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RESEARCH

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Low-regret control for a fractional wave equation with incomplete data

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Abstract

We investigate in this manuscript an optimal control problem for a fractional wave equation involving the fractional Riemann-Liouville derivative and with missing initial condition. For this purpose, we use the concept of no-regret and low-regret controls. Assuming that the missing datum belongs to a certain space we show the existence and the uniqueness of the low-regret control. Besides, its convergence to the no-regret control is discussed together with the optimality system describing the no-regret control.

Keywords: Riemann-Liouville fractional derivative; Caputo fractional derivative; optimal control; no-regret control; low-regret control

1 Introduction

Let us consider $N \in \mathbb{N}^*$ and Ω a bounded open subset of \mathbb{R}^N possessing the boundary $\partial\Omega$ of class C^2 . When the time $T > 0$, we consider $Q = \Omega \times]0, T[$ and $\Sigma = \partial\Omega \times]0, T[$ and we discuss the fractional wave equation:

$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = v(x, t), & (x, t) \in Q, \\ y(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ I^{2-\alpha} y(x, 0^+) = y^0, & x \in \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) = g, & x \in \Omega, \end{cases} \quad (1)$$

such that $3/2 < \alpha < 2$, $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $I^{2-\alpha} y(x, 0^+) = \lim_{t \rightarrow 0} I^{2-\alpha} y(x, t)$ and $\frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) = \lim_{t \rightarrow 0} \frac{\partial}{\partial t} I^{2-\alpha} y(x, t)$ where the fractional integral I^α of order α and the fractional derivative D_{RL}^α of order α are within the Riemann-Liouville sense. The function g is unknown and belongs to $L^2(\Omega)$ and the control $v \in L^2(Q)$.

Since the initial condition is unknown, the system (1) is a fractional wave equations with missing data. Such equations are used to model pollution phenomena. In this system g represents the pollution term.

According to the data, we know that system (1) admits a unique solution $y(v, g) = y(x, t; v, g)$ in $L^2((0, T); H_0^1(\Omega)) \subset L^2(Q)$ [1]. Hence, we can define the following functional:

$$J(v, g) = \|y(v, g) - z_d\|_{L^2(Q)}^2 + N \|v\|_{L^2(Q)}^2, \quad (2)$$

where $z_d \in L^2(Q)$ and $N > 0$.

48 In this manuscript, we discuss the optimal control problem, namely

49
 50
$$\inf_{v \in L^2(Q)} J(v, g), \quad \forall g \in L^2(\Omega). \tag{3}$$

51
 52 If the function g is given, namely $g = g_0 \in L^2(\Omega)$, then system (1) is completely determined
 53 and problem (3) becomes a classical optimal control problem [2]. Such a problem was
 54 studied by Mophou and Joseph [1] with a cost function defined with a final observation.
 55 Actually, the authors proved that one can approach the fractional integral of order $0 <$
 56 $2 - \alpha < 1/2$ of the state at final time by a desired state by acting on a distributed control.
 57 For more literature on fractional optimal control, we refer to [3–11] and the references
 58 therein.

59 Since the function g is unknown, the optimal control problem (3) has no sense because
 60 $L^2(\Omega)$ is of infinite dimension. So, to solve this problem, we proceed as Lions [12, 13] for
 61 the control of partial differential equations with integer time derivatives and missing data.
 62 This means that we use the notions of no-regret and low-regret controls. There are many
 63 works using these concepts in the literature. In [14] for instance, Nakoulima *et al.* utilized
 64 these concepts to control distributed linear systems possessing missing data. A generaliza-
 65 tion of this approach can be found in [15] for some nonlinear distributed systems possess-
 66 ing incomplete data. Jacob and Omrane used the notion of no-regret control to control
 67 a linear population dynamics equation with missing initial data [16]. Recently, Mophou
 68 [17] used these notions to control a fractional diffusion equation with unknown bound-
 69 ary condition. For more literature on such control we refer to [18–23] and the references
 70 therein.

71
 72 In our paper, we show that the low-regret control problem associated to (1) admits a
 73 unique solution which converges toward the no-regret control. We provide the singular
 74 optimality system for the no-regret control.

75 Below we present the organization of our manuscript. In the following section, we show
 76 briefly some results about fractional derivatives and preliminary results on the existence
 77 and uniqueness of solution to fractional wave equations. In Section 3, we investigate the
 78 no-regret and low-regret control problems corresponding to (1).

79 **2 Preliminaries**

80 Below, we give briefly some results about fractional calculus and some existence results
 81 about fractional wave equations.

82
 83 **Definition 2.1** [24, 25] If $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function on \mathbb{R}_+ , and $\alpha > 0$, then the
 84 expression of the Riemann-Liouville fractional integral of order α is

85
 86
$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0.$$

87
 88
 89 **Definition 2.2** [25, 26] The form of the left Riemann-Liouville fractional derivative of
 90 order $0 \leq n - 1 < \alpha < n$, $n \in \mathbb{N}$ of f is given by

91
 92
$$D_{RL}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-1-\alpha} f(s) ds, \quad t > 0.$$

95 **Definition 2.3** [25, 26] The left Caputo fractional derivative of order $0 \leq n - 1 < \alpha < n$,
 96 $n \in \mathbb{N}$ of f is given by

97
 98
$$D_C^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-1-\alpha} f^{(n)}(s) ds, \quad t > 0. \tag{4}$$

 99

100 We mention that in the above two definitions we consider $f : \mathbb{R}_+ \rightarrow \mathbb{R}$.

102 **Definition 2.4** [25–27] Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, $0 \leq n - 1 < \alpha < n$, $n \in \mathbb{N}$. Then the right Caputo
 103 fractional derivative of order α of f is

105
$$D_C^\alpha f(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^T (s - t)^{n-1-\alpha} f^{(n)}(s) ds, \quad 0 < t < T. \tag{5}$$

 106
 107

108 In all above definitions we assume that the integrals exist.

109
 110 **Lemma 2.5** [1] Let $y \in C^\infty(\overline{Q})$ and $\varphi \in C^\infty(\overline{Q})$. Then we have

111
 112
$$\int_Q (D_{RL}^\alpha y(x, t) - \Delta y(x, t)) \varphi(x, t) dx dt$$

 113
 114
$$= \int_\Omega \varphi(x, T) \frac{\partial}{\partial t} I^{2-\alpha} y(x, T) dx - \int_\Omega \varphi(x, 0) \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) dx$$

 115
 116
$$- \int_\Omega I^{2-\alpha} y(x, T) \frac{\partial \varphi}{\partial t}(x, T) dx + \int_\Omega I^{2-\alpha} y(x, 0) \frac{\partial \varphi}{\partial t}(x, 0) dx$$

 117
 118
$$+ \int_\Sigma y(\sigma, s) \frac{\partial \varphi}{\partial \nu}(\sigma, s) d\sigma dt - \int_\Sigma \frac{\partial y}{\partial \nu}(\sigma, s) \varphi(\sigma, s) d\sigma dt$$

 119
 120
$$+ \int_Q y(x, t) (D_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) dx dt. \tag{6}$$

 121
 122

123 In the following we give some results that will be use to prove the existence of the low-
 124 regret and no-regret controls.

125
 126 **Theorem 2.6** [1] Let $3/2 < \alpha < 2$, $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$ and $v \in L^2(Q)$. Then the
 127 *problem*

128
 129
$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = v(x, t), & (x, t) \in Q, \\ y(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ I^{2-\alpha} y(x, 0^+) = y^0, & x \in \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) = y^1, & x \in \Omega \end{cases} \tag{7}$$

 130
 131
 132

133 *has a unique solution $y \in L^2((0, T); H_0^1(\Omega))$. Moreover, the following estimates hold:*

134
 135
 136
$$\|y\|_{L^2((0, T); H_0^1(\Omega))} \leq \Delta (\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}), \tag{8}$$

137
$$\|I^{2-\alpha} y\|_{C([0, T]; H_0^1(\Omega))} \leq \Pi (\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}), \tag{9}$$

 138

139
$$\left\| \frac{\partial}{\partial t} I^{2-\alpha} y \right\|_{C([0, T]; L^2(\Omega))} \leq \Theta (\|y^0\|_{H^2(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}), \tag{10}$$

 140
 141

142 with

143

$$144 \quad \Delta = \max \left(C \sqrt{\frac{2T^{2\alpha-3}}{(2\alpha-3)}}, C \sqrt{\frac{T^{\alpha-1}}{(\alpha-1)}}, C \sqrt{\frac{2T^\alpha}{\alpha(\alpha-1)}} \right),$$

146

$$147 \quad \Pi = \sup \left(C\sqrt{2}, C\sqrt{2T^{2-\alpha}}, C\sqrt{\frac{2T^{3-\alpha}}{(3-\alpha)}} \right),$$

148

149 and

150

$$151 \quad \Theta = \max(\sqrt{2}CT^{\alpha-1}, \sqrt{2}C).$$

152

153

154 Consider the fractional wave equation involving the left Caputo fractional derivative of
 155 order $3/2 < \alpha < 2$:

$$156 \quad \begin{cases} D_C^\alpha y(x, t) - \Delta y(x, t) = f, & (x, t) \in Q, \\ y(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ y(x, 0) = 0, & x \in \Omega, \\ \frac{\partial y}{\partial t}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (11)$$

159

160 where $f \in L^2(Q)$.

162

163 **Theorem 2.7** Let $f \in L^2(Q)$. Then problem (11) has a unique solution $y \in C([0, T]; H_0^1(\Omega))$.
 164 Moreover, $\frac{\partial y}{\partial t} \in C([0, T]; L^2(\Omega))$ and there exists $C > 0$ in such a way that

165

$$166 \quad \|y\|_{C([0, T]; H_0^1(\Omega))} \leq C \sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|f\|_{L^2(Q)} \quad (12)$$

168

169 and

170

$$171 \quad \left\| \frac{\partial y}{\partial t} \right\|_{C([0, T]; L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|f\|_{L^2(Q)}. \quad (13)$$

172

173 *Proof* Below we proceed as was mentioned in [28]. □

174

175 **Corollary 2.8** Let $3/2 < \alpha < 2$ and $\phi \in L^2(Q)$. Consider the fractional wave equation:

176

$$177 \quad \begin{cases} D_C^\alpha \psi(x, t) - \Delta \psi(x, t) = \phi, & (x, t) \in Q, \\ \psi(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ \psi(x, T) = 0, & x \in \Omega, \\ \frac{\partial \psi}{\partial t}(x, T) = 0, & x \in \Omega, \end{cases} \quad (14)$$

180

181 where D_C^α is the right Caputo fractional derivative of order $1 < \alpha < 2$. Then (14) has a unique
 182 solution $\psi \in C([0, T]; H_0^1(\Omega))$. Moreover, $\frac{\partial \psi}{\partial t} \in C([0, T]; L^2(\Omega))$, and there exists $C > 0$ ful-
 183 filling

184

$$185 \quad \|\psi\|_{C([0, T]; H_0^1(\Omega))} \leq C \sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|\phi\|_{L^2(Q)} \quad (15)$$

186

187

188

189 and

190

$$191 \quad \left\| \frac{\partial \psi}{\partial t} \right\|_{C([0, T]; L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|\phi\|_{L^2(Q)}. \quad (16)$$

192

193
 194 *Proof* If we make the change of variable $t \rightarrow T - t$ in (14), then we conclude that $\hat{\psi}(t) =$
 195 $\psi(T - t)$ verifies

196

$$197 \quad \begin{cases} D_C^\alpha \hat{\psi} - \Delta \hat{\psi} = \hat{\phi} & \text{in } Q, \\ \hat{\psi} = 0 & \text{on } \Sigma, \\ \hat{\psi}(0) = 0 & \text{in } \Omega, \\ \frac{\partial \hat{\psi}}{\partial t}(0) = 0 & \text{in } \Omega, \end{cases} \quad (17)$$

200

201
 202 where $\hat{\phi}(t) = \phi(T - t)$ and D_C^α is the left Caputo fractional derivative of order $3/2 < \alpha < 2$.
 203 Because $T - t \in [0, T]$ when $t \in [0, T]$, we say that $\hat{\phi} \in L^2(Q)$ due to the fact that $\phi \in L^2(Q)$.
 204 It then suffices to use Theorem 2.7 to conclude. \square

205

206 We also need some trace results.

207

208 **Lemma 2.9** [1] *Let $f \in L^2(Q)$ and $y \in L^2(Q)$ such that $D_{RL}^\alpha y - \Delta y = f$. Then:*

209

- 210 (i) $y|_{\partial\Omega}$ and $\frac{\partial y}{\partial \nu}|_{\partial\Omega}$ exist and belong to $H^{-2}((0, T); H^{-1/2}(\partial\Omega))$ and
 211 $H^{-2}((0, T); H^{-3/2}(\partial\Omega))$ respectively.
- 212 (ii) $I^{2-\alpha}y \in C([0, T]; L^2(\Omega))$.
- 213 (iii) $\frac{\partial}{\partial t} I^{2-\alpha}y \in C([0, T]; H^{-1}(\Omega))$.

214

215 3 Existence and uniqueness of no-regret and low-regret controls

216 Below, we show the existence and the uniqueness of the no-regret control and the low-
 217 regret control problem for system (1).

218

219 **Lemma 3.1** *Let $v \in L^2(Q)$ and $g \in L^2(\Omega)$. Then we have*

220

$$221 \quad J(v, g) = J(0, g) + J(v, 0) - J(0, 0)$$

$$222 \quad + 2 \int_Q [y(0, g) - y(0, 0)][y(v, 0) - y(0, 0)] dt dx. \quad (18)$$

223

224
 225 Here J denotes the functional given by (2) and $y(v, g) = y(x, t; v, g) \in L^2(0, T; H_0^1(\Omega)) \subset L^2(Q)$
 226 is the solution of (1).

227

228 *Proof* Let us consider $y(v, 0) = y(x, t; v, 0)$, $y(0, g) = y(x, t; 0, g)$, and $y(0, 0) = y(x, t; 0, 0)$ be
 229 the solutions of

230

$$231 \quad \begin{cases} D_{RL}^\alpha y(v, 0) - \Delta y(v, 0) = v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I^{2-\alpha}y(0; v, 0) = y^0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha}y(0; v, 0) = 0 & \text{in } \Omega, \end{cases} \quad (19)$$

232

233

234

235

$$\begin{cases}
 D_{RL}^\alpha y(0, g) - \Delta y(0, g) = 0 & \text{in } Q, \\
 y = 0 & \text{on } \Sigma, \\
 I^{2-\alpha} y(0; 0, g) = y^0 & \text{in } \Omega, \\
 \frac{\partial}{\partial t} I^{2-\alpha} y(0; 0, g) = g & \text{in } \Omega,
 \end{cases} \tag{20}$$

240 and

$$\begin{cases}
 D_{RL}^\alpha y(0, 0) - \Delta y(0, 0) = 0 & \text{in } Q, \\
 y = 0 & \text{on } \Sigma, \\
 I^{2-\alpha} y(0; 0, 0) = y^0 & \text{in } \Omega, \\
 \frac{\partial}{\partial t} I^{2-\alpha} y(0; 0, 0) = 0 & \text{in } \Omega,
 \end{cases} \tag{21}$$

247 where $I^{2-\alpha} y(0; \nu, g) = \lim_{t \rightarrow 0^+} I^{2-\alpha} y(x, t; \nu, g)$ and $\frac{\partial}{\partial t} I^{2-\alpha} y(0; \nu, g) = \lim_{t \rightarrow 0^+} \frac{\partial}{\partial t} I^{2-\alpha} y(x, t; \nu, g)$.

248 Since $\nu \in L^2(Q)$, $y^0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $g \in L^2(\Omega)$, we see from Theorem 2.6 that $y(\nu, 0)$,
 249 $y(0, g)$ and $y(0, 0)$ belong to $L^2((0, T); H_0^1(\Omega))$.

250 Observing that

$$J(\nu, 0) = \int_Q [y(\nu, 0) - z_d][y(\nu, 0) - z_d] dt dx + N \| \nu \|_{L^2(Q)}^2, \tag{22a}$$

$$J(0, g) = \int_Q [y(0, g) - z_d][y(0, g) - z_d] dt dx, \tag{22b}$$

$$J(0, 0) = \int_Q [y(0, 0) - z_d][y(0, 0) - z_d] dt dx, \tag{22c}$$

259 and using the fact that

$$y(\nu, g) = y(\nu, 0) + y(0, g) - y(0, 0),$$

263 we have

$$\begin{aligned}
 J(\nu, g) &= \| y(\nu, g) - z_d \|_{L^2(Q)}^2 + N \| \nu \|_{L^2(Q)}^2 \\
 &= \| y(\nu, 0) + y(0, g) - y(0, 0) - z_d \|_{L^2(Q)}^2 + N \| \nu \|_{L^2(Q)}^2 \\
 &= J(\nu, 0) + 2 \int_Q [y(\nu, 0) - z_d][y(0, g) - y(0, 0)] dt dx \\
 &\quad + \| y(0, g) - y(0, 0) - z_d \|_{L^2(Q)}^2 \\
 &= J(\nu, 0) + 2 \int_Q [y(\nu, 0) - y(0, 0)][y(0, g) - y(0, 0)] dt dx \\
 &\quad + 2 \int_Q [y(0, 0) - z_d][y(0, g) - y(0, 0)] dt dx + \| y(0, g) - y(0, 0) - z_d \|_{L^2(Q)}^2.
 \end{aligned}$$

277 Using

$$\begin{aligned}
 \| y(0, g) - y(0, 0) - z_d \|_{L^2(Q)}^2 &= J(0, g) + J(0, 0) \\
 &\quad - 2 \int_Q [y(0, 0) - z_d][y(0, g) - y(0, 0)] dt dx - 2J(0, 0),
 \end{aligned}$$

282

283 we conclude that

284

285
$$J(v, g) = J(0, g) + J(v, 0) - J(0, 0)$$

286

287
$$+ 2 \int_{\Omega} \int_0^T [y(v, 0) - y(0, 0)][y(0, g) - y(0, 0)] dt dx. \quad \square$$

288

289

Lemma 3.2 *Let $v \in L^2(Q)$ and $g \in L^2(\Omega)$. Then we have*

290

291
$$J(v, g) = J(0, g) + J(v, 0) - J(0, 0) + 2 \int_{\Omega} g \zeta(x, 0; v) dx, \quad (23)$$

292

293 where $\zeta(v) = \zeta(x, t; v) \in C([0, T]; H_0^1(\Omega))$ be solution of

294

295
$$\begin{cases} \mathcal{D}_C^\alpha \zeta(v) - \Delta \zeta(v) = y(v, 0) - y(0, 0) & \text{in } Q, \\ \zeta = 0 & \text{on } \Sigma, \\ \zeta(x, T; v) = 0 & \text{in } \Omega, \\ \frac{\partial \zeta}{\partial t}(x, T; v) = 0 & \text{in } \Omega. \end{cases} \quad (24)$$

299

Proof Since $y(v, 0) - y(0, 0) \in L^2(Q)$, from Proposition 2.8, we know that the system (24) admits a unique solution $\zeta(v) \in C([0, T]; H_0^1(\Omega))$. Also, there exists $C > 0$ such that

300

301
$$\|\zeta(v)\|_{C([0, T]; H_0^1(\Omega))} \leq C \sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|y(v, 0) - y(0, 0)\|_{L^2(Q)} \quad (25)$$

304

and

305

306
$$\left\| \frac{\partial \zeta}{\partial t}(v) \right\|_{C([0, T]; L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|y(v, 0) - y(0, 0)\|_{L^2(Q)}. \quad (26)$$

308

309 Set $z = y(g, 0) - y(0, 0)$. Then z verifies

310

311
$$\begin{cases} D_{RL}^\alpha z - \Delta z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ I^{2-\alpha} z(0) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} z(0) = g & \text{in } \Omega. \end{cases} \quad (27)$$

315

Since $g \in L^2(\Omega)$, it follows from Theorem 2.6 that $z \in L^2([0, T]; H_0^1(\Omega))$, $I^{2-\alpha} z \in C([0, T], H_0^1(\Omega))$, and $\frac{\partial}{\partial t} I^{2-\alpha} z \in C([0, T], L^2(\Omega))$. So, if we multiply the first equation of (24) by z utilizing the fractional integration by parts provided by Lemma 2.5, we conclude

318

319
$$\begin{aligned} & \int_Q (y(v, 0) - y(0, 0)) z dt dx \\ &= \int_Q (\mathcal{D}_C^\alpha \zeta(v) - \Delta \zeta(v)) z dt dx \\ &= \int_{\Omega} \zeta(x, 0; v) \frac{\partial}{\partial t} I^{2-\alpha} z(0) dx. \end{aligned} \quad (28)$$

325

326 Thus, replacing z by $(y(0, g) - y(0, 0))$, we obtain

327

328
$$\int_Q [y(v, 0) - y(0, 0)][y(0, g) - y(0, 0)] dt dx = \int_{\Omega} \zeta(x, 0; v) g dx,$$

329

330 and (18) becomes

331

$$332 \quad J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2 \int_{\Omega} \zeta(x, 0; v)g \, dx. \quad \square$$

333

334 Now we consider the no-regret control problem:

335

$$336 \quad \inf_{v \in L^2(Q)} \sup_{g \in L^2(\Omega)} (J(v, g) - J(0, g)). \quad (29)$$

337

338 From (23), this problem is equivalent to the following one:

339

$$340 \quad \inf_{v \in L^2(Q)} \sup_{g \in L^2(\Omega)} \left[J(v, 0) - J(0, 0) + 2 \int_{\Omega} \zeta(x, 0; v)g \, dx \right]. \quad (30)$$

341

342 As the space $L^2(\Omega)$ is a vector space, the no-regret control exists only if

343

$$344 \quad \sup_{g \in L^2(\Omega)} \left(\int_{\Omega} \zeta(x, 0; v)g \, dx \right) = 0. \quad (31)$$

345

346 This implies that the no-regret control belongs to \mathcal{U} defined by

347

$$348 \quad \mathcal{U} = \left\{ v \in L^2(Q) \mid \left(\int_{\Omega} \zeta(x, 0; v)g \, dx \right) = 0, \forall g \in L^2(\Omega) \right\}.$$

349

350 As a result such control should be carefully investigated. So, we proceed by penalization.

351

352 For all $\gamma > 0$, we discuss the low-regret control problem:

353

$$354 \quad \inf_{v \in L^2(Q)} \sup_{g \in L^2(\Omega)} (J(v, g) - J(0, g) - \gamma \|g\|_{L^2(\Omega)}^2). \quad (32)$$

355

356 According to (23), the problem (32) is equivalent to the following problem:

357

$$358 \quad \inf_{v \in L^2(Q)} \left[J(v, 0) - J(0, 0) + 2 \sup_{g \in L^2(\Omega)} \left(\int_{\Omega} \zeta(x, 0; v)g \, dx - \frac{\gamma}{2} \|g\|_{L^2(\Omega)}^2 \right) \right].$$

359

360 Using the Legendre-Fenchel transform, we conclude

361

$$362 \quad 2\gamma \sup_{g \in L^2(\Omega)} \left(\int_{\Omega} \frac{1}{\gamma} \zeta(x, 0; v)g \, dx - \frac{1}{\gamma} \|g\|_{L^2(\Omega)}^2 \right) = \frac{1}{\gamma} \|\zeta(\cdot, 0; v)\|_{L^2(\Omega)}^2,$$

363

364 and problem (32) becomes: For any $\gamma > 0$, find $u^\gamma \in L^2(Q)$ such that

365

$$366 \quad J_\gamma(u^\gamma) = \inf_{v \in L^2(Q)} J_\gamma(v), \quad (33)$$

367

368 where

369

$$370 \quad J_\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|\zeta(\cdot, 0; v)\|_{L^2(\Omega)}^2. \quad (34)$$

371

372 **Proposition 3.3** *Let $\gamma > 0$. Then (33) has a unique solution u^γ , called a low-regret control.*

373

374

377 *Proof* We recall that

378
 379
$$J_\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|\zeta(\cdot, 0; v)\|_{L^2(\Omega)}^2 \geq -J(0, 0).$$

380
 381 Thus, we can say that $\inf_{v \in L^2(Q)} J_\gamma$ exists. Let $(v_n) \in L^2(Q)$ be a minimizing sequence such
 382 that

383
 384
$$\lim_{n \rightarrow +\infty} J_\gamma(v_n) = \inf_{v \in L^2(Q)} J_\gamma(v). \tag{35}$$

385
 386 Then $y_n = y(x, t; v_n, 0)$ is a solution of (19) and y_n satisfies

387
 388
$$D_{RL}^\alpha y_n(x, t) - \Delta y_n(x, t) = v_n(x, t) \quad \text{in } Q, \tag{36a}$$

389
$$y_n(x, t) = 0 \quad \text{on } \Sigma, \tag{36b}$$

390
 391
$$I^{2-\alpha} y_n(x, 0) = y^0 \quad \text{in } \Omega, \tag{36c}$$

392
 393
$$\frac{\partial}{\partial t} I^{2-\alpha} y_n(x, 0) = 0 \quad \text{in } \Omega. \tag{36d}$$

394
 395 It follows from (35) that there exists $C(\gamma) > 0$ independent of n such that

396
 397
$$0 \leq J(v_n, 0) + \frac{1}{\gamma} \|\zeta(\cdot, 0; v_n)\|_{L^2(\Omega)}^2 \leq C(\gamma) + J(0, 0) = C(\gamma).$$

398
 399 From the definition of $J(v_n, 0)$ we obtain

400
 401
$$\|v_n\|_{L^2(Q)} \leq C(\gamma), \tag{37a}$$

402
 403
$$\|\zeta(\cdot, 0; v_n)\|_{L^2(\Omega)} \leq \sqrt{\gamma} C(\gamma). \tag{37b}$$

404
 405 Therefore, from Theorem 2.6, we know that there exists a constant C independent of n
 406 such that

407
 408
$$\|y_n\|_{L^2((0, T); H_0^1(\Omega))} \leq C(\gamma), \tag{38a}$$

409
$$\|I^{2-\alpha} y_n\|_{L^2((0, T); H_0^1(\Omega))} \leq C(\gamma), \tag{38b}$$

410
 411
$$\left\| \frac{\partial}{\partial t} I^{2-\alpha} y_n \right\|_{L^2((0, T); L^2(\Omega))} \leq C(\gamma). \tag{38c}$$

412
 413 Moreover, from (36a) and (37a), we have

414
 415
$$\|D_{RL}^\alpha y_n - \Delta y_n\|_{L^2(Q)} \leq C(\gamma). \tag{39}$$

416
 417 Consequently, there exist $u^\gamma \in L^2(Q)$, $y^\gamma \in L^2((0, T); H_0^1(\Omega))$, $\delta \in L^2(Q)$, $\eta \in L^2((0, T);$
 418 $H_0^1(\Omega))$, $\theta \in L^2((0, T); L^2(\Omega))$ and we can extract subsequences of (v_n) and (y_n) (still called
 419 (v_n) and (y_n)) such that:

420
 421
$$v_n \rightharpoonup u^\gamma \quad \text{weakly in } L^2(Q), \tag{40a}$$

422
 423
$$D_{RL}^\alpha y_n - \Delta y_n \rightharpoonup \delta \quad \text{weakly in } L^2(Q), \tag{40b}$$

424 $y_n \rightharpoonup y^\gamma$ weakly in $L^2((0, T); H_0^1(\Omega))$, (40c)

425 $I^{2-\alpha} y_n \rightharpoonup \eta$ weakly in $L^2([0, T], H_0^1(\Omega))$, (40d)

426 $\frac{\partial}{\partial t} I^{2-\alpha} y_n \rightharpoonup \theta$ weakly in $L^2((0, T); L^2(\Omega))$. (40e)

429 The remaining part of the proof contains three steps.

430 *Step 1:* We show that (u^γ, y^γ) fulfills (19).

431 Set $\mathbb{D}(Q)$, the set of C^∞ function on Q with compact support and denote by $\mathbb{D}'(Q)$ its
 432 dual. Multiplying (36a) by $\varphi \in \mathbb{D}(Q)$ and using Lemma 2.5, (40a), and (40c), we prove as
 433 in [1] that

435 $D_{RL}^\alpha y_n - \Delta y_n \rightharpoonup D_{RL}^\alpha y^\gamma - \Delta y^\gamma$ weakly in $\mathbb{D}'(Q)$.

437 From (40b) and the uniqueness of the limit, we conclude

439 $D_{RL}^\alpha y^\gamma - \Delta y^\gamma = \delta \in L^2(Q)$. (41)

441 Hence,

443 $D_{RL}^\alpha y_n - \Delta y_n \rightharpoonup D_{RL}^\alpha y^\gamma - \Delta y^\gamma$ weakly in $L^2(Q)$. (42)

445 Then passing to the limit in (36a) and using (41) and (40a), we obtain

447 $D_{RL}^\alpha y^\gamma(x, t) - \Delta y^\gamma(x, t) = u^\gamma(x, t), \quad (x, t) \in Q$. (43)

449 On the other hand, we have

451
$$\int_Q I^{2-\alpha} y_n(x, t) \varphi(x, t) dt dx$$
 452
$$= \int_\Omega \int_0^T y_n(x, s) \left(\frac{1}{\Gamma(2-\alpha)} \int_s^T (t-s)^{1-\alpha} \varphi(x, t) dt \right) ds dx, \quad \forall \varphi \in \mathbb{D}(Q).$$

456 Thus using (40c) and (40d), while passing to the limit, we get

458
$$\int_Q \eta \varphi(x, t) dt dx = \int_\Omega \int_0^T y^\gamma(x, s) \left(\frac{1}{\Gamma(2-\alpha)} \int_s^T (t-s)^{1-\alpha} \varphi(x, t) dt \right) ds dx$$
 459
$$= \int_Q I^{2-\alpha} y^\gamma(x, t) \varphi(x, t) dt dx, \quad \forall \varphi \in \mathbb{D}(Q).$$

463 This implies that

465 $I^{2-\alpha} y^\gamma(x, t) = \eta$ in Q .

467 Thus, (40d) becomes

469 $I^{2-\alpha} y_n \rightharpoonup I^{2-\alpha} y^\gamma$ weakly in $L^2([0, T], H_0^1(\Omega))$. (44)

471 In view of (44), we have

472
 473
$$\frac{\partial}{\partial t} I^{2-\alpha} y_n \rightharpoonup \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma \quad \text{weakly in } \mathbb{D}'(Q),$$

 474

475 and as we have (40e), we obtain

476
 477
$$\frac{\partial}{\partial t} I^{2-\alpha} y_n \rightharpoonup \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma = \theta \quad \text{weakly in } L^2(Q). \tag{45}$$

 478

479 Since $y^\gamma \in L^2(Q)$ and $D_{\text{RL}}^\alpha y^\gamma - \Delta y^\gamma \in L^2(Q)$, in view of Lemma 2.9, we know that $y^\gamma|_{\partial\Omega}$ and
 480 $\frac{\partial y^\gamma}{\partial \nu}|_{\partial\Omega}$ exist and belong to $H^{-2}((0, T); H^{-1/2}(\partial\Omega))$ and $H^{-2}((0, T); H^{-3/2}(\partial\Omega))$, respectively.
 481 Moreover, we have $I^{2-\alpha} y^\gamma \in C([0, T]; L^2(\Omega))$ and $\frac{\partial}{\partial t} I^{2-\alpha} y^\gamma \in C([0, T]; H^{-1}(\Omega))$.

482 Now multiplying (36a) by a function $\varphi \in C^\infty(\bar{Q})$ such that $\varphi|_{\partial\Omega} = 0$ and $\varphi(x, T) =$
 483 $\frac{\partial \varphi}{\partial t}(x, T) = 0$ in Ω , and integrating by parts over Q , we obtain

484
 485
$$\int_Q v_n(x, t) \varphi(x, t) \, dx \, dt = \int_Q (D_{\text{RL}}^\alpha y_n(x, t) - \Delta y_n(x, t)) \varphi(x, t) \, dx \, dt$$

 486
 487
$$= \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) \, dx$$

 488
 489
$$+ \int_Q y_n(x, t) (D_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) \, dx \, dt$$

 490

491 because we have (36c) and (36d). Thus, using (40a) and (40c) while passing to the limit
 492 in the latter identity, we get

493
 494
$$\int_Q u^\gamma(x, t) \varphi(x, t) \, dx \, dt = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) \, dx$$

 495
 496
$$+ \int_Q y^\gamma(x, t) (D_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) \, dx \, dt,$$

 497

498
 499 $\forall \varphi \in C^\infty(\bar{Q})$ such that $\varphi|_{\partial\Omega} = 0$, $\varphi(T) = \frac{\partial \varphi}{\partial t}(T) = 0$ in Ω , which, according to Lemma 2.5,
 500 can be rewritten as

501
 502
$$\int_Q u^\gamma(x, t) \varphi(x, t) \, dx \, dt = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) \, dx$$

 503
 504
$$+ \int_Q (D_{\text{RL}}^\alpha y^\gamma(x, t) - \Delta y^\gamma(x, t)) \varphi(x, t) \, dx \, dt$$

 505
 506
$$+ \left\langle \varphi(x, 0), \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(x, 0) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)}$$

 507
 508
$$- \int_\Omega \frac{\partial \varphi}{\partial t}(x, 0), I^{2-\alpha} y^\gamma(x, 0) \, dx$$

 509
 510
$$- \left\langle y^\gamma(\sigma, t), \frac{\partial \varphi}{\partial \nu}(\sigma, t) \right\rangle_{H^{-2}((0, T); H^{-1/2}(\partial\Omega)), H_0^2((0, T); H^{1/2}(\partial\Omega))},$$

 511
 512

513 $\forall \varphi \in C^\infty(\bar{Q})$ such that $\varphi|_{\partial\Omega} = 0$, $\varphi(T) = \frac{\partial \varphi}{\partial t}(T) = 0$ in Ω .

514 Using (43), we obtain

515
 516
$$0 = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) \, dx$$

 517

$$\begin{aligned}
 & + \left\langle \varphi(x, 0), \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(x, 0) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
 & - \int_{\Omega} \frac{\partial \varphi}{\partial t}(x, 0), I^{2-\alpha} y^\gamma(x, 0) dx \\
 & - \left\langle y^\gamma(\sigma, t), \frac{\partial \varphi}{\partial v}(\sigma, t) \right\rangle_{H^{-2}((0, T); H^{-1/2}(\partial\Omega)), H_0^2((0, T); H^{1/2}(\partial\Omega))}, \tag{46}
 \end{aligned}$$

$\forall \varphi \in C^\infty(\overline{Q})$ such that $\varphi|_{\partial\Omega} = 0$, $\varphi(T) = \frac{\partial \varphi}{\partial t}(T) = 0$ in Ω .

Choosing successively in (46) φ such that $\varphi(x, 0) = \frac{\partial \varphi}{\partial t}(x, 0) = 0$ and $\varphi(x, 0) = 0$, we deduce that

$$y^\gamma(x, t) = 0, \quad (x, t) \in \Sigma, \tag{47}$$

$$I^{2-\alpha} y^\gamma(x, 0) = y^0, \quad x \in \Omega, \tag{48}$$

and then

$$\frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(x, 0) = 0, \quad x \in \Omega. \tag{49}$$

In view of (43), (47), (48), and (49), we see that $y^\gamma = y^\gamma(x, t; u^\gamma, 0)$ is a solution of (19).

Step 2: We show $\zeta_n = \zeta(x, t; v_n)$ converges to $\zeta^\gamma = \zeta(x, t; u^\gamma)$.

In view of (24), $\zeta_n = \zeta(x, t; v_n)$ verifies

$$\begin{cases}
 D_C^\alpha \zeta_n - \Delta \zeta_n = y(v_n, 0) - y(0, 0) & \text{in } Q, \\
 \zeta_n = 0 & \text{on } \Sigma, \\
 \zeta_n(T) = 0 & \text{in } \Omega, \\
 \frac{\partial}{\partial t} \zeta_n(T) = 0 & \text{in } \Omega.
 \end{cases} \tag{50}$$

Set $z_n = y(v_n, 0) - y(0, 0)$. In view of (19) and (21), z_n verifies

$$\begin{cases}
 D_{RL}^\alpha z_n - \Delta z_n = v_n & \text{in } Q, \\
 z_n = 0 & \text{on } \Sigma, \\
 I^{2-\alpha} z_n(0) = 0 & \text{in } \Omega, \\
 \frac{\partial}{\partial t} I^{2-\alpha} z_n(0) = 0 & \text{in } \Omega.
 \end{cases}$$

It follows, from Theorem 2.6 and (37a), that

$$\|z_n\|_{L^2((0, T); H_0^1(\Omega))} = \|y_n(v_n, 0) - y(0, 0)\|_{L^2((0, T); H_0^1(\Omega))} \leq C(\gamma).$$

Hence, from Corollary 2.8, we deduce that

$$\|\zeta_n\|_{C([0, T]; H_0^1(\Omega))} \leq C(\gamma), \tag{51}$$

$$\left\| \frac{\partial}{\partial t} \zeta_n \right\|_{C([0, T]; L^2(\Omega))} \leq C(\gamma). \tag{52}$$

Since the embedding of $C([0, T]; H_0^1(\Omega))$ into $L^2((0, T); H_0^1(\Omega))$ and the embedding of $C([0, T]; L^2(\Omega))$ into $L^2(Q)$ are continuous, we can conclude that there exists $\zeta^\gamma \in$

565 $L^2((0, T); H_0^1(\Omega))$ such that

566

567
$$\zeta_n \rightharpoonup \zeta^\gamma \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)). \tag{53}$$

568

569 Therefore,

570

571
$$\frac{\partial}{\partial t} \zeta_n \rightharpoonup \frac{\partial}{\partial t} \zeta^\gamma \quad \text{weakly in } \mathbb{D}'(Q)$$

572

573 and, consequently,

574

575
$$\frac{\partial}{\partial t} \zeta_n \rightharpoonup \frac{\partial}{\partial t} \zeta^\gamma \quad \text{weakly in } L^2(Q). \tag{54}$$

576

577 Since $\zeta^\gamma \in L^2((0, T); H_0^1(\Omega))$ and $\frac{\partial}{\partial t} \zeta^\gamma \in L^2(Q)$, we see that $\zeta^\gamma(0)$ and $\zeta^\gamma(T)$ belongs to $L^2(\Omega)$. In view of (50)₃, we have

578

580
$$\zeta^\gamma(T) = 0 \quad \text{in } \Omega \tag{55}$$

581

582

and in view of (50)₄ and (52), we set

583

584
$$\frac{\partial}{\partial t} \zeta^\gamma(T) = 0 \quad \text{in } \Omega. \tag{56}$$

585

587 From (37b), we deduce that there exists $\rho \in L^2(\Omega)$ such that

588

589
$$\zeta(\cdot, 0; v_n) \rightharpoonup \rho \quad \text{weakly in } L^2(\Omega). \tag{57}$$

590

591 Multiplying the first equation of (50) by $\phi \in \mathbb{D}(Q)$ then, using the integration by parts given by Lemma 2.5, we obtain

592

593
$$\int_Q (y(x, t; v_n, 0) - y(x, t; 0, 0)) \phi(x, t) dt dx$$

594

595
$$= \int_Q [D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t)] \zeta_n(x, t) dt dx.$$

596

598 Hence, using (40c) and (53) while passing to the limit in the latter identity, we have

599

600
$$\int_Q (y(x, t; u^\gamma, 0) - y(x, t; 0, 0)) \phi(x, t) dt dx$$

601

602
$$= \int_Q (D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t)) \zeta^\gamma(x, t) dt dx, \quad \forall \phi \in \mathbb{D}(Q), \tag{58}$$

603

605 which by using again Lemma 2.5 gives

606

607
$$\int_Q (y(x, t; u^\gamma, 0) - y(x, t; 0, 0)) \phi(x, t) dt dx$$

608

609
$$= \int_Q (D_C^\alpha \zeta^\gamma(x, t) - \Delta \zeta^\gamma(x, t)) \phi(x, t) dt dx, \quad \forall \phi \in \mathbb{D}(Q).$$

610

611

612 This implies that

613
 614
$$\mathcal{D}_C^\alpha \zeta^\gamma - \Delta \zeta^\gamma = y(u^\gamma, 0) - y(0, 0) \quad \text{in } Q. \tag{59}$$

615
 616 Now, if we multiply the first equation of (50) by $\phi \in C^\infty(\overline{Q})$ with $\phi|_{\partial\Omega} = 0$ and $I^{2-\alpha}\phi(0) = 0$
 617 in Ω and integrating by parts over Q , we obtain

618
 619
$$\int_Q (y_n(x, t) - y(x, t; 0, 0))\phi(x, t) dt dx$$

 620
 621
$$= \int_Q (\mathcal{D}_C^\alpha \zeta_n(x, t) - \Delta \zeta_n(x, t))\phi(x, t) dt dx$$

 622
 623
$$= \int_Q (D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t))\zeta_n(x, t) dt dx + \int_\Omega \zeta(x, 0, v_n) \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx.$$

 624

625 Using (40c), (53), and (57) while passing the latter identity to the limit, we obtain

626
 627
$$\int_Q (y^\gamma(x, t) - y(x, t; 0, 0))\phi(x, t) dt dx$$

 628
 629
$$= \int_Q (D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t))\zeta^\gamma(x, t) dt dx + \int_\Omega \rho \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx,$$

 630
 631
$$\forall \phi \in C^\infty(\overline{Q}) \text{ such that } \phi|_{\partial\Omega} = 0, I^{2-\alpha}\phi(0) = 0 \text{ in } \Omega, \tag{60}$$

 632

633 which by using again Lemma 2.5, (55), (56), and (59) gives

634
 635
$$- \int_\Sigma \frac{\partial}{\partial \nu} \phi(\sigma, t) \zeta^\gamma(\sigma, t) d\sigma dt + \int_\Omega \rho(x) \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx$$

 636
 637
$$= \int_\Omega \zeta^\gamma(0) \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx$$

 638
 639
$$\forall \phi \in C^\infty(\overline{Q}) \text{ such that } \phi|_{\partial\Omega} = 0, I^{2-\alpha}\phi(0) = 0 \text{ in } \Omega.$$

 640

641 Hence, choosing $\phi \in C^\infty(\overline{Q})$, such that $\phi|_{\partial\Omega} = 0, I^{2-\alpha}\phi(0) = \frac{\partial}{\partial t} I^{2-\alpha}\phi(0) = 0$, we get

642
 643
$$\zeta^\gamma = 0 \quad \text{on } \Sigma, \tag{61}$$

644
 645 and then

646
 647
$$\zeta^\gamma(0) = \rho \quad \text{in } \Omega. \tag{62}$$

648
 649 In view of (55), (56), (59), and (61), we see that $\zeta^\gamma = \zeta(u^\gamma)$ is a solution of

650
 651
$$\begin{cases} \mathcal{D}_C^\alpha \zeta^\gamma - \Delta \zeta^\gamma = y(u^\gamma, 0) - y(0, 0) & \text{in } Q, \\ \zeta^\gamma = 0 & \text{on } \Sigma, \\ \zeta^\gamma(T) = 0 & \text{in } \Omega \\ \frac{\partial \zeta^\gamma}{\partial t}(T) = 0 & \text{in } \Omega. \end{cases} \tag{63}$$

 652
 653
 654

655 Moreover, using (62), equation (57) becomes

656
 657
$$\zeta(\cdot, 0; v_n) \rightharpoonup \zeta^\gamma(0) = \zeta(\cdot, 0; u^\gamma) \quad \text{weakly in } L^2(\Omega). \tag{64}$$

 658

659 *Step 3:* The function $v \rightarrow J_\gamma(v)$ being lower semi-continuous, we have

660

$$661 \quad J_\gamma(u^\gamma) \leq \liminf_{n \rightarrow \infty} J_\gamma(v_n),$$

662

663

664 which in view of (35) implies that

665

$$666 \quad J_\gamma(u^\gamma) = \inf_{v \in L^2(Q)} J_\gamma(v).$$

667

668 The uniqueness of u^γ comes from the fact that the functional J_γ is strictly convex. \square

669

670 **Theorem 3.4** *For any $\gamma > 0$, let u^γ be the low-regret control. Then there exist $q^\gamma \in$
 671 $L^2((0, T); H_0^1(\Omega))$ and $p^\gamma \in C([0, T]; H_0^1(\Omega))$ such that $(u^\gamma, y^\gamma = y^\gamma(u^\gamma, 0), q^\gamma, p^\gamma)$ satisfies
 672 the following optimality system:
 673*

$$674 \quad \begin{cases} D_{RL}^\alpha y^\gamma - \Delta y^\gamma = u^\gamma & \text{in } Q, \\ y^\gamma = 0 & \text{on } \Sigma, \\ I^{2-\alpha} y^\gamma(0) = y^0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(0) = 0 & \text{in } \Omega, \end{cases} \quad (65)$$

$$679 \quad \begin{cases} D_{RL}^\alpha q^\gamma - \Delta q^\gamma = 0 & \text{in } Q, \\ q^\gamma = 0 & \text{on } \Sigma, \\ I^{2-\alpha} q^\gamma(0) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} q^\gamma(0) = \frac{1}{\sqrt{\gamma}} \zeta(0; u^\gamma) & \text{in } \Omega, \end{cases} \quad (66)$$

$$684 \quad \begin{cases} D_C^\alpha p^\gamma - \Delta p^\gamma = y^\gamma - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma & \text{in } Q, \\ p^\gamma = 0 & \text{on } \Sigma, \\ p^\gamma(T) = 0 & \text{in } \Omega, \\ \frac{\partial p^\gamma}{\partial t}(T) = 0 & \text{in } \Omega, \end{cases} \quad (67)$$

688

689 and

690

$$691 \quad Nu^\gamma + p^\gamma = 0 \quad \text{in } Q. \quad (68)$$

692

693 *Proof* Equations (43), (47), (48), and (49) give (65). To characterize the low-regret control
 694 u^γ , we use the Euler-Lagrange optimality conditions:
 695

$$696 \quad \frac{d}{dk} J_\gamma(u^\gamma + k(v - u^\gamma)) \Big|_{k=0} = 0, \quad \forall v \in L^2(Q). \quad (69)$$

698

699 After some calculations, we obtain

700

$$701 \quad \int_Q (y(u^\gamma, 0) - z_d)(y(v, 0) - y(u^\gamma, 0)) dt dx + \int_Q Nu^\gamma (v - u^\gamma) dt dx \\ 702 \quad + \frac{1}{\gamma} \int_\Omega (\zeta(x, 0; u^\gamma), \zeta(x, 0; v - u^\gamma)) dx = 0, \quad \forall v \in L^2(Q), \quad (70)$$

705

706 where from (24), $\zeta(v - u^\gamma) = \zeta(x, t; v - u^\gamma) \in C([0, T]; H_0^1(\Omega))$ is a solution of

707
 708
$$\begin{cases} D^\alpha \zeta(v - u^\gamma) - \Delta \zeta(v - u^\gamma) = y(v, 0) - y^\gamma(u^\gamma, 0) & \text{in } Q, \\ \zeta(v - u^\gamma) = 0 & \text{on } \Sigma, \\ \zeta(T; v - u^\gamma) = 0 & \text{in } \Omega, \\ \frac{\partial \zeta}{\partial t}(T; v - u^\gamma) = 0 & \text{in } \Omega. \end{cases} \quad (71)$$

712
 713 Let $z(v - u^\gamma) = y(x, t; v, 0) - y^\gamma(x, t; u^\gamma, 0)$ be the state associated to $(v - u^\gamma) \in L^2(Q)$. Then
 714 in view of (19), $z = z(v - u^\gamma) \in L^2((0, T); H_0^1(\Omega))$ is a solution of

715
 716
$$\begin{cases} D_{RL}^\alpha z - \Delta z = v - u^\gamma & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ I^{2-\alpha} z(0) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} z(0) = 0 & \text{in } \Omega. \end{cases} \quad (72)$$

720
 721 To interpret (70), we introduce $q^\gamma = q^\gamma(u^\gamma, 0)$ as a solution of equation (66). As
 722 $\frac{1}{\sqrt{\gamma}} \zeta(\cdot, 0; u^\gamma) \in L^2(\Omega)$, according to Theorem 2.6, q^γ is unique and belongs to $L^2((0, T);$
 723 $H_0^1(\Omega))$. Moreover,

724
 725
$$\|q^\gamma\|_{L^2((0, T); H_0^1(\Omega))} \leq \frac{C}{\sqrt{\gamma}} \|\zeta(0; u^\gamma)\|_{L^2(\Omega)}, \quad (73)$$

726
 727 where $C > 0$ is a positive constant independent of γ .

728 Multiplying the first equation of (71) by $\frac{1}{\sqrt{\gamma}} q^\gamma$ and using Lemma 2.5, we obtain

729
 730
$$\int_\Omega \frac{1}{\gamma} \zeta(x, 0; v - u^\gamma) \zeta(x, 0; u^\gamma) dx = \int_Q y(v, 0) - y(u^\gamma, 0) \frac{1}{\sqrt{\gamma}} q^\gamma dt dx,$$

731
 732 which combining with (70) gives

733
 734
$$\begin{aligned} & \int_Q [y(v, 0) - y(u^\gamma, 0)] \left[\left(y(u^\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma \right) \right] dt dx \\ & + \int_Q Nu^\gamma (v - u^\gamma) dt dx = 0, \quad \forall v \in L^2(Q). \end{aligned} \quad (74)$$

735
 736 Now, let p^γ verify (67). Then, in view of Corollary 2.8, $p^\gamma \in C([0, T]; H_0^1(\Omega))$, and $\frac{\partial}{\partial t} p^\gamma \in$
 741 $C([0, T]; L^2(\Omega))$ since $y^\gamma - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma \in L^2(Q)$.

742 Thus, multiplying the first equation of (72) by p^γ , a solution of (67), then, utilizing the
 743 fractional integration by parts provided by Lemma 2.5, we conclude

744
 745
$$\int_Q z(v - u^\gamma) (D_C^\alpha p^\gamma - \Delta p^\gamma) dx dt = \int_Q (v - u^\gamma) p^\gamma dx dt.$$

746
 747 Replacing in the latter identity $z(v - u^\gamma)$ by $y(x, t; v, 0) - y^\gamma(x, t; u^\gamma, 0)$, which is a solution
 748 of (72), we obtain

749
 750
$$\int_Q (v - u^\gamma) p^\gamma dx dt$$

751
 752

$$= \int_Q [y(x, t; v, 0) - y^\gamma(x, t; u^\gamma, 0)] \left[\left(y^\gamma(u^\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma \right) \right] dt dx,$$

which combining with (74) gives

$$\int_Q (Nu^\gamma + p^\gamma)(v - u^\gamma) dx dt = 0, \quad \forall v \in L^2(Q).$$

Consequently $Nu^\gamma + p^\gamma = 0$ in Q . □

Proposition 3.5 *For any $\gamma > 0$, let u^γ be the low-regret control. Then u^γ converges to u , a solution of the no-regret problem (30).*

Proof As u^γ is a solution of (33), we have

$$J_\gamma(u^\gamma) \leq J_\gamma(0) = 0,$$

because in view of (24), $\zeta(0) = \zeta(x, t; 0) = 0$ in Q . It then follows from the definition of J_γ given by (34) that

$$\begin{aligned} & \|y(u^\gamma, 0) - z_d\|_{L^2(Q)}^2 + N \|u^\gamma\|_{L^2(Q)}^2 + \frac{1}{\gamma} \|\zeta(\cdot, 0; u^\gamma)\|_{L^2(\Omega)}^2 \\ & \leq J(0, 0) = \|y(0, 0) - z_d\|_{L^2(Q)}^2. \end{aligned}$$

Therefore, we deduce that

$$\|y(u^\gamma, 0)\|_{L^2(Q)} \leq \|y(0, 0) - z_d\|_{L^2(Q)}, \tag{75a}$$

$$\|u^\gamma\|_{L^2(Q)} \leq \frac{1}{N} \|y(0, 0) - z_d\|_{L^2(Q)}, \tag{75b}$$

$$\|\zeta(\cdot, 0; u^\gamma)\|_{L^2(\Omega)} \leq \sqrt{\gamma} \|y(0, 0) - z_d\|_{L^2(Q)}. \tag{75c}$$

Hence from (75b) and (65)₁, we have

$$\|D_{RL}^\alpha y(u^\gamma, 0) - \Delta y(u^\gamma, 0)\|_{L^2(Q)} \leq \frac{1}{N} \|y(0, 0) - z_d\|_{L^2(Q)}. \tag{76}$$

Since $y(u^\gamma, 0)$ is solution of (65), we see from Theorem 2.6 that there exists a constant C independent of γ such that

$$\|y(u^\gamma, 0)\|_{L^2((0, T); H_0^1(\Omega))} \leq \frac{C}{N} \|y(0, 0) - z_d\|_{L^2(Q)}. \tag{77}$$

Thus there exist $u \in L^2(Q)$, $y \in L^2((0, T); H_0^1(\Omega))$, $\delta \in L^2(Q)$, and subsequences extracted of (u^γ) and (y^γ) (still called (u^γ) and (y^γ)) such that

$$u^\gamma \rightharpoonup u \quad \text{weakly in } L^2(Q), \tag{78a}$$

$$y^\gamma \rightharpoonup y \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)), \tag{78b}$$

$$D_{RL}^\alpha y^\gamma - \Delta y^\gamma \rightharpoonup \delta \quad \text{weakly in } L^2(Q). \tag{78c}$$

800 If we proceed as in pp.10 to 14, using (78a)-(78c), we show that $y = y(x, t; u, 0)$ is such that

801

$$\begin{cases}
 802 & D_{RL}^\alpha y - \Delta y = u & \text{in } Q, \\
 803 & y = 0 & \text{on } \Sigma, \\
 804 & I^{2-\alpha} y(x, 0) = y^0 & \text{in } \Omega, \\
 805 & \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0) = 0 & \text{in } \Omega,
 \end{cases} \tag{79}$$

806
 807 and $\zeta = \zeta(x, t; u) \in \mathcal{C}([0, T]; H_0^1(\Omega))$ is a solution of

808

$$\begin{cases}
 809 & \mathcal{D}^\alpha \zeta - \Delta \zeta = y(u, 0) - y(0, 0) & \text{in } Q, \\
 810 & \zeta = 0 & \text{on } \Sigma, \\
 811 & \zeta(T) = 0 & \text{in } \Omega, \\
 812 & \frac{\partial \zeta}{\partial t}(T) = 0 & \text{in } \Omega.
 \end{cases} \tag{80}$$

813 Moreover, in view of (75c), we have

814

$$\zeta(\cdot, 0; u^\gamma) \rightarrow \zeta(\cdot, 0; u) = 0 \quad \text{strongly in } L^2(\Omega). \tag{81}$$

815
816

817 Consequently, $\int_\Omega g \zeta(x, 0; u) dx = 0$.

818 This implies that u is solution of the no-regret control problem (30). □

819

820 **Theorem 3.6** *Let us consider $u = \lim_{\gamma \rightarrow 0} u^\gamma$ be the no-regret control corresponding to the*
 821 *state $y(u, 0)$. Then there exist $q \in L^2((0, T); H_0^1(\Omega))$ and $p \in \mathcal{C}([0, T]; H_0^1(\Omega))$ in such a way*
 822 *that $(u, y = y(u, 0), q, p)$ fulfills the following optimality system:*

823

$$\begin{cases}
 824 & D_{RL}^\alpha y - \Delta y = u & \text{in } Q, \\
 825 & y = 0 & \text{on } \Sigma, \\
 826 & I^{2-\alpha} y(0) = y^0 & \text{in } \Omega, \\
 827 & \frac{\partial}{\partial t} I^{2-\alpha} y(0) = 0 & \text{in } \Omega,
 \end{cases} \tag{82}$$

828

$$\begin{cases}
 829 & D_{RL}^\alpha q - \Delta q = 0 & \text{in } Q, \\
 830 & q = 0 & \text{on } \Sigma, \\
 831 & I^{2-\alpha} q(0) = 0 & \text{in } \Omega, \\
 832 & \frac{\partial}{\partial t} I^{2-\alpha} q(0) = \tau_1 & \text{in } \Omega,
 \end{cases} \tag{83}$$

833

$$\begin{cases}
 834 & \mathcal{D}_C^\alpha p - \Delta p = y(u, 0) - z_d + \tau_2 & \text{in } Q, \\
 835 & p = 0 & \text{on } \Sigma, \\
 836 & p(T) = 0 & \text{in } \Omega, \\
 & \frac{\partial p}{\partial t}(T) = 0 & \text{in } \Omega,
 \end{cases} \tag{84}$$

837
 838 and

839

$$Nu + p = 0 \quad \text{in } Q. \tag{85}$$

840

841 *Proof* We have (82) (see system (79)).

842 From (75c), we get

843

$$\left\| \frac{1}{\sqrt{\gamma}} \zeta(0; u^\gamma) \right\|_{L^2(\Omega)} \leq \|y(0, 0) - z_d\|_{L^2(Q)}.$$

844
845
846

847 Consequently, equation (73) becomes

848

$$849 \quad \|q^\gamma\|_{L^2((0,T);H_0^1(\Omega))} \leq C \|y(0,0) - z_d\|_{L^2(Q)}. \quad (86)$$

850

851 Thus, there exist $\tau_1 \in L^2(\Omega)$ and $q \in L^2(0, T; H_0^1(\Omega))$ such that

852

$$853 \quad \frac{1}{\sqrt{\gamma}} \zeta(\cdot, 0; u^\gamma) \rightharpoonup \tau_1 \quad \text{weakly in } L^2(\Omega), \quad (87)$$

854

$$855 \quad q^\gamma \rightharpoonup q \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)). \quad (88)$$

856

857 Using (87) and (88) while passing to the limit in (66), we show as for the convergence of
 858 $y_n = y(v_n, 0)$ (see pp.10 to 12) that q satisfies (83).

859 From (68) and (75b), we have

860

$$861 \quad \|p^\gamma\|_{L^2(Q)} \leq \|y(0,0) - z_d\|_{L^2(Q)}.$$

862

863 Therefore there exists $p \in L^2(Q)$ such that

864

$$865 \quad p^\gamma \rightharpoonup p \quad \text{weakly in } L^2(Q). \quad (89)$$

866

867 In view of (67) and (75a), we know that there exist $\tau_2 \in L^2(Q)$ such that

868

$$869 \quad \frac{1}{\sqrt{\gamma}} q^\gamma \rightharpoonup \tau_2 \quad \text{weakly in } L^2(Q). \quad (90)$$

870

871 Then we prove as for the convergence of $\zeta_n = \zeta(x, t; v_n)$ (see pp.12 to 14) that p is solution
 872 of (84). Using (75b) and (89) while passing to the limit in (68), we conclude (85). \square

873

874 4 Conclusions

875

876 We study an optimal control problem associated to a fractional wave equation involving
 877 Riemann-Liouville fractional derivative and with incomplete data. Actually, the initial con-
 878 dition is missing. In order to solve the problem, we assume that the missing data belongs
 879 to an infinite dimensional space. Using the notions of no-regret and low-regret controls,
 880 we show that when $3/2 \leq \alpha \leq 2$, such a control exists and is unique. Then we give the
 singular optimality system that characterizes the control.

881

882

883 Competing interests

The authors declare that they have no competing interests.

884

885 Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

886

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