_i$$

where each fracture $\Gamma_i \subset \Omega$, $i \in I$ is a planar polygonal simply connected open domain. Without restriction of generality, we will assume that the fractures may intersect exclusively at their boundaries (see Figure 1), that is for any $i, j \in I, i \neq j$ one has $\Gamma_i \cap \Gamma_j = \varnothing$, but not necessarily $\overline{\Gamma}_i \cap \overline{\Gamma}_j = \varnothing$.

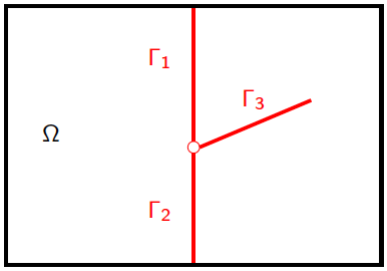



Figure 1: Example of a 2D domain $\Omega$ with 3 intersecting fractures $\Gamma_i$, $i = 1, 2, 3$.

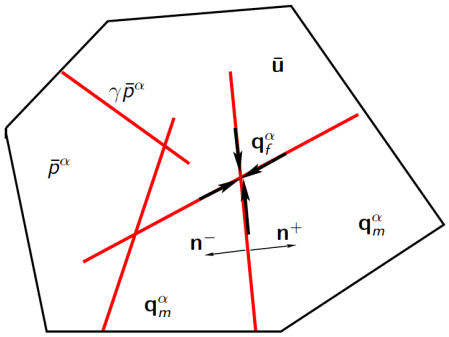



Figure 2: Example of a 2D domain $\Omega$ with its fracture network $\Gamma$, the unit normal vectors $\mathbf{n}^\pm$ to $\Gamma$, the phase pressures $\bar{p}^\alpha$ in the matrix and $\gamma\bar{p}^\alpha$ in the fracture network, the displacement vector field $\bar{\mathbf{u}}$, the matrix Darcy velocities $\mathbf{q}_m^\alpha$ and the fracture tangential Darcy velocities $\mathbf{q}_f^\alpha$ integrated along the fracture width.

The two sides of a given fracture of $\Gamma$ are denoted by $\pm$ in the matrix domain, with unit normal vectors $\mathbf{n}^\pm$ oriented outward of the sides $\pm$. We denote by $\gamma$ the trace operator on $\Gamma$ for functions in $H^1(\Omega)$, by $\gamma_{\partial\Omega}$ the trace operator for the same functions on $\partial\Omega$, and by $[\![\cdot]\!]$ the normal trace jump operator on $\Gamma$ for functions in $H_{\rm div}(\Omega\backslash\overline{\Gamma})$, defined by

$$[\![\bar{\mathbf{u}}]\!] = \bar{\mathbf{u}}^+ \cdot \mathbf{n}^+ + \bar{\mathbf{u}}^- \cdot \mathbf{n}^- \ \text{ for all } \ \bar{\mathbf{u}} \in H_{\rm div}(\Omega\backslash\overline{\Gamma}).$$

We denote by $\nabla_\tau$ the tangential gradient and by $\mathrm{div}_\tau$ the tangential divergence on the fracture network $\Gamma$. The symmetric gradient operator $\mathfrak{e}$ is defined such that $\mathfrak{e}(\bar{\mathbf{v}}) = \frac{1}{2}(\nabla\bar{\mathbf{v}} + {}^t(\nabla\bar{\mathbf{v}}))$ for a given vector field $\bar{\mathbf{v}} \in H^1(\Omega\backslash\overline{\Gamma})^d$.

Let us fix a continuous function $d_0 : \Gamma \to (0, +\infty)$ with zero limits at $\partial\Gamma\backslash(\partial\Gamma \cap \partial\Omega)$ (i.e. the tips of $\Gamma$) and strictly positive limits at $\partial\Gamma \cap \partial\Omega$. The fracture aperture, denoted by $\bar{d}_f$ and such that $\bar{d}_f = -[\![\bar{\mathbf{u}}]\!]$ for a displacement field $\bar{\mathbf{u}} \in H^1(\Omega\backslash\overline{\Gamma})^d$ will be assumed to satisfy the following open fracture condition

$$\bar{d}_f(\mathbf{x}) \geqslant d_0(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Gamma.$$

Let us introduce some relevant function spaces:

$$\mathbf{U}_0 = \{\bar{\mathbf{v}} \in (H^1(\Omega\backslash\overline{\Gamma}))^d \mid \gamma_{\partial\Omega}\bar{\mathbf{v}} = 0\} \tag{1}$$

for the displacement vector, and

$$V_0 = \{\bar{v} \in H^1_0(\Omega) \mid \gamma\bar{v} \in H^1_{d_0}(\Gamma)\} \tag{2}$$

for each phase pressure, where the space $H^1_{d_0}(\Gamma)$ is made of functions $v_\Gamma$ in $L^2(\Gamma)$, such that $d_0^{3/2}\nabla_\tau v_\Gamma$ is in $L^2(\Gamma)^{d-1}$, and whose traces are continuous at fracture intersections $\partial\Gamma_i \cap \partial\Gamma_j$, $(i,j) \in I \times I$ $(i \neq j)$ and vanish on the boundary $\partial\Gamma \cap \partial\Omega$.

The matrix and fracture rock types are denoted by the indices $\mathrm{rt} = m$ and $\mathrm{rt} = f$, respectively, and the non-wetting and wetting phases by the superscripts $\alpha = \mathrm{nw}$ and $\alpha = \mathrm{w}$, respectively.

The PDEs model reads: find the phase pressures $\bar{p}^\alpha$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, and the displacement vector field $\bar{\mathbf{u}}$, such that $\bar{p}_c = \bar{p}^{\mathrm{nw}} - \bar{p}^{\mathrm{w}}$ and, for $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$,

$$\begin{cases}
\partial_t\left(\bar{\phi}_m S_m^\alpha(\bar{p}_c)\right) + \mathrm{div}\left(\mathbf{q}_m^\alpha\right) = h_m^\alpha & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma},\\
\mathbf{q}_m^\alpha = -\eta_m^\alpha(S_m^\alpha(\bar{p}_c))\mathbb{K}_m\nabla\bar{p}^\alpha & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma},\\
\partial_t\left(\bar{d}_f S_f^\alpha(\gamma\bar{p}_c)\right) + \mathrm{div}_\tau(\mathbf{q}_f^\alpha) - [\![\mathbf{q}_m^\alpha]\!] = h_f^\alpha & \text{on } (0,T)\times\Gamma,\\
\mathbf{q}_f^\alpha = -\eta_f^\alpha(S_f^\alpha(\gamma\bar{p}_c))(\frac{1}{12}\bar{d}_f^3)\nabla_\tau\gamma\bar{p}^\alpha & \text{on } (0,T)\times\Gamma,\\
-\mathrm{div}\Big(\boldsymbol{\sigma}(\bar{\mathbf{u}}) - b\,\bar{p}_m^E\mathbb{I}\Big) = \mathbf{f} & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}\\
\boldsymbol{\sigma}(\bar{\mathbf{u}}) = 2\mu\,\mathfrak{e}(\bar{\mathbf{u}}) + \lambda\,\mathrm{div}(\bar{\mathbf{u}})\,\mathbb{I} & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma},
\end{cases} \tag{3}$$

with

$$\begin{cases} \partial_t \bar{\phi}_m = b\ \mathrm{div}\partial_t \bar{\mathbf{u}} + \dfrac{1}{M}\partial_t \bar{p}_m^E & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ (\varpi(\bar{\mathbf{u}}) - b\ \bar{p}_m^E \mathbb{I})\mathbf{n}^{\pm} = -\bar{p}_f^E \mathbf{n}^{\pm} & \text{on } (0,T)\times\Gamma, \\ \bar{d}_f = -[\![\bar{\mathbf{u}}]\!] & \text{on } (0,T)\times\Gamma, \end{cases} \tag{4}$$

and the initial conditions

$$\bar{p}^\alpha|_{t=0} = \bar{p}_0^\alpha, \quad \bar{\phi}_m|_{t=0} = \bar{\phi}_m^0.$$

Here, the equivalent pressures $\bar{p}_m^E$ and $\bar{p}_f^E$ are defined, following [15], by

$$\bar{p}_m^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} \bar{p}^\alpha\ S_m^\alpha(\bar{p}_c) - U_m(\bar{p}_c), \quad \bar{p}_f^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} \gamma\bar{p}^\alpha\ S_f^\alpha(\gamma\bar{p}_c) - U_f(\gamma\bar{p}_c),$$

where

$$U_{\mathrm{rt}}(\bar{p}_c) = \int_0^{\bar{p}_c} q\,(S_{\mathrm{rt}}^{\mathrm{nw}})'(q)\mathrm{d}q \tag{5}$$

is the capillary energy density function for each rock type $\mathrm{rt} \in \{m, f\}$. As already noticed in [37, 34], this is a key choice to obtain the energy estimates that are the starting point for the convergence analysis.

We make the following main assumptions on the data:

(H1) For each phase $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$ and rock type $\mathrm{rt} \in \{m, f\}$, the mobility function $\eta_{\mathrm{rt}}^\alpha$ is continuous, non-decreasing, and there exist $0 < \eta_{\mathrm{rt,min}}^\alpha \leqslant \eta_{\mathrm{rt,max}}^\alpha < +\infty$ such that $\eta_{\mathrm{rt,min}}^\alpha \leqslant \eta_{\mathrm{rt}}^\alpha(s) \leqslant \eta_{\mathrm{rt,max}}^\alpha$ for all $s \in [0,1]$.

(H2) For each rock type $\mathrm{rt} \in \{m, f\}$, the non-wetting phase saturation function $S_{\mathrm{rt}}^{\mathrm{nw}}$ is a non-decreasing Lipschitz continuous function with values in $[0,1]$, and $S_{\mathrm{rt}}^{\mathrm{w}} = 1 - S_{\mathrm{rt}}^{\mathrm{nw}}$.

(H3) $b \in [0,1]$ is the Biot coefficient, $M > 0$ is the Biot modulus, and $\lambda > 0$, $\mu > 0$ are the Lamé coefficients. These coefficients are assumed to be constant for simplicity.

(H4) The initial matrix porosity satisfies $\bar{\phi}_m^0 \in L^\infty(\Omega)$ and there exist $0 < \phi_{m,\mathrm{min}}^0 \leqslant \phi_{m,\mathrm{max}}^0 < 1$ such that $\phi_{m,\mathrm{min}}^0 \leqslant \bar{\phi}_m^0(\mathbf{x}) \leqslant \phi_{m,\mathrm{max}}^0$ for a.e. $\mathbf{x} \in \Omega$.

(H5) The initial fracture aperture $\bar{d}_f^0$ satisfies $\bar{d}_f^0(\mathbf{x}) \geqslant d_0(\mathbf{x})$ for a.e. $\mathbf{x} \in \Gamma$.

(H6) The initial pressures are such that $\bar{p}_0^\alpha \in V_0 \cap L^\infty(\Omega)$ and $\gamma\bar{p}_0^\alpha \in L^\infty(\Gamma)$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$.

(H7) The source terms satisfy $\mathbf{f} \in L^2(\Omega)^d$, $h_m^\alpha \in L^2((0,T)\times\Omega)$, and $h_f^\alpha \in L^2((0,T)\times\Gamma)$.

(H8) The matrix permeability tensor $\mathbb{K}_m$ is symmetric and uniformly elliptic on $\Omega$.

**Definition 2.1** (Weak solution of the model)**.** A *weak solution* of the model is given by $\bar{p}^\alpha \in L^2(0,T;V_0)$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, and $\bar{\mathbf{u}} \in L^\infty(0,T;\mathbf{U}_0)$, such that, for any $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, $\bar{d}_f^{3/2}\nabla_\tau\gamma\bar{p}^\alpha \in L^2((0,T)\times\Gamma))^{d-1}$ and, for all $\bar{\varphi}^\alpha \in C_c^\infty([0,T)\times\Omega)$ and all smooth functions $\bar{\mathbf{v}} : [0,T]\times(\Omega\backslash\overline{\Gamma}) \to \mathbb{R}^d$ vanishing on $\partial\Omega$ and admitting finite limits on each side of $\Gamma$,

$$\begin{aligned} &\int_0^T \int_\Omega \Big(-\bar{\phi}_m S_m^\alpha(\bar{p}_c)\partial_t\bar{\varphi}^\alpha + \eta_m^\alpha(S_m^\alpha(\bar{p}_c))\mathbb{K}_m\nabla\bar{p}^\alpha\cdot\nabla\bar{\varphi}^\alpha\Big)\mathrm{d}\mathbf{x}\mathrm{d}t \\ &+ \int_0^T \int_\Gamma \Big(-\bar{d}_f S_f^\alpha(\gamma\bar{p}_c)\partial_t\gamma\bar{\varphi}^\alpha + \eta_f^\alpha(S_f^\alpha(\gamma\bar{p}_c))\frac{\bar{d}_f^3}{12}\nabla_\tau\gamma\bar{p}^\alpha\cdot\nabla_\tau\gamma\bar{\varphi}^\alpha\Big)\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \\ &- \int_\Omega \bar{\phi}_m^0 S_m^\alpha(\bar{p}_c^0)\bar{\varphi}^\alpha(0,\cdot)\mathrm{d}\mathbf{x} - \int_\Gamma \bar{d}_f^0 S_f^\alpha(\gamma\bar{p}_c^0)\gamma\bar{\varphi}^\alpha(0,\cdot)\mathrm{d}\sigma(\mathbf{x}) \\ &= \int_0^T \int_\Omega h_m^\alpha\bar{\varphi}^\alpha\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma h_f^\alpha\gamma\bar{\varphi}^\alpha\ \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t, \end{aligned} \tag{6}$$

$$\int_0^T \int_\Omega \Big( \mathfrak{o}(\bar{\mathbf{u}}) : \mathfrak{e}(\bar{\mathbf{v}}) - b\, \bar{p}_m^E \mathrm{div}(\bar{\mathbf{v}}) \Big) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma \bar{p}_f^E \, [\![\bar{\mathbf{v}}]\!] \, \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}} \, \mathrm{d}\mathbf{x}\mathrm{d}t, \tag{7}$$

with $\bar{p}_c = \bar{p}^{\mathrm{nw}} - \bar{p}^{\mathrm{w}}$, $\bar{d}_f = -[\![\bar{\mathbf{u}}]\!]$, $\bar{\phi}_m - \bar{\phi}_m^0 = b\ \mathrm{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}^0) + \dfrac{1}{M}(\bar{p}_m^E - \bar{p}_m^{E,0})$, $\bar{d}_f^0 = -[\![\bar{\mathbf{u}}^0]\!]$, where $\bar{\mathbf{u}}^0$ is the solution of (7) without the time integral and using the initial equivalent pressures $\bar{p}_m^{E,0}$ and $\bar{p}_f^{E,0}$ obtained from the initial pressures $\bar{p}_0^\alpha$ and $\gamma \bar{p}_0^\alpha$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$.

*Remark* 2.2 (Regularity of the fracture aperture). Notice that, by the Sobolev–trace embeddings [1, Theorem 4.12], $\bar{\mathbf{u}} \in L^\infty(0,T;\mathbf{U}_0)$ implies that $\bar{d}_f = -[\![\bar{\mathbf{u}}]\!] \in L^\infty(0,T;L^4(\Gamma))$. All the integrals above are thus well-defined.

# 3 The gradient discretization method

The gradient discretization (GD) for the Darcy continuous pressure model, introduced in [11], is defined by a finite-dimensional vector space of discrete unknowns $X_{\mathcal{D}_p}^0$ and

- two discrete gradient linear operators on the matrix and fracture domains
$$\nabla_{\mathcal{D}_p}^m : X_{\mathcal{D}_p}^0 \to L^\infty(\Omega)^d, \qquad \nabla_{\mathcal{D}_p}^f : X_{\mathcal{D}_p}^0 \to L^\infty(\Gamma)^{d-1};$$
- two function reconstruction linear operators on the matrix and fracture domains
$$\Pi_{\mathcal{D}_p}^m : X_{\mathcal{D}_p}^0 \to L^\infty(\Omega), \qquad \Pi_{\mathcal{D}_p}^f : X_{\mathcal{D}_p}^0 \to L^\infty(\Gamma),$$
which are *piecewise constant* [20, Definition 2.12].

A consequence of the piecewise-constant property is the following: there is a basis $(\mathbf{e}_i)_{i\in I}$ of $X_{\mathcal{D}_p}^0$ such that, if $v = \sum_{i\in I} v_i \mathbf{e}_i$ and if, for a mapping $g : \mathbb{R} \to \mathbb{R}$, we define $g(v) = \sum_{i\in I} g(v_i)\mathbf{e}_i \in X_{\mathcal{D}_p}^0$ by applying $g$ component-wise, then $\Pi_{\mathcal{D}_p}^{\mathrm{rt}} g(v) = g(\Pi_{\mathcal{D}_p}^{\mathrm{rt}} v)$ for $\mathrm{rt} \in \{m, f\}$. Note that the basis $(\mathbf{e}_i)_{i\in I}$ is usually canonical and chosen in the design of $X_{\mathcal{D}_p}^0$. The vector space $X_{\mathcal{D}_p}^0$ is endowed with

$$\|v\|_{\mathcal{D}_p} \coloneqq \|\nabla_{\mathcal{D}_p}^m v\|_{L^2(\Omega)} + \|d_0^{3/2} \nabla_{\mathcal{D}_p}^f v\|_{L^2(\Gamma)},$$

assumed to define a norm on $X_{\mathcal{D}_p}^0$.

The gradient discretization for the mechanics is defined by a finite-dimensional vector space of discrete unknowns $X_{\mathcal{D}_\mathbf{u}}^0$ and

- a symmetric gradient linear operator $\mathfrak{e}_{\mathcal{D}_\mathbf{u}} : X_{\mathcal{D}_\mathbf{u}}^0 \to L^2(\Omega, \mathcal{S}_d(\mathbb{R}))$,
- a displacement function reconstruction linear operator $\Pi_{\mathcal{D}_\mathbf{u}} : X_{\mathcal{D}_\mathbf{u}}^0 \to L^2(\Omega)^d$,
- a normal jump function reconstruction linear operator $[\![\cdot]\!]_{\mathcal{D}_\mathbf{u}} : X_{\mathcal{D}_\mathbf{u}}^0 \to L^4(\Gamma)$,

where $\mathcal{S}_d(\mathbb{R})$ is the vector space of real symmetric matrices of size $d$. Let us define the divergence operator $\mathrm{div}_{\mathcal{D}_\mathbf{u}}(\cdot) = \mathrm{Trace}(\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\cdot))$, the stress tensor operator

$$\mathfrak{o}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) = 2\mu \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) + \lambda\, \mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) \mathbb{I},$$

and the fracture width $d_{f,\mathcal{D}_\mathbf{u}} = -[\![\mathbf{u}]\!]_{\mathcal{D}_\mathbf{u}}$. It is assumed that

$$\|\mathbf{v}\|_{\mathcal{D}_\mathbf{u}} \coloneqq \|\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v})\|_{L^2(\Omega, \mathcal{S}_d(\mathbb{R}))} \tag{8}$$

is a norm on $X^0_{\mathcal{D}_\mathbf{u}}$.

A spatial GD can be extended into a space-time GD by complementing it with

- a discretization $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$;
- interpolators $I_{\mathcal{D}_p} : V_0 \to X^0_{\mathcal{D}_p}$ and $I^m_{\mathcal{D}_p} : L^2(\Omega) \to X^0_{\mathcal{D}_p}$ of initial conditions.

For $n \in \{0, \ldots, N\}$, we denote by $\delta t^{n+\frac{1}{2}} = t_{n+1} - t_n$ the time steps, and by $\Delta t = \max_{n=0,\ldots,N} \delta t^{n+\frac{1}{2}}$ the maximum time step.

The spatial operators are extended into space-time operators as follows. Let $\chi$ represent either $p$ or $\mathbf{u}$. If $w = (w_n)_{n=0}^N \in (X^0_{\mathcal{D}_\chi})^{N+1}$, and $\Psi_{\mathcal{D}_\chi}$ is a spatial GDM operator, its space-time extension is defined by

$$\Psi_{\mathcal{D}_\chi} w(0, \cdot) = \Psi_{\mathcal{D}_\chi} w_0 \text{ and, } \forall n \in \{0, \ldots, N-1\}\,,\ \forall t \in (t_n, t_{n+1}],\ \ \Psi_{\mathcal{D}_\chi} w(t, \cdot) = \Psi_{\mathcal{D}_\chi} w_{n+1}.$$

For convenience, the same notation is kept for the spatial and space-time operators. Moreover, we define the discrete time derivative as follows: for $f : [0, T] \to L^1(\Omega)$ piecewise constant on the time discretization, with $f_n = f_{|(t_{n-1}, t_n]}$ and $f_0 = f(0)$, we set $\delta_t f(t) = \frac{f_{n+1} - f_n}{\delta t^{n+\frac{1}{2}}}$ for all $t \in (t_n, t_{n+1}]$, $n \in \{0, \ldots, N-1\}$.

Notice that the space of piecewise constant $X^0_{\mathcal{D}_\chi}$-valued functions $f$ on the time discretization together with the initial value $f_0 = f(0)$ can be identified with $(X^0_{\mathcal{D}_\chi})^{N+1}$. The same definition of discrete derivative can thus be given for an element $w \in (X^0_{\mathcal{D}_\chi})^{N+1}$. Namely, $\delta_t w \in (X^0_{\mathcal{D}_\chi})^N$ is defined by setting, for any $n \in \{0, \ldots, N-1\}$ and $t \in (t_n, t_{n+1}]$, $\delta_t w(t) = (\delta_t w)_{n+1} \coloneqq \frac{w_{n+1} - w_n}{\delta t^{n+\frac{1}{2}}}$. If $\Psi_{\mathcal{D}_\chi}(t, \cdot)$ is a space-time GDM operator, by linearity the following commutativity property holds: $\Psi_{\mathcal{D}_\chi} \delta_t w(t, \cdot) = \delta_t(\Psi_{\mathcal{D}_\chi} w(t, \cdot))$.

The gradient scheme for (3) consists in writing the weak formulation (6)–(7) with continuous spaces and operators substituted by their discrete counterparts, after a formal integration by part: find $p^\alpha \in (X^0_{\mathcal{D}_p})^{N+1}$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, and $\mathbf{u} \in (X^0_{\mathcal{D}_\mathbf{u}})^{N+1}$, such that for all $\varphi^\alpha \in (X^0_{\mathcal{D}_p})^{N+1}$, $\mathbf{v} \in (X^0_{\mathcal{D}_\mathbf{u}})^{N+1}$ and $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$,

$$\begin{aligned}
&\int_0^T \int_\Omega \Big( \delta_t \Big( \phi_\mathcal{D} \Pi^m_{\mathcal{D}_p} s^\alpha_m \Big) \Pi^m_{\mathcal{D}_p} \varphi^\alpha + \eta^\alpha_m (\Pi^m_{\mathcal{D}_p} s^\alpha_m) \mathbb{K}_m \nabla^m_{\mathcal{D}_p} p^\alpha \cdot \nabla^m_{\mathcal{D}_p} \varphi^\alpha \Big) \mathrm{d}\mathbf{x}\mathrm{d}t \\
&+ \int_0^T \int_\Gamma \delta_t \Big( d_{f,\mathcal{D}_\mathbf{u}} \Pi^f_{\mathcal{D}_p} s^\alpha_f \Big) \Pi^f_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \\
&+ \int_0^T \int_\Gamma \eta^\alpha_f (\Pi^f_{\mathcal{D}_p} s^\alpha_f) \frac{d^3_{f,\mathcal{D}_\mathbf{u}}}{12} \nabla^f_{\mathcal{D}_p} p^\alpha \cdot \nabla^f_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \\
&= \int_0^T \int_\Omega h^\alpha_m \Pi^m_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma h^\alpha_f \Pi^f_{\mathcal{D}_p} \varphi^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,
\end{aligned} \tag{9a}$$

$$\begin{aligned}
&\int_0^T \int_\Omega \Big( \mathfrak{o}_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) - b\ \Pi^m_{\mathcal{D}_p} p^E_m\ \mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) \Big) \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\qquad + \int_0^T \int_\Gamma \Pi^f_{\mathcal{D}_p} p^E_f\ [\![\mathbf{v}]\!]_{\mathcal{D}_\mathbf{u}} \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f} \cdot \Pi_{\mathcal{D}_\mathbf{u}} \mathbf{v}\ \mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned} \tag{9b}$$

with the closure equations

$$\begin{cases} p_c = p^{\mathrm{nw}} - p^{\mathrm{w}}, \quad s_m^\alpha = S_m^\alpha(p_c), \quad s_f^\alpha = S_f^\alpha(p_c), \\ p_m^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} p^\alpha s_m^\alpha - U_m(p_c), \quad p_f^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} p^\alpha s_f^\alpha - U_f(p_c), \\ \phi_{\mathcal{D}} - \Pi_{\mathcal{D}_p}^m \phi_m^0 = b\,\mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{u}-\mathbf{u}^0) + \frac{1}{M}\Pi_{\mathcal{D}_p}^m(p_m^E - p_m^{E,0}), \\ d_{f,\mathcal{D}_\mathbf{u}} = -[\![\mathbf{u}]\!]_{\mathcal{D}_\mathbf{u}}, \\ \varpi_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) = 2\mu \mathfrak{C}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) + \lambda\,\mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v})\mathbb{I}. \end{cases} \tag{9c}$$

The initial conditions are given by $p_0^\alpha = I_{\mathcal{D}_p}\bar{p}_0^\alpha$ ($\alpha \in \{\mathrm{nw},\mathrm{w}\}$), $\phi_m^0 = I_{\mathcal{D}_p}^m \bar{\phi}^0$, and the initial displacement $\mathbf{u}^0$ is the solution of (9b) with the equivalent pressures obtained from the initial pressures $(p_0^\alpha)_{\alpha\in\{\mathrm{nw},\mathrm{w}\}}$.

## 3.1 Properties of gradient discretizations

Let $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$ and $(\mathcal{D}_\mathbf{u}^l)_{l\in\mathbb{N}}$ be sequences of GDs. We state here the assumptions on these sequences which ensure that the solutions to the corresponding schemes converge. Most of these assumptions are adaptation of classical GDM assumptions [20], except for the chain-rule, product rule and cut-off properties used in Subsection 4.2 to obtain compactness properties; we note that all these assumptions hold for standard discretizations used in porous media flows.

Following [11], the spatial GD of the Darcy flow $\mathcal{D}_p = \left(X_{\mathcal{D}_p}^0, \nabla_{\mathcal{D}_p}^m, \nabla_{\mathcal{D}_p}^f, \Pi_{\mathcal{D}_p}^m, \Pi_{\mathcal{D}_p}^f\right)$ is assumed to satisfy the following coercivity, consistency, limit-conformity and compactness properties.

**Coercivity of $\mathcal{D}_p$.** Let $C_{\mathcal{D}_p} > 0$ be defined by

$$C_{\mathcal{D}_p} = \max_{0\neq v\in X_{\mathcal{D}_p}^0} \frac{\|\Pi_{\mathcal{D}_p}^m v\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_p}^f v\|_{L^2(\Gamma)}}{\|v\|_{\mathcal{D}_p}}. \tag{10}$$

Then, a sequence of spatial GDs $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$ is said to be *coercive* if there exists $\overline{C}_p > 0$ such that $C_{\mathcal{D}_p^l} \leqslant \overline{C}_p$ for all $l \in \mathbb{N}$.

**Consistency of $\mathcal{D}_p$.** Let $r > 8$ be given, and for all $w \in V_0$ and $v \in X_{\mathcal{D}_p}^0$ let us define

$$\begin{aligned} S_{\mathcal{D}_p}(w,v) = {} & \|\nabla_{\mathcal{D}_p}^m v - \nabla w\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}_p}^f v - \nabla_\tau \gamma w\|_{L^r(\Gamma)} \\ & + \|\Pi_{\mathcal{D}_p}^m v - w\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_p}^f v - \gamma w\|_{L^r(\Gamma)}, \end{aligned} \tag{11}$$

and $\mathcal{S}_{\mathcal{D}_p}(w) = \min_{v\in X_{\mathcal{D}_p}^0} S_{\mathcal{D}_p}(w,v)$. Then, a sequence of spatial GDs $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$ is said to be *consistent* if for all $w \in V_0$ one has $\lim_{l\to+\infty} \mathcal{S}_{\mathcal{D}_p^l}(w) = 0$. Moreover, if $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$ is a sequence of *space-time* GDs, then it is said to be consistent if the underlying sequence of spatial GDs is consistent as above, and if, for any $\varphi \in V_0$ and $\psi \in L^2(\Omega)$, as $l \to +\infty$,

$$\Delta t^l \to 0\,,\ \|\Pi_{\mathcal{D}_p^l}^m I_{\mathcal{D}_p^l}\varphi - \varphi\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_p^l}^f I_{\mathcal{D}_p^l}\varphi - \varphi\|_{L^2(\Gamma)} \to 0 \text{ and } \|\Pi_{\mathcal{D}_p^l}^m I_{\mathcal{D}_p^l}^m \psi - \psi\|_{L^2(\Omega)} \to 0. \tag{12}$$

*Remark* 3.1 (Consistency). In [11], the consistency is only considered for $r = 2$. We have here to adopt a slightly stronger assumption to deal with the coupling and non-linearity involving the fracture aperture $d_f$. Note that, under standard mesh regularity assumptions, this stronger consistency property is still satisfied for all classical GDs.

**Limit-conformity of $\mathcal{D}_p$.** For all $(\mathbf{q}_m, \mathbf{q}_f) \in C^\infty(\Omega\backslash\overline{\Gamma})^d \times C^\infty(\Gamma)^{d-1}$ and $v \in X^0_{\mathcal{D}_p}$, let us define

$$\begin{aligned} W_{\mathcal{D}_p}(\mathbf{q}_m, \mathbf{q}_f, v) = &\int_\Omega \Big(\mathbf{q}_m \cdot \nabla^m_{\mathcal{D}_p} v + \ \Pi^m_{\mathcal{D}_p} v \ \mathrm{div}(\mathbf{q}_m)\Big)\mathrm{d}\mathbf{x} \\ &+ \int_\Gamma \Big(\mathbf{q}_f \cdot \nabla^f_{\mathcal{D}_p} v + \ \Pi^f_{\mathcal{D}_p} v \ (\mathrm{div}_\tau(\mathbf{q}_f) - [\![\mathbf{q}_m]\!])\Big)\mathrm{d}\sigma(\mathbf{x}), \end{aligned} \tag{13}$$

and $\mathcal{W}_{\mathcal{D}_p}(\mathbf{q}_m, \mathbf{q}_f) = \max\limits_{0\neq v\in X^0_{\mathcal{D}_p}} \dfrac{|W_{\mathcal{D}_p}(\mathbf{q}_m, \mathbf{q}_f, v)|}{\|v\|_{\mathcal{D}_p}}$. Then, a sequence of spatial GDs $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$ is said to be *limit-conforming* if for all $(\mathbf{q}_m, \mathbf{q}_f) \in C^\infty(\Omega\backslash\overline{\Gamma})^d \times C^\infty_c(\Gamma)^{d-1}$ one has $\lim_{l\to+\infty} \mathcal{W}_{\mathcal{D}^l_p}(\mathbf{q}_m, \mathbf{q}_f) = 0$. Here $C^\infty_c(\Gamma)^{d-1}$ denotes the space of functions whose restriction to each $\Gamma_i$ is in $C^\infty(\Gamma_i)^{d-1}$ tangent to $\Gamma_i$, compactly supported away from the tips, and satisfying normal flux conservation at fracture intersections not located at the boundary $\partial\Omega$.

**(Local) compactness** of $\mathcal{D}_p$. A sequence of spatial GDs $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$ is said to be *locally compact* if for all sequences $(v^l)_{l\in\mathbb{N}} \in (X^0_{\mathcal{D}^l_p})_{l\in\mathbb{N}}$ such that $\sup_{l\in\mathbb{N}} \|v^l\|_{\mathcal{D}^l_p} < +\infty$ and all compact sets $K_m \subset \Omega$ and $K_f \subset \Gamma$, such that $K_f$ is disjoint from the intersections $(\overline{\Gamma}_i \cap \overline{\Gamma}_j)_{i\neq j}$, the sequences $(\Pi^m_{\mathcal{D}^l_p} v^l)_{l\in\mathbb{N}}$ and $(\Pi^f_{\mathcal{D}^l_p} v^l)_{l\in\mathbb{N}}$ are relatively compact in $L^2(K_m)$ and $L^2(K_f)$, respectively.

*Remark* 3.2 (Local compactness through estimates of space translates). For $K_m, K_f$ as above, set

$$T_{\mathcal{D}^l_p, K_m, K_f}(\xi, \eta) = \max_{v\in X^0_{\mathcal{D}^l_p}\backslash\{0\}} \frac{\|\Pi^m_{\mathcal{D}^l_p} v(\cdot+\xi) - \Pi^m_{\mathcal{D}^l_p} v\|_{L^2(K_m)} + \sum_{i\in I} \|\Pi^f_{\mathcal{D}^l_p} v(\cdot+\eta_i) - \Pi^f_{\mathcal{D}^l_p} v\|_{L^2(K_f\cap\Gamma_i)}}{\|v\|_{\mathcal{D}^l_p}},$$

where $\xi \in \mathbb{R}^d$, $\eta = (\eta_i)_{i\in I}$ with $\eta_i$ tangent to $\Gamma_i$; for $\xi$ and $\eta$ small enough, this expression is well defined since $K_m$ and $K_f$ are compact in $\Omega$ and $\Gamma$, respectively. Following [20, Lemma 2.21], An equivalent formulation of the local compactness property is: for all $K_m, K_f$ as above,

$$\lim_{\xi,\eta\to 0} \sup_{l\in\mathbb{N}} T_{\mathcal{D}^l_p, K_m, K_f}(\xi, \eta) = 0.$$

*Remark* 3.3 (Usual compactness property for GDs). The standard compactness property for GD is not *local* but *global*, that is, on the entire domain not any of its compact subsets (see, e.g., [20, Definition 2.8] and also below for $\mathcal{D}_{\mathbf{u}}$). Two reasons pushed us to consider here the weaker notion of local compactness: firstly, for standard GDs, the global compactness does not seem obvious to establish (or even true) in the fractures, because of the weight $d_0$ in the norm $\|\cdot\|_{\mathcal{D}_p}$, which prevents us from estimating the translates of the reconstructed function by the gradient near the fracture tips; secondly, we will only prove compactness on saturations, which are uniformly bounded by 1 and for which local and global compactness are therefore equivalent.

In the following, for brevity we refer to the local compactness of $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$ simply as the *compactness* of this sequence of GDs.

**Chain rule estimate on** $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$: for any Lipschitz-continuous function $F : \mathbb{R} \to \mathbb{R}$, there is $C_F \geqslant 0$ such that, for all $l \in \mathbb{N}$, $v \in X^0_{\mathcal{D}^l_p}$,

$$\|\nabla^m_{\mathcal{D}^l_p} F(v)\|_{L^2(\Omega)} \leqslant C_F \|\nabla^m_{\mathcal{D}^l_p} v\|_{L^2(\Omega)}.$$

**Product rule estimate on** $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$: there exists $C_P$ such that, for any $l \in \mathbb{N}$ and any $u^l, v^l \in X^0_{\mathcal{D}^l_p}$, it holds

$$\|\nabla^m_{\mathcal{D}_p}(u^l v^l)\|_{L^2(\Omega)} \leqslant C_P\Big(|u^l|_\infty \|\nabla^m_{\mathcal{D}_p} v^l\|_{L^2(\Omega)} + |v^l|_\infty \|\nabla^m_{\mathcal{D}_p} u^l\|_{L^2(\Omega)}\Big),$$

where $|w|_\infty \coloneqq \max_{i\in I} |w_i|$ whenever $w = \sum_{i\in I} w_i \mathbf{e}_i$ with $(\mathbf{e}_i)_{i\in I}$ the canonical basis of $X^0_{\mathcal{D}^l_p}$.

**Cut-off property of** $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$: for any compact set $K \subset \Omega\backslash\Gamma$, there exists $C_K \geqslant 0$ and $(\psi^l)_{l\in\mathbb{N}} \in (X^0_{\mathcal{D}_p^l})_{l\in\mathbb{N}}$ such that $(|\psi^l|_\infty)_{l\in\mathbb{N}}$ is bounded and, for $l$ large enough:

$$\Pi^m_{\mathcal{D}_p^l}\psi^l \geqslant 0 \text{ on } \Omega; \quad \Pi^m_{\mathcal{D}_p^l}\psi^l = 1 \text{ on } K; \quad \|\nabla^m_{\mathcal{D}_p^l}\psi^l\|_{L^2(\Omega)} \leqslant C_K$$
$$\Pi^f_{\mathcal{D}_p^l}(v^l\psi^l) = 0 \text{ and } \nabla^f_{\mathcal{D}_p^l}(v^l\psi^l) = 0 \quad \text{for all } v^l \in X^0_{\mathcal{D}_p^l}$$

**Coercivity of** $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$. Let $C_{\mathcal{D}_{\mathbf{u}}} > 0$ be defined by

$$C_{\mathcal{D}_{\mathbf{u}}} = \max_{\mathbf{0}\neq\mathbf{v}\in X^0_{\mathcal{D}_{\mathbf{u}}^l}} \frac{\|\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{v}\|_{L^2(\Omega)} + \|[\![\mathbf{v}]\!]_{\mathcal{D}_{\mathbf{u}}^l}\|_{L^4(\Gamma)}}{\|\mathbf{v}\|_{\mathcal{D}_{\mathbf{u}}^l}}. \tag{14}$$

Then, the sequence of spatial GDs $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$ is said to be *coercive* if there exists $\overline{C}_{\mathbf{u}} > 0$ such that $C_{\mathcal{D}_{\mathbf{u}}^l} \leqslant \overline{C}_{\mathbf{u}}$ for all $l \in \mathbb{N}$.

**Consistency of** $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$. For all $\mathbf{w} \in \mathbf{U}_0$, it holds $\lim_{l\to+\infty} \mathcal{S}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{w}) = 0$ where

$$\mathcal{S}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{w}) = \min_{\mathbf{v}\in X^0_{\mathcal{D}_{\mathbf{u}}^l}} \Big[\|\mathfrak{e}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{v}) - \mathfrak{e}(\mathbf{w})\|_{L^2(\Omega,\mathcal{S}_d(\mathbb{R}))} + \|\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{v} - \mathbf{w}\|_{L^2(\Omega)} + \big\|[\![\mathbf{v}]\!]_{\mathcal{D}_{\mathbf{u}}^l} - [\![\mathbf{w}]\!]\big\|_{L^4(\Gamma)}\Big]. \tag{15}$$

**Limit-conformity of** $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$. Let $C^\infty_\Gamma(\Omega\backslash\overline{\Gamma}, \mathcal{S}_d(\mathbb{R}))$ denote the vector space of smooth functions $\mathfrak{o} : \Omega\backslash\overline{\Gamma} \to \mathcal{S}_d(\mathbb{R})$ admitting finite limits on each side of $\Gamma$, and such that $\mathfrak{o}^+(\mathbf{x})\mathbf{n}^+ + \mathfrak{o}^-(\mathbf{x})\mathbf{n}^- = \mathbf{0}$ and $(\mathfrak{o}^+(\mathbf{x})\mathbf{n}^+)\times\mathbf{n}^+ = \mathbf{0}$ for a.e. $\mathbf{x} \in \Gamma$. For all $\mathfrak{o} \in C^\infty_\Gamma(\Omega\backslash\overline{\Gamma}, \mathcal{S}_d(\mathbb{R}))$, it holds $\lim_{l\to+\infty} \mathcal{W}_{\mathcal{D}_{\mathbf{u}}^l}(\mathfrak{o}) = 0$ where

$$\mathcal{W}_{\mathcal{D}_{\mathbf{u}}^l}(\mathfrak{o}) = \max_{\mathbf{0}\neq\mathbf{v}\in X^0_{\mathcal{D}_{\mathbf{u}}^l}} \frac{1}{\|\mathbf{v}\|_{\mathcal{D}_{\mathbf{u}}^l}} \left[\int_\Omega \Big(\mathfrak{o} : \mathfrak{e}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{v}) + \Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{v}\cdot\mathrm{div}(\mathfrak{o})\Big)\mathrm{d}\mathbf{x} - \int_\Gamma (\mathfrak{o}\mathbf{n}^+)\cdot\mathbf{n}^+[\![\mathbf{v}]\!]_{\mathcal{D}_{\mathbf{u}}^l}\mathrm{d}\sigma(\mathbf{x})\right].$$

**Compactness of** $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$. For any sequence $(\mathbf{v}^l)_{l\in\mathbb{N}} \in (X^0_{\mathcal{D}_{\mathbf{u}}^l})_{l\in\mathbb{N}}$ such that $\sup_{l\in\mathbb{N}} \|\mathbf{v}^l\|_{\mathcal{D}_{\mathbf{u}}^l} < +\infty$, the sequences $(\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{v}^l)_{l\in\mathbb{N}}$ and $([\![\mathbf{v}^l]\!]_{\mathcal{D}_{\mathbf{u}}^l})_{l\in\mathbb{N}}$ are relatively compact in $L^2(\Omega)^d$ and in $L^s(\Gamma)$ for all $s < 4$, respectively.

*Remark* 3.4 (Compactness through estimates of space translates). Similarly to Remark 3.2 (see also [20, Lemma 2.21]), the compactness of $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$ is equivalent to

$$\lim_{\xi,\eta\to 0} \sup_{l\in\mathbb{N}} T_{\mathcal{D}_{\mathbf{u}}^l,s}(\xi,\eta) = 0 \quad \forall s < 4,$$

where

$$T_{\mathcal{D}_{\mathbf{u}}^l,s}(\xi,\eta) = \max_{\mathbf{v}\in X^0_{\mathcal{D}_{\mathbf{u}}^l}\backslash\{0\}} \frac{\|\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{v}(\cdot+\xi) - \Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{v}\|_{L^2(\Omega)} + \sum_{i\in I}\big\|[\![\mathbf{v}^l]\!]_{\mathcal{D}_{\mathbf{u}}^l}(\cdot+\eta_i) - [\![\mathbf{v}^l]\!]_{\mathcal{D}_{\mathbf{u}}^l}\big\|_{L^s(\Gamma_i)}}{\|\mathbf{v}\|_{\mathcal{D}_{\mathbf{u}}^l}},$$

with $\xi \in \mathbb{R}^d$, $\eta = (\eta_i)_{i\in I}$ with $\eta_i$ tangent to $\Gamma_i$, and the functions extended by 0 outside their respective domain $\Omega$ or $\Gamma$.

# 4 Convergence analysis

The main result of this work is the following convergence theorem.

**Theorem 4.1.** *Let $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$, $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$, $\{(t_n^l)_{n=0}^{N^l}\}_{l\in\mathbb{N}}$, be sequences of space time GDs assumed to satisfy the coercivity, consistency, limit-conformity and compactness properties. Let $0 < \phi_{m,\min} \leqslant \phi_{m,\max} < +\infty$ and assume that, for each $l \in \mathbb{N}$, the gradient scheme (9a)–(9b) has a solution $p_l^\alpha \in (X^0_{\mathcal{D}_p^l})^{N^l+1}$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, $\mathbf{u}^l \in (X^0_{\mathcal{D}_{\mathbf{u}}^l})^{N^l+1}$ such that*

*(i)* $d_{f,\mathcal{D}_{\mathbf{u}}^l}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Gamma$,

*(ii)* $\phi_{m,\min} \leqslant \phi_{\mathcal{D}^l}(t,\mathbf{x}) \leqslant \phi_{m,\max}$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Omega$.

*Then, there exist* $\bar{p}^\alpha \in L^2(0,T;V_0)$, $\alpha \in \{\mathrm{nw},\mathrm{w}\}$, *and* $\bar{\mathbf{u}} \in L^\infty(0,T;\mathbf{U}_0)$ *satisfying the weak formulation* (6)–(7) *such that for* $\alpha \in \{\mathrm{nw},\mathrm{w}\}$ *and up to a subsequence*

$$
\begin{aligned}
&\Pi^m_{\mathcal{D}_p^l} p_l^\alpha \rightharpoonup \bar{p}^\alpha && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
&\Pi^f_{\mathcal{D}_p^l} p_l^\alpha \rightharpoonup \gamma\bar{p}^\alpha && \text{weakly in } L^2(0,T;L^2(\Gamma)),\\
&\Pi_{\mathcal{D}_{\mathbf{u}}^l} \mathbf{u}^l \rightharpoonup \bar{\mathbf{u}} && \text{weakly-}\star \text{ in } L^\infty(0,T;L^2(\Omega)^d),\\
&\phi_{\mathcal{D}^l} \rightharpoonup \bar{\phi}_m && \text{weakly-}\star \text{ in } L^\infty(0,T;L^2(\Omega)),\\
&d_{f,\mathcal{D}_{\mathbf{u}}^l} \to \bar{d}_f && \text{in } L^\infty(0,T;L^p(\Gamma)) \text{ for } 2\leqslant p<4,\\
&\Pi^m_{\mathcal{D}_p^l} S_m^\alpha(p_c^l) \to S_m^\alpha(\bar{p}_c) && \text{in } L^2(0,T;L^2(\Omega)),\\
&\Pi^f_{\mathcal{D}_p^l} S_f^\alpha(p_c^l) \to S_f^\alpha(\gamma\bar{p}_c) && \text{in } L^2(0,T;L^2(\Gamma)),
\end{aligned}
$$

*where* $\bar{\phi}_m = \bar{\phi}_m^0 + b\,\mathrm{div}(\bar{\mathbf{u}}-\bar{\mathbf{u}}^0) + \dfrac{1}{M}(\bar{p}_m^E - \bar{p}_m^{E,0})$, $\bar{d}_f = -[\![\bar{\mathbf{u}}]\!]$, *and* $\bar{p}_c = \bar{p}^{\mathrm{nw}} - \bar{p}^{\mathrm{w}}$.

We first present in Subsections 4.1 and 4.2 a sequence of intermediate results that will be useful for the proof of Theorem 4.1 detailed in Subsection 4.3.

## 4.1 Energy estimates

Using the phase pressures and velocity (time derivative of the displacement field) as test functions, the following *a priori* estimates can be inferred.

**Lemma 4.2** (A priori estimates)**.** *Let* $p^\alpha, \mathbf{u}$ *be a solution to problem* (9) *such that*

*(i)* $d_{f,\mathcal{D}_{\mathbf{u}}}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Gamma$,

*(ii)* $\phi_{\mathcal{D}}(t,\mathbf{x}) \geqslant \phi_{m,\min}$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Omega$, *where* $\phi_{m,\min} > 0$ *is a constant.*

*Under hypotheses (H1)–(H8), there exists a real number* $C>0$ *depending on the data, the coercivity constants* $C_{\mathcal{D}_p}$, $C_{\mathcal{D}_{\mathbf{u}}}$, *and* $\phi_{m,\min}$, *such that the following estimates hold:*

$$
\begin{aligned}
\|\nabla^m_{\mathcal{D}_p} p^\alpha\|_{L^2((0,T)\times\Omega)} &\leqslant C, & \|d^{3/2}_{f,\mathcal{D}_{\mathbf{u}}}\nabla^f_{\mathcal{D}_p} p^\alpha\|_{L^2((0,T)\times\Gamma)} &\leqslant C,\\
\|U_m(\Pi^m_{\mathcal{D}_p} p_c)\|_{L^\infty(0,T;L^1(\Omega))} &\leqslant C, & \|d_0 U_f(\Pi^f_{\mathcal{D}_p} p_c)\|_{L^\infty(0,T;L^1(\Gamma))} &\leqslant C,\\
\|\Pi^m_{\mathcal{D}_p} p_m^E\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant C, & \|\mathfrak{C}_{\mathcal{D}_{\mathbf{u}}}(\mathbf{u})\|_{L^\infty(0,T;L^2(\Omega,\mathcal{S}_d(\mathbb{R})))} &\leqslant C,\\
\|d_{f,\mathcal{D}_{\mathbf{u}}}\|_{L^\infty(0,T;L^4(\Gamma))} &\leqslant C.
\end{aligned}
\tag{16}
$$

*Proof.* For a piecewise constant function $v$ on $[0,T]$ with $v(t) = v_{n+1}$ for all $t \in (t_n, t_{n+1}]$, $n \in \{0,\dots,N-1\}$, and the initial value $v(0) = v_0$, we define the piecewise constant function $\hat{v}$ such that $\hat{v}(t) = v_n$ for all $t \in (t_n, t_{n+1}]$. We notice the following expression for the discrete derivative of the product of two such functions:

$$
\delta_t(uv)(t) = \hat{u}(t)\delta_t v(t) + v(t)\delta_t u(t). \tag{17}
$$

In (9a), upon choosing $\varphi^\alpha = p^\alpha$ we obtain $T_1 + T_2 + T_3 + T_4 = T_5 + T_6$, with

$$
\begin{aligned}
T_1 &= \int_0^T \int_\Omega \delta_t\Big(\phi_{\mathcal{D}} \Pi^m_{\mathcal{D}_p} s^\alpha_m\Big) \Pi^m_{\mathcal{D}_p} p^\alpha \mathrm{d}\mathbf{x}\mathrm{d}t, & T_2 &= \int_0^T \int_\Omega \eta^\alpha_m (\Pi^m_{\mathcal{D}_p} s^\alpha_m) \mathbb{K}_m \nabla^m_{\mathcal{D}_p} p^\alpha \cdot \nabla^m_{\mathcal{D}_p} p^\alpha \mathrm{d}\mathbf{x}\mathrm{d}t,\\
T_3 &= \int_0^T \int_\Gamma \delta_t\Big(d_{f,\mathcal{D}_\mathbf{u}} \Pi^f_{\mathcal{D}_p} s^\alpha_f\Big) \Pi^f_{\mathcal{D}_p} p^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t, & T_4 &= \int_0^T \int_\Gamma \eta^\alpha_f (\Pi^f_{\mathcal{D}_p} s^\alpha_f) \frac{d^3_{f,\mathcal{D}_\mathbf{u}}}{12} \nabla^f_{\mathcal{D}_p} p^\alpha \cdot \nabla^f_{\mathcal{D}_p} p^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,\\
T_5 &= \int_0^T \int_\Omega h^\alpha_m \Pi^m_{\mathcal{D}_p} p^\alpha \mathrm{d}\mathbf{x}\mathrm{d}t, & T_6 &= \int_0^T \int_\Gamma h^\alpha_f \Pi^f_{\mathcal{D}_p} p^\alpha \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t.
\end{aligned}
\tag{18}
$$

First, we focus on the matrix and fracture accumulation terms $T_1$ and $T_3$, respectively. Using (17) and the piecewise constant function reconstruction property of $\Pi^{\mathrm{rt}}_{\mathcal{D}_p}$, $\mathrm{rt} \in \{m, f\}$, we can write

$$
\begin{aligned}
\delta_t(\phi_{\mathcal{D}} S^\alpha_m(\Pi^m_{\mathcal{D}_p} p_c)) &= \hat{\phi}_{\mathcal{D}} \delta_t S^\alpha_m(\Pi^m_{\mathcal{D}_p} p_c) + S^\alpha_m(\Pi^m_{\mathcal{D}_p} p_c) \delta_t \phi_{\mathcal{D}},\\
\delta_t(d_{f,\mathcal{D}_\mathbf{u}} S^\alpha_f(\Pi^f_{\mathcal{D}_p} p_c)) &= \hat{d}_{f,\mathcal{D}_\mathbf{u}} \delta_t S^\alpha_f(\Pi^f_{\mathcal{D}_p} p_c) + S^\alpha_f(\Pi^f_{\mathcal{D}_p} p_c) \delta_t d_{f,\mathcal{D}_\mathbf{u}}.
\end{aligned}
$$

Summing on $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, we obtain

$$
\begin{aligned}
\sum_\alpha (T_1 + T_3) = \sum_\alpha \Big( &\int_0^T \int_\Omega \hat{\phi}_{\mathcal{D}} \Pi^m_{\mathcal{D}_p} p^\alpha \, \delta_t S^\alpha_m(\Pi^m_{\mathcal{D}_p} p_c) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega S^\alpha_m(\Pi^m_{\mathcal{D}_p} p_c) \Pi^m_{\mathcal{D}_p} p^\alpha \, \delta_t \phi_{\mathcal{D}} \mathrm{d}\mathbf{x}\mathrm{d}t\\
&+ \int_0^T \int_\Gamma \hat{d}_{f,\mathcal{D}_\mathbf{u}} \Pi^f_{\mathcal{D}_p} p^\alpha \, \delta_t S^\alpha_f(\Pi^f_{\mathcal{D}_p} p_c) \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t + \int_0^T \int_\Gamma S^\alpha_f(\Pi^f_{\mathcal{D}_p} p_c) \Pi^f_{\mathcal{D}_p} p^\alpha \, \delta_t d_{f,\mathcal{D}_\mathbf{u}} \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \Big).
\end{aligned}
$$

Now, for $\mathrm{rt} \in \{m, f\}$,

$$
\sum_\alpha \Pi^{\mathrm{rt}}_{\mathcal{D}_p} p^\alpha \, \delta_t S^\alpha_{\mathrm{rt}}(\Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_c) = \Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_c \, \delta_t S^{\mathrm{nw}}_{\mathrm{rt}}(\Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_c) \geqslant \delta_t U_{\mathrm{rt}}(\Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_c). \tag{19}
$$

Indeed, for $n \in \{0, \ldots, N-1\}$, by the definition (5) of the capillary energy $U_{\mathrm{rt}}$ and letting $\pi^{\mathrm{rt}}_{c,j} = \Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_{c,j}$, we have

$$
\begin{aligned}
\pi^{\mathrm{rt}}_{c,n+1}(S^{\mathrm{nw}}_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n+1}) - S^{\mathrm{nw}}_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n})) &= U_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n+1}) - U_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n}) + \int_{\pi^{\mathrm{rt}}_{c,n}}^{\pi^{\mathrm{rt}}_{c,n+1}} (S^{\mathrm{nw}}_{\mathrm{rt}}(q) - S^{\mathrm{nw}}_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n}))\mathrm{d}q\\
&\geqslant U_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n+1}) - U_{\mathrm{rt}}(\pi^{\mathrm{rt}}_{c,n}),
\end{aligned}
$$

where the last inequality holds since $S^{\mathrm{nw}}_{\mathrm{rt}}$ is a non-decreasing function. Replacing $p_{c,n}$ and $p_{c,n+1}$ by $\Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_{c,n}$ and $\Pi^{\mathrm{rt}}_{\mathcal{D}_p} p_{c,n+1}$, respectively, in the above estimate and dividing by $\delta t^{n+\frac{1}{2}}$, we get (19). Thus, we obtain

$$
\begin{aligned}
\sum_\alpha (T_1 + T_3) \geqslant &\int_0^T \int_\Omega \hat{\phi}_{\mathcal{D}} \delta_t U_m(\Pi^m_{\mathcal{D}_p} p_c) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma \hat{d}_{f,\mathcal{D}_\mathbf{u}} \delta_t U_f(\Pi^f_{\mathcal{D}_p} p_c) \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t\\
&+ \sum_\alpha \Big( \int_0^T \int_\Omega S^\alpha_m(\Pi^m_{\mathcal{D}_p} p_c) \Pi^m_{\mathcal{D}_p} p^\alpha \delta_t \phi_{\mathcal{D}} \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma S^\alpha_f(\Pi^f_{\mathcal{D}_p} p_c) \Pi^f_{\mathcal{D}_p} p^\alpha \delta_t d_{f,\mathcal{D}_\mathbf{u}} \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \Big).
\end{aligned}
$$

Applying again (17), we have

$$
\begin{aligned}
\hat{\phi}_{\mathcal{D}} \delta_t U_m(\Pi^m_{\mathcal{D}_p} p_c) &= \delta_t(\phi_{\mathcal{D}} U_m(\Pi^m_{\mathcal{D}_p} p_c)) - U_m(\Pi^m_{\mathcal{D}_p} p_c) \delta_t \phi_{\mathcal{D}},\\
\hat{d}_{f,\mathcal{D}_\mathbf{u}} \delta_t U_f(\Pi^f_{\mathcal{D}_p} p_c) &= \delta_t(d_{f,\mathcal{D}_\mathbf{u}} U_f(\Pi^f_{\mathcal{D}_p} p_c)) - U_f(\Pi^f_{\mathcal{D}_p} p_c) \delta_t d_{f,\mathcal{D}_\mathbf{u}}.
\end{aligned}
$$

In the light of the closure equations (9c), this allows us to infer that

$$\begin{aligned}\sum_\alpha (T_1+T_3) \geqslant & \int_0^T\int_\Omega \delta_t(\phi_{\mathcal{D}} U_m(\Pi^m_{\mathcal{D}_p} p_c))\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Gamma \delta_t(d_{f,\mathcal{D}_\mathbf{u}} U_f(\Pi^f_{\mathcal{D}_p} p_c))\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \\ & + \int_0^T\int_\Omega \frac{1}{2M}\delta_t\Big(\Pi^m_{\mathcal{D}_p} p^E_m\Big)^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Omega b\,\Pi^m_{\mathcal{D}_p} p^E_m \,\mathrm{div}_{\mathcal{D}_\mathbf{u}}(\delta_t\mathbf{u})\mathrm{d}\mathbf{x}\mathrm{d}t \\ & - \int_0^T\int_\Gamma \Pi^f_{\mathcal{D}_p} p^E_f \,[\![\delta_t\mathbf{u}]\!]_{\mathcal{D}_\mathbf{u}}\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,\end{aligned} \tag{20}$$

where we have used the fact that

$$v\delta_t v \geqslant \delta_t\left(\frac{v^2}{2}\right) \tag{21}$$

for $v$ piecewise constant on $[0,T]$. Then, taking into account assumptions (H1)–(H8) and (i) in the lemma, there exists a real number $C>0$ depending only on the data such that

$$\sum_\alpha (T_2+T_4) \geqslant C\Big(\int_0^T\int_\Omega \sum_\alpha |\nabla^m_{\mathcal{D}_p} p^\alpha|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Gamma \sum_\alpha |d^{3/2}_{f,\mathcal{D}_\mathbf{u}} \nabla^f_{\mathcal{D}_p} p^\alpha|^2 \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t\Big). \tag{22}$$

On the other hand, upon choosing $\mathbf{v}=\delta_t\mathbf{u}$ in (9b), we get $T_7+T_8+T_9=T_{10}$, with

$$\begin{aligned} T_7 &= \int_0^T\int_\Omega \varpi_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\delta_t\mathbf{u})\mathrm{d}\mathbf{x}\mathrm{d}t, \quad T_8 = -\int_0^T\int_\Omega b\,\Pi^m_{\mathcal{D}_p} p^E_m \,\mathrm{div}_{\mathcal{D}_\mathbf{u}}(\delta_t\mathbf{u})\mathrm{d}\mathbf{x}\mathrm{d}t \\ T_9 &= \int_0^T\int_\Gamma \Pi^f_{\mathcal{D}_p} p^E_f \,[\![\delta_t\mathbf{u}]\!]_{\mathcal{D}_\mathbf{u}}\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t, \quad T_{10} = \int_0^T\int_\Omega \mathbf{f}\cdot\Pi_{\mathcal{D}_\mathbf{u}}(\delta_t\mathbf{u})\mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned} \tag{23}$$

Using (21), we see that

$$T_7 \geqslant \int_0^T\int_\Omega \delta_t\Big(\frac{1}{2}\varpi_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})\Big)\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{24}$$

so that, all in all, taking into account that $\sum_\alpha(T_1+T_2+T_3+T_4)+T_7+T_8+T_9=\sum_\alpha(T_5+T_6)+T_{10}$ and inequalities (20)–(22)–(24), we obtain the following estimate for the solutions of (9): there is a real number $C>0$ depending on the data such that

$$\begin{aligned} & \int_0^T\int_\Omega \delta_t(\phi_{\mathcal{D}} U_m(\Pi^m_{\mathcal{D}_p} p_c))\,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Gamma \delta_t(d_{f,\mathcal{D}_\mathbf{u}} U_f(\Pi^f_{\mathcal{D}_p} p_c))\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \\ & + \int_0^T\int_\Omega \delta_t\left(\frac{1}{2}\varpi_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) + \frac{1}{2M}(\Pi^m_{\mathcal{D}_p} p^E_m)^2\right)\mathrm{d}\mathbf{x}\mathrm{d}t \\ & + \sum_\alpha \int_0^T\int_\Omega |\nabla^m_{\mathcal{D}_p} p^\alpha|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t + \sum_\alpha \int_0^T\int_\Gamma |d^{3/2}_{f,\mathcal{D}_\mathbf{u}} \nabla^f_{\mathcal{D}_p} p^\alpha|^2\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \\ & \leqslant C\left(\int_0^T\int_\Omega \mathbf{f}\cdot\delta_t\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{u}\,\mathrm{d}\mathbf{x}\mathrm{d}t + \sum_\alpha \int_0^T\int_\Omega h^\alpha_m \Pi^m_{\mathcal{D}_p} p^\alpha\,\mathrm{d}\mathbf{x}\mathrm{d}t \right. \\ & \qquad \left. + \sum_\alpha \int_0^T\int_\Gamma h^\alpha_f \Pi^f_{\mathcal{D}_p} p^\alpha\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t\right).\end{aligned} \tag{25}$$

Now, we have

$$\begin{aligned}\int_0^T\int_\Omega \mathbf{f}\cdot\delta_t\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{u}\,\mathrm{d}\mathbf{x}\mathrm{d}t &= \int_\Omega \mathbf{f}\cdot(\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{u}(T) - \mathbf{f}\cdot\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{u}(0))\mathrm{d}\mathbf{x} \\ &\leqslant C_{\mathcal{D}_\mathbf{u}}\|\mathbf{f}\|_{L^2(\Omega)}(\|\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})(T)\|_{L^2(\Omega,\mathcal{S}_d(\mathbb{R}))} + \|\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})(0)\|_{L^2(\Omega,\mathcal{S}_d(\mathbb{R}))}),\end{aligned}$$

$$\sum_\alpha\Big(\int_0^T\int_\Omega h_m^\alpha\Pi_{\mathcal{D}_p}^m p^\alpha\,d\mathbf{x}\mathrm{d}t+\int_0^T\int_\Gamma h_f^\alpha\Pi_{\mathcal{D}_p}^f p^\alpha\,d\sigma(\mathbf{x})\mathrm{d}t\Big)$$
$$\leqslant C_{\mathcal{D}_p}\sum_\alpha(\|h_m^\alpha\|_{L^2((0,T)\times\Omega)}+\|h_f^\alpha\|_{L^2((0,T)\times\Gamma)})(\|\nabla_{\mathcal{D}_p}^m p^\alpha\|_{L^2(0,T;L^2(\Omega))}+\|d_{f,\mathcal{D}_\mathbf{u}}^{3/2}\nabla_{\mathcal{D}_p}^f p^\alpha\|_{L^2(0,T;L^2(\Gamma))}),$$

where we have used the coercivity properties of the two gradient discretizations along with the Cauchy–Schwarz inequality and $d_0\leqslant d_{f,\mathcal{D}_\mathbf{u}}$. Using Young's inequality in the last two estimates as well as hypotheses (H1)–(H8) and (ii) in the lemma, it is then possible to infer from (25) the existence of a real number $C>0$ depending on the data and on $\phi_{m,\min}$ such that

$$\begin{aligned}&\|U_m(\Pi_{\mathcal{D}_p}^m p_c)(T)\|_{L^1(\Omega)}+\|d_0U_f(\Pi_{\mathcal{D}_p}^m p_c)(T)\|_{L^1(\Omega)}+\|(\Pi_{\mathcal{D}_p}^m p_m^E)(T)\|_{L^2(\Omega)}^2\\&+\|\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})(T)\|_{L^2(\Omega,\mathcal{S}_d(\mathbb{R}))}^2+\sum_\alpha\Big(\|\nabla_{\mathcal{D}_p}^m p^\alpha\|_{L^2(0,T;L^2(\Omega))}^2+\|d_{f,\mathcal{D}_\mathbf{u}}^{3/2}\nabla_{\mathcal{D}_p}^f p^\alpha\|_{L^2(0,T;L^2(\Gamma))}^2\Big)\\&\qquad\leqslant C\Big(\|\mathbf{f}\|_{L^2(\Omega)}^2+\sum_\alpha\Big(\|h_m^\alpha\|_{L^2((0,T)\times\Omega)}^2+\|h_f^\alpha\|_{L^2((0,T)\times\Gamma)}^2\Big)\Big).\end{aligned}$$

The above inequality, along with the fact that $T$ can be replaced by any $t\in(0,T]$ in the left-hand side, yields the a priori estimates (16) on $p^\alpha$, $p_c$, $p_m^E$ and $\mathbf{u}$. The estimate on $d_{f,\mathcal{D}_\mathbf{u}}$ follows from its definition and from the definition (14) of $C_{\mathcal{D}_\mathbf{u}}$. □

## 4.2 Compactness properties

### 4.2.1 Estimates on time translates

**Proposition 4.3.** *Let $\mathcal{D}_p$, $\mathcal{D}_\mathbf{u}$, $(t_n)_{n=0}^N$ be given space time GDs and $\phi_{m,\min}>0$. It is assumed that the gradient scheme (9a)–(9b) has a solution $p^\alpha\in(X_{\mathcal{D}_p}^0)^{N+1}$, $\alpha\in\{\mathrm{nw},\mathrm{w}\}$, $\mathbf{u}\in(X_{\mathcal{D}_\mathbf{u}}^0)^{N+1}$ such that $\phi_\mathcal{D}(t,\mathbf{x})\geqslant\phi_{m,\min}$ for a.e. $(t,\mathbf{x})\in(0,T)\times\Omega$ and $d_{f,\mathcal{D}_\mathbf{u}}(t,\mathbf{x})\geqslant d_0(\mathbf{x})$ for a.e. $(t,\mathbf{x})\in(0,T)\times\Gamma$. Let $\tau,\tau'\in(0,T)$ and let $n_t$ be such that $t\in(t_{n_t},t_{n_t+1}]$. For any $\varphi\in X_{\mathcal{D}_p}^0$, there exist $\xi_{\mathrm{rt}}^{(j),\alpha,n+1}$, $n\in\{0,\dots,N-1\}$, $\mathrm{rt}\in\{m,f\}$, $j\in\{1,2\}$, $\alpha\in\{\mathrm{nw},\mathrm{w}\}$ such that we have the following estimate*

$$\begin{aligned}&\Big|\langle[\phi_\mathcal{D}\Pi_{\mathcal{D}_p}^m s_m^\alpha](\tau)-[\phi_\mathcal{D}\Pi_{\mathcal{D}_p}^m s_m^\alpha](\tau'),\Pi_{\mathcal{D}_p}^m\varphi\rangle_{L^2(\Omega)}\\&\qquad+\langle[d_{f,\mathcal{D}_\mathbf{u}}\Pi_{\mathcal{D}_p}^f s_f^\alpha](\tau)-[d_{f,\mathcal{D}_\mathbf{u}}\Pi_{\mathcal{D}_p}^f s_f^\alpha](\tau'),\Pi_{\mathcal{D}_p}^f\varphi\rangle_{L^2(\Gamma)}\Big|\\&\lesssim\sum_{n=n_\tau+1}^{n_{\tau'}}\delta t^{n+\frac12}\Big(\xi_m^{(1),\alpha,n+1}\|\nabla_{\mathcal{D}_p}^m\varphi\|_{L^2(\Omega)}+\xi_f^{(1),\alpha,n+1}\|\nabla_{\mathcal{D}_p}^f\varphi\|_{L^8(\Gamma)}\\&\qquad+\xi_m^{(2),\alpha,n+1}\|\Pi_{\mathcal{D}_p}^m\varphi\|_{L^2(\Omega)}+\xi_f^{(2),\alpha,n+1}\|\Pi_{\mathcal{D}_p}^f\varphi\|_{L^2(\Gamma)}\Big),\end{aligned}\tag{26}$$

*with*

$$\sum_{n=0}^{N-1}\delta t^{n+\frac12}\Big(\xi_{\mathrm{rt}}^{(j),\alpha,n+1}\Big)^2\lesssim1\qquad\text{for }\mathrm{rt}\in\{m,f\},\ j\in\{1,2\},$$

*and*

$$\xi_m^{(1),\alpha,n+1}=\|\nabla_{\mathcal{D}_p}^m p_{n+1}^\alpha\|_{L^2(\Omega)}\quad\text{and}\quad\xi_f^{(1),\alpha,n+1}=\|(d_{f,\mathcal{D}_\mathbf{u}}^{n+1})^{3/2}\nabla_{\mathcal{D}_p}^f p_{n+1}^\alpha\|_{L^2(\Gamma)}\|d_{f,\mathcal{D}_\mathbf{u}}^{n+1}\|_{L^4(\Gamma)}^{3/2},$$
$$\xi_m^{(2),\alpha,n+1}=\Big\|\frac{1}{\delta t^{n+\frac12}}\int_{t_n}^{t_{n+1}}h_m^\alpha(t,\cdot)\mathrm{d}t\Big\|_{L^2(\Omega)}\quad\xi_f^{(2),\alpha,n+1}=\Big\|\frac{1}{\delta t^{n+\frac12}}\int_{t_n}^{t_{n+1}}h_f^\alpha(t,\cdot)\mathrm{d}t\Big\|_{L^2(\Gamma)}.$$

*Above, the hidden constants in the estimates are independent of $\varphi$ and depend only on the coercivity constants $C_{\mathcal{D}_p}$, $C_{\mathcal{D}_\mathbf{u}}$ of the spatial GDs and on the physical data.*

*Proof.* For any $\varphi \in X^0_{\mathcal{D}_p}$, writing the difference of piecewise-constant functions at times $\tau$ and $\tau'$ as the sum of their jumps between these two times, one has

$$\begin{aligned}
&\Big|\langle[\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau) - [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau'), \Pi^m_{\mathcal{D}_p}\varphi\rangle_{L^2(\Omega)} \\
&\qquad + \langle[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau) - [d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau'), \Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\Big| \\
&\leqslant \sum_{n=n_\tau+1}^{n_{\tau'}} \delta t^{n+\frac12}\Big|\langle\delta_t[\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}s^\alpha_m](t_{n+1}), \Pi^m_{\mathcal{D}_p}\varphi\rangle_{L^2(\Omega)} + \langle\delta_t[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](t_{n+1}), \Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\Big|.
\end{aligned} \tag{27}$$

From the gradient scheme discrete variational equation (9a), we deduce that

$$\begin{aligned}
&\Big|\langle\delta_t[\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}s^\alpha_m](t_{n+1}), \Pi^m_{\mathcal{D}_p}\varphi\rangle_{L^2(\Omega)} + \langle\delta_t[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](t_{n+1}), \Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\Big| \\
&\quad\lesssim \|\nabla^m_{\mathcal{D}_p}p^\alpha_{n+1}\|_{L^2(\Omega)}\,\|\nabla^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)} + \|(d^{n+1}_{f,\mathcal{D}_\mathbf{u}})^{3/2}\nabla^f_{\mathcal{D}_p}p^\alpha_{n+1}\|_{L^2(\Gamma)}\,\|(d^{n+1}_{f,\mathcal{D}_\mathbf{u}})^{3/2}\nabla^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)} \\
&\qquad + \Big\|\frac{1}{\delta t^{n+\frac12}}\int_{t_n}^{t_{n+1}} h^\alpha_m(t,\cdot)\mathrm{d}t\Big\|_{L^2(\Omega)}\,\|\Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)} \\
&\qquad + \Big\|\frac{1}{\delta t^{n+\frac12}}\int_{t_n}^{t_{n+1}} h^\alpha_f(t,\cdot)\mathrm{d}t\Big\|_{L^2(\Gamma)}\,\|\Pi^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)} \\
&\quad\lesssim \xi^{(1),\alpha,n+1}_m\|\nabla^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)} + \xi^{(1),\alpha,n+1}_f\|\nabla^f_{\mathcal{D}_p}\varphi\|_{L^8(\Gamma)} \\
&\qquad + \xi^{(2),\alpha,n+1}_m\|\Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)} + \xi^{(2),\alpha,n+1}_f\|\Pi^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)},
\end{aligned} \tag{28}$$

where the term $\|(d^{n+1}_{f,\mathcal{D}_\mathbf{u}})^{3/2}\nabla^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)}$ has been estimated using the generalized Hölder inequality with exponents $(8, 8/3)$, which satisfy $\frac18 + \frac38 = \frac12$. Hence the result follows from (27), (28), the a priori estimates of Lemma 4.2, and from the assumptions $h^\alpha_m \in L^2((0,T)\times\Omega)$, $h^\alpha_f \in L^2((0,T)\times\Gamma)$. □

*Remark* 4.4. Summing the estimate (26) on $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$ we obtain the following time translate estimates on $\phi_{\mathcal{D}}$ and $d_{f,\mathcal{D}_\mathbf{u}}$:

$$\begin{aligned}
&\Big|\langle\phi_{\mathcal{D}}(\tau) - \phi_{\mathcal{D}}(\tau'), \Pi^m_{\mathcal{D}_p}\varphi\rangle_{L^2(\Omega)} + \langle d_{f,\mathcal{D}_\mathbf{u}}(\tau) - d_{f,\mathcal{D}_\mathbf{u}}(\tau'), \Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\Big| \\
&\lesssim \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}}\sum_{n=n_\tau+1}^{n_{\tau'}} \delta t^{n+\frac12}\Big(\xi^{(1),\alpha,n+1}_m\|\nabla^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)} + \xi^{(1),\alpha,n+1}_f\|\nabla^f_{\mathcal{D}_p}\varphi\|_{L^8(\Gamma)} \\
&\qquad\qquad + \xi^{(2),\alpha,n+1}_m\|\Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)} + \xi^{(2),\alpha,n+1}_f\|\Pi^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)}\Big).
\end{aligned} \tag{29}$$

### 4.2.2 Compactness properties of $\Pi^m_{\mathcal{D}_p}s^\alpha_m$

**Proposition 4.5.** *Let $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$, $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$, $\{(t^l_n)_{n=0}^{N^l}\}_{l\in\mathbb{N}}$ be sequences of space time GDs assumed to satisfy the coercivity and compactness properties, and such that $\lim_{l\to+\infty}\Delta t^l = 0$. Let $\phi_{m,\min} > 0$ and assume that, for each $l \in \mathbb{N}$, the gradient scheme (9a)–(9b) has a solution $p^\alpha_l \in (X^0_{\mathcal{D}^l_p})^{N^l+1}$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, $\mathbf{u}^l \in (X^0_{\mathcal{D}^l_\mathbf{u}})^{N^l+1}$ such that $\phi_{\mathcal{D}^l}(t,\mathbf{x}) \geqslant \phi_{m,\min}$ for a.e. $(t,\mathbf{x}) \in (0,T)\times\Omega$ and $d_{f,\mathcal{D}^l_\mathbf{u}}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$ for a.e. $(t,\mathbf{x}) \in (0,T)\times\Gamma$. Then, the sequence $(\Pi^m_{\mathcal{D}_p}s^{\alpha,l}_m)_{l\in\mathbb{N}}$, with $s^{\alpha,l}_m = S^\alpha_m(p^l_c)$, is relatively compact in $L^2((0,T)\times\Omega)$.*

*Proof.* Let $K$ be a fixed compact set of $\Omega\backslash\Gamma$ and let us consider cut-off functions $\psi^l$ as defined in the cut-off property of the sequence of spatial GDs $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$. The superscript $l \in \mathbb{N}$ will be dropped in the proof, and assumed to be large enough. All hidden constants in the following estimates are

independent of $l$. Using that $\phi_{\mathcal{D}}(t,\mathbf{x}) \geqslant \phi_{m,\min}$ for a.e. $(t,\mathbf{x}) \in (0,T)\times\Omega$, the properties of the cut-off functions, and noting that $\Pi^m_{\mathcal{D}_p} s^{\alpha,l}_m = S^\alpha_m(\Pi^m_{\mathcal{D}_p} p^l_c) \in [0,1]$, we obtain

$$\begin{aligned}
&\int_0^T \|\Pi^m_{\mathcal{D}_p} s^\alpha_m(\cdot+\tau,\cdot) - \Pi^m_{\mathcal{D}_p} s^\alpha_m\|^2_{L^2(K)}\mathrm{d}t \\
&\lesssim \tau + \int_0^{T-\tau}\int_\Omega (\Pi^m_{\mathcal{D}_p}\psi)\ \phi_{\mathcal{D}}\left(\Pi^m_{\mathcal{D}_p} s^\alpha_m(\cdot+\tau,\cdot) - \Pi^m_{\mathcal{D}_p} s^\alpha_m\right)^2 \mathrm{d}\mathbf{x}\mathrm{d}t = \tau + T_1 + T_2,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \int_0^{T-\tau} \left|\langle [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s^\alpha_m](t+\tau) - [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s^\alpha_m](t), \Pi^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\rangle_{L^2(\Omega)}\right|\mathrm{d}t, \\
T_2 &= \int_0^{T-\tau} \left|\langle \phi_{\mathcal{D}}(t+\tau) - \phi_{\mathcal{D}}(t), \Pi^m_{\mathcal{D}_p}\chi^\alpha_m(t)\rangle_{L^2(\Omega)}\right|\mathrm{d}t,
\end{aligned}$$

with $\zeta^\alpha_m(t) = \left(s^\alpha_m(t+\tau) - s^\alpha_m(t)\right)\psi$ and $\chi^\alpha_m(t) = \zeta^\alpha_m(t)\ s^\alpha_m(t+\tau)$. From the cut-off property it results that $\Pi^f_{\mathcal{D}_p}\zeta^\alpha_m = 0$ and $\nabla^f_{\mathcal{D}_p}\zeta^\alpha_m = 0$. Then, in view of the estimates (26), we have

$$T_1 \lesssim \int_0^{T-\tau} \sum_{n=n_t+1}^{n_{(t+\tau)}} \delta t^{n+\frac{1}{2}} \left(\xi^{(1),\alpha,n+1}_m \|\nabla^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\|_{L^2(\Omega)} + \xi^{(2),\alpha,n+1}_m \|\Pi^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\|_{L^2(\Omega)}\right)\mathrm{d}t$$

$$\lesssim \int_0^{T-\tau} \sum_{n=n_t+1}^{n_{(t+\tau)}} \delta t^{n+\frac{1}{2}} \left((\xi^{(1),\alpha,n+1}_m)^2 + (\xi^{(2),\alpha,n+1}_m)^2 + \|\nabla^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\|^2_{L^2(\Omega)} + \|\Pi^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\|^2_{L^2(\Omega)}\right)\mathrm{d}t.$$

From Proposition 4.3, we have

$$\sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \left((\xi^{(1),\alpha,n+1}_m)^2 + (\xi^{(2),\alpha,n+1}_m)^2\right) \lesssim 1.$$

Using the a priori estimates of Lemma 4.2, $h^\alpha_m \in L^2((0,T)\times\Omega)$, the Lipschitz property of $S^\alpha_m$, the chain rule and product rule estimates on the sequence of GDs $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$, and the cut-off property, we obtain that

$$\int_0^{T-\tau} \left(\|\nabla^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\|^2_{L^2(\Omega)} + \|\Pi^m_{\mathcal{D}_p}\zeta^\alpha_m(t)\|^2_{L^2(\Omega)}\right)\mathrm{d}t \lesssim 1.$$

We deduce from [5, Lemma 4.1] that $T_1 \lesssim \tau + \Delta t$ with a hidden constant depending on $K$ but independent of $l$. Similarly, using the time translate estimate (29), one shows that $T_2 \lesssim \tau + \Delta t$, which provides the time translates estimates on $\Pi^m_{\mathcal{D}_p} s^\alpha_m$ in $L^2(0,T;L^2(K))$.

The space translates estimates for $\Pi^m_{\mathcal{D}_p} s^\alpha_m$ in $L^2(0,T;L^2(K))$ derive from the a priori estimates of Lemma 4.2, the Lipschitz properties of $S^\alpha_m$ and from the compactness property of the sequence of spatial GDs $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$ (cf. Remark 3.2). Combined with the time translate estimates, the Fréchet–Kolmogorov theorem implies that $\Pi^m_{\mathcal{D}_p} s^\alpha_m$ is relatively compact in $L^2(0,T;L^2(K))$ for any compact set $K$ of $\Omega\backslash\Gamma$. Since $\Pi^m_{\mathcal{D}_p} s^\alpha_m \in [0,1]$, it results that $\Pi^m_{\mathcal{D}_p} s^\alpha_m$ is relatively compact in $L^2((0,T)\times\Omega)$. □

#### 4.2.3 Uniform-in-time $L^2$-weak convergence of $\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s^\alpha_m$ and $\phi_{\mathcal{D}}$

**Proposition 4.6.** *Let $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$, $(\mathcal{D}^l_{\mathbf{u}})_{l\in\mathbb{N}}$, $\{(t^l_n)^{N^l}_{n=0}\}_{l\in\mathbb{N}}$ be sequences of space time GDs assumed to satisfy the coercivity and consistency properties. Let $0 < \phi_{m,\min} \leqslant \phi_{m,\max} < +\infty$ and assume that, for each $l \in \mathbb{N}$, the gradient scheme (9a)–(9b) has a solution $p^\alpha_l \in (X^0_{\mathcal{D}^l_p})^{N^l+1}$, $\alpha \in \{\mathrm{nw},\mathrm{w}\}$, $\mathbf{u}^l \in (X^0_{\mathcal{D}^l_{\mathbf{u}}})^{N^l+1}$ such that*

*(i)* $d_{f,\mathcal{D}_{\mathbf{u}}^l}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Gamma$,

*(ii)* $\phi_{m,\min} \leqslant \phi_{\mathcal{D}^l}(t,\mathbf{x}) \leqslant \phi_{m,\max}$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Omega$.

*Then, up to a subsequence, the sequences* $(\phi_{\mathcal{D}^l})_{l\in\mathbb{N}}$ *and* $(\phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p} s_m^{\alpha,l})_{l\in\mathbb{N}}$, *with* $s_m^{\alpha,l} = S_m^\alpha(p_c^l)$, *converge uniformly in time weakly in* $L^2(\Omega)$.

*Proof.* Let $K$ be a fixed compact set of $\Omega\backslash\Gamma$ and let $\psi^l$ be cut-off functions for this compact set, as defined in the cut-off property of $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$. The superscript $l\in\mathbb{N}$ will be dropped when not required for the clarity of the proof.

For $w\in V_0$ we let $P_{\mathcal{D}_p} w \in X^0_{\mathcal{D}_p}$ be the element that realises the minimum in $\mathcal{S}_{\mathcal{D}_p}(w)$, so that

$$\begin{aligned}\|\nabla^m_{\mathcal{D}_p} P_{\mathcal{D}_p} w - \nabla w\|_{L^2(\Omega)} + \|\nabla^f_{\mathcal{D}_p} P_{\mathcal{D}_p} w - \nabla_\tau \gamma w\|_{L^r(\Gamma)} \\ + \|\Pi^m_{\mathcal{D}_p} P_{\mathcal{D}_p} w - w\|_{L^2(\Omega)} + \|\Pi^f_{\mathcal{D}_p} P_{\mathcal{D}_p} w - \gamma w\|_{L^r(\Gamma)} = \mathcal{S}_{\mathcal{D}_p}(w). \quad (30)\end{aligned}$$

Let $\overline{\varphi}\in C_c^\infty(\Omega)$ and set $\varphi = P_{\mathcal{D}_p}\overline{\varphi}$. It results from the cut-off property that $\Pi^f_{\mathcal{D}_p}(\psi\varphi) = 0$ and $\nabla^f_{\mathcal{D}_p}(\psi\varphi) = 0$. Using the GD consistency property of $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$ and (30), we see that $\|\nabla^m_{\mathcal{D}_p}(\psi\varphi)\|_{L^2(\Omega)}$ and $\|\Pi^m_{\mathcal{D}_p}(\psi\varphi)\|_{L^2(\Omega)}$ are bounded by constants depending on $K$ and $\overline{\varphi}$ but independent of $l$. Then, from Proposition 4.3, we have with hidden constants independent of $l$ but possibly depending on $K$ and $\overline{\varphi}$, that

$$\begin{aligned}&\left|\langle \Pi^m_{\mathcal{D}_p}\psi \left([\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha](\tau) - [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha](\tau')\right), \Pi^m_{\mathcal{D}_p}\varphi\rangle_{L^2(\Omega)}\right| \\ &= \left|\langle [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha](\tau) - [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha](\tau'), \Pi^m_{\mathcal{D}_p}(\psi\varphi)\rangle_{L^2(\Omega)}\right| \\ &\lesssim \sum_{n=n_\tau+1}^{n_{\tau'}} \delta t^{n+\frac12}\left(\xi_m^{(1),\alpha,n+1}\|\nabla^m_{\mathcal{D}_p}(\psi\varphi)\|_{L^2(\Omega)} + \xi_m^{(2),\alpha,n+1}\|\Pi^m_{\mathcal{D}_p}(\psi\varphi)\|_{L^2(\Omega)}\right) \\ &\lesssim \left(\sum_{n=n_\tau+1}^{n_{\tau'}} \delta t^{n+\frac12}\left(\left(\xi_m^{(1),\alpha,n+1}\right)^2 + \left(\xi_m^{(2),\alpha,n+1}\right)^2\right)\right)^{\frac12}\left(\sum_{n=n_\tau+1}^{n_{\tau'}}\delta t^{n+\frac12}\right)^{\frac12} \\ &\lesssim |\tau-\tau'|^{\frac12} + \Delta t^{\frac12}.\end{aligned}$$

Using that $\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha \leqslant \phi_{m,\max}$, and that $\Pi^m_{\mathcal{D}_p}\psi$ is uniformly bounded, one has

$$\left|\langle \Pi^m_{\mathcal{D}_p}\psi \left([\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha](\tau) - [\phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p} s_m^\alpha](\tau')\right), \overline{\varphi}\rangle_{L^2(\Omega)}\right| \lesssim |\tau-\tau'|^{\frac12} + \Delta t^{\frac12} + \omega_{\mathcal{D}_p}, \quad (31)$$

with $\omega_{\mathcal{D}_p} = \|\overline{\varphi} - \Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)}$ a consistency error term such that $\lim_{l\to+\infty}\omega_{\mathcal{D}_p^l} = 0$. It follows from the discontinuous Ascoli-Arzelà theorem [20, Theorem C.11] that (up to a subsequence) the sequence $(\Pi^m_{\mathcal{D}_p}\psi)\phi_{\mathcal{D}}(\Pi^m_{\mathcal{D}_p} s_m^\alpha) = \phi_{\mathcal{D}}\Pi^m_{\mathcal{D}_p}(s_m^\alpha\psi)$ converges uniformly in time weakly in $L^2(\Omega)$.

Let us now take $w\in C_c^\infty(\Omega\backslash\Gamma)$ and let $K$ be the support of $w$. For $l$ large enough, by definition of $\psi^l$ we have $\left(\phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p^l} s_m^{\alpha,l}\right)|_K = \phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p^l}(\psi^l s_m^{\alpha,l})$. Hence,

$$\langle \phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p^l} s_m^{\alpha,l}, w\rangle_{L^2(\Omega)} \text{ converges uniformly with respect to } t\in[0,T]. \quad (32)$$

Since $(\phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p^l} s_m^{\alpha,l})_{l\in\mathbb{N}}$ is uniformly bounded (it takes values in $[\phi_{m,\min},\phi_{m,\max}]$), the density of $C_c^\infty(\Omega\backslash\Gamma)$ in $L^2(\Omega)$ shows that the convergence (32) is valid for any $w\in L^2(\Omega)$, which concludes the proof that the sequence $\phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p^l} s_m^{\alpha,l}$ converges uniformly in time, weakly in $L^2(\Omega)$.

We deduce that the sequence $\phi_{\mathcal{D}^l} = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}}\phi_{\mathcal{D}^l}\Pi^m_{\mathcal{D}_p^l} s_m^{\alpha,l}$ also converges uniformly in time, weakly in $L^2(\Omega)$. □

#### 4.2.4 Uniform-in-time $L^2$-weak convergence of $d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f$ and $d_{f,\mathcal{D}_\mathbf{u}}$

**Proposition 4.7.** *Let $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$, $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$, $\{(t^l_n)_{n=0}^{N^l}\}_{l\in\mathbb{N}}$ be sequences of space time GDs assumed to satisfy the coercivity and consistency properties. Let $0<\phi_{m,\min}\leqslant\phi_{m,\max}<+\infty$ and assume that, for each $l\in\mathbb{N}$, the gradient scheme (9a)–(9b) has a solution $p^\alpha_l\in(X^0_{\mathcal{D}^l_p})^{N^l+1}$, $\alpha\in\{\mathrm{nw},\mathrm{w}\}$, $\mathbf{u}^l\in(X^0_{\mathcal{D}^l_\mathbf{u}})^{N^l+1}$ such that*

*(i)* $d_{f,\mathcal{D}^l_\mathbf{u}}(t,\mathbf{x})\geqslant d_0(\mathbf{x})$ *for a.e.* $(t,\mathbf{x})\in(0,T)\times\Gamma$,

*(ii)* $\phi_{m,\min}\leqslant\phi_{\mathcal{D}^l}(t,\mathbf{x})\leqslant\phi_{m,\max}$ *for a.e.* $(t,\mathbf{x})\in(0,T)\times\Omega$.

*Then, the sequences $(d_{f,\mathcal{D}^l_\mathbf{u}})_{l\in\mathbb{N}}$ and $(d_{f,\mathcal{D}^l_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^{\alpha,l}_f)_{l\in\mathbb{N}}$, with $s^{\alpha,l}_f=S^\alpha_f(p^l_c)$ converge, up to a subsequence, uniformly in time weakly in $L^2(\Gamma)$.*

*Proof.* Let $K$ be a fixed compact set of $\Omega\backslash\Gamma$ and let us consider cut-off functions $\psi^l$ as defined in the cut-off property of $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$. In the following, the superscript $l\in\mathbb{N}$ is dropped when not required for the clarity of the proof, and the hidden constants are independent of $l$. Let $\overline{\varphi}\in C^\infty_c(\Omega)$ and set $\varphi=P_{\mathcal{D}_p}\overline{\varphi}$, with $P_{\mathcal{D}_p}$ characterised by (30). From Proposition 4.3 we have

$$
\begin{aligned}
&\left|\langle[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau)-[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau'),\Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\right|\\
&\lesssim\left|\left\langle\left([\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau)-[\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau')\right),\Pi^m_{\mathcal{D}_p}\varphi\right\rangle_{L^2(\Omega)}\right|\\
&\quad+\max\left(\|\nabla^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)},\|\nabla^f_{\mathcal{D}_p}\varphi\|_{L^8(\Gamma)},\|\Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)},\|\Pi^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)}\right)\\
&\qquad\times\left(\sum_{n=n_\tau+1}^{n_{\tau'}}\delta t^{n+\frac12}\left(\left(\xi^{(1),\alpha,n+1}_m\right)^2+\left(\xi^{(1),\alpha,n+1}_f\right)^2+\left(\xi^{(2),\alpha,n+1}_m\right)^2+\left(\xi^{(2),\alpha,n+1}_f\right)^2\right)\right)^{\frac12}\\
&\qquad\times\left(\sum_{n=n_\tau+1}^{n_{\tau'}}\delta t^{n+\frac12}\right)^{\frac12}\\
&\lesssim\left(|\tau-\tau'|^{\frac12}+\Delta t^{\frac12}\right)+\left|\left\langle\left([\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau)-[\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau')\right),\Pi^m_{\mathcal{D}_p}\varphi\right\rangle_{L^2(\Omega)}\right|.
\end{aligned}
$$

Using Proposition 4.6, $\phi_\mathcal{D}\leqslant\phi_{m,\max}$ and $\Pi^m_{\mathcal{D}_p}s^\alpha_m\in[0,1]$, we have

$$
\begin{aligned}
&\left|\left\langle\left([\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau)-[\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau')\right),\Pi^m_{\mathcal{D}_p}\varphi\right\rangle_{L^2(\Omega)}\right|\\
&\lesssim\phi_{m,\max}\|\bar\varphi-\Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)}+\left|\langle[\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau)-[\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau'),\bar\varphi\rangle_{L^2(\Omega)}\right|
\end{aligned}
$$

and

$$
\left|\langle[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau)-[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau'),\bar\varphi-\Pi^f_{\mathcal{D}_p}\varphi\rangle_{L^2(\Gamma)}\right|\lesssim\|d_{f,\mathcal{D}_\mathbf{u}}\|_{L^\infty(0,T;L^2(\Gamma))}\|\bar\varphi-\Pi^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)}.
$$

Using the a priori estimates of Lemma 4.2, and Proposition 4.6 stating the uniform-in-time $L^2(\Omega)$-weak convergence of $\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m$ (which implies the equi-continuity of the functions $\tau\mapsto\langle[\phi_\mathcal{D}\Pi^m_{\mathcal{D}_p}s^\alpha_m](\tau),\bar\varphi\rangle_{L^2(\Omega)}$), we deduce that

$$
\left|\langle[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau)-[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f](\tau'),\bar\varphi\rangle_{L^2(\Gamma)}\right|\lesssim\omega(|\tau-\tau'|)+\Delta t^{\frac12}+\varpi_{\mathcal{D}_p},
$$

with $\lim_{h\to0}\omega(h)=0$ and $\varpi_{\mathcal{D}_p}=\|\overline{\varphi}-\Pi^m_{\mathcal{D}_p}\varphi\|_{L^2(\Omega)}+\|\bar\varphi-\Pi^f_{\mathcal{D}_p}\varphi\|_{L^2(\Gamma)}$ a consistency error term such that $\lim_{l\to+\infty}\varpi_{\mathcal{D}^l_p}=0$. It follows from the discontinuous Ascoli-Arzelà theorem [20, Theorem C.11] that (up to a subsequence) the sequence $d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p}s^\alpha_f$ converges uniformly in time weakly in $L^2(\Gamma)$. Summing over $\alpha\in\{\mathrm{nw},\mathrm{w}\}$, we also deduce the uniform-in-time $L^2(\Gamma)$-weak convergence of $d_{f,\mathcal{D}_\mathbf{u}}$. □

#### 4.2.5 Strong convergence of $d_{f,\mathcal{D}_\mathbf{u}}$, $d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f$, and $\Pi^f_{\mathcal{D}_p} s^\alpha_f$

**Proposition 4.8.** *Let $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$, $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$, $\{(t^l_n)_{n=0}^{N^l}\}_{l\in\mathbb{N}}$ be sequences of space time GDs assumed to satisfy the coercivity, consistency and compactness properties. Let $0 < \phi_{m,\min} \leqslant \phi_{m,\max} < +\infty$ and assume that, for each $l \in \mathbb{N}$, the gradient scheme (9a)–(9b) has a solution $p^\alpha_l \in (X^0_{\mathcal{D}^l_p})^{N^l+1}$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, $\mathbf{u}^l \in (X^0_{\mathcal{D}^l_\mathbf{u}})^{N^l+1}$ such that*

*(i)* $d_{f,\mathcal{D}^l_\mathbf{u}}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Gamma$,

*(ii)* $\phi_{m,\min} \leqslant \phi_{\mathcal{D}^l}(t,\mathbf{x}) \leqslant \phi_{m,\max}$ *for a.e.* $(t,\mathbf{x}) \in (0,T)\times\Omega$.

*Then, the sequence $(d_{f,\mathcal{D}^l_\mathbf{u}})_{l\in\mathbb{N}}$ converges up to a subsequence in $L^\infty(0,T;L^p(\Gamma))$ for all $2 \leqslant p < 4$, and the sequences $(d_{f,\mathcal{D}^l_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^{\alpha,l}_f)_{l\in\mathbb{N}}$ and $(\Pi^f_{\mathcal{D}_p} s^{\alpha,l}_f)_{l\in\mathbb{N}}$, with $s^{\alpha,l}_f = S^\alpha_f(p^l_c)$ converge, up to a subsequence, in $L^4(0,T;L^2(\Gamma))$.*

*Proof.* By the characterization in Remark 3.4 of the compactness of $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$ and the estimate on $\mathfrak{c}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})$ in Lemma 4.2, we have, for all $i \in I$, all $\eta_i$ tangent to $\Gamma_i$, a.e. $t \in (0,T)$ and all $s < 4$,

$$\left\|d_{f,\mathcal{D}^l_\mathbf{u}}(t,\cdot+\eta_i) - d_{f,\mathcal{D}^l_\mathbf{u}}(t,\cdot)\right\|_{L^s(\Gamma_i)} \leqslant T_{\mathcal{D}^l_\mathbf{u},s}(0,\eta)\|\mathfrak{c}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})(t,\cdot)\|_{L^2(\Omega,\mathcal{S}_d(\mathbb{R}))} \lesssim T_{\mathcal{D}^l_\mathbf{u},s}(0,\eta),$$

where $\eta = (0,\dots,0,\eta_i,0,\dots,0)$ and $d_{f,\mathcal{D}^l_\mathbf{u}}$ has been extended by 0 in the hyperplane spanned by $\Gamma_i$. Together with the uniform-in-time $L^2(\Gamma)$-weak convergence of $d_{f,\mathcal{D}^l_\mathbf{u}}$ from Proposition 4.7, this shows that we can apply Lemma A.2 to $d_{f,\mathcal{D}^l_\mathbf{u}}$ with $p = +\infty$ and get the convergence of this sequence in $L^\infty(0,T;L^2(\Gamma))$. Since, from the a priori estimates of Lemma 4.2, this sequence $d_{f,\mathcal{D}^l_\mathbf{u}}$ is bounded in $L^\infty(0,T;L^4(\Gamma))$, it follows that it converges in $L^\infty(0,T;L^q(\Gamma))$ for all $2 \leqslant q < 4$.

For any compact set $K_f \subset \Gamma$ that is disjoint from the intersections $(\overline{\Gamma}_i \cap \overline{\Gamma}_j)_{i\neq j}$, using that $\Pi^f_{\mathcal{D}_p} s^\alpha_f \in [0,1]$, that $\|d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot)\|_{L^4(\Gamma)}$ is uniformly bounded in $t$, and the Lipschitz properties of $S^\alpha_f$, it follows that, for all $i \in I$ and $\eta_i$ tangent to $\Gamma_i$ small enough,

$$\begin{aligned}
&\|[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f](t,\cdot+\eta_i) - [d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f](t,\cdot)\|_{L^2(K_f\cap\Gamma_i)}\\
&\leqslant \|d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot+\eta_i) - d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot)\|_{L^2(K_f\cap\Gamma_i)}\\
&\qquad + \|\Pi^f_{\mathcal{D}_p} s^\alpha_f(t,\cdot+\eta_i) - \Pi^f_{\mathcal{D}_p} s^\alpha_f(t,\cdot)\|_{L^4(K_f\cap\Gamma_i)}\|d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot)\|_{L^4(K_f\cap\Gamma_i)}\\
&\lesssim \|d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot+\eta_i) - d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot)\|_{L^2(K_f\cap\Gamma_i)} + \|\Pi^f_{\mathcal{D}_p} s^\alpha_f(t,\cdot+\eta_i) - \Pi^f_{\mathcal{D}_p} s^\alpha_f(t,\cdot)\|^{\frac12}_{L^2(K_f\cap\Gamma_i)}\\
&\lesssim \|d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot+\eta_i) - d_{f,\mathcal{D}_\mathbf{u}}(t,\cdot)\|_{L^2(K_f\cap\Gamma_i)} + \|\Pi^f_{\mathcal{D}_p} p_c(t,\cdot+\eta_i) - \Pi^f_{\mathcal{D}_p} p_c(t,\cdot)\|^{\frac12}_{L^2(K_f\cap\Gamma_i)}.
\end{aligned}$$

From the compactness properties of $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$ and $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$ (see Remarks 3.2 and 3.4) it results that

$$\begin{aligned}
&\sum_{i\in I}\Big\|\sup_{|\eta_i|\leqslant\delta}\|[d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f](\cdot,\cdot+\eta_i) - [d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f](\cdot,\cdot)\|_{L^2(K_f\cap\Gamma_i)}\Big\|_{L^4(0,T)}\\
&\lesssim T_{K_f}(\delta)\Big(\|\mathfrak{c}_{\mathcal{D}_\mathbf{u}}(\mathbf{u})\|_{L^\infty(0,T;L^2(\Omega))} + \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}}(\|d_0^{3/2}\nabla^f_{\mathcal{D}_p} p^\alpha\|_{L^2(0,T;L^2(\Gamma))} + \|\nabla^m_{\mathcal{D}_p} p^\alpha\|_{L^2(0,T;L^2(\Omega))})\Big)
\end{aligned}$$

with $\lim_{\delta\to0} T_{K_f}(\delta) = 0$. From the a priori estimates of Lemma 4.2, and the uniform-in-time $L^2(\Gamma)$-weak convergence of $d_{f,\mathcal{D}_\mathbf{u}} s^\alpha_f$ of Proposition 4.7, it follows from Lemma A.2 that the sequence $d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f$ converges up to a subsequence in $L^4(0,T;L^2(K_f))$.

From the assumption $d_{f,\mathcal{D}_\mathbf{u}}(t,\mathbf{x}) \geqslant d_0(\mathbf{x})$, the sequence $d_{f,\mathcal{D}_\mathbf{u}}$ is bounded from below by a strictly positive constant on $K_f$. Writing that $\Pi^f_{\mathcal{D}_p} s^\alpha_f = \frac{1}{d_{f,\mathcal{D}_\mathbf{u}}}(d_{f,\mathcal{D}_\mathbf{u}}\Pi^f_{\mathcal{D}_p} s^\alpha_f)$, it follows that the sequence $\Pi^f_{\mathcal{D}_p} s^\alpha_f$ converges in $L^4(0,T;L^2(K_f))$. Since this is true for any $K_f$ compact in $\Gamma$ that does not touch the fractures intersections, and since $\Pi^f_{\mathcal{D}_p} s^\alpha_f \in [0,1]$, we deduce that the sequence $\Pi^f_{\mathcal{D}_p} s^\alpha_f$ converges in $L^4(0,T;L^2(\Gamma))$. □

### 4.3 Convergence to a weak solution

*Proof of Theorem 4.1.* The superscript $l$ will be dropped in the proof, and all convergences are up to appropriate subsequences. From Lemma 4.2 and Proposition 4.8, there exist $\bar{d}_f \in L^\infty(0,T;L^4(\Gamma))$ and $\bar{s}_f^\alpha \in L^\infty((0,T)\times\Gamma)$ such that

$$\begin{aligned} &d_{f,\mathcal{D}_\mathbf{u}} \to \bar{d}_f && \text{in } L^\infty(0,T;L^p(\Gamma)),\ 2\leqslant p<4,\\ &\Pi^f_{D_p} S_f^\alpha(p_c) \to \bar{s}_f^\alpha && \text{in } L^4(0,T;L^2(\Gamma)). \end{aligned} \tag{33}$$

From Proposition 4.5, there exists $\bar{s}_m^\alpha \in L^\infty((0,T)\times\Omega)$ such that

$$\Pi^m_{D_p} S_m^\alpha(p_c) \to \bar{s}_m^\alpha \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{34}$$

The identification of the limit [11, Lemma 5.5], resulting from the limit-conformity property, can easily be adapted to our definition of $V_0$, with weight $d_0^{3/2}$ and the use in the definition of limit-conformity of fracture flux functions that are compactly supported away from the tips. Using this lemma and the a priori estimates of Lemma 4.2, we obtain $\bar{p}^\alpha \in L^2(0,T;V_0)$ and $\mathbf{g}_f^\alpha \in L^2(0,T;L^2(\Gamma)^{d-1})$, such that the following weak limits hold

$$\begin{aligned} &\Pi^m_{\mathcal{D}_p} p^\alpha \rightharpoonup \bar{p}^\alpha && \text{in } L^2(0,T;L^2(\Omega)),\\ &\Pi^f_{\mathcal{D}_p} p^\alpha \rightharpoonup \gamma\bar{p}^\alpha && \text{in } L^2(0,T;L^2(\Gamma)),\\ &\nabla^m_{\mathcal{D}_p} p^\alpha \rightharpoonup \nabla\bar{p}^\alpha && \text{in } L^2(0,T;L^2(\Omega)^d),\\ &d_0^{3/2}\nabla^f_{\mathcal{D}_p} p^\alpha \rightharpoonup d_0^{3/2}\nabla_\tau\gamma\bar{p}^\alpha && \text{in } L^2(0,T;L^2(\Gamma)^{d-1}),\\ &d_{f,\mathcal{D}_\mathbf{u}}^{3/2}\nabla^f_{\mathcal{D}_p} p^\alpha \rightharpoonup \mathbf{g}_f^\alpha && \text{in } L^2(0,T;L^2(\Gamma)^{d-1}). \end{aligned} \tag{35}$$

Let $\boldsymbol{\varphi} \in C_c^0((0,T)\times\Gamma)^{d-1}$ whose support is contained in $(0,T)\times K$, with $K$ compact set not containing the tips of $\Gamma$. We have

$$\int_0^T\int_\Gamma d_{f,\mathcal{D}_\mathbf{u}}^{3/2}\nabla^f_{\mathcal{D}_p} p^\alpha\cdot\boldsymbol{\varphi}\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \to \int_0^T\int_\Gamma \mathbf{g}_f^\alpha\cdot\boldsymbol{\varphi}\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t.$$

On the other hand, it results from (35) and the fact that $d_0$ is bounded away from 0 on $K$ (because $d_0$ is continuous and does not vanish outside the tips of $\Gamma$) that $\nabla^f_{\mathcal{D}_p} p^\alpha \rightharpoonup \nabla_\tau\gamma\bar{p}^\alpha$ in $L^2(0,T;L^2(K)^{d-1})$. Combined with the convergence $d_{f,\mathcal{D}_\mathbf{u}}^{3/2}\boldsymbol{\varphi} \to (\bar{d}_f)^{3/2}\boldsymbol{\varphi}$ in $L^\infty(0,T;L^2(\Gamma)^{d-1})$ given by (33), we infer that

$$\int_0^T\int_\Gamma d_{f,\mathcal{D}_\mathbf{u}}^{3/2}\nabla^f_{\mathcal{D}_p} p^\alpha\cdot\boldsymbol{\varphi}\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t \to \int_0^T\int_\Gamma (\bar{d}_f)^{3/2}\nabla_\tau\gamma\bar{p}^\alpha\cdot\boldsymbol{\varphi}\,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t.$$

This shows that $\mathbf{g}_f^\alpha = (\bar{d}_f)^{3/2}\nabla_\tau\gamma\bar{p}^\alpha$ on $(0,T)\times\Gamma$.

Combining the strong convergence of $\Pi^m_{D_p} S_m^\alpha(p_c) = S_m^\alpha(\Pi^m_{D_p} p_c)$ (resp. of $\Pi^f_{D_p} S_f^\alpha(p_c) = S_f^\alpha(\Pi^f_{D_p} p_c)$), the weak convergence of $\Pi^m_{D_p} p_c$ (resp. $\Pi^f_{D_p} p_c$), and the monotonicity of $S_m^\alpha$ (resp. $S_f^\alpha$), it results from the Minty trick (see e.g. [24, Lemma 2.6]) that $\bar{s}_m^\alpha = S_m^\alpha(\bar{p}_c)$ (resp. $\bar{s}_f^\alpha = S_f^\alpha(\gamma\bar{p}_c)$) with $\bar{p}_c = \bar{p}^{\mathrm{nw}} - \bar{p}^{\mathrm{w}}$.

From the a priori estimates of Lemma 4.2 and the limit-conformity property of the sequence of GDs $(\mathcal{D}_\mathbf{u}^l)_{l\in\mathbb{N}}$ (see Lemma A.3), there exists $\bar{\mathbf{u}} \in L^\infty(0,T;\mathbf{U}_0)$, such that

$$\begin{aligned} &\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{u} \rightharpoonup \bar{\mathbf{u}} && \text{in } L^\infty(0,T;L^2(\Omega)^d) \text{ weak }\star,\\ &\mathfrak{C}_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) \rightharpoonup \mathfrak{C}(\bar{\mathbf{u}}) && \text{in } L^\infty(0,T;L^2(\Omega,\mathcal{S}_d(\mathbb{R}))) \text{ weak }\star,\\ &\mathrm{div}_{\mathcal{D}_\mathbf{u}}\mathbf{u} \rightharpoonup \mathrm{div}(\bar{\mathbf{u}}) && \text{in } L^\infty(0,T;L^2(\Omega)) \text{ weak }\star,\\ &d_{f,\mathcal{D}_\mathbf{u}} = -[\![\mathbf{u}]\!]_{\mathcal{D}_\mathbf{u}} \rightharpoonup -[\![\bar{\mathbf{u}}]\!] && \text{in } L^\infty(0,T;L^2(\Gamma)) \text{ weak }\star, \end{aligned} \tag{36}$$

from which we deduce that $\bar{d}_f = -[\![\bar{\mathbf{u}}]\!]$ and that $\varpi_{\mathcal{D}_u}(\mathbf{u})$ converges to $\varpi(\bar{\mathbf{u}})$ in $L^\infty(0,T;L^2(\Omega,\mathcal{S}_d(\mathbb{R})))$ weak $\star$.

From the a priori estimates and the closure equations (9c), there exist $\bar{\phi}_m \in L^\infty(0,T;L^2(\Omega)$ and $\bar{p}_m^E \in L^\infty(0,T;L^2(\Omega)$ such that

$$\begin{array}{ll} \phi_{\mathcal{D}} \rightharpoonup \bar{\phi}_m & \text{in } L^\infty(0,T;L^2(\Omega)) \text{ weak } \star, \\ \Pi_{\mathcal{D}_p}^m p_m^E \rightharpoonup \bar{p}_m^E & \text{in } L^\infty(0,T;L^2(\Omega)) \text{ weak } \star. \end{array} \tag{37}$$

Since $0 \leqslant U_{\rm rt}(p) = \int_0^p q(S_{\rm rt}^{\rm nw})'(q)\mathrm{d}q \leqslant 2|p|$ for ${\rm rt} \in \{m,f\}$, it results from the a priori estimates of Lemma 4.2 that there exist $\bar{p}_f^E \in L^2(0,T;L^2(\Gamma))$, $\bar{U}_f \in L^2(0,T;L^2(\Gamma))$ and $\bar{U}_m \in L^2(0,T;L^2(\Omega))$ such that

$$\begin{array}{ll} \Pi_{\mathcal{D}_p}^f p_f^E \rightharpoonup \bar{p}_f^E & \text{in } L^2(0,T;L^2(\Gamma)), \\ \Pi_{\mathcal{D}_p}^f U_f(p_c) \rightharpoonup \bar{U}_f & \text{in } L^2(0,T;L^2(\Gamma)), \\ \Pi_{\mathcal{D}_p}^m U_m(p_c) \rightharpoonup \bar{U}_m & \text{in } L^2(0,T;L^2(\Omega)). \end{array} \tag{38}$$

For ${\rm rt} \in \{{\rm nw},{\rm w}\}$, it is shown in [21] that $U_{\rm rt}(p) = B_{\rm rt}(S_{\rm rt}^{\rm nw}(p))$ where $s \in [0,1] \mapsto B_{\rm rt}(s) \in (-\infty,+\infty]$ is a convex lower semi-continuous function with finite limits at $s=0$ and $s=1$ (note that $B_{\rm rt}$ is therefore continuous with values in $(-\infty,+\infty]$, the infinite value being only possible at the endpoints). Since $\Pi_{\mathcal{D}_p}^m s_m^{\rm nw}$ converges strongly in $L^2((0,T)\times\Omega)$ to $S_m^{\rm nw}(\bar{p}_c)$, it converges a.e. in $(0,T)\times\Omega$. It results that $B_m(\Pi_{\mathcal{D}_p}^m s_m^{\rm nw})$ converges a.e. in $(0,T)\times\Omega$ to $B_m(S_m^{\rm nw}(\bar{p}_c))$, and hence that $\bar{U}_m = B_m(S_m^{\rm nw}(\bar{p}_c)) = U_m(\bar{p}_c)$. Similarly, $\bar{U}_f = B_f(S_f^{\rm nw}(\gamma\bar{p}_c)) = U_f(\gamma\bar{p}_c)$. We deduce that

$$\bar{p}_m^E = \sum_{\alpha\in\{\rm nw,w\}} \bar{p}^\alpha S_m^\alpha(\bar{p}_c) - U_m(\bar{p}_c) \quad \text{and} \quad \bar{p}_f^E = \sum_{\alpha\in\{\rm nw,w\}} \gamma\bar{p}^\alpha S_f^\alpha(\gamma\bar{p}_c) - U_f(\gamma\bar{p}_c).$$

Using the estimate

$$|U_{\rm rt}(p_2) - U_{\rm rt}(p_1)| = \left|\int_{p_1}^{p_2} q(S_{\rm rt}^{\rm nw})'(q)\mathrm{d}q\right| \leqslant |p_2 - p_1| + |p_2 S_{\rm rt}^{\rm nw}(p_2) - p_1 S_{\rm rt}^{\rm nw}(p_1)|,$$

the Lipschitz property of $S_{\rm rt}^{\rm nw}$, $\bar{p}_0^\alpha \in V_0 \cap L^\infty(\Omega)$, $\gamma\bar{p}_0^\alpha \in L^\infty(\Gamma)$, $\alpha \in \{{\rm nw},{\rm w}\}$, and the consistency of the sequence of GDs $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$, we deduce that

$$\begin{array}{ll} \Pi_{\mathcal{D}_p}^m p_m^{E,0} \to \bar{p}_m^{E,0} & \text{in } L^2(\Omega), \\ \Pi_{\mathcal{D}_p}^f p_f^{E,0} \to \bar{p}_f^{E,0} & \text{in } L^2(\Gamma). \end{array} \tag{39}$$

Then, from Proposition A.4 it holds that

$$\begin{array}{ll} {\rm div}_{\mathcal{D}_{\mathbf{u}}}(\mathbf{u}^0) \to {\rm div}(\bar{\mathbf{u}}^0) & \text{in } L^2(\Omega), \\ [\![\mathbf{u}^0]\!]_{\mathcal{D}_{\mathbf{u}}} \to [\![\bar{\mathbf{u}}^0]\!] = -\bar{d}_f^0 & \text{in } L^2(\Gamma). \end{array} \tag{40}$$

It results from (39) and (40) that

$$\bar{\phi}_m = \bar{\phi}_m^0 + b\ {\rm div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}^0) + \frac{1}{M}(\bar{p}_m^E - \bar{p}_m^{E,0}).$$

Let us now prove that the functions $\bar{p}^\alpha$, $\alpha \in \{{\rm nw},{\rm w}\}$, and $\bar{\mathbf{u}}$ satisfy the variational formulation (6)–(7) by passing to the limit in the gradient scheme (9).

For $\theta \in C_c^\infty([0,T))$ and $\psi \in C_c^\infty(\Omega)$ let us set, with $P_{\mathcal{D}_p}$ characterised by (30),

$$\varphi = (\varphi^1,\dots,\varphi^N) \in (X_{\mathcal{D}_p}^0)^N \text{ with } \varphi^i = \theta(t_{i-1})(P_{\mathcal{D}_p}\psi).$$

From the consistency properties of $(\mathcal{D}_p^l)_{l\in\mathbb{N}}$ with given $r > 8$, we deduce that

$$\begin{array}{llll} \Pi_{\mathcal{D}_p}^m P_{\mathcal{D}_p}\psi \to \psi & \text{in } L^2(\Omega), & \Pi_{\mathcal{D}_p}^f P_{\mathcal{D}_p}\psi \to \gamma\psi & \text{in } L^2(\Gamma), \\ \Pi_{\mathcal{D}_p}^m \varphi \to \theta\psi & \text{in } L^\infty(0,T;L^2(\Omega)), & \Pi_{\mathcal{D}_p}^f \varphi \to \theta\gamma\psi & \text{in } L^\infty(0,T;L^2(\Gamma)), \\ \nabla_{\mathcal{D}_p}^m \varphi \to \theta\nabla\psi & \text{in } L^\infty(0,T;L^2(\Omega)^d), & \nabla_{\mathcal{D}_p}^f \varphi \to \theta\nabla_\tau\gamma\psi & \text{in } L^\infty(0,T;L^r(\Gamma)^{d-1}). \end{array} \tag{41}$$

Setting

$$
\begin{aligned}
T_1 &= \int_0^T \int_\Omega \Big(\delta_t\Big(\phi_{\mathcal{D}} \Pi^m_{\mathcal{D}_p} s^\alpha_m\Big)\Pi^m_{\mathcal{D}_p}\varphi \,\mathrm{d}\mathbf{x}\mathrm{d}t\\
T_2 &= \int_0^T \int_\Omega \eta^\alpha_m(\Pi^m_{\mathcal{D}_p} s^\alpha_m)\mathbb{K}_m \nabla^m_{\mathcal{D}_p} p^\alpha \cdot \nabla^m_{\mathcal{D}_p}\varphi \,\mathrm{d}\mathbf{x}\mathrm{d}t\\
T_3 &= \int_0^T \int_\Gamma \delta_t\Big(d_{f,\mathcal{D}_\mathbf{u}} \Pi^f_{\mathcal{D}_p} s^\alpha_f\Big)\Pi^f_{\mathcal{D}_p}\varphi \,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t\\
T_4 &= \int_0^T \int_\Gamma \eta^\alpha_f(\Pi^f_{\mathcal{D}_p} s^\alpha_f)\frac{d^3_{f,\mathcal{D}_\mathbf{u}}}{12}\nabla^f_{\mathcal{D}_p} p^\alpha \cdot \nabla^f_{\mathcal{D}_p}\varphi \,\mathrm{d}\mathbf{x}\mathrm{d}t\\
T_5 &= \int_0^T \int_\Omega h^\alpha_m \Pi^m_{\mathcal{D}_p}\varphi \,d\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma h^\alpha_f \Pi^f_{\mathcal{D}_p}\varphi \,\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,
\end{aligned}
$$

the gradient scheme variational formulation (9a) states that

$$T_1 + T_2 + T_3 + T_4 = T_5.$$

For $\omega \in C^\infty_c([0,T))$ and a smooth function $\mathbf{w} : \Omega\backslash\overline{\Gamma} \to \mathbb{R}^d$ vanishing on $\partial\Omega$ and admitting finite limits on each side of $\Gamma$, let us set

$$\mathbf{v} = (\mathbf{v}^1, \dots, \mathbf{v}^N) \in (X^0_{\mathcal{D}_\mathbf{u}})^N \text{ with } \mathbf{v}^i = \omega(t_{i-1})(P_{\mathcal{D}_\mathbf{u}}\mathbf{w})$$

where $P_{\mathcal{D}_\mathbf{u}}\mathbf{w}$ realises the minimum in the definition (15) of $\mathcal{S}_{\mathcal{D}_\mathbf{u}}(\mathbf{w})$. From the consistency properties of $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$, we deduce that

$$
\begin{array}{ll}
\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{v} \to \omega\psi & \text{in } L^\infty(0,T;L^2(\Omega)^d),\\
\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) \to \omega\mathfrak{e}(\mathbf{w}) & \text{in } L^\infty(0,T;L^2(\Omega,\mathcal{S}_d(\mathbb{R}))),\\
[\![\mathbf{v}]\!]_{\mathcal{D}_\mathbf{u}} \to \omega[\![\mathbf{w}]\!] & \text{in } L^\infty(0,T;L^2(\Gamma)).
\end{array}
\tag{42}
$$

Setting

$$
\begin{aligned}
T_6 &= \int_0^T \int_\Omega \Big(\mathfrak{w}_{\mathcal{D}_u}(\mathbf{u}) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) - b(\Pi^m_{\mathcal{D}_p} p^E_m)\mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v})\Big)d\mathbf{x}\mathrm{d}t,\\
T_7 &= \int_0^T \int_\Gamma (\Pi^f_{\mathcal{D}_p} p^E_f)[\![\mathbf{v}]\!]_{\mathcal{D}_\mathbf{u}}\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,\\
T_8 &= \int_0^T \int_\Omega \mathbf{f}\cdot\Pi_{\mathcal{D}_\mathbf{u}}\mathbf{v} \,\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned}
$$

the gradient scheme variational formulation (9b) states that

$$T_6 + T_7 = T_8.$$

Using a discrete integration by part [20, Section D.1.7], we have $T_1 = T_{11} + T_{12}$ with

$$
\begin{aligned}
T_{11} &= -\int_0^T \int_\Omega \phi_{\mathcal{D}}(\Pi^m_{\mathcal{D}_p} s^\alpha_m)(\Pi^m_{\mathcal{D}_p} P_{\mathcal{D}_p}\psi)\theta'(t) \,\mathrm{d}\mathbf{x}\mathrm{d}t,\\
T_{12} &= -\int_\Omega (\Pi^m_{\mathcal{D}_p} I^m_{\mathcal{D}_p}\bar\phi^0)(\Pi^m_{\mathcal{D}_p} S^\alpha_m(P_{\mathcal{D}_p}\bar p^\alpha_0))(\Pi^m_{\mathcal{D}_p} P_{\mathcal{D}_p}\psi)\theta(0) \,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Using (41) and (37), and that $\Pi^m_{\mathcal{D}_p} s^\alpha_m \in [0,1]$ converges to $S^\alpha_m(\bar p_c)$ a.e. in $(0,T)\times\Omega$ (this follows from (34)), it holds that

$$T_{11} \to -\int_0^T \int_\Omega \bar\phi_m S^\alpha_m(\bar p_c)\psi\theta'(t) \,\mathrm{d}\mathbf{x}\mathrm{d}t.$$

Using (41), that $\Pi^m_{\mathcal{D}_p} I^m_{\mathcal{D}_p}\bar\phi^0$ converges in $L^2(\Omega)$ to $\bar\phi^0$ and that $\Pi^m_{\mathcal{D}_p} S^\alpha_m(P_{\mathcal{D}_p}\bar p^\alpha_0) \in [0,1]$ converges a.e. in $\Omega$ to $S^\alpha_m(\bar p^\alpha_0)$, we deduce that

$$T_{12} \to -\int_\Omega \bar\phi^0 S^\alpha_m(\bar p^\alpha_0)\psi\theta(0) \,\mathrm{d}\mathbf{x}.$$

Writing $T_3 = T_{31} + T_{32}$ with

$$T_{31} = -\int_0^T \int_\Gamma d_{f,\mathcal{D}_\mathbf{u}} (\Pi^f_{\mathcal{D}_p} s^\alpha_f)(\Pi^f_{\mathcal{D}_p} P_{\mathcal{D}_p}\psi)\theta'(t)\ \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,$$
$$T_{32} = \int_\Gamma [\![\mathbf{u}^0]\!]_{\mathcal{D}_\mathbf{u}} (\Pi^f_{\mathcal{D}_p} S^\alpha_f(I_{\mathcal{D}_p}\bar{p}^\alpha_0))(\Pi^f_{\mathcal{D}_p} P_{\mathcal{D}_p}\psi)\theta(0)\ \mathrm{d}\sigma(\mathbf{x}),$$

we obtain, using similar arguments and (40), that

$$T_{31} \to -\int_0^T \int_\Gamma \bar{d}_f S^\alpha_f(\gamma\bar{p}_c)\gamma\psi\theta'(t)\ \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,$$

and

$$T_{32} \to -\int_\Gamma \bar{d}^0_f S^\alpha_f(\gamma\bar{p}^\alpha_0)\gamma\psi\theta(0)\ \mathrm{d}\sigma(\mathbf{x}).$$

Using that $0 \leqslant \eta^\alpha_m(\Pi^m_{\mathcal{D}_p} s^\alpha_m) \leqslant \eta^\alpha_{\mathrm{m,max}}$, the continuity of $\eta^\alpha_m$, the convergence of $\Pi^m_{\mathcal{D}_p} s^\alpha_m$ a.e. in $(0,T)\times\Omega$ to $S^\alpha_m(\bar{p}_c)$, (35) and (41), it holds that

$$T_2 \to \int_0^T \int_\Omega \eta^\alpha_m(S^\alpha_m(\bar{p}_c))\mathbb{K}_m\nabla\bar{p}^\alpha\cdot\theta\nabla\psi\ \mathrm{d}\mathbf{x}\mathrm{d}t$$

The convergence

$$T_4 \to \int_0^T \int_\Gamma \eta^\alpha_f(S^\alpha_f(\gamma\bar{p}_c))\frac{\bar{d}^3_f}{12}\nabla_\tau\gamma\bar{p}^\alpha\cdot\nabla_\tau\gamma\bar{\varphi}^\alpha\mathrm{d}\sigma(\mathbf{x})\mathrm{d}t$$

is established using $0 \leqslant \eta^\alpha_f(\Pi^f_{\mathcal{D}_p} s^\alpha_f) \leqslant \eta^\alpha_{\mathrm{f,max}}$, the continuity of $\eta^\alpha_f$, the convergence of $\Pi^f_{\mathcal{D}_p} s^\alpha_f$ a.e. in $(0,T)\times\Gamma$ to $S^\alpha_f(\gamma\bar{p}_c)$, combined with the weak convergence of $d^{3/2}_{f,\mathcal{D}_\mathbf{u}}\nabla^f_{\mathcal{D}_p}p^\alpha$ to $\bar{d}^{3/2}_f\nabla_\tau\gamma\bar{p}^\alpha$ in $L^2((0,T)\times\Gamma)^{d-1}$, the strong convergence of $d^{3/2}_{f,\mathcal{D}_\mathbf{u}}$ to $\bar{d}^{3/2}_f$ in $L^s((0,T)\times\Gamma)$ for all $2 \leqslant s < \frac{8}{3}$ (resulting from (33)), and the strong convergence (41) of $\nabla^f_{\mathcal{D}_p}\varphi$ to $\theta\nabla_\tau\gamma\psi$ in $L^\infty(0,T;L^r(\Gamma))$ with $r > 8$.

The convergence

$$T_5 \to \int_0^T \int_\Omega h^\alpha_m\ \theta\psi\ \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Gamma h^\alpha_f\ \theta(\gamma\psi)\ \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t$$

is readily obtained from (41). The following convergences of $T_6$, $T_7$, $T_8$

$$T_6 \to \int_0^T \int_\Omega \Big(\mathfrak{o}(\bar{\mathbf{u}}) : \mathfrak{e}(\mathbf{w})\omega - b\bar{p}^E_m \mathrm{div}(\mathbf{w})\omega\Big)\ \mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$T_7 \to \int_0^T \int_\Gamma \bar{p}^E_f\ [\![\mathbf{w}]\!]\omega\ \mathrm{d}\sigma(\mathbf{x})\mathrm{d}t,$$
$$T_8 \to \int_0^T \int_\Omega \mathbf{f}\cdot\mathbf{w}\omega\ \mathrm{d}\mathbf{x}\mathrm{d}t$$

classically result from the strong convergences (42) combined with the weak convergences (36).

Using the above limits in $T_1 + T_2 + T_3 + T_4 = T_5$ and $T_6 + T_7 = T_8$ concludes the proof that $\bar{p}^\alpha$, $\alpha \in \{\mathrm{nw}, \mathrm{w}\}$, and $\bar{\mathbf{u}}$ satisfy the variational formulation (6)–(7). □

# 5 Two-dimensional numerical example

In order to test the convergence and the performance of the numerical approach, we consider the following example. We solve problem (3) in the square $\Omega = (0,L)^2$ lying in the $xy$-plane,

with $L = 100\,\mathrm{m}$, containing a cross-shaped fracture network $\Gamma$ made up of four fractures, each one of length $\frac{L}{8}$ intersecting at $(\frac{L}{2}, \frac{L}{2})$ and aligned with the coordinate axes: $\overline{\Gamma} = \bigcup_{i=1}^{4} \overline{\Gamma}_i$, where $\Gamma_1 = (\frac{3}{8}L, \frac{L}{2}) \times \{\frac{L}{2}\}$, $\Gamma_2 = (\frac{L}{2}, \frac{5}{8}L) \times \{\frac{L}{2}\}$, $\Gamma_3 = \{\frac{L}{2}\} \times (\frac{3}{8}L, \frac{L}{2})$, and $\Gamma_4 = \{\frac{L}{2}\} \times (\frac{L}{2}, \frac{5}{8}L)$ (cf. Figure 3). We select as primary unknowns for the fluid flows the non-wetting pressure and saturation in the matrix and fracture network, so that the complete list of unknowns is $(p_m^{\mathrm{nw}}, s_m^{\mathrm{nw}}, p_f^{\mathrm{nw}}, s_f^{\mathrm{nw}}, \mathbf{u})$. The matrix and fracture network have the following mobility laws: $\eta_m^\alpha(s^\alpha) = \frac{(s^\alpha)^2}{\mu^\alpha}$, $\eta_f^\alpha(s^\alpha) = \frac{s^\alpha}{\mu^\alpha}$, $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, where $\mu^{\mathrm{w}} = 10^{-3}\,\mathrm{Pa{\cdot}s}$ and $\mu^{\mathrm{nw}} = 1.851{\cdot}10^{-5}\,\mathrm{Pa{\cdot}s}$ are the dynamic viscosities of the wetting and non-wetting phases, respectively. Notice that $\eta_m^\alpha$ and $\eta_f^\alpha$ do not satisfy the assumptions of our analysis, as they are not bounded below by a strictly positive number; these choices are however physically relevant, and as the test shows, do not seem to impair the convergence of the numerical scheme. The saturation–capillary pressure relation is Corey's law:

$$s_{\mathrm{rt}}^{\mathrm{nw}} = S_{\mathrm{rt}}^{\mathrm{nw}}(p_c) = \max(1 - e^{-p_c/R_{\mathrm{rt}}}, 0), \quad \mathrm{rt} \in \{m, f\},$$

with $R_m = 10^4\,\mathrm{Pa}$ and $R_f = 10\,\mathrm{Pa}$. The matrix is homogeneous and isotropic, i.e. $\mathbb{K}_m = \Lambda_m \mathbb{I}$, characterized by a permeability $\Lambda_m = 3{\cdot}10^{-15}\,\mathrm{m}^2$, an initial porosity $\phi_m^0 = 0.2$, effective Lamé parameters $\lambda = 833\,\mathrm{MPa}$, $\mu = 1250\,\mathrm{MPa}$, effective (drained) bulk modulus[1] $K_{\mathrm{dr}} = \lambda + \mu = 2083\,\mathrm{MPa}$, and solid grain bulk modulus $K_{\mathrm{s}} = 11244\,\mathrm{MPa}$. From these, one can infer the values of the Biot coefficient $b = 1 - \frac{K_{\mathrm{dr}}}{K_{\mathrm{s}}} \simeq 0.81$, and of the Biot modulus $M = \frac{K_{\mathrm{s}}}{b - \phi_m^0} \simeq 18.4\,\mathrm{GPa}$ (notice that the fluid compressibility is neglected for both phases). The densities of the fluids are $\rho^{\mathrm{w}} = 1000\,\mathrm{kg/m}^3$ and $\rho^{\mathrm{nw}} = 800\,\mathrm{kg/m}^3$. Since we consider a horizontal domain, the effects of gravity do not appear here ($\mathbf{f} = \mathbf{0}$ in $\Omega$). The domain is assumed to be clamped all over its boundary, i.e. $\mathbf{u} = \mathbf{0}$ on $(0,T) \times \partial\Omega$; for the flows, we impose a wetting saturation $s_m^{\mathrm{w}} = 1$ on the upper side of the boundary $(0,T) \times ((0,L) \times \{L\})$, whereas the remaining part of the boundary is considered as impervious ($\mathbf{q}_m^\alpha \cdot \mathbf{n} = 0, \alpha \in \{\mathrm{nw}, \mathrm{w}\}$). The system is subject to the initial conditions $p_0^{\mathrm{nw}} = p_0^{\mathrm{w}} = 10^5\,\mathrm{Pa}$, which in turn results in an initial saturation $s_{0,\mathrm{rt}}^{\mathrm{nw}} = 0$, $\mathrm{rt} \in \{m, f\}$. The final time is set to $T = 1000\,\mathrm{days} = 8.64{\cdot}10^7\,\mathrm{s}$. The system is excited by the following source term, representing injection of non-wetting fluid at the center of the fracture network:

$$h_f^{\mathrm{nw}}(t, \mathbf{x}) = \frac{g(\mathbf{x})}{\displaystyle\int_\Gamma g(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{x})} \frac{V_{\mathrm{por}}}{T/5}, \quad (t, \mathbf{x}) \in (0,T) \times \Gamma,$$

where $V_{\mathrm{por}} = \displaystyle\int_\Omega \phi_m^0(\mathbf{x})\,\mathrm{d}\mathbf{x}$ is the initial porous volume and $g(\mathbf{x}) = e^{-\beta|(\mathbf{x}-\mathbf{x}_0)/L|^2}$, $\mathbf{x}_0 = (\frac{L}{2}, \frac{L}{2})$, with $\beta = 1000$ and $|\cdot|$ the Euclidean norm. The remaining source terms $h_f^{\mathrm{w}}$ and $h_m^\alpha$, $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, are all set to zero.

The flow part of (3) is discretized in space by a Two-Point Flux Approximation (TPFA) finite volume scheme using $mf$-linear $m$-upwind model for matrix-fracture interactions (cf. [2]). The mechanical part of the system (3) is discretized using the second-order finite elements ($\mathbb{P}_2$) for the displacement field in the matrix (see e.g. [16, 33]), adding supplementary unknowns on the fracture faces to account for the discontinuities. The computational domain $\Omega$ is decomposed using *admissible* triangular meshes for the TPFA scheme (cf. [23, Section 3.1.2]). Let $n \in \mathbb{N}^\star$ denote the time step index. The time stepping is adaptive, defined as

$$\delta t^{n+\frac{1}{2}} = \min\{\rho \delta t^{n-\frac{1}{2}}, \Delta t^{\max}\},$$

where $\delta t^{\frac{1}{2}}$ is the initial time step and $\rho = 1.1$. At each time step, the flow unknowns are computed by a Newton-Raphson algorithm. At each Newton-Raphson iteration, the Jacobian matrix is computed analytically and the linear system is solved using a GMRes iterative solver. The time step is reduced by a factor 2 whenever the Newton-Raphson algorithm does not converge within 50 iterations, with the stopping criteria defined by the relative residual norm lower than $10^{-5}$ or

[1] In general, $K_{\mathrm{dr}} = \lambda + 2\mu/d$, where $d \in \{2, 3\}$ is the space dimension.

a maximum normalized variation of the primary unknowns lower than $10^{-4}$. On the other hand, given the matrix and fracture equivalent pressures $p_m^E$ and $p_f^E$, the displacement field $\mathbf{u}$ is computed using the direct solver MA48 (see [22]). Following [9, 16, 33, 38, 40, 28], the coupling between the two-phase Darcy flow and the mechanical deformation is solved by means of a *fixed-point* algorithm. This algorithm computes the matrix porosity and the fracture aperture, using discrete versions of the coupling laws (4), at each time step and fixed-point iteration. The algorithm is summarized in the following scheme, where $k$ denotes the current fixed-point iteration and $n$ the current time step.

**Iterative coupling algorithm**

At each time step $n$, for $k = 1, \dots,$ until convergence, solve the following Darcy and mechanical subproblems:

(i) **Compute** $p_{\mathrm{rt}}^{\alpha,n,k}$, $s_{\mathrm{rt}}^{\alpha,n,k}$, $\alpha \in \{\mathrm{w}, \mathrm{nw}\}$, $\mathrm{rt} \in \{m, f\}$, solving the Darcy flow model using $d_f^{n,k-1}$ in the fracture conductivity and the following porosity and fracture aperture in the accumulation term:

$$\begin{cases} \phi_m^{n,k} - \phi_m^{n-1} = C_{r,m}(p_m^{E,n,k} - p_m^{E,n,k-1}) + b\,\mathrm{div}(\mathbf{u}^{n,k-1} - \mathbf{u}^{n-1}) + \dfrac{1}{M}(p_m^{E,n,k} - p_m^{E,n-1}), \\ d_f^{n,k} - d_f^{n-1} = C_{r,f}(p_f^{E,n,k} - p_f^{E,n,k-1}) + [\![\mathbf{u}^{n,k-1} - \mathbf{u}^{n-1}]\!]. \end{cases}$$

(ii) **Compute** the displacement field $\mathbf{u}^{n,k}$ using the equivalent pressures $p_m^{E,n,k}$ and $p_f^{E,n,k}$ computed at step (i).

**Initialization**

For given $n \geqslant 1$ and $k = 1$, set

$$\begin{cases} \dfrac{p_{\mathrm{rt}}^{E,n,0} - p_{\mathrm{rt}}^{E,n-1}}{\delta t^{n-\frac{1}{2}}} = \dfrac{p_{\mathrm{rt}}^{E,n-1,0} - p_{\mathrm{rt}}^{E,n-2}}{\delta t^{n-\frac{3}{2}}}, \quad \mathrm{rt} \in \{m, f\}, \\ \dfrac{\mathbf{u}^{n,0} - \mathbf{u}^{n-1}}{\delta t^{n-\frac{1}{2}}} = \dfrac{\mathbf{u}^{n-1} - \mathbf{u}^{n-2}}{\delta t^{n-\frac{3}{2}}}; \end{cases}$$

For $n = 1$, set

$$\begin{cases} p_{\mathrm{rt}}^{E,-1} = p_{\mathrm{rt}}^{E,0}, \quad \mathrm{rt} \in \{m, f\}, \\ \mathbf{u}^{-1} = \mathbf{u}^0. \end{cases}$$

Here, $C_{r,m}$ and $C_{r,f}$ are positive relaxation parameters mimicking the rock compressibility (see e.g. [9, 16, 33, 38, 40, 28]). For our numerical simulations, we choose $C_{r,m} = \frac{16b^2}{2\mu+2\lambda}$ (cf. [40]), and $C_{r,f} = \widetilde{d}_f C_{r,m}$ with $\widetilde{d}_f = 10^{-3}\,\mathrm{m}$.

To verify the convergence of the method, we take into account six sequentially refined admissible triangular nested grids. In particular, we consider a reference solution $(p_{m,\mathrm{ref}}^{\mathrm{nw}}, s_{m,\mathrm{ref}}^{\mathrm{nw}}, p_{f,\mathrm{ref}}^{\mathrm{nw}}, s_{f,\mathrm{ref}}^{\mathrm{nw}}, \mathbf{u}_{\mathrm{ref}})$ computed on the finest (sixth) grid, made up by 229376 cells, and used to showcase the time histories of the solution as well as to compute the time histories of the relative errors for each grid. Figure 4 shows the variation with respect to the curvilinear abscissa ($x$ or $y$, depending on the orientation) of the initial and final apertures for the fractures in the cross-shaped network, based on the reference solution. At time $t = 0$, the widths of both the $x$- and $y$-oriented fractures coincide. Figure 5 displays the final non-wetting matrix pressure and saturations computed on the finest grid, as well as the initial and final displacement field; as expected, the non-wetting phase flows through the fracture network tips and is attracted towards the upper side of the domain, where the wetting saturation is maximal ($s_m^{\mathrm{w}} = 1$); also, the intensity of the final displacement field turns out to be one order of magnitude larger than the initial one. Figure 6 showcases the time histories of the average of some relevant physical quantities computed based on the reference solution (the

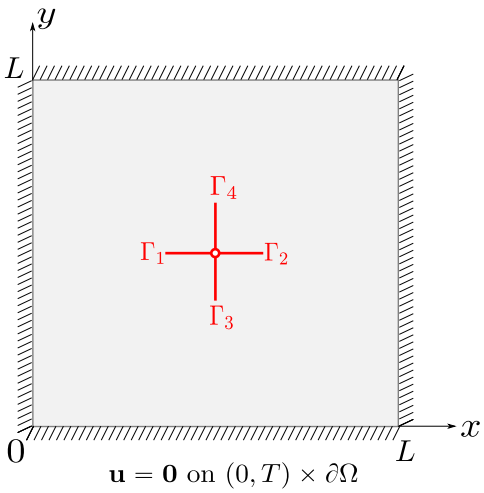



Figure 3: Computational domain.

average of $a$ is denoted by $a^\star$). In particular, we notice the increase in width for the fracture network as a result of the gas injection, followed by a decrease after attaining a maximum due to an increasing gas matrix mobility in the neighborhood of the fractures. The mean saturation in the fracture network, as expected, grows linearly with time until the gas front reaches the upper boundary. Finally, the variation of the mean matrix porosity over time is almost negligible (on the order of $10^{-6}$). Figure 7 shows the time histories of the relative errors with respect to the averaged physical quantities given by the reference solution. Computations are carried out, again, using averaged quantities ($a_N^\star$ denotes the average of quantity $a$ computed using $N$ triangular elements). As the number $N$ of elements increases, the relative errors decrease, which shows the convergence of the method. Finally, we give an insight into the performance of our method in Table 1, where

- NbCells is the number of cells of the mesh,
- $\mathrm{N}_{\Delta t}$ is the number of successful time steps,
- $\mathrm{N}_{\mathrm{Newton}}$ is the total number of Newton-Raphson iterations,
- $\mathrm{N}_{\mathrm{GMRes}}$ is the total number of GMRes iterations,
- $\mathrm{N}_{\mathrm{FixedPoint}}$ is the total number of fixed point iterations.

| NbCells | $\mathrm{N}_{\Delta t}$ | $\mathrm{N}_{\mathrm{Newton}}$ | $\mathrm{N}_{\mathrm{GMRes}}$ | $\mathrm{N}_{\mathrm{FixedPoint}}$ |
|---|---|---|---|---|
| 224 | 153 | 3298 | 13716 | 2351 |
| 896 | 153 | 1857 | 9751 | 1308 |
| 3584 | 153 | 1839 | 10792 | 1312 |
| 14336 | 153 | 1976 | 14062 | 1324 |
| 57344 | 153 | 2247 | 20330 | 1322 |
| 229376 | 153 | 2772 | 34799 | 1329 |

Table 1: Performance of the method in terms of the number of mesh elements, the number of successful time steps, the total number of Newton-Raphson iterations, the total number of GMRes iterations, and the total number of fixed-point iterations.

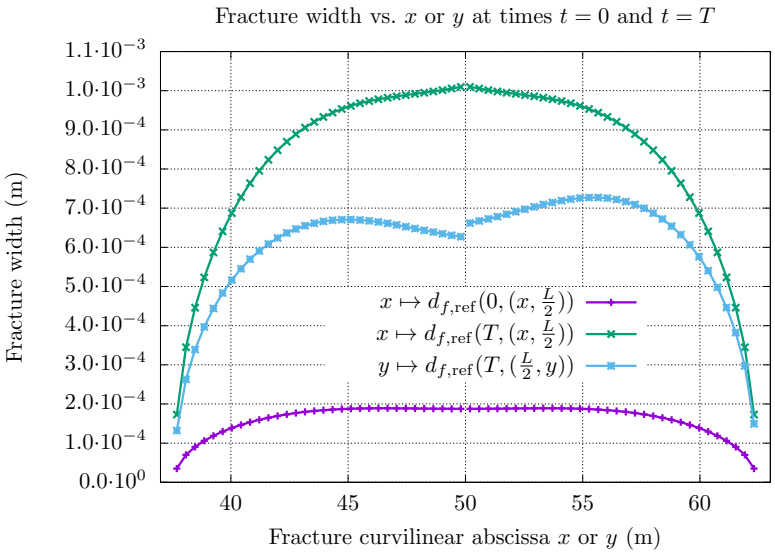


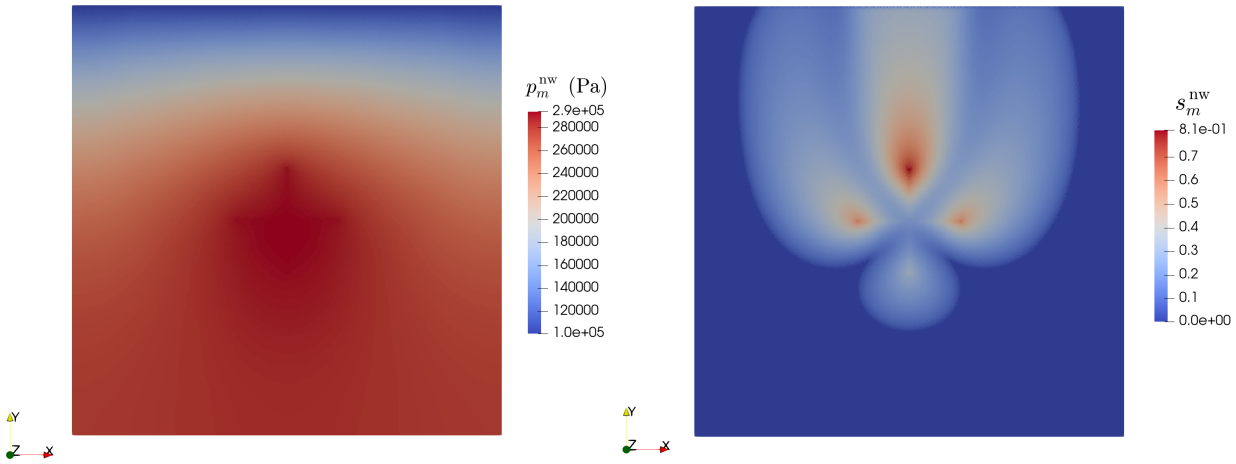

Figure 4: Initial and final widths of the $x$- and $y$-oriented fractures vs. corresponding curvilinear abscissae, computed using the finest grid (reference solution). The initial width for both the $x$- and $y$-oriented fractures is the same.

# 6 Conclusions

We developed, in the framework of the gradient discretization method, the numerical analysis of a two-phase flow model in deformable and fractured porous media. The model considers a linear elastic mechanical model with open fractures coupled with an hybrid-dimensional two-phase Darcy flow with continuous phase pressures at matrix fracture interfaces. The model accounts for a general network of planar fractures including immersed, non-immersed fractures and fracture intersections, and considers different rock types in the matrix and fracture network domains.

It is assumed, for the convergence analysis, that the porosity remains bounded from below and above by strictly positive constants and that the fracture aperture remains larger than a fixed non-negative continuous function vanishing only at the tips of the fracture network. In addition, the mobility functions are assumed to be bounded from below by strictly positive constants. However, unlike previous works, the fracture conductivity $d_f^3/12$ was not frozen and the complete non-linear coupling between the flow and mechanics equations was considered.

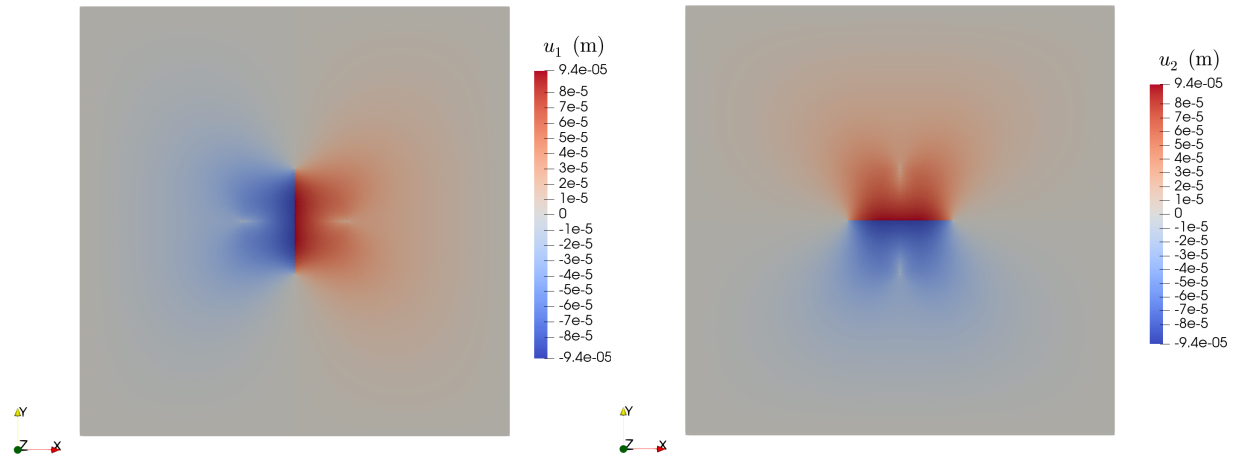
Assuming that the gradient discretization meet generic coercivity, consistency, limit-conformity and compactness properties, we proved the weak convergence of the phase pressures and displacement field to a weak continuous solution, as well as the strong convergence of the fracture aperture and of the matrix and fracture saturations. Numerical experiments carried out for a cross-shaped fracture network immersed in a two-dimensional porous medium and using a TPFA finite volume scheme for the flow combined with a $\mathbb{P}_2$ finite element method for the mechanics, confirmed the numerical convergence of the scheme.




(a) $p_m^{\mathrm{nw}}(T;x,y)$



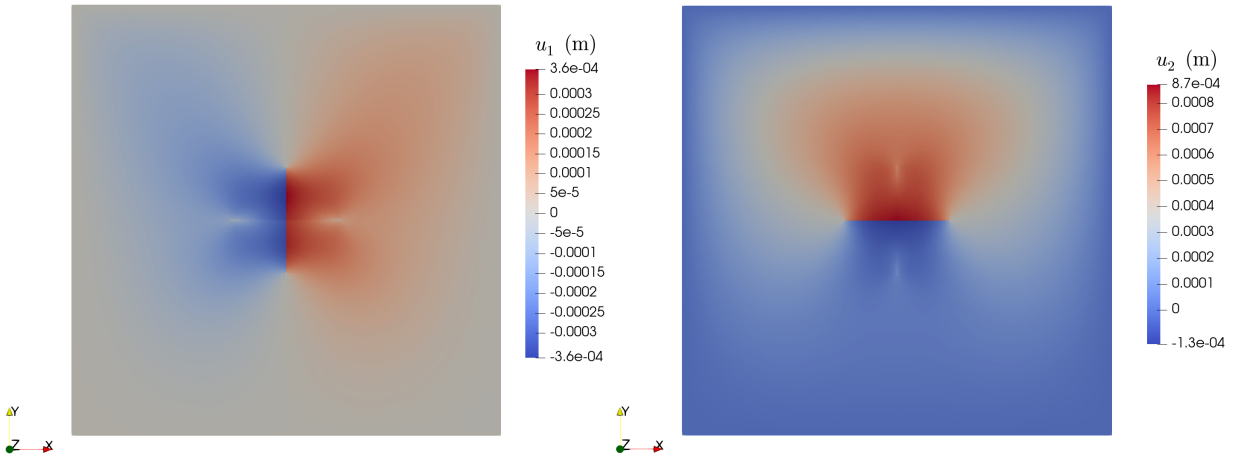

(b) $s_m^{\mathrm{nw}}(T;x,y)$




(c) $u_1(0;x,y)$




(d) $u_2(0;x,y)$




(e) $u_1(T;x,y)$




(f) $u_2(T;x,y)$

Figure 5: Final non-wetting matrix pressure and saturation (a)-(b), and initial (c)-(d) and final (e)-(f) displacement field.

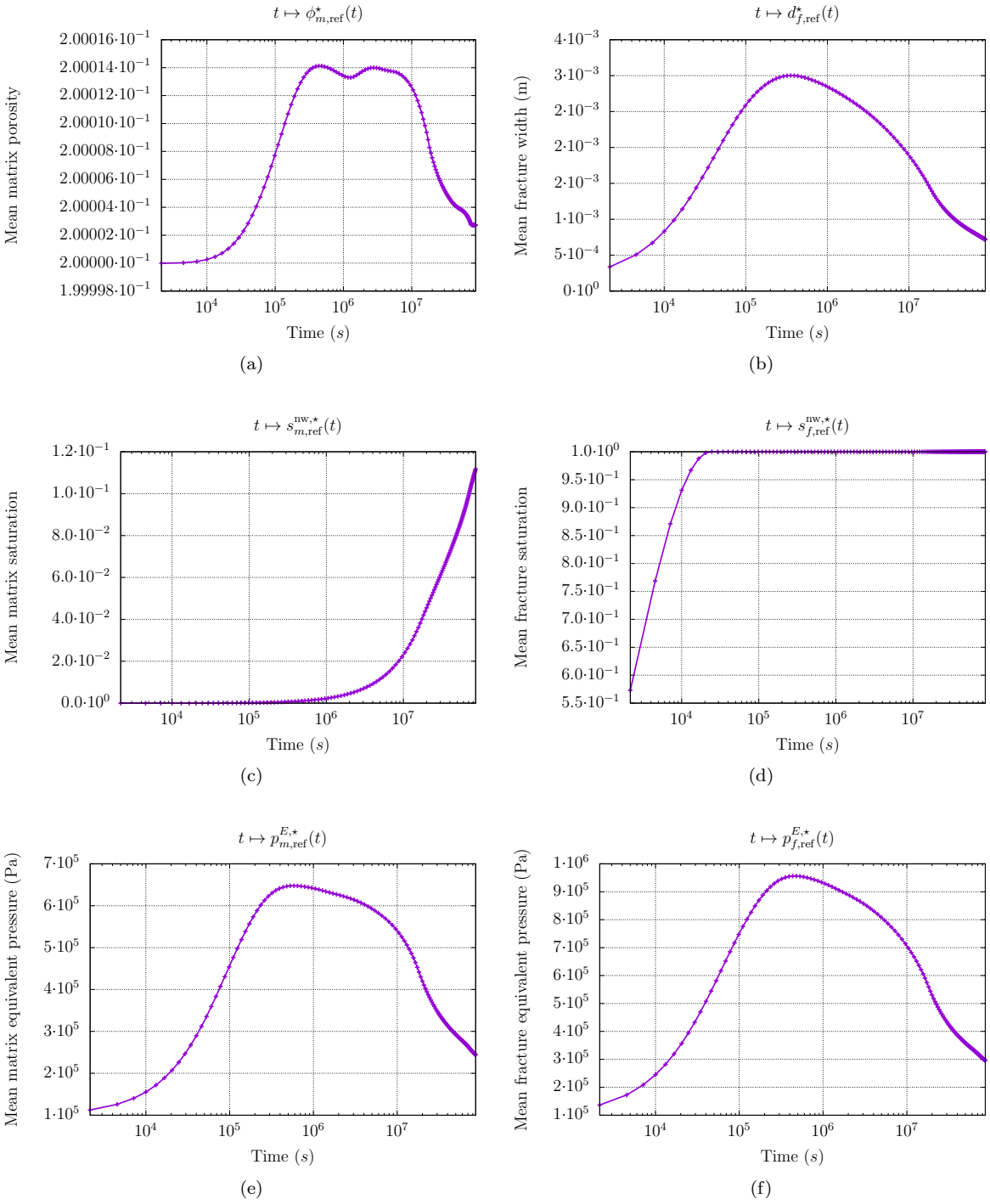



Figure 6: Time histories of the average of some physical quantities based on the reference solution ($a^{\star}$ denotes the mean of $a$).

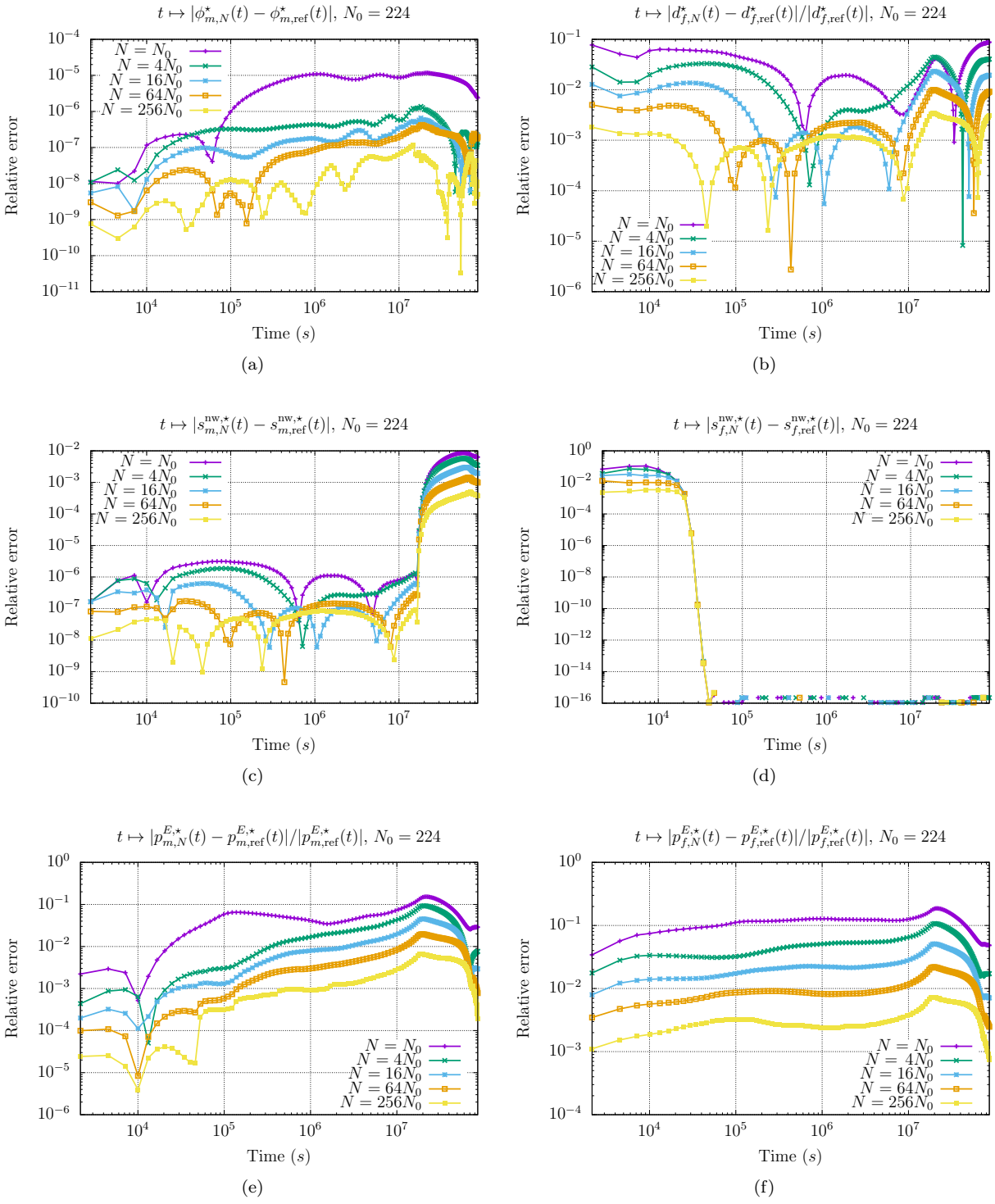



Figure 7: Time histories of the relative errors computed for five nested grids, with $N$ triangular elements, with respect to the corresponding reference time histories (cf. Figure 6).

# A Appendix

## A.1 Appendix 1

**Proposition A.1.** *Let $X \subset \mathbb{R}^d$ be bounded, $\delta > 0$ and let $\left(A_m^\delta\right)_{m\in M_\delta}$ be a covering of $X$ in disjoint cubes of length $\delta$. Let $R^\delta : L^2(\mathbb{R}^d) \to L^2(X)$ be such that, for any $v \in L^2(\mathbb{R}^d)$,*

$$\left(R^\delta v\right)|_{A_m^\delta \cap X} = \frac{1}{\delta^d}\int_{A_m^\delta} v(\mathbf{x})\ \mathrm{d}\mathbf{x} \quad \forall m \in M_\delta,$$

*Then, we have*

$$\|R^\delta v - v\|_{L^2(X)} \leqslant 2^{d/2} \sup_{|z|\leqslant\delta} \|v(\cdot + z) - v\|_{L^2(X)}.$$

*Proof.* The proof can be found in [19, p. 756]. Note that the assumption, in this reference, that $v$ is zero outside $X$ is actually not useful. □

**Lemma A.2.** *Let $X \subset \mathbb{R}^d$ be bounded, and $U$ be an open subset of $\mathbb{R}^d$ such that $\{\mathbf{x} \in \mathbb{R}^d : \mathrm{dist}(\mathbf{x}, X) < \delta_0\} \subset U$ for a given $\delta_0 > 0$, where the distance is considered for the supremum norm in $\mathbb{R}^d$. Let $(w_k)_{k\in\mathbb{N}}$ be a bounded sequence in $L^\infty(0,T;L^2(U))$ that converges uniformly in time and weakly in $L^2(U)$ to $w \in L^\infty(0,T;L^2(U))$. Let $p \in [1,+\infty]$ and let us define*

$$T(\delta) = \sup_k \left\| \sup_{|z|\leqslant\delta} \|w_k(\cdot,\cdot+z) - w_k(\cdot,\cdot)\|_{L^2(X)} \right\|_{L^p(0,T)}.$$

*If $\lim_{\delta\to 0} T(\delta) = 0$, then the sequence $(w_k)_{k\in\mathbb{N}}$ converges to $w$ in $L^p(0,T;L^2(X))$.*

*Proof.* For $0 < \delta < \delta_0$, let $\left(A_m^\delta\right)_{m\in M_\delta}$ be a covering of $X$ in disjoint cubes of length $\delta$ and let $R^\delta$ be the corresponding $L^2$ projection operator as defined in Proposition A.1. We write

$$w_k - w = (w_k - R^\delta w_k) + (R^\delta w_k - R^\delta w) + (R^\delta w - w)$$

and we establish the convergence to 0 of each bracketed term in the right-hand side. First, in view of Proposition A.1

$$\|w_k(t,\cdot) - R^\delta w_k(t,\cdot)\|_{L^2(X)} \lesssim \sup_{|z|\leqslant\delta} \|w_k(t,\cdot+z) - w_k(t,\cdot)\|_{L^2(X)}$$

implying that

$$\|w_k - R^\delta w_k\|_{L^p(0,T;L^2(X))} \lesssim T(\delta).$$

Setting $v_k = w_k - R^\delta w_k$, $k \in \mathbb{N}$, we have, if $p = \infty$, $\|v_k(t,\cdot)\|_{L^2(X)} \lesssim T(\delta)$ for a.e. $t \in (0,T)$. Using the weak convergence $v_k(t,\cdot) \rightharpoonup v(t,\cdot) \coloneqq w(t,\cdot) - R^\delta w(t,\cdot)$ in $L^2(X)$, this provides

$$\|w(t,\cdot) - R^\delta w(t,\cdot)\|_{L^2(X)} \leqslant \liminf_{k\to+\infty} \|v_k(t,\cdot)\|_{L^2(X)} \lesssim T(\delta).$$

For $p < \infty$, we have, using the above weak convergence of $(v_k(t,\cdot))_{k\in\mathbb{N}}$ and Fatou's lemma,

$$\int_0^T \|v(t,\cdot)\|_{L^2(X)}^p \mathrm{d}t \leqslant \int_0^T \liminf_{k\to+\infty} \|v_k(t,\cdot)\|_{L^2(X)}^p \mathrm{d}t \leqslant \liminf_{k\to+\infty} \int_0^T \|v_k(t,\cdot)\|_{L^2(X)}^p \mathrm{d}t \lesssim T^p(\delta).$$

Hence, for any $p$,

$$\|w - R^\delta w\|_{L^p(0,T;L^2(X))} \lesssim T(\delta).$$

Finally,

$$R^\delta w_k - R^\delta w = \sum_{m\in M_\delta} a_{km}^\delta(t) \mathbb{1}_{A_m^\delta \cap X}, \quad \text{with} \quad a_{km}^\delta(t) = \frac{1}{\delta^d}\int_{A_m^\delta} (w_k(t,\mathbf{x}) - w(t,\mathbf{x}))\mathrm{d}\mathbf{x},$$

Since the covering $\left(A_m^\delta\right)_{m\in M_\delta}$ is finite and since, for all $m \in M_\delta$, the term $a_{km}^\delta(t)$ converge uniformly in time to zero, it results that $(R^\delta w_k - R^\delta w)$ converges as $k \to +\infty$ to zero in $L^\infty((0,T)\times X)$.

Gathering the estimates, we have that

$$\|w_k - w\|_{L^p(0,T;L^2(X))} \lesssim 2T(\delta) + \|R^\delta w_k - R^\delta w\|_{L^p(0,T;L^2(X))}.$$

Passing to the superior limit as $k \to \infty$, we deduce that $\limsup_{k\to+\infty} \|w_k - w\|_{L^p(0,T;L^2(X))} \lesssim 2T(\delta)$ which yields, letting $\delta \to 0$, $\limsup_{k\to+\infty} \|w_k - w\|_{L^p(0,T;L^2(X))} = 0$. □

## A.2 Appendix 2

**Lemma A.3.** *Let $(\mathcal{D}_{\mathbf{u}}^l)_{l\in\mathbb{N}}$ be a sequence of GDs assumed to satisfy the coercivity and limit-conformity properties. Let $(\mathbf{u}^l)_{l\in\mathbb{N}}$ be a sequence of vectors with $\mathbf{u}^l \in X_{\mathcal{D}_{\mathbf{u}}^l}^0$ such that there exist $C$ independent of $l \in \mathbb{N}$ with $\|\mathbf{u}^l\|_{\mathcal{D}_{\mathbf{u}}} \leqslant C$. Then, there exists $\bar{\mathbf{u}} \in \mathbf{U}_0$ such that, up to a subsequence, the following weak limits hold:*

$$\begin{array}{ll}
\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{u}^l \rightharpoonup \bar{\mathbf{u}} & \text{in } L^2(\Omega)^d,\\
\mathfrak{e}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{u}^l) \rightharpoonup \mathfrak{e}(\bar{\mathbf{u}}) & \text{in } L^2(\Omega,\mathcal{S}_d(\mathbb{R})),\\
\mathrm{div}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{u}^l) \rightharpoonup \mathrm{div}(\bar{\mathbf{u}}) & \text{in } L^2(\Omega),\\
[\![\mathbf{u}^l]\!]_{\mathcal{D}_{\mathbf{u}}^l} \rightharpoonup [\![\bar{\mathbf{u}}]\!] & \text{in } L^2(\Gamma).
\end{array}$$

*Proof.* By assumption the sequence $(\|\mathfrak{e}_{\mathcal{D}_{\mathbf{u}}^l}\|_{L^2(\Omega,\mathcal{S}_d(\mathbb{R}))})_{l\in\mathbb{N}}$ is bounded which implies, from the coercivity property, that the sequences $(\|\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{u}^l\|_{L^2(\Omega)})_{l\in\mathbb{N}}$ and $(\|[\![\mathbf{u}^l]\!]_{\mathcal{D}_{\mathbf{u}}^l}\|_{L^2(\Gamma)})_{l\in\mathbb{N}}$ are also bounded. Hence there exist $\bar{\mathbf{u}} \in L^2(\Omega)^d$, $\bar{\mathfrak{e}} \in L^2(\Omega,\mathcal{S}_d(\mathbb{R}))$ and $\bar{g} \in L^2(\Gamma)$ such that, up to a subsequence, one has

$$\begin{array}{ll}
\Pi_{\mathcal{D}_{\mathbf{u}}^l}\mathbf{u}^l \rightharpoonup \bar{\mathbf{u}} & \text{in } L^2(\Omega)^d,\\
\mathfrak{e}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{u}^l) \rightharpoonup \bar{\mathfrak{e}} & \text{in } L^2(\Omega,\mathcal{S}_d(\mathbb{R})),\\
[\![\mathbf{u}^l]\!]_{\mathcal{D}_{\mathbf{u}}^l} \rightharpoonup \bar{g} & \text{in } L^2(\Gamma).
\end{array}$$

Passing to the limit in the definition of the limit-conformity yields, for any $\mathfrak{o} \in C_\Gamma^\infty(\Omega\backslash\overline{\Gamma},\mathcal{S}_d(\mathbb{R}))$,

$$\int_\Omega \Big(\mathfrak{o} : \bar{\mathfrak{e}} + \bar{\mathbf{u}}\cdot\mathrm{div}(\mathfrak{o})\Big)\mathrm{d}\mathbf{x} - \int_\Gamma (\mathfrak{o}\mathbf{n}^+)\cdot\mathbf{n}^+\bar{g}\,\mathrm{d}\sigma(\mathbf{x}) = 0.$$

Selecting first $\mathfrak{o}$ with a compact support in $\Omega\backslash\Gamma$, and then a generic $\mathfrak{o}$, it results that $\bar{\mathbf{u}} \in \mathbf{U}_0$ with $\bar{\mathfrak{e}} = \mathfrak{e}(\bar{\mathbf{u}})$ and $\bar{g} = [\![\bar{\mathbf{u}}]\!]$. Since $\mathrm{div}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{u}^l) = \mathrm{Trace}(\mathfrak{e}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{u}^l))$, it also holds that $\mathrm{div}_{\mathcal{D}_{\mathbf{u}}^l}(\mathbf{u}^l) \rightharpoonup \mathrm{div}(\bar{\mathbf{u}})$ in $L^2(\Omega)$.

□

Let us fix $\bar{p}^\alpha \in V_0$, $\alpha \in \{\mathrm{nw},\mathrm{w}\}$, $\mathbf{f} \in L^2(\Omega)^d$, and define

$$\bar{p}_m^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} \bar{p}^\alpha S_m^\alpha(\bar{p}_c) - U_m(\bar{p}_c) \quad \text{and} \quad \bar{p}_f^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} \gamma\bar{p}^\alpha S_f^\alpha(\gamma\bar{p}_c) - U_f(\gamma\bar{p}_c).$$

with $\bar{p}_c = \bar{p}^{\mathrm{nw}} - \bar{p}^{\mathrm{w}}$. We consider the solution $\bar{\mathbf{u}} \in \mathbf{U}_0$ of the following variational formulation

$$\int_\Omega \Big(\mathfrak{o}(\bar{\mathbf{u}}) : \mathfrak{e}(\bar{\mathbf{v}}) - b\,\bar{p}_m^E\mathrm{div}(\bar{\mathbf{v}})\Big)\mathrm{d}\mathbf{x} + \int_\Gamma \bar{p}_f^E\,[\![\bar{\mathbf{v}}]\!]\mathrm{d}\sigma(\mathbf{x}) = \int_\Omega \mathbf{f}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x}, \qquad \forall\bar{\mathbf{v}} \in \mathbf{U}_0. \tag{43}$$

Let us take $p^\alpha \in \mathcal{D}_p$, $\alpha \in \{\mathrm{nw},\mathrm{w}\}$, $p_c = p^{\mathrm{nw}} - p^{\mathrm{w}}$ and

$$p_m^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} p^\alpha S_m^\alpha(p_c) - U_m(p_c) \quad \text{and} \quad p_f^E = \sum_{\alpha\in\{\mathrm{nw},\mathrm{w}\}} p^\alpha S_f^\alpha(p_c) - U_f(p_c).$$

We consider the following gradient scheme for (43): Find $\mathbf{u} \in X^0_{\mathcal{D}_\mathbf{u}}$ such that, for all $\mathbf{v} \in X^0_{\mathcal{D}_\mathbf{u}}$,

$$\int_\Omega \Big( \varpi_{\mathcal{D}_u}(\mathbf{u}) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) - b\, (\Pi^m_{\mathcal{D}_p} p^E_m) \mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) \Big) \mathrm{d}\mathbf{x} + \int_\Gamma (\Pi^f_{\mathcal{D}_p} p^E_f) [\![\mathbf{v}]\!]_{\mathcal{D}_\mathbf{u}} \mathrm{d}\sigma(\mathbf{x}) = \int_\Omega \mathbf{f} \cdot \Pi_{\mathcal{D}_\mathbf{u}} \mathbf{v} \, \mathrm{d}\mathbf{x}. \quad (44)$$

The Lax Milgram theorem ensures that the exact solution $\bar{\mathbf{u}}$ and approximate solution $\mathbf{u}$ exist and are unique. The following proposition provides an error estimate.

**Proposition A.4.** *Let $\bar{\mathbf{u}} \in \mathbf{U}_0$ be the solution of (43) and $\mathbf{u} \in X^0_{\mathcal{D}_\mathbf{u}}$ the solution of the gradient scheme (44). Then, there exists a hidden constant depending only on the coercivity constant $C_{\mathcal{D}_\mathbf{u}}$ and on the physical data such that the following error estimate holds*

$$\begin{aligned} &\|\mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{u}) - \mathfrak{e}(\bar{\mathbf{u}})\|_{L^2(\Omega, \mathcal{S}_d(\mathbb{R}))} + \|\Pi_{\mathcal{D}_\mathbf{u}} \mathbf{u} - \bar{\mathbf{u}}\|_{L^2(\Omega)} + \|[\![\mathbf{u}]\!]_{\mathcal{D}_\mathbf{u}} - [\![\bar{\mathbf{u}}]\!]\|_{L^2(\Gamma)} \\ &\lesssim \mathcal{S}_{\mathcal{D}_\mathbf{u}}(\bar{\mathbf{u}}) + \mathcal{W}_{\mathcal{D}_\mathbf{u}}(\varpi(\bar{\mathbf{u}}) - b\,\bar{p}^E_m \mathbb{I}) + \|\bar{p}^E_m - \Pi^m_{\mathcal{D}_p} p^E_m\|_{L^2(\Omega)} + \|\bar{p}^E_f - \Pi^f_{\mathcal{D}_p} p^E_f\|_{L^2(\Gamma)}. \end{aligned} \quad (45)$$

*As a consequence, if $(\mathcal{D}^l_\mathbf{u})_{l\in\mathbb{N}}$ is a sequence of coercive, consistent and limit-conforming GDs, if $\mathbf{u}^l$ is the solution of (44) for $\mathcal{D}_\mathbf{u} = \mathcal{D}^l_\mathbf{u}$, if $(\mathcal{D}^l_p)_{l\in\mathbb{N}}$ is a sequence of GDs and $p^{\alpha,l} \in X^0_{\mathcal{D}^l_p}$, $l \in \mathbb{N}$, are such that $\Pi^m_{\mathcal{D}^l_p} p^{E,l}_m \to \bar{p}^E_m$ in $L^2(\Omega)$ and $\Pi^f_{\mathcal{D}^l_p} p^{E,l}_f \to \bar{p}^E_f$ in $L^2(\Gamma)$, then, as $l \to \infty$,*

$$\begin{aligned} \mathfrak{e}_{\mathcal{D}^l_\mathbf{u}}(\mathbf{u}^l) &\to \mathfrak{e}(\bar{\mathbf{u}}) \quad &&\text{in } L^2(\Omega, \mathcal{S}_d(\mathbb{R})), \\ \Pi_{\mathcal{D}^l_\mathbf{u}} \mathbf{u}^l &\to \bar{\mathbf{u}} \quad &&\text{in } L^2(\Omega)^d, \\ [\![\mathbf{u}^l]\!]_{\mathcal{D}^l_\mathbf{u}} &\to [\![\bar{\mathbf{u}}]\!] \quad &&\text{in } L^2(\Gamma). \end{aligned} \quad (46)$$

*Proof.* We note that even though $\mathcal{W}_{\mathcal{D}_\mathbf{u}}$ was considered, in the definition of limit-conformity of a sequence of GDs, only on $C^\infty_\Gamma(\Omega\backslash\overline{\Gamma}, \mathcal{S}_d(\mathbb{R}))$, it can be defined on

$$\begin{aligned} H_{\mathrm{div},\Gamma}(\Omega\backslash\overline{\Gamma}; \mathcal{S}_d(\mathbb{R})) \coloneqq \big\{ &\varpi \in L^2(\Omega; \mathcal{S}_d(\mathbb{R})) \,:\, \mathrm{div}(\varpi)|_{\Omega^\beta} \in L^2(\Omega^\beta)^d, \beta \in \Xi, \\ &\varpi^+ \mathbf{n}^+ + \varpi^- \mathbf{n}^- = \mathbf{0} \text{ on } \Gamma,\ (\varpi^+ \mathbf{n}^+) \times \mathbf{n}^+ = \mathbf{0} \text{ on } \Gamma \big\}, \end{aligned}$$

where $(\Omega^\beta)_{\beta\in\Xi}$ are the connected components of $\Omega\backslash\overline{\Gamma}$. Setting $\varpi = \varpi(\bar{\mathbf{u}}) - b\,\bar{p}^E_m \mathbb{I} \in H_{\mathrm{div},\Gamma}(\Omega\backslash\overline{\Gamma}; \mathcal{S}_d(\mathbb{R}))$ as an argument of $\mathcal{W}_{\mathcal{D}_\mathbf{u}}$ and using $\mathrm{div}\varpi = -\mathbf{f}$, we obtain that for all $\mathbf{v} \in X^0_{\mathcal{D}_\mathbf{u}}$

$$\begin{aligned} &\left| \int_\Omega \Big( (\varpi(\bar{\mathbf{u}}) - \varpi_{\mathcal{D}_\mathbf{u}}(\mathbf{u})) : \mathfrak{e}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) - b(\bar{p}^E_m - \Pi^m_{\mathcal{D}_p} p^E_m) \mathrm{div}_{\mathcal{D}_\mathbf{u}}(\mathbf{v}) \Big) \mathrm{d}\mathbf{x} + \int_\Gamma (\bar{p}^E_f - \Pi^f_{\mathcal{D}_p} p^E_f) [\![\mathbf{v}]\!]_{\mathcal{D}_\mathbf{u}} \mathrm{d}\sigma(\mathbf{x}) \right| \\ &\quad \leqslant \|\mathbf{v}\|_{\mathcal{D}_\mathbf{u}} \mathcal{W}_{\mathcal{D}_\mathbf{u}}(\varpi(\bar{\mathbf{u}}) - b\,\bar{p}^E_m \mathbb{I}). \end{aligned}$$

Setting $\mathbf{v} = P_{\mathcal{D}_\mathbf{u}} \bar{\mathbf{u}} - \mathbf{u}$, where $P_{\mathcal{D}_\mathbf{u}} \bar{\mathbf{u}}$ realises the minimum in $\mathcal{S}_{\mathcal{D}_\mathbf{u}}(\bar{\mathbf{u}})$, we infer

$$\|P_{\mathcal{D}_\mathbf{u}} \bar{\mathbf{u}} - \mathbf{u}\|_{\mathcal{D}_\mathbf{u}} \lesssim \mathcal{S}_{\mathcal{D}_\mathbf{u}}(\bar{\mathbf{u}}) + \mathcal{W}_{\mathcal{D}_\mathbf{u}}(\varpi(\bar{\mathbf{u}}) - b\,\bar{p}^E_m \mathbb{I}) + \|\bar{p}^E_m - \Pi^m_{\mathcal{D}_p} p^E_m\|_{L^2(\Omega)} + \|\bar{p}^E_f - \Pi^f_{\mathcal{D}_p} p^E_f\|_{L^2(\Gamma)}.$$

Combined with the definition of $C_{\mathcal{D}_\mathbf{u}}$, the estimates above establish (45).

Under the assumptions in the second part of the proposition, the hidden constant in (45) is independent of $l$, the last two terms in the left-hand side of this estimate converge to 0 as $l \to \infty$, as well as $S_{\mathcal{D}^l_\mathbf{u}}(\mathbf{u})$ by definition of the consistency of the sequence of GDs. When its argument $\varpi$ is in the vector space $C^\infty_\Gamma(\Omega\backslash\overline{\Gamma}, \mathcal{S}_d(\mathbb{R}))$, $\mathcal{W}_{\mathcal{D}^l_\mathbf{u}}(\varpi)$ also converges to 0 by definition of limit-conformity; since this space is dense in $H_{\mathrm{div},\Gamma}(\Omega\backslash\overline{\Gamma}; \mathcal{S}_d(\mathbb{R}))$, the arguments in [20, Lemma 2.17] show that this convergence also holds for the argument $\varpi = \varpi(\bar{\mathbf{u}}) - b\,\bar{p}^E_m \mathbb{I}$. Estimate (45) therefore yields the convergences (46). □

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