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# 3 Low-regret control for a fractional wave 4 equation with incomplete data

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## Abstract

We investigate in this manuscript an optimal control problem for a fractional wave equation involving the fractional Riemann-Liouville derivative and with missing initial condition. For this purpose, we use the concept of no-regret and low-regret controls. Assuming that the missing datum belongs to a certain space we show the existence and the uniqueness of the low-regret control. Besides, its convergence to the no-regret control is discussed together with the optimality system describing the no-regret control.

**Keywords:** Riemann-Liouville fractional derivative; Caputo fractional derivative; optimal control; no-regret control; low-regret control

## 1 Introduction

Let us consider  $N \in \mathbb{N}^*$  and  $\Omega$  a bounded open subset of  $\mathbb{R}^N$  possessing the boundary  $\partial\Omega$  of class  $C^2$ . When the time  $T > 0$ , we consider  $Q = \Omega \times ]0, T[$  and  $\Sigma = \partial\Omega \times ]0, T[$  and we discuss the fractional wave equation:

$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = v(x, t), & (x, t) \in Q, \\ y(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ I^{2-\alpha} y(x, 0^+) = y^0, & x \in \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) = g, & x \in \Omega, \end{cases} \quad (1)$$

such that  $3/2 < \alpha < 2$ ,  $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $I^{2-\alpha} y(x, 0^+) = \lim_{t \rightarrow 0} I^{2-\alpha} y(x, t)$  and  $\frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) = \lim_{t \rightarrow 0} \frac{\partial}{\partial t} I^{2-\alpha} y(x, t)$  where the fractional integral  $I^\alpha$  of order  $\alpha$  and the fractional derivative  $D_{RL}^\alpha$  of order  $\alpha$  are within the Riemann-Liouville sense. The function  $g$  is unknown and belongs to  $L^2(\Omega)$  and the control  $v \in L^2(Q)$ .

Since the initial condition is unknown, the system (1) is a fractional wave equations with missing data. Such equations are used to model pollution phenomena. In this system  $g$  represents the pollution term.

According to the data, we know that system (1) admits a unique solution  $y(v, g) = y(x, t; v, g)$  in  $L^2((0, T); H_0^1(\Omega)) \subset L^2(Q)$  [1]. Hence, we can define the following functional:

$$J(v, g) = \|y(v, g) - z_d\|_{L^2(Q)}^2 + N\|v\|_{L^2(Q)}^2, \quad (2)$$

where  $z_d \in L^2(Q)$  and  $N > 0$ .

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48 In this manuscript, we discuss the optimal control problem, namely  
 49

50 
$$\inf_{v \in L^2(Q)} J(v, g), \quad \forall g \in L^2(\Omega). \quad (3)$$
  
 51

52 If the function  $g$  is given, namely  $g = g_0 \in L^2(\Omega)$ , then system (1) is completely determined  
 53 and problem (3) becomes a classical optimal control problem [2]. Such a problem was  
 54 studied by Mophou and Joseph [1] with a cost function defined with a final observation.  
 55 Actually, the authors proved that one can approach the fractional integral of order  $0 <$   
 56  $2 - \alpha < 1/2$  of the state at final time by a desired state by acting on a distributed control.  
 57 For more literature on fractional optimal control, we refer to [3–11] and the references  
 58 therein.

59 Since the function  $g$  is unknown, the optimal control problem (3) has no sense because  
 60  $L^2(\Omega)$  is of infinite dimension. So, to solve this problem, we proceed as Lions [12, 13] for  
 61 the control of partial differential equations with integer time derivatives and missing data.  
 62 This means that we use the notions of no-regret and low-regret controls. There are many  
 63 works using these concepts in the literature. In [14] for instance, Nakoulima *et al.* utilized  
 64 these concepts to control distributed linear systems possessing missing data. A generaliza-  
 65 tion of this approach can be found in [15] for some nonlinear distributed systems posses-  
 66 ssing incomplete data. Jacob and Omrane used the notion of no-regret control to control  
 67 a linear population dynamics equation with missing initial data [16]. Recently, Mophou  
 68 [17] used these notions to control a fractional diffusion equation with unknown bound-  
 69 ary condition. For more literature on such control we refer to [18–23] and the references  
 70 therein.

71 In our paper, we show that the low-regret control problem associated to (1) admits a  
 72 unique solution which converges toward the no-regret control. We provide the singular  
 73 optimality system for the no-regret control.

74 Below we present the organization of our manuscript. In the following section, we show  
 75 briefly some results about fractional derivatives and preliminary results on the existence  
 76 and uniqueness of solution to fractional wave equations. In Section 3, we investigate the  
 77 no-regret and low-regret control problems corresponding to (1).

## 79 2 Preliminaries

80 Below, we give briefly some results about fractional calculus and some existence results  
 81 about fractional wave equations.

82 **Definition 2.1** [24, 25] If  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a continuous function on  $\mathbb{R}_+$ , and  $\alpha > 0$ , then the  
 83 expression of the Riemann-Liouville fractional integral of order  $\alpha$  is

84 
$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0.$$
  
 85

86 **Definition 2.2** [25, 26] The form of the left Riemann-Liouville fractional derivative of  
 87 order  $0 \leq n-1 < \alpha < n$ ,  $n \in \mathbb{N}$  of  $f$  is given by

88 
$$D_{RL}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-1-\alpha} f(s) ds, \quad t > 0.$$
  
 89

95    **Definition 2.3** [25, 26] The left Caputo fractional derivative of order  $0 \leq n - 1 < \alpha < n$ ,  
 96     $n \in \mathbb{N}$  of  $f$  is given by  
 97

98    
$$D_C^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} f^{(n)}(s) ds, \quad t > 0. \quad (4)$$

100    We mention that in the above two definitions we consider  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ .  
 101

102    **Definition 2.4** [25–27] Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $0 \leq n - 1 < \alpha < n$ ,  $n \in \mathbb{N}$ . Then the right Caputo  
 103    fractional derivative of order  $\alpha$  of  $f$  is  
 104

105    
$$D_C^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T (s-t)^{n-1-\alpha} f^{(n)}(s) ds, \quad 0 < t < T. \quad (5)$$

108    In all above definitions we assume that the integrals exist.  
 109

110    **Lemma 2.5** [1] Let  $y \in \mathcal{C}^\infty(\overline{Q})$  and  $\varphi \in \mathcal{C}^\infty(\overline{Q})$ . Then we have  
 111

112    
$$\begin{aligned} & \int_Q (D_{RL}^\alpha y(x, t) - \Delta y(x, t)) \varphi(x, t) dx dt \\ &= \int_\Omega \varphi(x, T) \frac{\partial}{\partial t} I^{2-\alpha} y(x, T) dx - \int_\Omega \varphi(x, 0) \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) dx \\ & \quad - \int_\Omega I^{2-\alpha} y(x, T) \frac{\partial \varphi}{\partial t}(x, T) dx + \int_\Omega I^{2-\alpha} y(x, 0) \frac{\partial \varphi}{\partial t}(x, 0) dx \\ & \quad + \int_\Sigma y(\sigma, s) \frac{\partial \varphi}{\partial \nu}(\sigma, s) d\sigma dt - \int_\Sigma \frac{\partial y}{\partial \nu}(\sigma, s) \varphi(\sigma, s) d\sigma dt \\ & \quad + \int_Q y(x, t) (D_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) dx dt. \end{aligned} \quad (6)$$

122    In the following we give some results that will be used to prove the existence of the low-regret and no-regret controls.  
 123

125    **Theorem 2.6** [1] Let  $3/2 < \alpha < 2$ ,  $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $y^1 \in L^2(\Omega)$  and  $v \in L^2(Q)$ . Then the  
 126    problem  
 127

128    
$$\begin{cases} D_{RL}^\alpha y(x, t) - \Delta y(x, t) = v(x, t), & (x, t) \in Q, \\ y(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ I^{2-\alpha} y(x, 0^+) = y^0, & x \in \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0^+) = y^1, & x \in \Omega \end{cases} \quad (7)$$

133    has a unique solution  $y \in L^2((0, T); H_0^1(\Omega))$ . Moreover, the following estimates hold:  
 134

135    
$$\|y\|_{L^2((0, T); H_0^1(\Omega))} \leq \Delta (\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}), \quad (8)$$

137    
$$\|I^{2-\alpha} y\|_{C([0, T]; H_0^1(\Omega))} \leq \Pi (\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}), \quad (9)$$

139    
$$\left\| \frac{\partial}{\partial t} I^{2-\alpha} y \right\|_{C([0, T]; L^2(\Omega))} \leq \Theta (\|y^0\|_{H^2(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}), \quad (10)$$

142 with

143

144  $\Delta = \max\left(C\sqrt{\frac{2T^{2\alpha-3}}{(2\alpha-3)}}, C\sqrt{\frac{T^{\alpha-1}}{(\alpha-1)}}, C\sqrt{\frac{2T^\alpha}{\alpha(\alpha-1)}}\right),$

145

146  $\Pi = \sup\left(C\sqrt{2}, C\sqrt{2T^{2-\alpha}}, C\sqrt{\frac{2T^{3-\alpha}}{(3-\alpha)}}\right),$

147

148 and

149

150  $\Theta = \max(\sqrt{2}CT^{\alpha-1}, \sqrt{2}C).$

151

152 Consider the fractional wave equation involving the left Caputo fractional derivative of  
 153 order  $3/2 < \alpha < 2$ :

154

155 
$$\begin{cases} D_C^\alpha y(x, t) - \Delta y(x, t) = f, & (x, t) \in Q, \\ y(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ y(x, 0) = 0, & x \in \Omega, \\ \frac{\partial y}{\partial t}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (11)$$

156

157 where  $f \in L^2(Q)$ .

158

159 **Theorem 2.7** Let  $f \in L^2(Q)$ . Then problem (11) has a unique solution  $y \in C([0, T]; H_0^1(\Omega))$ .  
 160 Moreover,  $\frac{\partial y}{\partial t} \in C([0, T]; L^2(\Omega))$  and there exists  $C > 0$  in such a way that

161

162  $\|y\|_{C([0, T]; H_0^1(\Omega))} \leq C\sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|f\|_{L^2(Q)} \quad (12)$

163

164 and

165

166  $\left\| \frac{\partial y}{\partial t} \right\|_{C([0, T]; L^2(\Omega))} \leq C\sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|f\|_{L^2(Q)}. \quad (13)$

167

168 *Proof* Below we proceed as was mentioned in [28]. □

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175 **Corollary 2.8** Let  $3/2 < \alpha < 2$  and  $\phi \in L^2(Q)$ . Consider the fractional wave equation:

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188

189 
$$\begin{cases} D_C^\alpha \psi(x, t) - \Delta \psi(x, t) = \phi, & (x, t) \in Q, \\ \psi(\sigma, t) = 0, & (\sigma, t) \in \Sigma, \\ \psi(x, T) = 0, & x \in \Omega, \\ \frac{\partial \psi}{\partial t}(x, T) = 0, & x \in \Omega, \end{cases} \quad (14)$$

190 where  $D_C^\alpha$  is the right Caputo fractional derivative of order  $1 < \alpha < 2$ . Then (14) has a unique  
 191 solution  $\psi \in C([0, T]; H_0^1(\Omega))$ . Moreover,  $\frac{\partial \psi}{\partial t} \in C([0, T]; L^2(\Omega))$ , and there exists  $C > 0$  ful-  
 192 filling

193  $\|\psi\|_{C([0, T]; H_0^1(\Omega))} \leq C\sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|\phi\|_{L^2(Q)} \quad (15)$

189 and

190

$$191 \quad \left\| \frac{\partial \psi}{\partial t} \right\|_{C([0,T];L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|\phi\|_{L^2(Q)}. \quad (16)$$

193

194 *Proof* If we make the change of variable  $t \rightarrow T - t$  in (14), then we conclude that  $\hat{\psi}(t) =$   
 195  $\psi(T - t)$  verifies

196

$$197 \quad \begin{cases} D_C^\alpha \hat{\psi} - \Delta \hat{\psi} = \hat{\phi} & \text{in } Q, \\ 198 \quad \hat{\psi} = 0 & \text{on } \Sigma, \\ 199 \quad \hat{\psi}(0) = 0 & \text{in } \Omega, \\ 200 \quad \frac{\partial \hat{\psi}}{\partial t}(0) = 0 & \text{in } \Omega, \end{cases} \quad (17)$$

201

202 where  $\hat{\phi}(t) = \phi(T - t)$  and  $D_C^\alpha$  is the left Caputo fractional derivative of order  $3/2 < \alpha < 2$ .

203 Because  $T - t \in [0, T]$  when  $t \in [0, T]$ , we say that  $\hat{\phi} \in L^2(Q)$  due to the fact that  $\phi \in L^2(Q)$ .

204 It then suffices to use Theorem 2.7 to conclude.  $\square$

205

206 We also need some trace results.

207

208 **Lemma 2.9** [1] Let  $f \in L^2(Q)$  and  $y \in L^2(Q)$  such that  $D_{RL}^\alpha y - \Delta y = f$ . Then:

- 209 (i)  $y|_{\partial\Omega}$  and  $\frac{\partial y}{\partial t}|_{\partial\Omega}$  exist and belong to  $H^{-2}((0, T); H^{-1/2}(\partial\Omega))$  and  
 210  $H^{-2}((0, T); H^{-3/2}(\partial\Omega))$  respectively.
- 211 (ii)  $I^{2-\alpha}y \in C([0, T]; L^2(\Omega))$ .
- 212 (iii)  $\frac{\partial}{\partial t} I^{2-\alpha}y \in C([0, T]; H^{-1}(\Omega))$ .

214

### 215 **3 Existence and uniqueness of no-regret and low-regret controls**

216 Below, we show the existence and the uniqueness of the no-regret control and the low-  
 217 regret control problem for system (1).

218

219 **Lemma 3.1** Let  $v \in L^2(Q)$  and  $g \in L^2(\Omega)$ . Then we have

220

$$221 \quad J(v, g) = J(0, g) + J(v, 0) - J(0, 0) \\ 222 \quad + 2 \int_Q [y(0, g) - y(0, 0)] [y(v, 0) - y(0, 0)] dt dx. \quad (18)$$

224

225 Here  $J$  denotes the functional given by (2) and  $y(v, g) = y(x, t; v, g) \in L^2(0, T; H_0^1(\Omega)) \subset L^2(Q)$   
 226 is the solution of (1).

227

228 *Proof* Let us consider  $y(v, 0) = y(x, t; v, 0)$ ,  $y(0, g) = y(x, t; 0, g)$ , and  $y(0, 0) = y(x, t; 0, 0)$  be  
 229 the solutions of

230

$$231 \quad \begin{cases} D_{RL}^\alpha y(v, 0) - \Delta y(v, 0) = v & \text{in } Q, \\ 232 \quad y = 0 & \text{on } \Sigma, \\ 233 \quad I^{2-\alpha}y(0; v, 0) = y^0 & \text{in } \Omega, \\ 234 \quad \frac{\partial}{\partial t} I^{2-\alpha}y(0; v, 0) = 0 & \text{in } \Omega, \end{cases} \quad (19)$$

235

236      
$$\begin{cases} D_{\text{RL}}^{\alpha}y(0, g) - \Delta y(0, g) = 0 & \text{in } Q, \\ 237 \quad y = 0 & \text{on } \Sigma, \\ 238 \quad I^{2-\alpha}y(0; 0, g) = y^0 & \text{in } \Omega, \\ 239 \quad \frac{\partial}{\partial t}I^{2-\alpha}y(0; 0, g) = g & \text{in } \Omega, \end{cases} \quad (20)$$

240      and  
 241

242      
$$\begin{cases} D_{\text{RL}}^{\alpha}y(0, 0) - \Delta y(0, 0) = 0 & \text{in } Q, \\ 243 \quad y = 0 & \text{on } \Sigma, \\ 244 \quad I^{2-\alpha}y(0; 0, 0) = y^0 & \text{in } \Omega, \\ 245 \quad \frac{\partial}{\partial t}I^{2-\alpha}y(0; 0, 0) = 0 & \text{in } \Omega, \end{cases} \quad (21)$$

247      where  $I^{2-\alpha}y(0; \nu, g) = \lim_{t \rightarrow 0^+} I^{2-\alpha}y(x, t; \nu, g)$  and  $\frac{\partial}{\partial t}I^{2-\alpha}y(0; \nu, g) = \lim_{t \rightarrow 0^+} \frac{\partial}{\partial t}I^{2-\alpha}y(x, t; \nu, g)$ .

248      Since  $\nu \in L^2(Q)$ ,  $y^0 \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $g \in L^2(\Omega)$ , we see from Theorem 2.6 that  $y(\nu, 0)$ ,  
 249       $y(0, g)$  and  $y(0, 0)$  belong to  $L^2((0, T); H_0^1(\Omega))$ .

250      Observing that  
 251

252      
$$J(\nu, 0) = \int_Q [y(\nu, 0) - z_d][y(\nu, 0) - z_d] dt dx + N \|\nu\|_{L^2(Q)}^2, \quad (22a)$$

253      
$$J(0, g) = \int_Q [y(0, g) - z_d][y(0, g) - z_d] dt dx, \quad (22b)$$

254      
$$J(0, 0) = \int_Q [y(0, 0) - z_d][y(0, 0) - z_d] dt dx, \quad (22c)$$

255      and using the fact that  
 256

257      
$$y(\nu, g) = y(\nu, 0) + y(0, g) - y(0, 0),$$

258      we have

259      
$$\begin{aligned} 260 \quad J(\nu, g) &= \|y(\nu, g) - z_d\|_{L^2(Q)}^2 + N \|\nu\|_{L^2(Q)}^2 \\ 261 \quad &= \|y(\nu, 0) + y(0, g) - y(0, 0) - z_d\|_{L^2(Q)}^2 + N \|\nu\|_{L^2(Q)}^2 \\ 262 \quad &= J(\nu, 0) + 2 \int_Q [y(\nu, 0) - z_d][y(0, g) - y(0, 0)] dt dx \\ 263 \quad &\quad + \|y(0, g) - y(0, 0) - z_d\|_{L^2(Q)}^2 \\ 264 \quad &= J(\nu, 0) + 2 \int_Q [y(\nu, 0) - y(0, 0)][y(0, g) - y(0, 0)] dt dx \\ 265 \quad &\quad + 2 \int_Q [y(0, 0) - z_d][y(0, g) - y(0, 0)] dt dx + \|y(0, g) - y(0, 0) - z_d\|_{L^2(Q)}^2. \end{aligned}$$

266      Using  
 267

268      
$$\begin{aligned} 269 \quad \|y(0, g) - y(0, 0) - z_d\|_{L^2(Q)}^2 &= J(0, g) + J(0, 0) \\ 270 \quad &\quad - 2 \int_Q [y(0, 0) - z_d][y(0, g) - y(0, 0)] dt dx - 2J(0, 0), \end{aligned}$$

283 we conclude that

284

285 
$$J(v, g) = J(0, g) + J(v, 0) - J(0, 0)$$
  
 286 
$$+ 2 \int_{\Omega} \int_0^T [y(v, 0) - y(0, 0)][y(0, g) - y(0, 0)] dt dx. \quad \square$$

288

289 **Lemma 3.2** Let  $v \in L^2(Q)$  and  $g \in L^2(\Omega)$ . Then we have

290

291 
$$J(v, g) = J(0, g) + J(v, 0) - J(0, 0) + 2 \int_{\Omega} g \zeta(x, 0; v) dx, \quad (23)$$

292

293 where  $\zeta(v) = \zeta(x, t; v) \in C([0, T]; H_0^1(\Omega))$  be solution of

294

295 
$$\begin{cases} D_C^\alpha \zeta(v) - \Delta \zeta(v) = y(v, 0) - y(0, 0) & \text{in } Q, \\ \zeta = 0 & \text{on } \Sigma, \\ \zeta(x, T; v) = 0 & \text{in } \Omega, \\ \frac{\partial \zeta}{\partial t}(x, T; v) = 0 & \text{in } \Omega. \end{cases} \quad (24)$$

299

300 *Proof* Since  $y(v, 0) - y(0, 0) \in L^2(Q)$ , from Proposition 2.8, we know that the system (24)  
 301 admits a unique solution  $\zeta(v) \in C([0, T]; H_0^1(\Omega))$ . Also, there exists  $C > 0$  such that

302

303 
$$\|\zeta(v)\|_{C([0, T]; H_0^1(\Omega))} \leq C \sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|y(v, 0) - y(0, 0)\|_{L^2(Q)} \quad (25)$$

304

305 and

306

307 
$$\left\| \frac{\partial \zeta}{\partial t}(v) \right\|_{C([0, T]; L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|y(v, 0) - y(0, 0)\|_{L^2(Q)}. \quad (26)$$

308

309 Set  $z = y(g, 0) - y(0, 0)$ . Then  $z$  verifies

310

311 
$$\begin{cases} D_{RL}^\alpha z - \Delta z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ I^{2-\alpha} z(0) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} z(0) = g & \text{in } \Omega. \end{cases} \quad (27)$$

315

316 Since  $g \in L^2(\Omega)$ , it follows from Theorem 2.6 that  $z \in L^2((0, T); H_0^1(\Omega))$ ,  $I^{2-\alpha} z \in C([0, T],$   
 317  $H_0^1(\Omega))$ , and  $\frac{\partial}{\partial t} I^{2-\alpha} z \in C([0, T], L^2(\Omega))$ . So, if we multiply the first equation of (24) by  $z$   
 318 utilizing the fractional integration by parts provided by Lemma 2.5, we conclude

319

320 
$$\begin{aligned} & \int_Q (y(v, 0) - y(0, 0)) z dt dx \\ &= \int_Q (D_C^\alpha \zeta(v) - \Delta \zeta(v)) z dt dx \\ &= \int_{\Omega} \zeta(x, 0; v) \frac{\partial}{\partial t} I^{2-\alpha} z(0) dx. \end{aligned} \quad (28)$$

325

326 Thus, replacing  $z$  by  $(y(0, g) - y(0, 0))$ , we obtain

327

328 
$$\int_Q [y(v, 0) - y(0, 0)][y(0, g) - y(0, 0)] dt dx = \int_{\Omega} \zeta(x, 0; v) g dx,$$

329

330 and (18) becomes

331

332 
$$J(\nu, g) - J(0, g) = J(\nu, 0) - J(0, 0) + 2 \int_{\Omega} \zeta(x, 0; \nu) g dx.$$

333

□

334 Now we consider the no-regret control problem:

335

336 
$$\inf_{\nu \in L^2(Q)} \sup_{g \in L^2(\Omega)} (J(\nu, g) - J(0, g)). \quad (29)$$

337

338

From (23), this problem is equivalent to the following one:

339

340 
$$\inf_{\nu \in L^2(Q)} \sup_{g \in L^2(\Omega)} \left[ J(\nu, 0) - J(0, 0) + 2 \int_{\Omega} \zeta(x, 0; \nu) g dx \right]. \quad (30)$$

341

342

As the space  $L^2(\Omega)$  is a vector space, the no-regret control exists only if

343

344 
$$\sup_{g \in L^2(\Omega)} \left( \int_{\Omega} \zeta(x, 0; \nu) g dx \right) = 0. \quad (31)$$

345

346

This implies that the no-regret control belongs to  $\mathcal{U}$  defined by

347

348 
$$\mathcal{U} = \left\{ \nu \in L^2(Q) \mid \left( \int_{\Omega} \zeta(x, 0; \nu) g dx \right) = 0, \forall g \in L^2(\Omega) \right\}.$$

349

350

As a result such control should be carefully investigated. So, we proceed by penalization.

351 For all  $\gamma > 0$ , we discuss the low-regret control problem:

352

353 
$$\inf_{\nu \in L^2(Q)} \sup_{g \in L^2(\Omega)} (J(\nu, g) - J(0, g) - \gamma \|g\|_{L^2(\Omega)}^2). \quad (32)$$

354

355

According to (23), the problem (32) is equivalent to the following problem:

356

357 
$$\inf_{\nu \in L^2(Q)} \left[ J(\nu, 0) - J(0, 0) + 2 \sup_{g \in L^2(\Omega)} \left( \int_{\Omega} \zeta(x, 0; \nu) g dx - \frac{\gamma}{2} \|g\|_{L^2(\Omega)}^2 \right) \right].$$

358

359

Using the Legendre-Fenchel transform, we conclude

360

361 
$$2\gamma \sup_{g \in L^2(\Omega)} \left( \int_{\Omega} \frac{1}{\gamma} \zeta(x, 0; \nu) g dx - \frac{1}{\gamma} \frac{\gamma}{2} \|g\|_{L^2(\Omega)}^2 \right) = \frac{1}{\gamma} \|\zeta(\cdot, 0; \nu)\|_{L^2(\Omega)}^2,$$

362

363

and problem (32) becomes: For any  $\gamma > 0$ , find  $u^\gamma \in L^2(Q)$  such that

364

365 
$$J_\gamma(u^\gamma) = \inf_{\nu \in L^2(Q)} J_\gamma(\nu), \quad (33)$$

366

367

where

368

369 
$$J_\gamma(\nu) = J(\nu, 0) - J(0, 0) + \frac{1}{\gamma} \|\zeta(\cdot, 0; \nu)\|_{L^2(\Omega)}^2. \quad (34)$$

370

371 **Proposition 3.3** Let  $\gamma > 0$ . Then (33) has a unique solution  $u^\gamma$ , called a low-regret control.

372

377 *Proof* We recall that

378

$$379 \quad J_\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|\xi(\cdot, 0; v)\|_{L^2(\Omega)}^2 \geq -J(0, 0).$$

380

381 Thus, we can say that  $\inf_{v \in L^2(Q)} J_\gamma$  exists. Let  $(v_n) \in L^2(Q)$  be a minimizing sequence such  
 382 that

383

$$384 \quad \lim_{n \rightarrow +\infty} J_\gamma(v_n) = \inf_{v \in L^2(Q)} J_\gamma(v). \quad (35)$$

385

386 Then  $y_n = y(x, t; v_n, 0)$  is a solution of (19) and  $y_n$  satisfies

387

$$388 \quad D_{RL}^\alpha y_n(x, t) - \Delta y_n(x, t) = v_n(x, t) \quad \text{in } Q, \quad (36a)$$

389

$$390 \quad y_n(x, t) = 0 \quad \text{on } \Sigma, \quad (36b)$$

391

$$392 \quad I^{2-\alpha} y_n(x, 0) = y^0 \quad \text{in } \Omega, \quad (36c)$$

393

$$394 \quad \frac{\partial}{\partial t} I^{2-\alpha} y_n(x, 0) = 0 \quad \text{in } \Omega. \quad (36d)$$

395

396 It follows from (35) that there exists  $C(\gamma) > 0$  independent of  $n$  such that

397

$$398 \quad 0 \leq J(v_n, 0) + \frac{1}{\gamma} \|\xi(\cdot, 0; v_n)\|_{L^2(\Omega)}^2 \leq C(\gamma) + J(0, 0) = C(\gamma).$$

399

400 From the definition of  $J(v_n, 0)$  we obtain

401

$$402 \quad \|v_n\|_{L^2(Q)} \leq C(\gamma), \quad (37a)$$

403

$$404 \quad \|\xi(\cdot, 0; v_n)\|_{L^2(\Omega)} \leq \sqrt{\gamma} C(\gamma). \quad (37b)$$

405

406 Therefore, from Theorem 2.6, we know that there exists a constant  $C$  independent of  $n$   
 407 such that

408

$$409 \quad \|y_n\|_{L^2((0, T); H_0^1(\Omega))} \leq C(\gamma), \quad (38a)$$

410

$$411 \quad \|I^{2-\alpha} y_n\|_{L^2((0, T); H_0^1(\Omega))} \leq C(\gamma), \quad (38b)$$

412

$$413 \quad \left\| \frac{\partial}{\partial t} I^{2-\alpha} y_n \right\|_{L^2((0, T); L^2(\Omega))} \leq C(\gamma). \quad (38c)$$

414

415 Moreover, from (36a) and (37a), we have

416

$$417 \quad \|D_{RL}^\alpha y_n - \Delta y_n\|_{L^2(Q)} \leq C(\gamma). \quad (39)$$

418

419 Consequently, there exist  $u^\gamma \in L^2(Q)$ ,  $y^\gamma \in L^2((0, T); H_0^1(\Omega))$ ,  $\delta \in L^2(Q)$ ,  $\eta \in L^2((0, T); H_0^1(\Omega))$ ,  $\theta \in L^2((0, T); L^2(\Omega))$  and we can extract subsequences of  $(v_n)$  and  $(y_n)$  (still called  
 420  $(v_n)$  and  $(y_n)$ ) such that:

421

$$422 \quad v_n \rightharpoonup u^\gamma \quad \text{weakly in } L^2(Q), \quad (40a)$$

423

$$424 \quad D_{RL}^\alpha y_n - \Delta y_n \rightharpoonup \delta \quad \text{weakly in } L^2(Q), \quad (40b)$$

424       $y_n \rightharpoonup y^\gamma$  weakly in  $L^2((0, T); H_0^1(\Omega))$ ,      (40c)

425       $I^{2-\alpha}y_n \rightharpoonup \eta$  weakly in  $L^2([0, T], H_0^1(\Omega))$ ,      (40d)

427       $\frac{\partial}{\partial t} I^{2-\alpha}y_n \rightharpoonup \theta$  weakly in  $L^2((0, T); L^2(\Omega))$ .      (40e)

429      The remaining part of the proof contains three steps.

430      Step 1: We show that  $(u^\gamma, y^\gamma)$  fulfills (19).

431      Set  $\mathbb{D}(Q)$ , the set of  $C^\infty$  function on  $Q$  with compact support and denote by  $\mathbb{D}'(Q)$  its  
 432      dual. Multiplying (36a) by  $\varphi \in \mathbb{D}(Q)$  and using Lemma 2.5, (40a), and (40c), we prove as  
 433      in [1] that

434

435       $D_{RL}^\alpha y_n - \Delta y_n \rightharpoonup D_{RL}^\alpha y^\gamma - \Delta y^\gamma$  weakly in  $\mathbb{D}'(Q)$ .

436

437      From (40b) and the uniqueness of the limit, we conclude

438

439       $D_{RL}^\alpha y^\gamma - \Delta y^\gamma = \delta \in L^2(Q)$ .      (41)

440

441      Hence,

442

443       $D_{RL}^\alpha y_n - \Delta y_n \rightharpoonup D_{RL}^\alpha y^\gamma - \Delta y^\gamma$  weakly in  $L^2(Q)$ .      (42)

444

445      Then passing to the limit in (36a) and using (41) and (40a), we obtain

446

447       $D_{RL}^\alpha y^\gamma(x, t) - \Delta y^\gamma(x, t) = u^\gamma(x, t), \quad (x, t) \in Q$ .      (43)

448

449      On the other hand, we have

450

$$\begin{aligned} 451 & \int_Q I^{2-\alpha}y_n(x, t)\varphi(x, t) dt dx \\ 452 &= \int_\Omega \int_0^T y_n(x, s) \left( \frac{1}{\Gamma(2-\alpha)} \int_s^T (t-s)^{1-\alpha} \varphi(x, t) dt \right) ds dx, \quad \forall \varphi \in \mathbb{D}(Q). \end{aligned}$$

453

454      Thus using (40c) and (40d), while passing to the limit, we get

455

$$\begin{aligned} 456 & \int_Q \eta\varphi(x, t) dt dx = \int_\Omega \int_0^T y^\gamma(x, s) \left( \frac{1}{\Gamma(2-\alpha)} \int_s^T (t-s)^{1-\alpha} \varphi(x, t) dt \right) ds dx \\ 457 &= \int_Q I^{2-\alpha}y^\gamma(x, t)\varphi(x, t) dt dx, \quad \forall \varphi \in \mathbb{D}(Q). \end{aligned}$$

458

459      This implies that

460

461       $I^{2-\alpha}y^\gamma(x, t) = \eta \quad \text{in } Q$ .

462

463      Thus, (40d) becomes

464

465       $I^{2-\alpha}y_n \rightharpoonup I^{2-\alpha}y^\gamma$  weakly in  $L^2([0, T], H_0^1(\Omega))$ .      (44)

466

471 In view of (44), we have

472

$$473 \quad \frac{\partial}{\partial t} I^{2-\alpha} y_n \rightharpoonup \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma \quad \text{weakly in } \mathbb{D}'(Q), \\ 474$$

475 and as we have (40e), we obtain

476

$$477 \quad \frac{\partial}{\partial t} I^{2-\alpha} y_n \rightharpoonup \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma = \theta \quad \text{weakly in } L^2(Q). \quad (45) \\ 478$$

479 Since  $y^\gamma \in L^2(Q)$  and  $D_{RL}^\alpha y^\gamma - \Delta y^\gamma \in L^2(Q)$ , in view of Lemma 2.9, we know that  $y^\gamma|_{\partial\Omega}$  and  
 480  $\frac{\partial y^\gamma}{\partial \nu}|_{\partial\Omega}$  exist and belong to  $H^{-2}((0, T); H^{-1/2}(\partial\Omega))$  and  $H^{-2}((0, T); H^{-3/2}(\partial\Omega))$ , respectively.

481 Moreover, we have  $I^{2-\alpha} y^\gamma \in C([0, T]; L^2(\Omega))$  and  $\frac{\partial}{\partial t} I^{2-\alpha} y^\gamma \in C([0, T]; H^{-1}(\Omega))$ .

482 Now multiplying (36a) by a function  $\varphi \in C^\infty(\bar{Q})$  such that  $\varphi|_{\partial\Omega} = 0$  and  $\varphi(x, T) =$   
 483  $\frac{\partial \varphi}{\partial t}(x, T) = 0$  in  $\Omega$ , and integrating by parts over  $Q$ , we obtain

484

$$485 \quad \int_Q v_n(x, t)\varphi(x, t) dx dt = \int_Q (D_{RL}^\alpha y_n(x, t) - \Delta y_n(x, t))\varphi(x, t) dx dt \\ 486 \\ 487 \quad = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) dx \\ 488 \\ 489 \quad + \int_Q y_n(x, t)(D_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) dx dt \\ 490$$

491 because we have (36c) and (36d). Thus, using (40a) and (40c) while passing to the limit  
 492 in the latter identity, we get

493

$$494 \quad \int_Q u^\gamma(x, t)\varphi(x, t) dx dt = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) dx \\ 495 \\ 496 \quad + \int_Q y^\gamma(x, t)(D_C^\alpha \varphi(x, t) - \Delta \varphi(x, t)) dx dt,$$

497

498  $\forall \varphi \in C^\infty(\bar{Q})$  such that  $\varphi|_{\partial\Omega} = 0$ ,  $\varphi(T) = \frac{\partial \varphi}{\partial t}(T) = 0$  in  $\Omega$ , which, according to Lemma 2.5,  
 499 can be rewritten as

500

$$501 \quad \int_Q u^\gamma(x, t)\varphi(x, t) dx dt = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) dx \\ 502 \\ 503 \quad + \int_Q (D_{RL}^\alpha y^\gamma(x, t) - \Delta y^\gamma(x, t))\varphi(x, t) dx dt \\ 504 \\ 505 \quad + \left\langle \varphi(x, 0), \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(x, 0) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ 506 \\ 507 \quad - \int_\Omega \frac{\partial \varphi}{\partial t}(x, 0), I^{2-\alpha} y^\gamma(x, 0) dx \\ 508 \\ 509 \quad - \left\langle y^\gamma(\sigma, t), \frac{\partial \varphi}{\partial \nu}(\sigma, t) \right\rangle_{H^{-2}((0, T); H^{-1/2}(\partial\Omega)), H_0^2((0, T); H^{1/2}(\partial\Omega))}, \\ 510 \\ 511$$

512

513  $\forall \varphi \in C^\infty(\bar{Q})$  such that  $\varphi|_{\partial\Omega} = 0$ ,  $\varphi(T) = \frac{\partial \varphi}{\partial t}(T) = 0$  in  $\Omega$ .

514 Using (43), we obtain

515

$$516 \quad 0 = \int_\Omega y^0 \frac{\partial \varphi}{\partial t}(x, 0) dx \\ 517$$

$$\begin{aligned}
& + \left\langle \varphi(x, 0), \frac{\partial}{\partial t} I^{2-\alpha} y^\nu(x, 0) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\
& - \int_{\Omega} \frac{\partial \varphi}{\partial t}(x, 0), I^{2-\alpha} y^\nu(x, 0) \, dx \\
& - \left\langle y^\nu(\sigma, t), \frac{\partial \varphi}{\partial \nu}(\sigma, t) \right\rangle_{H^{-2}((0, T); H^{-1/2}(\partial\Omega)), H_0^2((0, T); H^{1/2}(\partial\Omega))}, \tag{46}
\end{aligned}$$

$$525 \quad \forall \varphi \in C^\infty(\overline{Q}) \text{ such that } \varphi|_{\partial\Omega} = 0, \varphi(T) = \frac{\partial \varphi}{\partial t}(T) = 0 \text{ in } \Omega.$$

Choosing successively in (46)  $\varphi$  such that  $\varphi(x, 0) = \frac{\partial \varphi}{\partial t}(x, 0) = 0$  and  $\varphi(x, 0) = 0$ , we deduce that

$$y^\gamma(x, t) = 0, \quad (x, t) \in \Sigma, \quad (47)$$

$$I^{2-\alpha}y^\gamma(x, 0) = y^0, \quad x \in \Omega, \quad (48)$$

532 and then

$$\begin{aligned} & \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(x, 0) = 0, \quad x \in \Omega. \end{aligned} \tag{49}$$

In view of (43), (47), (48), and (49), we see that  $\gamma^\nu = \gamma^\nu(x, t; u^\nu, 0)$  is a solution of (19).

Step 2: We show  $\zeta_n \equiv \zeta(x, t; v_n)$  converges to  $\zeta^\gamma \equiv \zeta(x, t; v^\gamma)$ .

In view of (24),  $\zeta_n \equiv \zeta(x, t; v_n)$  verifies

$$\begin{aligned} & \left\{ \begin{array}{l} \mathcal{D}_C^\alpha \zeta_n - \Delta \zeta_n = y(v_n, 0) - y(0, 0) \quad \text{in } Q, \\ \zeta_n = 0 \quad \text{on } \Sigma, \\ \zeta_n(T) = 0 \quad \text{in } \Omega, \\ \frac{\partial}{\partial t} \zeta_n(T) = 0 \quad \text{in } \Omega. \end{array} \right. \end{aligned} \quad (50)$$

Set  $z_0 = \gamma(v_0, 0) - \gamma(0, 0)$ . In view of (19) and (21),  $z_0$  verifies

$$\begin{aligned} 547 \quad & \left\{ \begin{array}{l} D_{\text{RL}}^\alpha z_n - \Delta z_n = v_n \quad \text{in } Q, \\ 548 \quad z_n = 0 \quad \text{on } \Sigma, \\ 549 \quad I^{1-\alpha} z_n(0) = 0 \quad \text{in } \Omega, \\ 550 \quad \frac{\partial}{\partial t} I^{2-\alpha} z_n(0) = 0 \quad \text{in } \Omega. \end{array} \right. \end{aligned}$$

It follows from Theorem 2.6 and (37a) that

$$\|z_n\|_{L^2((0,T);H_0^1(\Omega))} = \|y_n(v_n, 0) - y(0, 0)\|_{L^2((0,T);H_0^1(\Omega))} \leq C(\gamma).$$

Hence, from Corollary 2.8, we deduce that

$$\|\zeta_n\|_{C([0,T]\cdot H^1(\Omega))} \leq C(\gamma), \quad (51)$$

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} \zeta_n \right\|_{C^{([0,T];L^2(\Omega))}} \leq C(\gamma). \end{aligned} \quad (52)$$

561 Since the embedding of  $C([0, T]; H_0^1(\Omega))$  into  $L^2((0, T); H_0^1(\Omega))$  and the embedding of  
 562  $C([0, T]; L^2(\Omega))$  into  $L^2(Q)$  are continuous, we can conclude that there exists  $\zeta^\gamma \in$

565  $L^2((0, T); H_0^1(\Omega))$  such that

566

567  $\zeta_n \rightharpoonup \zeta^\gamma$  weakly in  $L^2((0, T); H_0^1(\Omega))$ . (53)

568

569 Therefore,

570

571  $\frac{\partial}{\partial t} \zeta_n \rightharpoonup \frac{\partial}{\partial t} \zeta^\gamma$  weakly in  $\mathbb{D}'(Q)$

572

573 and, consequently,

574

575  $\frac{\partial}{\partial t} \zeta_n \rightharpoonup \frac{\partial}{\partial t} \zeta^\gamma$  weakly in  $L^2(Q)$ . (54)

576

577 Since  $\zeta^\gamma \in L^2((0, T); H_0^1(\Omega))$  and  $\frac{\partial}{\partial t} \zeta^\gamma \in L^2(Q)$ , we see that  $\zeta^\gamma(0)$  and  $\zeta^\gamma(T)$  belongs to  
 578  $L^2(\Omega)$ . In view of (50)<sub>3</sub>, we have

579

580  $\zeta^\gamma(T) = 0$  in  $\Omega$  (55)

581

582 and in view of (50)<sub>4</sub> and (52), we set

583

584  $\frac{\partial}{\partial t} \zeta^\gamma(T) = 0$  in  $\Omega$ . (56)

585

586 From (37b), we deduce that there exists  $\rho \in L^2(\Omega)$  such that

587

588  $\zeta(\cdot, 0; \nu_n) \rightharpoonup \rho$  weakly in  $L^2(\Omega)$ . (57)

589

590 Multiplying the first equation of (50) by  $\phi \in \mathbb{D}(Q)$  then, using the integration by parts  
 591 given by Lemma 2.5, we obtain

592

593 
$$\int_Q (y(x, t; \nu_n, 0) - y(x, t; 0, 0)) \phi(x, t) dt dx$$

594

595  $= \int_Q [D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t)] \zeta_n(x, t) dt dx.$

596

597 Hence, using (40c) and (53) while passing to the limit in the latter identity, we have

598

599 
$$\int_Q (y(x, t; u^\gamma, 0) - y(x, t; 0, 0)) \phi(x, t) dt dx$$

600

601  $= \int_Q (D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t)) \zeta^\gamma(x, t) dt dx, \quad \forall \phi \in \mathbb{D}(Q),$  (58)

602

603 which by using again Lemma 2.5 gives

604

605 
$$\int_Q (y(x, t; u^\gamma, 0) - y(x, t; 0, 0)) \phi(x, t) dt dx$$

606

607  $= \int_Q (D_C^\alpha \zeta^\gamma(x, t) - \Delta \zeta^\gamma(x, t)) \phi(x, t) dt dx, \quad \forall \phi \in \mathbb{D}(Q).$

608

612 This implies that

613

614  $\mathcal{D}_C^\alpha \zeta^\gamma - \Delta \zeta^\gamma = y(u^\gamma, 0) - y(0, 0) \quad \text{in } Q.$  (59)

615

616 Now, if we multiply the first equation of (50) by  $\phi \in \mathcal{C}^\infty(\overline{Q})$  with  $\phi|_{\partial\Omega} = 0$  and  $I^{2-\alpha}\phi(0) = 0$   
 617 in  $\Omega$  and integrating by parts over  $Q$ , we obtain

618

619 
$$\begin{aligned} & \int_Q (y_n(x, t) - y(x, t; 0, 0)) \phi(x, t) dt dx \\ 620 &= \int_Q (\mathcal{D}_C^\alpha \zeta_n(x, t) - \Delta \zeta_n(x, t)) \phi(x, t) dt dx \\ 622 &= \int_Q (D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t)) \zeta_n(x, t) dt dx + \int_\Omega \zeta(x, 0, v_n) \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx. \end{aligned}$$

623

624

625 Using (40c), (53), and (57) while passing the latter identity to the limit, we obtain

626

627 
$$\begin{aligned} & \int_Q (y^\gamma(x, t) - y(x, t; 0, 0)) \phi(x, t) dt dx \\ 628 &= \int_Q (D_{RL}^\alpha \phi(x, t) - \Delta \phi(x, t)) \zeta^\gamma(x, t) dt dx + \int_\Omega \rho \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx, \\ 631 & \forall \phi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \phi|_{\partial\Omega} = 0, I^{2-\alpha}\phi(0) = 0 \text{ in } \Omega, \end{aligned} \tag{60}$$

632

633 which by using again Lemma 2.5, (55), (56), and (59) gives

634

635 
$$\begin{aligned} & - \int_\Sigma \frac{\partial}{\partial \nu} \phi(\sigma, t) \zeta^\gamma(\sigma, t) d\sigma dt + \int_\Omega \rho(x) \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx \\ 636 &= \int_\Omega \zeta^\gamma(0) \frac{\partial}{\partial t} I^{2-\alpha} \phi(0) dx \\ 637 & \forall \phi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \phi|_{\partial\Omega} = 0, I^{2-\alpha}\phi(0) = 0 \text{ in } \Omega. \end{aligned}$$

638

639

640 Hence, choosing  $\phi \in \mathcal{C}^\infty(\overline{Q})$ , such that  $\phi|_{\partial\Omega} = 0, I^{2-\alpha}\phi(0) = \frac{\partial}{\partial t} I^{2-\alpha}\phi(0) = 0$ , we get

641

642

643  $\zeta^\gamma = 0 \quad \text{on } \Sigma,$  (61)

644

645 and then

646

647  $\zeta^\gamma(0) = \rho \quad \text{in } \Omega.$  (62)

648

649

In view of (55), (56), (59), and (61), we see that  $\zeta^\gamma = \zeta(u^\gamma)$  is a solution of

650

651

652

653

654

$$\begin{cases} \mathcal{D}_C^\alpha \zeta^\gamma - \Delta \zeta^\gamma = y(u^\gamma, 0) - y(0, 0) & \text{in } Q, \\ \zeta^\gamma = 0 & \text{on } \Sigma, \\ \zeta^\gamma(T) = 0 & \text{in } \Omega \\ \frac{\partial \zeta^\gamma}{\partial t}(T) = 0 & \text{in } \Omega. \end{cases} \tag{63}$$

655

656

Moreover, using (62), equation (57) becomes

657

$\zeta(\cdot, 0; v_n) \rightharpoonup \zeta^\gamma(0) = \zeta(\cdot, 0; u^\gamma) \quad \text{weakly in } L^2(\Omega).$  (64)

658

659     Step 3: The function  $v \rightarrow J_\gamma(v)$  being lower semi-continuous, we have  
 660

661     
$$J_\gamma(u^\gamma) \leq \liminf_{n \rightarrow \infty} J_\gamma(v_n),$$
  
 662

663     which in view of (35) implies that  
 664

665     
$$J_\gamma(u^\gamma) = \inf_{v \in L^2(Q)} J_\gamma(v).$$
  
 666

668     The uniqueness of  $u^\gamma$  comes from the fact that the functional  $J_\gamma$  is strictly convex.   □  
 669

670     **Theorem 3.4** For any  $\gamma > 0$ , let  $u^\gamma$  be the low-regret control. Then there exist  $q^\gamma \in$   
 671      $L^2((0, T); H_0^1(\Omega))$  and  $p^\gamma \in C([0, T]; H_0^1(\Omega))$  such that  $(u^\gamma, y^\gamma = y^\gamma(u^\gamma, 0), q^\gamma, p^\gamma)$  satisfies  
 672     the following optimality system:

674     
$$\begin{cases} D_{RL}^\alpha y^\gamma - \Delta y^\gamma = u^\gamma & \text{in } Q, \\ y^\gamma = 0 & \text{on } \Sigma, \\ I^{2-\alpha} y^\gamma(0) = y^0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} y^\gamma(0) = 0 & \text{in } \Omega, \end{cases} \quad (65)$$
  
 675

676     
$$\begin{cases} D_{RL}^\alpha q^\gamma - \Delta q^\gamma = 0 & \text{in } Q, \\ q^\gamma = 0 & \text{on } \Sigma, \\ I^{2-\alpha} q^\gamma(0) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} q^\gamma(0) = \frac{1}{\sqrt{\gamma}} \zeta(0; u^\gamma) & \text{in } \Omega, \end{cases} \quad (66)$$
  
 677

678     
$$\begin{cases} \mathcal{D}_C^\alpha p^\gamma - \Delta p^\gamma = y^\gamma - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma & \text{in } Q, \\ p^\gamma = 0 & \text{on } \Sigma, \\ p^\gamma(T) = 0 & \text{in } \Omega, \\ \frac{\partial p^\gamma}{\partial t}(T) = 0 & \text{in } \Omega, \end{cases} \quad (67)$$
  
 679

680     and

681     
$$N u^\gamma + p^\gamma = 0 \quad \text{in } Q. \quad (68)$$
  
 682

683     *Proof* Equations (43), (47), (48), and (49) give (65). To characterize the low-regret control  
 684      $u^\gamma$ , we use the Euler-Lagrange optimality conditions:  
 685

686     
$$\frac{d}{dk} J_\gamma(u^\gamma + k(v - u^\gamma))|_{k=0} = 0, \quad \forall v \in L^2(Q). \quad (69)$$
  
 687

688     After some calculations, we obtain  
 689

690     
$$\int_Q (y(u^\gamma, 0) - z_d)(y(v, 0) - y(u^\gamma, 0)) dt dx + \int_Q N u^\gamma (v - u^\gamma) dt dx$$
  
 691     
$$+ \frac{1}{\gamma} \int_\Omega (\zeta(x, 0; u^\gamma), \zeta(x, 0; v - u^\gamma)) dx = 0, \quad \forall v \in L^2(Q), \quad (70)$$
  
 692

693

706 where from (24),  $\zeta(v - u^\gamma) = \zeta(x, t; v - u^\gamma) \in C([0, T]; H_0^1(\Omega))$  is a solution of  
 707

708 
$$\begin{cases} \mathcal{D}_t^\alpha \zeta(v - u^\gamma) - \Delta \zeta(v - u^\gamma) = y(v, 0) - y^\gamma(u^\gamma, 0) & \text{in } Q, \\ \zeta(v - u^\gamma) = 0 & \text{on } \Sigma, \\ \zeta(T; v - u^\gamma) = 0 & \text{in } \Omega, \\ \frac{\partial \zeta}{\partial t}(T; v - u^\gamma) = 0 & \text{in } \Omega. \end{cases} \quad (71)$$
  
 711

713 Let  $z(v - u^\gamma) = y(x, t; v, 0) - y^\gamma(x, t; u^\gamma, 0)$  be the state associated to  $(v - u^\gamma) \in L^2(Q)$ . Then  
 714 in view of (19),  $z = z(v - u^\gamma) \in L^2((0, T); H_0^1(\Omega))$  is a solution of  
 715

716 
$$\begin{cases} D_{RL}^\alpha z - \Delta z = v - u^\gamma & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ I^{2-\alpha} z(0) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial t} I^{2-\alpha} z(0) = 0 & \text{in } \Omega. \end{cases} \quad (72)$$
  
 719

721 To interpret (70), we introduce  $q^\gamma = q^\gamma(u^\gamma, 0)$  as a solution of equation (66). As  
 722  $\frac{1}{\sqrt{\gamma}} \zeta(\cdot, 0; u^\gamma) \in L^2(\Omega)$ , according to Theorem 2.6,  $q^\gamma$  is unique and belongs to  $L^2((0, T);$   
 723  $H_0^1(\Omega))$ . Moreover,

724 
$$\|q^\gamma\|_{L^2((0, T); H_0^1(\Omega))} \leq \frac{C}{\sqrt{\gamma}} \|\zeta(0; u^\gamma)\|_{L^2(\Omega)}, \quad (73)$$
  
 726

727 where  $C > 0$  is a positive constant independent of  $\gamma$ .

728 Multiplying the first equation of (71) by  $\frac{1}{\sqrt{\gamma}} q^\gamma$  and using Lemma 2.5, we obtain  
 729

730 
$$\int_{\Omega} \frac{1}{\gamma} \zeta(x, 0; v - u^\gamma) \zeta(x, 0; u^\gamma) dx = \int_Q y(v, 0) - y(u^\gamma, 0) \frac{1}{\sqrt{\gamma}} q^\gamma dt dx,$$
  
 732

733 which combining with (70) gives  
 734

735 
$$\begin{aligned} & \int_Q [y(v, 0) - y(u^\gamma, 0)] \left[ \left( y(u^\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma \right) \right] dt dx \\ & + \int_Q N u^\gamma (v - u^\gamma) dt dx = 0, \quad \forall v \in L^2(Q). \end{aligned} \quad (74)$$
  
 739

740 Now, let  $p^\gamma$  verify (67). Then, in view of Corollary 2.8,  $p^\gamma \in C([0, T]; H_0^1(\Omega))$ , and  $\frac{\partial}{\partial t} p^\gamma \in$   
 741  $C([0, T]; L^2(\Omega))$  since  $y^\gamma - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma \in L^2(Q)$ .

742 Thus, multiplying the first equation of (72) by  $p^\gamma$ , a solution of (67), then, utilizing the  
 743 fractional integration by parts provided by Lemma 2.5, we conclude  
 744

745 
$$\int_Q z(v - u^\gamma) (\mathcal{D}_C^\alpha p^\gamma - \Delta p^\gamma) dx dt = \int_Q (v - u^\gamma) p^\gamma dx dt.$$
  
 746

747 Replacing in the latter identity  $z(v - u^\gamma)$  by  $y(x, t; v, 0) - y^\gamma(x, t; u^\gamma, 0)$ , which is a solution  
 748 of (72), we obtain  
 749

750 
$$\int_Q (v - u^\gamma) p^\gamma dx dt$$
  
 752

753  $= \int_Q [y(x, t; \nu, 0) - y^\gamma(x, t; u^\gamma, 0)] \left[ \left( y^\gamma(u^\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} q^\gamma \right) \right] dt dx,$   
 754

755 which combining with (74) gives  
 756

757  $\int_Q (Nu^\gamma + p^\gamma)(\nu - u^\gamma) dx dt = 0, \quad \forall \nu \in L^2(Q).$   
 758

759 Consequently  $Nu^\gamma + p^\gamma = 0$  in  $Q$ . □  
 760

761 **Proposition 3.5** For any  $\gamma > 0$ , let  $u^\gamma$  be the low-regret control. Then  $u^\gamma$  converges to  $u$ , a  
 762 solution of the no-regret problem (30).  
 763

764 *Proof* As  $u^\gamma$  is a solution of (33), we have  
 765

766  $J_\gamma(u^\gamma) \leq J_\gamma(0) = 0,$   
 767

768 because in view of (24),  $\zeta(0) = \zeta(x, t; 0) = 0$  in  $Q$ . It then follows from the definition of  $J_\gamma$   
 769 given by (34) that  
 770

771  $\|y(u^\gamma, 0) - z_d\|_{L^2(Q)}^2 + N \|u^\gamma\|_{L^2(Q)}^2 + \frac{1}{\gamma} \|\zeta(\cdot, 0; u^\gamma)\|_{L^2(\Omega)}^2$   
 772  
 773  $\leq J(0, 0) = \|y(0, 0) - z_d\|_{L^2(Q)}^2.$   
 774

775 Therefore, we deduce that  
 776

777  $\|y(u^\gamma, 0)\|_{L^2(Q)} \leq \|y(0, 0) - z_d\|_{L^2(Q)}, \tag{75a}$   
 778

779  $\|u^\gamma\|_{L^2(Q)} \leq \frac{1}{N} \|y(0, 0) - z_d\|_{L^2(Q)}, \tag{75b}$   
 780

781  $\|\zeta(\cdot, 0; u^\gamma)\|_{L^2(\Omega)} \leq \sqrt{\gamma} \|y(0, 0) - z_d\|_{L^2(Q)}. \tag{75c}$

782 Hence from (75b) and (65)<sub>1</sub>, we have  
 783

784  $\|D_{RL}^\alpha y(u^\gamma, 0) - \Delta y(u^\gamma, 0)\|_{L^2(Q)} \leq \frac{1}{N} \|y(0, 0) - z_d\|_{L^2(Q)}. \tag{76}$   
 785

786 Since  $y(u^\gamma, 0)$  is solution of (65), we see from Theorem 2.6 that there exists a constant  
 787  $C$  independent of  $\gamma$  such that  
 788

789  $\|y(u^\gamma, 0)\|_{L^2((0, T); H_0^1(\Omega))} \leq \frac{C}{N} \|y(0, 0) - z_d\|_{L^2(Q)}. \tag{77}$   
 790

792 Thus there exist  $u \in L^2(Q)$ ,  $y \in L^2((0, T); H_0^1(\Omega))$ ,  $\delta \in L^2(Q)$ , and subsequences extracted  
 793 of  $(u^\gamma)$  and  $(y^\gamma)$  (still called  $(u^\gamma)$  and  $(y^\gamma)$ ) such that  
 794

795  $u^\gamma \rightharpoonup u \text{ weakly in } L^2(Q), \tag{78a}$

796  $y^\gamma \rightharpoonup y \text{ weakly in } L^2((0, T); H_0^1(\Omega)), \tag{78b}$

798  $D_{RL}^\alpha y^\gamma - \Delta y^\gamma \rightharpoonup \delta \text{ weakly in } L^2(Q). \tag{78c}$

799

800 If we proceed as in pp.10 to 14, using (78a)-(78c), we show that  $y = y(x, t; u, 0)$  is such that  
 801

802 
$$\begin{cases} D_{\text{RL}}^\alpha y - \Delta y = u & \text{in } Q, \\ 803 y = 0 & \text{on } \Sigma, \\ 804 I^{2-\alpha} y(x, 0) = y^0 & \text{in } \Omega, \\ 805 \frac{\partial}{\partial t} I^{2-\alpha} y(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (79)$$

806  
 807 and  $\zeta = \zeta(x, t; u) \in \mathcal{C}([0, T]; H_0^1(\Omega))$  is a solution of  
 808

809 
$$\begin{cases} \mathcal{D}^\alpha \zeta - \Delta \zeta = y(u, 0) - y(0, 0) & \text{in } Q, \\ 810 \zeta = 0 & \text{on } \Sigma, \\ 811 \zeta(T) = 0 & \text{in } \Omega, \\ 812 \frac{\partial \zeta}{\partial t}(T) = 0 & \text{in } \Omega. \end{cases} \quad (80)$$

813 Moreover, in view of (75c), we have  
 814

815 
$$\zeta(\cdot, 0; u^\gamma) \rightarrow \zeta(\cdot, 0; u) = 0 \quad \text{strongly in } L^2(\Omega). \quad (81)$$
  
 816

817 Consequently,  $\int_\Omega g \zeta(x, 0; u) dx = 0$ .

818 This implies that  $u$  is solution of the no-regret control problem (30).  $\square$   
 819

820 **Theorem 3.6** Let us consider  $u = \lim_{\gamma \rightarrow 0} u^\gamma$  be the no-regret control corresponding to the  
 821 state  $y(u, 0)$ . Then there exist  $q \in L^2((0, T); H_0^1(\Omega))$  and  $p \in \mathcal{C}([0, T]; H_0^1(\Omega))$  in such a way  
 822 that  $(u, y = y(u, 0), q, p)$  fulfills the following optimality system:

823 
$$\begin{cases} D_{\text{RL}}^\alpha y - \Delta y = u & \text{in } Q, \\ 824 y = 0 & \text{on } \Sigma, \\ 825 I^{2-\alpha} y(0) = y^0 & \text{in } \Omega, \\ 826 \frac{\partial}{\partial t} I^{2-\alpha} y(0) = 0 & \text{in } \Omega, \end{cases} \quad (82)$$

827 
$$\begin{cases} D_{\text{RL}}^\alpha q - \Delta q = 0 & \text{in } Q, \\ 828 q = 0 & \text{on } \Sigma, \\ 829 I^{2-\alpha} q(0) = 0 & \text{in } \Omega, \\ 830 \frac{\partial}{\partial t} I^{2-\alpha} q(0) = \tau_1 & \text{in } \Omega, \end{cases} \quad (83)$$

831 
$$\begin{cases} \mathcal{D}_C^\alpha p - \Delta p = y(u, 0) - z_d + \tau_2 & \text{in } Q, \\ 832 p = 0 & \text{on } \Sigma, \\ 833 p(T) = 0 & \text{in } \Omega, \\ 834 \frac{\partial p}{\partial t}(T) = 0 & \text{in } \Omega, \end{cases} \quad (84)$$

835 and

836 
$$N u + p = 0 \quad \text{in } Q. \quad (85)$$
  
 837

838 *Proof* We have (82) (see system (79)).  
 839

840 From (75c), we get  
 841

842 
$$\left\| \frac{1}{\sqrt{\gamma}} \zeta(0; u^\gamma) \right\|_{L^2(\Omega)} \leq \|y(0, 0) - z_d\|_{L^2(Q)}.$$
  
 843

844

847 Consequently, equation (73) becomes

848

849  $\|q^\gamma\|_{L^2((0,T);H_0^1(\Omega))} \leq C \|y(0,0) - z_d\|_{L^2(Q)}.$  (86)

850

851 Thus, there exist  $\tau_1 \in L^2(\Omega)$  and  $q \in L^2(0, T; H_0^1(\Omega))$  such that

852

853  $\frac{1}{\sqrt{\gamma}} \zeta(\cdot, 0; u^\gamma) \rightharpoonup \tau_1 \quad \text{weakly in } L^2(\Omega),$  (87)

854

855  $q^\gamma \rightharpoonup q \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)).$  (88)

856

857 Using (87) and (88) while passing to the limit in (66), we show as for the convergence of  
858  $y_n = y(v_n, 0)$  (see pp.10 to 12) that  $q$  satisfies (83).

859 From (68) and (75b), we have

860

861  $\|p^\gamma\|_{L^2(Q)} \leq \|y(0,0) - z_d\|_{L^2(Q)}.$

862

863 Therefore there exists  $p \in L^2(Q)$  such that

864

865  $p^\gamma \rightharpoonup p \quad \text{weakly in } L^2(Q).$  (89)

866

867 In view of (67) and (75a), we know that there exist  $\tau_2 \in L^2(Q)$  such that

868

869  $\frac{1}{\sqrt{\gamma}} q^\gamma \rightharpoonup \tau_2 \quad \text{weakly in } L^2(Q).$  (90)

870

871 Then we prove as for the convergence of  $\zeta_n = \zeta(x, t; v_n)$  (see pp.12 to 14) that  $p$  is solution  
872 of (84). Using (75b) and (89) while passing to the limit in (68), we conclude (85).  $\square$

873

## 874 4 Conclusions

875

We study an optimal control problem associated to a fractional wave equation involving  
876 Riemann-Liouville fractional derivative and with incomplete data. Actually, the initial con-  
877 dition is missing. In order to solve the problem, we assume that the missing data belongs  
878 to an infinite dimensional space. Using the notions of no-regret and low-regret controls,  
879 we show that when  $3/2 \leq \alpha \leq 2$ , such a control exists and is unique. Then we give the  
880 singular optimality system that characterizes the control.

881

882

### 883 Competing interests

The authors declare that they have no competing interests.

884

### 885 Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

886

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889

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891

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