Complete integrability of the Benjamin-Ono equation on the multi-soliton manifolds

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Abstract This paper is dedicated to proving the complete integrability of the Benjamin–Ono (BO) equation on the line when restricted to every N-soliton manifold, denoted by $\mathcal{U}_N$. We construct generalized action–angle coordinates which establish a real analytic symplectomorphism from $\mathcal{U}_N$ onto some open convex subset of $\mathbb{R}^{2N}$ and allow to solve the equation by quadrature for any such initial datum. As a consequence, $\mathcal{U}_N$ is the universal covering of the manifold of N-gap potentials for the BO equation on the torus as described by Gérard–Kappeler [19]. The global well-posedness of the BO equation in $\mathcal{U}_N$ is given by a polynomial characterization and a spectral characterization of the manifold $\mathcal{U}_N$. Besides the spectral analysis of the Lax operator of the BO equation and the shift semigroup acting on some Hardy spaces, the construction of such coordinates also relies on the use of a generating functional, which encodes the entire BO hierarchy.

Keywords Benjamin–Ono equation, generalized action–angle coordinates, Lax pair, Hardy space, inverse spectral transform, multi-solitons, universal covering manifold

Throughout this paper, the main results of each section are stated at the beginning. Their proofs are left inside the corresponding subsections.

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Contents

1 Introduction

The Benjamin–Ono (BO) equation on the line reads as

$$\partial_t u = \mathrm{H}\partial_x^2 u - \partial_x(u^2), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1}$$

where u is real-valued and $\mathrm{H} = -i\mathrm{sign}(\mathrm{D}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the Hilbert transform, $\mathrm{D} = -i\partial_x$,

$$\widehat{\mathrm{H}f}(\xi) = -i\mathrm{sign}(\xi)\hat{f}(\xi), \qquad \forall f \in L^2(\mathbb{R}). \tag{1.2}$$

$\mathrm{sign}(\pm\xi) = \pm 1$, for all $\xi > 0$ and $\mathrm{sign}(0) = 0$, $\hat{f} \in L^2(\mathbb{R})$ denotes the Fourier–Plancherel transform of $f \in L^2(\mathbb{R})$. We adopt the convention $L^p(\mathbb{R}) = L^p(\mathbb{R},\mathbb{C})$. Its $\mathbb{R}$-subspace consisting of all real-valued L^p-functions is specially emphasized as $L^p(\mathbb{R},\mathbb{R})$ throughout this paper. Equipped with the inner product $(f,g) \in L^2(\mathbb{R})\times L^2(\mathbb{R}) \mapsto \langle f,g\rangle_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}x \in \mathbb{C}$, $L^2(\mathbb{R})$ is a $\mathbb{C}$-Hilbert space.

Derived by Benjamin [4] and Ono [49], this equation describes the evolution of weakly nonlinear internal long waves in a two-layer fluid. The BO equation is globally well-posed in every Sobolev spaces $H^s(\mathbb{R},\mathbb{R})$, $s \geq 0$. (see Tao [63] for $s \geq 1$, Burq–Planchon [8] for $s > \frac{1}{4}$, Ionescu–Kenig [33], Molinet–Pilod [43] and Ifrim–Tataru [29] for $s \geq 0$, etc.) Recall the scaling and translation invariances of equation (1.1): if $u = u(t,x)$ is a solution, so is $u_{c,y} : (t,x) \mapsto cu(c^2t, c(x-y))$. A smooth solution $u = u(t,x)$ is called a solitary wave of (1.1) if there exists $\mathcal{R} \in C^\infty(\mathbb{R})$ solving the following non local elliptic equation

$$\mathrm{H}\mathcal{R}' + \mathcal{R} - \mathcal{R}^2 = 0, \qquad \mathcal{R}(x) > 0 \tag{1.3}$$

and $u(t,x) = \mathcal{R}_c(x - y - ct)$, where $\mathcal{R}_c(x) = c\mathcal{R}(cx)$, for some $c > 0$ and $y \in \mathbb{R}$. The unique (up to translation) solution of equation (1.3) is given by the following formula

$$\mathcal{R}(x) = \frac{2}{1+x^2}, \qquad \forall x \in \mathbb{R}, \tag{1.4}$$

in Benjamin [4] and Amick–Toland [2] for the uniqueness statement. Inspired from the complete classification of solitary waves of the BO equation, we introduce the main object of this paper.

Definition 1.1. *A function of the form $u(x) = \sum_{j=1}^N \mathcal{R}_{c_j}(x - x_j)$ is called an N-soliton, for some positive integer $N \in \mathbb{N}_+ := \mathbb{Z}\bigcap(0,+\infty)$, where $c_j > 0$ and $x_j \in \mathbb{R}$, for every $j = 1, 2, \cdots, N$. Let $\mathcal{U}_N \subset L^2(\mathbb{R},\mathbb{R})$ denote the subset consisting of all the N-solitons.*

In the point of view of topology and differential manifolds, the subset $\mathcal{U}_N$ is a simply connected, real analytic, embedded submanifold of the $\mathbb{R}$-Hilbert space $L^2(\mathbb{R},\mathbb{R})$. It has real dimension $2N$. The tangent space to $\mathcal{U}_N$ at an arbitrary N-soliton is included in an auxiliary space

$$\mathcal{T} := \{h \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x) : h(\mathbb{R}) \subset \mathbb{R}, \quad \int_{\mathbb{R}} h = 0\}, \tag{1.5}$$

in which a 2-covector $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{T}^*)$ is well defined by $\boldsymbol{\omega}(h_1,h_2) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi}\mathrm{d}\xi$, for every $h_1, h_2 \in \mathcal{T}$, by Hardy's inequality. We define a translation-invariant 2-form $\omega : u \in \mathcal{U}_N \mapsto \boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{T}^*)$, endowed with which $\mathcal{U}_N$ is a symplectic manifold. The tangent space to $\mathcal{U}_N$ at $u \in \mathcal{U}_N$ is denoted by $\mathcal{T}_u(\mathcal{U}_N)$. For every smooth function $f : \mathcal{U}_N \to \mathbb{R}$, its Hamiltonian vector field $X_f \in \mathfrak{X}(\mathcal{U}_N)$ is given by

$$X_f : u \in \mathcal{U}_N \mapsto \partial_x \nabla_u f(u) \in \mathcal{T}_u(\mathcal{U}_N),$$

where $\nabla_u f(u)$ denotes the Fréchet derivative of f, i.e. $\mathrm{d}f(u)(h) = \langle h, \nabla_u f(u)\rangle_{L^2}$, for every $h \in \mathcal{T}_u(\mathcal{U}_N)$. The Poisson bracket of f and another smooth function $g : \mathcal{U}_N \to \mathbb{R}$ is defined by

$$\{f,g\} : u \in \mathcal{U}_N \mapsto \omega_u(X_f(u), X_g(u)) = \langle \partial_x \nabla_u f(u), \nabla_u g(u)\rangle_{L^2} \in \mathbb{R}.$$

Then the BO equation (1.1) in the N-soliton manifold $(\mathcal{U}_N, \omega)$ can be written in Hamiltonian form

$$\partial_t u = X_E(u), \qquad \text{where} \quad E(u) = \frac{1}{2}\langle |\mathrm{D}|u, u\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} - \frac{1}{3}\int_{\mathbb{R}} u^3. \tag{1.6}$$

The Cauchy problem of (1.6) is globally well-posed in the manifold $\mathcal{U}_N$ (see proposition 4.9). Inspired from the construction of Birkhoff coordinates of the space-periodic BO equation discovered by Gérard–Kappeler [19], we want to show the complete integrability of (1.6) in the Liouville sense.

Let $\Omega_N := \{(r^1, r^2, \cdots, r^N) \in \mathbb{R}^N : r^j < r^{j+1} < 0, \quad \forall j = 1, 2, \cdots, N-1\}$ denote the subset of actions and $\nu = \sum_{j=1}^N \mathrm{d}r^j \wedge \mathrm{d}\alpha^j$ denotes the canonical symplectic form on $\Omega_N \times \mathbb{R}^N$. The main result of this paper is stated as follows.

Theorem 1. *There exists a real analytic symplectomorphism $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ such that*

$$E \circ \Phi_N^{-1}(r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) = -\frac{1}{2\pi} \sum_{j=1}^{N} |r^j|^2. \tag{1.7}$$

Remark 1.2. *A consequence of theorem 1 is that $\mathcal{U}_N$ is simply connected. In fact the manifold $\mathcal{U}_N$ can be interpreted as the universal covering of the manifold of N-gap potentials for the Benjamin–Ono equation on the torus as described by Gérard–Kappeler in [19]. We refer to section **A** for a direct proof of these topological facts, independently of theorem 1.*

Remark 1.3. *Then $\Phi_N : u \in \mathcal{U}_N \mapsto (I_1(u), I_2(u), \cdots, I_N(u); \gamma_1(u), \gamma_2(u), \cdots, \gamma_N(u)) \in \Omega_N \times \mathbb{R}^N$ introduces the generalized action–angle coordinates of the BO equation in the N-soliton manifold, i.e.*

$$\{I_k, E\}(u) = 0, \qquad \{\gamma_k, E\}(u) = \frac{I_k(u)}{\pi}, \qquad \forall u \in \mathcal{U}_N. \tag{1.8}$$

Theorem 1 gives a complete description of the orbit structure of the flow of equation (1.6) up to real bi-analytic conjugacy. Let $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ denote the solution of equation (1.6), $r^k(t) = I_k \circ u(t)$ denotes action coordinates and $\alpha^k(t) = \gamma_k \circ u(t)$ denotes the generalized angle coordinates, then we have

$$r^k(t) = r^k(0), \qquad \alpha^k(t) = \alpha^k(0) - \frac{r^k(0)t}{\pi}, \qquad \forall k = 1, 2, \cdots, N. \tag{1.9}$$

We refer to definition 5.1 and theorem 5.2 for a precise description of Φ_N.

In order to establish the link between the action–angle coordinates and the translation–scaling parameters of an N-soliton, we introduce the inverse spectral matrix associated to Φ_N, denoted by

$$M : u \in \mathcal{U}_N \mapsto (M_{kj}(u))_{1 \le j,k \le N} \in \mathbb{C}^{N \times N}, \quad M_{kj}(u) = \begin{cases} \frac{2\pi i}{I_k(u) - I_j(u)} \sqrt{\frac{I_k(u)}{I_j(u)}}, & \text{if} \quad j \neq k, \\ \gamma_j(u) + \frac{\pi i}{I_j(u)}, & \text{if} \quad j = k, \end{cases} \tag{1.10}$$

where $I_k, \gamma_k : \mathcal{U} \to \mathbb{R}$ is given by remark 1.3. Then $\mathcal{U}_N$ has the following polynomial characterization.

Proposition 1.4. *A real-valued function $u \in \mathcal{U}_N$ if and only if there exists a monic polynomial $Q_u \in \mathbb{C}[X]$ of degree N, whose roots are contained in the lower half-plane $\mathbb{C}_-$ and $u = -2\mathrm{Im}\frac{Q_u'}{Q_u}$. Precisely, Q_u is unique and is the characteristic polynomial of the matrix $M(u) \in \mathbb{C}^{N \times N}$ defined by (1.10).*

An N-soliton is expressed by $u(x) = \sum_{j=1}^{N} \mathcal{R}_{c_j}(x - x_j)$ if and only if its translation–scaling parameters $\{x_j - c_j^{-1} i\}_{1 \le j \le N} \subset \mathbb{C}_-^N$ are the roots of the characteristic polynomial $Q_u(X) = \det(X - M(u))$, whose coefficients are expressed in terms of the action–angle coordinates $(I_j(u), \gamma_j(u))_{1 \le j \le N} \in \Omega_N \times \mathbb{R}^N$. Proposition 1.4 is restated with more details in proposition 4.1, formula (5.11) and theorem 4.8 which gives a spectral characterization of $\mathcal{U}_N$. If $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solves the BO equation (1.1), then we have the following explicit formula

$$u(t, x) = 2\mathrm{Im}\langle \big(M(u_0) - (x + \tfrac{t}{\pi}\mathfrak{V}(u_0))\big)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N}, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.11}$$

where the inner product of $\mathbb{C}^N$ is $\langle X, Y\rangle_{\mathbb{C}^N} = X^T \overline{Y}$, for every $u \in \mathcal{U}_N$, the matrix $\mathfrak{V}(u) \in \mathbb{C}^{N \times N}$ and the vectors $X(u), Y(u) \in \mathbb{C}^N$ are defined by

$$\sqrt{2\pi} X(u)^T = (\sqrt{|I_1(u)|}, \sqrt{|I_2(u)|}, \cdots, \sqrt{|I_N(u)|}),$$
$$\sqrt{2\pi}^{-1} Y(u)^T = (\sqrt{|I_1(u)|^{-1}}, \sqrt{|I_2(u)|^{-1}}, \cdots, \sqrt{|I_N(u)|^{-1}}),$$
$$\mathfrak{V}(u) = \begin{pmatrix} I_1(u) & & \\ & I_2(u) & \\ & & \ddots \\ & & & I_N(u) \end{pmatrix}.$$

1.1 Notation

Before outlining the construction of action–angle coordinates, we introduce some notations used in this paper. The indicator function of a subset $A \subset X$ is denoted by $\mathbf{1}_A$, i.e. $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \in X \backslash A$. Recall that $\mathrm{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the Hilbert transform given by (1.2). Set $\mathrm{Id}_{L^2(\mathbb{R})}(f) = f$, for every $f \in L^2(\mathbb{R})$. Let $\Pi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Szegő projector, defined by

$$\Pi := \frac{\mathrm{Id}_{L^2(\mathbb{R})} + i\mathrm{H}}{2} \Longleftrightarrow \widehat{\Pi f}(\xi) = \mathbf{1}_{[0,+\infty)}(\xi)\hat{f}(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall f \in L^2(\mathbb{R}). \tag{1.12}$$

If $\mathfrak{O}$ is an open subset of $\mathbb{C}$, we denote by $\mathrm{Hol}(\mathfrak{O})$ all holomorphic functions on $\mathfrak{O}$. Let the upper half-plane and the lower half-plane be denoted by $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$ respectively. For every $p \in (0, +\infty]$, we denote by L^p_+ to be the Hardy space of holomorphic functions on $\mathbb{C}_+$ such that $L^p_+ = \{g \in \mathrm{Hol}(\mathbb{C}_+) : \|g\|_{L^p_+} < +\infty\}$, where

$$\|g\|_{L^p_+} = \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^p \mathrm{d}x \right)^{\frac{1}{p}}, \qquad \text{if} \quad p \in (0, +\infty), \tag{1.13}$$

and $\|g\|_{L^\infty_+} = \sup_{z \in \mathbb{C}_+} |g(z)|$. A function $g \in L^\infty_+$ is called an *inner function* if $|g| = 1$ on $\mathbb{R}$. When $p = 2$, the Paley–Wiener theorem yields the identification between L^2_+ and $\Pi[L^2(\mathbb{R})]$:

$$L^2_+ = \mathcal{F}^{-1}[L^2(0, +\infty)] = \{f \in L^2(\mathbb{R}) : \mathrm{supp} \hat{f} \subset [0, +\infty)\} = \Pi(L^2(\mathbb{R})),$$

where $\mathcal{F} : f \in L^2(\mathbb{R}) \mapsto \hat{f} \in L^2(\mathbb{R})$ denotes the Fourier–Plancherel transform. Similarly, we set $L^2_- = (\mathrm{Id}_{L^2(\mathbb{R})} - \Pi)(L^2(\mathbb{R}))$. Let the filtered Sobolev spaces be denoted as $H^s_+ := L^2_+ \bigcap H^s(\mathbb{R})$ and $H^s_- := L^2_- \bigcap H^s(\mathbb{R})$, for every $s \geq 0$.

The domain of definition of an unbounded operator $\mathcal{A}$ on some Hilbert space $\mathcal{E}$ is denoted by $\mathbf{D}(\mathcal{A}) \subset \mathcal{E}$. Given another operator $\mathcal{B}$ on $\mathbf{D}(\mathcal{B}) \subset \mathcal{E}$ such that $\mathcal{A}(\mathbf{D}(\mathcal{A})) \subset \mathbf{D}(\mathcal{B})$ and $\mathcal{B}(\mathbf{D}(\mathcal{B})) \subset \mathbf{D}(\mathcal{A})$, their Lie bracket is an operator defined on $\mathbf{D}(\mathcal{A}) \bigcap \mathbf{D}(\mathcal{B}) \subset \mathcal{E}$, which is given by

$$[\mathcal{A}, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}. \tag{1.14}$$

If the operator $\mathcal{A}$ is self-adjoint, let $\sigma(\mathcal{A})$ denote its spectrum, $\sigma_{\mathrm{pp}}(\mathcal{A})$ denotes the set of its eigenvalues and $\sigma_{\mathrm{cont}}(\mathcal{A})$ denotes its continuous spectrum. Then $\sigma_{\mathrm{cont}}(\mathcal{A}) \bigcup \overline{\sigma_{\mathrm{pp}}(\mathcal{A})} = \sigma(A) \subset \mathbb{R}$. Given two $\mathbb{C}$-Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, let $\mathfrak{B}(\mathcal{E}_1, \mathcal{E}_2)$ denote the $\mathbb{C}$-Banach space of all bounded $\mathbb{C}$-linear transformations $\mathcal{E}_1 \to \mathcal{E}_2$, equipped with the uniform norm.

Given a smooth manifold $\mathbf{M}$ of real dimension N, let $C^\infty(\mathbf{M})$ denote all smooth functions $f : \mathbf{M} \to \mathbb{R}$ and the set of all smooth vector fields is denoted by $\mathfrak{X}(\mathbf{M})$. The tangent (resp. cotangent) space to $\mathbf{M}$ at $p \in \mathbf{M}$ is denoted by $\mathcal{T}_p(\mathbf{M})$ (resp. $\mathcal{T}^*_p(\mathbf{M})$). Given $k \in \mathbb{N}$, the $\mathbb{R}$-vector space of smooth k-forms on $\mathbf{M}$ is denoted by $\mathbf{\Omega}^k(\mathbf{M})$. Given a $\mathbb{R}$-vector space $\mathbb{V}$, we denote by $\mathbf{\Lambda}^k(\mathbb{V}^*)$ the vector space of all k-covectors on $\mathbb{V}$. Given a smooth covariant tensor field $\mathbf{A}$ on $\mathbf{M}$ and $X \in \mathfrak{X}(\mathbf{M})$, the Lie derivative of $\mathbf{A}$ with respect to X is denoted by $\mathscr{L}_X(\mathbf{A})$, which is also a smooth tensor field on $\mathbf{M}$. If $\mathbf{N}$ is another smooth manifold, $\mathbf{F} : \mathbf{N} \to \mathbf{M}$ is a smooth map and $\mathbf{A}$ is a smooth covariant k-tensor field on $\mathbf{M}$, the pullback of $\mathbf{A}$ by $\mathbf{F}$ is denoted by $\mathbf{F}^*\mathbf{A}$, which is a smooth k-tensor field on $\mathbf{N}$ defined by $\forall p \in \mathbf{N}$, $\forall j = 1, 2, \cdots, k$,

$$(\mathbf{F}^*\mathbf{A})_p(v_1, v_2, \cdots, v_k) = \mathbf{A}_{\mathbf{F}(p)} \left(\mathrm{d}\mathbf{F}(p)(v_1), \mathrm{d}\mathbf{F}(p)(v_2), \cdots, \mathrm{d}\mathbf{F}(p)(v_k) \right), \quad \forall v_j \in \mathcal{T}_p(\mathbf{N}). \tag{1.15}$$

Given a positive integer N, let $\mathbb{C}_{\leq N-1}[X]$ denote the $\mathbb{C}$-vector space of all polynomials with complex coefficients whose degree is no greater than $N-1$ and $\mathbb{C}_N[X] = \mathbb{C}_{\leq N}[X]\backslash\mathbb{C}_{\leq N-1}[X]$ consists of all polynomials of degree exactly N. $\mathbb{R}_+ = [0,+\infty)$ and $\mathbb{R}_+^* = (0,+\infty)$. $D(z,r) \subset \mathbb{C}$ denotes the open disc of radius $r>0$, whose center is $z \in \mathbb{C}$.

1.2 Organization of this paper

The construction of action–angle coordinates for the BO equation (1.6) mainly relies on the Lax pair formulation $\partial_t L_u = [B_u, L_u]$, discovered by Nakamura [45] and Bock–Kruskal [6]. Section **2** is dedicated to the spectral analysis of the Lax operator $L_u : h \in H_+^1 \mapsto -i\partial_x h - \Pi(uh) \in L_+^2$ given by definition 2.1 for general symbol $u \in L^2(\mathbb{R},\mathbb{R})$, where Π denotes the Szegő projector given in (1.12) and the Hardy space L_+^2 is defined in (1.13). L_u is an unbounded self-adjoint operator on L_+^2 that is bounded from below, it has essential spectrum $\sigma_{\mathrm{ess}}(L_u) = [0,+\infty)$. If $x \mapsto xu(x) \in L^2(\mathbb{R})$ in addition, every eigenvalue is negative and simple, thanks to an identity firstly found by Wu [65]. Then we introduce a generating function which encodes the entire BO hierarchy,

$$\mathcal{H}_\lambda(u) = \langle (L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}, \qquad \text{if} \quad \lambda \in \mathbb{C}\backslash\sigma(-L_u), \tag{1.16}$$

in definition 2.9. It provides a sequence of conservation laws controlling every Sobolev norms.

In section **3**, we study the shift semigroup $(S(\eta)^*)_{\eta\geq 0}$ acting on the Hardy space L_+^2, where $S(\eta)f = e_\eta f$ and $e_\eta(x) = e^{i\eta x}$. Then a weak version of Beurling–Lax theorem can be obtained by solving a linear differential equation with constant coefficients. Every N-dimensional subspace of L_+^2 that is invariant under its infinitesimal generator $G = i\frac{\mathrm{d}}{\mathrm{d}\eta}\big|_{\eta=0^+} S(\eta)^*$ is of the form $\frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, for some monic polynomial Q whose roots are contained in the lower half-plane $\mathbb{C}_-$.

In section **4**, the real analytic structure and symplectic structure of the N-soliton subset $\mathcal{U}_N$ are established at first. Then we continue the spectral analysis of the Lax operator L_u, $\forall u \in \mathcal{U}_N$. L_u has N simple eigenvalues $\lambda_1^u < \lambda_2^u < \cdots < \lambda_N^u < 0$ and the Hardy space L_+^2 splits as

$$L_+^2 = \mathscr{H}_{\mathrm{cont}}(L_u) \bigoplus \mathscr{H}_{\mathrm{pp}}(L_u), \quad \mathscr{H}_{\mathrm{cont}}(L_u) = \mathscr{H}_{\mathrm{ac}}(L_u) = \Theta_u L_+^2, \quad \mathscr{H}_{\mathrm{pp}}(L_u) = \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u}. \tag{1.17}$$

where Q_u denotes the characteristic polynomial of u given by proposition 1.4 and $\Theta_u = \frac{\overline{Q}_u}{Q_u}$ is an inner function on the upper half-plane $\mathbb{C}_+$. Proposition 1.4 is proved by identifying $M(u)$ in (1.10) as the matrix of the restriction $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$ associated to the spectral basis $\{\varphi_1^u, \varphi_2^u, \cdots, \varphi_N^u\}$, where $\varphi_j^u \in \mathrm{Ker}(\lambda_j^u - L_u)$ such that $\|\varphi_j^u\|_{L^2} = 1$ and $\int_{\mathbb{R}} u\varphi_j^u > 0$. The generating function $\mathcal{H}_\lambda$ in (1.16) can be identified as the Borel–Cauchy transform of the spectral measure of L_u associated to the vector Πu, which yields the invariance of $\mathcal{U}_N$ under the BO flow in $H^\infty(\mathbb{R},\mathbb{R})$. Hence (1.6) is a globally well-posed Hamiltonian system on $\mathcal{U}_N$.

Section **5** is dedicated to completing the proof of theorem 1. The generalized angle-variables are the real parts of the diagonal elements of the matrix $M(u)$, i.e. $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2} \in \mathbb{R}$ and the action-variables are $I_j : u \in \mathcal{U}_N \mapsto 2\pi\lambda_j^u \in \mathbb{R}$. Thanks to the Lax pair formulation $\mathrm{d}L(u)(X_{\mathcal{H}_\lambda}(u)) = [B_u^\lambda, L_u]$, where $L : u \in \mathcal{U}_N \mapsto L_u \in \mathfrak{B}(H_+^1, L_+^2)$ is $\mathbb{R}$-affine and B_u^λ is some skew-adjoint operator on L_+^2, we have the following formulas of Poisson brackets,

$$2\pi\{\lambda_j, \gamma_k\} = \mathbf{1}_{j=k}, \qquad \{\gamma_j, \gamma_k\} = 0 \quad \text{on} \quad \mathcal{U}_N, \quad 1 \leq j,k \leq N. \tag{1.18}$$

which implies that $\Phi_N : u \in \mathcal{U}_N \mapsto (I_1(u), I_2(u), \cdots, I_N(u); \gamma_1(u), \gamma_2(u), \cdots, \gamma_N(u)) \in \Omega_N \times \mathbb{R}^N$ is a real analytic immersion. The diffeomorphism property of Φ_N is given by Hadamard's global inverse theorem. The inverse spectral formula $\Pi u = \frac{Q_u'}{Q_u}$ with $Q_u(X) = \det(X - G|_{\mathscr{H}_{\rm pp}(L_u)})$, which is restated as formula (5.11), implies the explicit formula (1.11) of all multi-soliton solutions of the BO equation (1.1) and (5.11) provides an alternative proof of the injectivity of Φ_N. Finally, we show that $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is a symplectomorphism by restricting the 2-form $\omega - \Phi_N^* \nu$ to a special Lagrangian submanifold $\Lambda_N := \bigcap_{j=1}^N \gamma_j^{-1}(0) \subset \mathcal{U}_N$.

In appendix **A**, we establish the simple connectedness of $\mathcal{U}_N$ and a covering map from $\mathcal{U}_N$ to the manifold of N-gap potentials from their constructions without using the integrability theorems.

1.3 Related work

The BO equation has been extensively studied for nearly sixty years in the domain of partial differential equations. We refer to Saut [60] for an excellent account of these results. Besides the global well-posedness problem, various properties of its multi-soliton solutions has been investigated in details. Matsuno [41] has found the explicit expression of multi-soliton solutions of (1.1) by following the bilinear method of Hirota [26]. The multi-phase solutions (periodic multi-solitons) have been constructed by Satsuma–Ishimori [58] at first. We point out the work of Amick–Toland [2] on the characterization of 1-soliton solutions which can also be revisited by theorem 1 and proposition 1.4. In Dobrokhotov–Krichever [10], the multi-phase solutions are constructed by finite zone integration and they have also established an inversion formula for multi-phase solutions. Compared to their work, we give a geometric description of the inverse spectral transform by proving the real bi-analyticity and the symplectomorphism property of the action–angle map. Furthermore, the inverse spectral formula

$$\Pi u(x) = i\frac{Q_u'(x)}{Q_u(x)}, \qquad Q_u(x) = \det(x - G|_{\mathscr{H}_{\rm pp}(L_u)}) = \det(x - M(u)), \qquad \forall x \in \mathbb{R}. \tag{1.19}$$

provides a spectral connection between the Lax operator L_u and the infinitesimal generator G. The idea of introducing generating function $\mathcal{H}_\lambda$ has also been used for the quantum BO equation in Nazarov–Sklyanin [46]. Their method has also been developed by Moll [44] for the classical BO equation. The asymptotic stability of soliton solutions and of solutions starting with sums of widely separated soliton profiles is obtained by Kenig–Martel [34].

Concerning the investigation of integrability for the BO equation on $\mathbb{R}$ besides the discovery of Lax pair formulation, we mention the pioneering work of Ablowitz–Fokas [1], Coifman–Wickerhauser [9], Kaup–Matsuno [35] and Wu [65, 66] for the inverse scattering transform. In the space-periodic regime, the BO equation on the torus $\mathbb{T}$ admits global Birkhoff coordinates on $L^2_{r,0}(\mathbb{T}) := \{v \in L^2(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} v = 0\}$ in Gérard–Kappeler [19]. We refer to Gérard–Kappeler–Topalov [20] to see that the Birkhoff coordinates of the BO equation on the torus can be extended to a larger Sobolev space $H^s_{r,0}(\mathbb{T}) := \{v \in H^s(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} v = 0\}$, for every $-\frac{1}{2} < s < 0$. We point out that both Korteweg–de Vries equation on $\mathbb{T}$ (see Kappeler–Pöschel [30]) and the defocusing cubic Schödinger equation on $\mathbb{T}$ (see Grébert–Kappeler [24]) admit global Birkhoff coordinates. The theory of finite-dimensional Hamiltonian system is transferred to the BO, KdV and dNLS equation on $\mathbb{T}$ through the submanifolds of corresponding finite-gap potentials, which are introduced to solve the periodic KdV initial problem. We refer to Matveev [42] for details.

Moreover, the cubic Szegő equation both on $\mathbb{T}$ (see Gérard–Grellier [15, 16, 17, 18]) and on $\mathbb{R}$ (see Pocovnicu [51, 52]) admit global (generalized) action–angle coordinates on all finite-rank generic rational func-

tion manifolds, denoted respectively by $\mathcal{M}(N)^{\mathbb{T}}_{\rm gen}$ and $\mathcal{M}(N)^{\mathbb{R}}_{\rm gen}$. Moreover, the cubic Szegő equation both on $\mathbb{T}$ and on $\mathbb{R}$ have inverse spectral formulas which permit the Szegő flows to be expressed explicitly in terms of time-variables and initial data without using action–angle coordinates. The shift semigroup $(S(\eta)^*)_{\eta\geq 0}$ and its infinitesimal generator G are also used in Pocovnicu [52] to establish the integrability of the cubic Szegő equation on the line.

The BO equation admits an infinite hierarchy of conservation laws controlling every H^s-norm (see Ablowitz–Fokas [1], Coifman–Wickerhauser [9] in the case $2s\in\mathbb{N}$ and Talbut [62] in the case $-\frac{1}{2}<s<0$ and conservation law controlling Besov norms etc.), so does the KdV equation and the NLS equation (see Killip–Vişan–Zhang [37], Koch–Tataru [36], Faddeev–Takhtajan [11], Gérard [14] and Sun [61] etc.)

Throughout this paper, the main results of each section are stated at the beginning. Their proofs are left inside the corresponding subsections.

2 The Lax operator

This section is dedicated to studying the Lax operator L_u in the Lax pair formulation of the BO equation (1.1), discovered by Nakamura [45] and Bock–Kruskal [6]. Then we describe the location and revisit the simplicity of eigenvalues of L_u. At last, we introduce a generating functional $\mathcal{H}_\lambda$ which encodes the entire BO hierarchy. The equation $\partial_t u=\partial_x\nabla_u\mathcal{H}_\lambda(u)$ also enjoys a Lax pair structure with the same Lax operator L_u.

Definition 2.1. *Given $u\in L^2(\mathbb{R},\mathbb{R})$, its associated Lax operator L_u is an unbounded operator on L^2_+, given by $L_u:=\mathrm{D}-T_u$, where $\mathrm{D}:h\in H^1_+\mapsto -i\partial_x h\in L^2_+$ and T_u denotes the Toeplitz operator of symbol u, defined by $T_u:h\in H^1_+\mapsto\Pi(uh)\in L^2_+$, where the Szegő projector $\Pi:L^2(\mathbb{R})\to L^2_+$ is given by (1.12). We set $B_u:=i(T_{|\mathrm{D}|u}-T_u^2)$.*

Both D and T_u are densely defined symmetric operators on L^2_+ and $\|T_u(h)\|_{L^2}\leq\|u\|_{L^2}\|h\|_{L^\infty}$, for every $h\in H^1_+$ and $u\in L^2(\mathbb{R},\mathbb{R})$. Moreover, the Fourier–Plancherel transform implies that D is a self-adjoint operator on L^2_+, whose domain of definition is H^1_+.

Proposition 2.2. *If $u\in L^2(\mathbb{R},\mathbb{R})$, then L_u is an unbounded self-adjoint operator on L^2_+, whose domain of definition is $\mathbf{D}(L_u)=H^1_+$. Moreover, L_u is bounded from below. The essential spectrum of L_u is $\sigma_{\rm ess}(L_u)=\sigma_{\rm ess}(\mathrm{D})=[0,+\infty)$ and its pure point spectrum satisfies $\sigma_{\rm pp}(L_u)\subset[-\frac{C^2}{4}\|u\|^2_{L^2},+\infty)$, where $C=\inf_{f\in H^1_+\setminus\{0\}}\frac{\||\mathrm{D}|^{\frac{1}{4}}f\|_{L^2}}{\|f\|_{L^4}}$ denotes the Sobolev constant.*

Thanks to an identity firstly found by Wu [65] in the negative eigenvalue case, we show the simplicity of the pure point spectrum $\sigma_{\rm pp}(L_u)$, if $u\in L^2(\mathbb{R},(1+x^2)\mathrm{d}x)$ is real-valued.

Proposition 2.3. *Assume that $u\in L^2(\mathbb{R};\mathbb{R})$ and $x\mapsto xu(x)\in L^2(\mathbb{R})$. For every $\lambda\in\mathbb{R}$ and $\varphi\in\mathrm{Ker}(\lambda-L_u)$, we have $\widehat{u\varphi}\in C^1(\mathbb{R})\bigcap H^1(\mathbb{R})$ and the following identity holds,*

$$\Big|\int_{\mathbb{R}}u\varphi\Big|^2=-2\pi\lambda\int_{\mathbb{R}}|\varphi|^2. \tag{2.1}$$

Thus $\sigma_{\rm pp}(L_u)\subset(-\infty,0)$ and for every $\lambda\in\sigma_{\rm pp}(L_u)$, we have

$$\mathrm{Ker}(\lambda-L_u)\subset\{\varphi\in H^1_+:\hat{\varphi}_{|\mathbb{R}_+}\in C^1(\mathbb{R}_+)\bigcap H^1(\mathbb{R}_+)\quad\text{and}\quad\xi\mapsto\xi[\hat{\varphi}(\xi)+\partial_\xi\hat{\varphi}(\xi)]\in L^2(\mathbb{R}_+)\}. \tag{2.2}$$

Corollary 2.4. *Assume that $u \in L^2(\mathbb{R};\mathbb{R})$ and $x \mapsto xu(x) \in L^2(\mathbb{R})$. Then every eigenvalue of L_u is simple. If $u \in L^\infty(\mathbb{R})$ in addition, then $\sigma_{\rm pp}(L_u)$ is a finite subset of $[-\frac{C^2\|u\|_{L^2}^2}{4}, 0)$.*

Proof. Fix $\lambda \in \sigma_{\rm pp}(L_u)$ and set $V_\lambda = \mathrm{Ker}(\lambda - L_u)$, then $\dim_{\mathbb{C}}(V_\lambda) \geq 1$. We define a linear form $A : V_\lambda \to \mathbb{C}$ such that

$$A(\varphi) := \int_{\mathbb{R}} u\varphi$$

Then identity (2.1) yields that $\mathrm{Ker}(A) = \{0\}$. Thus $V \cong V/\mathrm{Ker}(A) \cong \mathrm{Im}(A) \hookrightarrow \mathbb{C}$. So we have $\dim_{\mathbb{C}}(V_\lambda) = 1$. When $u \in L^\infty(\mathbb{R})$ in addition, the finiteness of $\sigma_{\rm pp}(L_u) \bigcap (-\infty, 0)$ is given by Theorem 1.2 of Wu [65]. □

We recall some known results of global well-posedness of the BO equation on the line.

Proposition 2.5. *For every $s \geq 0$, the Fréchet space $C(\mathbb{R}, H^s(\mathbb{R}))$ is endowed with the topology of uniform convergence on every compact subset of $\mathbb{R}$. There exists a unique continuous mapping $u_0 \in H^s(\mathbb{R}) \mapsto u \in C(\mathbb{R}, H^s(\mathbb{R}))$ such that u solves the BO equation (1.1) with initial datum $u(0) = u_0$.*

Proof. See Tao [63], Burq–Planchon [8], Ionescu–Kenig [33], Molinet–Pilod [43], Ifrim–Tataru [29] etc. □

Proposition 2.6. *For every $n \in \mathbb{N}$, if $u_0 \in H^{\frac{n}{2}}(\mathbb{R},\mathbb{R})$, let $u : t \in \mathbb{R} \mapsto u(t) \in H^{\frac{n}{2}}(\mathbb{R},\mathbb{R})$ solves equation (1.1) with initial datum $u(0) = u_0$, then $C(\|u_0\|_{H^{\frac{n}{2}}}) := \sup_{t\in\mathbb{R}} \|u(t)\|_{H^{\frac{n}{2}}} < +\infty$.*

Proof. See Ablowitz–Fokas [1], Coifman–Wickerhauser [9]. □

When $u \in H^2(\mathbb{R},\mathbb{R})$, the Toeplitz operators $T_{|\mathrm{D}|u}$ and T_u are bounded both on L^2_+ and on H^1_+. So B_u is a bounded skew-adjoint operator both on L^2_+ and on H^1_+.

Proposition 2.7. *Let $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R},\mathbb{R})$ denote the unique solution of equation (1.1), then*

$$\partial_t L_{u(t)} = [B_{u(t)}, L_{u(t)}] \in \mathfrak{B}(H^1_+, L^2_+), \qquad \forall t \in \mathbb{R}. \tag{2.3}$$

Let $U : t \mapsto U(t) \in \mathfrak{B}(L^2_+) := \mathfrak{B}(L^2_+, L^2_+)$ denote the unique solution of the following equation

$$U'(t) = B_{u(t)}U(t), \qquad U(0) = \mathrm{Id}_{L^2_+}, \tag{2.4}$$

if $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R},\mathbb{R})$ denote the unique solution of equation (1.1). The system (2.4) is globally well-posed in $\mathfrak{B}(L^2_+)$, thanks to proposition 2.6, the following estimate

$$\|B_u(h)\|_{L^2} \lesssim (\|u\|_{H^2} + \|u\|_{H^1}^2)\|h\|_{L^2}, \qquad \forall h \in L^2_+, \quad \forall u \in H^2(\mathbb{R},\mathbb{R}).$$

and a classical Cauchy theorem (see for instance lemma 7.2 of Sun [61]). Since $B_u^* = -B_u$, the operator $U(t)$ is unitary for every $t \in \mathbb{R}$. Thus, the Lax pair formulation (2.3) of the BO equation (1.1) is equivalent to the unitary equivalence between $L_{u(t)}$ and $L_{u(0)}$,

$$L_{u(t)} = U(t)L_{u(0)}U(t)^* \in \mathfrak{B}(H^1_+, L^2_+). \tag{2.5}$$

On the one hand, the spectrum of L_u is invariant under the BO flow. In particular, we have $\sigma_{\rm pp}(L_{u(t)}) = \sigma_{\rm pp}(L_{u(0)})$. On the other hand, there exists a sequence of conservation laws controlling every Sobolev norms $H^{\frac{n}{2}}(\mathbb{R})$, $n \geq 0$. Furthermore, the Lax operator in the Lax pair formulation is not unique. If $f \in L^\infty(\mathbb{R})$ and p is a polynomial with complex coefficients, then

$$f(L_{u(t)}) = U(t)f(L_{u(0)})U(t)^* \in \mathfrak{B}(L^2_+), \qquad p(L_{u(t)}) = U(t)p(L_{u(0)})U(t)^* \in \mathfrak{B}(H^N_+, L^2_+), \tag{2.6}$$

where N is the degree of the polynomial p.

Proposition 2.8. *Given $n \in \mathbb{N}$, let $u : t \in \mathbb{R} \mapsto u(t) \in H^{\frac{n}{2}}(\mathbb{R},\mathbb{R})$ denote the solution of equation (1.1), we set*

$$E_n(u) := \langle L_u^n \Pi u, \Pi u\rangle_{H^{-\frac{n}{2}},H^{\frac{n}{2}}}. \tag{2.7}$$

Then $E_n(u(t)) = E_n(u(0))$, for every $t \in \mathbb{R}$. In particular, $E_1 = E$ on $H^{\frac{1}{2}}(\mathbb{R},\mathbb{R})$, where the energy functional E is given by (1.6).

Definition 2.9. *Given $u \in L^2(\mathbb{R},\mathbb{R})$ and $\lambda \in \mathbb{C}\backslash\sigma(-L_u)$, the $\mathbb{C}$-linear transformation $\lambda + L_u$ is invertible in $\mathfrak{B}(H^1_+, L^2_+)$ and the generating function is defined by $\mathcal{H}_\lambda(u) = \langle (L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}$. The subset $\mathcal{X} := \{(\lambda,u) \in \mathbb{R}\times L^2(\mathbb{R},\mathbb{R}) : 4\lambda > C^2\|u\|^2_{L^2}\}$ is open in the $\mathbb{R}$-Banach space $\mathbb{R}\times L^2(\mathbb{R},\mathbb{R})$, where the Sobolev constant is given by $C = \inf_{f\in H^1_+\backslash\{0\}} \frac{\||\mathrm{D}|^{\frac{1}{4}} f\|_{L^2}}{\|f\|_{L^4}}$ and we have $\sigma(L_u) \subset [-\frac{C^2\|u\|^2_{L^2}}{4}, +\infty)$ by proposition 2.2.*

The map $(\lambda,u) \in \mathcal{X} \mapsto \mathcal{H}_\lambda(u) = \langle (L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2} \in \mathbb{R}$ is real analytic.

Proposition 2.10. *Let $u : t \in \mathbb{R} \mapsto u(t) \in L^2(\mathbb{R},\mathbb{R})$ denote the solution of the BO equation (1.1) and we choose $\lambda > \frac{C^2\|u(0)\|^2_{L^2}}{4}$, then $\mathcal{H}_\lambda(u(t)) = \mathcal{H}_\lambda(u(0))$, for every $t \in \mathbb{R}$.*

Given $(\lambda,u) \in \mathcal{X}$, there exists a neighbourhood of u in $L^2(\mathbb{R},\mathbb{R})$, denoted by $\mathcal{V}_u$ such that the restriction $\mathcal{H}_\lambda : v \in \mathcal{V}_u \mapsto \mathcal{H}_\lambda(v) \in \mathbb{R}$ is real analytic. The Fréchet derivative of $\mathcal{H}_\lambda$ at u is computed as follows,

$$\mathrm{d}\mathcal{H}_\lambda(u)(h) = \langle w_\lambda, \Pi h\rangle_{L^2} + \overline{\langle w_\lambda, \Pi h\rangle}_{L^2} + \langle T_h w_\lambda, w_\lambda\rangle_{L^2} = \langle h, w_\lambda + \overline{w}_\lambda + |w_\lambda|^2\rangle_{L^2}, \quad \forall h \in L^2(\mathbb{R},\mathbb{R}).$$

where $w_\lambda \in H^1_+$ is given by $w_\lambda \equiv w_\lambda(u) \equiv w_\lambda(x,u) = [(L_u+\lambda)^{-1}\circ\Pi]u(x)$, for every $x \in \mathbb{R}$. Then

$$\nabla_u \mathcal{H}_\lambda(u) = |w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u). \tag{2.8}$$

Given $(\lambda,u_0) \in \mathcal{X}$ fixed, the pseudo-Hamiltonian equation associated to H_λ is defined by

$$\partial_t u = \partial_x \nabla_u \mathcal{H}_\lambda(u) = \partial_x\left(|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u)\right), \qquad u(0) = u_0. \tag{2.9}$$

There exists an open subset $\mathcal{V}_{u_0}$ of $L^2(\mathbb{R},\mathbb{R})$ such that $v \in \mathcal{V}_{u_0} \mapsto \partial_x\left(|w_\lambda(v)|^2 + w_\lambda(v) + \overline{w}_\lambda(v)\right) \in L^2_+$ is real analytic and $u_0 \in \mathcal{V}_{u_0}$. Hence (2.9) admits a local solution by Cauchy–Lipschitz theorem.

Remark 2.11. *The word 'pseudo-Hamiltonian' is used here because no symplectic form has been defined on $L^2(\mathbb{R},\mathbb{R})$ until now. In section 4, we show that $\partial_x\nabla f(u)$ is exactly the Hamiltonian vector field of the smooth function $f : \mathcal{U}_N \to \mathbb{R}$ with respect to the symplectic form ω on the N-soliton manifold $\mathcal{U}_N$ defined in (4.2).*

Proposition 2.12. *Given $(\lambda,u_0) \in \mathcal{X}$ fixed, there exists $\varepsilon > 0$ such that $(\lambda,u(t)) \in \mathcal{X}$, for every $t \in (-\varepsilon,\varepsilon)$, where $u : t \in (-\varepsilon,+\varepsilon) \mapsto u(t) \in L^2(\mathbb{R},\mathbb{R})$ denotes the local solution of (2.9) with initial datum $u(0) = u_0$. We have*

$$\partial_t L_{u(t)} = [B^\lambda_{u(t)}, L_{u(t)}], \quad \text{where} \quad B^\lambda_v := i(T_{w_\lambda(v)}T_{\overline{w}_\lambda(v)} + T_{w_\lambda(v)} + T_{\overline{w}_\lambda(v)}), \quad \text{if} \quad (\lambda,v) \in \mathcal{X}. \tag{2.10}$$

i.e. (L_u, B^λ_u) is a Lax pair of equation (2.9).

Remark 2.13. *The Toeplitz operators $T_{w_\lambda(v)}$ and $T_{\overline{w}_\lambda(v)}$ are bounded both on L^2_+ and on H^1_+, so is the skew-adjoint operator B^λ_v, if $(\lambda,v) \in \mathcal{X}$.*

For every $u \in H^\infty(\mathbb{R},\mathbb{R})$ and $\epsilon \in (0, \frac{4}{C^2\|u\|_{L^2}^2})$, we set $\tilde{\mathcal{H}}_\epsilon(u) := \frac{1}{\epsilon}\mathcal{H}_{\frac{1}{\epsilon}}(u)$ and $\tilde{B}_{\epsilon,u} := \frac{1}{\epsilon}B_u^{\frac{1}{\epsilon}}$. Recall that $E_n(u) = \langle L_u^n \Pi u, \Pi u\rangle_{L^2}$, we have the following Taylor expansion

$$\tilde{\mathcal{H}}_\epsilon(u) = \sum_{k=0}^{M}(-\epsilon)^n E_n(u) - (-\epsilon)^M \langle (L_u + \tfrac{1}{\epsilon})^{-1}\Pi u, L_u^M \Pi u\rangle_{L^2}, \quad \forall M \in \mathbb{N}. \tag{2.11}$$

Proposition 2.12 then leads to a Lax pair formulation for the equations corresponding to the conservation laws in the BO hierarchy,

$$\partial_t L_u = [\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\Big|_{\epsilon=0}\tilde{B}_{\epsilon,u}, L_u],$$

where now u evolves according to the pseudo-Hamiltonian flow of $E_n = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\big|_{\epsilon=0}\tilde{\mathcal{H}}_\epsilon$. In the case $n = 1$, we have $E_1 = E$ and $B_u = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0}\tilde{B}_{\epsilon,u}$.

This section is organized as follows. In subsection **2.1**, we recall some basic facts concerning unitarily equivalent self-adjoint operators on different Hilbert spaces. The subsection **2.2** is dedicated to the proofs of proposition 2.2 and 2.3. Proposition 2.8 and 2.10 that concern the conservation laws are proved in subsection **2.3**. Proposition 2.7 and proposition 2.12 that indicate the Lax pair structures are proved in subsection **2.4**.

2.1 Unitary equivalence

Generally, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two Hilbert spaces, let $\mathcal{A}$ be a self-adjoint operator defined on $\mathbf{D}(\mathcal{A}) \subset \mathcal{E}_1$ and $\mathcal{B}$ be a self-adjoint operator defined on $\mathbf{D}(\mathcal{B}) \subset \mathcal{E}_2$. Both $\mathcal{A}$ and $\mathcal{B}$ have spectral decompositions

$$\mathcal{E}_1 = \mathscr{H}_{\mathrm{ac}}(\mathcal{A}) \bigoplus \mathscr{H}_{\mathrm{sc}}(\mathcal{A}) \bigoplus \mathscr{H}_{\mathrm{pp}}(\mathcal{A}), \qquad \mathcal{E}_2 = \mathscr{H}_{\mathrm{ac}}(\mathcal{B}) \bigoplus \mathscr{H}_{\mathrm{sc}}(\mathcal{B}) \bigoplus \mathscr{H}_{\mathrm{pp}}(\mathcal{B}). \tag{2.12}$$

If $\mathcal{A}$ and $\mathcal{B}$ are unitarily equivalent i.e. there exists a unitary operator $\mathcal{U} : \mathcal{E}_1 \to \mathcal{E}_2$ such that

$$\mathcal{B} = \mathcal{U}\mathcal{A}\mathcal{U}^*, \qquad \mathbf{D}(\mathcal{B}) = \mathcal{U}\mathbf{D}(\mathcal{A}), \tag{2.13}$$

then we have the following identification result.

Proposition 2.14. *The operators $\mathcal{A}$ and $\mathcal{B}$ have the same spectrum and $\mathcal{U}\mathscr{H}_{\mathrm{xx}}(\mathcal{A}) = \mathscr{H}_{\mathrm{xx}}(\mathcal{B})$, for every $\mathrm{xx} \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}\}$. Moreover, for every bounded borel function $f : \mathbb{R} \to \mathbb{C}$, $f(\mathcal{A})$ is a bounded operator on $\mathcal{E}_1$, $f(\mathcal{B})$ is a bounded operator on $\mathcal{E}_2$, we have $f(\mathcal{B}) = \mathcal{U}f(\mathcal{A})\mathcal{U}^*$.*

Proof. If f is a bounded Borel function, $\psi \in \mathcal{E}_1$, consider the spectral measure of $\mathcal{A}$ associated to the vector $\psi \in \mathcal{E}_1$, denoted by $\mu_\psi^{\mathcal{A}}$. Similarly, we denote by $\mu_{\mathcal{U}\psi}^{\mathcal{B}}$ the spectral measure of $\mathcal{B}$ associated to the vector $\mathcal{U}\psi \in \mathcal{E}_2$. Clearly, we have

$$\mathrm{supp}(\mu_\psi^{\mathcal{A}}) \subset \sigma(\mathcal{A}) \subset \mathbb{R}, \qquad \mathrm{supp}(\mu_{\mathcal{U}\psi}^{\mathcal{B}}) \subset \sigma(\mathcal{B}) \subset \mathbb{R}.$$

For every $\lambda \in \mathbb{C}\backslash\sigma(\mathcal{A}) = \mathbb{C}\backslash\sigma(\mathcal{B})$, formula (2.13) implies that $\mathcal{U}(\lambda - \mathcal{A})^{-1}\mathcal{U}^* = (\lambda - \mathcal{B})^{-1}$. So the Borel–Cauchy transforms of these two spectral measures are the same.

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi^{\mathcal{A}}(\xi)}{\lambda - \xi} = \langle(\lambda - \mathcal{A})^{-1}\psi, \psi\rangle_{\mathcal{E}_1} = \langle(\lambda - \mathcal{B})^{-1}\mathcal{U}\psi, \mathcal{U}\psi\rangle_{\mathcal{E}_2} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_{\mathcal{U}\psi}^{\mathcal{B}}(\xi)}{\lambda - \xi}.$$

Both of these two spectral measures have finite total variations : $\mu_\psi^{\mathcal{A}}(\mathbb{R}) = \mu_{\mathcal{U}\psi}^{\mathcal{B}}(\mathbb{R}) = \|\psi\|_{\mathcal{E}_1}^2$. Since every finite Borel measure is uniquely determined by its Borel–Cauchy transform (see Theorem 3.21 of Teschl [64] page 108), we have $\mu_\psi^{\mathcal{A}} = \mu_{\mathcal{U}\psi}^{\mathcal{B}}$. So the restriction $\mathcal{U}|_{\mathscr{H}_{\rm xx}(\mathcal{A})} : \mathscr{H}_{\rm xx}(\mathcal{A}) \to \mathscr{H}_{\rm xx}(\mathcal{B})$ is a linear isomorphism, for every $\mathrm{xx} \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}\}$. Finally, we use the definition of the spectral measures to obtain

$$\langle f(\mathcal{A})\psi, \psi\rangle_{\mathcal{E}_1} = \int_{\mathbb{R}} f(\xi)\mathrm{d}\mu_\psi^{\mathcal{A}}(\xi) = \int_{\mathbb{R}} f(\xi)\mathrm{d}\mu_{\mathcal{U}\psi}^{\mathcal{B}}(\xi) = \langle f(\mathcal{B})\mathcal{U}\psi, \mathcal{U}\psi\rangle_{\mathcal{E}_2}$$

We may assume that f is real-valued, so that $f(\mathcal{A})$ is self-adjoint. The polarization identity implies that $\langle f(\mathcal{A})\psi, \phi\rangle_{\mathcal{E}_1} = \langle f(\mathcal{B})\mathcal{U}\psi, \mathcal{U}\phi\rangle_{\mathcal{E}_2}$, for every $\psi, \phi \in \mathcal{E}_1$. So we obtain $f(\mathcal{B}) = \mathcal{U}f(\mathcal{A})\mathcal{U}^*$ in the case f is real-valued bounded Borel function. In the general case, it suffices to use $f = \mathrm{Re}f + i\mathrm{Im}f$.

□

2.2 Spectral analysis I

In this subsection, we study the essential spectrum and discrete spectrum of the Lax operator L_u by proving proposition 2.2 and 2.3. The spectral analysis of L_u such that u is a multi-soliton in definition 1.1, will be continued in subsection **4.2**.

Proof of proposition 2.2. For every $h \in L^2_+$, let μ_h^{D} denote the spectral measure of D associated to h, then

$$\langle f(\mathrm{D})h, h\rangle_{L^2} = \int_0^{+\infty} \hat{f}(\xi)\frac{|\hat{h}(\xi)|^2}{2\pi}\mathrm{d}\xi \Longrightarrow \mathrm{d}\mu_h^{\mathrm{D}}(\xi) = \frac{\mathbf{1}_{[0,+\infty)}(\xi)|\hat{h}(\xi)|^2}{2\pi}\mathrm{d}\xi.$$

Thus we have $\sigma(\mathrm{D}) = \sigma_{\mathrm{ess}}(\mathrm{D}) = \sigma_{\mathrm{ac}}(\mathrm{D}) = [0, +\infty)$. If $u \in L^2(\mathbb{R}, \mathbb{R})$, we claim that $\mathcal{P}_u := T_u \circ (\mathrm{D} + i)^{-1}$ is a Hilbert–Schmidt operator on L^2_+.

Recall that $\mathbb{R}_+^* = (0, +\infty)$. In fact, let $\mathscr{F} : h \in L^2_+ \mapsto \frac{\hat{h}}{\sqrt{2\pi}} \in L^2(\mathbb{R}_+^*)$ denotes the renormalized Fourier–Plancherel transform, then $\mathcal{A}_u := \mathscr{F} \circ \mathcal{P}_u \circ \mathscr{F}^{-1}$ is an operator on $L^2(\mathbb{R}_+^*)$. Then we have

$$\mathcal{A}_u g(\xi) = \int_0^{+\infty} K_u(\xi, \eta)g(\eta)\mathrm{d}\eta, \qquad K_u(\xi, \eta) := \frac{\hat{u}(\xi - \eta)}{2\pi(\eta + i)}, \quad \forall \xi, \eta \in \mathbb{R}_+^*.$$

Hence its Hilbert–Schmidt norm $\|\mathcal{A}_u\|_{\mathcal{HS}(L^2(\mathbb{R}_+^*))} \leq \|K\|_{L^2(\mathbb{R}_+^* \times \mathbb{R}_+^*)} \leq \frac{\|u\|_{L^2}}{2}$. Since $\mathcal{P}_u$ is unitarily equivalent to $\mathcal{A}_u$, we have $\|\mathcal{P}_u\|^2_{\mathcal{HS}(L^2_+)} = \sum_{\lambda\in\sigma(\mathcal{P}_u)} \lambda^2 = \sum_{\lambda\in\sigma(\mathcal{A}_u)} \lambda^2 = \|\mathcal{A}_u\|^2_{\mathcal{HS}(L^2(\mathbb{R}_+^*))} \leq \frac{\|u\|^2_{L^2}}{4}$.

Then the symmetric operator T_u is relatively compact with respect to D and Weyl's essential spectrum theorem (Theorem XIII.14 of Reed–Simon [54]) yields that $\sigma_{\mathrm{ess}}(L_u) = \sigma_{\mathrm{ess}}(\mathrm{D})$ and L_u is self-adjoint with $\mathbf{D}(L_u) = \mathbf{D}(\mathrm{D}) = H^1_+$. An alternative proof of the self-adjointness of L_u can be given by Kato–Rellich theorem (Theorem X.12 of Reed–Simon [53]) and the following estimate, for every $f \in H^1_+$,

$$2\pi\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1} \leq \|\hat{f}\|_{L^2}\sqrt{A} + \|\widehat{\partial_x f}\|_{L^2}\sqrt{A^{-1}} \leq 2\left(\|\hat{f}\|_{L^2}\|\widehat{\partial_x f}\|_{L^2}\right)^{\frac{1}{2}}, \qquad A = \sqrt{\frac{\|\partial_x f\|_{L^2}}{\|f\|_{L^2}}}.$$

So $\|T_u(f)\|_{L^2} \leq \|u\|_{L^2}\|f\|_{L^\infty} \leq \frac{2}{\pi}\|\partial_x f\|_{L^2} + \frac{\|u\|^2_{L^2}}{4}\|f\|_{L^2}$.

Moreover, $|\langle T_u f, f\rangle_{L^2}| = |\int_{\mathbb{R}} u|f|^2| \leq \|u\|_{L^2}\|f\|_{L^4}^2 \leq C\|u\|_{L^2}\|f\|_{L^2}\||\mathrm{D}|^{\frac{1}{2}}f\|_{L^2}$ holds by Sobolev embedding $\|f\|_{L^4} \leq C\||\mathrm{D}|^{\frac{1}{4}}f\|_{L^2}$, for every $f \in H^1_+$. Then L_u is bounded from below, precisely

$$\langle L_u f, f\rangle_{L^2} = \||\mathrm{D}|^{\frac{1}{2}}f\|_{L^2}^2 - \langle T_u f, f\rangle_{L^2} \geq -\frac{C^2\|u\|_{L^2}^2\|f\|_{L^2}^2}{4}.$$

When $\lambda < -\frac{C^2\|u\|_{L^2}^2}{4}$, the map $L_u - \lambda : H^1_+ \to L^2_+$ is injective. Hence $\sigma_{\mathrm{pp}}(L_u) \subset [-\frac{C^2}{4}\|u\|_{L^2}^2, +\infty)$. □

Before the proof of proposition 2.3, we recall a lemma concerning the regularity of convolutions.

Lemma 2.15. *For every $p \in (1, +\infty)$ and $m, n \in \mathbb{N}$, we have*

$$W^{m,p}(\mathbb{R}) * W^{n,\frac{p}{p-1}}(\mathbb{R}) \hookrightarrow C^{m+n}(\mathbb{R}) \bigcap W^{m+n,+\infty}(\mathbb{R}). \tag{2.14}$$

For every $f \in W^{m,p}(\mathbb{R}) * W^{n,\frac{p}{p-1}}(\mathbb{R})$, we have $\lim_{|x|\to+\infty} \partial_x^\alpha f(x) = 0$, for every $\alpha = 0, 1, \cdots, m+n$.

Proof. In the case $m = n = 0$, it suffices use Hölder's inequality and the density argument of the Schwartz class $\mathscr{S}(\mathbb{R}) \subset W^{m,p}(\mathbb{R})$. In the case $m = 0$ and $n = 1$, recall that a continuous function whose weak-derivative is continuous is of class C^1 and $\langle f, \varphi\rangle_{\mathscr{D}(\mathbb{R})',\mathscr{D}(\mathbb{R})} = f * \check{\varphi}(0)$, we use the density argument of the test function class $\mathscr{D}(\mathbb{R}) \subset L^p(\mathbb{R})$. We conclude by induction on $n \geq 1$ and $m \in \mathbb{N}$. □

Remark 2.16. *Identity (2.1) was firstly found by Wu [65] in the case $\lambda < 0$. We show that (2.1) still holds in the case $\lambda \geq 0$. Hence the operator L_u has no eigenvalues in $[0, +\infty)$.*

Proof of proposition 2.3. We choose $u \in L^2(\mathbb{R}; (1+x^2)\mathrm{d}x)$ such that $u(\mathbb{R}) \subset \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\varphi \in L^2_+$ such that $L_u(\varphi) = \lambda\varphi$. Applying the Fourier–Plancherel transform, we obtain

$$\widehat{u\varphi}(\xi)\mathbf{1}_{\xi\geq 0} = (\xi - \lambda)\hat{\varphi}(\xi) =: g_\lambda(\xi). \tag{2.15}$$

Since $\hat{u} \in H^1(\mathbb{R})$ and $\hat{\varphi} \in L^2(\mathbb{R})$, their convolution $\widehat{u\varphi} = \frac{1}{2\pi}\hat{u} * \hat{\varphi} \in C^1(\mathbb{R}) \bigcap C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ denotes the uniform closure of $C_c(\mathbb{R})$ with respect to the $L^\infty(\mathbb{R})$-norm, by lemma 2.15. Recall $\mathbb{R}_+ = [0, +\infty)$.

We claim that

$$\begin{cases} \text{if} \quad \lambda < 0, & \text{then} \quad \hat{\varphi} \in C^1(\mathbb{R}_+); \\ \text{if} \quad \lambda \geq 0, & \text{then} \quad \hat{\varphi} \in C(\mathbb{R}_+) \bigcap C^1(\mathbb{R}_+ \backslash \{\lambda\}). \end{cases}$$

In fact, if $\lambda \geq 0$, we have $g_\lambda(\lambda) = 0$. Otherwise, λ would be a singular point of $\hat{\varphi}$ that prevents $\hat{\varphi}$ from being a L^2 function on $\mathbb{R}_+$, because $\xi \to \frac{1}{\xi-\lambda} \notin L^2(\mathbb{R}_+)$. By using the fact $g \in C^1(\mathbb{R}_+)$ (g is right differentiable at $\xi = 0$ and the derivative g' is right continuous at $\xi = 0$), we have

$$\hat{\varphi}(\xi) = \frac{g_\lambda(\xi) - g_\lambda(\lambda)}{\xi - \lambda} \to \begin{cases} g'_\lambda(\lambda), & \text{if} \quad \lambda > 0; \\ g'_\lambda(0^+), & \text{if} \quad \lambda = 0; \end{cases}$$

when $\xi \to \lambda$. So $\hat{\varphi} \in C(\mathbb{R}_+)$ and $\lim_{\xi\to+\infty} \hat{\varphi}(\xi) = 0$. Then we derive formula (2.15) with respect to ξ to get the following

$$-i\widehat{xu} * \hat{\varphi}(\xi) = g'_\lambda(\xi) = (\widehat{u\varphi})'(\xi) = \hat{\varphi}(\xi) + (\xi - \lambda)(\hat{\varphi})'(\xi), \qquad \forall \xi \in [0, +\infty)\backslash\{\lambda\}. \tag{2.16}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[(\xi - \lambda)|\hat{\varphi}(\xi)|^2] = |\hat{\varphi}(\xi)|^2 + 2\mathrm{Re}[((\xi - \lambda)(\hat{\varphi})'(\xi))\overline{\hat{\varphi}}(\xi)] = 2\mathrm{Re}[(\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)] - |\hat{\varphi}(\xi)|^2. \tag{2.17}$$

When $\lambda < 0$, it suffices to integrate equation (2.17) on $[0, +\infty)$ and use the Plancherel formula

$$\int_0^{+\infty} (\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi = -2\pi i \int_{\mathbb{R}} xu(x)|\varphi(x)|^2\mathrm{d}x.$$

We also use the fact $(\xi-\lambda)|\hat{\varphi}(\xi)|^2 = \widehat{u\varphi}(\xi)\overline{\hat{\varphi}}(\xi) \to 0$, as $\xi \to +\infty$. Thus,

$$\lambda|\hat{\varphi}(0)|^2 = \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}\xi}[(\xi-\lambda)|\hat{\varphi}(\xi)|^2]\mathrm{d}\xi = 4\pi\mathrm{Im}\int_{\mathbb{R}} xu(x)|\varphi(x)|^2\mathrm{d}x - \int_0^{+\infty}|\hat{\varphi}(\xi)|^2\mathrm{d}\xi = -2\pi\|\varphi\|^2_{L^2(\mathbb{R})}.$$

When $\lambda > 0$, there may be some problem of derivability of $\hat{\varphi}$ at $\xi = \lambda$. We replace the integral $\int_0^{+\infty}$ by two integrals $\int_0^{\lambda-\epsilon}$ and $\int_{\lambda+\epsilon}^{+\infty}$, for some $\epsilon \in (0, \lambda)$. Set

$$\begin{aligned}\mathcal{I}(\epsilon) :=& \lambda|\hat{\varphi}(0)|^2 - \epsilon|\hat{\varphi}(\lambda-\epsilon)|^2 - \epsilon|\hat{\varphi}(\lambda+\epsilon)|^2\\ =& 2\mathrm{Re}\left(\int_0^{+\infty}(\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi - \int_{\lambda-\epsilon}^{\lambda+\epsilon}(\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi\right) - \int_0^{+\infty}|\hat{\varphi}(\xi)|^2\mathrm{d}\xi + \int_{\lambda-\epsilon}^{\lambda+\epsilon}|\hat{\varphi}(\xi)|^2\mathrm{d}\xi\end{aligned}$$

Thanks to the continuity of $\hat{\varphi}$ on $\mathbb{R}_+$, we have $\lambda|\hat{\varphi}(0)|^2 = \lim_{\epsilon\to 0^+}\mathcal{I}(\epsilon) = -2\pi\|\varphi\|^2_{L^2(\mathbb{R})}$.

When $\lambda = 0$, we use the same idea and integrate (2.17) over interval $[\epsilon, +\infty)$, for some $\epsilon > 0$. Then

$$\mathcal{J}(\epsilon) := -\epsilon|\hat{\varphi}(\epsilon)|^2 = 2\mathrm{Re}\int_\epsilon^{+\infty}(\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi - \int_\epsilon^{+\infty}|\hat{\varphi}(\xi)|^2\mathrm{d}\xi \to 0,$$

as $\epsilon \to 0$. So we always have

$$-2\pi\|\varphi\|^2_{L^2(\mathbb{R})} = \lambda|\hat{\varphi}(0)|^2, \qquad \text{if} \quad \varphi \in \mathrm{Ker}(\lambda - L_u). \tag{2.18}$$

As a consequence L_u has only negative eigenvalues, if the real-valued function $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$. Finally we use $\widehat{u\varphi}(0) = -\lambda\hat{\varphi}(0)$ to get identity (2.1). If $\lambda \in \sigma_{\mathrm{pp}}(L_u)$ and $\varphi \in \mathrm{Ker}(\lambda - L_u)\backslash\{0\}$, we want to prove that

$$\xi \mapsto (1+|\xi|)\partial_\xi\hat{\varphi}(\xi) \in L^2(0, +\infty). \tag{2.19}$$

In fact, since $\varphi \in H^1_+ \hookrightarrow L^\infty(\mathbb{R})$ and $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$, we have $\widehat{u\varphi} = \frac{\hat{u}*\hat{\varphi}}{2\pi} \in H^1(\mathbb{R})$. Formula (2.15) yields that $\xi \mapsto (|\lambda|+\xi)\hat{\varphi}(\xi) \in L^2(\mathbb{R})$ and we have $\hat{\varphi} \in L^1(\mathbb{R})$. The hypothesis $u \in L^2(\mathbb{R}, x^2\mathrm{d}x)$ implies that the convolution term $\widehat{xu} * \hat{\varphi} \in L^2(\mathbb{R})$. Since $\lambda < 0$, we obtain (2.19) by using formula (2.16). □

2.3 Conservation laws

Proposition 2.8 and 2.10 are proved in this subsection. We begin with the following proposition.

Proposition 2.17. *If $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ denotes the unique solution of the BO equation (1.1), then we have*

$$\partial_t\Pi u(t) = B_{u(t)}(\Pi u(t)) + iL^2_{u(t)}(\Pi u(t)) \in L^2_+. \tag{2.20}$$

Proof. For every $u \in H^2(\mathbb{R}, \mathbb{R})$ is real-valued, B_u is a bounded operator on both L^2_+ and H^1_+, $\Pi u \in \mathbf{D}(L_u) = H^1_+$. We have $\hat{u}(-\xi) = \overline{\hat{u}(\xi)}$, $u = \Pi u + \overline{\Pi u}$ and $|\mathrm{D}|u = \mathrm{D}\Pi u - \mathrm{D}\overline{\Pi u}$. Since $\mathrm{D}\overline{\Pi u} \in L^2_-$, we have $\Pi(\Pi u\mathrm{D}\overline{\Pi u}) = \Pi(u\mathrm{D}\overline{\Pi u})$. Thus the following two formulas hold,

$$\begin{aligned}&B_u(\Pi u) = i(T_{|\mathrm{D}|u} - T_u^2)(\Pi u) = i(\Pi u)(\mathrm{D}\Pi u) - i\Pi(u\mathrm{D}\overline{\Pi u}) - iT_u^2(\Pi u) = \Pi u\partial_x\Pi u - \Pi(u\partial_x\overline{\Pi u}) - iT_u^2(\Pi u),\\ &iL_u^2(\Pi u) = i\mathrm{D}^2\Pi u - iT_u(\mathrm{D}\Pi u) - i\mathrm{D}\circ T_u(\Pi u) + iT_u^2(\Pi u) = -i\partial_x^2\Pi u - T_u(\partial_x\Pi u) - \partial_x[T_u(\Pi u)] + iT_u^2(\Pi u).\end{aligned}$$

Then we add them together to get the following

$$B_u(\Pi u) + iL_u^2(\Pi u) = -i\partial_x^2 \Pi u - 2\Pi[\Pi u \partial_x \Pi u + \Pi u \partial_x \overline{\Pi u} + \overline{\Pi u}\partial_x \Pi u]$$

Finally we replace u by $u(t)$, where $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ solves equation (1.1) to obtain (2.20). □

Proof of proposition 2.8. It suffices to prove (2.7) in the case $u_0 \in H^\infty(\mathbb{R}, \mathbb{R})$. Then we use the density argument and the continuity of the flow map

$$u_0 \in H^s(\mathbb{R}) \mapsto u \in C([-T, T]; H^s(\mathbb{R})) \quad \text{with} \quad T > 0, \quad s \geq 0,$$

in proposition 2.5. We choose $u = u(t) \in H^\infty(\mathbb{R}, \mathbb{R}) = \bigcap_{s\geq 0} H^s(\mathbb{R}, \mathbb{R})$, so the functions $L_u^n \Pi u$, $\partial_t \Pi u$ and $\partial_t(L_u^n)\Pi u = [B_u, L_u^n]\Pi u$ are in $H^\infty(\mathbb{R}, \mathbb{C})$. Thus

$$\partial_t E_n(u) = 2\mathrm{Re}\langle L_u^n \Pi u, \partial_t \Pi u\rangle_{L^2} + \langle \partial_t(L_u^n)\Pi u, \Pi u\rangle_{L^2}.$$

Since $B_u + iL_u^2$ is skew-adjoint, we use formula (2.20) to get the following

$$2\mathrm{Re}\langle L_u^n \Pi u, \partial_t \Pi u\rangle_{L^2} = \langle [L_u^n, B_u + iL_u^2]\Pi u, \Pi u\rangle_{L^2} = \langle [L_u^n, B_u]\Pi u, \Pi u\rangle_{L^2}.$$

Since (L_u^n, B_u) is also a Lax pair of the Benjamin–Ono equation (1.1), we have

$$\partial_t E_n(u) = \langle ([L_u^n, B_u] + \partial_t(L_u^n))\Pi u, \partial_t \Pi u\rangle_{L^2} = 0.$$

In the case $n = 1$, we assume that $u \in H^1(\mathbb{R}, \mathbb{R})$. Since $u = \Pi u + \overline{\Pi u}$, $|\mathrm{D}|u = \mathrm{D}\Pi u - \mathrm{D}\overline{\Pi u}$ and $\int_{\mathbb{R}} (\Pi u)^3 = 0$, we have $\langle |\mathrm{D}|u, u\rangle_{L^2} = 2\langle \mathrm{D}\Pi u, \Pi u\rangle_{L^2}$ and $\int_{\mathbb{R}} u^3 = 3\int_{\mathbb{R}} (\Pi u + \overline{\Pi u})|\Pi u|^2 = 3\int_{\mathbb{R}} u|\Pi u|^2$. In the general case $u \in H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R})$, we use the density argument.

□

Proof of proposition 2.10. It suffices to prove the case $u(0) \in H^\infty(\mathbb{R}, \mathbb{R})$ and we use the density argument. Let $u : t \mapsto u(t) \in H^\infty(\mathbb{R}, \mathbb{R})$ solve equation (1.1). Since $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ by proposition 2.8 and $4\lambda > C^2\|u(0)\|_{L^2}^2$, we have $(\lambda, u(t)) \in \mathcal{X}$, $\partial_t L_{u(t)} = [B_{u(t)}, L_{u(t)} + \lambda]$ and

$$\partial_t \mathcal{H}_\lambda(u) = 2\mathrm{Re}\langle (L_u + \lambda)^{-1}\Pi u, \partial_t \Pi u\rangle_{L^2} - \langle (L_u + \lambda)^{-1}\partial_t L_u (L_u + \lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}. \tag{2.21}$$

Formula (2.20) yields that

$$2\mathrm{Re}\langle (L_u + \lambda)^{-1}\Pi u, \partial_t \Pi u\rangle_{L^2} = \langle [(L_u + \lambda)^{-1}, B_u + iL_u^2]\Pi u, \Pi u\rangle_{L^2} = \langle [(L_u + \lambda)^{-1}, B_u]\Pi u, \Pi u\rangle_{L^2},$$

$$\begin{aligned}\langle [(L_u + \lambda)^{-1}, B_u]\Pi u, \Pi u\rangle_{L^2} =& \langle B_u \Pi u, (L_u + \lambda)^{-1}\Pi u\rangle_{L^2} + \langle (L_u + \lambda)B_u(L_u + \lambda)^{-1}\Pi u, (L_u + \lambda)^{-1}\Pi u\rangle_{L^2} \\ =& \langle (L_u + \lambda)^{-1}[B_u, L_u + \lambda](L_u + \lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}.\end{aligned}$$

Then (2.21) yields that $\partial_t H_\lambda(u(t)) = 0$. In the general case $u(t) \in L^2(\mathbb{R}, \mathbb{R})$, we proceed as in the proof of proposition 2.8 and use the continuity of the generating functional

$$\mathcal{H}_\lambda : u \in \{v \in L^2(\mathbb{R}, \mathbb{R}) : \|v\|_{L^2} < \tfrac{2\sqrt{\lambda}}{C}\} \mapsto \mathcal{H}_\lambda(u) \in \mathbb{R}.$$

□

2.4 Lax pair formulation

In this subsection, we prove proposition 2.12 and 2.7. The Hankel operators whose symbols are in $L^2(\mathbb{R})\bigcup L^\infty(\mathbb{R})$ will be used to calculate the commutators of Toeplitz operators. We notice that the Hankel operators are $\mathbb{C}$-anti-linear and the Toeplitz operators are $\mathbb{C}$-linear. For every symbol $v\in L^2(\mathbb{R})\bigcup L^\infty(\mathbb{R})$, we define its associated Hankel operator to be $H_v(h)=T_{\overline{h}}v=\Pi(v\overline{h})$, for every $h\in H^1_+$. If $v\in L^\infty(\mathbb{R})$, then $H_v: L^2_+\to L^2_+$ is a bounded operator. If $v\in L^2(\mathbb{R})$, then H_v may be an unbounded operator on L^2_+ whose domain of definition contains H^1_+. For every $b\in H^1(\mathbb{R})$, we have $\|T_b(h)\|_{H^1}+\|H_b(h)\|_{H^1}\lesssim\|b\|_{H^1}\|h\|_{H^1}$, for every $h\in H^1_+$, so both T_b and H_b are bounded on L^2_+ and on H^1_+.

Lemma 2.18. *For every $v,w\in L^2_+\bigcap L^\infty(\mathbb{R})$ and $u\in L^2(\mathbb{R})$, we have*

$$[T_v,T_{\overline{w}}]=-H_v\circ H_w\in\mathfrak{B}(L^2_+). \tag{2.22}$$

If $w\in H^1_+$ in addition, then we have $T_u(w)\in L^2_+$ and

$$H_{T_uw}=T_w\circ H_{\Pi u}+H_w\circ T_{\overline{u}}=T_u\circ H_w+H_{\Pi u}\circ T_{\overline{w}}\in\mathfrak{B}(H^1_+,L^2_+). \tag{2.23}$$

Proof. For every $v,w\in L^2_+\bigcap L^\infty(\mathbb{R})$ and $h\in L^2_+$, we have $\overline{w}h=\Pi(\overline{w}h)+\overline{\Pi(w\overline{h})}\in L^2_+$. Thus,

$$[T_v,T_{\overline{w}}]h=\Pi(v\Pi(\overline{w}h)-\overline{w}\Pi(vh))=\Pi(v\overline{w}h-v\overline{\Pi(w\overline{h})}-v\overline{w}h)=-\Pi(v\overline{\Pi(w\overline{h})})=-H_v\circ H_w(h)\in L^2_+.$$

Given $u\in L^2(\mathbb{R})$ and $w\in H^1_+$, for every $h\in H^1_+$, we have $w\overline{h}=\Pi(w\overline{h})+\overline{\Pi(\overline{w}h)}\in H^1(\mathbb{R})$ and $H_w(h),T_{\overline{w}}(h)\in H^1_+$. So $\Pi(u\overline{\Pi(\overline{w}h)})=\Pi(\overline{\Pi(\overline{w}h)}\Pi u)=H_{\Pi u}\circ T_{\overline{w}}(h)\in L^2_+$ and we have

$$H_{T_uw}(h)=\Pi(\Pi(uw)\overline{h})=\Pi(uw\overline{h})=\Pi(u\Pi(w\overline{h})+u\overline{\Pi(\overline{w}h)})=(T_u\circ H_w+H_{\Pi u}\circ T_{\overline{w}})(h)\in L^2_+.$$

Similarly, we have $u\overline{h}=\Pi(u\overline{h})+\overline{\Pi(\overline{u}h)}\in L^2(\mathbb{R})$ and $\Pi(u\overline{h})=\Pi(\overline{h}\Pi u)=H_{\Pi u}(h)\in L^2_+$. Thus,

$$H_{T_uw}(h)=\Pi(wu\overline{h})=\Pi(w\Pi(u\overline{h})+w\overline{\Pi(\overline{u}h)})=(T_w\circ H_{\Pi u}+H_w\circ T_{\overline{u}})(h)\in L^2_+.$$

□

Lemma 2.19. *Given $(\lambda,u)\in\mathcal{X}$ given in definition 2.9, set $w_\lambda(u)=(L_u+\lambda)^{-1}\circ\Pi(u)\in H^1_+$, then*

$$[\mathrm{D}-T_u,T_{w_\lambda(u)}T_{\overline{w}_\lambda(u)}+T_{w_\lambda(u)}+T_{\overline{w}_\lambda(u)}]=T_{\mathrm{D}[|w_\lambda(u)|^2+w_\lambda(u)+\overline{w}_\lambda(u)]}\in\mathfrak{B}(H^1_+,L^2_+). \tag{2.24}$$

Proof. We use abbreviation $w_\lambda:=w_\lambda(u)\in H^1_+$, then $\overline{w}_\lambda\in H^1_-$. If $f^+,g^+\in H^1_+$ and $f^-,g^-\in H^1_-$, then we have $[T_{f^+},T_{g^+}]=[T_{f^-},T_{g^-}]=0$, because for every $h\in L^2_+$, we have

$$T_{f^+}[T_{g^+}(h)]=f^+g^+h=T_{g^+}[T_{f^+}(h)],\qquad\forall h\in L^2_+.$$

and $T_{f^-}[T_{g^-}(h)]=\Pi(f^-\Pi(g^-h))=\Pi(f^-g^-h)=\Pi(g^-\Pi(f^-h))=T_{g^-}[T_{f^-}(h)]$. Since $\Pi u\in L^2_+$ and $\overline{\Pi u}\in L^2_-$, we use Leibnitz's rule and formula (2.22) to obtain that

$$\begin{aligned}[\mathrm{D}-T_u,T_{w_\lambda}+T_{\overline{w}_\lambda}]=&T_{\mathrm{D}w_\lambda}+T_{\mathrm{D}\overline{w}_\lambda}-[T_u,T_{w_\lambda}]-[T_u,T_{\overline{w}_\lambda}]\\=&T_{\mathrm{D}w_\lambda}+T_{\mathrm{D}\overline{w}_\lambda}-[T_{\overline{\Pi u}},T_{w_\lambda}]-[T_{\Pi u},T_{\overline{w}_\lambda}]\\=&T_{\mathrm{D}w_\lambda}+T_{\mathrm{D}\overline{w}_\lambda}-H_{w_\lambda}H_{\Pi u}+H_{\Pi u}H_{w_\lambda}.\end{aligned} \tag{2.25}$$

Similarly, formula (2.22) implies that

$$\begin{aligned}[T_u, T_{w_\lambda} T_{\overline{w}_\lambda}] =& [T_u, T_{w_\lambda}] T_{\overline{w}_\lambda} + T_{w_\lambda} [T_u, T_{\overline{w}_\lambda}] \\ =& [T_{\overline{\Pi u}}, T_{w_\lambda}] T_{\overline{w}_\lambda} + T_{w_\lambda} [T_{\Pi u}, T_{\overline{w}_\lambda}] \\ =& H_{w_\lambda} H_{\Pi u} T_{\overline{w}_\lambda} - T_{w_\lambda} H_{\Pi u} H_{w_\lambda}.\end{aligned} \tag{2.26}$$

For every $h \in H^1_+$, since $\overline{w}_\lambda, \mathrm{D}\overline{w}_\lambda \in L^2_-$, we have

$$\begin{aligned}[\mathrm{D}, T_{\overline{w}_\lambda} T_{w_\lambda}] h =& [\mathrm{D}, T_{\overline{w}_\lambda}] T_{w_\lambda} h + T_{\overline{w}_\lambda} [\mathrm{D}, T_{w_\lambda}] h \\ =& T_{\mathrm{D}\overline{w}_\lambda} (T_{w_\lambda} h) + T_{\overline{w}_\lambda} (T_{\mathrm{D}w_\lambda} h) \\ =& \Pi[\mathrm{D}\overline{w}_\lambda \Pi(w_\lambda h) + \overline{w}_\lambda \Pi(\mathrm{D}w_\lambda h)] = \Pi[(w_\lambda \mathrm{D}\overline{w}_\lambda + \overline{w}_\lambda \mathrm{D}w_\lambda) h] \in L^2_+.\end{aligned}$$

So $[\mathrm{D}, T_{\overline{w}_\lambda} T_{w_\lambda}] = T_{\mathrm{D}|w_\lambda|^2} \in \mathfrak{B}(H^1_+, L^2_+)$. We use formula (2.22) and Leibnitz's Rule to obtain that

$$[\mathrm{D}, T_{w_\lambda} T_{\overline{w}_\lambda}] = [\mathrm{D}, T_{\overline{w}_\lambda} T_{w_\lambda}] - [\mathrm{D}, H^2_{w_\lambda}] = T_{\mathrm{D}|w_\lambda|^2} - H_{\mathrm{D}w_\lambda} H_{w_\lambda} + H_{w_\lambda} H_{\mathrm{D}w_\lambda} \tag{2.27}$$

Recall that $w_\lambda = (\lambda + L_u)^{-1} \Pi u$, then we have

$$\mathrm{D}w_\lambda = T_u(w_\lambda) - \lambda w_\lambda + \Pi u. \tag{2.28}$$

The formula (2.23) and (2.28) yield that

$$H_{\mathrm{D}w_\lambda} - T_{w_\lambda} H_{\Pi u} = H_{T_u w_\lambda} - \lambda H_{w_\lambda} + H_{\Pi u} - T_{w_\lambda} H_{\Pi u} = H_{w_\lambda} T_u - \lambda H_{w_\lambda} + H_{\Pi u} \tag{2.29}$$

and

$$H_{\mathrm{D}w_\lambda} - H_{\Pi u} T_{\overline{w}_\lambda} = H_{T_u w_\lambda} - \lambda H_{w_\lambda} + H_{\Pi u} - H_{\Pi u} T_{\overline{w}_\lambda} = T_u H_{w_\lambda} - \lambda H_{w_\lambda} + H_{\Pi u}. \tag{2.30}$$

We use formulas (2.26), (2.27), (2.29) and (2.30) to get the following formula

$$\begin{aligned}&[\mathrm{D} - T_u, T_{w_\lambda} T_{\overline{w}_\lambda}] \\ =& T_{\mathrm{D}|w_\lambda|^2} - (H_{\mathrm{D}w_\lambda} - T_{w_\lambda} H_{\Pi u}) H_{w_\lambda} + H_{w_\lambda} (H_{\mathrm{D}w_\lambda} - H_{\Pi u} T_{\overline{w}_\lambda}) \\ =& T_{\mathrm{D}|w_\lambda|^2} - (H_{w_\lambda} T_u H_{w_\lambda} - \lambda H^2_{w_\lambda} + H_{\Pi u} H_{w_\lambda}) + (H_{w_\lambda} T_u H_{w_\lambda} - \lambda H^2_{w_\lambda} + H_{w_\lambda} H_{\Pi u}) \\ =& T_{\mathrm{D}|w_\lambda|^2} - H_{\Pi u} H_{w_\lambda} + H_{w_\lambda} H_{\Pi u}\end{aligned} \tag{2.31}$$

At last, we combine formulas (2.25) and (2.31) to obtain formula (2.24). □

End of the proof of proposition 2.12. Since $L : u \in L^2(\mathbb{R}, \mathbb{R}) \mapsto L_u = \mathrm{D} - T_u \in \mathfrak{B}(H^1_+, L^2_+)$ is $\mathbb{R}$-affine, for every $u \in L^2_+$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(L \circ u)(t) = -T_{\partial_t u(t)} = -iT_{\mathrm{D}(w_\lambda(u(t))\overline{w}_\lambda(u(t)) + w_\lambda(u(t)) + \overline{w}_\lambda(u(t)))}.$$

Thus the Lax equation (2.10) is equivalent to identity (2.24) in lemma 2.19. □

The proof of proposition 2.7 can be found in Gérard–Kappeler [19], Wu [65] etc. In order to make this paper self contained, we recall it here.

Proof of proposition 2.7. Since the Lax map $L : u \in H^2(\mathbb{R},\mathbb{R}) \mapsto \mathrm{D} - T_u \in \mathfrak{B}(H^1_+, L^2_+)$ is $\mathbb{R}$-affine,

$$\frac{\mathrm{d}}{\mathrm{d}t}(L\circ u)(t) = -T_{\partial_t u(t)} = -T_{\mathrm{H}\partial_x^2 u(t) - \partial_x(u(t)^2)}.$$

It suffices to prove $[B_u, L_u] + T_{\mathrm{H}\partial_x^2 u - \partial_x(u^2)} = 0$ for every $u \in H^2(\mathbb{R},\mathbb{R})$.

In fact, u is real-valued, we have $\hat{u}(-\xi) = \overline{\hat{u}(\xi)}$, $u = \Pi u + \overline{\Pi u}$ and $|\mathrm{D}|u = \mathrm{D}\Pi u - \mathrm{D}\overline{\Pi u}$. Since both T_u and B_u are bounded operators $L^2_+ \to L^2_+$ and bounded operators $H^1_+ \to H^1_+$, their Lie Bracket $[B_u, L_u]$ is given by

$$\begin{aligned}[B_u, L_u]f =& -\Pi(f\partial_x|\mathrm{D}|u) + i\Pi[u\Pi(f|\mathrm{D}|u) - |\mathrm{D}|u\Pi(uf)] + \Pi[\partial_x u\Pi(uf) + u\Pi(f\partial_x u)]\\ =& -\Pi(fH\partial_x^2 u) + \mathcal{I}_1 + \mathcal{I}_2 \in L^2_+,\end{aligned} \tag{2.32}$$

for every $f \in H^1_+$, where the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by

$$\begin{aligned}\mathcal{I}_1 :=& i\Pi[u\Pi(f|\mathrm{D}|u) - |\mathrm{D}|u\Pi(uf)]\\ =& \Pi[f\overline{\Pi u}\partial_x\Pi u + f\Pi u\partial_x\overline{\Pi u}] - \Pi u\Pi(f\partial_x\overline{\Pi u}) - \Pi(f\overline{\Pi u})\partial_x\Pi u + \Pi[\Pi(f\overline{\Pi u})\partial_x\overline{\Pi u} - \overline{\Pi u}\Pi(f\partial_x\overline{\Pi u})],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_2 :=& \Pi[\partial_x u\Pi(uf) + u\Pi(f\partial_x u)] = \Pi(f\overline{\Pi u})\partial_x\Pi u + \Pi u\Pi(f\partial_x\overline{\Pi u}) + \Pi(\overline{\Pi u}\Pi(f\partial_x\overline{\Pi u}))\\ &+ 2f\Pi u\partial_x\Pi u + \Pi[f\Pi u\partial_x\overline{\Pi u} + f\overline{\Pi u}\partial_x\Pi u + \Pi(f\overline{\Pi u})\partial_x\overline{\Pi u}].\end{aligned}$$

If $h_1 \in H^1_-$ and $h_2 \in L^2_-$, then $h_1h_2 \in L^2_-$. Since $\partial_x\overline{\Pi u} \in L^2_-$, we have $\Pi[\Pi(f\overline{\Pi u})\partial_x\overline{\Pi u}] = \Pi[f\overline{\Pi u}\partial_x\overline{\Pi u}]$. Thus

$$\mathcal{I}_1 + \mathcal{I}_2 = 2f\Pi u\partial_x\Pi u + 2\Pi[f\Pi u\partial_x\overline{\Pi u} + f\overline{\Pi u}\partial_x\Pi u + \Pi(f\overline{\Pi u})\partial_x\overline{\Pi u}] = \Pi[f\partial_x(u^2)] \in H^1_+. \tag{2.33}$$

Formulas (2.32) and (2.33) yield that $[B_u, L_u]f = \Pi[f(\partial_x(u^2) - \mathrm{H}\partial_x^2 u)]$. Thus equation (2.3) holds along the evolution of equation (1.1). □

Remark 2.20. *As indicated in Gérard–Kappeler [19], there are many choices of the operator B_u. We can replace B_u by any operator of the form $B_u + P_u$ such that P_u is a skew-adjoint operator commuting with L_u. For instance, we set $C_u := B_u + iL_u^2$ and we obtain $C_u = i\mathrm{D}^2 - 2i\mathrm{D}T_u + 2iT_{\mathrm{D}\Pi u}$. So (L_u, C_u) is also a Lax pair of the BO equation (1.1). The advantage of the operator $B_u = i(T_{|\mathrm{D}|u} - T_u^2)$ is that $B_u : L^2_+ \to L^2_+$ is bounded if u is sufficiently regular. For instance, $u \in H^2(\mathbb{R},\mathbb{R})$.*

3 The action of the shift semigroup

In this section, we introduce the semigroup of shift operators $(S(\eta)^*)_{\eta\geq 0}$ acting on the Hardy space L^2_+ and classify all finite-dimensional translation-invariant subspaces of L^2_+.

For every $\eta \geq 0$, we define the operator $S(\eta) : L^2_+ \to L^2_+$ such that $S(\eta)f = e_\eta f$, where $e_\eta(x) = e^{i\eta x}$. Its adjoint is given by $S(\eta)^* = T_{e_{-\eta}}$. We have

$$S(\eta)^* \circ L_u \circ S(\eta) = L_u + \eta\mathrm{Id}_{L^2_+}, \qquad \forall \eta \geq 0.$$

Since $\|S(\eta)^*\|_{\mathfrak{B}(L^2_+)} = \|S(\eta)\|_{\mathfrak{B}(L^2_+)} = 1$, $(S(\eta)^*)_{\eta\geq 0}$ is a contraction semi-group. Let $-iG$ be its infinitesimal generator, i.e. $Gf = i\frac{\mathrm{d}}{\mathrm{d}\eta}\Big|_{\eta=0^+} S(\eta)^* f \in L^2_+$, $\forall f \in \mathbf{D}(G)$, where

$$\mathbf{D}(G) := \{f \in L^2_+ : \hat{f}_{|\mathbb{R}_+} \in H^1(0,+\infty)\}, \tag{3.1}$$

because $\lim_{\epsilon\to 0} \|\frac{\psi-\tau_\epsilon\psi}{\epsilon} - \partial_x\psi\|_{L^2(0,+\infty)} = 0$, where $\tau_\epsilon\psi(x) = \psi(x-\epsilon)$ and $\psi \in H^1(0,+\infty)$. Every function $f \in \mathbf{D}(G)$ has bounded Hölder continuous Fourier transform by Morrey's inequality and Sobolev extension operator yields the existence of $\hat{f}(0^+) := \lim_{\xi\to 0^+} \hat{f}(\xi)$. The operator G is densely defined and closed. The Fourier transform of Gf is given by

$$\widehat{Gf}(\xi) = i\partial_\xi \hat{f}(\xi), \qquad \forall f \in \mathbf{D}(G), \qquad \forall \xi > 0. \tag{3.2}$$

In accordance with the Hille–Yosida theorem, we have

$$(-\infty, 0) \subset \rho(iG), \qquad \|(G-\lambda i)^{-1}\|_{\mathfrak{B}(L^2_+)} \leq \lambda^{-1}, \quad \forall \lambda > 0. \tag{3.3}$$

Lemma 3.1. *For every $b \in L^2(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$, we have $T_b(\mathbf{D}(G)) \subset \mathbf{D}(G)$ and the following identity*

$$[G, T_b]\varphi = \frac{i\hat{\varphi}(0^+)}{2\pi}\Pi b \tag{3.4}$$

holds for every $\varphi \in \mathbf{D}(G)$.

Proof. For every $\eta > 0$ and $\varphi \in \mathbf{D}(G)$, both $S(\eta)^*$ and T_b are bounded operators, so we have

$$\Big([\tfrac{S(\eta)^*}{\eta}, T_b]\varphi\Big)^\wedge(\xi) = \frac{\hat{b} * \hat{\varphi}(\xi+\eta) - \hat{b} * [\mathbf{1}_{\mathbb{R}_+}(\tau_{-\eta}\hat{\varphi})](\xi)}{2\pi\eta} = \frac{1}{2\pi\eta}\int_\xi^{\xi+\eta} \hat{b}(\zeta)\hat{\varphi}(\xi+\eta-\zeta)\mathrm{d}\zeta, \quad \forall \xi > 0,$$

where $\tau_{-\eta}\hat{\varphi}(x) = \hat{\varphi}(x+\eta)$, for every $x \in \mathbb{R}$. Then we change the variable $\zeta = \xi + t\eta$, for $0 \leq t \leq 1$,

$$\Big([\tfrac{S(\eta)^*-\mathrm{Id}_{L^2_+}}{\eta}, T_b]\varphi\Big)^\wedge(\xi) = \frac{1}{2\pi}\int_0^1 \hat{b}(\xi+t\eta)\hat{\varphi}((1-t)\eta)\mathrm{d}\zeta = a_\eta\widehat{b}(\xi) + \widehat{\phi_\eta}(\xi), \quad \forall \xi > 0, \tag{3.5}$$

where $a_\eta := \frac{1}{2\pi}\int_0^1 \hat{\varphi}((1-t)\eta)\mathrm{d}\zeta \in \mathbb{C}$ and $\phi_\eta \in L^2_+$ such that

$$\widehat{\phi_\eta}(\xi) := \frac{1}{2\pi}\int_0^1 [\hat{b}(\xi+t\eta) - \hat{b}(\xi)]\hat{\varphi}((1-t)\eta)\mathrm{d}t, \qquad \forall \xi > 0.$$

Since $\hat{\varphi}|_{\mathbb{R}_+} \in H^1(0,+\infty)$, $\hat{\varphi}$ is bounded and $\lim_{\eta\to 0^+}\hat{\varphi}(\eta) = \hat{\varphi}(0^+)$, Lebesgue's dominated convergence theorem yields that $\lim_{\eta\to 0^+} a_\eta = \frac{\hat{\varphi}(0^+)}{2\pi}$. Since $b \in L^2(\mathbb{R})$, we have $\lim_{\epsilon\to 0}\|\tau_\epsilon\hat{b} - \hat{b}\|_{L^2} = 0$. By using Cauchy–Schwarz inequality and Fubini's theorem, we have

$$\|\phi_\eta\|^2_{L^2} \lesssim \|\hat{\varphi}\|^2_{L^\infty}\int_0^1\int_0^{+\infty} |\hat{b}(\xi+t\eta) - \hat{b}(\xi)|^2\mathrm{d}\xi\mathrm{d}t = \|\hat{\varphi}\|^2_{L^\infty}\int_0^1 \|\tau_{-t\eta}\hat{b} - \hat{b}\|^2_{L^2}\mathrm{d}t \to 0,$$

when $\eta \to 0^+$, by Lebesgue's dominated convergence theorem. Thus (3.5) implies that

$$[\tfrac{S(\eta)^*-\mathrm{Id}_{L^2_+}}{\eta}, T_b]\varphi = a_\eta\Pi b + \phi_\eta \to \frac{\hat{\varphi}(0^+)}{2\pi}\Pi b, \qquad \text{in} \quad L^2_+, \qquad \text{when} \quad \eta \to 0^+.$$

Since $\varphi \in \mathbf{D}(G)$ and T_b is bounded, we have $\frac{1}{\eta}T_b[(S(\eta)^* - \mathrm{Id}_{L^2_+})\varphi] \to (T_bG)\varphi$ in L^2_+, consequently

$$\tfrac{1}{\eta}(S(\eta)^* - \mathrm{Id}_{L^2_+})(T_b\varphi) \to (T_bG)\varphi + \frac{\hat{\varphi}(0^+)}{2\pi} \qquad \text{in} \quad L^2_+, \qquad \text{when} \quad \eta \to 0^+.$$

So $T_b\varphi \in \mathbf{D}(G)$ and (3.4) holds. □

The following scalar representation theorem of Lax [39] allows to classify all translation-invariant subspaces of the Hardy space L^2_+, which plays the same role as Beurling's theorem in the case of Hardy space on the circle (see Theorem 17.21 of Rudin [56]).

Theorem 3.2 (Beurling–Lax)**.** *Every nonempty closed subspace of L^2_+ that is invariant under the semigroup of shift operators $(S(\eta))_{\eta\geq 0}$ is of the form ΘL^2_+, where Θ is a holomorphic function in the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$. We have $|\Theta(z)| \leq 1$, for all $z \in \mathbb{C}_+$ and $|\Theta(x)| = 1$, $\forall x \in \mathbb{R}$. Moreover, Θ is uniquely determined up to multiplication by a complex constant of absolute value 1.*

The following lemma classifies all finite-dimensional subspaces that are invariant under the semi-group $(S(\eta)^*)_{\eta\geq 0}$, which is a weak version of theorem 3.2.

Lemma 3.3. *Let M be a subspace of L^2_+ of finite dimension $N = \dim_{\mathbb{C}} M \geq 1$ and $G(M) \subset M$. Then there exists a unique monic polynomial $Q \in \mathbb{C}_N[X]$ such that $Q^{-1}(0) \subset \mathbb{C}_-$ and $M = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, where $\mathbb{C}_{\leq N-1}[X]$ denotes all the polynomials whose degrees are at most $N-1$. Q is the characteristic polynomial of the operator $G|_M$.*

Proof. We set $\hat{M} = \{\hat{f} \in L^2(0,+\infty) : f \in M\}$, then $\dim_{\mathbb{C}} \hat{M} = N$. Since $\widehat{Gf} = i\partial_\xi \hat{f}$ on $\mathbb{R}\backslash\{0\}$, the restriction $G|_M$ is unitarily equivalent to $i\partial_\xi|_{\hat{M}}$ by the renormalized Fourier–Plancherel transformation. So the characteristic polynomial $Q \in \mathbb{C}_N[X]$ of $i\partial_\xi|_{\hat{M}}$ is well defined, let $\{\overline{\beta}_1, \overline{\beta}_2, \cdots, \overline{\beta}_n\} \subset \mathbb{C}$ denote the distinct roots of Q and m_j denote the multiplicity of $\overline{\beta}_j$, we have $\sum_{j=1}^n m_j = N$ and

$$Q(z) = \det(z - i\partial_\xi|_{\hat{M}}) = \prod_{j=1}^{n}(z - \overline{\beta}_j)^{m_j} = z^d + \sum_{k=0}^{N-1} c_k z^k, \qquad c_k \in \mathbb{C}.$$

The Cayley–Hamilton theorem implies that $Q(i\partial_\xi) = 0$ on the subspace $\hat{M}$. If $\psi \in \hat{M} \subset L^2(0,+\infty)$, then ψ is a weak-solution of the following differential equation

$$i^{-N}Q(-\mathrm{D})\psi = \partial_\xi^N\psi + \sum_{k=0}^{N-1} i^{k-N}c_k\partial_\xi^k\psi = 0 \quad \text{on} \quad (0,+\infty), \qquad \psi \equiv 0 \quad \text{on} \quad (-\infty,0). \tag{3.6}$$

The differential operator $Q(-\mathrm{D})$ is elliptic is on the open interval $(0,+\infty)$ in the following sense: the symbol of the principal part of $Q(-\mathrm{D})$, denoted by $a_Q : (x,\xi) \in (0,+\infty)\times\mathbb{R} \mapsto (-\xi)^N$, does not vanish except for $\xi = 0$. Theorem 8.12 of Rudin [57] yields that ψ is a smooth function. The solution space

$$\mathrm{Sol}(3.6) = \mathrm{Span}_{\mathbb{C}}\{\hat{f}_{j,l}\}_{0\leq l\leq m_j-1, 1\leq j\leq n}, \qquad \hat{f}_{j,l}(\xi) = \xi^l e^{-i\overline{\beta}_j\xi}\mathbf{1}_{\mathbb{R}_+}. \tag{3.7}$$

has complex dimension $\sum_{j=1}^n m_j = N$ so we have $\mathrm{Sol}(3.6) = \hat{M} \subset L^2_+$ and $\mathrm{Im}\beta_j = \mathrm{Re}(i\overline{\beta}_j) > 0$ and $Q^{-1}(0) \subset \mathbb{C}_-$. At last, we have $M = \mathrm{Span}_{\mathbb{C}}\{f_{j,l}\}_{0\leq l\leq m_j-1, 1\leq j\leq n} = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, where

$$f_{j,l}(x) = \frac{l!}{2\pi[(-i)(x-\overline{\beta}_j)]^{l+1}}, \quad \forall x \in \mathbb{R}. \tag{3.8}$$

The uniqueness is obtained by identifying all the roots. □

4 The manifold of multi-solitons

This section is dedicated to a geometric description of the multi-soliton subsets in definition 1.1. We give at first a polynomial characterization then a spectral characterization for the real analytic symplectic manifold of N-solitons in order to prove the global well-posedness of the BO equation with N-soliton solutions (1.6).

Recall that every N-soliton has the form $u(x) = \sum_{j=1}^{N} \mathcal{R}_{\eta_j^{-1}}(x - x_j) = \sum_{j=1}^{N} \frac{2\eta_j}{(x-x_j)^2+\eta_j^2}$ with $x_j \in \mathbb{R}$ and $\eta_j > 0$, then we have the following polynomial characterization of the N-solitons.

Proposition 4.1. *The N-soliton subset $\mathcal{U}_N \subset H^\infty(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$ and $\mathcal{U}_N \bigcap \mathcal{U}_M = \emptyset$, for every $M \neq N$. Moreover, each of the following three properties implies the others:*

(a). *$u \in \mathcal{U}_N$.*
(b). *There exists a unique monic polynomial $Q_u \in \mathbb{C}_N[X]$ whose roots are contained in the lower half-plane $\mathbb{C}_-$ such that $\Pi u = i\frac{Q_u'}{Q_u}$.*
(c). *There exists $Q \in \mathbb{C}_N[X]$ such that $Q^{-1}(0) \subset \mathbb{C}_-$ and $\Pi u = i\frac{Q'}{Q}$.*

Proof. We only prove the uniqueness in (a) $\Rightarrow$ (b). If $\Pi u = i\frac{Q_u'}{Q_u} = i\frac{P'}{P}$, then we have $\left(\frac{P}{Q_u}\right)' \equiv 0$ on $\mathbb{R}$. Since P and Q_u are monic polynomials, we have $P = Q_u$. The other assertions are consequences of $u = \Pi u + \overline{\Pi u}$. □

Definition 4.2. *For every $u \in \mathcal{U}_N$, the unique monic polynomial $Q_u \in \mathbb{C}_N[X]$ given by proposition 4.1 is called the characteristic polynomial of u. Its roots are denoted by $z_j = x_j - i\eta_j \in \mathbb{C}_-$, for $1 \leq j \leq N$ (not necessarily all distinct). The unordered N-uplet $\mathbf{cl}(z_1, z_2, \cdots, z_N) \in \mathbb{C}_-^N/\mathrm{S}_N$ is called the translation–scaling parameters of u, where $\mathbb{C}_-^N/\mathrm{S}_N$ denotes the orbit space of the action (A.3) of symmetric group S_N on $\mathbb{C}_-^N$.*

The real analytic structure of $\mathcal{U}_N$ is given in the next proposition.

Proposition 4.3. *Equipped with the subspace topology of $L^2(\mathbb{R},\mathbb{R})$, the subset $\mathcal{U}_N$ is a connected, real analytic, embedded submanifold of the $\mathbb{R}$-Hilbert space $L^2(\mathbb{R},\mathbb{R})$ and $\dim_{\mathbb{R}} \mathcal{U}_N = 2N$. For every $u \in \mathcal{U}_N$, its translation–scaling parameters are denoted by $\mathbf{cl}(x_1 - i\eta_1, x_2 - i\eta_2, \cdots, x_N - i\eta_N)$ for some $x_j \in \mathbb{R}$ and $\eta_j > 0$, then the tangent space to $\mathcal{U}_N$ at u is given by*

$$\mathcal{T}_u(\mathcal{U}_N) = \bigoplus_{j=1}^{N} (\mathbb{R} f_j^u \bigoplus \mathbb{R} g_j^u), \qquad \text{where} \quad f_j^u(x) = \tfrac{2[(x-x_j)^2-\eta_j^2]}{[(x-x_j)^2+\eta_j^2]^2}, \quad g_j^u(x) = \tfrac{4\eta_j(x-x_j)}{[(x-x_j)^2+\eta_j^2]^2}. \tag{4.1}$$

Every tangent space $\mathcal{T}_u(\mathcal{U}_N)$ is contained in the auxiliary space $\mathcal{T}$ defined by (1.5) in which the global 2-covector $\boldsymbol{\omega} \in \mathbf{\Lambda}^2(\mathcal{T}^*)$ is well defined. Recall that the nondegenerate 2-form ω on $\mathcal{U}_N$ is given by

$$\omega_u(h_1, h_2) = \boldsymbol{\omega}(h_1, h_2) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi} \mathrm{d}\xi, \quad \forall h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N). \tag{4.2}$$

It provides the symplectic structure of the manifold $\mathcal{U}_N$.

Proposition 4.4. *The nondegenerate real analytic 2-form ω is closed on $\mathcal{U}_N$. Endowed with the symplectic form ω, the real analytic manifold $(\mathcal{U}_N, \omega)$ is a symplectic manifold.*

For every smooth real-valued function $f : \mathcal{U}_N \to \mathbb{R}$, let $X_f \in \mathfrak{X}(\mathcal{U}_N)$ denote its Hamiltonian vector field, defined as follows: for every $u \in \mathcal{U}_N$ and $h \in \mathcal{T}_u(\mathcal{U}_N)$,

$$\mathrm{d}f(u)(h) = \langle h, \nabla_u f(u)\rangle_{L^2} = \frac{i}{2\pi}\int_0^{+\infty} \frac{\hat{h}(\xi)}{\xi}\overline{i\xi(\nabla_u f(u))\hat{}(\xi)}\mathrm{d}\xi = \omega_u(h, X_f(u)).$$

Then we have

$$X_f(u) = \partial_x \nabla_u f(u) \in \mathcal{T}_u(\mathcal{U}_N), \qquad \forall u \in \mathcal{U}_N. \tag{4.3}$$

Remark 4.5. *There are several ways to prove the simple connectedness of $\mathcal{U}_N$. Firstly, it is irrelevant to the proof of proposition 5.16. In subsection 5.4, we show that the real analytic manifold $\mathcal{U}_N$ is diffeomorphic to some open convex subset of $\mathbb{R}^{2N}$, hence $\mathcal{U}_N$ is homotopy equivalent to a one-point space. On the other hand, the simple connectedness of the Kähler manifold $\Pi(\mathcal{U}_N)$ can be directly obtained from its construction (see proposition A.5).*

Then, we return back to spectral analysis in order to establish a spectral characterization of the manifold $\mathcal{U}_N$. For every monic polynomial $Q \in \mathbb{C}_N[X]$ with roots in $\mathbb{C}_-$, we set $\Theta = \Theta_Q := \frac{\overline{Q}}{Q} \in \mathrm{Hol}(\mathbb{C}_+)$, where

$$\overline{Q}(x) := \sum_{j=0}^{N-1} \overline{a}_j x^j + x^N, \qquad \text{if} \qquad Q(x) = \sum_{j=0}^{N-1} a_j x^j + x^N.$$

Then Θ is an inner function on the upper half-plane $\mathbb{C}_+$, because $|\Theta| \leq 1$ on $\mathbb{C}_+$ and $|\Theta| = 1$ on $\mathbb{R}$. Recall the shift operator $S(\eta) : L^2_+ \to L^2_+$ defined in section 3, we have $S(\eta)[\Theta h] = \Theta[S(\eta)h]$, for every $h \in L^2_+$, so ΘL^2_+ is a closed subspace of L^2_+ that is invariant by the semigroup $(S(\eta))_{\eta\geq 0}$ (see also the Beurling–Lax theorem 3.2 of the complete classification of the translation-invariant subspaces of the Hardy space L^2_+). We define K_Θ to be the orthogonal complement of ΘL^2_+, thus

$$L^2_+ = \Theta L^2_+ \bigoplus K_\Theta, \quad S(\eta)^*(K_\Theta) \subset K_\Theta \quad \text{and} \quad G(\mathbf{D}(G)\bigcap K_\Theta) \subset K_\Theta. \tag{4.4}$$

where the infinitesimal generator G is defined in (3.2). Recall that the $\mathbb{C}$-vector space $\mathbb{C}_{\leq N-1}[X]$ consists of all polynomials with complex coefficients of degree at most $N-1$. So $\frac{\mathbb{C}_{\leq N-1}[X]}{Q}$ is an N-dimensional subspace of L^2_+.

The Lax map $L : u \in L^2(\mathbb{R},\mathbb{R}) \mapsto L_u = \mathrm{D} - T_u \in \mathfrak{B}(H^1_+, L^2_+)$ is $\mathbb{R}$-affine. Defined on $\mathbf{D}(L_u) = H^1_+$, the unbounded self-adjoint operator L_u has the following spectral decomposition

$$L^2_+ = \mathscr{H}_{\mathrm{ac}}(L_u) \bigoplus \mathscr{H}_{\mathrm{sc}}(L_u) \bigoplus \mathscr{H}_{\mathrm{pp}}(L_u). \tag{4.5}$$

The following proposition gives an identification of these subspaces in the spectral decomposition (4.5).

Proposition 4.6. *If $u \in \mathcal{U}_N$, then L_u has exactly N simple negative eigenvalues. Let Q_u denote the characteristic polynomial of the N-soliton u given in definition 4.2 and $\Theta_u := \Theta_{Q_u} = \frac{\overline{Q}_u}{Q_u}$ denote the associated inner function. Then we have the following identification,*

$$\mathscr{H}_{\mathrm{ac}}(L_u) = \Theta_u L^2_+, \qquad \mathscr{H}_{\mathrm{sc}}(L_u) = \{0\}, \qquad \mathscr{H}_{\mathrm{pp}}(L_u) = K_{\Theta_u} = \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u}. \tag{4.6}$$

For every $u \in \mathcal{U}_N$, we have the following spectral decomposition of L_u:

$$\sigma(L_u) = \sigma_{\rm ac}(L_u) \bigcup \sigma_{\rm sc}(L_u) \bigcup \sigma_{\rm pp}(L_u), \qquad \text{where} \quad \sigma_{\rm ac}(L_u) = [0, +\infty), \quad \sigma_{\rm sc}(L_u) = \emptyset \tag{4.7}$$

and $\sigma_{\rm pp}(L_u) = \{\lambda_1^u, \lambda_2^u, \cdots, \lambda_N^u\}$ consists of all eigenvalues of L_u. Proposition 2.2 yields that L_u is bounded from below and $-\frac{C^2}{4}\|u\|_{L^2}^2 \leq \lambda_1^u < \cdots < \lambda_N^u < 0$, where $C = \inf_{f \in H_+^1 \setminus \{0\}} \frac{\||{\rm D}|^{\frac{1}{4}} f\|_{L^2}}{\|f\|_{L^4}}$ denotes the Sobolev constant. Hence the min-max principle (Theorem XIII.1 of Reed–Simon [54]) yields that

$$\lambda_n^u = \sup_{\dim_{\mathbb{C}} F = n-1} \mathfrak{I}(F, L_u), \qquad \mathfrak{I}(F, L_u) = \inf\{\langle L_u h, h\rangle_{L^2} : h \in H_+^1 \bigcap F^\perp, \|h\|_{L^2} = 1\} \tag{4.8}$$

where, the above supremum, F describes all subspaces of L_+^2 of complex dimension n, $1 \leq n \leq N$. When $n \geq N+1$, $\sup_{\dim_{\mathbb{C}} F = n} \mathfrak{I}(F, L_u) = \inf \sigma_{\rm ess}(L_u) = 0$. Proposition 2.3 and corollary 2.4 yield that there exist eigenfunctions $\varphi_j : u \in \mathcal{U}_N \mapsto \varphi_j^u \in \mathscr{H}_{\rm pp}(L_u)$ such that

$$\mathrm{Ker}(\lambda_j^u - L_u) = \mathbb{C}\varphi_j^u, \qquad \|\varphi_j^u\|_{L^2} = 1, \qquad \langle \varphi_j^u, u\rangle_{L^2} = \int_{\mathbb{R}} u\varphi_j^u = \sqrt{2\pi|\lambda_j^u|}, \tag{4.9}$$

for every $j = 1, 2, \cdots, N$. Then $\{\varphi_1^u, \varphi_2^u, \cdots, \varphi_N^u\}$ is an orthonormal basis of the subspace $\mathscr{H}_{\rm pp}(L_u)$. We have the following result.

Proposition 4.7. *For every $j = 1, 2, \cdots, N$, the j th eigenvalue $\lambda_j : u \in \mathcal{U}_N \mapsto \lambda_j^u \in \mathbb{R}$ is real analytic.*

We refer to proposition 4.14 and formula (4.4) to see that the subspace $\mathscr{H}_{\rm pp}(L_u) \subset \mathbf{D}(G)$ is invariant by G. The matrix representation of $G|_{\mathscr{H}_{\rm pp}(L_u)}$ with respect to the orthonormal basis $\{\varphi_1^u, \varphi_2^u, \cdots, \varphi_N^u\}$ is given in proposition 5.4. Then the following theorem gives the spectral characterization for N-solitons.

Theorem 4.8. *A function $u \in \mathcal{U}_N$ if and only if $u \in L^2(\mathbb{R}, (1+x^2){\rm d}x)$ is real-valued, $\dim_{\mathbb{C}} \mathscr{H}_{\rm pp}(L_u) = N$ and $\Pi u \in \mathscr{H}_{\rm pp}(L_u)$. Moreover, we have the following inversion formula*

$$\Pi u(x) = i\frac{\frac{\rm d}{{\rm d}x}\det(x - G|_{\mathscr{H}_{\rm pp}(L_u)})}{\det(x - G|_{\mathscr{H}_{\rm pp}(L_u)})}, \qquad \forall x \in \mathbb{R}. \tag{4.10}$$

Then Q_u in definition 4.2 is the characteristic polynomial of $G|_{\mathscr{H}_{\rm pp}(L_u)}$. The translation–scaling parameters of u can be identified as the spectrum of $G|_{\mathscr{H}_{\rm pp}(L_u)}$. Finally the invariance of $\mathcal{U}_N$ under the BO flow is obtained by its spectral characterization, so we have the global well-posedness of the BO equation in the N-soliton manifold (1.6).

Proposition 4.9. *If $u_0 \in \mathcal{U}_N$, we denote by $u : t \in \mathbb{R} \mapsto u(t) \in H^\infty(\mathbb{R}, \mathbb{R})$ the solution of the BO equation (1.1) with initial datum $u(0) = u_0$. Then $u(t) \in \mathcal{U}_N$, for every $t \in \mathbb{R}$.*

This section is organized as follows. The real analytic structure and the symplectic structure are given in subsection **4.1**. Then the spectral decomposition of the Lax operator L_u and the real analyticity of its eigenvalues are given in subsection **4.2**, for every $u \in \mathcal{U}_N$. The characterization theorem 4.8 is proved in subsection **4.3**. Finally, we show the stability of $\mathcal{U}_N$ under the BO flow in subsection **4.4**.

4.1 Differential structure

The construction of real analytic structure and symplectic structure of $\mathcal{U}_N$ is divided into three steps. Firstly, we describe the complex structure of $\Pi(\mathcal{U}_N)$. Then the Hermitian metric $\mathfrak{H}$ for the complex manifold $\Pi(\mathcal{U}_N)$ is introduced in (4.15) and we establish a real analytic diffeomorphism between $\mathcal{U}_N$ and $\Pi(\mathcal{U}_N)$. The third step is to prove $\mathrm{d}\omega = 0$ on $\mathcal{U}_N$. Since $\omega = -\Pi^*(\mathrm{Im}\mathfrak{H})$, $(\Pi(\mathcal{U}_N), \mathfrak{H})$ is a Kähler manifold.

Step **I**. The Viète map $\mathbf{V} : (\beta_1, \beta_2, \cdots, \beta_N) \in \mathbb{C}^N \mapsto (a_0, a_1, \cdots, a_{N-1}) \in \mathbb{C}^N$ is defined as follows

$$\prod_{j=1}^{N}(X - \beta_j) = \sum_{k=0}^{N-1} a_k X^k + X^N. \tag{4.11}$$

Both addition and multiplication of two complex numbers are open continuous maps $\mathbb{C}^2 \to \mathbb{C}$, the Viète map $\mathbf{V} : \mathbb{C}^N \to \mathbb{C}^N$ is an open quotient map. So $\mathbf{V}(\mathbb{C}^N_-)$ is an open connected subset of $\mathbb{C}^N$ (see also proposition A.5). With the subspace topology and the Hermitian form $\mathfrak{H}_{\mathbb{C}^N}(X, Y) = \langle X, Y\rangle_{\mathbb{C}^N} = X^T\overline{Y}$, the subset $(\mathbf{V}(\mathbb{C}^N_-), \mathfrak{H}_{\mathbb{C}^N})$ is a connected Kähler manifold of complex dimension N.

Lemma 4.10. *Equipped with the subspace topology of L^2_+, the subset $\Pi(\mathcal{U}_N)$ is a connected topological manifold of complex dimension N and it has a unique complex analytic structure making it into an embedded submanifold of the $\mathbb{C}$-Hilbert space L^2_+. For every $u \in \mathcal{U}_N$, its translation–scaling parameters are denoted by $\mathbf{cl}(x_1 - i\eta_1, x_2 - i\eta_2, \cdots, x_N - i\eta_N)$, for some $x_j \in \mathbb{R}$ and $\eta_j > 0$, then the tangent space to $\Pi(\mathcal{U}_N)$ at Πu is given by*

$$\mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N)) = \bigoplus_{j=1}^{N} \mathbb{C}h^u_j, \qquad \text{where} \quad h^u_j(x) = \frac{1}{(x - x_j + \eta_j i)^2}. \tag{4.12}$$

Proof. We define $\Gamma_N : \mathbf{a} = (a_0, a_1, \cdots, a_{N-1}) \in \mathbf{V}(\mathbb{C}^N_-) \mapsto \Pi u = i\frac{Q'}{Q} \in \Pi(\mathcal{U}_N) \subset L^2_+$ such that

$$Q(X) = \sum_{k=0}^{N-1} a_k X^k + X^N.$$

The surjectivity of Γ_N is given by the definition of $\mathcal{U}_N$. Since the monic polynomial Q is uniquely determined by $u \in \mathcal{U}_N$, the map Γ_N is injective. For every $\mathbf{h} = (h_0, h_1, \cdots, h_{N-1}) \in \mathbb{C}^N$, we have

$$\mathrm{d}\Gamma_N(a_0, a_1, \cdots, a_{N-1})\mathbf{h} = i\frac{QH' - Q'H}{Q^2}, \qquad \text{where} \quad H(X) = \sum_{k=0}^{N-1} h_k X^k.$$

If $\mathrm{d}\Gamma_N(a_0, a_1, \cdots, a_{N-1})\mathbf{h} = 0$, then $(\frac{H}{Q})' \equiv 0$. Since $\deg H \le \deg Q - 1$, we have $H = 0$. Thus $\Gamma_N : \mathbf{V}(\mathbb{C}^N_-) \to L^2_+$ is a complex analytic immersion. We claim that Γ_N is a topological embedding.

In fact we set $\mathbf{a}^{(n)} = (a^{(n)}_0, a^{(n)}_1, \cdots, a^{(n)}_{N-1}) \in \mathbf{V}(\mathbb{C}^N_-)$ such that

$$\frac{\partial_x Q_n}{Q_n} \to \frac{\partial_x Q}{Q} \quad \text{in} \quad L^2_+, \qquad \text{as} \quad n \to +\infty, \qquad \text{where} \quad Q_n(x) = \sum_{j=0}^{N-1} a^{(n)}_j x^j + x^N, \quad \forall x \in \mathbb{R}.$$

Since $\mathbf{a}^{(n)} \in \mathbf{V}(\mathbb{C}^N_-)$, we have $a^{(n)}_0 = Q_n(0) \neq 0$. For every $x \in \mathbb{R}$, we have

$$\frac{Q_n(x)}{Q_n(0)} = \exp\Big(\int_0^x \frac{\partial_y Q_n(y)}{Q_n(y)}\mathrm{d}y\Big) \to \exp\Big(\int_0^x \frac{\partial_y Q(y)}{Q(y)}\mathrm{d}y\Big) = \frac{Q(x)}{Q(0)}, \qquad \text{as} \quad n \to +\infty. \tag{4.13}$$

Every coefficient of $\frac{Q_n}{Q_n(0)}$ converges to the corresponding coefficient of $\frac{Q(x)}{Q(0)}$. Since Q_n, Q are monic, we have $\lim_{n\to+\infty}\frac{1}{Q_n(0)}=\frac{1}{Q(0)}$ and $\lim_{n\to+\infty}\mathbf{a}^{(n)}=\mathbf{a}$. Then $\Gamma_N^{-1}:\Pi(\mathcal{U}_N)\subset L^2_+\to\mathbf{V}(\mathbb{C}^N_-)$ is continuous. Since Γ_N is a complex analytic embedding, with the subspace topology of L^2_+, there exists a unique complex analytic structure making $\Pi(\mathcal{U}_N)=\Gamma_N\circ\mathbf{V}(\mathbb{C}^N_-)$ into an embedded complex analytic submanifold of L^2_+. The map $\Gamma_N:\mathbf{V}(\mathbb{C}^N_-)\to\Pi(\mathcal{U}_N)$ is biholomorphic. Set $u(x)=\sum_{j=1}^N\frac{2\eta_j}{(x-x_j)^2+\eta_j^2}$ for some $x_j=x_j(u)\in\mathbb{R}$ and $\eta_j=\eta_j(u)>0$. Then every $h\in\mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N))$ is identified as the velocity of the smooth curve $c:t\in(-1,1)\to\Pi(\mathcal{U}_N)$ such that $c(0)=\Pi u$ at $t=0$. If we choose

$$c(t,x)=\sum_{j=1}^N\frac{i}{x-x_j(t)+\eta_j(t)i}\qquad\text{where}\quad x_j(t)\in\mathbb{R},\quad \eta_j(t)>0.$$

Then we have $x_j(0)=x_j$, $\eta_j(0)=\eta_j$ and

$$h(x)=\partial_t\big|_{t=0}c(t,x)=\sum_{j=1}^N\frac{\eta_j'(0)+ix_j'(0)}{(x-x_j+\eta_j i)^2}.\tag{4.14}$$

We have $h_j^u=\Pi f_j^u=-i\Pi g_j^u$ and $(h_j^u)^\wedge(\xi)=-2\pi\mathbf{1}_{\xi\geq0}\xi e^{-(ix_j(u)+\eta_j(u))\xi}$. For every $h\in\mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N))$, we have $\xi\mapsto\xi^{-1}\hat{h}(\xi)\in L^2(\mathbb{R})$ (see also Hardy's inequality (4.18)). □

Step **II**. Given $u\in\mathcal{U}_N$, the Hermitian metric $\mathfrak{H}_{\Pi u}$ is defined as follows

$$\mathfrak{H}_{\Pi u}(h_1,h_2)=\int_0^{+\infty}\frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\pi\xi}\mathrm{d}\xi,\qquad\forall h_1,h_2\in\mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N)).\tag{4.15}$$

The sesquilinear form $\mathfrak{H}_{\Pi u}$ is positive definite because $\mathfrak{H}_{\Pi u}(h,h)=\int_0^{+\infty}\frac{|\hat{h}(\xi)|^2}{\pi\xi}\mathrm{d}\xi>0$, if $h\neq0$. Hence the smooth symmetric covariant 2-tensor field $\mathrm{Re}\mathfrak{H}$ is positive definite on $\Pi(\mathcal{U}_N)$, so $(\Pi(\mathcal{U}_N),\mathrm{Re}\mathfrak{H})$ is a Riemannian manifold of real dimension $2N$.

We consider the $\mathbb{R}$-linear isomorphism between the Hilbert spaces

$$\Pi:u\in L^2(\mathbb{R},\mathbb{R})\mapsto\Pi u\in L^2_+,\qquad f\in L^2_+\mapsto2\mathrm{Re}f\in L^2(\mathbb{R},\mathbb{R}).$$

Then $\Pi\circ2\mathrm{Re}=\mathrm{Id}_{L^2_+}$ and $2\mathrm{Re}\circ\Pi=\mathrm{Id}_{L^2(\mathbb{R},\mathbb{R})}$ and $\|u\|_{L^2}=\sqrt{2}\|\Pi u\|_{L^2}$. Then $\mathcal{U}_N=2\mathrm{Re}\circ\Pi(\mathcal{U}_N)$ is a real analytic manifold of real dimension $2N$. Furthermore we have $f_j^u=2\mathrm{Re}h_j^u$, $g_j^u=2i\mathrm{Re}h_j^u$ and

$$2\mathrm{Re}:\mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N))\to\mathcal{T}_u(\mathcal{U}_N)\tag{4.16}$$

is an $\mathbb{R}$-linear isomorphism. Since $\mathfrak{H}$ is Hermitian, the 2-form $\omega=-\Pi^*(\mathrm{Im}\mathfrak{H})$ is nondegenerate on $\mathcal{U}_N$.

Step **III**. We set $\mathcal{E}:=L^2(\mathbb{R},\mathbb{R})\bigcap L^2(\mathbb{R},x^2\mathrm{d}x)$, $\mathcal{E}_c:=\{u\in\mathcal{E}:\int_{\mathbb{R}}u=c\}$, for every $c\in\mathbb{R}$. Then

$$\mathcal{U}_N\subset\mathcal{E}_{2\pi N},\qquad\mathcal{T}_u(\mathcal{U}_N)\subset\mathcal{T}:=\mathcal{E}_0,\quad\forall u\in\mathcal{U}_N.$$

The nondegenerate 2-form ω can be extended to a 2-covector of the subspace $\mathcal{T}$. Recall that

$$\boldsymbol{\omega}(h_1,h_2)=\frac{i}{2\pi}\int_{\mathbb{R}}\frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi}\mathrm{d}\xi,\qquad\forall h_1,h_2\in\mathcal{T}.\tag{4.17}$$

If $h \in \mathcal{T}$, then we have $\hat{h}(0) = 0$ and $\hat{h} \in H^1(\mathbb{R})$. Hence the Hardy's inequality (see Brezis [7], Bahouri–Chemin–Danchin [3] etc.) yields that

$$\int_{\mathbb{R}} \frac{|\hat{h}(\xi)|^2}{|\xi|^2} \mathrm{d}\xi \leq 4\|\partial_\xi \hat{h}\|_{L^2}^2 \Longrightarrow \xi \mapsto \frac{\hat{h}(\xi)}{\xi} \in L^2(\mathbb{R}), \tag{4.18}$$

so the 2-covector $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{T}^*)$ is well defined and $\omega_u(h_1, h_2) = \boldsymbol{\omega}(h_1, h_2)$. For every smooth vector field $X \in \mathfrak{X}(\mathcal{U}_N)$, let $X \lrcorner \omega \in \boldsymbol{\Omega}^1(\mathcal{U}_N)$ denote the interior multiplication by X, i.e. $(X \lrcorner \omega)(Y) = \omega(X, Y)$, for every $Y \in \mathfrak{X}(\mathcal{U}_N)$. We shall prove that $\mathrm{d}\omega = 0$ on $\mathcal{U}_N$ by using Cartan's formula:

$$\mathscr{L}_X \omega = X \lrcorner (\mathrm{d}\omega) + \mathrm{d}(X \lrcorner \omega). \tag{4.19}$$

Proof of proposition 4.4. For any smooth vector field $X \in \mathfrak{X}(\mathcal{U}_N)$, let ϕ denote the smooth maximal flow of X. If t is sufficiently close to 0, then $\phi_t : u \in \mathcal{U}_N \mapsto \phi(t, u) \in \mathcal{U}_N$ is a local diffeomorphism by the fundamental theorem on flows (see Theorem 9.12 of Lee [40]). For every $u \in \mathcal{U}_N$, $h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N)$, we compute the Lie derivative of ω with respect to X,

$$\begin{aligned}(\mathscr{L}_X \omega)_u(h_1, h_2) &= \lim_{t \to 0} \frac{\omega_{\phi_t(u)}(\mathrm{d}\phi_t(u)h_1, \mathrm{d}\phi_t(u)h_2) - \omega_u(h_1, h_2)}{t} \\ &= \lim_{t \to 0} \boldsymbol{\omega}\left(\frac{\mathrm{d}\phi_t(u)h_1 - h_1}{t}, \mathrm{d}\phi_t(u)h_2\right) + \lim_{t \to 0} \boldsymbol{\omega}\left(h_1, \frac{\mathrm{d}\phi_t(u)h_2 - h_2}{t}\right).\end{aligned}$$

Since $\lim_{t \to 0} \frac{\mathrm{d}\phi_t(u)h_j - h_j}{t} = \mathrm{d}X(u)h_j \in \mathcal{T}_u(\mathcal{U}_N)$, for every $j = 1, 2$, we have

$$(\mathscr{L}_X \omega)_u(h_1, h_2) = \boldsymbol{\omega}(\mathrm{d}X(u)h_1, h_2) + \boldsymbol{\omega}(h_1, \mathrm{d}X(u)h_2) = (h_1 \omega(X, h_2))(u) - (h_2 \omega(X, h_1))(u).$$

We choose (V, x^i) a smooth local chart for $\mathcal{U}_N$ such that $u \in V$ and the tangent vector h_k has the coordinate expression $h_k = \sum_{j=1}^{2N} h_k^{(j)} \frac{\partial}{\partial x^j}\big|_u$, for some $h_k^{(j)} \in \mathbb{R}$, $j = 1, 2 \cdots, 2N$ and $k = 1, 2$. The tangent vector h_k can be identified as some locally constant vector field $Y_k \in \mathfrak{X}(\mathcal{U}_N)$ defined by

$$Y_k : v \in V \mapsto \sum_{j=1}^{2N} h_k^{(j)} \frac{\partial}{\partial x^j}\Big|_v \in \mathcal{T}_v(\mathcal{U}_N), \quad Y_k : u \mapsto (Y_k)_u = h_k, \qquad k = 1, 2.$$

Then the vector field $[Y_1, Y_2]$ vanishes in the open subset V. The exterior derivative of the 1-form $\beta = X \lrcorner \omega$ is computed as $\mathrm{d}\beta(Y_1, Y_2) = Y_1(\beta(Y_2)) - Y_2(\beta(Y_1)) + \beta([Y_1, Y_2])$. Thus

$$\mathrm{d}(X \lrcorner \omega)_u(h_1, h_2) = h_1 \omega_u(X_u, h_2) - h_2 \omega_u(X_u, h_1) + \omega_u(X_u, [Y_1, Y_2]_u) = (\mathscr{L}_X \omega)_u(h_1, h_2).$$

Then Cartan's formula (4.19) yields that $X \lrcorner (\mathrm{d}\omega) = 0$. Since $X \in \mathfrak{X}(\mathcal{U}_N)$ is arbitrary, we have $\mathrm{d}\omega = 0$. As a consequence, the real analytic 2-form $\omega : u \in \mathcal{U}_N \mapsto \boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{T}^*)$ is a symplectic form. □

Since $\mathrm{Im}\mathfrak{H} = (-2\mathrm{Re})^*\omega$, where $-2\mathrm{Re} : \Pi(\mathcal{U}_N) \to \mathcal{U}_N$ is a real analytic diffeomorphism, the associated 2-form $\mathrm{Im}\mathfrak{H}$ is closed. So $(\Pi(\mathcal{U}_N), \mathfrak{H})$ is a Kähler manifold. The simple connectedness of $\Pi(\mathcal{U}_N)$ is proved in subsection **A.1**.

4.2 Spectral analysis II

We continue to study the spectrum of the Lax operator L_u introduced in definition 2.1. The general cases $u \in L^2(\mathbb{R},\mathbb{R})$ and $u \in L^2(\mathbb{R},(1+x^2)\mathrm{d}x)$ have been studied in subsection **2.2**. We restrict our study to the case $u \in \mathcal{U}_N$ in this subsection. Let $Q = Q_u$ denote the characteristic polynomial of u and $\Theta := \frac{\overline{Q}}{Q}$, $K_\Theta = (\Theta L^2_+)^\perp$. Since L_u is an unbounded self-adjoint operator of L^2_+, we have the following

$$L^2_+ = \Theta L^2_+ \bigoplus K_\Theta = \mathscr{H}_{\mathrm{ac}}(L_u) \bigoplus \mathscr{H}_{\mathrm{sc}}(L_u) \bigoplus \mathscr{H}_{\mathrm{pp}}(L_u).$$

We shall at first identify those subspaces by proving proposition 4.6 and formula (4.7). Then we turn to study the real analyticity of each eigenvalue $\lambda_j : u \in \mathcal{U}_N \mapsto \lambda_j^u \in \mathbb{R}$.

Proof of proposition 4.6. The first step is to prove $K_\Theta = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$. In fact, for every $h \in L^2_+$ and $f = \frac{P}{Q} \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, for some $P \in \mathbb{C}_{\leq N-1}[X]$, we have

$$\langle f, \Theta h\rangle_{L^2} = \int_{\mathbb{R}} \frac{P(x)\overline{\Theta}(x)\overline{h}(x)}{Q(x)}\mathrm{d}x = \int_{\mathbb{R}} \frac{P(x)\overline{h}(x)}{\overline{Q}(x)}\mathrm{d}x = \langle \frac{P}{\overline{Q}}, h\rangle_{L^2}.$$

Since $\overline{Q}(x) = \prod_{j=1}^N (x-\alpha_j)$ with $\mathrm{Im}(\alpha_j) > 0$, the meromorphic function $\frac{P}{\overline{Q}}$ has poles in $\mathbb{C}_+$, so $\frac{P}{\overline{Q}} \in L^2_-$. Thus $\langle f, \Theta h\rangle_{L^2} = \langle \frac{P}{\overline{Q}}, h\rangle_{L^2} = 0$. Thus $\frac{\mathbb{C}_{\leq N-1}[X]}{Q} \subset (\Theta L^2_+)^\perp = K_\Theta$.

Conversely, if $f \in K_\Theta$, then $\langle \Theta^{-1} f, h\rangle_{L^2} = \langle f, \Theta h\rangle_{L^2} = 0$, for every $h \in L^2_+$. Thus $g := \frac{Q}{\overline{Q}} f \in L^2_-$. It suffices to prove that $P := Qf = \overline{Q}g \in \mathbb{C}[X]$. In fact,

$$\widehat{Qf} = Q(i\partial_\xi)\hat{f} \qquad \text{and} \qquad \mathrm{supp}(\hat{f}) \subset [0,+\infty) \Longrightarrow \mathrm{supp}(\widehat{Qf}) \subset [0,+\infty).$$

Similarly, $\mathrm{supp}((\overline{Q}g)^\wedge) \subset (-\infty, 0]$. Thus $\mathrm{supp}(\hat{P}) \subset \{0\}$ and P is a polynomial. Since $f = \frac{P}{Q} \in L^2(\mathbb{R})$, we have $\deg P \leq N-1$. So $K_\Theta \subset \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$.

The second step is to prove $L_u(\Theta L^2_+) \subset \Theta L^2_+$. Precisely, we have

$$L_u(\Theta h) = \Theta \mathrm{D} h, \qquad \forall h \in L^2_+. \tag{4.20}$$

Since $\frac{\mathbb{C}_{\leq N-1}[X]}{Q} \subset L^2_+$, $\Theta = \frac{\overline{Q}}{Q}$ and $\frac{\mathrm{D}\Theta}{\Theta} = \frac{\mathrm{D}\overline{Q}}{\overline{Q}} - \frac{\mathrm{D}Q}{Q} = i\frac{Q'}{Q} - i\frac{\overline{Q}'}{\overline{Q}} = \Pi u + \overline{\Pi u} = u$ on $\mathbb{R}$, we have

$$L_u(\Theta h) = (\mathrm{D} - T_u)(\Theta h) = \Theta \mathrm{D} h + h\left(\mathrm{D}\Theta - i\frac{Q'}{Q}\Theta + i\frac{\overline{Q}'}{Q}\right) = \Theta \mathrm{D} h + h\Theta\left(\frac{\mathrm{D}\Theta}{\Theta} - i\frac{Q'}{Q} + \frac{\overline{Q}'}{\overline{Q}}\right) = \Theta \mathrm{D} h.$$

Recall that $L_u = L_u^*$, so we have $L_u(K_\Theta) \subset K_\Theta$. Since $\dim_{\mathbb{C}} K_\Theta = N$, corollary 2.4 yields that the Hermitian matrix $L_{u|K_\Theta}$ has exactly N distinct eigenvalues. Hence $K_\Theta \subset \mathscr{H}_{\mathrm{pp}}(L_u)$.

On the other hand, we set $U_\Theta : L^2_+ \to \Theta L^2_+$ such that $U_\Theta h = \Theta h$. Thus $\|U_\Theta\|_{\mathfrak{B}(L^2_+, \Theta L^2_+)} = 1$ and

$$U_\Theta^{-1} = U_\Theta^* : g \in \Theta L^2_+ \mapsto \Theta^{-1} g \in L^2_+.$$

So $U_\Theta : L^2_+ \to \Theta L^2_+$ is a unitary operator. $U_\Theta(H^1_+) = \Theta H^1_+ = H^1_+ \bigcap \Theta L^2_+$. Formula (4.20) yields that

$$U_\Theta^* L_{u|\Theta L^2_+} U_\Theta = \mathrm{D}, \qquad U_\Theta[\mathbf{D}(\mathrm{D})] = \Theta H^1_+ = H^1_+ \bigcap \Theta L^2_+ = \mathbf{D}(L_{u|\Theta L^2_+}).$$

For every bounded Borel function $f : \mathbb{R} \to \mathbb{C}$, we have $f(L_u)U_\Theta = U_\Theta f(\mathrm{D})$ by proposition 2.14. We denote by $\mu_\psi = \mu_\psi^{L_u}$ the spectral measure of L_u associated to $\psi \in L^2_+$, then $\forall h \in L^2_+$, we have

$$\int_{\mathbb{R}} f(\xi)\mathrm{d}\mu_{\Theta h}(\xi) = \langle f(L_u)U_\Theta h, U_\Theta h\rangle_{L^2} = \langle \Theta f(\mathrm{D})h, \Theta h\rangle_{L^2} = \langle f(\mathrm{D})h, h\rangle_{L^2} = \frac{1}{2\pi}\int_0^{+\infty} f(\xi)|\hat{h}(\xi)|^2\mathrm{d}\xi.$$

So $\mathrm{d}\mu_{\Theta h}(\xi) = \frac{\mathbf{1}_{\mathbb{R}_+}|\hat{h}(\xi)|^2}{2\pi}\mathrm{d}\xi$. The spectral measure $\mu_{\Theta h}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Thus $\Theta L^2_+ \subset \mathscr{H}_{\mathrm{ac}}(L_u) \subset \mathscr{H}_{\mathrm{cont}}(L_u) = (\mathscr{H}_{\mathrm{pp}}(L_u))^\perp \subset \Theta L^2_+$ and (4.6) is obtained. We have $\mathrm{supp}(\mu_{\Theta h}) \subset [0, +\infty)$, for every $h \in L^2_+$. $\forall \xi \geq 0$, there exists $h \in L^2_+$ such that $\hat{h}(\xi) \neq 0$. So we have $\sigma_{\mathrm{ess}}(L_u) = \sigma_{\mathrm{cont}}(L_u) = \sigma_{\mathrm{ac}}(L_u) = [0, +\infty)$. □

Before proving the real analyticity of each eigenvalue, we show its continuity at first.

Lemma 4.11. *For every $j = 1, 2, \cdots, N$, the j th eigenvalue $\lambda_j : u \in \mathcal{U}_N \mapsto \lambda_j^u \in \mathbb{R}$ is Lipschitz continuous on every compact subset of $\mathcal{U}_N$.*

Proof. For every $f \in H^1(\mathbb{R})$, the Sobolev embedding $\|f\|_{L^4} \leq C\||\mathrm{D}|^{\frac{1}{4}}f\|_{L^2}$ yields that $\forall u, v \in \mathcal{U}_N$,

$$\left|\langle L_u h, h\rangle_{L^2} - \langle L_v h, h\rangle_{L^2}\right| \leq \|u - v\|_{L^2}\|h\|^2_{L^4} \leq C\|u - v\|_{L^2}\||\mathrm{D}|^{\frac{1}{2}}h\|_{L^2}\|h\|_{L^2}, \quad \forall h \in H^1_+. \tag{4.21}$$

Given $j = 1, 2, \cdots, N$ and a subspace $F \subset L^2_+$ with complex dimension $j - 1$, we choose

$$h \in F^\perp \bigcap \bigoplus_{k=1}^{j} \mathrm{Ker}(\lambda_k^u - L_u) \subset H^1_+, \qquad \|h\|_{L^2} = 1, \qquad h = \sum_{k=1}^{j} h_k\varphi_k^u.$$

Then $\langle L_u h, h\rangle_{L^2} = \sum_{k=1}^j |h_k|^2\lambda_k^u \leq \lambda_j^u < 0$, because $\lambda_k^u < \lambda_{k+1}^u$. We have the following estimate

$$\||\mathrm{D}|^{\frac{1}{2}}h\|^2_{L^2} = \langle \mathrm{D}h, h\rangle_{L^2} = \langle L_u h, h\rangle_{L^2} + \langle uh, h\rangle_{L^2} \leq \lambda_j^u + \|u\|_{L^2}\|h\|^2_{L^4} \leq C\|u\|_{L^2}\||\mathrm{D}|^{\frac{1}{2}}h\|_{L^2}\|h\|_{L^2}. \tag{4.22}$$

So estimates (4.21) and (4.22) yield that $\langle L_v h, h\rangle_{L^2} \leq \lambda_j^u + C^2\|u\|_{L^2}\|u - v\|_{L^2}$. Since F is arbitrary, the max–min formula (4.8) implies that

$$|\lambda_j^u - \lambda_j^v| \leq C^2(\|u\|_{L^2} + \|v\|_{L^2})\|u - v\|_{L^2}.$$

Every compact subset $K \subset \mathcal{U}_N$ is bounded in $L^2(\mathbb{R}, \mathbb{R})$. Hence $u \in K \mapsto \lambda_j^u \in \mathbb{R}$ is Lipschitz continuous. □

Proof of proposition 4.7. For every $u \in \mathcal{U}_N$, the Lax operator L_u has N negative simple eigenvalues, denoted by $\lambda_1^u < \lambda_2^u < \cdots < \lambda_N^u < 0$. Let $\mathbb{P}_u^j$ denotes the Riesz projector of the eigenvalue λ_j^u and

$$D(z, \epsilon) = \{\eta \in \mathbb{C} : |\eta - z| < \epsilon\}, \quad \mathscr{C}(z, \epsilon) = \partial D(z, \epsilon) = \{\eta \in \mathbb{C} : |\eta - z| = \epsilon\}, \quad \forall z \in \mathbb{C}, \quad \epsilon > 0.$$

Then there exists $\epsilon_0 > 0$ such that the family of closed discs $\{\overline{D}(\lambda_j^u, \epsilon_0)\}_{1\leq j\leq N} \bigcup \{\overline{D}(0, \epsilon_0)\}$ is mutually disjoint and for every $j, k = 1, 2\cdots, N$ and any closed path $\mathbf{\Gamma}_j^u$ (piecewise C^1 closed curve) in $D(\lambda_j^u, \epsilon_0)$ with respect to which the eigenvalue λ_j^u has winding number 1, we have

$$\mathbb{P}_u^j = \frac{1}{2\pi i}\oint_{\mathbf{\Gamma}_j^u} (\zeta - L_u)^{-1}\mathrm{d}\zeta, \qquad \mathbb{P}_u^j \circ \mathbb{P}_u^j = \mathbb{P}_u^j, \qquad \mathbb{P}_u^j\varphi_k^u = \mathbf{1}_{j=k}\varphi_k^u. \tag{4.23}$$

by Theorem XII.5 of Reed–Simon [54]. We choose $\mathbf{\Gamma}_j^u$ to be the counterclockwise-oriented circle $\mathscr{C}(\lambda_j^u, \epsilon)$ in (4.23) for some $\epsilon \in (0, \epsilon_0)$. We claim that $\mathrm{Im}\mathbb{P}_u^j = \mathrm{Ker}(\lambda_j^u - L_u) = \mathbb{C}\varphi_j^u$.

It suffices to show that $\mathbb{P}_u^j|_{\mathscr{H}_{\mathrm{ac}}(L_u)} = 0$. In fact the operator $\mathbb{P}_u^j = g_{\lambda_j^u}(L_u)$ is self-adjoint by Theorem VIII.6 of Reed–Simon [55], where the real-valued bounded Borel function $g_\lambda : \mathbb{R} \to \mathbb{R}$ is given by

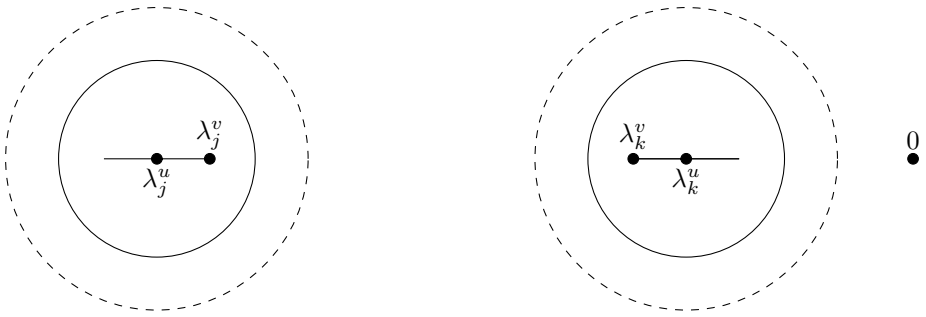
$$g_\lambda(x) := \frac{1}{2\pi i} \oint_{\mathscr{C}(\lambda,\epsilon)} (\zeta - x)^{-1} \mathrm{d}\zeta = \mathbf{1}_{(\lambda-\epsilon,\lambda+\epsilon)}(x), \qquad \text{a.e.} \quad \text{on} \quad \mathbb{R},$$

for every $\lambda \in \mathbb{R}$. Since $\mathbb{P}_u^j(\mathscr{H}_{\mathrm{pp}}(L_u)) \subset \mathbb{C}\varphi_j^u \subset \mathscr{H}_{\mathrm{pp}}(L_u)$, we have $\mathbb{P}_u^j(\mathscr{H}_{\mathrm{ac}}(L_u)) \subset \mathscr{H}_{\mathrm{ac}}(L_u)$. Let $\mu_\psi = \mu_\psi^{L_u}$ denote the spectral measure of L_u associated to the function $\psi \in \mathscr{H}_{\mathrm{ac}}(L_u)$, whose support is included in $[0, +\infty)$ by formula (4.7), we have

$$\langle \mathbb{P}_u^j \psi, \psi \rangle_{L^2} = \frac{1}{2\pi i} \oint_{\mathscr{C}(\lambda_j^u,\epsilon)} \langle (\zeta - L_u)^{-1} \psi, \psi \rangle_{L^2} \mathrm{d}\zeta = \frac{1}{2\pi i} \int_0^{+\infty} \left(\oint_{\mathscr{C}(\lambda_j^u,\epsilon)} (\zeta - \xi)^{-1} \mathrm{d}\zeta \right) \mathrm{d}\mu_\psi(\xi) = 0.$$

Set $\tilde{\psi} = \mathbb{P}_u^j \psi \in \mathscr{H}_{\mathrm{ac}}(L_u)$, then $\|\tilde{\psi}\|_{L^2}^2 = \langle \mathbb{P}_u^j \tilde{\psi}, \tilde{\psi} \rangle_{L^2} = 0$. So the claim is obtained.

For every fixed $j = 1, 2, \cdots N$, we have $\lambda_j^u = \mathrm{Tr}(L_u \circ \mathbb{P}_u^j)$. Since every eigenvalue $\lambda_k : v \in \mathcal{U}_N \mapsto \lambda_k^v \in \mathbb{R}$ is continuous, there exists an open subset $\mathcal{V} \subset \mathcal{U}_N$ containing u such that $\sup_{v\in\mathcal{V}} \sup_{1\le k\le N} |\lambda_k^v - \lambda_k^u| < \frac{\epsilon_0}{3}$. We set $\epsilon = \frac{2\epsilon_0}{3}$, then $\lambda_j^v \in D(\lambda_j^u, \epsilon) \backslash \overline{D}(\lambda_k^u, \epsilon_0)$, for every $v \in \mathcal{V}$ and $k \neq j$. For example, in the next picture, the dashed circles denote respectively $\mathscr{C}(\lambda_j^u, \epsilon_0)$ and $\mathscr{C}(\lambda_k^u, \epsilon_0)$; the smaller circles denote respectively $\mathscr{C}(\lambda_j^u, \epsilon)$ and $\mathscr{C}(\lambda_k^u, \epsilon)$ with $j < k$. The segments inside small circles denote the possible positions of λ_j^v and λ_k^v.



Then $\sigma(L_v) \bigcap D(\lambda_j^u, \epsilon_0) = \{\lambda_j^v\}$ and $\mathscr{C}(\lambda_j^u, \epsilon)$ is a closed path in $D(\lambda_j^u, \epsilon_0)$ with respect to which λ_j^v has winding number 1. Thus,

$$\mathbb{P}_v^j = \frac{1}{2\pi i} \oint_{\mathscr{C}(\lambda_j^u,\epsilon)} (\zeta - L_v)^{-1} \mathrm{d}\zeta, \qquad \lambda_j^v = \mathrm{Tr}(L_v \circ \mathbb{P}_v^j), \qquad \forall v \in \mathcal{V}. \tag{4.24}$$

Since $v \in \mathcal{V} \mapsto L_v \in \mathfrak{B}(H_+^1, L_+^2)$ is $\mathbb{R}$-affine and $\mathbf{i} : \mathcal{A} \in \mathfrak{B}_{\mathfrak{I}}(H_+^1, L_+^2) \mapsto \mathcal{A}^{-1} \in \mathfrak{B}(L_+^2, H_+^1)$ is complex analytic, where $\mathfrak{B}_{\mathfrak{I}}(H_+^1, L_+^2) \subset \mathfrak{B}(H_+^1, L_+^2)$ denotes the open subset of all bijective bounded $\mathbb{C}$-linear transformations $H_+^1 \to L_+^2$, we have the real analyticity of the following map

$$(\zeta, v) \in \left(D(\lambda_j^u, \frac{3}{4}\epsilon_0) \backslash \overline{D}(\lambda_j^u, \frac{1}{2}\epsilon_0) \right) \times \mathcal{V} \mapsto (\zeta - L_v)^{-1} \in \mathfrak{B}(L_+^2, H_+^1). \tag{4.25}$$

Hence the maps $\mathbb{P}^j : v \in \mathcal{V} \mapsto \mathbb{P}^j_v \in \mathfrak{B}(L^2_+, H^1_+)$ and $\lambda_j : v \in \mathcal{V} \mapsto \mathrm{Tr}(L_v \circ \mathbb{P}^j_v) \in \mathbb{R}$ are both real analytic by composing (4.24) and (4.25).

□

Recall that $\mathscr{H}_{\mathrm{pp}}(L_u) = \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u}$, where Q_u denotes the characteristic polynomial of $u \in \mathcal{U}_N$ whose zeros are contained in $\mathbb{C}_-$, so $\mathscr{H}_{\mathrm{pp}}(L_u) \subset \mathbf{D}(G)$ is given by (3.7). We have the following consequence.

Corollary 4.12. *For every $j = 1, 2, \cdots, N$, the map $\mho_j : u \in \mathcal{U}_N \mapsto \langle G\varphi^u_j, \varphi^u_j\rangle_{L^2} \in \mathbb{C}$ is real analytic.*

Proof. For every $u, v \in \mathcal{U}_N$, we have $\mathbb{P}^j_v \varphi^u_j = \langle \varphi^u_j, \varphi^v_j\rangle_{L^2} \varphi^v_j$. Since the Riesz projector $\mathbb{P}^j : v \in \mathcal{U}_N \mapsto \mathbb{P}^j_v \in \mathfrak{B}(L^2_+, H^1_+)$ is real analytic in the proof of proposition 4.7 and $\|\mathbb{P}^j_u \varphi^u_j\|_{L^2} = 1$, there exists a neighbourhood of u, denoted by $\mathcal{V}$, such that $\|\mathbb{P}^j_v \varphi^u_j\|_{L^2} > \frac{1}{2}$ for every $v \in \mathcal{V}$ and $\mathbb{P}^j : v \in \mathcal{V} \mapsto \mathbb{P}^j_v \in \mathfrak{B}(L^2_+, H^1_+)$ can be expressed by power series. Then

$$\varphi^v_j = \frac{\mathbb{P}^j_v \varphi^u_j}{\langle \varphi^u_j, \varphi^v_j\rangle_{L^2}}, \qquad \mho_j(v) = \frac{\langle G \circ \mathbb{P}^j_v(\varphi^u_j), \mathbb{P}^j_v(\varphi^u_j)\rangle_{L^2}}{\|\mathbb{P}^j_v(\varphi^u_j)\|^2_{L^2}}.$$

Hence the restriction $\mho_j : v \in \mathcal{V} \mapsto \|\mathbb{P}^j_v(\varphi^u_j)\|^{-2}_{L^2} \langle G \circ \mathbb{P}^j_v(\varphi^u_j), \mathbb{P}^j_v(\varphi^u_j)\rangle_{L^2} \in \mathbb{C}$ is real analytic. □

4.3 Characterization theorem

The characterization theorem 4.8 is proved in this subsection. The direct sense is given by proposition 4.1 and proposition 4.6. Before proving the converse sense of theorem 4.8, we need the following lemmas to prove the invariance of $\mathscr{H}_{\mathrm{pp}}(L_u)$ under G, if $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$ is real-valued, $\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u)$ and $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_u) = N \geq 1$. The following lemma gives another version of formula of commutators (see also lemma 3.1).

Lemma 4.13. *For $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$, $\varphi \in \mathrm{Ker}(\lambda - L_u)$ for some $\lambda \in \sigma_{\mathrm{pp}}(L_u)$, then we have $\varphi, T_u\varphi, L_u\varphi \in \mathbf{D}(G)$ and*

$$[G, T_u]\varphi = \frac{i\hat{\varphi}(0^+)}{2\pi}\Pi u, \qquad [G, L_u]\varphi = i\varphi - \frac{i\hat{\varphi}(0^+)}{2\pi}\Pi u. \tag{4.26}$$

where $\Theta = \Theta_u = \frac{\overline{Q}_u}{Q_u}$ with Q_u the characteristic polynomial of u.

Proof. In proposition 2.3, we have shown that $\widehat{u\varphi} \in H^1(\mathbb{R})$, so $(T_u\varphi)^\wedge = \widehat{u\varphi}\mathbf{1}_{\mathbb{R}_+} \in H^1(0, +\infty)$ and $T_u\varphi \in \mathbf{D}(G)$. We recall the regularity of eigenfunctions (2.2)

$$\mathrm{Ker}(\lambda - L_u) \subset \{\varphi \in H^1_+ : \hat{\varphi}_{|\mathbb{R}_+} \in C^1(\mathbb{R}_+) \bigcap H^1(\mathbb{R}_+) \quad \text{and} \quad \xi \mapsto \xi[\hat{\varphi}(\xi) + \partial_\xi \hat{\varphi}(\xi)] \in L^2(\mathbb{R}_+)\}. \tag{4.27}$$

So $G\varphi \in H^1_+ = \mathbf{D}(L_u) = \mathbf{D}(T_u)$. Moreover, we have $\hat{\varphi}$ is right-continuous at $\xi = 0^+$ and $\hat{\varphi} \in C^1(0, +\infty)$. The weak-derivative of $\hat{\varphi}$ is denoted by $\partial^w_\xi \hat{\varphi}$, δ_0 denotes the Dirac measure with support $\{0\}$, then

$$\partial^w_\xi \hat{\varphi} = \mathbf{1}_{\mathbb{R}^*_+} \frac{\mathrm{d}}{\mathrm{d}\xi}\hat{\varphi} + \hat{\varphi}(0^+)\delta_0, \qquad \partial_\xi(\hat{u} * \hat{\varphi}) = \partial^w_\xi(\hat{u} * \hat{\varphi}) = \hat{u} * \partial^w_\xi \hat{\varphi} \tag{4.28}$$

by lemma 2.15. Since $\hat{\varphi} = \mathbf{1}_{\mathbb{R}^*_+}\hat{\varphi}$ a.e. in $\mathbb{R}$ and $\hat{u} \in H^1(\mathbb{R})$, we have $\hat{u} * \widehat{G\varphi}(\xi) = \hat{u} * [\mathbf{1}_{\mathbb{R}^*_+}\widehat{G\varphi}](\xi)$, for every $\xi > 0$ and $([G, T_u]\varphi)^\wedge(\xi) = \frac{i}{2\pi}\partial_\xi(\hat{u} * \hat{\varphi})(\xi) - \frac{i}{2\pi}\hat{u} * [\mathbf{1}_{\mathbb{R}^*_+}\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{f}](\xi) = \frac{i}{2\pi}\hat{\varphi}(0^+)\hat{u}(\xi)$. Together with

(5.9), the first formula of (4.26) is obtained. Since $L_u = \mathrm{D} - T_u$, we claim that $\mathrm{D}\varphi \in \mathbf{D}(G)$. In fact, $\partial_\xi(\mathrm{D}\varphi)^\wedge(\xi) = \hat{\varphi}(\xi) + \xi\partial_\xi\hat{\varphi}(\xi)$, $\forall \xi > 0$. Thus (4.27) implies that $\widehat{\mathrm{D}\varphi} \in H^1(0,+\infty)$. Then

$$([G,\mathrm{D}]\varphi)^\wedge(\xi) = i\partial_\xi(\xi\hat{\varphi})(\xi) - \xi \cdot i\partial_\xi\hat{\varphi}(\xi) = i\hat{\varphi}(\xi), \qquad \forall \xi > 0. \tag{4.29}$$

So we have $[\partial_x, G] = \mathrm{Id}_{L^2_+}$. The second formula of (4.26) holds. □

Proposition 4.14. *If $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$ is real-valued, $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_u) = N \geq 1$ and $\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u)$, then we have $\mathscr{H}_{\mathrm{pp}}(L_u) \subset \mathbf{D}(G)$ and $G(\mathscr{H}_{\mathrm{pp}}(L_u)) \subset \mathscr{H}_{\mathrm{pp}}(L_u)$.*

Proof. There exists an orthonormal basis of $\mathscr{H}_{\mathrm{pp}}(L_u)$, denoted by $\{\psi_1, \psi_2, \cdots, \psi_N\}$, such that

$$L_u\psi_j = \lambda_j\psi_j, \quad \text{where} \quad \sigma_{\mathrm{pp}}(L_u) = \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subset (-\infty, 0), \quad \lambda_j < \lambda_{j+1}.$$

Since (4.27) implies that $\mathscr{H}_{\mathrm{pp}}(L_u) \subset G^{-1}(H^1_+) \bigcap \mathbf{D}(G)$, formula (4.26) gives that

$$f_j := [L_u, G]\psi_j = -i\psi_j + \frac{i\hat{\psi}_j(0^+)}{2\pi}\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u), \qquad \forall j = 1, 2, \cdots, N.$$

So we have $\langle f_j, \psi_j\rangle_{L^2} = \langle G\psi_j, L_u\psi_j\rangle_{L^2} - \langle GL_u\psi_j, \psi_j\rangle_{L^2} = \lambda(\langle G\psi_j, \psi_j\rangle_{L^2} - \langle G\psi_j, \psi_j\rangle_{L^2}) = 0$.

For every $j = 1, 2, \cdots, N$, we set $g_j := \sum_{1\leq k\leq N, k\neq j} \frac{\langle f_j, \psi_k\rangle_{L^2}}{\lambda_k - \lambda_j}\psi_k$. Since $f_j = \sum_{1\leq k\leq N, k\neq j}\langle f_j, \psi_k\rangle_{L^2}\psi_k$, we have $(L_u - \lambda_j)g_j = f_j = (L_u - \lambda_j)G\psi_j$. Then $G\psi_j - g_j \in \mathrm{Ker}(L_u - \lambda_j) = \mathbb{C}\psi_j$ and

$$G\psi_j \in g_j + \mathbb{C}\psi_j \subset \mathscr{H}_{\mathrm{pp}}(L_u).$$

We conclude by $\mathscr{H}_{\mathrm{pp}}(L_u) = \mathrm{Span}_{\mathbb{C}}\{\psi_1, \psi_2, \cdots, \psi_N\}$. (see also formulas (4.4) and (4.6)) □

Now, we perform the proof of converse sense of theorem 4.8 give the explicit formula of Q_u.

End of the proof of theorem 4.8. ⇐: Proposition 4.14 yields that $G(\mathscr{H}_{\mathrm{pp}}(L_u)) \subset \mathscr{H}_{\mathrm{pp}}(L_u)$. Let Q denote the characteristic polynomial of the operator $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$, then we have $\mathscr{H}_{\mathrm{pp}}(L_u) = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$ by lemma 3.3. So $\Pi u = \frac{\mathrm{P}_0}{Q}$, for some $\mathrm{P}_0 \in \mathbb{C}[X]$ such that $\deg \mathrm{P}_0 \leq N-1$. It remains to show that $\mathrm{P}_0 = iQ'$. Since $\mathscr{H}_{\mathrm{pp}}(L_u)$ is invariant under L_u, for every $P \in \mathbb{C}_{\leq N-1}[X]$, we have

$$L_u(\frac{P}{Q}) = (\mathrm{D} - T_{\frac{\mathrm{P}_0}{Q}} - T_{\frac{\overline{\mathrm{P}_0}}{\overline{Q}}})(\frac{P}{Q}) = \frac{\mathrm{D}P}{Q} - \Pi(\frac{\overline{\mathrm{P}}_0 P}{\overline{Q}Q}) + \frac{(iQ' - \mathrm{P}_0)P}{Q^2} \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}.$$

Partial-fraction decomposition implies that $\Pi(\frac{\overline{\mathrm{P}}_0 P}{\overline{Q}Q}) \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$. So $\frac{(iQ' - \mathrm{P}_0)P}{Q} \in \mathbb{C}_{\leq N-1}[X]$ for every $P \in \mathbb{C}_{\leq N-1}[X]$. Choose $P = \mathbf{1}$, since $\deg(iQ' - \mathrm{P}_0) \leq N-1$, we have $\mathrm{P}_0 = iQ'$, so $u \in \mathcal{U}_N$. Since $Q \in \mathbb{C}_N[X]$ is monic and $Q^{-1}(0) \subset \mathbb{C}_-$, we have $Q_u(x) = Q(x) = \det(x - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)})$. □

4.4 The stability under the Benjamin–Ono flow

Finally we prove proposition 4.9 in this subsection. Two lemmas will be proved at first in order to obtain the invariance of the property $x \mapsto xu(x) \in L^2(\mathbb{R})$ under the BO flow.

Lemma 4.15. *If $u_0 \in H^2(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$, let $u = u(t,x)$ solves the BO equation (1.1) with initial datum $u(0) = u_0$, then $u(t) \in L^2(\mathbb{R}, x^2\mathrm{d}x)$, for every $t \in \mathbb{R}$.*

Remark 4.16. *This result can be strengthened by replacing the assumption $u_0 \in H^2(\mathbb{R},\mathbb{R})$ by a weaker assumption $u_0 \in H^{\frac{3}{2}+}(\mathbb{R},\mathbb{R}) = \bigcup_{s>\frac{3}{2}} H^s(\mathbb{R},\mathbb{R})$, because one can construct the conservation law of BO equation controlling the H^s-norm for every $s > -\frac{1}{2}$ by using the method of perturbation of determinants. We refer to Talbut [62] to see details and Killip–Vişan–Zhang [37] for the KdV and the NLS cases (see also Koch–Tataru [36]). It suffices to use lemma 4.15 to prove proposition 4.9.*

Before proving lemma 4.15, we need some commutator estimates used in Gérard–Lenzmann–Pocovnicu–Raphaël [21], we recall it here.

Lemma 4.17. *For a general locally Lipschitz function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\partial_x\chi, \partial_x^3\chi, \partial_x^5\chi \in L^1(\mathbb{R})$, then we have the following commutator estimates*

$$\begin{aligned} &\|[|\mathrm{D}|,\chi]g\|_{L^2} + \|[\partial_x,\chi]g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}\|g\|_{L^2}, && \forall g \in L^2(\mathbb{R}),\\ &\||\mathrm{D}|[\partial_x,\chi]g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}\|\partial_x g\|_{L^2} + (\|\partial_x\chi\|_{L^1}\|\partial_x^5\chi\|_{L^1})^{\frac{1}{2}}\|g\|_{L^2}, && \forall g \in H^1(\mathbb{R}). \end{aligned} \tag{4.30}$$

Proof. We use $\big||\xi| - |\eta|\big| \le |\xi-\eta|$ to estimate the Fourier modes of $[|\mathrm{D}|,\chi]g$.

$$2\pi\Big|\left([|\mathrm{D}|,\chi]g\right)^\wedge(\xi)\Big| \le \int_{\eta\in\mathbb{R}} \big||\xi|-|\eta|\big||\hat{\chi}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta \le \int_{\eta\in\mathbb{R}} |\xi-\eta||\hat{\chi}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta = |\widehat{\partial_x\chi}| * |\hat{g}|(\xi).$$

Then Young's convolution inequality yields that $\|[|\mathrm{D}|,\chi]g\|_{L^2} \lesssim \||\widehat{\partial_x\chi}| * |\hat{g}|\|_{L^2} \lesssim \|\widehat{\partial_x\chi}\|_{L^1}\|g\|_{L^2}$. In order to estimate $\|\widehat{\partial_x\chi}\|_{L^1}$, we divide the integral as two parts. Wet set $\mathcal{R}_1 = \|\partial_x\chi\|_{L^1}^{-\frac{1}{2}}\|\partial_x^3\chi\|_{L^1}^{\frac{1}{2}}$, so

$$\|\widehat{\partial_x\chi}\|_{L^1} \le \|\widehat{\partial_x\chi}\|_{L^\infty}\int_{|\xi|\le\mathcal{R}_1}\mathrm{d}\xi + \int_{|\xi|>\mathcal{R}_1}\frac{\|\widehat{\partial_x^3\chi}\|_{L^\infty}}{|\xi|^2}\mathrm{d}\xi \lesssim \|\partial_x\chi\|_{L^1}\mathcal{R}_1 + \frac{\|\partial_x^3\chi\|_{L^1}}{\mathcal{R}_1} = (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}.$$

Similarly, we have $\|[\partial_x,\chi]g\|_{L^2} \lesssim \|\widehat{\partial_x\chi}\|_{L^1}\|g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}$. Thus (4.30) is obtained.

$$\begin{aligned} 2\pi\Big|\left(|\mathrm{D}|[\partial_x,\chi]g\right)^\wedge(\xi)\Big| \le& |\xi|\int_{\eta\in\mathbb{R}}|\xi-\eta||\hat{\chi}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta\\ \le& \int_{\eta\in\mathbb{R}}|\xi-\eta|^2|\hat{\chi}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta + \int_{\eta\in\mathbb{R}}|\xi-\eta||\hat{\chi}(\xi-\eta)||\eta||\hat{g}(\eta)|\mathrm{d}\eta\\ =& |\widehat{\partial_x^2\chi}| * |\hat{g}|(\xi) + |\widehat{\partial_x\chi}| * |\widehat{\partial_x g}|(\xi) \end{aligned}$$

So we have $\||\mathrm{D}|[\partial_x,\chi]g\|_{L^2} \lesssim \||\widehat{\partial_x^2\chi}| * |\hat{g}|\|_{L^2} + \||\widehat{\partial_x\chi}| * |\widehat{\partial_x g}|\|_{L^2} \lesssim \|\widehat{\partial_x^2\chi}\|_{L^1}\|g\|_{L^2} + \|\widehat{\partial_x\chi}\|_{L^1}\|\partial_x g\|_{L^2}$. Then we use the same idea to estimate $\|\widehat{\partial_x^2\chi}\|_{L^1}$, we set $\mathcal{R}_2 := \|\partial_x\chi\|_{L^1}^{-\frac{1}{4}}\|\partial_x^5\chi\|_{L^1}^{\frac{1}{4}}$. Thus,

$$\|\widehat{\partial_x^2\chi}\|_{L^1} \le \|\widehat{\partial_x\chi}\|_{L^\infty}\int_{|\xi|\le\mathcal{R}_1}|\xi|\mathrm{d}\xi + \int_{|\xi|>\mathcal{R}_1}\frac{\|\widehat{\partial_x^5\chi}\|_{L^\infty}}{|\xi|^3}\mathrm{d}\xi \lesssim \|\partial_x\chi\|_{L^1}\mathcal{R}_2^2 + \frac{\|\partial_x^5\chi\|_{L^1}}{\mathcal{R}_2^2} = (\|\partial_x\chi\|_{L^1}\|\partial_x^5\chi\|_{L^1})^{\frac{1}{2}}.$$

Finally, we add them together to get the second estimate in (4.30). □

Now we prove the invariance of the property $x \mapsto xu(x) \in L^2(\mathbb{R})$ is invariant under the BO flow.

Proof of lemma 4.15. We choose a cut-off function $\chi \in C_c^\infty(\mathbb{R})$ such that χ decreases in $[0, +\infty)$, χ is even and

$$0 \leq \chi \leq 1, \qquad \chi \equiv 1 \quad \text{on} \quad [-1, 1], \qquad \text{supp}(\chi) \subset [-2, 2]. \tag{4.31}$$

If $u_0 \in H^2(\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2 \mathrm{d}x)$, we claim that there exists a constant $\mathcal{C} = \mathcal{C}(\|u(0)\|_{H^1})$ such that

$$I(R,t) := \int_{\mathbb{R}} \chi^2(\tfrac{x}{R})|x|^2|u(t,x)|^2 \mathrm{d}x \leq \mathcal{C}e^{|t|}\Big(\int_{\mathbb{R}} |x|^2|u(0,x)|^2 \mathrm{d}x + 1\Big), \qquad \forall t \in \mathbb{R}, \quad \forall R > 1, \tag{4.32}$$

if u solves the BO equation $\partial_t u = \mathrm{H}\partial_x^2 u - \partial_x(u^2) = |\mathrm{D}|\partial_x u - 2u\partial_x u$.

In fact, we define $\rho(x) := x\chi(x)$. For every $R > 0$, we set $\rho_R(x) := R\rho(\frac{x}{R}) = x\chi(\frac{x}{R})$. Thus

$$\partial_t I(R,t) = 2\mathrm{Re}\langle \rho_R^2 \partial_t u(t), u(t)\rangle_{L^2} = 2\mathrm{Re}\langle \rho_R^2 |\mathrm{D}|\partial_x u(t) - 2\rho_R^2 u(t)\partial_x u(t), u(t)\rangle_{L^2} = \mathcal{J}_1(u(t)) + \mathcal{J}_2(u(t)),$$

where for every $u \in H^2(\mathbb{R})$, we define

$$\mathcal{J}_1(u) := -4\mathrm{Re}\langle \rho_R^2 u\partial_x u, u\rangle_{L^2} \Longrightarrow |\mathcal{J}_1(u)| \leq 4\|\partial_x u\|_{L^\infty}\|\rho_R u\|_{L^2}^2 \lesssim \|u\|_{H^2}\|\rho_R u\|_{L^2}^2 \tag{4.33}$$

and

$$\mathcal{J}_2(u) := 2\mathrm{Re}\langle \rho_R^2 |\mathrm{D}|\partial_x u, u\rangle_{L^2} = \langle [\rho_R^2, |\mathrm{D}|\partial_x]u, u\rangle_{L^2},$$

because $|\mathrm{D}|\partial_x = -(|\mathrm{D}|\partial_x)^*$ is an unbounded skew-adjoint operator on $L^2(\mathbb{R})$, whose domain of definition is $H^2(\mathbb{R})$, $u \mapsto \rho_R u$ is a bounded self-adjoint operator on $H^s(\mathbb{R})$, for every $s \geq 0$. Since

$$[\rho_R^2, |\mathrm{D}|\partial_x] = \rho_R[\rho_R, |\mathrm{D}|\partial_x] + [\rho_R, |\mathrm{D}|\partial_x]\rho_R, \quad [\rho_R, |\mathrm{D}|\partial_x] = [\rho_R, |\mathrm{D}|\partial_x]^* = [\rho_R, |\mathrm{D}|]\partial_x + |\mathrm{D}|[\rho_R, \partial_x],$$

we have

$$\begin{aligned} \mathcal{J}_2(u) =& \langle \rho_R[\rho_R, |\mathrm{D}|\partial_x]u + [\rho_R, |\mathrm{D}|\partial_x]\rho_R u, u\rangle_{L^2} \\ =& 2\mathrm{Re}\langle [\rho_R, |\mathrm{D}|\partial_x]u, \rho_R u\rangle_{L^2} \\ =& 2\mathrm{Re}\langle [\rho_R, |\mathrm{D}|]\partial_x u, \rho_R u\rangle_{L^2} + 2\mathrm{Re}\langle |\mathrm{D}|[\rho_R, \partial_x]u, \rho_R u\rangle_{L^2}. \end{aligned} \tag{4.34}$$

Since $\|\partial_x \rho_R\|_{L^1} = R\|\partial_x \rho\|_{L^1}$, $\|\partial_x^3 \rho_R\|_{L^1} = R^{-1}\|\partial_x \rho\|_{L^1}$ and $\|\partial_x^5 \rho_R\|_{L^1} = R^{-3}\|\partial_x \rho\|_{L^1}$, the commutator estimates (4.30) yield that if $u \in H^2(\mathbb{R})$, then

$$\begin{aligned} |\mathcal{J}_2(u)| \leq& 2\|\rho_R u\|_{L^2}^2 + \|[\rho_R, |\mathrm{D}|]\partial_x u\|_{L^2}^2 + \||\mathrm{D}|[\rho_R, \partial_x]u\|_{L^2}^2 \\ \lesssim& \|\rho_R u\|_{L^2}^2 + \|\partial_x \rho_R\|_{L^1}\|\partial_x^3 \rho_R\|_{L^1}\|\partial_x u\|_{L^2}^2 + \|\partial_x \rho_R\|_{L^1}\|\partial_x^5 \rho_R\|_{L^1}\|u\|_{L^2}^2 \\ \lesssim& \|\rho_R u\|_{L^2}^2 + \|\partial_x \rho\|_{L^1}\|\partial_x^3 \rho\|_{L^1}\|\partial_x u\|_{L^2}^2 + R^{-2}\|\partial_x \rho\|_{L^1}\|\partial_x^5 \rho\|_{L^1}\|u\|_{L^2}^2 \\ \lesssim& \|\rho_R u\|_{L^2}^2 + \|u\|_{H^1}^2 \end{aligned} \tag{4.35}$$

for every $R \geq 1$. Proposition 2.6 and 2.8 yield that there exists a conservation law of (1.1) controlling H^2-norm of the solution. Let $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R})$ denote the solution of the BO equation (1.1). Then $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^2} \lesssim_{\|u_0\|_{H^2}} 1$. Since $I(R,t) = \|\rho_R u(t)\|_{L^2}^2$, estimates (4.33) and (4.35) imply that

$$|\partial_t I(R,t)| \leq \mathcal{C}(I(R,t) + 1), \qquad t \in \mathbb{R},$$

for some constant $\mathcal{C} = \mathcal{C}(\|u_0\|_{H^2})$. Thus (4.32) is obtained by Gronwall's inequality. Let $R \to +\infty$, we conclude by using Lebesgue's monotone convergence theorem. □

Since the generating function $\lambda \in \mathbb{C}\backslash\sigma(-L_u) \mapsto \mathcal{H}_\lambda(u) \in \mathbb{C}$ is the Borel–Cauchy transform of the spectral measure of L_u, the invariance of the $N-$soliton manifold $\mathcal{U}_N$ under BO flow is obtained by using the inverse spectral transform.

End of the proof of proposition 4.9. If $u_0 \in \mathcal{U}_N \subset H^\infty(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$, let $u = u(t,x)$ be the unique solution of the BO equation (1.1) with initial datum $u(0) = u_0$, then $u(t) \in H^\infty(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$ by proposition 2.5 and lemma 4.15. Recall the generating function $\mathcal{H}_\lambda : u \in L^2(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ defined as

$$\mathcal{H}_\lambda(u) = \langle (\lambda + L_u)^{-1}\Pi u, \Pi u\rangle_{L^2} = \int_{\mathbb{R}} \frac{\mathrm{d}\mathbf{m}_u(\xi)}{\xi+\lambda}, \quad \mathbf{m}_u := \mu^{L_u}_{\Pi u}, \qquad \forall \lambda \in \mathbb{C}\backslash\sigma(-L_u), \tag{4.36}$$

where $\mu^{L_u}_\psi$ denotes the spectral measure of L_u associated to the function $\psi \in L^2_+$. So the holomorphic function $\lambda \in \mathbb{C}\backslash\sigma(-L_u) \mapsto \mathcal{H}_\lambda u$ is the Borel–Cauchy transform of the positive Borel measure $\mathbf{m}_u$. We recall that the total variation $\mathbf{m}_u(\mathbb{R}) = \|\Pi u\|^2_{L^2}$ is a conservation law of the BO equation (1.1) by proposition 2.8 and formula (2.20). Every finite Borel measure is uniquely determined by its Borel–Cauchy transform (see Theorem 3.21 of Teschl [64] page 108), precisely for every $a \leq b$ real numbers, we use Stieltjes inversion formula to obtain that

$$\frac{1}{2}\mathbf{m}_u((a,b)) + \frac{1}{2}\mathbf{m}_u([a,b]) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_a^b \mathrm{Im}\mathcal{H}_{x+i\epsilon}(u)\mathrm{d}x.$$

For every $t \in \mathbb{R}$, proposition 2.10 yields that $\mathcal{H}_\lambda[u(t)] = \mathcal{H}_\lambda[u(0)]$, $\forall \lambda \in \mathbb{C}\backslash\sigma_{\mathrm{pp}}(L_{u(0)}) = \mathbb{C}\backslash\sigma_{\mathrm{pp}}(L_{u(t)})$. Since $u(0) \in \mathcal{U}_N$, we have $\Pi[u(0)] \in \mathscr{H}_{\mathrm{pp}}(L_{u(0)})$ by proposition 4.6 and there exist $c_1, c_2, \cdots, c_N \in \mathbb{R}_+$ such that

$$\mu^{L_{u(t)}}_{\Pi[u(t)]} = \mathbf{m}_{u(t)} = \mathbf{m}_{u(0)} = \mu^{L_{u(0)}}_{\Pi[u(0)]} = \sum_{j=1}^N c_j \delta_{\lambda_j^{u(0)}}.$$

The spectral measure $\mu^{L_{u(t)}}_{\Pi[u(t)]}$ is purely point, so $\Pi[u(t)] \in \mathscr{H}_{\mathrm{pp}}(L_{u(t)})$ for every $t \in \mathbb{R}$. The Lax pair structure yields the unitary equivalence between $L_{u(t)}$ and $L_{u(0)}$. So $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_{u(t)}) = \dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_{u(0)}) = N$ is given by proposition 2.14. We conclude by theorem 4.8. □

5 The generalized action–angle coordinates

In this section, we construct the (generalized) action–angle coordinates Φ_N in theorem 1 of the BO equation (1.6) with solutions in the real analytic symplectic manifold $(\mathcal{U}_N, \omega)$ of real dimension $2N$ given in proposition 4.3. The goal of this section is to establish the diffeomorphism property and the symplectomorphism property of Φ_N.

Recall that the BO equation with N-soliton solutions is identified as a globally well-posed Hamiltonian system reading as

$$\partial_t u(t) = X_E(u(t)), \qquad u(t) \in \mathcal{U}_N, \tag{5.1}$$

whose energy functional $E(u) = \langle L_u \Pi u, \Pi u\rangle_{L^2}$ is well defined on $\mathcal{U}_N$ and the Hamiltonian vector field $X_E : u \in \mathcal{U}_N \mapsto X_E(u) = \partial_x(|\mathrm{D}|u - u^2) \in \mathcal{T}_u(\mathcal{U}_N)$ coincides with the definition (4.3). The Poisson bracket of two smooth functions $f, g : \mathcal{U}_N \to \mathbb{R}$ is given by

$$\{f,g\} : u \in \mathcal{U}_N \mapsto \omega_u(X_f(u), X_g(u)) = \langle \partial_x \nabla_u f(u), \nabla_u g(u)\rangle_{L^2} \in \mathbb{R}. \tag{5.2}$$

Given $u \in \mathcal{U}_N$, proposition 4.6 yields that there exist $\lambda_1^u < \lambda_2^u < \cdots < \lambda_N^u < 0$ and $\varphi_j^u \in \mathrm{Ker}(\lambda_j^u - L_u) \subset \mathbf{D}(G)$ such that $\|\varphi_j^u\|_{L^2} = 1$ and $\langle u, \varphi_j^u\rangle_{L^2} = \sqrt{2\pi|\lambda_j^u|}$, thanks to the spectral analysis in subsection **4.2**.

Definition 5.1. *For every $j = 1, 2, \cdots, N$, the map $I_j : u \in \mathcal{U}_N \mapsto 2\pi\lambda_j^u \in \mathbb{R}$ is called the j th action. The map $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2} \in \mathbb{R}$ is called the j th (generalized) angle.*

Set $\Omega_N := \{(r^1, r^2, \cdots, r^N) \in \mathbb{R}^N : r^1 < r^2 < \cdots < r^N < 0\} \subset \mathbb{R}^N$, the canonical symplectic form on $\mathbb{R}^{2N} = \{(r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) : \forall r^j, \alpha^j \in \mathbb{R}\}$ is given by $\nu = \sum_{j=1}^N \mathrm{d}r^j \wedge \mathrm{d}\alpha^j$. Endowed with the subspace topology and the embedded real analytic structure of $\mathbb{R}^{2N}$, the submanifold $(\Omega_N \times \mathbb{R}^N, \nu)$ is a symplectic manifold of real dimension $2N$. The action–angle map is defined by

$$\Phi_N : u \in \mathcal{U}_N \mapsto (I_1(u), I_2(u), \cdots, I_N(u); \gamma_1(u), \gamma_2(u), \cdots, \gamma_N(u)) \in \Omega_N \times \mathbb{R}^N. \tag{5.3}$$

Theorem 1 is restated here.

Theorem 5.2. *The map Φ_N has following properties:*

(a). *The map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a real analytic diffeomorphism.*
(b). *The pullback of ν by Φ_N is ω, i.e. $\Phi_N^* \nu = \omega$.*
(c). *We have $E \circ \Phi_N^{-1} : (r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) \in \Omega_N \times \mathbb{R}^N \mapsto -\frac{1}{2\pi}\sum_{j=1}^N |r^j|^2 \in (-\infty, 0)$.*

Remark 5.3. *The real analyticity of $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is given by proposition 4.7 and corollary 4.12. The symplectomorphism property* (b) *is equivalent to the following Poisson bracket characterization (see proposition 5.24)*

$$\{I_j, I_k\} = 0, \quad \{I_j, \gamma_k\} = \mathbf{1}_{j=k}, \quad \{\gamma_j, \gamma_k\} = 0 \quad \text{on} \quad \mathcal{U}_N, \qquad \forall j, k = 1, 2, \cdots, N. \tag{5.4}$$

The family $(X_{I_1}, X_{I_2}, \cdots, X_{I_N}; X_{\gamma_1}, X_{\gamma_2}, \cdots, X_{\gamma_N})$ is linearly independent in $\mathfrak{X}(\mathcal{U}_N)$ and we have

$$\mathrm{d}\Phi_N(u) : X_{I_k}(u) \mapsto \frac{\partial}{\partial \alpha^k}\Big|_{\Phi_N(u)}, \qquad \mathrm{d}\Phi_N(u) : X_{\gamma_k}(u) \mapsto -\frac{\partial}{\partial r^k}\Big|_{\Phi_N(u)}.$$

The assertion (c) *is obtained by a direct calculus: $\Pi u = \sum_{j=1}^N \langle \Pi u, \varphi_j^u\rangle_{L^2}\varphi_j^u$, formula (4.9) yields that*

$$E(u) = \langle L_u(\Pi u), \Pi u\rangle_{L^2} = \sum_{j=1}^N |\langle \Pi u, \varphi_j^u\rangle_{L^2}|^2 \lambda_j^u = -\sum_{j=1}^N \frac{I_j(u)^2}{2\pi}.$$

Thus theorem 5.2 introduces (generalized) action–angle coordinates of the BO equation (5.1) in the sense of (1.8), i.e. $\{I_j, E\}(u) = 0$ and $\{\gamma_j, E\}(u) = 2\lambda_j^u$, for every $u \in \mathcal{U}_N$.

This section is organized as follows. The matrix associated to $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$ is expressed in terms of actions and angles in subsection **5.1**. Then the injectivity of Φ_N is given by inversion formulas in subsection **5.2**. In subsection **5.3**, the Poisson brackets of actions and angles are used to show the local diffeomorphism property of Φ_N. The surjectivity of Φ_N is obtained by Hadamard's global inverse theorem in subsection **5.4**. Finally, we use subsection **5.5** and subsection **5.6** to prove that $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ preserves the symplectic structure.

5.1 The associated matrix

We continue to study the infinitesimal generator G defined in (3.2) when restricted to the invariant subspace $\mathscr{H}_{\rm pp}(L_u)$ with complex dimension N. Let $M(u)=(M_{kj}(u))_{1\leq k,j\leq N}$ denote the matrix associated to the operator $G|_{\mathscr{H}_{\rm pp}(L_u)}$ with respect to the basis $\{\varphi_1^u,\varphi_2^u,\cdots,\varphi_N^u\}$. Then we state a general linear algebra lemma that describes the location of eigenvalues of the matrix $M(u)$.

Proposition 5.4. *For every $u\in\mathcal{U}_N$, the coefficients of matrix $M(u)=(M_{kj}(u))_{1\leq k,j\leq N}$ are given by*

$$M_{kj}(u)=\langle G\varphi_j^u,\varphi_k^u\rangle_{L^2}=\begin{cases}\frac{i}{\lambda_k^u-\lambda_j^u}\sqrt{\frac{|\lambda_k^u|}{|\lambda_j^u|}}, & \text{if}\quad j\neq k,\\ \gamma_j(u)-\frac{i}{2|\lambda_j^u|}, & \text{if}\quad j=k.\end{cases}\tag{5.5}$$

Proof. Since L_u is a self-adjoint operator on L^2_+ and $\mathscr{H}_{\rm pp}(L_u)\subset\mathbf{D}(G)$, we have

$$(\lambda_j^u-\lambda_k^u)M_{kj}(u)=\langle GL_u\varphi_j^u,\varphi_k^u\rangle_{L^2}-\langle G\varphi_j^u,L_u\varphi_k^u\rangle_{L^2}=\langle[G,L_u]\varphi_j^u,\varphi_k^u\rangle_{L^2}.$$

Since formulas (2.15) and (4.9) imply that $-\lambda_j^u\widehat{\varphi_j^u}(0)=\widehat{u\varphi_j^u}(0)=\sqrt{2\pi|\lambda_j^u|}$, we use (4.26) to obtain

$$(\lambda_j^u-\lambda_k^u)M_{kj}(u)=\langle i\varphi_j^u-\frac{i}{2\pi}\widehat{\varphi_j^u}(0^+)\Pi u,\varphi_k^u\rangle_{L^2}=-\frac{i}{2\pi}\widehat{\varphi_j^u}(0^+)\overline{\widehat{u\varphi_k^u}}(0)=-i\sqrt{\frac{|\lambda_k^u|}{|\lambda_j^u|}}.$$

In the case $k=j$, we use Plancherel formula and integration by parts to calculate

$$\langle G^*f,g\rangle_{L^2}=\langle f,Gg\rangle_{L^2}=-\tfrac{i}{2\pi}\int_0^{+\infty}\hat f(\xi)\partial_\xi\overline{\hat g}(\xi)\mathrm{d}\xi=\tfrac{i}{2\pi}\left[\hat f(0^+)\overline{\hat g}(0^+)+\int_0^{+\infty}\partial_\xi\hat f(\xi)\overline{\hat g}(\xi)\mathrm{d}\xi\right]$$

Thus we have $\langle G^*f,g\rangle_{L^2}=\langle Gf,g\rangle_{L^2}+\frac{i}{2\pi}\hat f(0^+)\overline{\hat g}(0^+)$, for every $f,g\in\mathscr{H}_{\rm pp}(L_u)$. Then

$$\mathrm{Im}M_{jj}(u)=\frac{1}{2i}(\langle G\varphi_j^u,\varphi_j^u\rangle_{L^2}-\langle G^*\varphi_j^u,\varphi_j^u\rangle_{L^2})=-\frac{|\widehat{\varphi_j^u}(0)|^2}{4\pi}=-\frac{1}{2|\lambda_j^u|}.$$

We conclude by $\gamma_j(u)=\mathrm{Re}\mathfrak{V}_j(u)=\langle G\varphi_j^u,\varphi_j^u\rangle_{L^2}$ defined in corollary 4.12. □

Then we state a linear algebra lemma that describe the location of spectrum of all matrices of the form defined as (5.5).

Lemma 5.5. *For every $N\in\mathbb{N}_+$, we choose N negative numbers $\lambda_1<\lambda_2<\cdots<\lambda_N<0$ and N real numbers $\gamma_1,\gamma_2,\cdots,\gamma_N\in\mathbb{R}$. The matrix $\mathcal{M}=(\mathcal{M}_{kj})_{1\leq k,j\leq N}\in\mathbb{C}^{N\times N}$ is defined as*

$$\mathcal{M}_{kj}=\begin{cases}\frac{i}{\lambda_k-\lambda_j}\sqrt{\frac{|\lambda_k|}{|\lambda_j|}}, & \text{if}\quad k\neq j,\\ \gamma_j-\frac{i}{2|\lambda_j|}, & \text{if}\quad k=j.\end{cases}\tag{5.6}$$

Then $\mathrm{Im}\mathcal{M}=\frac{\mathcal{M}-\mathcal{M}^*}{2i}$ is negative semi-definite and $\sigma_{\rm pp}(\mathcal{M})\subset\mathbb{C}_-$. Furthermore, the map

$$(\lambda_1,\lambda_2,\cdots,\lambda_N;\gamma_1,\gamma_2,\cdots,\gamma_N)\mapsto\mathcal{M}=(\mathcal{M}_{kj})_{1\leq k,j\leq N}$$

defined as (5.6) is real analytic on $\Omega_N\times\mathbb{R}^N$.

Proof. The vector $V_\lambda \in \mathbb{R}^N$ is defined as $V_\lambda^T := ((2|\lambda_1|)^{-\frac{1}{2}}, (2|\lambda_2|)^{-\frac{1}{2}}, \cdots, (2|\lambda_N|)^{-\frac{1}{2}})$. So we have

$$\mathrm{Im}\mathcal{M} = \left(-\frac{1}{2\sqrt{|\lambda_j||\lambda_k|}}\right)_{1\leq k,j\leq N} = -V_\lambda \cdot V_\lambda^T.$$

Recall that $\langle X, Y\rangle_{\mathbb{C}^N} := X^T \cdot \overline{Y}$, thus $\langle(\mathrm{Im}\mathcal{M})X, X\rangle_{\mathbb{C}^N} = -|\langle X, V_\lambda\rangle_{\mathbb{C}^N}|^2 \leq 0$. So $\mathrm{Im}\mathcal{M}$ is a negative semi-definite matrix. If $\mu \in \sigma_{\mathrm{pp}}(\mathcal{M})$ and $V \in \mathrm{Ker}(\mu - \mathcal{M})\backslash\{0\}$, it suffices to show that $\mathrm{Im}\mu < 0$.

$$-|\langle V, V_\lambda\rangle_{\mathbb{C}^N}|^2 = \langle(\mathrm{Im}\mathcal{M})V, V\rangle_{\mathbb{C}^N} = \mathrm{Im}\mu \|V\|^2_{\mathbb{C}^N}, \qquad \text{where} \quad \|V\|^2_{\mathbb{C}^N} = \langle V, V\rangle_{\mathbb{C}^N} > 0. \tag{5.7}$$

So we have $\mathrm{Im}\mu \leq 0$. Assume that $\mu \in \mathbb{R}$, then formula (5.7) yields that $V \perp V_\lambda$. Moreover, we have $(\mathcal{M} - \mathcal{M}^*)V = -2i\langle V, V_\lambda\rangle_{\mathbb{C}^N} V_\lambda = 0$. We set $D^\lambda \in \mathbb{C}^{N\times N}$ to be the diagonal matrix whose diagonal elements are $\lambda_1, \lambda_2, \cdots, \lambda_N$, i.e. $D^\lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$. Then we have the following formula

$$[\mathcal{M}, D^\lambda] = i(\mathrm{I}_N + 2D^\lambda V_\lambda V_\lambda^T). \tag{5.8}$$

So $[\mathcal{M}, D^\lambda]V = iV$ by (5.8). Recall that $\mathcal{M}^*V = \mathcal{M}V = \mu V$. Finally,

$$i\|V\|^2_{\mathbb{C}^N} = \langle[\mathcal{M}, D^\lambda]V, V\rangle_{\mathbb{C}^N} = \langle(\mathcal{M} - \mu)D^\lambda V, V\rangle_{\mathbb{C}^N} = \langle D^\lambda V, (\mathcal{M}^* - \mu)V\rangle_{\mathbb{C}^N} = 0$$

contradicts the fact that $V \neq 0$. Consequently, we have $\mu \in \mathbb{C}_-$. □

Corollary 5.6. *For every $u \in \mathcal{U}_N$, let $M(u) = (M_{kj}(u))_{1\leq k,j\leq N} \in \mathbb{C}^{N\times N}$ denote the matrix defined by formula (5.5), then $\mathrm{Im}M(u) = \frac{M(u)-M(u)^*}{2i}$ is negative semi-definite and $\sigma_{\mathrm{pp}}(M(u)) \subset \mathbb{C}_-$.*

Remark 5.7. *The fact $\sigma_{\mathrm{pp}}(M(u)) \subset \mathbb{C}_-$ can also be given by using the inversion formula (4.10) and proposition 4.1. The characteristic polynomial $Q_u(x) = \det(x - M(u))$ has zeros in $\mathbb{C}_-$.*

5.2 Inverse spectral formulas

The injectivity of Φ_N is proved in this subsection by using inverse spectral formulas. The following lemma describes the relation between the Fourier transform of an eigenfunction $\varphi \in \mathscr{H}_{\mathrm{pp}}(L_u)$ and the inner function associated to u defined by $\Theta_u = \frac{\overline{Q}_u}{Q_u}$ with $Q_u(x) = \det(x - M(u))$.

Lemma 5.8. *For every monic polynomial $Q \in \mathbb{C}_N[X]$ such that $Q^{-1}(0) \subset \mathbb{C}_-$, the associated inner function is defined by $\Theta = \frac{\overline{Q}}{Q}$. The following identity holds for every $\varphi \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$,*

$$\hat{\varphi}(\xi) = \langle S(\xi)^*\varphi, 1 - \Theta\rangle_{L^2}. \tag{5.9}$$

In particular, $\hat{\varphi}(0^+) = \langle\varphi, 1 - \Theta\rangle_{L^2}$.

Proof. Since $\varphi = \frac{P}{Q}$, for some $P \in \mathbb{C}_{\leq N-1}[X]$ and $Q^{-1}(0) \subset \mathbb{C}_-$, recall that $Q(x) = \prod_{j=1}^n (x - z_j)^{m_j}$ with $\mathrm{Im}z_j < 0$, $z_1, z_2, \cdots, z_N$ are all distinct and $\sum_{j=1}^n m_j = N$. Formulas (3.7) and (3.8) imply that

$$f_{j,l}(x) = \frac{l!}{2\pi[(-i)(x - z_j)]^{l+1}} \Longrightarrow \hat{f}_{j,l}(\xi) = \xi^l e^{-iz_j\xi}\mathbf{1}_{\mathbb{R}_+}(\xi).$$

Since $\varphi \in \mathrm{Span}_{\mathbb{C}}\{f_{j,l}\}_{1\leq j\leq m_j, 1\leq j\leq n}$, partial-fractional decomposition implies that $\hat{\varphi} \in C^1(\mathbb{R}_+^*)$, and the right limit $\hat{\varphi}(0^+) = \lim_{\xi\to 0^+}\hat{\varphi}(\xi)$ exists. Recall that $\Theta = \frac{\overline{Q}}{Q}$, so we have $\overline{\Theta}\varphi = \frac{Q}{\overline{Q}}\frac{P}{Q} = \frac{P}{\overline{Q}} \in L^2_-$. Since $\Theta(x) = 1 + 2i\sum_{j=1}^N \frac{\mathrm{Im} z_j}{x-z_j} + \mathcal{O}(\frac{1}{x^2})$, when $x \to +\infty$, we have $1-\Theta \in L^2_+$. Then

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(y)(1-\overline{\Theta(y)})e^{-iy\xi}\mathrm{d}y = \langle \varphi, S(\xi)(1-\Theta)\rangle_{L^2} = \langle S(\xi)^*\varphi, 1-\Theta\rangle_{L^2}, \qquad \forall \xi \geq 0.$$

□

Proposition 5.9. *For every $u \in \mathcal{U}_N$, we set $Q_u \in \mathbb{C}_N[X]$ to be the characteristic polynomial of u and we define the associated inner function as $\Theta_u = \frac{\overline{Q}_u}{Q_u}$. Then the following inversion formula holds,*

$$f(z) = \frac{1}{2\pi i}\langle (G-z)^{-1}f, 1-\Theta_u\rangle_{L^2}, \qquad f \in \mathscr{H}_{\mathrm{pp}}(L_u), \qquad \forall z \in \mathbb{C}_+. \tag{5.10}$$

Proof. If $f \in \mathscr{H}_{\mathrm{pp}}(L_u) = \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u}$, then formula (5.9) yields that

$$\hat{f}(\xi) = \langle S(\xi)^* f, 1-\Theta_u\rangle_{L^2} = \langle e^{-i\xi G}f, 1-\Theta_u\rangle_{L^2}.$$

Since $\mathrm{Im}G := \frac{G-G^*}{2i}$ is a negative semi-definite operator on $\mathscr{H}_{\mathrm{pp}}(L_u)$ by proposition 5.4 and lemma 5.5, the operator $\mathrm{Re}(i(z-G))_{|\mathscr{H}_{\mathrm{pp}}(L_u)} = (\mathrm{Im}G - \mathrm{Im}z)_{|\mathscr{H}_{\mathrm{pp}}(L_u)}$ is negative definite, for every $z \in \mathbb{C}_+$. So

$$f(z) = \frac{1}{2\pi}\int_0^{+\infty} \langle e^{i\xi(z-G)}f, 1-\Theta_u\rangle_{L^2}\mathrm{d}\xi = \frac{1}{2\pi i}\langle (G-z)^{-1}f, 1-\Theta_u\rangle_{L^2}.$$

□

Recall that $\langle \Pi u, \varphi_j^u\rangle_{L^2} = \sqrt{2\pi|\lambda_j^u|}$ and $\langle 1-\Theta, \varphi_j^u\rangle_{L^2} = \sqrt{\frac{2\pi}{|\lambda_j^u|}}$, for every $j = 1, 2, \cdots, N$, by (2.15) and (4.9). Since $\Pi u \in \mathrm{Hol}(\{z \in \mathbb{C} : \mathrm{Im}z > -\epsilon\})$, for some $\epsilon > 0$, we have the following inversion formula

$$\Pi u(x) = \tfrac{1}{2\pi i}\langle (G-x)^{-1}\Pi u, 1-\Theta\rangle_{L^2} = -i\langle (M(u)-x)^{-1}X(u), Y(u)\rangle_{\mathbb{C}^N}, \qquad \forall x \in \mathbb{R}, \tag{5.11}$$

where the two vectors $X(u), Y(u) \in \mathbb{R}^N$ are defined as

$$X(u)^T = (\sqrt{|\lambda_1^u|}, \sqrt{|\lambda_2^u|}, \cdots, \sqrt{|\lambda_N^u|}), \qquad Y(u)^T = (\sqrt{|\lambda_1^u|^{-1}}, \sqrt{|\lambda_2^u|^{-1}}, \cdots, \sqrt{|\lambda_N^u|^{-1}}), \tag{5.12}$$

and $M(u)$ is the $N \times N$ matrix of the infinitesimal generator G associated to the orthonormal basis $\{\varphi_1^u, \varphi_1^u, \cdots, \varphi_N^u\}$, defined in (5.4). A consequence of the inverse spectral formula (5.11) is the explicit formula of the BO flow with N-soliton solutions as described by formula (1.11).

Corollary 5.10. *The map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is injective.*

Proof. If $\Phi_N(u) = \Phi_N(v)$ for some $u, v \in \mathcal{U}_N$, then $\lambda_j^u = \lambda_j^v$ and $\gamma_j(u) = \gamma_j(v)$, for every j. So

$$M(u) = M(v), \qquad X(u) = X(v), \qquad Y(u) = Y(v).$$

Then the inversion formula (5.11) gives that $\Pi u = \Pi v$. Thus, $u = 2\mathrm{Re}\Pi u = 2\mathrm{Re}\Pi v = v$. □

At last we show the equivalence between the inversion formulas (4.10) and (5.11).

Revisiting formula (4.10). For every $k, j = 1, 2, \cdots, N$, let $K^u_{kj}(x)$ denote the $(N-1)\times(N-1)$ submatrix obtained by deleting the k th column and j th row of the matrix $M(u) - x$, for every $x \in \mathbb{R}$. So the inversion formula (5.11) and the Cramer's rule imply that

$$i\Pi u(x) = \sum_{1\le k,j\le N} \frac{(-1)^{k+j}\det(K^u_{kj}(x))}{\det(M(u)-x)}\sqrt{\frac{\lambda^u_k}{\lambda^u_j}} = \frac{\sum_{j=1}^N \det(K^u_{jj}(x)) + R}{\det(M(u)-x)}, \tag{5.13}$$

where $R := \sum_{1\le k\ne j\le N}(-1)^{k+j}\det(K^u_{kj}(x))\sqrt{\frac{\lambda^u_k}{\lambda^u_j}}$. The coefficients of the matrix $M(u) - x$ satisfies that

$$(M(u)-x)_{kj} = M_{kj}(u) = \tfrac{i}{\lambda^u_k - \lambda^u_j}\sqrt{\tfrac{\lambda^u_k}{\lambda^u_j}}, \qquad \text{if} \quad 1 \le j \ne k \le N,$$

by formula (5.5). Using expansion by minors, we have

$$iR = \sum_{1\le k,j\le N}(-1)^{k+j}(\lambda^u_k - \lambda^u_j)(M(u)-x)_{kj}\det(K^u_{kj}(x)) = (\sum_{k=1}^N \lambda^u_k - \sum_{j=1}^N \lambda^u_j)\det(M(u)-x) = 0.$$

Finally, let Q denote the characteristic polynomial of the operator $G|_{\mathscr{H}_{\mathrm{pp}(L_u)}}$, so

$$Q(x) = \det(x - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}) = \det(x - M(u)), \qquad Q'(x) = (-1)^N \sum_{j=1}^N \det(K^u_{jj}(x)).$$

□

5.3 Poisson brackets

In this subsection, the Poisson bracket defined in (5.2) is generalized in order to obtain the first two formulas of (5.4). It can be defined between a smooth function from $\mathcal{U}_N$ to an arbitrary Banach space and another smooth function from $\mathcal{U}_N$ to $\mathbb{R}$.

The N-soliton subset $(\mathcal{U}_N, \omega)$ is a real analytic symplectic manifold of real dimension $2N$, where

$$\omega_u(h_1, h_2) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi}\mathrm{d}\xi, \qquad \forall h_1, h_2 \in \mathcal{T}_u(U_N), \qquad \forall u \in \mathcal{U}_N.$$

For every smooth function $f : \mathcal{U}_N \to \mathbb{R}$, its Hamiltonian vector field $X_f \in \mathfrak{X}(\mathcal{U}_N)$ is given by (4.3). Recall that $X_f(u) = \partial_x \nabla_u f(u)$ and $\mathrm{d}f(u)(h) = \omega(h, X_f(u))$, $\forall h \in \mathcal{T}_u(\mathcal{U}_N)$. For any Banach space $\mathcal{E}$ and any smooth map $F : u \in \mathcal{U}_N \mapsto F(u) \in \mathcal{E}$, we define the Poisson bracket of f and F as follows

$$\{f, F\} : u \in \mathcal{U}_N \mapsto \{f, F\}(u) := \mathrm{d}F(u)(X_f(u)) \in \mathcal{T}_{F(u)}(\mathcal{E}) = \mathcal{E}. \tag{5.14}$$

If $\mathcal{E} = \mathbb{R}$, then the definition in formula (5.14) coincide with (5.2) and we recall it here,

$$\{f, F\}(u) = \mathrm{d}F(u)(X_f(u)) = \omega_u(X_f(u), X_F(u)). \tag{5.15}$$

For every $\lambda \in \mathbb{C}\backslash\sigma(-L_u)$, the generating function $\mathcal{H}_\lambda(u) = \langle (L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}$ is well defined. Since $\Pi u = \sum_{j=1}^N \langle \Pi u, \varphi_j^u\rangle_{L^2}\varphi_j^u$, we have

$$\mathcal{H}_\lambda(u) = \sum_{j=1}^N \frac{|\langle \Pi u, \varphi_j^u\rangle_{L^2}|^2}{\lambda+\lambda_j^u} = -\sum_{j=1}^N \frac{2\pi\lambda_j^u}{\lambda+\lambda_j^u}. \tag{5.16}$$

The analytical continuation allow to extend the generating function $\lambda \mapsto \mathcal{H}_\lambda(u)$ to the domain $\mathbb{C}\backslash\sigma_{\mathrm{pp}}(-L_u)$, and it has simple poles at every $\lambda = -\lambda_j^u$. Proposition 2.2 yields that $-\frac{C^2}{4}\|u\|_{L^2}^2 \leq \lambda_1^u < \cdots < \lambda_N^u < 0$, where $C = \inf_{f\in H_+^1\backslash\{0\}} \frac{\||\mathrm{D}|^{\frac{1}{4}} f\|_{L^2}}{\|f\|_{L^4}}$ denotes the Sobolev constant. So we introduce

$$\mathcal{Y} = \{(\lambda, u) \in \mathbb{R}\times\mathcal{U}_N : 4\lambda > C^2\|u\|_{L^2}^2\} = \mathcal{X}\bigcap(\mathbb{R}\times\mathcal{U}_N), \tag{5.17}$$

where $\mathcal{X}$ is given by definition 2.9. Then the subset $\mathcal{Y}$ is open in $\mathbb{R}\times\mathcal{U}_N$ and the map $\mathcal{H} : (\lambda, u) \in \mathcal{Y} \mapsto -\sum_{j=1}^N \frac{2\pi\lambda_j^u}{\lambda+\lambda_j^u} \in \mathbb{R}$ is real analytic by proposition 4.7. Recall that the Fréchet derivative (2.8) is given by

$$\mathrm{d}\mathcal{H}_\lambda(u)(h) = \langle w_\lambda, \Pi h\rangle_{L^2} + \overline{\langle w_\lambda, \Pi h\rangle}_{L^2} + \langle T_h w_\lambda, w_\lambda\rangle_{L^2} = \langle h, w_\lambda + \overline{w}_\lambda + |w_\lambda|^2\rangle_{L^2}, \quad \forall h \in \mathcal{T}_u(\mathcal{U}_N).$$

where $w_\lambda \in H_+^1$ is given by $w_\lambda \equiv w_\lambda(u) \equiv w_\lambda(x,u) = [(L_u+\lambda)^{-1}\circ\Pi]u(x)$, for every $x \in \mathbb{R}$. Thus

$$X_{\mathcal{H}_\lambda}(u) = \partial_x\nabla_u\mathcal{H}_\lambda(u) = \partial_x(|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u)), \qquad \forall(\lambda,u)\in\mathcal{Y}. \tag{5.18}$$

by (4.3). The Lax map $L : u \in \mathcal{U}_N \mapsto L_u = \mathrm{D} - T_u \in \mathfrak{B}(H_+^1, L_+^2)$ is $\mathbb{R}$-affine, hence real analytic. The following proposition restates the Lax pair structure of the Hamiltonian equation associated to $\mathcal{H}_\lambda$. Even though the stability of $\mathcal{U}_N$ under the Hamiltonian flow of $\mathcal{H}_\lambda$ remains as an open problem, the Poisson bracket defined in (5.14) provides an algebraic method to obtain the first two formulas of (5.4).

Proposition 5.11. *Given $(\lambda, u) \in \mathcal{Y}$ defined by (5.17), we have $\{\mathcal{H}_\lambda, L\}(u) = [B_\lambda^u, L_u]$ and*

$$\{\mathcal{H}_\lambda, \lambda_j\}(u) = 0, \qquad \{\mathcal{H}_\lambda, \gamma_j\}(u) = \mathrm{Re}\langle [G, B_u^\lambda]\varphi_j^u, \varphi_j^u\rangle_{L^2} = -\frac{\lambda}{(\lambda+\lambda_j^u)^2}, \tag{5.19}$$

for every $j = 1, 2, \cdots, N$, where $B_\lambda^u = i(T_{w_\lambda(u)}T_{\overline{w}_\lambda(u)} + T_{w_\lambda(u)} + T_{\overline{w}_\lambda(u)})$.

Proof. Since $L : u \in L^2(\mathbb{R},\mathbb{R}) \mapsto L_u = \mathrm{D} - T_u \in \mathfrak{B}(H_+^1, L_+^2)$, for every $u \in L_+^2$, we have

$$\mathrm{d}L(u)(h) = -T_h, \qquad \forall h \in L_+^2.$$

If $(\lambda, u) \in \mathcal{Y}$, then the $\mathbb{C}$-linear transformation $L_u + \lambda \in \mathfrak{B}(H_+^1, L_+^2)$ is bijective. So formula (5.18) yields that $\{\mathcal{H}_\lambda, L\}(u) = \mathrm{d}L(u)(X_{\mathcal{H}_\lambda}(u)) = -T_{\mathrm{D}(|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u))}$. Then identity (2.24) yields the Lax equation for the Hamiltonian flow of the generating function $\mathcal{H}_\lambda$, i.e.

$$\{\mathcal{H}_\lambda, L\}(u) = [B_\lambda^u, L_u] \in \mathfrak{B}(H_+^1, L_+^2). \tag{5.20}$$

Consider the map $L\varphi_j : u \in \mathcal{U}_N \mapsto L_u\varphi_j^u = \lambda_j^u\varphi_j^u \in H_+^1$, for every $(\lambda, u) \in \mathcal{Y}$, we have

$$\{\mathcal{H}_\lambda, L\}(u)\varphi_j^u + L_u(\{\mathcal{H}_\lambda, \varphi_j\}(u)) = \lambda_j^u\{\mathcal{H}_\lambda, \varphi_j\}(u) + \{\mathcal{H}_\lambda, \lambda_j\}(u)\varphi_j^u$$

with $\{\mathcal{H}_\lambda, \varphi_j\}(u) \in H_+^1$ and $\{\mathcal{H}_\lambda, \lambda_j\}(u) \in \mathbb{R}$. Then (5.20) yields that

$$(\lambda_j^u - L_u)\left(B_u^\lambda\varphi_j^u - \{\mathcal{H}_\lambda, \varphi_j\}(u)\right) = \{\mathcal{H}_\lambda, \lambda_j\}(u)\varphi_j^u.$$

Since $\varphi_j^u \in \mathrm{Ker}(\lambda_j^u - L_u)$ and $\|\varphi_j^u\|_{L^2} = 1$ by the definition in (4.9), we have

$$\{\mathcal{H}_\lambda, \lambda_j\}(u) = \langle (\lambda_j^u - L_u)\left(B_u^\lambda \varphi_j^u - \{\mathcal{H}_\lambda, \varphi_j\}(u)\right), \varphi_j^u\rangle_{L^2} = 0.$$

Let $\mathcal{N}_2 : \varphi \in L^2 \mapsto \|\varphi\|_{L^2}^2$, then we have $\mathcal{N}_2 \circ \varphi_j \equiv 1$ on $\mathcal{U}_N$. Then we have

$$0 = \mathrm{d}(\mathcal{N}_2 \circ \varphi_j)(u) = 2\mathrm{Re}\langle \varphi_j^u, \{\mathcal{H}_\lambda, \lambda_j\}(u)\rangle_{L^2}. \tag{5.21}$$

So there exists $r \in \mathbb{R}$ such that $B_u^\lambda \varphi_j^u - \{\mathcal{H}_\lambda, \varphi_j\}(u) = ir\varphi_j^u$ because $\mathrm{Ker}(\lambda_j^u - L_u) = \mathbb{C}\varphi_j^u$ by corollary 2.4 and formula (5.21). Recall that B_u^λ is skew-adjoint and $\gamma_j = \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2}$, we have

$$\{\mathcal{H}_\lambda, \gamma_j\}(u) = \mathrm{Re}\left(\langle G\{\mathcal{H}_\lambda, \varphi_j\}(u), \varphi_j^u\rangle_{L^2} + \langle G\varphi_j^u, \{\mathcal{H}_\lambda, \varphi_j\}(u)\rangle_{L^2}\right) = \mathrm{Re}\langle [G, B_u^\lambda]\varphi_j^u, \varphi_j^u\rangle_{L^2}.$$

Furthermore, for every $(\lambda, u) \in \mathcal{Y}$, formula (3.4) implies that $[G, T_{\overline{w}_\lambda(u)}] = 0$ and

$$[G, B_u^\lambda]f = i[G, T_{w_\lambda(u)}](T_{\overline{w}_\lambda(u)}(f) + f) = -\tfrac{1}{2\pi}[(\overline{w}_\lambda(u)f)\hat{}(0^+) + \hat{f}(0^+)]w_\lambda(u), \quad \forall f \in \mathbf{D}(G). \tag{5.22}$$

Since $(\overline{w}_\lambda(u)\varphi_j^u)\hat{}(0^+) = \langle \varphi_j^u, w_\lambda(u)\rangle_{L^2} = (\lambda + \lambda_j^u)^{-1}\overline{\langle u, \varphi_j^u\rangle}_{L^2}$ and $\overline{\langle u, \varphi_j^u\rangle}_{L^2} = -\lambda_j^u \widehat{\varphi_j^u}(0^+)$, we replace f by φ_j^u in formula (5.22) to obtain the following

$$\langle [G, B_u^\lambda]\varphi_j^u, \varphi_j^u\rangle_{L^2} = \frac{\overline{\langle u, \varphi_j^u\rangle}_{L^2}}{2\pi}\left(\frac{1}{\lambda_j^u} - \frac{1}{\lambda + \lambda_j^u}\right)\langle w_\lambda(u), \varphi_j^u\rangle_{L^2} = -\frac{\lambda}{(\lambda + \lambda_j^u)^2}, \quad \forall (\lambda, u) \in \mathcal{Y}.$$

□

Remark 5.12. *Recall that* $\tilde{\mathcal{H}}_\epsilon = \frac{1}{\epsilon}\mathcal{H}_{\frac{1}{\epsilon}}$ *and* $\tilde{B}_{\epsilon,u} := \frac{1}{\epsilon}B_u^{\frac{1}{\epsilon}}$ *for every* $(\epsilon^{-1}, u) \in \mathcal{Y}$. *In general, the identity*

$$\{E_n, \gamma_j\}(u) = \mathrm{Re}\langle [G, \tfrac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\big|_{\epsilon=0}\tilde{B}_{\epsilon,u}]\varphi_j^u, \varphi_j^u\rangle_{L^2}, \qquad 1 \le j \le N$$

holds for every conservation law $E_n = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\big|_{\epsilon=0}\tilde{\mathcal{H}}_\epsilon$ *in the BO hierarchy.*

Corollary 5.13. *For every* $j, k = 1, 2, \cdots, N$, *we have*

$$2\pi\{\lambda_j, \gamma_k\}(u) = \mathbf{1}_{j=k}, \qquad \{\lambda_k, \lambda_j\}(u) = 0, \qquad \forall u \in \mathcal{U}_N. \tag{5.23}$$

Proof. Given $u \in \mathcal{U}_N$, for every $\lambda > \frac{C^2\|u\|_{L^2}^2}{4}$ then $(\lambda, u) \in \mathcal{Y}$, then (5.16) and (5.19) imply that

$$-\frac{\lambda}{(\lambda + \lambda_j^u)^2} = \{\mathcal{H}_\lambda, \gamma_j\}(u) = 2\pi\sum_{k=1}^N \{\frac{\lambda}{\lambda + \lambda_k^u}, \gamma_j\}(u) = -2\pi\lambda\sum_{k=1}^N \frac{\{\lambda_k, \gamma_j\}(u)}{(\lambda + \lambda_k^u)^2},$$

and $0 = \{\mathcal{H}_\lambda, \lambda_j\}(u) = 2\pi\lambda\sum_{k=1}^N \frac{\{\lambda_k, \lambda_j\}(u)}{(\lambda + \lambda_k^u)^2}$, for every $j = 1, 2, \cdots, N$. The uniqueness of analytic continuation yields that the following formula holds for every $z \in \mathbb{C}\backslash\mathbb{R}$,

$$-\frac{z}{(z + \lambda_j^u)^2} = -2\pi z\sum_{k=1}^N \frac{\{\lambda_k, \gamma_j\}(u)}{(z + \lambda_k^u)^2}, \qquad \sum_{k=1}^N \frac{\{\lambda_k, \lambda_j\}(u)}{(z + \lambda_k^u)^2} = 0.$$

□

Recall that the actions $I_j : u \in \mathcal{U}_N \mapsto 2\pi\lambda_j^u$ and the generalized angles $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2}$ are both real analytic functions by proposition 4.7 and corollary 4.12.

Proposition 5.14. *For every $u \in \mathcal{U}_N$, the family of differentials*

$$\{\mathrm{d}I_1(u), \mathrm{d}I_2(u), \cdots \mathrm{d}I_N(u); \mathrm{d}\gamma_1(u), \mathrm{d}\gamma_2(u), \cdots \mathrm{d}\gamma_N(u)\}$$

is linearly independent in the cotangent space $\mathcal{T}_u^*(\mathcal{U}_N)$.

Proof. For every $a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N \in \mathbb{R}$ such that

$$\left(\sum_{j=1}^N a_j \mathrm{d}I_j(u) + b_j \mathrm{d}\gamma_j(u)\right)(h) = 0, \qquad \forall h \in \mathcal{T}_u(\mathcal{U}_N). \tag{5.24}$$

Formula of Poisson brackets (5.23) yields that for every $j, k = 1, 2, \cdots, N$, we have

$$\mathrm{d}I_j(u)(X_{I_k}(u)) = \{I_k, I_j\}(u) = 0, \qquad \mathrm{d}\gamma_j(u)(X_{I_k}(u)) = \{I_k, \gamma_j\}(u) = \mathbf{1}_{j=k}$$

We replace h by $X_{I_k}(u)$ in (5.24) to obtain that $b_k = 0$, $\forall k = 1, 2, \cdots, N$. Then set $h = X_{\gamma_k}(u)$

$$-a_k = \sum_{j=1}^N a_j \{\gamma_k, I_j\}(u) = \left(\sum_{j=1}^N a_j \mathrm{d}I_j(u)\right)(X_{\gamma_k}(u)) = 0, \qquad \forall k = 1, 2, \cdots, N.$$

□

As a consequence, $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a local diffeomorphism. Moreover, since all the actions $(I_j)_{1\leq j\leq N}$ are in evolution by (5.23) and the differentials $(\mathrm{d}I_j(u))_{1\leq j\leq N}$ are linearly independent for every $u \in \mathcal{U}_N$, for every $r = (r^1, r^2, \cdots, r^N) \in \Omega_N$, the level set

$$\mathcal{L}_r = \bigcap_{j=1}^N I_j^{-1}(r^j), \qquad \text{where} \quad r = (r^1, r^2, \cdots, r^N)$$

is a smooth Lagrangian submanifold of U_N and $\mathcal{L}_r$ is invariant under the Hamiltonian flow of I_j, for every $j = 1, 2, \cdots, N$, by the Liouville–Arnold theorem (see Theorem 5.5.21 of Katok–Hasselblatt [32], see also Fiorani–Giachetta–Sardanashvily [12] and Fiorani–Sardanashvily [13] for the non-compact invariant manifold case).

5.4 The diffeomorphism property

This subsection is dedicated to proving the real bi-analyticity of $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$. It remains to show the surjectivity. Its proof is based on Hadamard's global inverse theorem 5.18.

Lemma 5.15. *The map $\Phi : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is proper.*

Proof. If K is compact in $\Omega_N \times \mathbb{R}^N$, we choose $u_n \in \Phi_N^{-1}(K)$, so

$$\Phi_N(u_n) = (2\pi\lambda_1^{u_n}, 2\pi\lambda_2^{u_n}, \cdots, 2\pi\lambda_N^{u_n}; \gamma_1(u_n), \gamma_2(u_n), \cdots, \gamma_N(u_n)) \in K, \qquad \forall n \in \mathbb{N}.$$

We assume that there exists $(2\pi\lambda_1, 2\pi\lambda_2, \cdots, 2\pi\lambda_N; \gamma_1, \gamma_2, \cdots, \gamma_N) \in K$ such that $\lambda_j^{u_n} \to \lambda_j$ and $\gamma_j(u_n) \to \gamma_j$ up to a subsequence. So $(M(u_n))_{n\in\mathbb{N}}$ converges to some matrix $M \in \mathbb{C}^{N\times N}$ whose coefficients are defined as follows

$$M_{kj} = \begin{cases} \frac{i}{\lambda_k - \lambda_j}\sqrt{\frac{|\lambda_k|}{|\lambda_j|}}, & \text{if} \quad k \neq j, \\ \gamma_j - \frac{i}{2|\lambda_j|}, & \text{if} \quad k = j. \end{cases}$$

Lemma 5.5 yields that $\sigma_{\mathrm{pp}}(M) \subset \mathbb{C}_-$. We set $Q(x) := \det(x - M)$ and $u = i\frac{Q'}{Q} - i\frac{\overline{Q}'}{\overline{Q}} \in \mathcal{U}_N$. The Viète map $\mathbf{V}$ is defined in (4.11) and $\mathbf{V}(\mathbb{C}_-^N)$ is open in $\mathbb{C}^N$. Then there exists

$$\mathbf{a}^{(n)} = (a_0^{(n)}, a_1^{(n)}, \cdots, a_{N-1}^{(n)}), \quad \mathbf{a} = (a_0, a_1, \cdots, a_{N-1}) \in \mathbf{V}(\mathbb{C}_-^N)$$

such that $Q_n(x) = \det(x - M(u_n)) = \sum_{j=0}^{N-1} a_j^{(n)} x^j + x^N$ and $Q(x) = \sum_{j=0}^{N-1} a_j x^j + x^N$. We have

$$\lim_{n\to+\infty} Q_n(x) = Q(x), \quad \forall x \in \mathbb{R} \quad \Longrightarrow \quad \lim_{n\to+\infty} \mathbf{a}^{(n)} = \mathbf{a}$$

The continuity of the map $\Gamma_N : \mathbf{a} = (a_0, a_1, \cdots, a_{N-1}) \in \mathbf{V}(\mathbb{C}_-^N) \mapsto \Pi u = i\frac{Q'}{Q} \in L^2_+$ yields that

$$\Pi u_n = i\frac{Q_n'}{Q_n} = \Gamma_N(\mathbf{a}^{(n)}) \to \Gamma_N(\mathbf{a}) = i\frac{Q'}{Q} = \Pi u \qquad \text{in} \quad L^2_+, \qquad \text{as} \quad n \to +\infty.$$

Since $\mathcal{U}_N$ inherits the subspace topology of $L^2(\mathbb{R}, \mathbb{R})$, we have $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathcal{U}_N$. The continuity of the map Φ_N shows that $\Phi_N(u) = (2\pi\lambda_1, 2\pi\lambda_2, \cdots, 2\pi\lambda_N; \gamma_1, \gamma_2, \cdots, \gamma_N) \in K$.

□

Proposition 5.16. *The map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is bijective and both Φ_N and its inverse Φ_N^{-1} are real analytic.*

Proof. The analyticity of Φ_N is given by proposition 4.7 and corollary 4.12. The injectivity is given by corollary 5.10. Proposition 5.14 yields that $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a local diffeomorphism by inverse function theorem for manifolds. So Φ_N is an open map. Since every proper continuous map to locally compact space is closed, Φ_N is also a closed map by lemma 5.15. Since the target space $\Omega_N \times \mathbb{R}^N$ is connected, we have $\Phi_N(\mathcal{U}_N) = \Omega_N \times \mathbb{R}^N$ and $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a real analytic diffeomorphism.

□

Remark 5.17. *We establish the relation between $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ and $\Gamma_N : \mathbf{V}(\mathbb{C}_-^N) \to \Pi(\mathcal{U}_N)$ introduced in proposition 4.10. We set $\mathcal{M} : \Omega_N \times \mathbb{R}^N \to \mathbb{C}^{N\times N}$ to be the matrix-valued real analytic function $\mathcal{M}(\eta_1, \eta_2, \cdots, \eta_N; \theta_1, \theta_2, \cdots, \theta_N) = (\mathcal{M}_{kj})_{1\le k,j\le N}$ with coefficients defined as*

$$\mathcal{M}_{kj} = \begin{cases} \frac{2\pi i}{\eta_k - \eta_j}\sqrt{\frac{\eta_k}{\eta_j}}, & \text{if} \quad k \neq j, \\ \theta_j + \frac{\pi i}{\eta_j}, & \text{if} \quad k = j. \end{cases}$$

Then, we set $\mathcal{C} : M \in \mathbb{C}^{N\times N} \mapsto (a_0, a_1, \cdots, a_{N-1}) \in \mathbb{C}^N$ such that

$$Q(x) := \sum_{j=0}^{N-1} a_j x^j + x^N = \det(x - M). \tag{5.25}$$

Since $(-1)^{n-j}a_j = \mathrm{Tr}(\mathbf{\Lambda}^{n-j}M)$ *is the sum of all principle minors of* M *of size* $(N-j)\times(N-j)$*, for every* $j = 1, 2, \cdots, N$*, the map* $\mathcal{C}$ *is real analytic on* $\mathbb{C}^{N\times N}$ *and* $\mathcal{C}\circ\mathcal{M}(\Omega_N \times \mathbb{R}^N) \subset \mathbf{V}(\mathbb{C}_-^N)$ *by lemma 5.5, where* $\mathbf{V}$ *denotes the Viète map defined as (4.11). In lemma 4.10, we have shown that the map* $\Gamma_N : \mathbf{a} = (a_0, a_1, \cdots, a_{N-1}) \in \mathbf{V}(\mathbb{C}_-^N) \mapsto \Pi u = i\frac{Q'}{Q} \in \Pi(\mathcal{U}_N)$ *is biholomorphic, where the polynomial* Q *is defined as (5.25). We conclude by the following identity*

$$\Phi_N^{-1} = 2\mathrm{Re}\circ\Gamma_N\circ\mathcal{C}\circ\mathcal{M} \tag{5.26}$$

The smooth manifolds $\Pi(\mathcal{U}_N)$ and $\mathbf{V}(\mathbb{C}_-^N)$ are both diffeomorphic to the convex open subset $\Omega_N \times \mathbb{R}^N$, so they are simply connected (see also proposition A.5). At last, we recall Hadamard's global inverse theorem.

Theorem 5.18. *Suppose* X *and* Y *are connected smooth manifolds, then every proper local diffeomorphism* $F : X \to Y$ *is surjective. If* Y *is simply connected in addition, then every proper local diffeomorphism* $F : X \to Y$ *is a diffeomorphism.*

Proof. For the surjectivity, see Nijenhuis–Richardson [47] and the proof of proposition 5.16. If the target space is simply connected, see Gordon [23] for the injectivity. □

Remark 5.19. *Since the target space* $\Omega_N \times \mathbb{R}^N$ *is convex, there is another way to show the injectivity of* Φ_N *without using the inversion formulas in subsection* **5.2**. *It suffices to use the simple connectedness of* $\Omega_N \times \mathbb{R}^N$ *and Hadamard's global inverse theorem 5.18.*

5.5 A Lagrangian submanifold

In general, the symplectomorphism property of Φ_N is equivalent to its Poisson bracket characterization (5.4), which will be proved in proposition 5.24. The first two formulas of (5.4) given in corollary 5.13, lead us to focusing on the study of a special Lagrangian submanifold of $\mathcal{U}_N$, denoted by

$$\Lambda_N := \{u \in \mathcal{U}_N : \gamma_j(u) = 0, \quad \forall j = 1, 2, \cdots, N\}, \tag{5.27}$$

where the generalized angles $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2}$ are defined in (5.1). A characterization lemma of Λ_N is given at first.

Lemma 5.20. *For every* $u \in \mathcal{U}_N$*, then each of the following four properties implies the others:*

(a). $u \in \Lambda_N$.
(b). *For every* $x \in \mathbb{R}$*, we have* $\overline{\Pi u}(x) = \Pi u(-x)$.
(c). u *is an even function* $\mathbb{R} \to \mathbb{R}$.
(d). *The Fourier transform* $\hat{u}$ *is real-valued.*

Then every element $u \in \Lambda_N$ has translation–scaling parameter in $(i\mathbb{R})^N/\mathrm{S}_N$ i.e. $u(x) = \sum_{j=1}^N \frac{2\eta_j}{x^2+\eta_j^2}$, for some $\eta_j > 0$.

Proof. (a) ⇒ (b): If $u \in \Lambda_N$, then the matrix $M(u)$ defined in (5.5) is an $N \times N$ matrix with purely imaginary coefficients. Recall the definition of $X(u), Y(u) \in \mathbb{R}^N$ in (5.12):

$$X(u)^T = (\sqrt{|\lambda_1^u|}, \sqrt{|\lambda_2^u|}, \cdots, \sqrt{|\lambda_N^u|}), \qquad Y(u)^T = (\sqrt{|\lambda_1^u|^{-1}}, \sqrt{|\lambda_2^u|^{-1}}, \cdots, \sqrt{|\lambda_N^u|^{-1}}).$$

The inversion formula (5.11) yields that

$$\overline{\Pi u}(x) = i\langle(\overline{M}(u) - x)^{-1}X(u), Y(u)\rangle_{\mathbb{C}^N} = -i\langle(M(u) + x)^{-1}X(u), Y(u)\rangle_{\mathbb{C}^N} = \Pi u(-x).$$

(b) $\Rightarrow$ (c) is given by the formula $u = \Pi u + \overline{\Pi u}$. (c) $\Rightarrow$ (d) is given by $\overline{u}(x) = u(x) = u(-x)$.

(d) $\Rightarrow$ (a): Choose $\lambda \in \sigma_{\mathrm{pp}}(L_u) = \{\lambda_1^u, \lambda_2^u, \cdots, \lambda_N^u\}$ and $\varphi \in \mathrm{Ker}(\lambda - L_u)$. Since both u and its Fourier transform $\hat{u}$ are real-valued, we have $[(\overline{\varphi})^\vee]^\wedge(\xi) = \overline{\hat{\varphi}(\xi)}$, where $(\overline{\varphi})^\vee(x) := \overline{\varphi(-x)}$, $\forall x, \xi \in \mathbb{R}$. Thus,

$$T_u((\overline{\varphi})^\vee) = (\overline{T_u\varphi})^\vee \Longrightarrow (\overline{\varphi})^\vee \in \mathrm{Ker}(\lambda - L_u).$$

We choose the orthonormal basis $\{\varphi_1^u, \varphi_2^u, , \cdots, \varphi_N^u\}$ in $\mathscr{H}_{\mathrm{pp}}(L_u)$ as in formula (4.9). Proposition 2.4 yields that $\dim_{\mathbb{C}} \mathrm{Ker}(\lambda - L_u) = 1$. For every $j = 1, 2, \cdots, N$, there exists $\tilde{\theta}_j \in \mathbb{R}$ such that

$$(\overline{\varphi_j^u})^\vee = e^{i\tilde{\theta}_j}\varphi_j^u \Longleftrightarrow \overline{(\varphi_j^u)^\wedge}(\xi) = e^{i\tilde{\theta}_j}(\varphi_j^u)^\wedge(\xi), \quad \forall \xi \in \mathbb{R}.$$

So we set $\phi_j^u := \exp(\frac{i\tilde{\theta}_j}{2})\varphi_j^u$, then its Fourier transform $(\phi_j^u)^\wedge$ is a real-valued function. Recall the definition of G in (3.2) and γ_j in (5.5), then we have

$$\gamma_j(u) = \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2(\mathbb{R})} = \mathrm{Re}\langle G\phi_j^u, \phi_j^u\rangle_{L^2(\mathbb{R})} = -\frac{1}{2\pi}\mathrm{Im}\langle\partial_\xi[(\phi_j^u)^\wedge], (\phi_j^u)^\wedge\rangle_{L^2(0,+\infty)} = 0.$$

by using Plancherel formula.

□

Lemma 5.21. *The level set Λ_N is a real analytic Lagrangian submanifold of $(\mathcal{U}_N, \omega)$.*

Proof. The map $\gamma : u \in \mathcal{U}_N \mapsto (\gamma_1(u), \gamma_2(u), \cdots, \gamma_N(u)) \in \mathbb{R}^N$ is a real analytic submersion by proposition 5.14. So the level set Λ_N is a properly embedded real analytic submanifold of $\mathcal{U}_N$ and $\dim_{\mathbb{R}} \Lambda_N = N$. The classification of the tangent space $\mathcal{T}_u(\mathcal{U}_N)$ is given by formula (4.1). If $u(x) = \sum_{j=1}^N \frac{2\eta_j}{x^2+\eta_j^2}$, for some $\eta_j > 0$, every tangent vector $h \in \Lambda_N$ is an even function by lemma 5.20. So $\hat{h}$ is real valued and we have

$$\mathcal{T}_u(\Lambda_N) = \bigoplus_{j=1}^N \mathbb{R}f_j^u, \qquad \text{where} \quad f_j^u(x) = \tfrac{2[x^2-\eta_j^2]}{[x^2+\eta_j^2]^2}. \tag{5.28}$$

We have $(f_j^u)^\wedge(\xi) = -2\pi|\xi|e^{-\eta_j|\xi|}$. Then by definition of ω, we have

$$\omega_u(h_1, h_2) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi}\mathrm{d}\xi = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\hat{h}_2(\xi)}{\xi}\mathrm{d}\xi \in i\mathbb{R}, \qquad \forall h_1, h_2 \in \mathcal{T}_u(\Lambda_N). \tag{5.29}$$

Since the symplectic form ω is real-valued, we have $\omega_u(h_1, h_2) = 0$, for every $h_1, h_2 \in \mathcal{T}_u(\Lambda_N)$. Since $\dim_{\mathbb{R}}(\Lambda_N) = N = \frac{1}{2}\dim_{\mathbb{R}}\mathcal{U}_N$, Λ_N is a Lagrangian submanifold of $\mathcal{U}_N$. □

5.6 The symplectomorphism property

Finally, we prove the assertion (b) in theorem 5.2, i.e. the map $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is symplectic, where $\omega(h_1, h_2) := \frac{i}{2\pi} \int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi} \mathrm{d}\xi$, for every $h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N)$ and

$$\Omega_N \times \mathbb{R}^N = \{(r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) \in \mathbb{R}^{2N} : r^1 < r^2 < \cdots < r^N < 0\}, \quad \nu = \sum_{j=1}^{N} \mathrm{d}r^j \wedge \mathrm{d}\alpha^j.$$

We set $\Psi_N = \Phi_N^{-1} : \Omega_N \times \mathbb{R}^N \to \mathcal{U}_N$, let $\Psi_N^* \omega$ denote the pullback of the symplectic form ω by Ψ_N, i.e. for every $p = (r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) \in \Omega_N \times \mathbb{R}^N$, set $u = \Psi_N(p) \in \mathcal{U}_N$,

$$\left(\Psi_N^* \omega\right)_p (V_1, V_2) = \omega_u(\mathrm{d}\Psi_N(p)(V_1), \mathrm{d}\Psi_N(p)(V_2)), \tag{5.30}$$

for every $V_1, V_2 \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$. The goal is to prove that

$$\tilde{\nu} := \Psi_N^* \omega - \nu = 0. \tag{5.31}$$

Recall that the coordinate vectors $\frac{\partial}{\partial r^1}\Big|_p, \frac{\partial}{\partial r^2}\Big|_p, \cdots, \frac{\partial}{\partial r^N}\Big|_p; \frac{\partial}{\partial \alpha^1}\Big|_p, \frac{\partial}{\partial \alpha^2}\Big|_p, \cdots, \frac{\partial}{\partial \alpha^N}\Big|_p$ form a basis for the tangent space $\mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$. We have the following lemma.

Lemma 5.22. *For every $u \in \mathcal{U}_N$, set $p = \Phi_N(u) \in \Omega_N \times \mathbb{R}^N$. Then we have*

$$\mathrm{d}\Phi_N(u)(X_{I_k}(u)) = \frac{\partial}{\partial \alpha^k}\Big|_p, \qquad \forall k = 1, 2, \cdots, N. \tag{5.32}$$

Proof. Fix $u \in \mathcal{U}_N$ and $p = \Phi_N(u)$, for every $h \in \mathcal{T}_u(\mathcal{U}_N)$, we have $\mathrm{d}\Phi_N(u)(h) \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$. For every smooth function $f : \mathbf{p} = (r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) \in \Omega_N \times \mathbb{R}^N \mapsto f(\mathbf{p}) \in \mathbb{R}$, then

$$\left(\mathrm{d}\Phi_N(u)(h)\right) f = \mathrm{d}(f \circ \Phi_N)(u)(h) = \sum_{j=1}^{N} \left(\mathrm{d}I_j(u)(h) \frac{\partial f}{\partial r^j}\Big|_p + \mathrm{d}\gamma_j(u)(h) \frac{\partial f}{\partial \alpha^j}\Big|_p \right). \tag{5.33}$$

For every $k = 1, 2, \cdots, N$, we replace h by $X_{I_k}(u)$, where X_{I_k} denotes the Hamiltonian vector field of the k th action I_k defined in (5.1), thus the Poisson bracket formulas (5.23) yield that

$$\frac{\partial f}{\partial \alpha^k}\Big|_p = \sum_{j=1}^{N} \left(\{I_k, I_j\}(u) \frac{\partial f}{\partial r^j}\Big|_p + \{I_k, \gamma_j\}(u) \frac{\partial f}{\partial \alpha^j}\Big|_p \right) = \left(\mathrm{d}\Phi_N(u)(X_{I_k}(u))\right) f.$$

□

Lemma 5.23. *For every $1 \leq j < k \leq N$, there exists a smooth function $c_{jk} \in C^\infty(\Omega_N \times \mathbb{R}^N)$ such that*

$$\tilde{\nu} = \sum_{1 \leq j < k \leq N} c_{jk} \mathrm{d}r^j \wedge \mathrm{d}r^k, \qquad \frac{\partial c_{jk}}{\partial \alpha^l}\Big|_p = 0, \quad \forall j, k, l = 1, 2, \cdots, N, \tag{5.34}$$

for every $p = (r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) \in \Omega_N \times \mathbb{R}^N$.

Proof. The proof is divided into three steps. The first step is to prove that for every $p \in \Omega_N \times \mathbb{R}^N$ and every $V \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$,

$$\tilde{\nu}_p(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = 0, \qquad \forall l = 1, 2, \cdots, N. \tag{5.35}$$

In fact, let $u = \Psi_N(p) \in \mathcal{U}_N$ and $p = (r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N)$, so $r^l = r^l(p) = I_l \circ \Psi_N(p)$. Then

$$(\Psi_N^* \omega)_p(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = \omega_u(\mathrm{d}\Psi_N(p)\left(\frac{\partial}{\partial \alpha^l}\Big|_p\right), \mathrm{d}\Psi_N(p)(V)) = \omega_u(X_{I_l}(u), \mathrm{d}\Psi_N(p)(V))$$

by (5.32). Thus $(\Psi_N^*\omega)_p(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = -\mathrm{d}I_l(u)(\mathrm{d}\Psi_N(p)(V)) = -\mathrm{d}(I_l \circ \Psi_N)(p)(V)$. On the other hand,

$$\nu_p(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = \sum_{j=1}^{N} (\mathrm{d}r^j \wedge \mathrm{d}\alpha^j)\left(\frac{\partial}{\partial \alpha^l}\Big|_p, V\right) = -\mathrm{d}r^l(p)(V)$$

Thus (5.35) is obtained by $\tilde{\nu} = \Psi_N^*\omega - \nu$.

Since $\tilde{\nu}$ is a smooth 2-form on $\Omega_N \times \mathbb{R}^N$, we have

$$\tilde{\nu} = \sum_{1 \le j < k \le N} (a_{jk}\mathrm{d}\alpha^j \wedge \mathrm{d}\alpha^k + b_{jk}\mathrm{d}r^j \wedge \mathrm{d}\alpha^k + c_{jk}\mathrm{d}r^j \wedge \mathrm{d}r^k),$$

for some smooth functions $a_{jk}, b_{jk}, c_{jk} \in C^\infty(\Omega_N \times \mathbb{R}^N)$, $1 \le j < k \le N$. The second step is to prove that $a_{jk} = b_{jk} = 0$ on $\Omega_N \times \mathbb{R}^N$, for every $1 \le j < k \le N$. In fact, we have $\mathrm{d}r^j \wedge \mathrm{d}r^k(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = 0$,

$$\mathrm{d}r^j \wedge \mathrm{d}\alpha^k(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = -\mathbf{1}_{k=l}\mathrm{d}r^j(p)(V) \quad \text{and} \quad \mathrm{d}\alpha^j \wedge \mathrm{d}\alpha^k(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = \mathbf{1}_{j=l}\mathrm{d}\alpha^k(p)(V) - \mathbf{1}_{k=l}\mathrm{d}\alpha^j(p)(V).$$

Then, let $l \in \{2, \cdots, N\}$ be fixed, for every $1 \le j < k \le N$, we have

$$\sum_{1 \le l < k \le N} a_{lk}\mathrm{d}\alpha^k(p)(V) - \sum_{1 \le j < l \le N} (a_{jl}\mathrm{d}\alpha^j(p)(V) + b_{jl}\mathrm{d}r^j(p)(V)) = \tilde{\nu}_p(\frac{\partial}{\partial \alpha^l}\Big|_p, V) = 0. \tag{5.36}$$

Then we replace V by $\frac{\partial}{\partial r^j}\Big|_p$ and $\frac{\partial}{\partial \alpha^j}\Big|_p$ respectively in formula (5.36), then $a_{jl} = b_{jl} = 0$, for every $j = 1, 2, \cdots, l-1$.

It remains to show that c_{jk} depends on $r^1, r^2, \cdots, r^N$, for every $1 \le j < k \le N$. The symplectic form ω is closed by proposition 4.4 and $\nu = \mathrm{d}\kappa$ is exact, where $\kappa = \sum_{j=1}^N r^j \mathrm{d}\alpha^j$. So

$$\mathrm{d}\tilde{\nu} = \mathrm{d}(\Psi_N^*\omega) - \mathrm{d}\nu = \Psi_N^*(\mathrm{d}\omega) = 0.$$

The exterior derivative of $\tilde{\nu} = \sum_{1 \le j < k \le N} c_{jk}\mathrm{d}r^j \wedge \mathrm{d}r^k$ is computed as following

$$0 = \sum_{1 \le j < k \le N} \sum_{l=1}^{N} \left(\frac{\partial c_{jk}}{\partial \alpha^l}\mathrm{d}\alpha^l \wedge \mathrm{d}r^j \wedge \mathrm{d}r^k + \frac{\partial c_{jk}}{\partial r^l}\mathrm{d}r^l \wedge \mathrm{d}r^j \wedge \mathrm{d}r^k\right).$$

Since the family $\{\mathrm{d}r^j \wedge \mathrm{d}r^k \wedge \mathrm{d}\alpha^l\}_{1 \le j < k \le N, 1 \le l \le N} \bigcup \{\mathrm{d}r^j \wedge \mathrm{d}r^k \wedge \mathrm{d}r^l\}_{1 \le j < k < l \le N}$ is linearly independent in $\mathbf{\Omega}^3(\mathcal{U}_N)$, we have $\frac{\partial c_{jk}}{\partial \alpha^l} = 0$, for every $1 \le j < k \le N$ and $l = 1, 2, \cdots, N$. □

Since the 2-form $\tilde{\nu}$ is independent of $\alpha^1, \alpha^2, \cdots, \alpha^N$, it suffices to consider points $p = (r, \alpha) \in \Omega_N \times \mathbb{R}^N$ with $\alpha = 0$. We shall prove that $\tilde{\nu} = 0$ by introducing the following Lagrangian submanifold of $\Omega_N \times \mathbb{R}^N$,

$$\Omega_N \times \{0_{\mathbb{R}^N}\} = \{(r^1, r^2, \cdots, r^N; 0, 0, \cdots, 0) \in \mathbb{R}^{2N} : r^1 < r^2 < \cdots < r^N < 0\}.$$

End of the proof of formula (5.31). The submersion level set theorem implies that $\Omega_N \times \{0_{\mathbb{R}^N}\}$ is a properly embedded N-dimensional submanifold of $\Omega_N \times \mathbb{R}^N$. We have $\Omega_N \times \{0_{\mathbb{R}^N}\} = \Phi_N(\Lambda_N)$, where Λ_N is the Lagrangian submanifold of $(\mathcal{U}_N, \omega)$ defined by (5.27). For every $q \in \Omega_N \times \{0_{\mathbb{R}^N}\}$, set $v = \Psi_N(q) \in \Lambda_N$, we claim at first that

$$\mathcal{T}_q(\Omega_N \times \{0_{\mathbb{R}^N}\}) = \bigoplus_{j=1}^{N} \mathbb{R} \frac{\partial}{\partial r^j}\Big|_q = \mathrm{d}\Phi_N(v)(\mathcal{T}_v(\Lambda_N)). \tag{5.37}$$

In fact, every tangent vector $V \in \mathcal{T}_q(\Omega_N \times \{0_{\mathbb{R}^N}\})$ is the velocity at $t = 0$ of some smooth curve $\xi : t \in (-1, 1) \mapsto \xi(t) = (\xi_1(t), \xi_2(t), \cdots, \xi_N(t); 0, 0, \cdots, 0) \in \Omega_N \times \{0_{\mathbb{R}^N}\}$ such that $\xi(0) = q$, i.e.

$$Vf = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f \circ \xi) = \sum_{j=1}^{N} \xi_j'(0) \frac{\partial f}{\partial r^j}\Big|_q, \qquad \forall f \in C^\infty(\Omega_N \times \mathbb{R}^N). \tag{5.38}$$

So the first equality of (5.37) is obtained. Then we set $\eta(t) = \Psi_N \circ \xi(t)$, $\forall t \in (-1, 1)$. For every $g \in C^\infty(\mathcal{U}_N)$, we replace f by $g \circ \Psi_N \in C^\infty(\Omega_N \times \mathbb{R}^N)$ in (5.38) to obtain that

$$\mathrm{d}\Psi_N(q)(V)g = V(g \circ \Psi_N) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(g \circ \eta) = \eta'(0)g.$$

Since η is a smooth curve in the Lagrangian section Λ_N such that $\eta(0) = v$, we have $\mathrm{d}\Psi_N(q)(V) = \eta'(0) \in \mathcal{T}_v(\Lambda_N)$. So formula (5.37) holds. Since $\nu = \sum_{j=1}^{N} \mathrm{d}r^j \wedge \mathrm{d}\alpha^j$, the submanifold $\Omega_N \times \{0_{\mathbb{R}^N}\}$ is Lagrangian.

For every $p = (r^1, r^2, \cdots, r^N; \alpha^1, \alpha^2, \cdots, \alpha^N) \in \Omega_N \times \mathbb{R}^N$ and every $V_1, V_2 \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$, where

$$V_m = \sum_{j=1}^{N} \left(a_j^{(m)} \frac{\partial}{\partial r^j}\Big|_p + b_j^{(m)} \frac{\partial}{\partial \alpha^j}\Big|_p \right), \qquad a_j^{(m)}, b_j^{(m)} \in \mathbb{R}, \quad m = 1, 2,$$

we choose $q = (r^1, r^2, \cdots, r^N; 0, 0, \cdots, 0) \in \Omega_N \times \{0_{\mathbb{R}^N}\}$ and $W_1, W_2 \in \mathcal{T}_q(\Omega_N \times \{0_{\mathbb{R}^N}\})$, where

$$W_m = \sum_{j=1}^{N} a_j^{(m)} \frac{\partial}{\partial r^j}\Big|_p, \qquad m = 1, 2.$$

We set $v = \Psi_N(q) \in \Lambda_N$. We have proved that $c_{jk}(p) = c_{jk}(q)$, then (5.34) yields that

$$\tilde{\nu}_p(V_1, V_2) = \sum_{1 \le j < k \le N} a_j^{(1)} a_k^{(2)} c_{jk}(p) = \tilde{\nu}_q(W_1, W_2) = \omega_v(\mathrm{d}\Psi_N(v)(W_1), \mathrm{d}\Psi_N(v)(W_2)),$$

because $\nu_q(W_1, W_2) = 0$. The identification (5.37) yields that $h_m := \mathrm{d}\Psi_N(v)(W_m) \in \mathcal{T}_v(\Lambda_N)$, for $m = 1, 2$. Consequently, we have $\tilde{\nu}_p(V_1, V_2) = \omega_v(h_1, h_2) = 0$. □

Formula (5.31) is equivalent to $\Phi_N^* \nu = \omega$, so $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is a symplectomorphism. Finally, we recall a basic property in symplectic geometry: the three formulas in (5.4) are equivalent to the symplectomorphism property of Φ_N.

Proposition 5.24. *If $\tilde{\Phi}_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is a diffeomorphism,*

$$\tilde{\Phi}_N(u) = (\tilde{I}_1(u), \tilde{I}_2(u), \cdots, \tilde{I}_N(u); \tilde{\gamma}_1(u), \tilde{\gamma}_2(u), \cdots, \tilde{\gamma}_N(u)), \qquad \forall u \in \mathcal{U}_N,$$

for some smooth functions $\tilde{I}_j, \tilde{\gamma}_j$ on $\mathcal{U}_N$, then each of the following three properties implies the others:

(a). *$\tilde{\Phi}_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is a symplectomorphism, i.e. $\tilde{\Phi}_N^* \nu = \omega$.*
(b). *For every $j, k = 1, 2, \cdots, N$, we have $\{\tilde{I}_j, \tilde{I}_k\} = \{\tilde{\gamma}_j, \tilde{\gamma}_k\} = 0$ and $\{\tilde{I}_j, \tilde{\gamma}_k\} = \mathbf{1}_{j=k}$ on $\mathcal{U}_N$.*
(c). *For every $k = 1, 2, \cdots, N$, we have*

$$\mathrm{d}\tilde{\Phi}_N(u)(X_{\tilde{I}_k}(u)) = \frac{\partial}{\partial \alpha^k}\Big|_{\tilde{\Phi}_N(u)}, \qquad \mathrm{d}\tilde{\Phi}_N(u)(X_{\tilde{\gamma}_k}(u)) = -\frac{\partial}{\partial r^k}\Big|_{\tilde{\Phi}_N(u)}, \qquad \forall u \in \mathcal{U}_N.$$

Proof. (a) $\Rightarrow$ (b). For any smooth function $f : \Omega_N \times \mathbb{R}^N \to \mathbb{R}$, its Hamiltonian vector field is given by

$$X_f(p) = \sum_{j=1}^N \frac{\partial f}{\partial r^j}(p) \frac{\partial}{\partial \alpha^j}\Big|_p - \frac{\partial f}{\partial \alpha^j}(p) \frac{\partial}{\partial r^j}\Big|_p, \qquad \forall p \in \Omega_N \times \mathbb{R}^N. \tag{5.39}$$

If $\tilde{\Phi}_N^* \nu = \omega$, then $X_{f \circ \tilde{\Phi}_N}(u) = \mathrm{d}\tilde{\Psi}_N(p) \circ X_f(p)$, if $p = \tilde{\Phi}_N(u)$, where $\tilde{\Psi}_N = \tilde{\Phi}_N^{-1}$. The Poisson bracket of two smooth functions f, g on $\Omega_N \times \mathbb{R}^N$ is given by

$$\{f, g\}^\nu(p) = (X_f g)\Big|_p = \nu_p(X_f(p), X_g(p)) = \sum_{j=1}^N \frac{\partial f}{\partial r^j}(p) \frac{\partial g}{\partial \alpha^j}(p) - \frac{\partial f}{\partial \alpha^j}(p) \frac{\partial g}{\partial r^j}(p). \tag{5.40}$$

Then $\{f \circ \tilde{\Phi}_N, g \circ \tilde{\Phi}_N\} = \{f, g\}^\nu \circ \tilde{\Phi}_N$ on $\mathcal{U}_N$. It suffices to choose $f, g \in \{\tilde{I}_j \circ \tilde{\Psi}_N, \tilde{\gamma}_j \circ \tilde{\Psi}_N\}_{1 \leq j \leq N}$.

(b) $\Rightarrow$ (c). We do the same calculus as in lemma 5.22 to obtain that

$$\mathrm{d}\tilde{\Phi}_N(u)(X_{\tilde{I}_k}(u)) = \frac{\partial}{\partial \alpha^k}\Big|_{\tilde{\Phi}_N(u)}, \qquad \mathrm{d}\tilde{\Phi}_N(u)(X_{\tilde{\gamma}_k}(u)) = -\frac{\partial}{\partial r^k}\Big|_{\tilde{\Phi}_N(u)}, \qquad \forall u \in \mathcal{U}_N. \tag{5.41}$$

(c) $\Rightarrow$ (a). Formula (5.41) implies that $\{X_{\tilde{I}_1}, X_{\tilde{I}_2}, \cdots, X_{\tilde{I}_N}; X_{\tilde{\gamma}_1}, X_{\tilde{\gamma}_2}, \cdots, X_{\tilde{\gamma}_N}\}$ forms a basis in $\mathfrak{X}(\mathcal{U}_N)$. Since the 2-covectors $(\tilde{\Phi}_N^* \nu)_u$ and ω_u coincide at every couple of elements of this basis, they are the same, so $\tilde{\Phi}_N^* \nu = \omega$. □

A Appendices

We establish several topological properties of the N-soliton manifold $\mathcal{U}_N$ without using the action–angle map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$. The Viète map $\mathbf{V} : (\beta_1, \beta_2, \cdots, \beta_N) \in \mathbb{C}^N \mapsto (a_0, a_1, \cdots, a_{N-1}) \in \mathbb{C}^N$ is defined by

$$\prod_{j=1}^N (X - \beta_j) = \sum_{k=0}^{N-1} a_k X^k + X^N. \tag{A.1}$$

Proposition A.1. *Endowed with the Hermitian form $\mathfrak{H}$ introduced in (4.15), $(\Pi(\mathcal{U}_N), \mathfrak{H})$ is a simply connected Kähler manifold which is biholomorphically equivalent to $\mathbf{V}(\mathbb{C}_-^N)$.*

Proposition A.2. *The N-soliton manifold $\mathcal{U}_N$ is a universal covering manifold of the following N-gap potential manifold for the BO equation on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as described by Gérard–Kappeler [19],*

$$U_N^{\mathbb{T}} = \{v = h + \overline{h} \in L^2(\mathbb{T},\mathbb{R}) : \quad h : y \in \mathbb{T} \mapsto -e^{iy}\frac{\mathfrak{Q}'(e^{iy})}{\mathfrak{Q}(e^{iy})} \in \mathbb{C}, \quad \mathfrak{Q} \in \mathbb{C}_N^+[X]\}, \tag{A.2}$$

where $\mathbb{C}_N^+[X]$ consists of all monic polynomial $\mathfrak{Q} \in \mathbb{C}[X]$ of degree N, whose roots are contained in the annulus $\mathscr{A} := \{z \in \mathbb{C} : |z| > 1\}$. The fundamental group of $U_N^{\mathbb{T}}$ is $(\mathbb{Z}, +)$.

Remark A.3. *The real analytic symplectic manifold $U_N^{\mathbb{T}}$ is mapped real bi-analytically onto $\mathbb{C}^{N-1} \times \mathbb{C}^*$ by the restriction of the Birkhoff map constructed in Gérard–Kappeler [19]. The union of all finite gap potentials $\bigcup_{N\geq 0} U_N^{\mathbb{T}}$ is dense in $L^2_{r,0}(\mathbb{T}) = \{v \in L^2(\mathbb{T},\mathbb{R}) : \int_{\mathbb{T}} v = 0\}$. However $\bigcup_{N\geq 1}\mathcal{U}_N$ is not dense in $L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$. We refer to Coifman–Wickerhauser [9] to see solutions with sufficiently small initial data and the case of non-existence of rapidly decreasing solitons.*

The simple connectedness of $\mathcal{U}_N$ is proved in subsection **A.1**. Then we establish a real analytic covering map $\mathcal{U}_N \to U_N^{\mathbb{T}}$ in subsection **A.2**.

A.1 The simple connectedness of $\mathcal{U}_N$

Thanks to the biholomorphical equivalence between the Kähler manifolds $\Pi(\mathcal{U}_N)$ and $\mathbf{V}(\mathbb{C}_-^N)$ established in lemma 4.10, it suffices to prove the simple connectedness of the subset $\mathbf{V}(\mathbb{C}_-^N)$, where $\mathbf{V}$ denotes the Viète map defined by (A.1). Since every fiber of the Viète map is invariant under the permutation of components, we introduce the following group action. Equipped with the discrete topology, the symmetric group S_N acts continuously on $\mathbb{C}^N$ by permuting the components of every vector:

$$\boldsymbol{\sigma} : (\beta_0, \beta_1, \cdots, \beta_{N-1}) \in \mathbb{C}^N \mapsto (\beta_{\sigma(0)}, \beta_{\sigma(1)}, \cdots, \beta_{\sigma(N-1)}) \in \mathbb{C}^N, \qquad \forall \sigma \in \mathrm{S}_N. \tag{A.3}$$

A subset $A \subset \mathbb{C}^N$ is said to be *stable under* S_N if $\bigcup_{\sigma\in\mathrm{S}_N} \boldsymbol{\sigma}(A) = A$. We recall the basic property of the Viète map $\mathbf{V}$ and the action of symmetric group S_N.

Lemma A.4. *The Viète map $\mathbf{V} : \mathbb{C}^N \to \mathbb{C}^N$ is a both open and closed quotient map. For every $A \subset \mathbb{C}^N$, A is stable under S_N if and only if A is saturated with respect to $\mathbf{V}$, the quotient space A/S_N is homeomorphic to $\mathbf{V}(A)$.*

We set $\Delta := \{(\beta, \beta, \cdots, \beta) \in \mathbb{C}^N : \forall \beta \in \mathbb{C}\}$. The goal of this subsection is to prove the following result.

Proposition A.5. *For every open simply connected subset $A \subset \mathbb{C}^N$, if A is stable under the symmetric group S_N and $A \bigcap \Delta \neq \emptyset$, then $\mathbf{V}(A)$ is an open simply connected subset of $\mathbb{C}^N$.*

Proof. Let $A \subset \mathbb{C}^N$ be a nonempty open simply connected subset that is stable by S_N. The subset $B := \mathbf{V}(A)$ is open, connected and locally simply connected, then it admits a universal covering space E and a covering map $\pi : E \to B$. The triple (E, π, B) is identified as a fiber bundle over B whose model fiber $\mathbf{F}$ is discrete. The target is to show that $\mathbf{F}$ has cardinality 1.

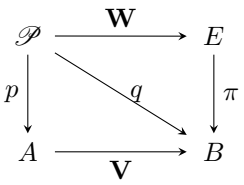
Let $(\mathscr{P}, q, B)$ denote the fiber product (Husemöller [28]) of bundles $(A, \mathbf{V}, B)$ and (E, π, B), defined by

$$\mathscr{P} = A \times_B E := \{(\beta, e) \in A \times E : \pi(e) = \mathbf{V}(\beta)\}, \qquad q : (\beta, e) \in \mathscr{P} \mapsto \mathbf{V}(\beta) = \pi(e) \in B. \tag{A.4}$$

The total space $\mathscr{P}$ is equipped with the subspace topology of the product space $A \times E$ and projections onto the first factor and onto the second factor are denoted respectively by

$$p : (\beta, e) \in \mathscr{P} \mapsto \beta \in A, \qquad \mathbf{W} : (\beta, e) \in \mathscr{P} \mapsto e \in E. \tag{A.5}$$

Both p and $\mathbf{W}$ are continuous functions on $\mathscr{P}$ and the following diagram commutes.



We claim two properties concerning the projections p and $\mathbf{W}$.

i. $\mathbf{W} : \mathscr{P} \to E$ is an open quotient map and $p : \mathscr{P} \to A$ is a covering map whose model fiber is $\mathbf{F}$.
ii. Equipped with the discrete topology, the symmetric group S_N acts continuously on $\mathscr{P}$ by permuting components of the first factor

$$\overline{\boldsymbol{\sigma}} : (\beta, e) \in \mathscr{P} \mapsto (\boldsymbol{\sigma}(\beta), e) \in \mathscr{P}, \qquad \forall \sigma \in \mathrm{S}_N,$$

where $\boldsymbol{\sigma} \in \mathrm{GL}_N(\mathbb{C})$ is defined by (A.3). Hence the quotient map $\mathbf{W} : \mathscr{P} \to E$ is closed.

Thanks to the simple connectedness of the base space A, the covering space $\mathscr{P}$ is the disjoint union of its connected components $(\mathscr{A}_k)_{k\in\mathbf{F}}$ and the restriction of the covering map $p|_{\mathscr{A}_k} : \mathscr{A}_k \to A$ is a homeomorphism. Since $\mathscr{P}$ is locally path-connected, every component $\mathscr{A}_k$ is both open and closed, then $\mathbf{W}|_{\mathscr{A}_k} : \mathscr{A}_k \to E$ an open closed quotient map. So is the lift $g_k := \mathbf{W}|_{\mathscr{A}_k} \circ (p|_{\mathscr{A}_k})^{-1} : A \to E$. Note that $\pi \circ g_k = \mathbf{V}$ and S_N stabilizes every element of Δ. We choose $\beta \in A \bigcap \Delta$ and $b := \mathbf{V}(\beta)$. Since the fiber $\mathbf{V}^{-1}(b) = \{\beta\}$ is a singleton, so is the fiber $\pi^{-1}(b)$. Hence $|\mathbf{F}| = 1$ and the universal covering map $p : E \to B$ is a homeomorphism. So B is simply connected. □

Remark A.6. *Let F be a closed submanifold of a smooth connected manifold M without boundary of finite dimension. If $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} F \geq 3$, then the inclusion map $i : M\backslash F \to M$ induces an isomorphism between the fundamental groups $i_* : \pi_1(M\backslash F, x) \to \pi_1(M, x)$, for every $x \in M\backslash F$ (see Théorème 2.3 in P.146 of Godbillon [22]). Note that the closed submanifold $\Delta \subset \mathbb{C}^N$ has real dimension 2. When $N \geq 3$, the condition $A \bigcap \Delta \neq \emptyset$ cannot be deduced by the other three conditions in the hypothesis of proposition A.5: A is open, simply connected and stable by S_N.*

As a consequence, $\mathbf{V}(\mathbb{C}^N_-)$ is open and simply connected because $\mathbb{C}^N_-$ is an open convex subset of $\mathbb{C}^N$ which is stable under the symmetric group S_N and $\Delta \bigcap \mathbb{C}^N_- = \{(z, z, \cdots, z) \in \mathbb{C}^N : \mathrm{Im} z < 0\}$. Together with lemma 4.10, we finish the proof of proposition A.1.

A.2 Covering manifold

The Szegő projector on $L^2(\mathbb{T}, \mathbb{C})$ is given by $\Pi^{\mathbb{T}} v(x) = \sum_{n\geq 0} v_n e^{inx}$, for every $v \in L^2(\mathbb{T}, \mathbb{C})$ such that $v(x) = \sum_{n\in\mathbb{Z}} v_n e^{inx}$ with $v_n = \frac{1}{2\pi}\int_0^{2\pi} v(x) e^{-inx} \mathrm{d}x$. Equipped with the subspace topology of $\Pi^{\mathbb{T}}(L^2(\mathbb{T}, \mathbb{C}))$ and the Hermitian form

$$\mathfrak{H}^{\mathbb{T}}(v_1, v_2) = \langle \mathrm{D}^{-1}\Pi^{\mathbb{T}} v_1, \Pi^{\mathbb{T}} v_2 \rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{D}^{-1}\Pi^{\mathbb{T}} v_1(x)\overline{\Pi^{\mathbb{T}} v_2(x)}\mathrm{d}x,$$

the subset $\Pi^{\mathbb{T}}(U_N^{\mathbb{T}})$ is a Kähler manifold, which is mapped biholomorphically onto $\mathbf{V}(\mathscr{A}^N)$ with $\mathscr{A} = \mathbb{C}\backslash\overline{D}(0,1) = \{z \in \mathbb{C} : |z| > 1\}$ in Gérard–Kappeler [19].

Proposition A.7. *There exists a covering map $\boldsymbol{\pi} : \mathbf{V}(\mathbb{C}^N_-) \to \mathbf{V}(\mathscr{A}^N)$.*

Remark A.8. *Consider the cubic Szegő equation on the torus (see Gérard–Grellier [15, 16, 17, 18])*

$$i\partial_t w^{\mathbb{T}} = \Pi^{\mathbb{T}}(|w^{\mathbb{T}}|^2 w^{\mathbb{T}}), \qquad (t,x) \in \mathbb{R}\times\mathbb{T}, \tag{A.6}$$

and the cubic Szegő equation on the line (see Pocovnicu [51, 52]), we set $\Pi^{\mathbb{R}} := \Pi$ *in* (1.12),

$$i\partial_t w^{\mathbb{R}} = \Pi^{\mathbb{R}}(|w^{\mathbb{R}}|^2 w^{\mathbb{R}}), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{A.7}$$

The manifold of N-solitons for the cubic Szegő equation on the line is not simply connected. Let $\mathcal{M}(N)^{\mathbb{R}}$ denote all rational functions of the form $w^{\mathbb{R}} : x \in \mathbb{R} \mapsto \frac{P(x)}{Q(x)} \in \mathbb{C}$ where $P \in \mathbb{C}_{\leq N-1}[X]$ and $Q \in \mathbb{C}_N[X]$ is a monic polynomial such that $Q^{-1}(0) \subset \mathbb{C}_-$ and P, Q have no common factors. Then $\mathcal{M}(N)^{\mathbb{R}}$ is a Kähler manifold of complex dimension $2N$. So is the subset $\mathcal{M}(N)^{\mathbb{T}}$ consisting of all rational functions of the form $w^{\mathbb{T}} : x \in \mathbb{T} \mapsto \frac{P(e^{ix})}{Q(e^{ix})} \in \mathbb{C}$ where $P \in \mathbb{C}_{\leq N-1}[X]$ and $Q \in \mathbb{C}_N[X]$ is a monic polynomial such that $Q^{-1}(0) \subset \mathscr{A}$ and P, Q have no common factors. Both of them have rank characterization of Hankel operators by Kronecker-type theorem (see Lemma 8.12 in Chapter 1 of Peller [50], p. 54). So the manifold $\mathcal{M}(N)^{\mathbb{R}}$ (resp. $\mathcal{M}(N)^{\mathbb{T}}$) is invariant under the flow of equation (A.7) (resp. of equation (A.6)) and the (generalized) action–angle coordinates of equation (A.7) (resp. of equation (A.6)) are defined in some open dense subset of $\mathcal{M}(N)^{\mathbb{R}}$ (resp. of $\mathcal{M}(N)^{\mathbb{T}}$). Moreover, if $N \geq 2$ then $\mathcal{M}(N)^{\mathbb{R}}$ is simply connected by proposition A.5 and remark A.6. There exists a holomorphic covering map $\mathcal{M}(N)^{\mathbb{R}} \to \mathcal{M}(N)^{\mathbb{T}}$ by following the construction in proposition A.7. The manifold of N-solitons for the cubic Szegő equation on the line is an open dense subset of $\mathcal{M}(N)^{\mathbb{R}}$

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