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# The Geometry of Uniqueness and Model Selection of Penalized Estimators including SLOPE, LASSO and Basis Pursuit

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## Abstract

We provide a necessary and sufficient condition for the uniqueness of penalized least-squares estimators with a penalty term consisting a norm whose unit ball is given by a polytope. The condition is given by a geometric criterion involving how the row span of the design matrix intersects the faces of the dual norm unit cube. This criterion also provides information about the model selection properties of the corresponding estimation method. Our analyses cover LASSO, the related method of basis pursuit, as well as the SLOPE estimator.

## 1 Introduction

The linear regression model  $Y = X\beta + \varepsilon$ , where  $X \in \mathbb{R}^{n \times p}$  is a fixed matrix,  $\beta \in \mathbb{R}^p$  is an unknown parameter vector, and  $\varepsilon$  is a centered random error term in  $\mathbb{R}^n$ , is one of the most important models in statistics. When  $\ker(X) = \{0\}$ , the ordinary least-squares estimator  $\hat{\beta}^{\text{ols}} = (X'X)^{-1}X'Y$ , which minimizes the residual sum of squares  $\|Y - Xb\|_2^2$  with respect to  $b \in \mathbb{R}^p$ , is the usual estimator of  $\beta$ . In high dimensions, when  $p > n$ , and thus  $\ker(X) \neq \{0\}$ , the ordinary least squares estimator is no longer well-defined, as then the function  $b \in \mathbb{R}^p \mapsto \|Y - Xb\|_2^2$  does not have a unique minimizer.

In this case, typically, a penalty term is added to the residual sum of squares to provide an alternative to ordinary least-squares estimation. In some cases, also the minimizer of the penalized least-squares optimization problem is not unique. In this paper, we provide a necessary and sufficient condition for uniqueness based on a geometric criterion for a wide class of penalties. Moreover, the geometry involved in this condition also yields results for model selection, i.e. sparsity, and related properties of SLOPE, LASSO and basis pursuit.

### 1.1 Penalized least-squares estimators

The Ridge estimator, which minimizes the function  $b \in \mathbb{R}^p \mapsto \frac{1}{2} \|Y - Xb\|_2^2 + \lambda \|b\|_2^2$ , where  $\lambda > 0$  is a so-called tuning parameter, was the first penalized estimator to appear in the statistics literature (Hoerl and Kennard, 1970; Golub et al., 1979). Due to the strict convexity of the function  $b \mapsto \|b\|_2^2$  the minimizer is always unique and is given by  $\hat{\beta}^{\text{ridge}} = (X'X + \lambda \mathbb{I}_p)^{-1}X'Y$ . This estimator is not

sparse, meaning that it does almost surely not set components equal to zero. Especially when  $p$  is large, this can make the estimator more difficult to interpret compared to other sparse methods such as LASSO or SLOPE which are described in the following.

The Least Absolute Shrinkage and Selection Operator (LASSO) (Chen and Donoho, 1994; Alliney and Ruzinsky, 1994; Tibshirani, 1996) is the  $l_1$ -penalized least-squares estimator defined as

$$\hat{\beta}^{\text{lasso}} = \arg \min_{b \in \mathbb{R}^p} \frac{1}{2} \|Y - Xb\|_2^2 + \lambda \|b\|_1, \text{ where } \lambda > 0.$$

When  $\ker(X) = \{0\}$ , the function  $b \in \mathbb{R}^p \mapsto \|Y - Xb\|_2^2$  is strictly convex, immediately implying the uniqueness of the LASSO minimizer. In high dimensions,  $\ker(X) \neq \{0\}$  and the function  $b \in \mathbb{R}^p \mapsto \|Y - Xb\|_2^2$  is not strictly convex, thus uniqueness of  $\hat{\beta}^{\text{lasso}}$  is not guaranteed. The condition of the columns of the design matrix  $X$  being in general position, first outlined by Rosset et al. (2004) and later investigated by Tibshirani (2013) and Ali and Tibshirani (2019), provides a sufficient condition under which, for all  $Y \in \mathbb{R}^n$ , the LASSO minimizer is unique and subsequently  $\hat{\beta}^{\text{lasso}}$  is well-defined. This condition was recently relaxed by Ewald and Schneider (2020) to a criterion that is both sufficient and necessary. As mentioned above, the LASSO estimator is a sparse method that generally sets components equal to zero with positive probability, entailing that the estimator also performs what is called model selection. In particular, if the solution is unique and  $p > n$ ,  $\hat{\beta}^{\text{lasso}}$  contains at least  $p - n$  null components.

A strongly related procedure is basis pursuit, which first appeared in compressed sensing (Chen and Donoho, 1994) and is defined as

$$\hat{\beta}^{\text{bp}} = \arg \min \|b\|_1 \text{ subject to } Y = Xb.$$

In the noiseless case, this method allows to recover a sparse vector  $\beta$  (see e.g. Candès et al., 2006; Cohen et al., 2009). In the noisy case, when  $\varepsilon$  is no longer zero, the basis pursuit estimator can be viewed as the LASSO when the tuning parameter  $\lambda > 0$  becomes infinitely small. This estimator sometimes used for sign recovery of  $\beta$  (see e.g. Saligrama and Zhao, 2011; Tardivel and Bogdan, 2018). Naturally, basis pursuit shares lot of properties with the LASSO estimator. For example, general position of the columns of the design matrix  $X$  is also a sufficient condition for uniqueness of  $\hat{\beta}^{\text{bp}}$  (see e.g. Tardivel and Bogdan, 2018) for all  $Y \in \mathbb{R}^n$ . However, to the best of our knowledge, a necessary and sufficient condition has previously been unknown.

Our results also cover Sorted L-One Penalized Estimation (SLOPE) (Zeng and Figueiredo, 2014; Bogdan et al., 2015) which is the penalized estimator given by

$$\hat{\beta}^{\text{slope}} = \arg \min_{b \in \mathbb{R}^p} \frac{1}{2} \|Y - Xb\|_2^2 + \sum_{j=1}^p w_j |b|_{(j)},$$

where  $w_1 > 0$  and  $w_1 \geq \dots \geq w_p \geq 0$  and  $|b|_{(1)} \geq \dots \geq |b|_{(p)}$ . Note that penalty term gives rise to the so-called SLOPE norm. A special case of this estimator, the Octagonal Shrinkage and Clustering Algorithm for Regression (OSCAR), has already been introduced in Bondell and Reich (2008). The

SLOPE estimator is well-defined once the corresponding minimizer is unique and, similarly to the LASSO, uniqueness is obvious when  $\ker(X) = \{0\}$ . However, in contrast to the LASSO, no condition guaranteeing uniqueness has been established.

The SLOPE estimator is a sparse method that also exhibits a clustering property in that some components may be equal in absolute value with positive probability. This can be learned from the explicit expressions one obtains in case the columns of  $X$  are orthogonal (Tardivel et al., 2020), and this property also holds in the non-orthogonal case.

In certain applications, this clustering phenomenon – which is not shared by the LASSO – may be of particular relevance (Kremer et al., 2019, 2020). To the best of our knowledge, an exhaustive description of the clusters induced by SLOPE has not been provided before.

## 1.2 Uniqueness and polytope unit balls

We study the problem of uniqueness of penalized estimators in a general setting, where the penalty term is not restricted the  $l_1$ - or the SLOPE norm. Let  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$ , and  $\|\cdot\|$  be a norm on  $\mathbb{R}^p$ . Consider the solution set  $S_{X, \|\cdot\|}(y)$  to the penalized least-squares problem

$$S_{X, \|\cdot\|}(y) = \arg \min_{b \in \mathbb{R}^p} \frac{1}{2} \|y - Xb\|_2^2 + \|b\|.$$

Note that  $S_{X, \|\cdot\|}(y)$  is non-empty since the function  $b \in \mathbb{R}^p \mapsto \frac{1}{2} \|y - Xb\|_2^2 + \|b\|$  is continuous and unbounded when  $\|b\|$  becomes large. The penalty term may include a positive tuning parameter which can be viewed as part of the norm, for instance  $\|\cdot\| = \lambda \|\cdot\|_1$  for the LASSO estimator. When  $\|\cdot\|$  is a norm for which  $\|b + \tilde{b}\| = \|b\| + \|\tilde{b}\|$  holds if and only if  $b = t\tilde{b}$  where  $t \geq 0$ , such as the  $l_2$ -norm, then  $S_{X, \|\cdot\|}(y)$  is a singleton for all  $y \in \mathbb{R}^n$  and for all  $X \in \mathbb{R}^{n \times p}$ . This statement is a straightforward consequence of the following facts. When  $\hat{\beta}, \tilde{\beta} \in S_{X, \|\cdot\|}(y)$  we have

- i)  $X\hat{\beta} = X\tilde{\beta}$  (see Lemma 2 in the appendix).
- ii) Since  $(\hat{\beta} + \tilde{\beta})/2 \in S_{X, \|\cdot\|}(y)$  also,  $\|(\hat{\beta} + \tilde{\beta})/2\| = (\|\hat{\beta}\| + \|\tilde{\beta}\|)/2$  follows.

Geometrically, such a norm  $\|\cdot\|$  possess a unit ball  $\{x \in \mathbb{R}^p : \|x\| = 1\}$  with no edges. Subsequently, the problem of uniqueness is only relevant when the unit ball of the norm under consideration contains an edge. More concretely, we restrict our attention to norms for which the unit ball  $B = \{x \in \mathbb{R}^p : \|x\| \leq 1\}$  is given by a polytope. Note that this is the case for the  $l_1$ -norm, the  $l_\infty$ -norm, and the SLOPE norm. Our results also cover methods with a mixed  $l_1, l_\infty$ -norm penalty term (Negahban and Wainwright, 2008; Bach et al., 2012).

## 1.3 Accessible models and sign estimation

The sign vector  $\text{sign}(\beta) = (\text{sign}(\beta_1), \dots, \text{sign}(\beta_p))'$  (where  $\text{sign}(t) = 1$  if  $t > 0$ ,  $\text{sign}(t) = -1$  if  $t < 0$  or 0 otherwise) is accessible when there exists  $Y \in \mathbb{R}^n$  for which  $\text{sign}(\hat{\beta}^{\text{lasso}}) = \text{sign}(\beta)$ . Under the assumption that LASSO minimizer is unique, Sepehri and Harris (2017) provide a geometrical characterization of accessible sign vectors. Of course, when  $\text{sign}(\beta)$  is not accessible then  $\mathbb{P}(\text{sign}(\hat{\beta}^{\text{lasso}}) = \text{sign}(\beta)) = 0$ . Note that the irrepresentable condition on  $\beta$  (which is a necessary

condition for  $\mathbb{P}(\text{sign}(\hat{\beta}^{\text{lasso}}) = \text{sign}(\beta)) > 1/2$ , see Wainwright, 2009) implies that  $\text{sign}(\beta)$  is accessible. To our knowledge, there is no theoretical work for clusters recovery by SLOPE estimator. Generalizing the notion of accessible sign vectors for LASSO estimator to accessible models for SLOPE estimator should be a first improvement for the development of such a theory.

## 1.4 Related geometrical works

Most articles providing geometrical properties for penalized estimators concern the LASSO and one recent article provides geometrical results for SLOPE, which we discuss below. For basis pursuit, geometrical properties mainly deal with the  $l_1$ -recovery in the noiseless case, en route to derive the phase-transition curve (Donoho and Tanner, 2009), which is not directly related to our work.

Concerning the LASSO, Tibshirani and Taylor (2012) show that the LASSO residual  $Y - X\hat{\beta}^{\text{lasso}}$  is the projection of  $Y$  onto the polytope  $\{z \in \mathbb{R}^n : \|X'z\|_\infty \leq \lambda\}$ . From this result, the authors derive an explicit formula for the Stein's unbiased risk estimate which allows to select  $\lambda$  in order to minimize the  $L_2$  risk:  $\mathbb{E}(\|X\hat{\beta}^{\text{lasso}} - X\beta\|_2^2)$ . This geometrical result also lays the groundwork for selective inference (Lee et al., 2016), for deriving screening procedures (Ghaoui et al., 2010; Wang et al., 2013), and to describe the accessible LASSO models in Sepehri and Harris (2017). Concerning SLOPE, very recent work of Minami (2020) generalizes some results of Tibshirani and Taylor (2012) and shows that the number of non-null clusters ( $\|\text{mdl}(\hat{\beta}^{\text{slope}})\|_\infty$  in our article) appears in the Stein's unbiased risk estimate for SLOPE estimator.

Our article provides properties for penalized estimators for 1) the uniqueness of the minimizer, 2) accessible models and 3) model selection based on the null polytope. We apply our approach to SLOPE, LASSO and Basis Pursuit (BP). Table 1 summarizes results already known in the literature.

	BP	LASSO	SLOPE
Sufficient condition for uniqueness	YES <sup>1</sup>	YES <sup>2,3,4</sup>	NO
Necessary and sufficient condition for uniqueness	NO	YES <sup>5</sup>	NO
Accessible models (characterization through dual unit ball in $\mathbb{R}^p$ )	NO	NO	NO
Accessible models (characterization through null polytope in $\mathbb{R}^n$ )	NO	YES <sup>6</sup>	NO

Table 1: Summary of results known in the literature. 1: Tardivel and Bogdan (2018), 2: Rosset et al. (2004), 3: Tibshirani (2013), 4: Ali and Tibshirani (2019), 5: Ewald and Schneider (2020), 6: Sepehri and Harris (2017).

## 1.5 Notation and structure

We use the following notation in this article. We denote the set  $\{1, \dots, k\}$  by  $[k]$  and use  $|I|$  for the cardinality of a set  $I$ . For a matrix  $A$ , the symbols  $\text{col}(A)$  and  $\text{row}(A)$  stand for the column and row space of  $A$ , respectively, and  $\text{conv}(A)$  is the convex hull of the columns of  $A$ . As used in the introduction, for a number  $t$ ,  $\text{sign}(t)$  is given by 1,  $-1$  or 0 if  $t > 0$ ,  $t < 0$ , or  $t = 0$ , respectively. For a vector  $x$ ,  $\text{sign}(x)$  is the vector containing the signs of the components of  $x$ . The symbols  $\|\cdot\|_1$ ,  $\|\cdot\|_2$ ,  $\|\cdot\|_\infty$ , and  $\|\cdot\|_w$  represent the  $l_1$ ,  $l_2$ , supremum, and the SLOPE-norm, respectively. Finally, the set  $\mathcal{S}_p$  contains all permutations on the set  $[p]$ .

The remainder of this article is organized as follows. Section 2 contains the general theorem of uniqueness for penalized least-squares estimators, as well as the analogous necessary and sufficient uniqueness condition for basis pursuit. In Section 3, we investigate the model selection properties related to the geometric condition introduced in Section 2 for LASSO, BP, and SLOPE estimators, including a characterization of the SLOPE’s clustering property. This section also contains results on model selection in connection with the null polytope. All proofs are relegated to the appendix, which also contains a remainder of basic facts of subdifferentials and polytopes.

## 2 A necessary and sufficient condition for uniqueness of penalized problems

For a norm  $\|\cdot\|$  on  $\mathbb{R}^p$ , the dual norm  $\|\cdot\|^*$  is defined by

$$\|x\|^* = \sup_{s \in \mathbb{R}^p: \|s\| \leq 1} s'x.$$

If the unit ball  $B = \{x \in \mathbb{R}^p : \|x\| \leq 1\}$  is of polytope shape, the polar dual of  $B$  is given by  $B^* = \{x \in \mathbb{R}^p : \|x\|^* \leq 1\}$ , the unit ball of the dual norm, which is, again, a polytope. In that case, there is a strong connection between the subdifferentials  $\partial_{\|\cdot\|}(\cdot)$  of the norm  $\|\cdot\|$  and the faces of the polytope  $B^*$ , the unit ball of the dual norm  $\|\cdot\|^*$ , see Appendices A.1-A.3 for more details. This connection is the basis for (the proof of) the main theorem:

**Theorem 1** (Necessary and sufficient condition for uniqueness). *Let  $X \in \mathbb{R}^{n \times p}$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^p$  whose unit ball  $B$  is given by a polytope. Consider the penalized optimization problem*

$$S_{X, \|\cdot\|}(y) = \arg \min_{b \in \mathbb{R}^p} \frac{1}{2} \|y - Xb\|^2 + \|b\|, \quad (1)$$

where  $y \in \mathbb{R}^n$ . Let  $B^*$  denote the unit ball of the dual norm  $\|\cdot\|^*$ . There exists  $y \in \mathbb{R}^n$  with  $|S_{X, \|\cdot\|}(y)| > 1$  if and only if  $\text{row}(X)$  intersects a face of the dual unit ball  $B^*$  whose codimension is larger than  $\text{rk}(X)$ .

The notion of uniqueness considered in Theorem 1 is strong in the sense that it involves uniqueness for a given design matrix  $X$  for all values  $y \in \mathbb{R}^n$ . Note that if the norm  $\|\cdot\|$  involves a tuning parameter  $\lambda$ , uniqueness of the corresponding penalized problem does not depend on the particular choice of  $\lambda$ : The parameter simply scales the unit ball  $B$  and subsequently the dual unit ball  $B^*$  which does not influence which faces are intersected by the vector space  $\text{row}(X)$ .

Theorem 1 generalizes Theorem 14 given in Ewald and Schneider (2020) which provides a necessary and sufficient condition for the uniqueness of the LASSO minimizer: All LASSO solutions are unique if and only if  $\text{row}(X)$  only intersects faces of the unit cube  $[-1, 1]^p$  whose codimension is less than or equal to  $\text{rk}(X)$ . Note that the unit cube is, indeed, the corresponding dual to the unit ball of the  $l_1$ -norm.

**Example.** We illustrate the criterion from Theorem 1 for  $\|\cdot\| = \|\cdot\|_\infty$ , the supremum norm. Let  $X = (1 \ 0)$ . The unit dual ball  $B^*$  is given by the unit cross polytope  $\text{conv}\{\pm(1, 0)', \pm(0, 1)'\}$  and

we have  $\text{rk}(X) = 1$ . Clearly, the vertex  $(1, 0)'$  with codimension  $p - 0 = 2 > 1 = \text{rk}(X)$  intersects  $\text{row}(X)$ , so that one can pick  $y \in \mathbb{R}$  for which the set of minimizers  $S_{X, \|\cdot\|_\infty}(y)$  is not a singleton. In Figure 1, we illustrate that  $S_{X, \|\cdot\|_\infty}(2)$  is not a singleton by plotting the contour lines of the function  $\phi(b_1, b_2) = 0.5(2 - b_1)^2 + \max\{|b_1|, |b_2|\}$ .

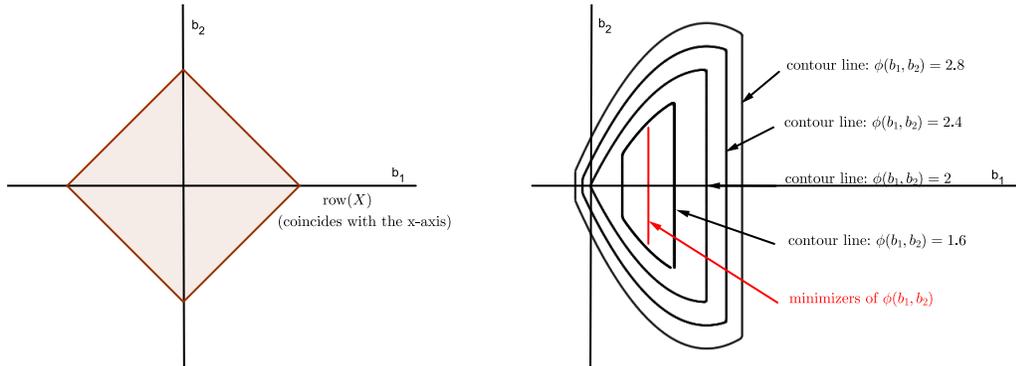


Figure 1: Illustration of Theorem 1 for the supremum norm and  $X = (1 \ 0)$ . On the left-hand side, we see that  $\text{row}(X)$  intersects a vertex of the cross polytope whose codimension is 2 and thus is larger than  $\text{rk}(X) = 1$ . Therefore, by Theorem 1, there exists  $y \in \mathbb{R}$  for which  $S_{X, \|\cdot\|_\infty}(y)$  is not a singleton. On the right-hand side, the contour lines of the function  $\phi(b_1, b_2) = 0.5(2 - b_1)^2 + \max\{|b_1|, |b_2|\}$  illustrate that the set  $S_{X, \text{bp}}(2)$  (in red) is, indeed, not a singleton.

Also consider  $X = (1 \ 1)$ . Because  $\text{row}(X)$  does not intersect any vertex of  $\text{conv}\{\pm(1, 0)', \pm(0, 1)'\}$ , the solution set  $S_{X, \|\cdot\|_\infty}(y)$  is always a singleton. In Figure 2, we illustrate that in this example,  $S_{X, \|\cdot\|_\infty}(2)$  is a singleton through the contour lines of the function  $\phi(b_1, b_2) = 0.5(2 - b_1 - b_2)^2 + \max\{|b_1|, |b_2|\}$ .

## 2.1 The related problem of basis pursuit

As mentioned before, the methods of LASSO and basis pursuit (BP) are closely related, as the BP problem can be thought of a LASSO problem with vanishing tuning parameter. More concretely, the setting for BP is the following. Let  $X \in \mathbb{R}^{n \times p}$  and let  $y \in \text{col}(X)$ . The set  $S_{X, \text{bp}}(y)$  of BP minimizers is defined as

$$S_{X, \text{bp}}(y) = \arg \min \|b\|_1 \text{ subject to } Xb = y.$$

The following theorem shows that, indeed, as BP is a limiting case of the LASSO, the corresponding uniqueness condition – which is independent of the choice of tuning parameter as discussed above – carries over to the BP problem.

**Theorem 2.** *Let  $X \in \mathbb{R}^{n \times p}$ . There exists  $y \in \text{col}(X)$  for which  $|S_{X, \text{bp}}(y)| > 1$  if and only if  $\text{row}(X)$  intersects a face of the unit cube  $[-1, 1]^p$  whose codimension is larger than  $\text{rk}(X)$ .*

We illustrate Theorem 2 in Figures 4 and 3.

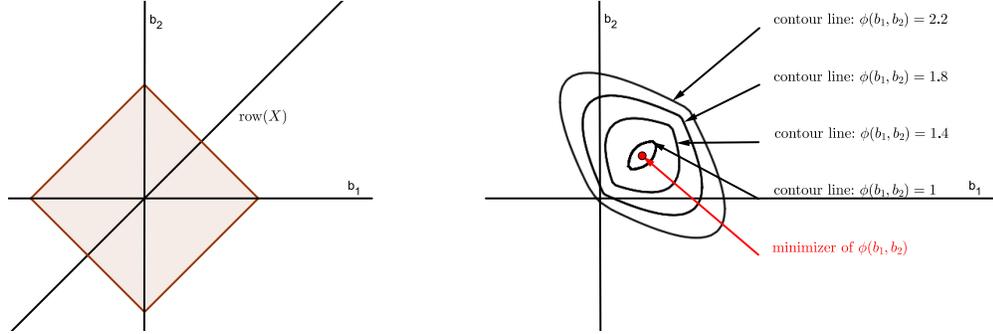


Figure 2: Illustration of Theorem 1 for the supremum norm and  $X = \begin{pmatrix} 1 & 1 \end{pmatrix}$ . On the left-hand side, we see that  $\text{row}(X)$  does not intersect a face of the cross polytope whose codimension is larger than  $\text{rk}(X) = 1$  (which are the vertices in this example). Therefore, by Theorem 1, the set  $S_{X, \|\cdot\|_\infty}(y)$  is a singleton for all  $y \in \mathbb{R}$ . On the right-hand side, the contour lines of the function  $\phi(b_1, b_2) = 0.5(2 - b_1)^2 + \max\{|b_1|, |b_2|\}$  illustrate that the set  $S_{X, \|\cdot\|_\infty}(2)$  (in red) is a singleton in this example.

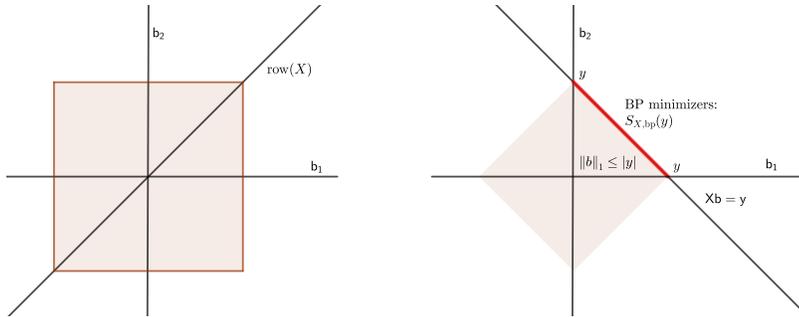


Figure 3: Illustration of Theorem 2 with  $X = \begin{pmatrix} 1 & 1 \end{pmatrix}$ . On the left-hand side, we see that  $\text{row}(X)$  intersects a face of the unit square whose codimension 2 is larger than  $\text{rk}(X) = 1$  (which are the vertices in this example). Therefore, by Theorem 2, there exists  $y \in \mathbb{R}$  for which the BP minimizer is not unique. The right-hand side illustrates that, indeed, for an arbitrary  $y \in \mathbb{R} \setminus \{0\}$ , the set  $S_{X, \text{bp}}(y)$  (the red segment) is not a singleton.

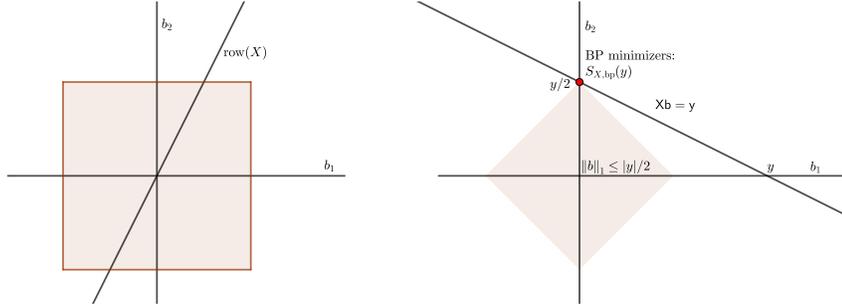


Figure 4: Illustration of Theorem 2 with  $X = \begin{pmatrix} 1 & 2 \end{pmatrix}$ . On the left-hand side, we see that  $\text{row}(X)$  does not intersect any face of the unit square whose codimension is larger than  $\text{rk}(X) = 1$  (which are the vertices in this example). Therefore, by Theorem 2, the BP minimizer is unique for all  $y \in \mathbb{R}$ . The right-hand side illustrates that for an arbitrary  $y \in \mathbb{R}$ , the set  $S_{X,\text{bp}}(y)$  (in red) is, indeed, a singleton.

In the following proposition, we show that the necessary and sufficient condition given in Theorem 1 and, therefore also the one given in Theorem 2, is weak. More precisely, we establish that the set of  $X \in \mathbb{R}^{n \times p}$  for which the necessary and sufficient condition given in Theorem 1 does not hold, is negligible with respect to the Lebesgue measure.

**Proposition 1.** *Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}^{n \times p}$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^p$  whose unit ball is given by a polytope. The following equality holds*

$$\mu(\{X \in \mathbb{R}^{n \times p} : \exists y \in \mathbb{R}^n \text{ with } |S_{X,\|\cdot\|}(y)| > 1\}) = 0.$$

The following corollary is then straightforward given the fact that LASSO, covered by Theorem 1), and BP share the same the characterization for uniqueness.

**Corollary 1.** *Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}^{n \times p}$ , then the following equality holds*

$$\mu(\{X \in \mathbb{R}^{n \times p} : \exists y \in \mathbb{R}^n \text{ with } |S_{X,\text{bp}}(y)| > 1\}) = 0.$$

By taking the appropriate norms in Proposition 1, and by Corollary 1, one may deduce that the necessary and sufficient conditions for uniqueness of BP, LASSO, and SLOPE are weak. However, one should be aware that Proposition 1 does not mean that this condition always occurs in practice! For example, for BP (or LASSO), when  $p > n$  and  $X \in \{-1, 1\}^{n \times p}$ , one can always pick  $y \in \text{col}(X)$  for which the set of minimizers  $S_{X,\text{bp}}(y)$  is not a singleton (or, for any  $\lambda > 0$ , one can pick  $y \in \mathbb{R}^n$  for which the set of minimizers  $S_{X,\lambda\|\cdot\|_1}(y)$  is not a singleton). Matrices having entries in  $\{-1, 1\}$  appear in several theoretical works, such as Rauhut (2010) and Tardivel et al. (2018), and are used for applications in radar and wireless communication (see e.g. Romberg, 2009; Haupt et al., 2010).

### 3 Model selection properties

We now investigate the geometrical aspects of the model selection features of certain penalized estimators.

#### 3.1 Accessible sign vectors for LASSO and BP

We start by introducing the notion of accessible sign vectors for LASSO and BP problems.

**Definition 1** (Accessible sign vectors for LASSO and BP). *Let  $X \in \mathbb{R}^{n \times p}$ ,  $\sigma \in \{-1, 0, 1\}^p$ , and  $\lambda > 0$ . We say that  $\sigma$  is an accessible sign vector for LASSO (or BP), if there exists  $y \in \mathbb{R}^n$  and  $\hat{\beta} \in S_{X, \lambda \|\cdot\|_1}(y)$  (or there exists  $y \in \text{col}(X)$  and  $\hat{\beta} \in S_{X, \text{bp}}(y)$ , respectively), such that  $\text{sign}(\hat{\beta}) = \sigma$ .*

The following theorem provides a geometric characterization of accessible sign vectors for LASSO and BP based on faces of the unit cube  $[-1, 1]^p$  and the vector space  $\text{row}(X)$ . First, note that subdifferential calculus of the  $l_1$ -norm at  $\sigma \in \{-1, 0, 1\}^p$  gives

$$\partial_{\|\cdot\|_1}(\sigma) = E_1 \times \cdots \times E_p \text{ with } E_j = \begin{cases} \{\sigma_j\} & |\sigma_j| = 1 \\ [-1, 1] & \sigma_j = 0, \end{cases}$$

where  $\partial_{\|\cdot\|_1}(x)$  denotes the subdifferential of the  $l_1$ -norm at  $x \in \mathbb{R}^p$ , see Appendices A.1 and A.3 for more details. Therefore, the mapping  $\sigma \mapsto \partial_{\|\cdot\|_1}(\sigma)$  is a bijection between sign vectors in  $\{-1, 0, 1\}^p$  and faces of the unit cube in  $\mathbb{R}^p$ . We let  $F(\sigma) = \partial_{\|\cdot\|_1}(\sigma)$  in the following.

**Theorem 3** (Characterization of accessible LASSO and BP sign vectors). *Let  $X \in \mathbb{R}^{n \times p}$  and  $\lambda > 0$ . A sign vector  $\sigma \in \{-1, 0, 1\}^p$  is accessible for LASSO (or BP, respectively) if and only if  $\text{row}(X)$  intersects the face  $F(\sigma)$ .*

Note that Theorem 3 shows that whether a sign vector is accessible for LASSO does not depend on the value of the tuning parameter  $\lambda$ . We also point out that Theorems 1 and 3 allow to deduce that the number of non-null components of the LASSO minimizer is always less than or equal to  $\text{rk}(X)$  if the solutions are unique. Indeed, when the LASSO minimizer is unique, according to Theorem 1,  $\text{row}(X)$  does not intersect a face of  $[-1, 1]^p$  associated to a sign vector having more than  $\text{rk}(X)$  non-null components, i.e., a face whose codimension is larger than  $\text{rk}(X)$ . This implies that only models (sign vectors) with at most  $\text{rk}(X)$  non-null components are accessible. For the LASSO, this is a refined version of the well-known fact that, in case the estimator is unique, at most  $n$  components can be non-zero (see e.g. Tibshirani, 2013; Osborne et al., 2000). A similar approach for SLOPE is developed in the following, geometrically characterizing why the number of non-null clusters of SLOPE minimizers is less than or equal to  $\text{rk}(X)$  in case of uniqueness.

#### 3.2 Accessible models for SLOPE

We investigate geometric features of the model selection properties of SLOPE. For the remainder of Section 3, we assume that the weight vector of the SLOPE norm  $\|\cdot\|_w$  satisfies

$$w_1 > \cdots > w_p > 0,$$

i.e., that all components non-zero and strictly decreasing.

**Definition 2.** We say that a vector  $m \in \mathbb{Z}^p$  is a SLOPE model, if either  $m = 0$ , or, if for all  $l \in [||m||_\infty]$ , there exists  $j \in [p]$  such that  $|m_j| = l$ . We denote the set of all SLOPE models with  $p$  by  $\mathcal{M}_p$ . Moreover, for  $x \in \mathbb{R}^p$ , we define  $\text{mdl}(x) \in \mathcal{M}_p$  through the following.

- 1)  $\text{sign}(x) = \text{sign}(\text{mdl}(x))$
- 2)  $|x_i| = |x_j| \implies |\text{mdl}(x)_i| = |\text{mdl}(x)_j|$
- 3)  $|x_i| > |x_j| \implies |\text{mdl}(x)_i| > |\text{mdl}(x)_j|$

**Example.** For  $x = (3.1, -1.2, 0, -3.1)'$ , we have  $\text{mdl}(x) = (2, -1, 0, -2)'$ . For  $x \in \mathbb{R}^4$  with  $\text{mdl}(x) = (0, 2, 1, -2)'$ , we have  $\text{sign}(x) = (0, 1, 1, -1)'$  and  $|x_2| = |x_4| > |x_3| > x_1 = 0$ . The set of all SLOPE models in  $\mathbb{R}^2$  is given by

$$\begin{aligned} \mathcal{M}_2 = \{ & (0, 0)', (1, 0)', (-1, 0)', (0, 1)', (0, -1)', (1, 1)', (1, -1)', (-1, 1)', (-1, -1)', \\ & (2, 1)', (-2, 1)', (2, -1)', (-2, -1)', (1, 2)', (-1, 2)', (1, -2)', (-1, -2)'\}. \end{aligned}$$

In Theorem 4 we prove there is a one-to-one relationship between SLOPE models and the so-called sign permutahedron, which is the unit ball of the dual of the SLOPE norm (see Proposition 8 in Appendix A.7). First, we define this polytope. The sign permutahedron  $P_w^\pm$  is defined as

$$P_w^\pm = \text{conv} \{ (\sigma_1 w_{\pi(1)}, \dots, \sigma_p w_{\pi(p)})' : \sigma_1, \dots, \sigma_p \in \{-1, 1\}, \pi \in \mathcal{S}_p \}.$$

The shape of this polytope is illustrated in Figure 5 in two dimensions, and in Figure 6 in three dimensions. We denote the set of non-empty faces of the sign permutahedron by  $\mathcal{F}_0(P_w^\pm)$ . The permutahedron  $P_w$ , which is a face of the sign permutahedron, is defined as the following polytope.

$$P_w = \text{conv} \{ (w_{\pi(1)}, \dots, w_{\pi(p)})' : \pi \in \mathcal{S}_p \}.$$

We denote the subdifferential of the SLOPE norm at  $x \in \mathbb{R}^p$  by  $\partial_{\|\cdot\|_w}(x)$  (for the definition of subdifferential, see Appendix A.1). In fact,  $\partial_{\|\cdot\|_w}(x)$  is a face of  $P_w^\pm$ , which we shall denote by  $F_w(x)$  in the following.

SLOPE models having only positive components can be interpreted as an ordered partition of  $[p]$ , where the the smallest and largest element of this partition is the set  $\{j : m_j = 1\}$  and the set  $\{j : m_j = ||m||_\infty\}$ , respectively. It is well known that there is a one-to-one relationship between the elements of an ordered partition and the faces of the permutahedron (see e.g. Maes and Kappen, 1992; Simion, 1997; Ziegler, 2012). Instigated by this, we show in Theorem 4 that this result can, indeed, be extended to a one-to-one relationship between all SLOPE models and the faces of the sign permutahedron.

**Theorem 4.** The mapping  $m \in \mathcal{M}_p \mapsto F_w(m) = \partial_{\|\cdot\|_w}(m)$  is a bijection between the SLOPE models  $\mathcal{M}_p$  and  $\mathcal{F}_0(P_w^\pm)$ , the non-empty faces of the sign permutahedron  $P_w^\pm$ . In addition, the following holds.

- 1) The codimension of  $F_w(m)$  is given by  $||m||_\infty$ .

2) We have  $F_w(x) = F_w(\text{mdl}(x))$ .

The assumption that components of  $w$  are strictly decreasing and non-zero is important. For example, if  $w_1 = \dots = w_p > 0$ , the sign permutahedron is just a cube and clearly, there is no one-to-one relationship between the set SLOPE models and the set of faces of the cube. A similar situation arises if  $w$  contains zero components. As can be seen when  $p = 2$  and  $w_2 = 0$ , the SLOPE norm is the supremum norm and the corresponding dual unit ball is the unit cross polytope in  $\mathbb{R}^2$ .

**Example.** We now describe the faces  $F_w(m)$ ,  $m \in \mathcal{M}_2$ , of the sign permutahedron  $P_w^\pm$  when  $w = (3.5, 1.5)'$ . In the following, we use the fact that – up to an orthogonal transformation described in Lemma 4 –  $F_w(m)$  is equal to  $F_w(\tilde{m})$  for some  $\tilde{m}$ , a non-negative and non-increasing SLOPE model. The relationship between the SLOPE models  $m \in \mathcal{M}_2$  and faces of the sign permutahedron  $P_w^\pm$  are listed below and illustrated in Figure 5. Note that  $\text{codim}(F_w(m)) = \|m\|_\infty$ .

model $\tilde{m}$	face $F_w(\tilde{m})$	codim.	faces $F_w(m)$ isometric to $F_w(\tilde{m})$
$\tilde{m} = (0, 0)'$	sign permutahedron: $P_w^\pm$	0	$\emptyset$
$\tilde{m} = (1, 0)'$	segment: $\{3.5\} \times [-1.5, 1.5]$	1	$m \in \{(-1, 0)', \pm(0, 1)'\}$
$\tilde{m} = (1, 1)'$	permutahedron: $P_w$	1	$m \in \{(-1, -1)', \pm(1, -1)'\}$
$\tilde{m} = (2, 1)'$	point: $(3.5, 1.5)'$	2	$m \in \{(-2, -1)', \pm(2, -1)', \pm(1, 2)', \pm(1, -2)'\}$

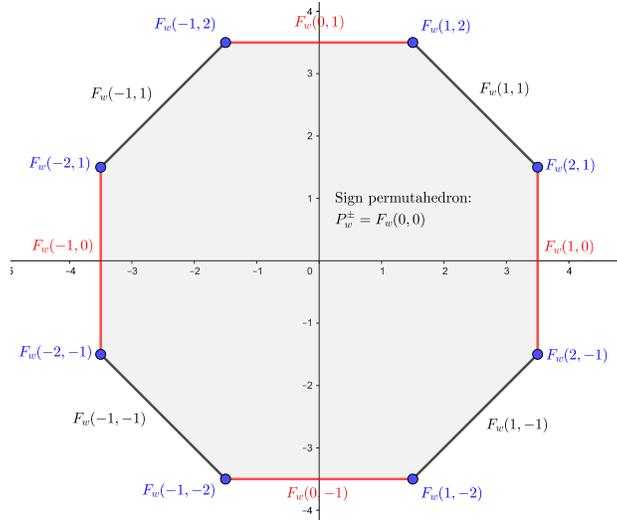


Figure 5: Illustration of the relationship between the SLOPE models and the faces of the sign permutahedron  $P_w^\pm$  for  $w = (3.5, 1.5)'$  through subdifferential calculus, see Proposition 6 in Appendix A.3 and Proposition 8 in Appendix A.7. Note that  $F_w(m) = \partial_{\|\cdot\|_w}(m)$ . Faces having the same color are isometric. One may notice that  $\text{codim}(F_w(m)) = \|m\|_\infty$ .

**Definition 3** (Accessible SLOPE model). Let  $X \in \mathbb{R}^{n \times p}$  and  $m \in \mathcal{M}_p$ . We say that  $m$  is an accessible SLOPE model if

$$\exists y \in \mathbb{R}^n \text{ and } \exists \hat{\beta} \in S_{X, \|\cdot\|_w}(y) \text{ such that } \text{mdl}(\hat{\beta}) = m.$$

**Theorem 5** (Characterization of accessible SLOPE models). Let  $X \in \mathbb{R}^{n \times p}$ . A SLOPE model  $m \in \mathcal{M}_p$  is accessible if and only if  $\text{row}(X)$  intersects the face  $F_w(m)$ .

Note that the set of accessible SLOPE models is invariant by scaling  $w$ , since  $\text{row}(X)$  intersects  $F_w(m)$  if and only if  $\text{row}(X)$  intersects  $F_{\lambda w}(m)$  with  $\lambda > 0$ . The following corollary, which confirms the very recent Theorem 2.1 given in Kremer et al. (2019), is a straightforward consequence of Theorems 1, 4 and 5.

**Corollary 2.** Let  $X \in \mathbb{R}^{n \times p}$ . If  $\text{row}(X)$  does not intersect any face of  $P_w^\pm$  with codimension larger than  $\text{rk}(X)$ , then for all  $y \in \mathbb{R}^n$ ,  $\hat{\beta}_w(y)$ , the unique element of  $S_{X, \|\cdot\|_w}(y)$ , satisfies  $\|\text{mdl}(\hat{\beta}_w(y))\|_\infty \leq \text{rk}(X)$ .

Corollary 2 generalizes the well known fact that, when uniqueness occurs, the LASSO minimizer has less than  $\text{rk}(X)$  non-null components. Indeed, the above corollary shows that when the SLOPE minimizer is unique, the number of non-null clusters is less than or equal to  $\text{rk}(X)$ .

**Example.** We illustrate the criterion for accessible SLOPE models for  $w = (5.5, 3.5, 1.5)'$  and  $X$  given by

$$X = \begin{pmatrix} 8 & 5 & 8 \\ 10 & 1.25 & -6 \end{pmatrix}.$$

The table below lists all accessible SLOPE models, the geometric illustrations is shown in Figure 6.

colour	type	intersection $\neq \emptyset$	face intersected isometric to	SLOPE models
orange	segments	$\text{row}(X) \cap F_w(\pm(1, 0, 0))$	$\{5.5\} \times P_{(3.5, 1.5)}^\pm$	$\pm(1, 0, 0)$
red	segments	$\text{row}(X) \cap F_w(\pm(1, 1, 1))$	$P_{(5.5, 3.5, 1.5)}$	$\pm(1, 1, 1)$
black	segments	$\text{row}(X) \cap F_w(\pm(0, 0, 1))$	$\{5.5\} \times P_{(3.5, 1.5)}^\pm$	$\pm(0, 0, 1)$
pink	segments	$\text{row}(X) \cap F_w(\pm(-1, 0, 1))$	$P_{(5.5, 3.5)} \times [-1.5, 1.5]$	$\pm(-1, 0, 1)$
purple	points	$\text{row}(X) \cap F_w(\pm(2, 0, -1))$	$\{5.5\} \times \{3.5\} \times [-1.5, 1.5]$	$\pm(2, 0, -1)$
green	points	$\text{row}(X) \cap F_w(\pm(2, 1, 1))$	$\{5.5\} \times P_{(3.5, 1.5)}$	$\pm(2, 1, 1)$
blue	points	$\text{row}(X) \cap F_w(\pm(1, 1, 2))$	$\{5.5\} \times P_{(3.5, 1.5)}$	$\pm(1, 1, 2)$
yellow	points	$\text{row}(X) \cap F_w(\pm(-1, 0, 2))$	$\{5.5\} \times \{3.5\} \times [-1.5, 1.5]$	$\pm(-1, 0, 2)$

### 3.3 The SLOPE null polytope

Above we gave description of accessible SLOPE models based on the intersection of  $\text{row}(X)$  and the sign permutahedron  $P_w^\pm$ . Given an accessible model  $m \in \mathcal{M}_p$ , our aim is to describe the set of  $y \in \mathbb{R}^n$  for which there exists  $\hat{\beta} \in S_{X, \|\cdot\|_w}(y)$  such that  $\text{mdl}(\hat{\beta}) = m$ . In other words, we want to describe the set

$$A_w(m) = \{y \in \mathbb{R}^n : \exists \hat{\beta} \in S_{X, \|\cdot\|_w}(y) \text{ where } \text{mdl}(\hat{\beta}) = m\}.$$

Note that when the SLOPE minimizer is unique, the sets  $A_w(m)$  and  $A_w(\tilde{m})$  are disjoint for  $m \neq \tilde{m}$ , whereas possibly  $A_w(m) \cap A_w(\tilde{m}) \neq \emptyset$ , when the SLOPE minimizer is not unique. Clearly,  $m = 0$  is

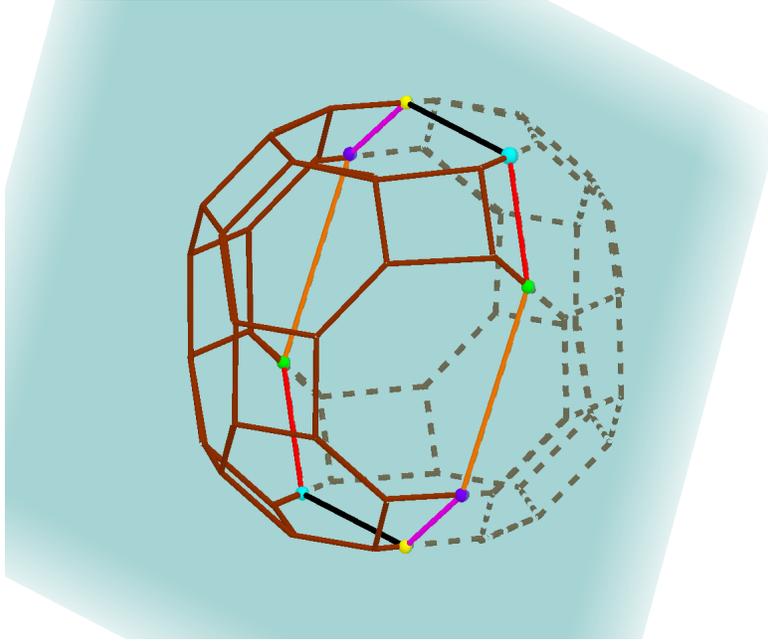


Figure 6: Illustration of the sign permutahedron  $P_w^\pm$  (in brown) and the plane  $\text{row}(X)$  (in light blue). Because  $\text{rk}(X) = 2$  and  $\text{row}(X)$  does not intersect any vertex of  $P_w^\pm$ , i.e., the faces of  $P_w^\pm$  with codimension equal to 3, one may notice that  $S_{X, \|\cdot\|_w}(y)$  is a singleton with unique element  $\hat{\beta}_w(y)$  for all values of  $y \in \mathbb{R}^2$ . Colored segments and points are the intersections between  $\text{row}(X)$  and the faces of  $P_w^\pm$ , determining the accessible SLOPE models shown in the table below. For example,  $m = (2, 1, 1)'$  is an accessible SLOPE model which implies that there exists  $y \in \mathbb{R}^2$  for which the SLOPE minimizer  $\hat{\beta}_w(y)$  satisfies 1)  $\text{sign}(\hat{\beta}_w(y)) = (1, 1, 1)'$  and 2)  $\hat{\beta}_w(y)_1 > \hat{\beta}_w(y)_2 = \hat{\beta}_w(y)_3 > 0$ . In addition, since  $(2, 1, 0)'$  is not an accessible model, one cannot pick  $y \in \mathbb{R}^2$  for which the SLOPE minimizer satisfies 1)  $\text{sign}(\hat{\beta}_w(y)) = (1, 1, 0)'$  and 2)  $\hat{\beta}_w(y)_1 > \hat{\beta}_w(y)_2 > \hat{\beta}_w(y)_3 = 0$ .

accessible. The set  $A_w(0)$ , called the *SLOPE null polytope*, given by

$$A_w(0) = \{y \in \mathbb{R}^n : \|X'y\|_w^* \leq 1\}$$

by Proposition 7, namely the set of all  $y$  such that  $X'y \in P_w^\pm$ , which is again a polytope. The proposition below shows that the faces of this polytope are given by  $N_w(m) = \{f \in \mathbb{R}^n : X'f \in F_w(m)\}$  for the *accessible* SLOPE models  $m \in M_p$ . These faces are the cornerstone to describe the sets  $A_w(m)$ .

**Proposition 2.** *Let  $X \in \mathbb{R}^{n \times p}$ . The SLOPE model  $m \in M_p$  is an accessible SLOPE model if and only if  $N_m = \{f \in \mathbb{R}^n : X'f \in F_w(m)\} \neq \emptyset$ . In that case, the set  $A_w(m)$  is given by*

$$A_w(m) = \{y = f + Xb : f \in N_w(m), \text{mdl}(b) = m\}.$$

If  $\text{rk}(X) = n$  and we are given the intersection between  $\text{row}(X)$  and  $F_w(m)$  for some accessible SLOPE model  $m$ , we have that  $A_w(m) = (XX')^{-1}X(\text{row}(X) \cap F_w(m))$  since

$$f \in N_w(m) \iff X'f \in \text{row}(X) \cap F_w(m) \iff f \in (XX')^{-1}X(\text{row}(X) \cap F_w(m)).$$

**Example.** *Figure 6 illustrates the accessible SLOPE models from Theorem 5 for  $w = (5.5, 3.5, 1.5)'$  and*

$$X = \begin{pmatrix} 8 & 5 & 8 \\ 10 & 1.25 & -6 \end{pmatrix}.$$

*Now, for every accessible SLOPE model (which are elements of  $\mathbb{R}^p$ ), Figure 7 below provides the set  $A_m = \{y \in \mathbb{R}^2 : \exists \hat{\beta} \in S_{X, \|\cdot\|_w}(y) \text{ where } \text{mdl}(\hat{\beta}) = m\}$  and the SLOPE null polytope (which are geometric objects in  $\mathbb{R}^n$ ).*

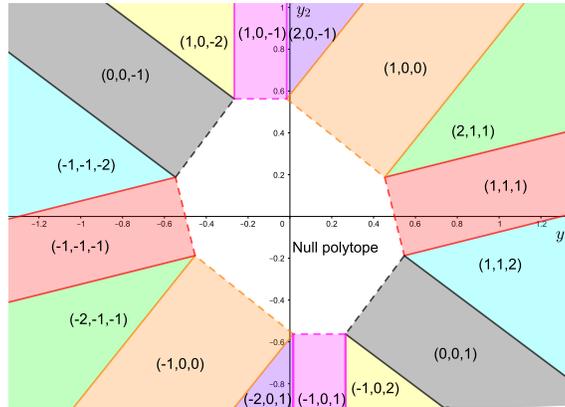


Figure 7: Illustration of the null polytope and the accessible models  $\{\pm(1, 0, 0), \pm(1, 1, 1), \pm(0, 0, 1), \pm(-1, 0, 1), \pm(2, 0, -1), \pm(2, 1, 1), \pm(1, 1, 2), \pm(-1, 0, 2)\}$ , associated to the faces of the polytope. Depicted also are the sets  $A_m = \{y \in \mathbb{R}^2 : \exists \hat{\beta} \in S_{X, \|\cdot\|_w}(y) \text{ where } \text{mdl}(\hat{\beta}) = m\}$  for each accessible model.

Note that the SLOPE null polytope  $A_w(0)$  can also be interpreted as the set of SLOPE residuals in

the sense that  $\hat{u} = y - X\hat{\beta}$  is the projection of  $y$  onto  $A_w(0)$  whenever  $\hat{\beta} \in S_{X, \|\cdot\|_w}(y)$  (Minami, 2020). Or put differently again, the decomposition  $y = X\hat{\beta} + \hat{u}$ , where  $X\hat{\beta}$  is the SLOPE fit and  $\hat{u} \in A_w(0)$ , the set of all values that lead to a zero SLOPE minimizer.

This property is well known also for the LASSO, (c.f. Tibshirani and Taylor, 2012). In fact, it is straightforward to see from Proposition 7 that the same considerations hold for all problems as defined in (1). For completeness, we summarize this in the following proposition.

**Proposition 3.** *Let  $X \in \mathbb{R}^{n \times p}$  and  $y \in \mathbb{R}^n$  and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^p$ . Define the null polytope  $A_\emptyset = \{u \in \mathbb{R}^n : \|X'u\|^* \leq 1\}$ . We then have  $S_{X, \|\cdot\|}(u) = \{0\}$  for all  $u \in A_\emptyset$ , and any  $\hat{\beta} \in S_{X, \|\cdot\|}(y)$  satisfies  $y = X\hat{\beta} + \hat{u}$  with  $\hat{u} \in A_\emptyset$ . Moreover,  $\hat{u}$  is the projection of  $y$  onto  $A_\emptyset$ .*

## A Appendix – Proofs

In the appendix, we additionally make use of the following notation. Let  $A$  be a matrix. We use the symbol  $A_j$  to denote the  $j$ -th column of  $A$ . For an index set  $I$ ,  $A_I$  is the matrix containing columns with indices in  $I$  only. For a vector  $x$ ,  $\text{supp}(x)$  contains the indices of the non-zero components of  $x$ . The symbol  $|x|_{(j)}$  denotes the  $j$ -th order statistic of the absolute values of the components of  $x$ , i.e.,  $|x|_{(1)} \geq |x|_{(2)} \geq \dots$ . Let  $l, k \in \mathbb{N}$  with  $l \leq k$ , then  $[l : k]$  denotes the set  $\{l, l+1, \dots, k\}$ . We let  $\mathbf{1}_m$  stand for the vector  $(1, \dots, 1)' \in \mathbb{R}^m$ . All inequalities involving vectors are understood componentwise.

### A.1 Facts about subdifferentials

We remind the reader of some definitions and facts on subgradients and subdifferentials. The following can, for instance, be found in Hiriart-Urruty and Lemarechal (1993). For a function  $f : \mathbb{R}^p \rightarrow \mathbb{R}$ , a vector  $s \in \mathbb{R}^p$  is a *subgradient* of  $f$  at  $x \in \mathbb{R}^p$  if

$$f(z) \geq f(x) + s'(z - x) \quad \forall z \in \mathbb{R}^p. \quad (2)$$

The set of all subgradients of  $f$  at  $x$ , which is a convex set, is called the *subdifferential* of  $f$  at  $x$ , denoted by  $\partial_f(x)$ . It is straightforward to characterize the minimizer of a function in the following way

$$x^* \in \arg \min f \iff 0 \in \partial_f(x^*). \quad (3)$$

While convexity of  $f$  is not necessary for the above statement, the treatment of subdifferentials is an especially important tool when this is the case. Given that  $f$  is convex, subdifferentiability is also a local property in the sense that for any  $\delta > 0$ , we have

$$s \in \partial_f(x) \iff f(x+h) \geq f(x) + s'h \text{ for all } h : \|h\|_\infty \leq \delta. \quad (4)$$

### A.2 Facts about polytopes

We report some basic definitions and facts on polytopes, which we will use throughout the article and, in particular, in proofs in the subsequent sections. The following can, for instance, be found in the excellent textbooks by Gruber (2007) and Ziegler (2012).

A set  $P_{\mathcal{V}} \subseteq \mathbb{R}^p$  is called a  $\mathcal{V}$ -polytope, if it is the convex hull of a finite set of points in  $\mathbb{R}^p$ , namely,

$$P_{\mathcal{V}} = \text{conv}(V_1, \dots, V_k) = \text{conv}(V),$$

for  $V = (V_1 \dots V_k) \in \mathbb{R}^{p \times k}$ . A set  $P_{\mathcal{H}} \subseteq \mathbb{R}^p$  is called an  $\mathcal{H}$ -polyhedron, if it is the intersection of a finite number of half-spaces, namely,

$$P_{\mathcal{H}} = \bigcap_{l=1}^m \{x \in \mathbb{R}^p : A'_l x \leq b_l\} = \{x \in \mathbb{R}^p : A'x \leq b\},$$

for some  $A = (A_1 \dots A_m) \in \mathbb{R}^{p \times m}$  and  $b \in \mathbb{R}^m$ . A bounded  $\mathcal{H}$ -polyhedron is called  $\mathcal{H}$ -polytope. A set  $P \subseteq \mathbb{R}^p$  is an  $\mathcal{H}$ -polytope if and only if it is a  $\mathcal{V}$ -polytope. We therefore simply use the term *polytope* in the following. The *dimension*  $\dim(P)$  of a polytope is given by the dimension of the affine subspace spanned by  $P$ , and its codimension by  $\text{codim}(P) = p - \dim(P)$ . A *face*  $F$  of  $P$  is any subset  $F \subseteq P$  that satisfies

$$F = \{x \in P : a'x = b_0\}, \text{ where } P \subseteq \{x : a'x \leq b_0\},$$

for some  $a \in \mathbb{R}^p$  and  $b_0 \in \mathbb{R}$ . Such an inequality  $a'x \leq b_0$  is called a *valid inequality* of  $P$ . Note that  $F = \emptyset$  and  $F = P$  are faces of  $P$  and that any face  $F$  is again a polytope. A face  $F \neq P$  is called *proper*. A face of dimension 0 is called *vertex*, and we denote the set of all vertices of  $P$  by  $\text{vert}(P)$ , which satisfy that  $\text{vert}(P) \subseteq \{V_1, \dots, V_k\}$ , where  $P = \text{conv}(V_1, \dots, V_k)$ . A point  $x_0 \in P$  lies in  $\text{relint}(P)$ , the *relative interior* of  $P$ , if  $x_0$  is not contained in a proper face of  $P$ . Finally, the (*polar*) *dual* of  $P$  is defined as

$$P^* = \{s \in \mathbb{R}^p : s'x \leq 1 \forall x \in P\},$$

which is again a polytope. We now list a number of useful facts about polytopes involving the above definitions, which are used throughout the article. These properties can either be found explicitly or as a straightforward consequence of properties listed in the references mentioned above.

**Proposition 4.** *Let  $P \in \mathbb{R}^p$  be a polytope given by  $P = \text{conv}(V)$ , where  $V = (V_1, \dots, V_k) \in \mathbb{R}^{p \times k}$ , and denote by  $P^*$  the dual of  $P$ . For simplicity, we assume that  $\text{vert}(P) = \{V_1, \dots, V_k\}$ . Moreover, let  $0 \in P$ . The following properties hold.*

- 1) *If  $F$  and  $\tilde{F}$  are faces of  $P$ , then so is  $F \cap \tilde{F}$ .*
- 2) *For any face  $F$  of  $P$ ,  $F = \text{conv}(\text{vert}(P) \cap F)$ .*
- 3) *We have  $x_0 \in \text{relint}(P)$  if and only if any valid inequality  $a'x \leq b_0$  of  $P$  with  $a'x_0 = b_0$  implies that  $a'x = b_0$  for all  $x \in P$ .*
- 4) *Let  $D$  be an affine line contained in the affine span of  $P$ . If  $D \cap \text{relint}(P) \neq \emptyset$  then  $D$  intersects a proper face of  $P$ .*
- 5) *We can write  $P^* = \{s \in \mathbb{R}^p : V's \leq \mathbf{1}_k\}$ .*
- 6) *Any face  $F^*$  of  $P^*$  can be written as  $F^* = \{s \in P^* : V'_I s = \mathbf{1}_{|I|}\}$  for some  $I \subseteq [k]$ .*

7) Let  $I \subseteq [k]$ .  $F = \text{conv}(V_I)$  is a face of  $P \iff F^* = \{s \in P^* : V_I' s = \mathbf{1}_{|I|}\}$  is a face of  $P^*$ , where  $I$  is the maximal index set in this representation.

In this case,  $F^*$  is the dual of  $F$  (and vice versa), and  $\text{codim}(F^*) = \text{rk}(V_I)$ .

### A.3 Facts on subdifferentials of norms with polytope unit balls

We now consider subdifferentials of norms and list several properties in the following. In particular, we show in Proposition 5 that the subdifferential of a norm evaluated at zero is simply given by the unit ball of the corresponding dual norm, a fact that will be used throughout subsequent proofs. Proposition 6 then shows that all faces of this dual norm unit ball can be represented by a subdifferential of the original norm, provided that this norm is such that its unit ball, and therefore also the unit ball of its dual norm, are given by a polytope. Lemma 1 contains a technical result needed for the proof of Theorem 1.

A version of the following proposition – which holds independently of the shape of the unit ball of the norm under consideration – can also be found in Hiriart-Urruty and Lemarechal (1993).

**Proposition 5.** *Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^p$ , and let  $\|\cdot\|^*$  denote the dual norm. Then the following holds.*

1) *The subdifferential of  $\|\cdot\|$  at 0 is given by*

$$\partial_{\|\cdot\|}(0) = \{s \in \mathbb{R}^p : \|s\|^* \leq 1\}.$$

2) *The subdifferential of  $\|\cdot\|$  at  $x$  is given by*

$$\partial_{\|\cdot\|}(x) = \{s \in \mathbb{R}^p : \|s\|^* \leq 1, s'x = \|x\|\}.$$

*Proof.* 1) According to the definition of a subdifferential in (2), we have

$$\begin{aligned} \partial_{\|\cdot\|}(0) &= \{s \in \mathbb{R}^p : \|v\| \geq s'v \ \forall v \in \mathbb{R}^p\} = \{s \in \mathbb{R}^p : v's \leq 1 \ \forall v \in \mathbb{R}^p \text{ with } \|v\| \leq 1\} \\ &= \{s \in \mathbb{R}^p : \|v\|^* \leq 1\}. \end{aligned}$$

2) By definition, we have

$$\partial_{\|\cdot\|}(x) = \{s \in \mathbb{R}^p : \|v\| \geq \|x\| + s'(v - x) \ \forall v \in \mathbb{R}^p\}$$

Take  $s \in \partial_{\|\cdot\|}(x)$ . When  $v = 0$ , we get  $s'x \geq \|x\|$ . When  $v = 2x$ , we may deduce that  $s'x \leq \|x\|$ , implying that  $s'x = \|x\|$  must hold. This also implies  $\|v\| \geq s'v$  for all  $v \in \mathbb{R}^p$ , so that  $s \in B^*$ , yielding

$$\partial_{\|\cdot\|}(x) \subseteq \{s \in B^* : s'x = \|x\|\}.$$

To see that also the converse is true, take any  $s \in B^*$  satisfying  $s'x = \|x\|$ . Now, take any  $v \in \mathbb{R}^p$ .

Clearly  $\|v\| \geq s'v = \|x\| + s'(v - x)$ , implying that

$$\{s \in B^* : s'x = \|x\|\} \subseteq \partial_{\|\cdot\|}(x).$$

□

**Proposition 6.** *Let  $\|\cdot\|$  be a norm whose unit ball  $B$  is the polytope  $\text{conv}(V)$  for some  $V = (V_1 \dots V_k) \in \mathbb{R}^{p \times k}$ . Let  $F \subseteq B^*$ , where  $B^*$  is the dual norm unit ball, with  $F \neq \emptyset$ . Then*

$$F \text{ is a face of } B^* \iff F = \partial_{\|\cdot\|}(x) \text{ for some } x \in \mathbb{R}^p.$$

*Proof.* ( $\implies$ ) If  $F = B^*$ , then  $x = 0$  by Proposition 5. If  $F$  is a proper face, then  $F = \{s \in B^* : V'_l s = \mathbf{1}_{|I|}\}$  for some  $I \subseteq [k]$ , where  $I$  is the maximal set satisfying this. Let  $x = \sum_{l \in I} V_l$ . Since  $x/|I| \in \text{conv}(V_I)$ , a proper and non-empty face of  $B$ , we have  $\|x\| = |I|$ . Note that for  $s \in B^*$ , we have  $s'V_l \leq 1$ , so that

$$s \in \partial_{\|\cdot\|}(x) \iff s'x = \sum_{l \in I} V'_l s = \|x\| = |I| \iff V'_l s = 1 \quad \forall l \in I \iff s \in F.$$

( $\impliedby$ ) If  $F = \partial_{\|\cdot\|}(x)$ , then  $F = \{s \in B^* : s'x = \|x\|\}$  by Proposition 5. Since  $(x/\|x\|)'s \leq 1$  clearly is a valid inequality for all  $s \in B^*$ ,  $F$  is a face of  $B^*$ . □

**Lemma 1.** *Let  $\|\cdot\|$  be a norm whose unit ball  $B$  is the polytope  $\text{conv}(V)$  for some  $V = (V_1 \dots V_k) \in \mathbb{R}^{p \times k}$ . Let  $F = \{s \in B^* : V'_l s = \mathbf{1}_{|I|}\}$  be a face of  $B^*$ , the dual norm unit ball, and let  $I$  be the maximal set satisfying this. Then the following holds.*

$$F \subseteq \partial_{\|\cdot\|}(b) \implies b \in \text{col}(V_I).$$

*Proof.* Since  $b/\|b\| \in B = \text{conv}(V)$ , we can write  $b = \sum_{l=1}^k \alpha_l V_l$  with  $\alpha_l \geq 0$  and  $\sum_{l=1}^k \alpha_l = \|b\|$ . Since  $\partial_{\|\cdot\|}(b) = \{s \in B^* : s'b = \|b\|\}$  and  $s'V_l \leq 1$ , we have for  $A = \text{supp}(\alpha)$  and any  $s \in \partial_{\|\cdot\|}(b)$

$$\|b\| = s'b = \sum_{l \in A} \alpha_l s'V_l \leq \sum_{l \in A} \alpha_l = \|b\|.$$

This implies that  $s'V_l = 1$  for all  $l \in \text{supp}(\alpha)$ , which, since  $F \subseteq \partial_{\|\cdot\|}(b)$ , yields  $\text{supp}(\alpha) \subseteq I$ . □

## A.4 Proofs of Theorems 1 and 2

The proofs of Theorems 1 and 2 follow a similar outline, with the proof of Theorem 2 being more accessible. We therefore start with the latter one.

### A.4.1 Characterization of BP minimizers and proof of Theorem 2

The following characterization of BP minimizers can be found in Zhang et al. (2015) and Gilbert (2017), and will prove to be useful in the following.

Let  $y \in \text{col}(X)$  and let  $\hat{\beta}$  satisfy  $X\hat{\beta} = y$  then,  $\hat{\beta} \in S_{X,\text{bp}}(y)$  if and only if

$$\exists z \in \mathbb{R}^n \text{ such that } \begin{cases} \|X'z\|_\infty \leq 1, \\ X'_j z = \text{sign}(\hat{\beta}_j) \quad \forall j \in \text{supp}(\hat{\beta}). \end{cases} \quad (5)$$

*Proof of Theorem 2.*

( $\Leftarrow$ ) Let us assume that  $\text{row}(X)$  intersects a face  $F$  of  $[-1, 1]^p$  whose codimension is larger than  $\text{rk}(X)$ . We show that one can find some  $y \in \text{col}(X)$  for which  $S_{X, \text{bp}}(y)$  is not a singleton.

The face  $F$  can be written as  $F = E_1 \times \cdots \times E_p$ , where  $E_j \in \{\{-1\}, \{1\}, [-1, 1]\}$  for  $j \in [p]$ . Now, let  $J = \{j \in [p] : |E_j| = 1\}$ , the set of indices of sets  $E_j$  that are singletons. We have  $\text{codim}(F) = |J|$  and, by assumption,  $|J| > \text{rk}(X)$ . Now define  $\hat{\beta} \in \mathbb{R}^p$  by setting

$$\hat{\beta}_j = \begin{cases} 1 & E_j = \{1\} \\ -1 & E_j = \{-1\} \\ 0 & j \notin J. \end{cases}$$

Clearly,  $\text{supp}(\hat{\beta}) = J$ . Set  $y = X\hat{\beta}$ . Since  $\text{row}(X)$  intersects  $F$ , there exists  $z \in \mathbb{R}^n$  such that  $X'z \in F$ . This implies that  $\|X'z\|_\infty \leq 1$  and  $X'_j z = \hat{\beta}_j = \text{sign}(\hat{\beta}_j)$  for any  $j \in \text{supp}(\hat{\beta}) = J$ . Therefore, by (5),  $\hat{\beta} \in S_{X, \text{bp}}(y)$ .

To show that  $\hat{\beta}$  is not a unique minimizer, we provide  $\tilde{\beta} \in \mathbb{R}^p$  with  $\tilde{\beta} \neq \hat{\beta}$ ,  $X\tilde{\beta} = y$  and  $\|\tilde{\beta}\|_1 = \|\hat{\beta}\|_1$ . Since  $|J| > \text{rk}(X)$ , the columns of  $X_J$  are linearly dependent, so that we can pick  $h \in \ker(X)$ ,  $h \neq 0$  such that  $\text{supp}(h) \subseteq J$  and  $\|h\|_\infty < 1$ . Since  $\|h\|_\infty < 1$ ,  $\text{sign}(\hat{\beta} + h) = \text{sign}(\hat{\beta}) = \hat{\beta}$ . Let  $\tilde{\beta} = \hat{\beta} + h$ . Note that  $X\tilde{\beta} = X\hat{\beta} = y$  and that

$$\begin{aligned} \|\tilde{\beta}\|_1 &= \sum_{j=1}^p \text{sign}(\hat{\beta}_j + h_j)(\hat{\beta}_j + h_j) = t \sum_{j=1}^p \text{sign}(\hat{\beta}_j)\hat{\beta}_j + \sum_{j \in J} \hat{\beta}_j h_j = \|\hat{\beta}\|_1 + \sum_{j \in J} (X'z)_j h_j \\ &= \|\hat{\beta}\|_1 + z'Xh = \|\hat{\beta}\|_1, \end{aligned}$$

implying that  $\tilde{\beta} \in S_{X, \text{bp}}(y)$  also.

( $\Rightarrow$ ) We assume that  $\hat{\beta}, \tilde{\beta} \in S_{X, \text{bp}}(y)$  with  $\hat{\beta} \neq \tilde{\beta}$  for some  $y \in \text{col}(X)$ . We need to show that there exists a face  $F$  of  $[-1, 1]^p$  with  $F \cap \text{row}(X) \neq \emptyset$  and  $\text{codim}(F) > \text{rk}(X)$ . Consider  $F = E_1 \times \cdots \times E_p$  and  $\tilde{F} = \tilde{E}_1 \times \cdots \times \tilde{E}_p$  with

$$E_j = \begin{cases} \{\text{sign}(\hat{\beta}_j)\} & \text{if } j \in \text{supp}(\hat{\beta}) \\ [-1, 1] & \text{if } j \notin \text{supp}(\hat{\beta}) \end{cases} \quad \text{and} \quad \tilde{E}_j = \begin{cases} \{\text{sign}(\tilde{\beta}_j)\} & \text{if } j \in \text{supp}(\tilde{\beta}) \\ [-1, 1] & \text{if } j \notin \text{supp}(\tilde{\beta}). \end{cases}$$

Note that for any two minimizers  $\hat{\beta}$  and  $\tilde{\beta}$ , we have  $\hat{\beta}_j \tilde{\beta}_j \geq 0$  for all  $j \in [p]$ , since otherwise  $\check{\beta} = (\hat{\beta} + \tilde{\beta})/2$  satisfies  $X\check{\beta} = X\hat{\beta} = X\tilde{\beta}$  as well as  $\|\check{\beta}\|_1 < \|\hat{\beta}\|_1 = \|\tilde{\beta}\|_1$ , which would lead to a contradiction. We therefore have  $\text{supp}(\check{\beta}) = \text{supp}(\hat{\beta}) \cup \text{supp}(\tilde{\beta})$ . Note that by a convexity argument,  $\check{\beta} \in S_{X, \text{bp}}(y)$  also, so that by (5), there exists  $\check{z} \in \mathbb{R}^n$  with  $\|X'\check{z}\|_\infty \leq 1$  and  $X'_j \check{z} = \text{sign}(\check{\beta}_j)$  for all  $j \in \text{supp}(\check{\beta})$ . Moreover,  $X'\check{z} \in F \cap \tilde{F}$  holds. Now, let  $F_0$  be a face of the face  $F \cap \tilde{F}$  of smallest dimension that still intersects  $\text{row}(X)$ . We write  $F_0 = E_{0,1} \times \cdots \times E_{0,p}$  and let  $J_0 = \{j \in [p] : |E_{0,j}| = 1\}$ . Note that  $\text{row}(X)$  must

intersect  $F_0$  in its relative interior  $\text{relint}(F_0)$  where

$$\text{relint}(F_0) = \text{relint}(E_{0,1}) \times \cdots \times \text{relint}(E_{0,p}) \quad \text{where } \text{relint}(E_{0,j}) = \begin{cases} E_{0,j} & j \in J_0 \\ (-1, 1) & j \notin J_0, \end{cases}$$

since otherwise  $\text{row}(X)$  intersects a proper face of  $F_0$ , which contradicts the assumption that  $F_0$  is of minimal dimension. We now need to show that  $\text{codim}(F_0) = |J_0| > \text{rk}(X)$ . Assume that  $|J_0| \leq \text{rk}(X)$ . The columns of  $X_{J_0}$  are linearly dependent since  $X_{J_0} \hat{\beta}_{J_0} = X \hat{\beta} = X \tilde{\beta} = X_{J_0} \tilde{\beta}_{J_0}$  with  $\hat{\beta}_{J_0} \neq \tilde{\beta}_{J_0}$ , since both  $\text{supp}(\hat{\beta})$  and  $\text{supp}(\tilde{\beta})$  are subsets of  $\text{supp}(\tilde{\beta}) \subseteq J_0$ . We therefore have

$$\dim(\text{col}(X_{J_0})) < |J_0| \leq \text{rk}(X) = \dim(\text{col}(X)) \quad \text{and} \quad \text{col}(X)^\perp \subsetneq \text{col}(X_{J_0})^\perp.$$

This implies that we can pick  $u \in \text{col}(X_{J_0})^\perp \setminus \text{col}(X)^\perp$  so that  $X'_{J_0} u = 0$ , but  $X' u \neq 0$ . Pick  $z_0 \in \mathbb{R}^n$  with  $X' z_0 \in \text{relint}(F_0)$ . The affine line  $\{X'(z_0 + tu) : t \in \mathbb{R}\} \subseteq \text{row}(X)$  intersects the relative interior  $\text{relint}(F_0)$  and is included in the affine span of  $F_0$  by construction of  $u$ . Therefore, by Proposition 4,  $\text{row}(X)$  intersects a proper face of  $F_0$ , yielding a contradiction.  $\square$

#### A.4.2 Characterization of penalized minimizers and proof of Theorem 1

In the particular and well-studied case in which the norm of the penalized problem is the  $l_1$ -norm, the solutions to the corresponding optimization problem can be characterized by the Karush-Kuhn-Tucker (KKT) conditions for the LASSO, which can be summarized as follows, see for instance, Bühlmann and Van de Geer (2011).

$$\begin{aligned} \hat{\beta} \in S_{X, \lambda \|\cdot\|_1}(y) &\iff \|X'(y - X\hat{\beta})\|_\infty \leq \lambda \text{ and } X'_j(y - X\hat{\beta}) = \lambda \text{sign}(\hat{\beta}_j) \quad \forall j \in \text{supp}(\hat{\beta}) \quad (6) \\ &\iff \|X'(y - X\hat{\beta})\|_\infty \leq \lambda \text{ and } \hat{\beta}' X'(y - X\hat{\beta}) = \lambda \|\hat{\beta}\|_1 \end{aligned}$$

In the above, the supremum-norm is the dual to the  $l_1$ -norm. We can generalize the above characterization for solutions to the penalized problem from (1) in the following proposition. Note that in our notation, the tuning parameter  $\lambda$  is part of the norm  $\|\cdot\|$ .

**Proposition 7.** *Let  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$ . We have  $\hat{\beta} \in S_{X, \|\cdot\|}(y)$  if and only if*

$$\|X'(y - X\hat{\beta})\|^* \leq 1 \text{ and } \hat{\beta}' X'(y - X\hat{\beta}) = \|\hat{\beta}\|.$$

*Proof of Proposition 7.* Using subdifferential calculus, the proof is a straightforward consequence of (3) and Proposition 5.

$$\begin{aligned} \hat{\beta} \in S_{X, \|\cdot\|}(y) &\iff 0 \in X'(X\hat{\beta} - y) + \partial_{\|\cdot\|}(\hat{\beta}) \iff X'(y - X\hat{\beta}) \in \partial_{\|\cdot\|}(\hat{\beta}) \\ &\iff \|X'(y - X\hat{\beta})\|^* \leq 1 \text{ and } \hat{\beta}' X'(y - X\hat{\beta}) = \|\hat{\beta}\|. \end{aligned}$$

$\square$

Before finally showing Theorem 1, the following lemma states that the fitted values are unique over

all solutions of the penalized problem for a given  $y$ . It is a generalization of Lemma 1 in Tibshirani (2013), who proves this fact for the special case of the LASSO.

**Lemma 2.** *Let  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$ . Then  $X\hat{\beta} = X\tilde{\beta}$  for all  $\hat{\beta}, \tilde{\beta} \in S_{X, \|\cdot\|}(y)$ .*

*Proof.* Assume that  $X\hat{\beta} \neq X\tilde{\beta}$  for some  $\hat{\beta}, \tilde{\beta} \in S_{X, \|\cdot\|}(y)$  and let  $\check{\beta} = (\hat{\beta} + \tilde{\beta})/2$ . Because the function  $\mu \in \mathbb{R}^n \mapsto \|y - \mu\|_2^2$  is strictly convex, one may deduce that

$$\|y - X\check{\beta}\|_2^2 < \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \frac{1}{2}\|y - X\tilde{\beta}\|_2^2.$$

Consequently,

$$\frac{1}{2}\|y - X\check{\beta}\|_2^2 + \|\check{\beta}\| < \frac{1}{2}\left(\|y - X\hat{\beta}\|_2^2 + \|\hat{\beta}\| + \|y - X\tilde{\beta}\|_2^2 + \|\tilde{\beta}\|\right),$$

which contradicts both  $\hat{\beta}$  and  $\tilde{\beta}$  being minimizers.  $\square$

*Proof of Theorem 1.*

Throughout the proof, let  $B = \text{conv}(V)$  with  $V = (V_1 \dots V_k) \in \mathbb{R}^{p \times k}$ .

( $\Leftarrow$ ) Assume that there exists a face  $F$  of  $B^*$  that intersects  $\text{row}(X)$  (so that  $F$  is non-empty) and satisfies  $\text{codim}(F) > \text{rk}(X)$  (so that  $F$  is proper). This implies that there exists  $I \subseteq [k]$  such that

$$F = \{s \in B^* : V_I' s = \mathbf{1}_{|I|}\},$$

where  $I$  is the maximal index set satisfying this relationship. Moreover, this implies that  $\text{conv}(V_I)$  is a proper, non-empty face of  $B$  and that we have  $\|s\|^* = 1$  for all  $s \in F$  and  $\|v\| = 1$  for all  $v \in \text{conv}(V_I)$ . We show that non-unique solutions exist. Define  $\hat{\beta} = \sum_{l \in I} V_l$  and observe that  $\|\hat{\beta}\| = \|I\| \sum_{l \in I} \|V_l\| / |I| = |I|$ . Pick  $z \in \mathbb{R}^n$  with  $X'z$ , which exists by assumption, and set  $y = X\hat{\beta} + z$ . Then  $\hat{\beta} \in S_{X, \|\cdot\|}(y)$  by Proposition 7, since

$$\|X'(y - X\hat{\beta})\|^* = \|X'z\|^* = 1 \quad \text{and} \quad \hat{\beta}'(X'(y - X\hat{\beta})) = \hat{\beta}'X'z = \sum_{l \in I} V_l' X'z = |I| = \|\hat{\beta}\|.$$

We now construct  $\tilde{\beta} \in S_{X, \|\cdot\|}(y)$  with  $\tilde{\beta} \neq \hat{\beta}$ . Since  $\text{codim}(F_I) = \dim(\text{col}(V_I)) > \text{rk}(X)$ , we can pick  $h \in \text{col}(V_I) \cap \ker(X)$  with  $h \neq 0$ . Scale  $h$  such that for  $h = \sum_{l \in I} c_l V_l$ , we have  $\max_{l \in I} |c_l| < 1$ , and define  $\tilde{\beta} = \hat{\beta} + h \neq \hat{\beta}$ . Clearly, we have  $X\tilde{\beta} = X\hat{\beta}$ . Note that  $1 + c_l \geq 0$  and let  $\gamma = \sum_{l \in I} (1 + c_l) > 0$ . We also have

$$\|\tilde{\beta}\| = \gamma \left\| \sum_{l \in I} \frac{1 + c_l}{\gamma} V_l \right\| = \gamma = \sum_{l \in I} (1 + c_l) = |I| + \sum_{l \in I} c_l (X'z)' V_l = |I| + (X'z)' h = |I| = \|\hat{\beta}\|,$$

proving that  $\tilde{\beta} \in S_{X, \|\cdot\|}(y)$  also.

( $\Rightarrow$ ) Let us assume that there exists  $y \in \mathbb{R}^n$  and  $\hat{\beta}, \tilde{\beta} \in S_{X, \|\cdot\|}(y)$  with  $\hat{\beta} \neq \tilde{\beta}$ . We then have

$$X'(y - X\hat{\beta}) \in \partial_{\|\cdot\|}(\hat{\beta}) \quad \text{and} \quad X'(y - X\tilde{\beta}) \in \partial_{\|\cdot\|}(\tilde{\beta}).$$

Because  $X\hat{\beta} = X\tilde{\beta}$  by Lemma 2, one may deduce that  $\text{row}(X)$  intersects the face  $\partial_{\|\cdot\|}(\hat{\beta}) \cap \partial_{\|\cdot\|}(\tilde{\beta})$ . Now, let  $F^*$  be a face of  $\partial_{\|\cdot\|}(\hat{\beta}) \cap \partial_{\|\cdot\|}(\tilde{\beta})$  of smallest dimension that intersects  $\text{row}(X)$  and write

$$F^* = \{s \in B^* : V_I' s = \mathbf{1}_{|I|}\},$$

where  $I$  is the largest index set  $I \subseteq [k]$  satisfying this relationship. If  $\text{codim}(F) = \dim(\text{col}(V_I)) \leq \text{rk}(X)$ , consider the following. Note that we can pick  $u \in \mathbb{R}^n$  for which  $X'u \neq 0$  and  $X'u \in \text{col}(V_I)^\perp$ . For this, let  $I_0 \subseteq I$  be such that the columns of  $V_{I_0}$  are linearly independent, and  $\text{col}(V_{I_0}) = \text{col}(V_I)$ . By Lemma 1, we have  $\hat{\beta}, \tilde{\beta} \in \text{col}(V_{I_0})$ , so that we get

$$XV_{I_0}\gamma = X\beta = X\tilde{\beta} = XV_{I_0}\tilde{\gamma}$$

with  $\gamma \neq \tilde{\gamma}$ , implying that the columns of  $XV_{I_0}$  are linearly dependent. But this means that

$$\text{rk}(XV_I) = \dim(\text{col}(XV_I)) = \dim(\text{col}(XV_{I_0})) < |I_0| = \dim(\text{col}(V_{I_0})) = \dim(\text{col}(V_I)) \leq \text{rk}(X).$$

Therefore,  $\text{col}(XV_I) \subsetneq \text{col}(X)$  and, consequently,  $\text{col}(X)^\perp \subsetneq \text{col}(XV_I)^\perp$ , so that we can pick  $u \in \text{col}(XV_I)^\perp \setminus \text{col}(X)^\perp$  for which  $X'u \neq 0$  and  $X'u \in \text{col}(V_I)^\perp$ . Also note that  $X'z \in F^*$  for some  $z \in \mathbb{R}^n$  and that  $X'z$  lies in the relative interior  $\text{relint}(F^*)$ , as otherwise,  $\text{row}(X)$  would intersect a face of  $\partial_{\|\cdot\|}(\hat{\beta}) \cap \partial_{\|\cdot\|}(\tilde{\beta})$  of smaller dimension. The affine line  $\{X'(z + tu) : t \in \mathbb{R}\} \subseteq \text{row}(X)$  intersects  $\text{relint}(F^*)$  and is included in the affine span of  $F^*$  by construction. Therefore, by Proposition 4,  $\text{row}(X)$  intersects a proper face of  $F^*$ , yielding a contradiction.  $\square$

## A.5 Proof of Proposition 1

Note that a set is negligible with respect to the Lebesgue measure on  $\mathbb{R}^{n \times p}$  if and only if it is negligible with respect to the standard Gaussian measure on  $\mathbb{R}^{n \times p}$ . Therefore, to establish Proposition 1, it is enough to prove the following equality

$$\mathbb{P}_Z(\exists y \in \mathbb{R}^n, |S_{Z, \|\cdot\|}(y)| > 1) = 0, \text{ where } Z \in \mathbb{R}^{n \times p} \text{ has iid } \mathcal{N}(0, 1) \text{ entries.} \quad (7)$$

Not that  $\text{rk}(Z) = \min\{n, p\}$  almost surely. Thus, when  $n \geq p$ ,  $\ker(Z) = 0$  almost surely and  $S_{Z, \|\cdot\|}(y)$  is a singleton almost surely. We will use the following lemma to establish (7), where  $\mathbb{N}$  stands for the (positive) natural numbers.

**Lemma 3.** *Let  $n \in \mathbb{N}$ ,  $q \geq n+1$ , and  $v \in \mathbb{R}^q$  where  $v \neq 0$  is a fixed vector. If  $Z = (Z_1, \dots, Z_n) \in \mathbb{R}^{q \times n}$  has iid  $\mathcal{N}(0, 1)$  entries, then  $\mathbb{P}_Z(v \in \text{col}(Z)) = 0$ .*

*Proof.* We first prove the result for  $q = n+1$ . If  $v \in \text{col}(Z)$  then

$$\det(Z_1, \dots, Z_n, v) = 0 \iff \det(Z_1/\|Z_1\|_2, \dots, Z_n/\|Z_n\|_2, v/\|v\|_2) = 0.$$

Now, because the columns  $Z_1/\|Z_1\|_2, \dots, Z_n/\|Z_n\|_2$  follow a uniform distribution on the  $l_2$  unit sphere, one may deduce that the distribution of the random variable  $\det(Z_1/\|Z_1\|_2, \dots, Z_n/\|Z_n\|_2, v/\|v\|_2)$  is

equal to the distribution of  $\det(Z_1/\|Z_1\|_2, \dots, Z_n/\|Z_n\|_2, \zeta/\|\zeta\|_2)$  where  $\zeta$  has a  $\mathcal{N}(0, \mathbb{I}_{n+1})$  distribution, independent from  $Z_1, \dots, Z_n$ , because conditioning on  $\zeta = v$  does not change the distribution. Finally, the random variable

$$\det(Z_1/\|Z_1\|_2, \dots, Z_n/\|Z_n\|_2, \zeta/\|\zeta\|_2) = \frac{1}{\|Z_1\|_2 \times \dots \times \|Z_n\|_2 \times \|\zeta\|_2} \det(Z_1, \dots, Z_n, \zeta)$$

is non-zero almost surely. Consequently,  $\mathbb{P}_Z(v \in \text{col}(Z)) = 0$ . When  $q > n + 1$ , let  $I \subseteq [q]$  with  $|I| = n + 1$  and  $v_I \neq 0$ . Consequently,  $v_I \in \text{col}(\tilde{Z})$ , where  $\tilde{Z} \in \mathbb{R}^{(n+1) \times n}$  is obtained by keeping the rows of  $Z$  with indices in  $I$ . Therefore,  $P_Z(v \in \text{col}(Z)) \leq P_{\tilde{Z}}(v_I \in \text{col}(\tilde{Z})) = 0$ , which concludes the proof.  $\square$

*Proof of Proposition 1.* If  $n \leq p$ , we are done. Therefore, assume that  $p > n$ , and let  $F_0$  be a proper face of  $B^*$  such that  $\text{codim}(F_0) = q > n$  (note that  $0 \notin \text{aff}(F_0)$ , the affine space spanned by  $F_0$ ). There exists  $A \in \mathbb{R}^{q \times p}$ , where rows of  $A$  are orthonormal ( $AA' = \mathbb{I}_q$ ) and  $v \in \mathbb{R}^q$ ,  $v \neq 0$  such that  $\text{aff}(F_0) = \{x \in \mathbb{R}^p : Ax = v\}$ . Since  $AA' = \mathbb{I}_q$ ,  $AZ' \in \mathbb{R}^{q \times n}$  has iid  $\mathcal{N}(0, 1)$  entries. Thus, by Lemma 3, we have

$$\mathbb{P}_Z(\text{row}(Z) \cap F_0 \neq \emptyset) \leq \mathbb{P}_Z(\text{row}(Z) \cap \text{aff}(F_0) \neq \emptyset) = \mathbb{P}_Z(v \in \text{col}(AZ')) = 0. \quad (8)$$

According to Theorem 1 and since  $\text{rk}(Z) = n$  almost surely, the following equalities hold.

$$\begin{aligned} \mathbb{P}_Z(\exists y \in \mathbb{R}^n, |S_{Z, \|\cdot\|}(y)| > 1) &= \mathbb{P}_Z \left( \bigcup_{\substack{F \in \mathcal{F}(P) \\ \text{codim}(F) > \text{rk}(Z)}} \{\text{row}(Z) \cap F \neq \emptyset\} \right) \\ &= \mathbb{P}_Z \left( \bigcup_{\substack{F \in \mathcal{F}(P) \\ \text{codim}(F) > n}} \{\text{row}(Z) \cap F \neq \emptyset\} \right) = 0. \end{aligned}$$

The last equality is a consequence of (8).  $\square$

## A.6 Proof of Theorem 3

*Proof.* ( $\implies$ ) Let  $\sigma$  be an accessible sign vector for LASSO. Then there exists  $y \in \mathbb{R}^n$  and  $\hat{\beta} \in S_{X, \lambda, \|\cdot\|_1}(y)$  such that  $\text{sign}(\hat{\beta}) = \sigma$ . According to the characterization of LASSO minimizers in (6), by setting  $z = (y - X\hat{\beta})/\lambda$ , one may deduce that  $X'z \in F(\sigma)$ . If  $\sigma$  is an accessible sign vector for BP, there exists  $y \in \text{col}(X)$  and  $\hat{\beta} \in S_{X, \text{bp}}(y)$  with  $\text{sign}(\hat{\beta}) = \sigma$ . According to the characterization of BP minimizers in (5), there exists  $z \in \mathbb{R}^n$  such that  $X'z \in F(\sigma)$ .

( $\impliedby$ ) If  $\text{row}(X)$  intersects the face  $F(\sigma)$ , then there exists  $f \in F(\sigma)$  and  $z \in \mathbb{R}^n$  such that  $X'z = f$ . Note that  $j \in \text{supp}(\sigma)$  implies that  $f_j = \sigma_j = \text{sign}(\sigma_j)$ . Set  $y = \lambda z + X\sigma$ . We show that

$\sigma \in S_{X, \|\cdot\|_1}(y)$ . We have

$$\begin{cases} \|X'(y - X\sigma)\|_\infty = \lambda \|X'z\|_\infty \leq \lambda, \\ X'_j(y - X\sigma) = \lambda X'_j z = \lambda f_j = \lambda \sigma_j = \lambda \text{sign}(\sigma_j) \quad \forall j \in \text{supp}(\sigma), \end{cases}$$

so that according to the characterization of LASSO minimizers in (6), we have  $\sigma \in S_{X, \|\cdot\|_1}(y)$ , implying that  $\sigma$  is accessible for LASSO. For BP, set  $y = X\sigma$  and note that, according to the characterization of BP minimizers in (5),  $\sigma \in S_{X, \text{bp}}(y)$ , implying that  $\sigma$  is also accessible for BP.  $\square$

## A.7 Proof of Theorem 4

Theorem 4 states that there is a bijection between the SLOPE models and the faces of the sign permutahedron. The basis for proving this is the fact that the sign permutahedron is the dual of the SLOPE norm unit ball, and that any face of it is given by a subdifferential of the SLOPE norm by Proposition 6.

We start by proving the following proposition which shows that the subdifferential of the SLOPE norm at zero is, indeed, the sign permutahedron, and also characterizes the subdifferential of the SLOPE norm for certain values of  $x$ .

**Proposition 8.** *The subdifferential  $F_w(x) = \partial_{\|\cdot\|_w}(x)$  of the SLOPE norm exhibits the following properties.*

- 1) We have  $F_w(0) = P_w^\pm$ .
- 2) For any  $x \in \mathbb{R}^p$  with  $x_1 = \dots = x_p > 0$ , we have  $F_w(x) = P_w$ .
- 3) For any  $x \in \mathbb{R}^p$  with  $x_1 \geq \dots \geq x_k > x_{k+1} \geq \dots \geq x_p \geq 0$ , we have

$$F_w(x) = F_{w_{[k]}}(x_{[k]}) \times F_{w_{[k+1:p]}}(x_{[k+1:p]}).$$

- 4) Let  $0 < k_1 < \dots < k_l < p$  be an arbitrary subdivision of  $[0 : p]$ , then for any  $x \in \mathbb{R}^p$  with  $x_1 = \dots = x_{k_1} > x_{k_1+1} = \dots = x_{k_2} > \dots > x_{k_l+1} = \dots = x_p \geq 0$ , we have  $\text{codim}(F_w(\text{mdl}(x))) = \|\text{mdl}(x)\|_\infty$  and

$$F_w(x) = F_w(\text{mdl}(x)) = \begin{cases} P_{w_{[k_1]}} \times \dots \times P_{w_{[k_{l-1}+1:k_l]}} \times P_{w_{[k_l+1:p]}} & \text{if } x_p > 0 \\ P_{w_{[k_1]}} \times \dots \times P_{w_{[k_{l-1}+1:k_l]}} \times P_{w_{[k_l+1:p]}}^\pm & \text{if } x_p = 0. \end{cases}$$

*Proof.* 1) By Proposition 5, we may show that  $P_w^\pm = B^*$ .

( $\subseteq$ ) Take any vertex  $W = (\sigma_1 w_{\pi(1)}, \dots, \sigma_p w_{\pi(p)})'$  of  $P_w^\pm$  and any  $x \in \mathbb{R}^p$  with  $\|x\|_w \leq 1$ . We have

$$W'x = \sum_{j=1}^p \sigma_j w_{\pi(j)} x_j \leq \sum_{j=1}^p |x_j| w_{\pi(j)} \leq \sum_{j=1}^p w_j |x_{(j)}| = \|x\|_w \leq 1$$

and therefore  $W \in B^*$ . By convexity,  $P_w^\pm \subseteq B^*$  follows.

( $\supseteq$ ) Let  $a'x \leq b_0$  for some  $a \in \mathbb{R}^p$  and  $b_0 \in \mathbb{R}$  be a valid inequality of  $P_w^\pm$ . We show that this is a valid inequality of  $B^*$  also: Let  $W$  be the vertex of  $P_w^\pm$  defined by  $W_j = \text{sign}(a_j)w_{\pi^{-1}(j)}$ , where the permutation  $\pi$  satisfies  $|a_{\pi(1)}| \geq \dots \geq |a_{\pi(p)}|$ . For any  $s \in B^*$ , we have

$$a's \leq \|a\|_w = \sum_{j=1}^p |a_{\pi(j)}|w_j = \sum_{j=1}^p \text{sign}(a_j)a_jw_{\pi^{-1}(j)} = a'W \leq b_0.$$

Since  $P_w^\pm$  can be written as the (finite) intersection of half-spaces,  $P_w^\pm \supseteq B^*$  follows.

2) According to Proposition 5 and 1), we have

$$F_w(x) = \left\{ s \in P_w^\pm : \sum_{j=1}^p s_j = \sum_{j=1}^p w_j \right\}.$$

A vertex  $W = (\sigma_1 w_{\pi(1)}, \dots, \sigma_p w_{\pi(p)})'$  of  $P_w^\pm$  with  $\sigma \in \{-1, 1\}^p$  and  $\pi \in \mathcal{S}_p$  then fulfills  $W \in F_w$  if and only if  $\sigma_1 = \dots = \sigma_p = 1$ . Convexity then yields  $F_w(x) = P_w$ .

3) ( $\subseteq$ ) Let  $s \in F_w(x)$ . We show that  $s_{[k]} \in F_{w_{[k]}}(x_{[k]})$  and  $s_{[k+1:p]} \in F_{w_{[k+1:p]}}(x_{[k+1:p]})$ . Let  $e = \frac{x_k - x_{k+1}}{2} > 0$  and  $h \in \mathbb{R}^p$  with  $\|h\|_\infty < e$ . Since the  $k$  largest components of  $x+h$  are  $\{x_j + h_j\}_{j \in [k]}$ , we have

$$\|x+h\|_w = \|(x+h)_{[k]}\|_{w_{[k]}} + \|(x+h)_{[k+1:p]}\|_{w_{[k+1:p]}}.$$

Now, take  $h \in \mathbb{R}^p$  such that  $\|h\|_\infty < e$  and  $h_{k+1} = \dots = h_p = 0$ . Using the above identity and the definition of  $F_w(x)$ , one may deduce that

$$\begin{aligned} \|(x+h)_{[k]}\|_{w_{[k]}} &= \|x+h\|_w - \|x_{[k+1:p]}\|_{w_{[k+1:p]}} \\ &\geq \|x\|_w + s'h - \|x_{[k+1:p]}\|_{w_{[k+1:p]}} = \|x_{[k]}\|_{w_{[k]}} + \sum_{j=1}^k s_j h_j. \end{aligned}$$

We therefore obtain that

$$\|x_{[k]} + h\|_{w_{[k]}} \geq \|x_{[k]}\|_{w_{[k]}} + s'_{[k]}h$$

for all  $h \in \mathbb{R}^k$  satisfying  $\|h\|_\infty < e$ . By (4), we conclude  $s_{[k]} \in F_{w_{[k]}}(x_{[k]})$ . To show that  $s_{[k+1:p]} \in F_{w_{[k+1:p]}}(x_{[k+1:p]})$ , one can proceed in a similar manner.

( $\supseteq$ ) For  $s \in F_{w_{[k]}}(x_{[k]}) \times F_{w_{[k+1:p]}}(x_{[k+1:p]})$ , we clearly have

$$s'x = \sum_{i=1}^k s_i x_i + \sum_{i=k+1}^p s_i x_i = \|x_{[k]}\|_{w_{[k]}} + \|x_{[k+1:p]}\|_{w_{[k+1:p]}} = \|x\|_w,$$

so that  $s \in F_w(x)$  follows.

4) For  $x \in \mathbb{R}^p$  with  $x_1 = \dots = x_{k_1} > \dots > x_{k_l+1} = \dots = x_p$ ,  $\text{mdl}(x)$  is clearly given by

$$\begin{cases} \text{mdl}(x)_1 = \dots = \text{mdl}(x)_{k_1} = l + 1 > \dots > \text{mdl}(x)_{k_l+1} = \dots = \text{mdl}(x)_p = 1 & \text{if } x_p > 0 \\ \text{mdl}(x)_1 = \dots = \text{mdl}(x)_{k_1} = l > \dots > \text{mdl}(x)_{k_l+1} = \dots = \text{mdl}(x)_p = 0 & \text{if } x_p = 0. \end{cases}$$

According to 1), 2) and 3), it is clear that

$$F_w(x) = F_w(\text{mdl}(x)) = \begin{cases} P_{w_{[k_1]}} \times \dots \times P_{w_{[k_{l-1}+1:k_l]}} \times P_{w_{[k_l+1:p]}} & \text{if } x_p > 0 \\ P_{w_{[k_1]}} \times \dots \times P_{w_{[k_{l-1}+1:k_l]}} \times P_{w_{[k_l+1:p]}}^\pm & \text{if } x_p = 0. \end{cases}$$

Since the codimension of a permutahedron is equal to 1 (see Maes and Kappen, 1992; Simion, 1997), the one of sign permutahedron is equal to 0, and since the (co-)dimensions of the individual (sign) permutahedra can simply be added up, we have  $\text{codim}(F_w(x)) = \|\text{mdl}(x)\|_\infty$ .  $\square$

Proposition 8 lays the groundwork by essentially proving Theorem 4 for all SLOPE models with non-negative and non-decreasing components. We denote this set of models by  $\mathcal{M}_p^{\geq,+}$ , given by

$$\mathcal{M}_p^{\geq,+} = \{m \in \mathcal{M}_p : m_1 \geq \dots \geq m_p \geq 0\}.$$

In order to extend this proposition to all SLOPE models in  $\mathcal{M}_p$ , we introduce the following group of linear transformations.

**Definition 4.** Let  $\sigma \in \{-1, 1\}^p$ , let  $\pi \in \mathcal{S}_p$ . We define the map

$$\phi_{\sigma,\pi} : x \in \mathbb{R}^p \mapsto (\sigma_1 x_{\pi(1)}, \dots, \sigma_p x_{\pi(p)})'$$

and denote by  $\mathcal{G} = \{\phi_{\sigma,\pi} : \sigma \in \{-1, 1\}^p, \pi \in \mathcal{S}_p\}$ .

The set  $\mathcal{G}$  is a finite sub-group of the group of orthogonal transformations on  $\mathbb{R}^p$ . We list a number of straight-forward properties of  $\mathcal{G}$  in the following lemma.

**Lemma 4.** Let  $x, v \in \mathbb{R}^p$ ,  $\phi \in \mathcal{G}$ , and let  $\sigma \in \{-1, 1\}^p$  and  $\pi \in \mathcal{S}_p$ . Then the following holds.

- 1)  $x'v = \phi(x)'\phi(v)$
- 2)  $\|x\|_w = \|\phi(x)\|_w$
- 3)  $\|x\|_\infty = \|\phi(x)\|_\infty$
- 4)  $\phi(\mathcal{M}_p) = \mathcal{M}_p$  and  $\phi(P_w^\pm) = P_w^\pm$
- 5)  $\text{mdl}(\phi(x)) = \phi(\text{mdl}(x))$
- 6)  $\phi_{\sigma,\pi}^{-1} = \phi_{\sigma,\pi^{-1}} \in \mathcal{G}$
- 7) If, for  $m \in \mathcal{M}_p$ ,  $|m_{\pi(1)}| \geq \dots \geq |m_{\pi(p)}|$  and  $\sigma_j m_{\pi(j)} = |m_{\pi(j)}|$  for all  $j \in [p]$ , then  $\phi_{\sigma,\pi}(m) \in \mathcal{M}_p^{\geq,+}$ .

**Lemma 5.** *Let  $\phi \in \mathcal{G}$  and  $x \in \mathbb{R}^p$ . We then have*

$$\phi^{-1}(F_w(\phi(x))) = F_w(x) \quad \text{and} \quad F_w(\phi(x)) = \phi(F_w(x)).$$

*Proof.* The two statements are equivalent, we show the second one. Let  $s \in P_w^\pm$ . Then

$$\begin{aligned} s \in F_w(\phi(x)) &\iff s' \phi(x) = \|\phi(x)\|_w \iff \phi^{-1}(s)' x = \|x\|_w \\ &\iff \phi^{-1}(s) \in F_w(x) \iff s \in \phi(F_w(x)) \end{aligned}$$

by Proposition 5 and Lemma 4. □

We are now equipped to prove Theorem 4.

*Proof of Theorem 4.*

We start by proving 1) and 2) before showing that the map is a bijection.

1) Let  $m \in \mathcal{M}_p$  and let  $\phi \in \mathcal{G}$  such that  $\phi(m) \in \mathcal{M}_p^{\geq,+}$ . According to Lemma 5, and because  $\phi$  is an isomorphism on  $\mathbb{R}^p$ , we have

$$\text{codim}(F_w(m)) = \text{codim}(\phi^{-1}(F_w(\phi(m)))) = \text{codim}(F_w(\phi(m))) = \|\phi(m)\|_\infty = \|m\|_\infty.$$

2) Let  $x \in \mathbb{R}^p$  and let  $\phi \in \mathcal{M}_p$  such that  $\phi(x)_1 \geq \dots \geq \phi(x)_p \geq 0$ . According to Lemma 5 and Proposition 8, the following equalities hold

$$F_w(x) = \phi^{-1}(F_w(\phi(x))) = \phi^{-1}(F_w(\text{mdl}(\phi(x)))) = \phi^{-1}(F_w(\phi(\text{mdl}(x)))) = F_w(\text{mdl}(x)).$$

We now show that the mapping under consideration is indeed a bijection between  $\mathcal{M}_p$  and  $\mathcal{F}_0$ .

(surjection) According to Proposition 6, a non-empty face of  $P_w^\pm$  can be expressed as  $F_w(x)$  for some  $x \in \mathbb{R}^p$ . According to 2) above, we have  $F_w(x) = F_w(\text{mdl}(x))$  for  $\text{mdl}(x) \in \mathcal{M}_p$ .

(injection) Note that Proposition 8 shows that the mapping is injective on  $\mathcal{M}_p^{\geq,+}$ . To prove that it remains injective on all of  $\mathcal{M}_p$ , we show that  $|\mathcal{M}_p| \leq |\mathcal{F}_0|$ . For this, we need several definitions. For  $m \in \mathcal{M}_p$ , let  $\text{stab}_{\mathcal{G}}(m) = \{\phi \in \mathcal{G} : \phi(m) = m\}$  and  $\text{orb}_{\mathcal{G}}(m) = \{\phi(m) : \phi \in \mathcal{G}\}$ , the stabilizer and orbit of  $m$ , respectively, with respect to  $\mathcal{G}$ . For  $m \in \mathcal{M}_p$ , there exists  $\phi \in \mathcal{G}$  such that  $\phi(m) \in \mathcal{M}_p^{\geq,+}$ . Therefore, the orbit-stabilizer formula gives

$$\mathcal{M}_p = \bigcup_{m \in \mathcal{M}_p^{\geq,+}} \text{orb}_{\mathcal{G}}(m) \implies |\mathcal{M}_p| \leq \sum_{m \in \mathcal{M}_p^{\geq,+}} |\text{orb}_{\mathcal{G}}(m)| = \sum_{m \in \mathcal{M}_p^{\geq,+}} \frac{|\mathcal{G}|}{|\text{stab}_{\mathcal{G}}(m)|}.$$

We also look at stabilizer and orbit when  $\mathcal{G}$  operates on  $\mathcal{F}_0$ . For a face  $F \in \mathcal{F}_0$ , let  $\text{stab}_{\mathcal{G}}(F) = \{\phi \in \mathcal{G} : \phi(F) = F\}$  and  $\text{orb}_{\mathcal{G}}(F) = \{\phi(F) : \phi \in \mathcal{G}\}$ . We first show that if  $\text{orb}_{\mathcal{G}}(F_w(m)) \cap \text{orb}_{\mathcal{G}}(F_w(\tilde{m})) \neq \emptyset$  for some  $m, \tilde{m} \in \mathcal{M}_p^{\geq,+}$ ,  $m = \tilde{m}$  follows. Let us assume that  $F_w(\tilde{m}) = \phi(F_w(m))$  for some  $\phi \in \mathcal{G}$ . Note that  $\phi(F_w(m)) = F_w(\phi(m))$  by Lemma 5. Since  $w \in F_w(m)$  and  $w \in F_w(\tilde{m}) = F_w(\phi(m))$ , we have

$$w'm = \|m\|_w = \|\phi(m)\|_w = w'\phi(m),$$

where the second-last equality holds by Lemma 4 and the last equality holds since  $m \in \mathcal{M}_p^{\geq,+}$ . Now, if  $\phi(m) \neq m$ ,  $\phi(m)'m < \|m\|_w$  follows since the components of  $w$  are positive and strictly decreasing. But that would contradict the above, so  $\phi(m) = m$  must hold. Consequently,  $F_w(\tilde{m}) = F_w(m)$ , which in turn implies  $\tilde{m} = m$  by Proposition 8.

Now, let  $m \in \mathcal{M}_p^{\geq,+}$  and let us show that  $\text{stab}_{\mathcal{G}}(m) = \text{stab}_{\mathcal{G}}(F_w(m))$ . The inclusion  $\text{stab}_{\mathcal{G}}(m) \subseteq \text{stab}_{\mathcal{G}}(F_w(m))$  immediately follows from

$$\begin{aligned} \phi \in \text{stab}_{\mathcal{G}}(m) &\implies F_w(m) = \phi^{-1}(F_w(\phi(m))) = \phi^{-1}(F_w(m)) \implies \phi(F_w(m)) = F_w(m) \\ &\implies \phi \in \text{stab}_{\mathcal{G}}(F_w(m)). \end{aligned}$$

To show  $\text{stab}_{\mathcal{G}}(F_w(m)) \subseteq \text{stab}_{\mathcal{G}}(m)$ , let  $\phi \in \text{stab}_{\mathcal{G}}(F_w(m))$  and note that  $F_w(m) = \phi(F_w(m)) = F_w(\phi(m))$ . Since  $m \in \mathcal{M}_p^{\geq,+}$ , this implies that  $w \in F_w(m) = F_w(\phi(m))$ , so that the same reasoning as above yields  $m = \phi(m)$  and  $\phi \in \text{stab}_{\mathcal{G}}(m)$ .

To conclude, note that since the orbits  $\text{orb}_{\mathcal{G}}(F_w(m))$  with  $m \in \mathcal{M}_p^{\geq,+}$  are disjoint, and since  $\text{stab}_{\mathcal{G}}(m) = \text{stab}_{\mathcal{G}}(F_w(m))$ , we may deduce that

$$|\mathcal{M}_p| \leq \sum_{m \in \mathcal{M}_p^{\geq,+}} \frac{|\mathcal{G}|}{|\text{stab}_{\mathcal{G}}(F_w(m))|} = \sum_{m \in \mathcal{M}_p^{\geq,+}} |\text{orb}_{\mathcal{G}}(F_w(m))| = \left| \bigcup_{m \in \mathcal{M}_p^{\geq,+}} \text{orb}_{\mathcal{G}}(F_w(m)) \right| \leq |\mathcal{F}_0|.$$

□

## A.8 Proof of Theorem 5

*Proof.* ( $\implies$ ) If  $m$  is an accessible SLOPE model, then

$$\exists y \in \mathbb{R}^n, \exists \hat{\beta} \in S_{X, \|\cdot\|_w}(y) \text{ such that } \text{mdl}(\hat{\beta}) = m.$$

By Theorem 4, we may deduce that  $\partial_{\|\cdot\|_w}(\hat{\beta}) = F_w(\hat{\beta}) = F_w(m)$ . Consequently,

$$0 \in X'(X\hat{\beta} - y) + \partial_{\|\cdot\|_w}(\hat{\beta}) = F_m \implies X'(y - X\hat{\beta}) \in F_w(m).$$

Therefore,  $\text{row}(X)$  intersects  $F_w(m)$ .

( $\impliedby$ ) If  $\text{row}(X)$  intersects the face  $F_w(m)$ , then there exists  $z \in \mathbb{R}^n$  such that  $X'z = f \in F_w(m)$ . We set  $y = z + Xm$  and show that  $m \in S_{X, \|\cdot\|_w}(y)$ . We have

$$\|X'(y - Xm)\|_w^* = \|f\|_w^* \leq 1 \text{ and } m'X'(y - Xm) = m'f = \|m\|_w,$$

which, by Proposition 7, yields  $m \in S_{X, \|\cdot\|_w}(y)$ . □

## A.9 Proof of Proposition 2

*Proof.* By Theorem 5, we know that

$$m \in \mathcal{M}_p \text{ is accessible} \iff \text{row}(X) \cap F_w(m) \neq \emptyset \iff \exists f \in \mathbb{R}^n : X'f \in F_w(m) \iff f \in N_w(m),$$

which proves the first statement. Now, let  $y = f + Xb$ , where  $f \in N_w(m)$  and  $b \in \mathbb{R}^p$  such that  $\text{mdl}(b) = m$ . Note that

$$\|X'(y - Xb)\|_w^* = \|X'f\|_w^* \leq 1 \text{ and } b'X'(y - Xb) = b'X'f = \|b\|_w^*,$$

where the first inequality holds since  $X'f \in F_w(m)$ , a face of  $P_w^\pm$ , and the latter one by applying Proposition 6 after noticing that  $X'f \in F_w(m) = F_w(b) = \partial_{\|\cdot\|_w}(b)$  by Theorem 4. Proposition 7 then yields  $b \in S_{X, \|\cdot\|_w}(y)$ , so that  $y \in A_w(m)$ .

Conversely, let  $y \in A_w(m)$  and let  $\hat{\beta} \in S_{X, \|\cdot\|_w}(y)$  so that  $\text{mdl}(\hat{\beta}) = m$ . Then  $y - X\hat{\beta} \in N_w(m)$  since by Proposition 7, we have

$$X'(y - X\hat{\beta}) \in \partial_{\|\cdot\|_w}(\hat{\beta}) = F_w(m),$$

where the last equality holds by Theorem 4. □

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