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Control of Stackelberg for coupled parabolic equations

Moumini KÉRÉ * Michelle MERCAN † Gisèle MOPHOU ‡

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Abstract

We consider the Stackelberg problem for coupled parabolic equations with a finite number of constraints on one of the states. This notion assumes that we have two controls to determine. The first control is supposed to bring the solution of the coupled system subjected to a finite number of constraints at rest at time zero while the second expresses that the states do not move too far from given states. The results are achieved by means of an observability inequality of Carleman adapted to the constraints.

Key-words : Null-controllability, Coupled system, Carleman inequalities, observability inequality.

AMS Subject Classification 35K05, 35K15, 35K20, 49J20, 93B05

1 Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of \mathbb{R}^N with boundary Γ of class \mathcal{C}^2 . For $T > 0$, we set $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$ and $\omega_T = \omega \times (0, T)$ where ω is a sub-domain compactly embedded in Ω . We consider the following system:

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$$\begin{cases} \frac{\partial y_1}{\partial t} - \Delta y_1 + ay_1 + by_2 = l + k\chi_\omega & \text{in } Q, \\ \frac{\partial y_2}{\partial t} - \Delta y_2 + cy_2 + dy_1 = 0 & \text{in } Q, \\ y_1 = y_2 = 0 & \text{on } \Sigma, \\ y_1(0) = y_2(0) = 0 & \text{in } \Omega, \end{cases} \quad (1)$$

where the controls l and k belong to $L^2(Q)$, χ_ω denotes the characteristic function of the set ω and

$$a, b, c, d \in L^\infty(Q), \text{ with } d \geq \nu > 0 \text{ in } \omega_T. \quad (2)$$

Under the assumptions on the data, we know that system (1) has an unique solution $(y_1, y_2) = (y_1(l, k), y_2(l, k)) \in (C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)))^2$ (See. [20]). Moreover, if we set $\|a, b, c, d\|_{L^\infty(Q)}^2 = \|a\|_{L^\infty(Q)}^2 + \|b\|_{L^\infty(Q)}^2 + \|c\|_{L^\infty(Q)}^2 + \|d\|_{L^\infty(Q)}^2$, there exists a positive constant $C = C(\Omega, T, \|a, b, c, d\|_{L^\infty(Q)})$ such that

$$\|y_1\|_{L^2(0, T, H_0^1(\Omega))} + \|y_2\|_{L^2(0, T, H_0^1(\Omega))} \leq C \left(\|l\|_{L^2(Q)} + \|k\|_{L^2(\omega_T)} \right). \quad (3)$$

The Stackelberg leadership model is a strategic game in economics in which two firms compete on the market of the same product. The first to act must integrate the reaction of the other company in the choices it makes in the amount of product that it decides to put on the market. Following this notion, we want to control system (1) by acting with two controls. To this end, we set

$$H_\theta = \{h \in L^2(Q) \text{ such that } \theta h \in L^2(Q)\}, \quad (4)$$

where θ is positive function precisely defined later on by (14). Then, it easy to prove that H_θ endowed with the norm

$$\|h\|_{H_\theta}^2 = (\theta h, \theta h)_{L^2(Q)} = \int_Q |\theta h|^2 dx dt, \quad \forall h \in H_\theta \quad (5)$$

is a Hilbert space. We consider the following problems:

Problem 1.1 *Given $a, c, b, d \in L^\infty(Q)$ with $d \geq \nu > 0$ in ω_T and $e_i \in L^2(Q)$, $1 \leq i \leq M$ such that*

$$(e_i \chi_\omega)_{1 \leq i \leq M} \text{ are linearly independent.} \quad (6)$$

For any $l \in H_\theta$, find a control $k = k(l) \in L^2(\omega_T)$ such that (y_1, y_2) solution of (1) satisfies

$$y_1(T) = y_2(T) = 0 \text{ in } \Omega \quad (7)$$

and

$$\int_Q y_2 e_i dx dt = 0, \quad 1 \leq i \leq M. \quad (8)$$

Problem 1.2 Assume that the control $k = k(l)$, solution of Problem 1.1 exists. Let $y = (y_1(l, k(l)), y_2(l, k(l)))$ be the solution of (1) corresponding to $k = k(l)$. Let also \mathcal{U}_{ad} be a nonempty closed convex subset of H_θ and $(z_d^1, z_d^2) \in (L^2(Q))^2$. Find $\hat{l} \in \mathcal{U}_{ad}$ such that

$$J(\hat{l}) = \inf_{l \in \mathcal{U}_{ad}} J(l), \quad (9)$$

where

$$J(l) = \frac{1}{2} \|y_1(l, k(l)) - z_d^1\|_{L^2(Q)}^2 + \frac{1}{2} \|y_2(l, k(l)) - z_d^2\|_{L^2(Q)}^2 + \frac{N}{2} \|l\|_{H_\theta}^2. \quad (10)$$

If the controls l and k exist, then the linear system (1) is controllable in the sense of Stackelberg. Following this strategy, the control l is the Leader and k is the Follower.

Problem 1.2 is a classical optimal control problem [3] whereas Problem 1.1 is a simultaneous controllability problem with state constraints.

There are some works on simultaneous controllability. In [17], D.L. Russell studied a simultaneous controllability for two wave equations; one with boundary control of Dirichlet type, the other of Neumann type in order to controlling the time evolution of electromagnetic fields independent of the axial coordinate. Using the Hilbert Uniqueness Method (HUM) developed by J. L. Lions in [5], G.O. Antunes et al. in [18] proved under suitable condition on final time that the boundary simultaneous controllability for a system of equations that constitutes a model of dynamical elasticity for incompressible materials holds. Following the same method, B.V. Kapitonov and G.P. Menzala in [19] proved the exact null simultaneous controllability for Maxwell equations system and for a wave equation vector with a pressure term. In [21], F. Ammar Khodja et al. studied the local simultaneous null controllability of a coupled system of semi-linear heat equations. They showed the existence of simultaneous control for a linear coupled system which is derived from the initial system. Then, by a argument of the fixed point, they established the existence of the solution of local simultaneous null controllability problems for the semi linear coupled system.

Recently, C. Louis-Rose [15] studied the simultaneous controllability problem with constraint on the control of a coupled system of linear heat equations that each had the same control. She initially turned the system into an equivalent system with the control acting only in a single equation, using a suitable change of variable. Then she solved the problem of simultaneous controllability of this new system using a Carleman inequality adapted to the constraint. This result was generalized to simultaneous null controllability for a coupled system of reaction-diffusion with a finite number of state constraints by Peng Gao [22].

The notion of controllability in the sense of Stackelberg was initiated by O. Nakoulima [8]. Actually, the author was interested in the concept of hierarchical control for a backward heat equation. This concept assumes to have two controls to determine: one of null controllability type with constraint on the control, called Follower, and the other of optimal control type, called Leader. The results

were achieved by means of a Carleman inequality adapted to the constraint and were applied to a problem of discriminating sentinels (see [4] for the notion of sentinel). In [9, 10], M. Mercan revisited the notion of controllability in the sense of Stackelberg given by O. Nakoulima [8] by choosing the Follower of minimal norm. This new notion is then applied by M. Mercan and O. Nakoulima in [13] on the controllability of a two-stroke problem with constraint on the states. The results were obtained by means of Carleman inequality adapted to the constraints. In this paper, motivated by all the above works, we investigate the controllability in the sense of Stackelberg to a system of coupled parabolic equations with constraints on one state. The main results is as follows:

Theorem 1.3 *Let Ω be a bounded open subset of \mathbb{R}^N with boundary Γ of class C^2 and ω be a sub-domain compactly embedded in Ω . Let $a, c, b, d \in L^\infty(Q)$ with $d \geq \nu > 0$ in ω_T and $e_i \in L^2(Q), 1 \leq i \leq M$ be such that (6) holds. Let also H_θ be the Hilbert space defined by (4) and (5). Then there exists a positive real function θ (a precise definition of θ will be given later on) such that for any function $l \in H_\theta$, there exists a unique control $k = k(l) \in L^2(Q)$ such that $(k(l), y = (y_1(l, k(l)), y_2(l, k(l))))$ is the solution of the null controllability problem with constraints on the state (1), (7), (8). Moreover,*

$$k = P\rho_1\chi_\omega - \rho_1\chi_\omega - u_0\chi_\omega,$$

where P is the orthogonal projection operator from $L^2(\omega_T)$ into K (a precise definition of K will be given later on), $u_0 = u_0(l) \in \frac{1}{\theta}K$ and $(\rho_1, \rho_2) = (\rho_1(l), \rho_2(l))$ satisfies:

$$\begin{cases} -\frac{\partial \rho_1}{\partial t} - \Delta \rho_1 + a\rho_1 + d\rho_2 = 0 & \text{in } Q, \\ -\frac{\partial \rho_2}{\partial t} - \Delta \rho_2 + c\rho_2 + b\rho_1 = 0 & \text{in } Q, \\ \rho_1 = \rho_2 = 0 & \text{on } \Sigma. \end{cases}$$

Theorem 1.4 *Assume that the hypotheses of Theorem 1.3 holds. Then there exists $(p_1, p_2) \in (C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)))^2$ such that the minimization problem (9) admits a unique solution \hat{l} which is characterized by the following optimality condition*

$$\left(\Lambda^{-1} \left(\frac{1}{\theta} p_1 + \frac{1}{\theta} F^* (p_1 \chi_\omega) \right) + N \hat{l}, l - \hat{l} \right)_{H_\theta} \geq 0, \quad \forall l \in \mathcal{U}_{ad},$$

where F is a linear and continuous application from H_θ to $L^2(Q)$ precisely defined later on by (101), F^* is the adjoint of F and (p_1, p_1) is solution of

$$\begin{cases} -\frac{\partial p_1}{\partial t} - \Delta p_1 + a p_1 + d p_2 = \hat{y}_1 - z_d^1 & \text{in } Q, \\ -\frac{\partial p_2}{\partial t} - \Delta p_2 + c p_2 + b p_1 = \hat{y}_2 - z_d^2 & \text{in } Q, \\ p_1 = p_2 = 0 & \text{on } \Sigma, \\ p_1(T) = 0; \quad p_2(T) = 0 & \text{in } \Omega, \end{cases}$$

with $(\hat{y}_1, \hat{y}_2) \in (C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)))^2$, the solution to (1) corresponding to the control \hat{l} .

The paper is organized as follows. Section 2 is devoted to the study of Problem 1.1. In this section, we prove the equivalence between the null controllability problem with constraints on the state and a null controllability with constraint on the control. Then, we establish a Carleman inequality adapted to the constraints and prove Theorem 1.3. We prove Theorem 1.4 in section 3.

2 Study of Problem 1

We first prove that the null controllability with constraint on the state is equivalent to a null controllability with constraint on the control. Then we solve the equivalent problem using an adapted Carleman inequality and a penalization method. This allows to prove Theorem 1.3.

2.1 Equivalence between the null controllability with constraint on the state and a null controllability with constraint on the control.

In this section, we show the equivalence between the null controllability problem with state constraints and a null controllability problem with constraint on the control. But before going further, we need some preliminary results.

So, let $\omega' \Subset \omega$ and $\psi \in C^2(\bar{\Omega})$ a function such that

$$\begin{cases} \min \left\{ |\nabla \psi(x)|, x \in \overline{\Omega \setminus \omega'} \right\} > 0 \\ \frac{\partial \psi}{\partial \nu} \leq 0 \text{ on } \Gamma, \end{cases} \quad (11)$$

where ν denotes the outward unit to Γ . Suppose moreover that ψ satisfies

$$\min \left\{ \psi(x), x \in \bar{\Omega} \right\} \geq \max \left\{ \frac{3}{4} \|\psi\|_{L^\infty(\Omega)}, \ln 3 \right\}.$$

Such function ψ was considered by A. Fursikov and O. Imanuvilov [1]. For all $\lambda > 0$, $\tau > 0$, we set

$$\rho(x, t) = \frac{e^{\lambda \psi(x)}}{t(T-t)}, \quad (x, t) \in Q, \quad (12)$$

$$\phi(x, t) = \tau \frac{e^{\frac{4}{3} \lambda \|\psi\|_{L^\infty(\Omega)}} - e^{\lambda \psi(x)}}{t(T-t)}, \quad (x, t) \in Q \quad (13)$$

and

$$\frac{1}{\theta^2}(x, t) = e^{-2\phi(x, t)}, \quad (x, t) \in Q. \quad (14)$$

Then $\frac{1}{\theta} \in L^\infty(Q)$ because $\phi(x, t) > 0$ for all $(x, t) \in Q$.

Lemma 2.1 *Let θ be defined by (14) and the functions $(e_i)_{1 \leq i \leq M} \in (L^2(Q))^M$ be such that (6) holds. Let $a, c, b, d \in L^\infty(Q)$ with $d \geq \nu > 0$ in ω_T . Let also $(p_i, q_i) \in (C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)))^2$ be solution of:*

$$\begin{cases} -\frac{\partial p_i}{\partial t} - \Delta p_i + ap_i + dq_i = 0 & \text{in } Q, \\ -\frac{\partial q_i}{\partial t} - \Delta q_i + cq_i + bp_i = e_i & \text{in } Q, \\ p_i = q_i = 0 & \text{on } \Sigma, \\ p_i(T) = q_i(T) = 0 & \text{in } \Omega. \end{cases} \quad (15)$$

Then the families $(p_1\chi_\omega, p_2\chi_\omega, \dots, p_M\chi_\omega)$ and $\left(\frac{1}{\theta}p_1\chi_\omega, \frac{1}{\theta}p_2\chi_\omega, \dots, \frac{1}{\theta}p_M\chi_\omega\right)$ are linearly independent.

Proof. Let $(\alpha_i)_{i=1, \dots, M} \in \mathbb{R}^M$ be such that $k = \sum_{i=1}^M \alpha_i p_i = 0$ in ω_T . Let $z = \sum_{i=1}^M \alpha_i q_i$ in ω_T . From (15) we have

$$\begin{cases} -\frac{\partial k}{\partial t} - \Delta k + ak + dz = 0 & \text{in } Q, \\ -\frac{\partial z}{\partial t} - \Delta z + cz + bk = \sum_{i=1}^M \alpha_i e_i & \text{in } Q, \\ k = z = 0 & \text{on } \Sigma, \\ k(T) = z(T) = 0 & \text{in } \Omega. \end{cases} \quad (16)$$

As $k = 0$ in ω_T and $d \neq 0$ in ω_T , equality (16)₁ give us $z = 0$ in ω_T . Then, using the fact that $k = z = 0$ in ω_T in (16)₂, we obtain $0 = \sum_{i=1}^M \alpha_i e_i$ in ω_T . Consequently, assumption (6) allows us to conclude that $\alpha_i = 0$ for all $1 \leq i \leq M$. Analogously, we prove that the family $\left(\frac{1}{\theta}p_i\chi_\omega\right)_{i=1, \dots, M}$ is linearly independent. \blacksquare

We set

$$K = \text{Span}(p_1\chi_\omega, p_2\chi_\omega, \dots, p_M\chi_\omega), \quad (17)$$

$$K_\theta = \frac{1}{\theta}K. \quad (18)$$

Lemma 2.2 *Let θ be defined by (14) and $(e_i)_{1 \leq i \leq M} \in (L^2(Q))^M$ such that (6) holds. Let also (p_i, q_i) be defined by (15). Set*

$$A_\theta = \left(\int_Q \frac{1}{\theta} p_i p_j \chi_\omega \right)_{1 \leq i, j \leq M}. \quad (19)$$

Then, there exists $\delta > 0$ such that

$$(A_\theta X, X)_{\mathbb{R}^M} \geq \delta \|X\|_{\mathbb{R}^M}^2, \quad (20)$$

where

$$(A_\theta X, X)_{\mathbb{R}^M} = \int_{\omega_T} \frac{1}{\theta} \left(\sum_{i=1}^M X_i p_i \right) \left(\sum_{i=1}^M X_j p_j \right) dxdt$$

and

$$X = (X_1, \dots, X_M) \in \mathbb{R}^M.$$

Proof. One proceeds as in [22]. ■

Proposition 2.1 *Let θ be defined by (14) and H_θ be defined by (4) and (5). Let also K and K_θ be defined by (17) and (18) respectively. Then the null controllability problem with constraints on the state (1), (7), (8) is equivalent to the following null controllability problem with constraint on control: Given $a, c, b, d \in L^\infty(Q)$ with $d \geq \nu > 0$ in ω_T , $l \in H_\theta$ and $u_0 = u_0(l) \in K_\theta$, find*

$$v = v(l) \in K^\perp \quad (21)$$

such that (y_1, y_2) , solution of

$$\begin{cases} \frac{\partial y_1}{\partial t} - \Delta y_1 + ay_1 + by_2 = l + (v - u_0) \chi_\omega & \text{in } Q, \\ \frac{\partial y_2}{\partial t} - \Delta y_2 + cy_2 + dy_1 = 0 & \text{in } Q, \\ y_1 = y_2 = 0 & \text{on } \Sigma, \\ y_1(0) = y_2(0) = 0 & \text{in } \Omega \end{cases} \quad (22)$$

satisfies

$$y_1(T) = y_2(T) = 0 \text{ in } \Omega. \quad (23)$$

Proof. Assume that (y_1, y_2) is solution of (1), (7), (8). Multiplying (15)₁ by y_1 and (15)₂ by y_2 and integrating by parts over Q , we get

$$\begin{aligned} \int_Q p_i \left(\frac{\partial y_1}{\partial t} - \Delta y_1 + ay_1 \right) dxdt + \int_Q dq_i y_1 dxdt &= \int_Q e_i y_2 dxdt, \\ \int_Q q_i \left(\frac{\partial y_2}{\partial t} - \Delta y_2 + cy_2 \right) dxdt + \int_Q bp_i y_2 dxdt &= 0. \end{aligned}$$

Adding the resulting identities and using (8), we obtain that

$$- \int_Q lp_i dxdt = \int_{\omega_T} kp_i dxdt. \quad (24)$$

Since the matrix A_θ defined by (19) is symmetric positive definite, there exists a unique $u_0 = u_0(l) \in K_\theta$ such that

$$\int_Q lp_i dxdt = \int_{\omega_T} u_0 p_i dxdt. \quad (25)$$

Combining (24) and (25) we deduce that

$$\int_{\omega_T} u_0 p_i dxdt = - \int_{\omega_T} k p_i dxdt. \quad (26)$$

Therefore

$$\int_{\omega_T} (u_0 + k) p_i dxdt = 0, \quad 1 \leq i \leq M.$$

This means that $u_0 + k = v \in K^\perp$. It then follows that $k = v - u_0$. Hence, replacing k by $v - u_0$ in (1) we obtain (22).

Conversely, assume that $(v, (y_1, y_2))$ is solution of (21), (22) and (23). Multiplying (15)₁ by y_1 and (15)₂ by y_2 and integrate by parts over Q , we get

$$\begin{aligned} \int_Q p_i \left(\frac{\partial y_1}{\partial t} - \Delta y_1 + a y_1 \right) dxdt + \int_Q d q_i y_1 dxdt &= 0, \\ \int_Q q_i \left(\frac{\partial y_2}{\partial t} - \Delta y_2 + c y_2 \right) dxdt + \int_Q b p_i y_2 dxdt &= \int_Q e_i y_2 dxdt. \end{aligned}$$

Adding the resulting identities, we obtain

$$\int_Q p_i (l + (v - u_0) \chi_\omega) dxdt = \int_Q y_2 e_i dxdt.$$

Since $v \in K^\perp$, we have

$$\int_Q p_i (l - u_0 \chi_\omega) dxdt = \int_Q y_2 e_i dxdt,$$

which in view of (25), yields

$$\int_Q y_2 e_i dxdt = 0.$$

We thus have proved that the pair (k, y) with $k = v - u_0$ and $y = (y_1, y_2)$ satisfies the null controllability problem with state constraints (1), (7), (8). ■

From now on, we use $C(X)$ to denote a positive constant whose value varies from a line to another but depends on X .

Proposition 2.2 *Let θ be defined by (14) and $l \in H_\theta$. Let also p_i and u_0 be respectively defined by (15) and (25). Then there exists a positive constant*

$$C = C \left(\Omega, \|(a, b, c, d)\|_\infty, \sum_{i=1}^M \|e_i\|_{L^2(Q)}^2 \right) \text{ such that}$$

$$\|\theta u_0\|_{L^2(\omega_T)} \leq C \|l\|_{H_\theta}, \quad (27)$$

$$\|u_0\|_{L^2(\omega_T)} \leq C \|l\|_{H_\theta}. \quad (28)$$

Proof. From (25), we have

$$\int_{\omega_T} u_0 p_i \, dxdt = \int_Q l p_i \, dxdt, \quad 1 \leq i \leq M. \quad (29)$$

Since $u_0(l) \in \text{Span}\{\frac{1}{\theta} p_1 \chi_\omega, \dots, \frac{1}{\theta} p_M \chi_\omega\}$, there exists $\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{R}^M$ such that

$$u_0 = \sum_{j=1}^M \alpha_j \frac{1}{\theta} p_j \chi_\omega. \quad (30)$$

Thus, replacing u_0 by $\sum_{j=1}^M \alpha_j \frac{1}{\theta} p_j \chi_\omega$ in (29), we obtain:

$$\int_{\omega_T} \sum_{j=1}^M \alpha_j \frac{1}{\theta} p_j p_i \, dxdt = \int_Q l p_i \, dxdt, \quad 1 \leq i \leq M.$$

Consequently,

$$\int_{\omega_T} \left(\sum_{j=1}^M \alpha_j \frac{1}{\theta} p_j \right) \left(\sum_{i=1}^M \alpha_i p_i \right) \, dxdt = \int_Q \theta l \left(\sum_{i=1}^M \alpha_i \frac{1}{\theta} p_i \right) \, dxdt,$$

which in view of the definition of the matrix A_θ given by (19) can be rewritten as

$$A_\theta(X, X) = \int_Q \theta l \left(\sum_{i=1}^M \alpha_i \frac{1}{\theta} p_i \right) \, dxdt,$$

with $X = (\alpha_i)_{1 \leq i \leq M}$. It then follows from Lemma 2.2 and the Cauchy-Schwartz inequality that

$$\delta \|\alpha\|^2 \leq \|l\|_{H_\theta} \sum_{i=1}^M |\alpha_i| \left\| \frac{1}{\theta} p_i \right\|_{L^2(Q)}.$$

Since $\frac{1}{\theta}$ is bounded, we obtain that

$$\delta \|\alpha\|^2 \leq C(\theta) \|l\|_{H_\theta} \sum_{i=1}^M |\alpha_i| \|p_i\|_{L^2(Q)}. \quad (31)$$

Because p_i is solution of (15), we have that for $1 \leq i \leq M$,

$$\|p_i\|_{L^2(Q)} \leq C(\Omega, \|a, b, c, d\|_\infty, T) \|e_i\|_{L^2(Q)},$$

which combined with (31) and the fact that $\delta > 0$ gives

$$\|\alpha\|_{\mathbb{R}^M} \leq \delta^{-1} C(\Omega, \theta, \|a, b, c, d\|_\infty, T) \|l\|_{H_\theta} \sqrt{\sum_{i=1}^M \|e_i\|_{L^2(Q)}^2}. \quad (32)$$

Finally, from (30), we have

$$\begin{aligned}
\|\theta u_0\|_{L^2(\omega_T)} &\leq \sum_{j=1}^M |\alpha_j| \|p_j\|_{L^2(\omega_T)} \\
&\leq C(\Omega, \theta, \|a, b, c, d\|_\infty, T) \sum_{j=1}^M |\alpha_j| \|e_j\|_{L^2(Q)} \\
&\leq C(\Omega, \theta, \|a, b, c, d\|_\infty, T) \|\alpha\|_{\mathbb{R}^M} \left(\sum_{i=1}^M \|e_i\|_{L^2(Q)}^2 \right)^{\frac{1}{2}},
\end{aligned}$$

on the one hand, and on the other hand,

$$\|u_0\|_{L^2(\omega_T)} \leq C(\Omega, \theta, \|a, b, c, d\|_\infty, T) \left\| \frac{1}{\theta} \right\|_\infty \|\alpha\|_{\mathbb{R}^M} \left(\sum_{i=1}^M \|e_i\|_{L^2(Q)}^2 \right)^{\frac{1}{2}}.$$

Hence, using (32) and the fact that $\frac{1}{\theta}$ is bounded in $L^\infty(Q)$, and setting

$$C = C(\Omega, \theta, \|a, b, c, d\|_\infty, T, \sum_{i=1}^M \|e_i\|_{L^2(Q)}) = \delta^{-1} C(\Omega, \theta, \|a, b, c, d\|_\infty, T)^2 \sum_{i=1}^M \|e_i\|_{L^2(Q)},$$

we deduce (27) and (28). \blacksquare

Lemma 2.3 *Assume that (2) and (6) hold. If (z_1, z_2) satisfies*

$$\begin{cases} -\frac{\partial z_1}{\partial t} - \Delta z_1 + a z_1 + d z_2 = 0 & \text{in } \omega_T, \\ -\frac{\partial z_2}{\partial t} - \Delta z_2 + c z_2 + b z_1 = 0 & \text{in } \omega_T, \\ z_1|_\omega \in K. \end{cases} \quad (33)$$

Then z_1 and z_2 are identically zero in ω_T .

Proof. Since $z_1|_\omega \in K$, there exists $(\alpha_i)_{1 \leq i \leq M} \in \mathbb{R}^M$ such that $z_1|_\omega = \sum_{i=1}^M \alpha_i p_i \chi_\omega$. In view of (15)₁, this gives

$$-\frac{\partial z_1}{\partial t} - \Delta z_1 + a z_1 + d \sum_{i=1}^M \alpha_i q_i = 0 \text{ in } \omega_T. \quad (34)$$

Since $d \neq 0$ in ω_T , we deduce from (33)₁ and (34) that

$$z_2 = \sum_{i=1}^M \alpha_i q_i \text{ in } \omega_T.$$

Then, it follows from (15)₂ and (33)₂ that

$$\sum_{i=1}^M \alpha_i e_i = -\frac{\partial z_2}{\partial t} - \Delta z_2 + cz_2 + bz_1 = 0 \text{ in } \omega_T.$$

Assumption (6) allows us to conclude that $\alpha_i = 0, 1 \leq i \leq M$. Hence, $z_1 = z_2 = 0$ in ω_T . \blacksquare

To solve Problem (21)–(23), we need a Carleman inequality adapted to the constraint on the control.

2.2 Adapted Carleman inequality

In this section we use the global Carleman due to A. Fursikov in [2] (Theorem 7.1 p. 288) to establish an adapted Carleman inequality necessary for resolution of the null controllability with constraint on the control (21)-(23).

Theorem 2.1 *Let ψ, ρ and ϕ be defined by (11), (12) and (13) respectively. Then there exist $\lambda_0 = \lambda_0(\Omega, \omega') > 0$, $\tau_0 = \tau_0(\Omega, \omega', T) > 0$, and a positive constant $C_0 = C_0(\Omega, \omega')$ such that, for any $\lambda \geq \lambda_0, \tau \geq \tau_0$ and $s \geq -3$, the following inequality*

$$\begin{aligned} & \int_Q \left(\frac{1}{\lambda} \left| \frac{\partial z}{\partial t} \right|^2 + \frac{1}{\lambda} |\Delta z|^2 + \lambda \tau^2 \rho^2 |\nabla z|^2 + \lambda^4 \tau^4 \rho^4 |z|^2 \right) \rho^{2s-1} e^{-2\phi} dxdt \\ & \leq C_0 \left(\tau \int_Q \left| \frac{\partial z}{\partial t} \pm \Delta z \right|^2 \rho^{2s} e^{-2\phi} dxdt + \lambda^4 \tau^4 \int_0^T \int_{\omega'} |z|^2 \rho^{2s+3} e^{-2\phi} dxdt \right) \end{aligned} \quad (35)$$

holds for any z satisfying the Dirichlet homogeneous condition and such that the right-hand side of the identity (35) is finite.

We adopt the following notations:

$$\left\{ \begin{array}{l} \mathcal{W} = \{(\varphi_1, \varphi_2) \in C^\infty(\overline{Q}) \times C^\infty(\overline{Q}), \varphi_1 = \varphi_2 = 0 \text{ on } \Sigma\}, \\ L_0 \varphi = \frac{\partial \varphi}{\partial t} - \Delta \varphi, \\ L_0^* \varphi = -\frac{\partial \varphi}{\partial t} - \Delta \varphi, \\ \mathcal{M}(\varphi, \sigma) = L_0^* \varphi + a\varphi + d\sigma, \\ \mathcal{N}(\varphi, \sigma) = L_0^* \sigma + c\sigma + b\varphi. \end{array} \right.$$

Proposition 2.3 *Let C_0 be the constant given by theorem 2.1. Assume that (2) holds, and that $\tau_1 = \frac{T^2}{4} \left(\frac{4C_0}{\lambda_0^4} \right)^{1/3} \|(a, b, c, d)\|_\infty^{2/3} \geq 1$. Then, for any $\lambda \geq \lambda_0, \tau \geq \tau_1, s \geq -3$ and $r \in [0, 2]$, there exists a positive constant $C =$*

$C(T, \|(a, b, c, d)\|_\infty, r, \nu)$ such that for any $\varphi = (\varphi_1, \varphi_2) \in \mathcal{W}$

$$\begin{aligned} \int_Q \frac{1}{\theta^2} (|\varphi_1|^2 + |\varphi_2|^2) dxdt &\leq C \left(\int_{\omega_T} |\varphi_1|^2 e^{-r\phi} dxdt \right. \\ &\quad \left. + \int_Q \frac{1}{\theta^2} (|\mathcal{M}(\varphi)|^2 + |\mathcal{N}(\varphi)|^2) dxdt \right). \end{aligned} \quad (36)$$

Proof. The proof is deduced from Lemma 2.3 and Theorem 2.4 in [15]. ■

As $\frac{1}{\theta^2}$ and $e^{-r\phi}$ are bounded in $L^\infty(Q)$, we get this next observability inequality for all $\varphi = (\varphi_1, \varphi_2) \in \mathcal{W}$:

$$\begin{aligned} \int_Q \frac{1}{\theta^2} (|\varphi_1|^2 + |\varphi_2|^2) dxdt &\leq C \int_{\omega_T} |\varphi_1|^2 dxdt \\ &\quad + C \int_Q (|\mathcal{M}(\varphi)|^2 + |\mathcal{N}(\varphi)|^2) dxdt, \end{aligned} \quad (37)$$

where $C = C(T, \|(a, b, c, d)\|_\infty, r, \nu) > 0$.

Remark 1 *The latter inequality differs from that of C. Louis-Rose [15] because we use the fact that $e^{-r\phi}$ is bounded. This let us obtain the following adapted Carleman inequality without defining $e^{(2-r)\phi}$ as in Proposition 2.6 in [15].*

Proposition 2.4 *Assume that (2) and (6) hold. Assume also that the hypotheses of Proposition 2.3 hold. Then there exists a positive constant C such that for any $\varphi = (\varphi_1, \varphi_2) \in \mathcal{W}$,*

$$\begin{aligned} \int_Q \frac{1}{\theta^2} (|\varphi_1|^2 + |\varphi_2|^2) dxdt &\leq C \left(\int_{\omega_T} |(\varphi_1 - P\varphi_1)\chi_\omega|^2 dxdt \right. \\ &\quad \left. + \int_Q (|\mathcal{M}(\varphi)|^2 + |\mathcal{N}(\varphi)|^2) dxdt \right) \end{aligned} \quad (38)$$

Proof. The proof uses a well-known compactness-uniqueness argument and inequality (37). Indeed, suppose that (38) does not hold. Then, there exists $(\varphi_n = (\varphi_{1n}, \varphi_{2n}))_{n \in \mathbb{N}^*} \subset \mathcal{W}$ such that

$$\int_Q \frac{1}{\theta^2} (|\varphi_{1n}|^2 + |\varphi_{2n}|^2) dxdt = 1, \quad (39a)$$

$$\int_{\omega_T} |(\varphi_{1n} - P\varphi_{1n})\chi_\omega|^2 dxdt \leq \frac{1}{n}, \quad (39b)$$

$$\int_Q |\mathcal{M}(\varphi_n)|^2 dxdt \leq \frac{1}{n}, \quad (39c)$$

$$\int_Q |\mathcal{N}(\varphi_n)|^2 dxdt \leq \frac{1}{n}. \quad (39d)$$

We have

$$\begin{aligned} \int_{\omega_T} \frac{1}{\theta^2} |P\varphi_{1n}\chi_\omega|^2 dxdt &= \int_{\omega_T} \frac{1}{\theta^2} |(P\varphi_{1n} - \varphi_{1n} + \varphi_{1n})\chi_\omega|^2 dxdt \\ &\leq 2 \left(\int_{\omega_T} \frac{1}{\theta^2} |(P\varphi_{1n} - \varphi_{1n})\chi_\omega|^2 dxdt + \int_{\omega_T} \frac{1}{\theta^2} |\varphi_{1n}\chi_\omega|^2 dxdt \right). \end{aligned}$$

Since $\frac{1}{\theta^2}$ is bounded, it follows from (39b) and (39a) that

$$\int_{\omega_T} \frac{1}{\theta^2} |P\varphi_{1n}\chi_\omega|^2 dxdt \leq C.$$

Because K is finite dimensional, there exists a positive real C' such that

$$\int_{\omega_T} |P\varphi_{1n}\chi_\omega|^2 dxdt \leq C'. \quad (40)$$

As we can write,

$$\|\varphi_{1n}\|_{L^2(\omega_T)} \leq \|\varphi_{1n} - P\varphi_{1n}\|_{L^2(\omega_T)} + \|P\varphi_{1n}\|_{L^2(\omega_T)},$$

using (40) and (39b), we deduce that the subsequence $(\varphi_{1n}\chi_\omega)_{n \in \mathbb{N}^*}$ is bounded in $L^2(\omega_T)$. Consequently, we can extract a subsequence of $(\varphi_{1n})_{n \in \mathbb{N}^*}$, still denoted $(\varphi_{1n})_{n \in \mathbb{N}^*}$, such that

$$\varphi_{1n} \rightharpoonup \varphi_1 \text{ weakly in } L^2(\omega_T). \quad (41)$$

Hence, from the continuity and the compactness of P , we deduce that

$$P\varphi_{1n} \rightarrow P\varphi_1 \text{ strongly in } L^2(\omega_T). \quad (42)$$

In view of (39b), we have that

$$(\varphi_{1n} - P\varphi_{1n}) \rightarrow 0 \text{ strongly in } L^2(\omega_T). \quad (43)$$

It then follows from (42) and (43) that

$$\varphi_{1n} \rightarrow P\varphi_1 \text{ strongly in } L^2(\omega_T), \quad (44)$$

which combining with (41) implies

$$P\varphi_1 = \varphi_1.$$

This means on the one hand that

$$\varphi_1 \in K, \quad (45)$$

and on the other hand that

$$\varphi_{1n} \rightarrow \varphi_1 \text{ strongly in } L^2(\omega_T) \quad (46)$$

since (44) holds.

Now, in view of (39a) we have

$$\int_Q \frac{1}{\theta^2} |\varphi_{in}|^2 dxdt \leq 1, \quad i = 1, 2.$$

Using the definition of $\frac{1}{\theta^2}$, we deduce that $(\varphi_{in})_n$, $i = 1, 2$ is bounded in $L^2((\mu, T - \mu) \times \Omega)$, $\forall \mu > 0$. Therefore, there exists $(\varphi_1, \varphi_2) \in [L^2((\mu, T - \mu) \times \Omega)]^2$ such that

$$\varphi_{in} \rightharpoonup \varphi_i \text{ weakly in } L^2((\mu, T - \mu) \times \Omega), \quad i = 1, 2.$$

Consequently,

$$\varphi_{in} \rightarrow \varphi_i \text{ in } \mathcal{D}'(Q), \quad i = 1, 2.$$

This implies that

$$\mathcal{M}(\varphi_n) \rightarrow M(\varphi_1, \varphi_2) \text{ in } \mathcal{D}'(Q),$$

and

$$\mathcal{N}(\varphi_n) \rightarrow N(\varphi_1, \varphi_2) \text{ in } \mathcal{D}'(Q).$$

From (39c) and (39d), we have that

$$\mathcal{M}(\varphi_n) \rightarrow 0 \text{ strongly in } L^2(Q) \quad (47)$$

and

$$\mathcal{N}(\varphi_n) \rightarrow 0 \text{ strongly in } L^2(Q). \quad (48)$$

It then follows from the uniqueness of the limit that

$$\mathcal{M}(\varphi_1, \varphi_2) = \mathcal{N}(\varphi_1, \varphi_2) = 0 \text{ in } Q. \quad (49)$$

Using (45), (49) and lemma 2.3, we deduce that $\varphi_1 = \varphi_2 = 0$ in ω_T . Finally, (46) allows us to conclude that

$$\varphi_{1n} \rightarrow 0 \text{ strongly in } L^2(\omega_T). \quad (50)$$

Since $\varphi_n = (\varphi_{1n}, \varphi_{2n}) \in \mathcal{W}$, it follows from the observability inequality (37) that

$$\begin{aligned} \int_Q \frac{1}{\theta^2} (|\varphi_{1n}|^2 + |\varphi_{2n}|^2) dxdt &\leq C \int_{\omega_T} |\varphi_{1n}|^2 dxdt \\ &+ \int_Q (|\mathcal{M}(\varphi_n)|^2 + |\mathcal{N}(\varphi_n)|^2) dxdt. \end{aligned}$$

Then, (47), (48) and (50) yield that $\int_Q \frac{1}{\theta^2} (|\varphi_{1n}|^2 + |\varphi_{2n}|^2) dxdt \rightarrow 0$ when $n \rightarrow +\infty$. The contradiction occurs thanks to condition (39a). The proof is complete. \blacksquare

2.3 Solution of the null controllability problem with constraint on the control

We proceed in three steps.

Step 1. We prove there exists a solution $(v_\theta, y_\theta = (y_{1\theta}, y_{2\theta}))$ to the null controllability problem with constraint on the control.

Consider the following symmetric bilinear form

$$\begin{aligned} \mathcal{B}(\varphi, \sigma) &= \int_Q \mathcal{M}(\varphi) M(\sigma) dxdt + \int_Q N(\varphi) \mathcal{N}(\sigma) dxdt \\ &+ \int_{\omega_T} (\varphi_1 - P\varphi_1) (\sigma_1 - P\sigma_1) dxdt, \end{aligned} \quad (51)$$

for any $\varphi = (\varphi_1, \varphi_2)$ and $\sigma = (\sigma_1, \sigma_2) \in \mathcal{W}$.

From (38), \mathcal{B} is a scalar product on \mathcal{W} . Let

$$W = \overline{\mathcal{W}}^{\mathcal{B}} \quad (52)$$

be the completion of \mathcal{W} under the norm

$$\varphi \mapsto \sqrt{\mathcal{B}(\varphi, \varphi)}. \quad (53)$$

Then the space $W = \overline{\mathcal{W}}^{\mathcal{B}}$ is a Hilbert space.

Let H_θ be a Hilbert space defined by (4) and $l \in H_\theta$. Let also θ and u_0 be respectively defined by (14) and (25). Then, using the estimate of θu_0 given by (27), the Carleman estimate (38), the definition of the norm on W given by (53) and the Cauchy-Schwartz inequality, we prove that the linear form \mathcal{L} defined on W by

$$\mathcal{L} : (\varphi_1, \varphi_2) \mapsto \int_Q (l - u_0 \chi_\omega) \varphi_1 dxdt \quad (54)$$

is continuous on W .

Hence, the Riesz representation theorem allows to say that there exists a unique $\varphi_\theta = (\varphi_{1\theta}, \varphi_{2\theta}) \in W$ such that

$$\mathcal{B}(\varphi_\theta, \varphi) = \mathcal{L}(\varphi), \quad \forall \varphi \in W. \quad (55)$$

Proposition 2.5 *Assume that the hypotheses of Proposition 2.4 holds. Let $\varphi_\theta \in W$ be the unique solution of (55). Set*

$$y_{1\theta} = \mathcal{M}(\varphi_\theta), \quad (56a)$$

$$y_{2\theta} = \mathcal{N}(\varphi_\theta), \quad (56b)$$

$$v_\theta = -(\varphi_{1\theta} - P\varphi_{1\theta}). \quad (56c)$$

Then $(v_\theta, y_{1\theta}, y_{2\theta})$ is solution of (21) – (23).

Moreover,

$$\|\varphi_\theta\|_W \leq C\|l\|_{H_\theta}, \quad (57a)$$

$$\|y_{1\theta}\|_{L^2(Q)} \leq C\|l\|_{H_\theta}, \quad (57b)$$

$$\|y_{2\theta}\|_{L^2(Q)} \leq C\|l\|_{H_\theta}, \quad (57c)$$

$$\|v_\theta\|_{L^2(G)} \leq C\|l\|_{H_\theta}, \quad (57d)$$

where $C = C\left(\Omega, \|(a, b, c, d)\|_\infty, \sum_{i=1}^M \|e_i\|_{L^2(Q)}^2\right) > 0$.

Proof. Since $\varphi_{1\theta} \in L^2(\omega_T)$ and $v_\theta = -(\varphi_{1\theta} - P\varphi_{1\theta})$, we have $v_\theta \in K^\perp$. In view of (55),

$$\begin{aligned} & \int_Q \mathcal{M}\varphi_\theta \mathcal{M}\varphi dxdt + \int_Q \mathcal{N}\varphi_\theta \mathcal{N}\varphi dxdt + \int_{\omega_T} (\varphi_{1\theta} - P\varphi_{1\theta})(\varphi_1 - P\varphi_1) dxdt \\ &= \int_Q (l - u_0\chi_\omega) \varphi_1 dxdt, \quad \forall \varphi = (\varphi_1, \varphi_2) \in W. \end{aligned}$$

Replacing in this latter identity $\mathcal{M}\varphi_\theta$ by $y_{1\theta}$, $\mathcal{N}\varphi_\theta$ by $y_{2\theta}$ and $(-\varphi_1 + P\varphi_1)\chi_\omega$ by v_θ , we deduce that

$$\begin{aligned} & \int_Q y_{1\theta} \left(-\frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 + a\varphi_1 + d\varphi_2 \right) dxdt + \int_Q y_{2\theta} \left(-\frac{\partial \varphi_2}{\partial t} - \Delta \varphi_2 + c\varphi_2 + b\varphi_1 \right) dxdt \\ & - \int_{\omega_T} v_\theta \varphi_1 dxdt = \int_Q (l - u_0\chi_\omega) \varphi_1 dxdt, \quad \forall \varphi = (\varphi_1, \varphi_2) \in W. \end{aligned} \quad (58)$$

Taking successively in (58) $\varphi_2 = 0$ and $\varphi_1 \in \mathcal{D}(Q)$ and then $\varphi_1 = 0$ and $\varphi_2 \in \mathcal{D}(Q)$, we successively obtain

$$\int_Q y_{1\theta} \left(-\frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 + a\varphi_1 \right) dxdt + \int_Q by_{2\theta} \varphi_1 dxdt = \int_Q (l + (v_\theta - u_0)\chi_\omega) \varphi_1 dxdt, \quad \forall \varphi_1 \in \mathcal{D}(Q) \quad (59)$$

and

$$\int_Q y_{2\theta} \left(-\frac{\partial \varphi_2}{\partial t} - \Delta \varphi_2 + c\varphi_2 \right) dxdt + \int_Q dy_{1\theta} \varphi_2 dxdt = 0, \quad \forall \varphi_2 \in \mathcal{D}(Q), \quad (60)$$

which after integration by parts over Q gives,

$$\begin{aligned} & \int_Q \varphi_1 \left(\frac{\partial y_{1\theta}}{\partial t} - \Delta y_{1\theta} + ay_{1\theta} + by_{2\theta} \right) dxdt = \\ & \int_Q (l + (v_\theta - u_0)\chi_\omega) \varphi_1 dxdt, \quad \forall \varphi_1 \in \mathcal{D}(Q) \end{aligned}$$

and

$$\int_Q \varphi_2 \left(\frac{\partial y_{2\theta}}{\partial t} - \Delta y_{2\theta} + cy_{2\theta} + dy_{1\theta} \right) dxdt = 0, \quad \forall \varphi_2 \in \mathcal{D}(Q).$$

It then follows that

$$\frac{\partial y_{1\theta}}{\partial t} - \Delta y_{1\theta} + ay_{1\theta} + by_{2\theta} = l + (v_\theta - u_0) \chi_\omega \text{ in } Q \quad (61)$$

and

$$\frac{\partial y_{2\theta}}{\partial t} - \Delta y_{2\theta} + cy_{2\theta} + dy_{1\theta} = 0 \text{ in } Q. \quad (62)$$

As $y_{1\theta}, y_{2\theta} \in L^2(Q)$, we have $\frac{\partial y_{1\theta}}{\partial t} \in H^{-1}((0, T); L^2(\Omega))$. Thus,

$$\Delta y_{1\theta} = \frac{\partial y_{1\theta}}{\partial t} + ay_{1\theta} + by_{2\theta} - l - (v_\theta - u_0) \chi_\omega \in H^{-1}((0, T); L^2(\Omega))$$

since $l + (v_\theta - u_0) \chi_\omega \in L^2(Q)$. Consequently,

$$(y_{1\theta}, y_{2\theta})|_\Sigma \in H^{-1}((0, T); H^{-1/2}(\Gamma)) \times H^{-1}((0, T); H^{-1/2}(\Gamma))$$

and

$$\left(\frac{\partial y_{1\theta}}{\partial \nu}, \frac{\partial y_{2\theta}}{\partial \nu} \right)_{|\Sigma} \in H^{-1}((0, T); H^{-3/2}(\Gamma)) \times H^{-1}((0, T); H^{-3/2}(\Gamma)) \text{ (see [6]).}$$

Also, as $y_{1\theta} \in L^2(Q), y_{2\theta} \in L^2(Q)$, we have $\Delta y_{1\theta} \in L^2((0, T); H^{-2}(\Omega))$. Thus, we have

$$\frac{\partial y_{1\theta}}{\partial t} = \Delta y_{1\theta} - ay_{1\theta} - by_{2\theta} + l + (\gamma v_\theta - u_0) \chi_\omega \in L^2((0, T); H^{-2}(\Omega)).$$

Consequently,

$$(y_{1\theta}(0), y_{2\theta}(0)) \in (H^{-1}(\Omega))^2 \text{ and } (y_{1\theta}(T), y_{2\theta}(T)) \in (H^{-1}(\Omega))^2. \quad (63)$$

Multiplying (61) and (62) respectively by φ_1 and $\varphi_2 \in \mathcal{C}^\infty(\overline{Q})$ such that $(\varphi_1, \varphi_2) = (0, 0)$ on Σ , then, integrating by parts over Q , we obtain

$$\begin{aligned} \int_Q (l + (v_\theta - u_0) \chi_\omega) \varphi_1 dxdt &= \int_Q by_{2\theta} \varphi_1 dxdt \\ &+ \int_Q y_{1\theta} \left(-\frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 + a\varphi_1 \right) dxdt \\ &+ \langle \varphi_1(T), y_{1\theta}(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &- \langle \varphi_1(0), y_{1\theta}(0) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &+ \langle y_{1\theta}, \frac{\partial \varphi_1}{\partial \nu} \rangle_{H^{-1}((0, T); H^{-1/2}(\Gamma)), H_0^1((0, T); H^{1/2}(\Gamma))}, \\ &\forall \varphi_1 \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi_1|_\Sigma = 0 \end{aligned}$$

and

$$\begin{aligned}
0 &= \int_Q dy_{1\theta} \varphi_2 dx dt \\
&+ \int_Q y_{2\theta} \left(-\frac{\partial \varphi_2}{\partial t} - \Delta \varphi_2 + c \varphi_2 \right) dx dt \\
&+ \langle \varphi_2(T), y_{2\theta}(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
&- \langle \varphi_2(0), y_{2\theta}(0) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
&+ \langle y_{2\theta}, \frac{\partial \varphi_2}{\partial \nu} \rangle_{H^{-1}((0,T); H^{-1/2}(\Gamma)), H_0^1((0,T); H^{1/2}(\Gamma))}, \\
&\forall \varphi_2 \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi_2|_\Sigma = 0,
\end{aligned}$$

which in view of relations (59) and (60) give

$$\begin{aligned}
0 &= \langle \varphi_1(T), y_{1\theta}(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
&- \langle \varphi_1(0), y_{1\theta}(0) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
&+ \langle y_{1\theta}, \frac{\partial \varphi_1}{\partial \nu} \rangle_{H^{-1}((0,T); H^{-1/2}(\Gamma)), H_0^1((0,T); H^{1/2}(\Gamma))}, \\
&\forall \varphi_1 \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi_1|_\Sigma = 0
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
0 &= \langle \varphi_2(T), y_{2\theta}(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
&- \langle \varphi_2(0), y_{2\theta}(0) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
&+ \langle y_{2\theta}, \frac{\partial \varphi_2}{\partial \nu} \rangle_{H^{-1}((0,T); H^{-1/2}(\Gamma)), H_0^1((0,T); H^{1/2}(\Gamma))}, \\
&\forall \varphi_2 \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi_2|_\Sigma = 0.
\end{aligned} \tag{65}$$

Choosing successively in (64) and (65) φ_i , such that $\varphi_i(0) = \varphi_i(T) = 0$ in Ω , then $\varphi_i(0) = 0$ in Ω , for $i=1,2$, we obtain

$$y_{1\theta} = 0, \quad y_{2\theta} = 0 \text{ on } \Sigma, \tag{66}$$

$$y_{1\theta}(T) = 0, \quad y_{2\theta}(T) = 0 \text{ in } \Omega \tag{67}$$

and then

$$y_{1\theta}(0) = 0, \quad y_{2\theta}(0) = 0 \text{ in } \Omega. \tag{68}$$

It then follows from (61), (62), (66)-(68) that $(v_\theta, y_{1\theta}, y_{2\theta})$ is solution of problem (21) – (23).

Set $\varphi = \varphi_\theta = (\varphi_{1\theta}, \varphi_{2\theta})$ in (55), then using definitions of the bilinear form \mathcal{B} , the linear form \mathcal{L} and the norm on W given respectively by (51), (54) and (53), we have

$$\begin{aligned}
\|\varphi_\theta\|_W^2 &= \int_Q (l - u_0 \chi_\omega) \varphi_{1\theta} dx dt \\
&= \int_Q \theta (l - u_0 \chi_\omega) \frac{1}{\theta} \varphi_{1\theta} dx dt \\
&\leq \|\theta (l - u_0 \chi_\omega)\|_{L^2(Q)} \left\| \frac{1}{\theta} \varphi_{1\theta} \right\|_{L^2(Q)} \\
&\leq \left(\|\theta l\|_{L^2(Q)} + \|\theta u_0\|_{L^2(\omega_T)} \right) \left\| \frac{1}{\theta} \varphi_{1\theta} \right\|_{L^2(Q)},
\end{aligned}$$

which in view of (27) and the definition of the norm on H_θ gives

$$\|\varphi_\theta\|_W^2 \leq C \|l\|_{H_\theta} \left\| \frac{1}{\theta} \varphi_{1\theta} \right\|_{L^2(Q)}, \quad (69)$$

where $C = C \left(\Omega, \|(a, b, c, d)\|_\infty, \sum_{i=1}^M \|e_i\|_{L^2(Q)}^2 \right)$.

In view of (38), it comes that

$$\left\| \frac{1}{\theta} \varphi_{1\theta} \right\|_{L^2(Q)} \leq C \|\varphi_\theta\|_W. \quad (70)$$

Combining (69) and (70), we deduce that

$$\|\varphi_\theta\|_W \leq C \|l\|_{H_\theta}. \quad (71)$$

In view of the definition of the norm on W given by (53), we can write

$$\|y_{1\theta}\|_{L^2(Q)}^2 + \|y_{2\theta}\|_{L^2(Q)}^2 + \|v_\theta\|_{L^2(Q)}^2 = \|\varphi_\theta\|_W^2.$$

Therefore, using (71), we obtain that

$$\|y_{1\theta}\|_{L^2(Q)}^2 + \|y_{2\theta}\|_{L^2(Q)}^2 + \|v_\theta\|_{L^2(Q)}^2 \leq C^2 \|l\|_{H_\theta}^2.$$

from which we deduce (57b), (57c) and (57d). ■

The adapted observability inequality(38) shows that the choice of the scalar product on \mathcal{W} is not unique. Thus there exists infinitely many controls functions v such that (21) – (23) hold.

Set

$$\mathcal{E} = \{v \in K^\perp \text{ such that } (v, y_1(v), y_2(v)) \text{ satisfies (21) – (23)}\}.$$

Then \mathcal{E} is non empty and it is clearly convex and closed in $L^2(\omega_T)$. Therefore, there exists a unique control variable \hat{v} such that

$$\|\hat{v}\|_{L^2(\omega_T)}^2 = \min_{v \in \mathcal{E}} \|v\|_{L^2(\omega_T)}^2. \quad (72)$$

Particularly, we have

$$\|\hat{v}\|_{L^2(\omega_T)}^2 \leq \|v_\theta\|_{L^2(\omega_T)}^2,$$

which in view of (57d) implies that

$$\|\hat{v}\|_{L^2(\omega_T)} \leq C \|l\|_{H_\theta}. \quad (73)$$

In the sequel, we characterize the control \hat{v} using the penalization method.

Step 2. We prove by means of penalization method that there exists a unique $(\hat{v}, \hat{y}_1, \hat{y}_2)$ solution to problem of controllability with constraint on the control (21)-(23).

So, for $\varepsilon > 0$, let us consider the following optimal control problem. If $v \in K^\perp$ let $(y_1 = y_1(v), y_2 = y_2(v))$ be the solution of (22). We know that $(y_1 = y_1(v), y_2 = y_2(v)) \in (C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)))^2$ and we can define the functional

$$J_\varepsilon(v) = \frac{1}{2} \|v\|_{L^2(\omega_T)}^2 + \frac{1}{2\varepsilon} \|y_1(T)\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon} \|y_2(T)\|_{L^2(\Omega)}^2, \quad (74)$$

The optimal control problem is then to find $u_\varepsilon \in K^\perp$ such that

$$J_\varepsilon(u_\varepsilon) = \inf_{v \in K^\perp} J_\varepsilon(v). \quad (75)$$

As K^\perp is a closed convex of $L^2(\omega_T)$, it is classical to show that there exists a unique solution u_ε to (75) (see for instance [3], Chapter 5, Section 3). If we write $y_\varepsilon = (y_{1\varepsilon}, y_{2\varepsilon})$ the solution to (22), state associated to u_ε , using an adjoint state $\rho_\varepsilon = (\rho_{1\varepsilon}, \rho_{2\varepsilon}) \in (C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)))^2$, the triplet $(u_\varepsilon, y_\varepsilon, \rho_\varepsilon)$ is solution of the following first order optimality system:

$$\left\{ \begin{array}{ll} \frac{\partial y_{1\varepsilon}}{\partial t} - \Delta y_{1\varepsilon} + ay_{1\varepsilon} + by_{2\varepsilon} = l + (u_\varepsilon - u_0) \chi_\omega & \text{in } Q, \\ \frac{\partial y_{2\varepsilon}}{\partial t} - \Delta y_{2\varepsilon} + cy_{2\varepsilon} + dy_{1\varepsilon} = 0 & \text{in } Q, \\ y_{1\varepsilon} = y_{2\varepsilon} = 0 & \text{on } \Sigma, \\ y_{1\varepsilon}(0) = y_{2\varepsilon}(0) = 0 & \text{in } \Omega, \end{array} \right. \quad (76)$$

$$\left\{ \begin{array}{ll} -\frac{\partial \rho_{1\varepsilon}}{\partial t} - \Delta \rho_{1\varepsilon} + a\rho_{1\varepsilon} + d\rho_{2\varepsilon} = 0 & \text{in } Q, \\ -\frac{\partial \rho_{2\varepsilon}}{\partial t} - \Delta \rho_{2\varepsilon} + c\rho_{2\varepsilon} + b\rho_{1\varepsilon} = 0 & \text{in } Q, \\ \rho_{1\varepsilon} = \rho_{2\varepsilon} = 0 & \text{on } \Sigma, \\ \rho_{1\varepsilon}(T) = \frac{1}{\varepsilon} y_{1\varepsilon}(T) & \text{in } \Omega, \\ \rho_{2\varepsilon}(T) = \frac{1}{\varepsilon} y_{2\varepsilon}(T) & \text{on } \Omega \end{array} \right. \quad (77)$$

and

$$u_\varepsilon = (P\rho_{1\varepsilon} - \rho_{1\varepsilon})\chi_{\omega_T} \in K^\perp. \quad (78)$$

Multiplying the state equations (76)₁ and (76)₂ respectively by $\rho_{1\varepsilon}$ and $\rho_{2\varepsilon}$ solutions of (77), then integrating by parts over Q and adding the resulting identities, we obtain

$$\frac{1}{\varepsilon} \|y_{1\varepsilon}(T)\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|y_{2\varepsilon}(T)\|_{L^2(\Omega)}^2 = \int_Q l\rho_{1\varepsilon} dx dt + \int_{\omega_T} (u_\varepsilon - u_0)\rho_{1\varepsilon} dx dt,$$

so that

$$\frac{1}{\varepsilon} \|y_{1\varepsilon}(T)\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|y_{2\varepsilon}(T)\|_{L^2(\Omega)}^2 + \|u_\varepsilon\|_{L^2(\omega_T)}^2 = \int_Q l\rho_{1\varepsilon} dx dt + \int_{\omega_T} u_\varepsilon P\rho_{1\varepsilon} \chi_\omega dx dt - \int_{\omega_T} u_0 \rho_{1\varepsilon} dx dt.$$

Since $u_\varepsilon \in K^\perp$, this gives

$$2J_\varepsilon(u_\varepsilon) = \int_Q l\rho_{1\varepsilon} dx dt - \int_{\omega_T} u_0\rho_{1\varepsilon} dx dt.$$

In view of (78),

$$u_\varepsilon = -(I - P)\rho_{1\varepsilon}\chi_\omega.$$

Thus,

$$\begin{aligned} \frac{1}{\varepsilon}\|y_{1\varepsilon}(T)\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\|y_{2\varepsilon}(T)\|_{L^2(\Omega)}^2 + \|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)}^2 = \\ \int_Q l\rho_{1\varepsilon} dx dt - \int_{\omega_T} u_0\rho_{1\varepsilon} dx dt. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{\varepsilon}\|y_{1\varepsilon}(T)\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\|y_{2\varepsilon}(T)\|_{L^2(\Omega)}^2 + \|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)}^2 \\ \leq [\|l\|_{H_\theta} + \|\theta u_0\|_{L^2(\omega_T)}] \|\frac{1}{\theta}\rho_{1\varepsilon}\|_{L^2(Q)}. \end{aligned} \quad (79)$$

Because $\rho_\varepsilon = (\rho_{1\varepsilon}, \rho_{2\varepsilon})$ solution to (77) belongs to W , we have from (38) that

$$\left\| \frac{1}{\theta}\rho_{1\varepsilon} \right\|_{L^2(Q)}^2 \leq C\|\rho_{1\varepsilon} - P\rho_{1\varepsilon}\|_{L^2(\omega_T)}^2 dx dt, \quad (80)$$

which combining with (79) gives

$$\begin{aligned} \frac{1}{\varepsilon}\|y_{1\varepsilon}(T)\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\|y_{2\varepsilon}(T)\|_{L^2(\Omega)}^2 + \|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)}^2 \\ \leq C(\|l\|_{H_\theta} + \|\theta u_0\|_{L^2(\omega_T)}) \|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)}. \end{aligned}$$

It then follows from (27) and (28) that

$$\frac{1}{\varepsilon}\|y_{1\varepsilon}(T)\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\|y_{2\varepsilon}(T)\|_{L^2(\Omega)}^2 + \|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)}^2 \leq C\|l\|_{H_\theta} \|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)}.$$

Hence, we deduce that

$$\|(I - P)\rho_{1\varepsilon}\|_{L^2(\omega_T)} \leq C\|l\|_{H_\theta}, \quad (81a)$$

$$\|y_{1\varepsilon}(T)\|_{L(\Omega)} \leq C\sqrt{\varepsilon}\|l\|_{H_\theta}, \quad (81b)$$

$$\|y_{2\varepsilon}(T)\|_{L(\Omega)} \leq C\sqrt{\varepsilon}\|l\|_{H_\theta}, \quad (81c)$$

$$\|u_\varepsilon\|_{L^2(\omega_T)}^2 \leq C\|l\|_{H_\theta}, \quad (81d)$$

where $C > 0$ is independent of ε . Then from the properties of the system (76) we get

$$\|y_\varepsilon\|_{(L^2((0,T);H_0^1(\Omega)))^2} \leq C\|l\|_{H_\theta}. \quad (82)$$

From (80), (81a), we have on the one hand,

$$\left\| \frac{1}{\theta} \rho_{1\varepsilon} \right\|_{L^2(Q)} \leq C \|l\|_{H_\theta}, \quad (83)$$

and on the other hand, because $\frac{1}{\theta} \in L^\infty(Q)$,

$$\left\| \frac{1}{\theta} P \rho_{1\varepsilon} \right\|_{L^2(\omega_T)} \leq C \|l\|_{H_\theta}.$$

Therefore, K being a finite dimensional vector subspace of $L^2(\omega_T)$, we deduce that

$$\|P \rho_{1\varepsilon}\|_{L^2(\omega_T)} \leq C \|l\|_{H_\theta}. \quad (84)$$

Using again (81a) we obtain

$$\|\rho_{1\varepsilon}\|_{L^2(\omega_T)} \leq C \|l\|_{H_\theta}. \quad (85)$$

From the definition of the norm on W given by (53), we can write

$$\|\rho_\varepsilon\|_W^2 = \|\rho_{1\varepsilon} - P \rho_{1\varepsilon}\|_{L^2(\omega_T)}^2,$$

because (77)₁ and (77)₂ hold. Hence, using (81a), we deduce that

$$\|\rho_\varepsilon\|_W \leq C \|l\|_{H_\theta}. \quad (86)$$

Step 3. We pass to the limit in the first order optimality system (76)-(78).

In view of (81) and (82), we can extract subsequences of (v_ε) and $(y_\varepsilon) = ((y_{1\varepsilon}), (y_{2\varepsilon}))$ (still denoted (v_ε) and $(y_\varepsilon) = ((y_{1\varepsilon}), (y_{2\varepsilon}))$) such that

$$u_\varepsilon \rightharpoonup v_0 \text{ weakly in } L^2(\omega_T), \quad (87)$$

$$(y_{1\varepsilon}(T), y_{2\varepsilon}(T)) \rightarrow (0, 0) \text{ strongly in } L^2(\Omega) \times L^2(\Omega), \quad (88)$$

$$y_{1\varepsilon} \rightharpoonup y_{10} \text{ weakly in } L^2((0, T), H_0^1(\Omega)), \quad (89)$$

$$y_{2\varepsilon} \rightharpoonup y_{20} \text{ weakly in } L^2((0, T), H_0^1(\Omega)). \quad (90)$$

Proceeding as for $y_\theta = (y_{1\theta}, y_{2\theta})$ at the pages 17 and 18, we prove that $(v_0, (y_{10}, y_{20}))$ is solution of (21)-(23).

Actually, $(v_0, y_0) = (\hat{v}, \hat{y})$ where \hat{v} is the optimal control solution of (72) and $\hat{y} = \hat{y}(\hat{v})$ is the solution to (22) corresponding to \hat{v} . Indeed, from the expression of J_ε given by (74), we can write

$$\frac{1}{2} \|u_\varepsilon\|_{L^2(\omega_T)}^2 \leq J_\varepsilon(u_\varepsilon).$$

Because (\hat{v}, \hat{y}) is solution of (21)-(23), it follows from (75) that

$$J_\varepsilon(u_\varepsilon) \leq \frac{1}{2} \|\hat{v}\|_{L^2(\omega_T)}^2.$$

We then deduce from these two latter inequalities that

$$\frac{1}{2} \|u_\varepsilon\|_{L^2(\omega_T)}^2 \leq J_\varepsilon(u_\varepsilon) \leq \frac{1}{2} \|\hat{v}\|_{L^2(\omega_T)}^2. \quad (91)$$

Then, using (87) while passing in the limit in (91), we obtain that

$$\frac{1}{2} \|v_0\|_{L^2(\omega_T)}^2 \leq \liminf_{n \rightarrow \infty} J_\varepsilon(u_\varepsilon) \leq \frac{1}{2} \|\hat{v}\|_{L^2(\omega_T)}^2.$$

Consequently,

$$\|v_0\|_{L^2(\omega_T)}^2 \leq \|\hat{v}\|_{L^2(\omega_T)}^2.$$

As \hat{v} is solution of (72), we deduce that $\hat{v} = v_0$. And, since (22) admits a unique solution, we have $y_0 = \hat{y}$. This means that (\hat{v}, \hat{y}) satisfies

$$\begin{cases} \frac{\partial \hat{y}_1}{\partial t} - \Delta \hat{y}_1 + a \hat{y}_1 + b \hat{y}_2 = l + (\hat{v} - u_0) \chi_\omega & \text{in } Q, \\ \frac{\partial \hat{y}_2}{\partial t} - \Delta \hat{y}_2 + c \hat{y}_2 + d \hat{y}_1 = 0 & \text{in } Q, \\ \hat{y}_1 = \hat{y}_2 = 0 & \text{on } \Sigma, \\ \hat{y}_1(0) = \hat{y}_2(0) = 0 & \text{in } \Omega. \end{cases} \quad (92)$$

Then from (88), we have that

$$\hat{y}_1(T) = \hat{y}_2(T) = 0 \text{ in } \Omega. \quad (93)$$

Now, in view of (83) and (86), we can extract subsequences of $(\rho_{1\varepsilon})$ and $(\rho_{2\varepsilon})$ (still denoted $(\rho_{1\varepsilon})$ and $(\rho_{2\varepsilon})$) such that

$$\rho_{1\varepsilon} \rightharpoonup \rho_1 \text{ weakly in } D'(Q), \quad (94a)$$

$$\rho_{2\varepsilon} \rightharpoonup \rho_2 \text{ weakly in } D'(Q), \quad (94b)$$

$$\rho_\varepsilon \rightharpoonup \rho \text{ weakly in } W. \quad (94c)$$

From (94a) and (94b), we have that

$$\begin{aligned} -\frac{\partial \rho_{1\varepsilon}}{\partial t} - \Delta \rho_{1\varepsilon} + a \rho_{1\varepsilon} + d \rho_{2\varepsilon} &\rightharpoonup -\frac{\partial \rho_1}{\partial t} - \Delta \rho_1 + a \rho_1 + d \rho_2 \text{ weakly in } D'(Q), \\ -\frac{\partial \rho_{2\varepsilon}}{\partial t} - \Delta \rho_{2\varepsilon} + c \rho_{2\varepsilon} + b \rho_{1\varepsilon} &\rightharpoonup -\frac{\partial \rho_2}{\partial t} - \Delta \rho_2 + c \rho_2 + b \rho_1 \text{ weakly in } D'(Q). \end{aligned}$$

It then follows from (77)₁, (77)₂ and (94c) that $\rho = (\rho_1, \rho_2) \in W$ satisfies

$$\begin{cases} -\frac{\partial \rho_1}{\partial t} - \Delta \rho_1 + a \rho_1 + d \rho_2 = 0 & \text{in } Q, \\ -\frac{\partial \rho_2}{\partial t} - \Delta \rho_2 + c \rho_2 + b \rho_1 = 0 & \text{in } Q, \\ \rho_1 = \rho_2 = 0 & \text{on } \Sigma. \end{cases} \quad (95)$$

In view of (85), we can write

$$\rho_{1\varepsilon} \rightharpoonup \rho_1 \text{ weakly in } L^2(\omega_T).$$

As P is a continuous and compact operator, we deduce that

$$P\rho_{1\varepsilon} \rightarrow P\rho_1 \text{ weakly in } L^2(\omega_T).$$

Consequently,

$$u_\varepsilon = (P - I)\rho_{1\varepsilon}\chi_\omega \rightharpoonup (P - I)\rho_1\chi_\omega \text{ weakly in } L^2(\omega_T).$$

Thus

$$\hat{v} = -(I - P)\rho_1\chi_\omega \in K^\perp. \quad (96)$$

2.4 Proof of Theorem 1.3

We showed that there exists a unique control $\hat{v} = (P - I)\rho_1\chi_\omega \in K^\perp$ such that (\hat{v}, \hat{y}) verifies the null controllability problem with constraint on the control (21)-(23). Consequently, Proposition 2.1 let us say that the control $k = -u_0 + (P - I)\rho_1\chi_\omega$ is solution of the null controllability problem with state constraints (1), (7), (8). Furthermore, as $k = \hat{v} - u_0$, using estimates (28) and (73), we obtain

$$\|k\|_{L^2(\omega_T)} \leq C\|l\|_{H_\theta}. \quad (97)$$

3 Study of Problem 2

We need some preliminary results, useful for the proof of Theorem 1.4.

3.1 Preliminary results

Lemma 3.1 *Let u_0 be defined by relation (25). Then $u_0 \in H_\theta$ and the map F_1 defined by*

$$\begin{aligned} F_1 : H_\theta &\rightarrow L^2(\omega_T) \\ l &\mapsto F_1(l) = u_0(l)\chi_\omega \end{aligned} \quad (98)$$

is linear and continuous.

Proof. Because $u_0 \in K_\theta \subset L^2(\omega_T)$, the application F_1 is well defined. From relation (25) we have that the application F_1 is linear, and continuous since (28) holds. ■

Lemma 3.2 *Let H_θ be defined by (4). Then for any $l \in H_\theta$, the map*

$$\begin{aligned} F_2 : H_\theta &\rightarrow L^2(\omega_T) \\ l &\mapsto F_2(l) = -(\rho_1 - P\rho_1)\chi_\omega, \end{aligned} \quad (99)$$

where $\rho = (\rho_1, \rho_2) = (\rho_1(l), \rho_2(l))$ is solution of (95), is linear and continuous.

Proof. Let W be the Hilbert space defined by (52). Set

$$W_0 = \{\varphi = (\varphi_1, \varphi_2) \in W = V \times V \text{ such that } \mathcal{M}(\varphi) = \mathcal{N}(\varphi) = 0\}.$$

Then W_0 is a Hilbert space since W_0 is a closed subspace of W .

Multiplying the state equations (92)₁ and (92)₂ respectively by φ_1 and φ_2 , integrating them by parts over Q and adding the resulting identities, we obtain

$$\begin{aligned} \int_Q \varphi_1(l + (\hat{v}(l) - u_0(l))\chi_\omega) dxdt &= \int_\Omega \hat{y}_1(T)\varphi_1(T)dx \\ &+ \int_\Omega \hat{y}_2(T)\varphi_2(T)dx, \quad \forall (\varphi_1, \varphi_2) \in W_0, \end{aligned}$$

which in view of (93) gives

$$\int_{\omega_T} \varphi_1 \hat{v}(l) dxdt = \int_Q \varphi_1(u_0(l)\chi_\omega - l) dxdt, \quad \forall (\varphi_1, \varphi_2) \in W_0. \quad (100)$$

On the other hand, using the fact that $\hat{v}(l) \in K^\perp$ and the definition of the bilinear form \mathcal{B} given by (51), we have

$$\begin{aligned} \int_{\omega_T} \varphi_1 \hat{v}(l) dxdt &= \int_{\omega_T} (P\rho_1(l) - \rho_1(l))\varphi_1 dxdt \\ &= \int_{\omega_T} (\rho_1 - P\rho_1)(\varphi_1 - P\varphi_1) dxdt \\ &= \mathcal{B}(\rho, \varphi), \quad \forall \varphi = (\varphi_1, \varphi_2) \in W_0. \end{aligned}$$

As the linear form \mathcal{L} defined by (54) is also continuous on W_0 , we deduce that there exists a unique $\rho = (\rho_1, \rho_2) \in W_0$ solution of the variational problem:

$$\mathcal{B}(\rho, \varphi) = \int_Q (l - u_0\chi_\omega)\varphi_1 dxdt, \quad \forall \varphi = (\varphi_1, \varphi_2) \in W_0.$$

Consequently, the map $l \mapsto \rho(l)$ is linear from H_θ into $L^2(\omega_T)$ and because the operator $I - P$ defined from $L^2(\omega_T)$ into $K^\perp \subset L^2(\omega_T)$ is linear, we deduce that F_2 is linear. The continuity of F_2 is straightforward from estimate (73). ■

Remark 2 *It follows from the definitions of the maps F_1 and F_2 given respectively by (98) and (99) that the map*

$$\begin{aligned} F : H_\theta &\rightarrow L^2(\omega_T) \\ l &\mapsto F(l) = (\hat{v}(l) - u_0(l))\chi_\omega, \end{aligned} \quad (101)$$

is linear and continuous.

3.2 Proof of Theorem 1.4

Let H_θ be defined by (4) and $(\hat{v}, \hat{y} = (\hat{y}_1, \hat{y}_2))$ be the solution of (92), (93) and (96) (or equivalently (1), (7) and (8)).

Consider the cost function J defined by

$$J(l) = \frac{1}{2} \|y_1(l) - z_d^1\|_{L^2(Q)}^2 + \frac{1}{2} \|y_2(l) - z_d^2\|_{L^2(Q)}^2 + \frac{N}{2} \|l\|_{H_\theta}^2$$

from which we associate the minimization problem

$$\inf_{l \in \mathcal{U}_{ad}} J(l), \quad (102)$$

where \mathcal{U}_{ad} is a non empty closed convex subspace of H_θ .

From the properties of the map F defined in Remark 2, we have that J is strictly convex, continuous and coercive. Consequently, the optimal control problem (102) has a unique control $\hat{l} \in \mathcal{U}_{ad}$. To characterize this optimal control, we write the Euler-Lagrange conditions:

$$\frac{d}{d\lambda} J(\hat{l} + \lambda(l - \hat{l}))|_{\lambda=0} \geq 0, \quad \forall l \in \mathcal{U}_{ad}.$$

After some calculations, we obtain

$$\begin{aligned} & \left(\hat{y}_1 - z_d^1, y_1(l - \hat{l}) \right)_{L^2(Q)} + \left(\hat{y}_2 - z_d^2, y_2(l - \hat{l}) \right)_{L^2(Q)} \\ & + N \left(\hat{l}, l - \hat{l} \right)_{H_\theta} \geq 0, \quad \forall l \in \mathcal{U}_{ad}, \end{aligned} \quad (103)$$

where

$w = l - \hat{l}$ and the corresponding state $(q_1, q_2) = (y_1(l - \hat{l}), y_2(l - \hat{l}))$ satisfy

$$\begin{cases} \frac{\partial q_1}{\partial t} - \Delta q_1 + a q_1 + b q_2 = w + (\hat{v}(w) - u_0(w)) \chi_\omega & \text{in } Q, \\ \frac{\partial q_2}{\partial t} - \Delta q_2 + c q_2 + d q_1 = 0 & \text{in } Q, \\ q_1 = q_2 = 0 & \text{on } \Sigma, \\ q_1(0) = q_2(0) = 0 & \text{in } \Omega. \end{cases} \quad (104)$$

To interpret (103), we use the adjoint system

$$\begin{cases} -\frac{\partial p_1}{\partial t} - \Delta p_1 + a p_1 + d p_2 = \hat{y}_1 - z_d^1 & \text{in } Q, \\ -\frac{\partial p_2}{\partial t} - \Delta p_2 + c p_2 + b p_1 = \hat{y}_2 - z_d^2 & \text{in } Q, \\ p_1 = p_2 = 0 & \text{on } \Sigma, \\ p_1(T) = p_2(T) = 0 & \text{in } \Omega. \end{cases} \quad (105)$$

Because $(\hat{y}_1 - z_d^1, \hat{y}_2 - z_d^2) \in [L^2(Q)]^2$, system (105) has a unique solution $p = (p_1, p_2) \in [C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega))]^2$.

Multiplying (105)₁ and (105)₂ respectively by q_1 and q_2 , integrating them by parts over Q and adding the resulting identities, we obtain

$$\begin{aligned} \int_Q q_1 (\hat{y}_1 - z_d^1) dxdt + \int_Q q_2 (\hat{y}_2 - z_d^2) dxdt &= \int_Q p_1 \left(\frac{\partial q_1}{\partial t} - \Delta q_1 + a q_1 + b q_2 \right) dxdt \\ &\quad + \int_Q p_2 \left(\frac{\partial q_2}{\partial t} - \Delta q_2 + c q_2 + d q_1 \right) dxdt, \end{aligned}$$

which in view of (104) gives

$$\begin{aligned} \int_Q q_1 (\hat{y}_1 - z_d^1) dxdt + \int_Q q_2 (\hat{y}_2 - z_d^2) dxdt &= \int_Q p_1 (w + (\hat{v}(w) - u_0(w))\chi_\omega) dxdt \\ &= \int_Q p_1 w dxdt + \int_{\omega_T} p_1 F(w) dxdt, \end{aligned}$$

where F is the application defined in Remark 2.

Let H_θ' be the dual of Hilbert space H_θ . Let also Λ^{-1} be the isometric isomorphism from H_θ' to H_θ .

We have

$$\begin{aligned} \int_Q p_1 w dxdt &= \int_Q \left(\frac{1}{\theta} p_1 \right) (\theta w) dxdt \\ &= \left\langle \frac{1}{\theta} p_1, w \right\rangle_{H_\theta', H_\theta} \\ &= \left(\Lambda^{-1} \left(\frac{1}{\theta} p_1 \right), w \right)_{H_\theta} \end{aligned}$$

and

$$\begin{aligned} \int_{\omega_T} p_1 F(w) dxdt &= \int_Q F^*(p_1 \chi_{\omega_T}) w dxdt \\ &= \left\langle \frac{1}{\theta} F^*(p_1 \chi_{\omega_T}), w \right\rangle_{H_\theta', H_\theta} \\ &= \left(\Lambda^{-1} \left(\frac{1}{\theta} F^*(p_1 \chi_{\omega_T}) \right), w \right)_{H_\theta}. \end{aligned}$$

Hence

$$\int_Q q_1 (\hat{y}_1 - z_d^1) dxdt + \int_Q q_2 (\hat{y}_2 - z_d^2) dxdt = \left(\Lambda^{-1} \left(\frac{1}{\theta} p_1 \right) + \Lambda^{-1} \left(\frac{1}{\theta} F^*(p_1 \chi_{\omega_T}) \right), w \right)_{H_\theta}. \quad (106)$$

Finally, (103) and (106) give

$$\left(\Lambda^{-1} \left(\frac{1}{\theta} p_1 + \frac{1}{\theta} F^*(p_1 \chi_{\omega_T}) \right) + N\hat{l}, l - \hat{l} \right)_{H_\theta} \geq 0, \quad \forall l \in \mathcal{U}_{ad}.$$

■

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