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# Instantaneous optimal control of friction dominated flow in a gas-network

Günter Leugering and Gisèle Mophou

**Abstract.** We consider optimal control problems for the flow of gas in a pipe network. The equations of motions are taken to be represented by a nonlinear model derived from asemi-linear approximation of the fully nonlinear isothermal Euler gas equations. We formulate an optimal control problem on a given network and introduce a time discretization thereof. We then study the well-posedness of the corresponding time-discrete optimal control problem. In order to further reduce the complexity, we consider an instantaneous control strategy. This involves a p-Laplace-type problem on the graph with  $p = \frac{3}{2}$ . We prove well-posedness, existence of optimal controls and derive a first order optimality condition.

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**Keywords.** Optimal control, gas networks, p-Laplace problem on a graph, optimality system.

## 1. Introduction

### 1.1. Modeling of gas flow in a single pipe

The Euler equations are given by a system of nonlinear hyperbolic partial differential equations (PDEs) which represent the motion of a compressible non-viscous fluid or a gas. They consist of the continuity equation, the balance of moments and the energy equation. The full set of equations is given by (siehe [4, 9, 10, 12]) Let  $\rho$  denote the density,  $v$  the velocity of the

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gas and  $p$  the pressure. We further denote  $g$  the gravitational constant,  $\lambda$  the friction coefficient of the pipe,  $D$  the diameter,  $a$  the area of the cross section. The state variables of the system are  $\rho$ , the flux  $q = \rho v$ . We also denote  $c$  the the speed of sound, i.e.  $c^2 = \frac{\partial p}{\partial \rho}$  (for constant entropy). For natural gas we have  $340 \frac{\text{m}}{\text{s}}$ . In particular, in the subsonic case ( $|v| < c$ ), the one which we consider in the sequel, two boundary conditions have to be imposed on the left end and one at the right end of the pipe. We consider here the isothermal case only. Thus, for horizontal pipes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) &= -\frac{\lambda}{2D}\rho v |v|. \end{aligned} \quad (1.1)$$

In the particular case, where the we have a constant speed of sound  $c = \sqrt{\frac{p}{\rho}}$ , for small velocities  $|v| \ll c$ , we arrive at the semi-linear model

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial p}{\partial x} &= -\frac{\lambda}{2D}\rho v |v|. \end{aligned} \quad (1.2)$$

If we further neglect the inertia with respect to the flux and introduce  $q = \rho v a$ , we arrive at

$$\begin{aligned} \frac{\partial p}{\partial t} + \frac{c^2}{a} \frac{\partial}{\partial x} q &= 0 \\ \frac{\partial p^2}{\partial x} &= -\frac{\lambda c^2}{D a^2} q |q| =: -\gamma^2 q |q|. \end{aligned} \quad (1.3)$$

We now set  $y := p^2$  and obtain from the second equation in (1.3)

$$q = -\frac{1}{\gamma} \frac{\frac{\partial y}{\partial x}}{\sqrt{\left| \frac{\partial y}{\partial x} \right|}}.$$

With  $\alpha := \frac{\gamma a}{c}$  we obtain

$$\alpha \frac{\partial}{\partial t} \frac{y}{\sqrt{|y|}} - \frac{\partial}{\partial x} \frac{\frac{\partial y}{\partial x}}{\sqrt{\left| \frac{\partial y}{\partial x} \right|}} = 0. \quad (1.4)$$

We introduce the monotone function  $\beta(s) := \frac{s}{\sqrt{|s|}}$ . With this (1.4) reads as

$$\alpha \frac{\partial}{\partial t} \beta(y) - \frac{\partial}{\partial x} \beta\left(\frac{\partial y}{\partial x}\right) = 0. \quad (1.5)$$

It is also possible to write this down in the p-Laplace format: (1.4) reads as

$$\alpha \frac{\partial}{\partial t} (|y|^{p-2}y) - \frac{\partial}{\partial x} \left( \left| \frac{\partial y}{\partial x} \right|^{p-2} \frac{\partial y}{\partial x} \right) = 0, \quad (1.6)$$

where  $p = \frac{3}{2}$ . Equation (1.6) has come to be known as doubly nonlinear parabolic equation of p-Laplace type. See [11]. Notice that  $p < 2$  and that the system is, therefore, singular for  $\frac{\partial}{\partial x}y(x) = 0$ . For  $p > 2$  such equations exhibit instead degeneration. Equations similar to (1.5) have been considered in the literature, see e.g. [2, 3]. In this contribution, we aim at a discussion of such equations together with optimal control problems on networks. A more recent study of doubly nonlinear parabolic equations in the context of friction dominated flow has been provided in [1]. Equations of the type (1.5) are known to exhibit positive solutions, finite speed of propagation and satisfy a maximum principle. As a matter of fact, to the best knowledge of the authors, there are no studies on optimal control of such systems on general graphs available from the literature. Therefore, these notes can be seen as the first attempt in that direction. We refer to a forthcoming publication, where the additional properties mentioned, full proofs and numerical analysis are provided [8].

## 1.2. Network modeling

Let  $G = (V, E)$  denote the graph of the gas network with vertices (nodes)  $V = \{n_1, n_2, \dots, n_{|V|}\} = \{n_j | j \in \mathcal{J}\}$  and edges  $E = \{e_1, e_2, \dots, e_{|E|}\} = \{e_i | i \in \mathcal{I}\}$ . For the sake of uniqueness, we associate to each edge a direction.

$$d_{ij} = \begin{cases} -1, & \text{if node } n_j \text{ if the the edge } e_i \text{ starts at node } n_j e_i, \\ +1, & \text{if node } n_j \text{ if the edges } e_i \text{ end at node } n_j e_i, \\ 0, & \text{else.} \end{cases}$$

The pressure variables  $y_i(n_j)$  coincide for all  $i \in \mathcal{I}_j := \{i \in 1, \dots, E | d_{ij} \neq 0\}$ . We express the transmission conditions at the nodes in the following way. We introduce the edge degree  $d_j := |\mathcal{I}_j|$ . Then the continuity conditions read as follows

$$y_i(n_j, t) = y_k(n_j, t), \quad \forall i, k \in \mathcal{I}_j, \quad d_j > 1. \quad (1.7)$$

The nodal balance equation for the fluxes can be written as the classical Kirchhoff-type condition

$$\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j, t)) = 0, \quad d_j > 1. \quad (1.8)$$

We use the  $d_j$  in order to decompose the index set for nodes  $\mathcal{J}$  into  $\mathcal{J} = \mathcal{J}^M \cup \mathcal{J}^S$ , where  $\mathcal{J}^M = \{j \in \mathcal{J} | d_j > 1\}$  represents the multiple nodes and  $\mathcal{J}^S = \{j \in \mathcal{J} | d_j = 1\}$  the simple nodes. According to Dirichlet or

Neumann boundary conditions at the simple nodes, we further decompose  $\mathcal{J}^S = \mathcal{J}_D^S \cup \mathcal{J}_N^S$ . We summarize the equations as follows:

$$\begin{aligned}
\alpha_i \partial_t \beta(y_i(x, t)) + \partial_x (\beta(\partial_x y_i(x, t))) &= 0, \quad i \in \mathcal{I}, \quad x \in (0, \ell_i), \quad t \in (0, T) \\
y_i(n_j, t) &= y_k(n_j, t), \quad \forall i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M, \quad t \in (0, T) \\
\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j, t)) &= 0, \quad j \in \mathcal{J}^M, \quad t \in (0, T) \\
y_i(n_j, t) &= 0, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S, \quad t \in (0, T) \\
d_{ij} \beta(\partial_x y_i(n_j, t)) &= u_j(t), \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S, \quad t \in (0, T) \\
p_i(x, 0) &= p_{i,0}(x), \quad q_i(x, 0) = q_{i,0}(x), \quad x \in (0, \ell_i)
\end{aligned} \tag{1.9}$$

To the best knowledge of the authors, problem (1.9), no published result seems to be available.

**Optimal control problems and outline.** We are now in the position to formulate optimal control problems on the level of the gas networks. We first describe the general format for an optimal control problem associated with the semi-linear model equations. This involves a cost function that assigns to admissible each pair  $(y, u)$  a 'cost'  $I(y, u)$ , which is represented on each individual edge by a contribution on the state  $I_i(y)$  and the controls acting at simple nodes. The typical example, the one that we will use in the sequel is given by

$$\begin{aligned}
I_i(y_i)(x) &:= \frac{1}{p} |y_i(x) - y_i^d(x)|^p, \quad x \in (0, \ell_i), \quad p \in \left\{ \frac{3}{2}, 2 \right\}. \\
\min_{(y, u) \in \Xi} I(y, u) &:= \sum_{i \in \mathcal{I}} \int_0^T \int_0^{\ell_i} I_i(y_i) dx dt + \frac{\nu}{2} \sum_{j \in \mathcal{J}_N^S} \int_0^T |u_j(t)|^2 dt \\
&\text{s.t.} \\
&(y, u) \text{ satisfies (1.9),}
\end{aligned} \tag{1.10}$$

$$\Xi := \{(y, u) : \underline{y}_i \leq y_i \leq \bar{y}_i, \quad i \in \mathcal{I}, \underline{u}_j \leq u_j \leq \bar{u}_j, \quad j \in \mathcal{J}_N^S\}. \tag{1.11}$$

In (1.11), the quantities  $\underline{y}_i, \bar{y}_i$  are given constants that determine the feasible pressures and flows in the pipe  $i$ , while  $\underline{u}_i, \bar{u}_i$  describe control constraints. In the continuous-time case the inequalities are considered as being satisfied for all times and everywhere along the pipes. Due to limitations in space, we will not consider state- and control constraints here and refer instead to a forthcoming article [8]. Instead, we just penalize the control costs using  $\nu > 0$ . Moreover, we will restrict ourselves to time discretizations of (1.10) and, in fact, to the instantaneous control regime that has come to be known also as rolling horizon problem.

### 1.3. Time discretization

We, therefore, consider the time discretization (1.9) such that  $[0, T]$  is decomposed into break points  $t_0 = 0 < t_1 < \dots < t_N = T$  with widths  $\Delta t_n := t_{n+1} - t_n, n = 0, \dots, N - 1$  (we use  $N + 1$  as the number of break points which is not related to  $N$  as indicating Neumann conditions). Accordingly, we denote  $p_i(x, t_n) := p_{i,n}(x), q_i(x, t_n) := q_{i,n}(x), n = 0, \dots, N - 1$ . We consider a semi-implicit Euler scheme which takes  $p_i$  in the friction term in an explicit manner.

$$\begin{aligned}
 \frac{1}{\Delta t} \beta(y_{i,n+1})(x) - \partial_x (\beta(\partial_x y_{i,n+1}(x))) &= \frac{1}{\Delta t} \beta(y_{i,n})(x), \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
 y_{i,n+1}(n_j) &= y_{k,n+1}(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M \\
 \sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_{i,n+1})(n_j) &= 0, \quad j \in \mathcal{J}^M \\
 \beta(\partial_x y_{i,n+1})(n_j) &= u_{j,n+1}, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
 y_{i,n+1}(n_j) &= 0, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S \\
 y_{i,0}(x) &= y_{i,0}(x), \quad i \in \mathcal{I}, \quad x \in (0, \ell_i), \quad n = 1, \dots, N - 1.
 \end{aligned} \tag{1.12}$$

We then obtain the optimal control problem on the time-discrete level:

$$\begin{aligned}
 \min_{(y,u)} I(y, u) &:= \sum_{i \in \mathcal{I}} \sum_{n=1}^N \int_0^{\ell_i} I_i(y_{i,n}) dx + \frac{\nu}{2} \sum_{j \in \mathcal{J}_N^S} \sum_{n=1}^N |u_j(n)|^2 \\
 \text{s.t.} & \\
 (y, u) &\text{ satisfies (1.12)}
 \end{aligned} \tag{1.13}$$

It is clear that (1.13) involves all time steps in the cost functional. We would like to reduce the complexity of the problem even further. To this aim we consider the *instantaneous control* regime. This amounts to reducing the sums in the cost function of (1.13) to the time-level  $t_{n+1}$ . This strategy has also come to be known as *rolling horizon* approach, the simplest case of the *moving horizon* paradigm. Thus, for each  $n = 1, \dots, N - 1$  and given  $y_{i,n}$ , we consider the problems

$$\begin{aligned}
 \min_{(y,u)} I(y, u) &:= \sum_{i \in \mathcal{I}} \int_0^{\ell_i} I_i(y_i) dx + \frac{\nu}{2} \sum_{j \in \mathcal{J}_N^S} |u_j|^2 \\
 \text{s.t.} & \\
 (y, u) &\text{ satisfies (1.12) at time level } n + 1.
 \end{aligned} \tag{1.14}$$

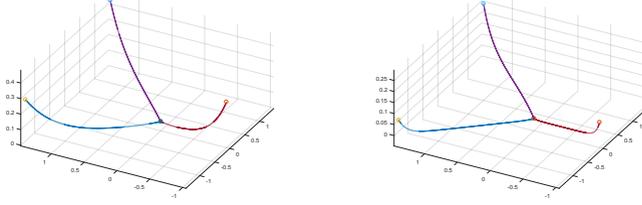


FIGURE 1. left:  $\alpha = 10$ ,  $\gamma = 1$ ,  $f_i = 10$ ; right:  $\alpha = 10$ ,  $\gamma = 100$ ,  $f_i = 10$

It is now convenient to discard the actual time level  $n + 1$  and redefine the states at the former time as input data. To this end, we replace  $\alpha_i := \frac{1}{\Delta t}$ ,  $f_i^1 := \alpha_i \beta(y_{i,n})$ , rewrite (1.12) as

$$\begin{aligned}
 \alpha_i \beta(y_i)(x) - \partial_x (\beta(\partial_x y_i(x))) &= f_i^1(x), \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
 y_i(n_j) &= y_k(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M \\
 \sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i)(n_j) &= 0, \quad j \in \mathcal{J}^M \\
 \beta(\partial_x y_i)(n_j) &= u_j, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
 y_i(n_j) &= 0, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S
 \end{aligned} \tag{1.15}$$

*Example.* We consider a star-graph with three edges (tripod). We take four numerical examples in order to illustrate the behavior of the system with respect to parameter changes. In Fig. 1.1, we take the parameters  $\alpha_i = 10$ ,  $\gamma_i = 1$ ,  $f_i = 10$ ; right:  $\alpha_i = 10$ ,  $\gamma_i = 100$ ,  $f_i = 10$ , where  $\gamma$  puts a weight on the nonlinearity. In the second Fig. 1.2, we take all parameters equal and set  $\alpha = 100$ ,  $\gamma = 1$ ,  $f_i = 10$ ; right:  $\alpha = 1$ ,  $\gamma = 0.1$ ,  $f_i = 10$ . The calculations are done with the routine *bvp4c.m* from MATLAB (R2017a).

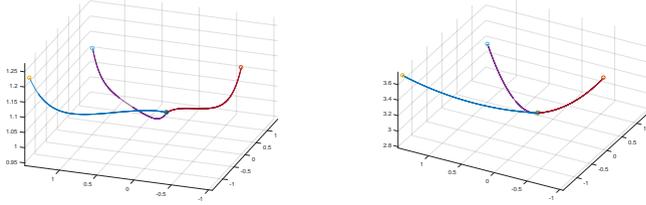


FIGURE 2. left:  $\alpha = 100$ ,  $\gamma = 1$ ,  $f_i = 10$ ; right:  $\alpha = 1$ ,  $\gamma = 0.1$ ,  $f_i = 10$

We now consider in the rest of the paper the following optimal control problem:

$$\begin{aligned}
 \min_{(y,u)} I(y,u) &:= \sum_{i \in \mathcal{I}} \int_0^{\ell_i} I_i(y_i) dx + \frac{\nu}{2} \sum_{j \in \mathcal{J}_N^S} |u_j|^2 \\
 &\text{s.t.} \\
 (y,u) &\text{ satisfies (1.15)}.
 \end{aligned} \tag{1.16}$$

## 2. Well-posedness

We begin our analysis of (1.15) with a Lagrange identity. To this end, we multiply the equation by a test function  $\phi$  and integrate by parts.

$$0 = \sum_{i \in \mathcal{I}} \int_0^{\ell_i} (\alpha_i \beta(y_i) - \partial_x(\beta(\partial_x y_i)) - f_i) \phi_i dx \quad (2.1)$$

$$\begin{aligned} &= \sum_{i \in \mathcal{I}} \int_0^{\ell_i} (\alpha_i \beta(y_i) \phi_i + (\beta(\partial_x y_i)) \partial_x \phi_i - f_i \phi_i) \phi_i dx \\ &\quad - \sum_{j \in \mathcal{J}^M} \sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j)) \phi_i(n_j) - \sum_{j \in \mathcal{J}_N^S} \sum_{i \in \mathcal{I}_j} u_j \phi_i(n_j) \end{aligned} \quad (2.2)$$

If the test function satisfies the nodal conditions of order zero, we arrive at the variational equation which serves for the proper definition of weak solutions

$$\sum_{i \in \mathcal{I}} \int_0^{\ell_i} (\alpha_i \beta(y_i) \phi_i + (\beta(\partial_x y_i)) \partial_x \phi_i) \phi_i dx = \sum_{j \in \mathcal{J}_N^S} \sum_{i \in \mathcal{I}_j} u_j \phi_i(n_j) + \sum_{i \in \mathcal{I}} \int_0^{\ell_i} f_i \phi_i. \quad (2.3)$$

For the sake of simpler notation, we switch freely between the formulation involving  $\beta(\cdot)$  and the explicit definition of  $\beta(s) = \frac{s}{\sqrt{|s|}}$ . Therefore, in our analysis  $p = \frac{3}{2}$ , even though, in many ways the results produced here are valid for general  $p \leq 2$ . For the general case, we refer to [8]. It is clear then that the dual space of  $L^p(0, \ell_i)$  is  $L^q(0, \ell_i) = L^3(0, \ell_i)$ . The mapping  $\beta(\cdot)$  satisfies

$$\beta : L^p(0, \ell_i) \rightarrow L^q(0, \ell_i), \text{ is bounded.}$$

Indeed,  $\|\beta(\phi)\|_{L^q(0, \ell_i)} \leq \|\phi\|_{L^p(0, \ell_i)}^{\frac{1}{2}}$ . This applies of course also to  $\beta(\partial_x \phi)$ . We now define the strong version of the guiding operator  $\mathcal{A}$ :

$$(\mathcal{A}y)_{i \in \mathcal{I}} := (\alpha_i \beta(y_i) - \partial_x(\beta(\partial_x y_i)))_{i \in \mathcal{I}} \quad (2.4)$$

$$D(\mathcal{A}) := \left\{ (y_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} W^{1,p}(0, \ell_i) \mid \partial_x \beta(\partial_x y_i) \in L^q(0, \ell_i), \right.$$

$$\left. \sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j)) \phi_i(n_j) = 0, \right.$$

$$y_i(n_j) = y_k(n_j), i, k \in \mathcal{I}_j, j \in \mathcal{J}^M, \left. \right.$$

$$\left. y_i(n_j) = 0, i \in \mathcal{I}_j, j \in \mathcal{J}_D^S, \beta(\partial_x y_i(n_j)) = 0, i \in \mathcal{I}_j, j \in \mathcal{J}_N^S \right\}$$

As for the weak formulation, we define the energy space

$$V := \left\{ (y_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} W^{1,p}(0, \ell_i) \mid y_i(n_j) = y_k(n_j), i, k \in \mathcal{I}_j, j \in \mathcal{J}^M, \right. \\ \left. y_i(n_j) = 0, i \in \mathcal{I}_j, j \in \mathcal{J}_D^S \right\}, \quad (2.5)$$

where  $\|\phi\|_V := \sum_{i \in \mathcal{I}} \|\phi_i\|_{W^{1,p}(0, \ell_i)}$ . Thus, for  $y \in D(\mathcal{A})$  we obtain

$$\langle \mathcal{A}(y), \phi \rangle := \sum_{i \in \mathcal{I}} \int_0^{\ell_i} (\alpha_i \beta(y_i) - \partial_x(\beta(\partial_x y_i))) \phi_i dx \quad (2.6)$$

$$= \sum_{i \in \mathcal{I}_j} \int_0^{\ell_i} (\alpha_i \beta(y_i) \phi_i + (\beta(\partial_x y_i)) \partial_x \phi_i) \phi_i dx \\ = \langle \mathcal{A}(y), \phi \rangle_{V^*, V}, \quad \forall \phi \in V \quad (2.7)$$

Using the boundedness of  $\beta(\cdot)$ , we infer that  $\mathcal{A} : V \rightarrow V^*$  is bounded as a nonlinear operator. Indeed, for an open neighbourhood  $U(0)$  of 0 in  $V$ ,

$$\langle \mathcal{A}(y), \phi \rangle_{V^*, V} \\ \leq C \sum_{i \in \mathcal{I}} \|y_i\|_{W^{1,p}(0, \ell_i)} \|\phi\|_{W^{1,p}(0, \ell_i)} \leq C \|\phi\|_V, \quad \forall y \in U(0) \subset V, \phi \in V.$$

We obtain the energy form

$$\langle \mathcal{A}(y), y \rangle = \sum_{i \in \mathcal{I}_j} \int_0^{\ell_i} \alpha_i |y_i|^p + |\partial_x y_i|^p dx \geq \underline{\alpha} \|y\|_V^p. \quad (2.8)$$

In order to further investigate the properties of  $\mathcal{A}$ , we recall ([5], [7])

$$(|a|^p a - |b|^p b)(a - b) \geq (1 + |a| + |b|)^{p-2} |a - b|^2, \quad 1 \leq p \leq 2.$$

From this it follows

$$\langle \mathcal{A}(y) - \mathcal{A}(z), y - z \rangle_{V^*, V} \geq c \|y - z\|_V^2 > 0, \quad \forall y, z \in V, \quad (2.9)$$

for some  $c > 0$ . The inequality (2.9) implies that  $\mathcal{A}$  is strictly monotone. We also easily verify

$$\frac{\langle \mathcal{A}(y), y \rangle_{V^*, V}}{\|y\|_V} \rightarrow \infty, \quad \text{as } \|y\|_V \rightarrow \infty,$$

i.e. the coercivity of  $\mathcal{A}$ . We have

$$\langle \mathcal{A}(y^k) - \mathcal{A}(y^0), \phi \rangle_{V^*, V} \\ = \sum_{i \in \mathcal{I}_j} \int_0^{\ell_i} \alpha_i (\beta(y_i^k) - \beta(y_i^0)) \phi_i + (\beta(\partial_x y_i^k) - \beta(\partial_x y_i^0)) \partial_x \phi_i dx \rightarrow 0,$$

as  $y^k \rightarrow y^0$  in  $V$  for  $k \rightarrow \infty$ . This shows that  $\mathcal{A}$  is demi-continuous. Summarizing, we have shown that  $\mathcal{A}$  is demi-continuous, coercive and strictly, monotone. Applying the classical Brezis' result (see e.g. [11]), we infer

$$\forall f \in V^* \exists! y \in V : \mathcal{A}(y) = f.$$

The sense of being a solution is weak, as  $y \in V$ . Clearly, more regular data imply strong solutions. If we consider the boundary controls as

$$f_u(\phi) := \sum_{j \in \mathcal{J}_N^S} \sum_{i \in \mathcal{I}_j} u_j \delta_{n_j}(\phi),$$

then  $f_u \in V^*$  and we obtain the following

**Theorem 2.1.** *For  $f \in \Pi_{i \in \mathcal{I}} L^3(0, \ell_i)$ ,  $u \in \mathbb{R}^{|\mathcal{J}_N^S|}$ , problem (1.15) admits a unique weak solution  $y \in V$ .*

Even though, the mapping  $\beta(\cdot)$  is differentiable in  $\mathbb{R} \setminus \{0\}$ , the control-to-state-mapping  $u \rightarrow y^u$  is not Gâteaux differentiable for  $p < 2$ . This has already been observed in [6]. However, the control-to-state-map is continuous. Indeed, let  $(u^k)_k$  be a sequence of controls that converges to  $u^0$  in  $\mathbb{R}^{|\mathcal{J}_N^S|}$ . For the corresponding solutions  $y^k := y^{u^k}$ ,  $y^0 := y^{u^0}$ , we obtain

$$\begin{aligned} & \sum_{i \in \mathcal{I}} \int_0^{\ell_i} (|\beta(y_i^k) - \beta(y_i^0)|(y_i^k - y_i^0) + \beta(\partial_x y_i^k) - \beta(\partial_x y_i^0))(\partial_x y_i^k - \partial_x y_i^0) dx \\ &= \sum_{j \in \mathcal{J}_N^S} \sum_{i \in \mathcal{I}_j} (u_j^k - u_j^0)(y_i^k - y_i^0)(n_j). \end{aligned}$$

By the continuity of  $\beta(\cdot)$  and the strong convergence of  $y^k$  to  $y^0$  in  $V$ , we obtain

**Theorem 2.2.** *The mapping  $u \rightarrow y^u$ , where  $y^u$  solves (1.15) is continuous between  $\mathbb{R}^{|\mathcal{J}_N^S|}$  and  $V$ .*

### 3. Optimal control

We are now in the position to prove existence of optimal pairs  $(y, u)$  for the optimal control problem (1.16). Indeed, the control-to-state-map is continuous and the cost function is strictly convex and lower-semi-continuous with respect to the strong topologies.

**Theorem 3.1.** *The optimal control problem (1.16) admits a unique solution  $(\bar{y}, \bar{u}) \in V \times \mathbb{R}^{|\mathcal{J}_N^S|}$ .*

The proof follows classical arguments and is omitted here. It becomes now important to investigate optimality conditions. Clearly, this involves adjoint states, satisfying the adjoint problem. However, the adjoint problem, in turn, involves the linearization of the state equation along the optimal trajectory. As the nonlinearities of the state equation are governed by the mapping  $\beta(\cdot)$  and the derivative of this mapping  $\beta'(s) = \frac{1}{2\sqrt{|s|}}$  is unbounded in the neighbourhood of  $s = 0$ , we need an argument that the optimal solution  $y_i$  together with its derivative  $\partial_x y_i$  stays away from 0. We begin with a regularization of the problem, as proposed in [5, 6]. The regularization is made in order to bring us back into the standard  $p = 2$  setting.

$$\begin{aligned}
\epsilon y_i(x) + \alpha_i \beta(y_i)(x) - \partial_x (\epsilon \partial_x y_i(x) + \beta(\partial_x y_i(x))) &= f_i^1(x), \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
y_i(n_j) &= y_k(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} (\epsilon \partial_x y_i(n_j) + \beta(\partial_x y_i)(n_j)) &= 0, \quad j \in \mathcal{J}^M \\
\epsilon \partial_x y_i(n_j) + \beta(\partial_x y_i)(n_j) &= u_j, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
y_i(n_j) &= 0, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S.
\end{aligned} \tag{3.1}$$

We formulate the  $\epsilon$ -perturbed optimal control problem as follows.

$$\begin{aligned}
\min_{(y^\epsilon, u^\epsilon)} I_\epsilon(y^\epsilon, u^\epsilon) &:= \sum_{i \in \mathcal{I}} \int_0^{\ell_i} I_i(y_i^\epsilon) dx + \frac{\nu}{2} \sum_{j \in \mathcal{J}^S} |u_j^\epsilon|^2 + \frac{\nu}{2} \sum_{j \in \mathcal{J}^S} |u_j^\epsilon - \bar{u}_j|^2 \\
&\text{s.t.} \\
(y^\epsilon, u^\epsilon) &\text{ satisfies (3.1),}
\end{aligned} \tag{3.2}$$

where  $\bar{u}$  denotes the optimal control for the unperturbed problem. Problem (3.2) is a standard optimal control problem, where the control-to-state-map is now continuously differentiable. The following theorem, therefore, can be stated without dwelling on the proof.

**Theorem 3.2.** *For each  $\epsilon > 0$ , there exists an optimal pair  $y^\epsilon, u^\epsilon$  and an adjoint state  $p^\epsilon$  such that  $y^\epsilon, p^\epsilon$  stay in*

$$W := \{y \in \Pi_{i \in \mathcal{I}} H^1(0, \ell_i) \mid y_i(n_j) = y_k(n_j), \quad i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M, \quad y_i(n_j) = 0, \\
i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S.\}$$

and satisfy the following optimality system.

$$\begin{aligned}
\epsilon y_i + \alpha_i \beta(y_i) - \partial_x (\epsilon \partial_x y_i + \beta(\partial_x y_i)) &= f_i, \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
\epsilon p_i + \alpha_i \beta'(y_i) p_i - \partial_x (\epsilon \partial_x p_i + \beta'(\partial_x y_i) \partial_x p_i) \\
&= \kappa (|y_i - y_i^d|^{p-2} (y_i - y_i^d)), \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
y_i(n_j) &= y_k(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad p_i(n_j) = p_k(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} (\epsilon \partial_x y_i(n_j) + \beta(\partial_x y_i)(n_j)) &= 0, \quad j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} (\epsilon \partial_x p_i(n_j) + \beta'(\partial_x y_i)(n_j) \partial_x p_i(n_j)) &= 0, \quad j \in \mathcal{J}^M \\
\epsilon \partial_x y_i(n_j) + \beta(\partial_x y_i)(n_j) &= u_j, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
\epsilon \partial_x p_i(n_j) + \beta'(\partial_x y_i)(n_j) \partial_x p_i(n_j) &= 0, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
y_i(n_j) = 0, p_i(n_j) = 0 & \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S \\
p_i^\epsilon(n_j) + \nu u_j^\epsilon + (u_j^\epsilon - \bar{u}_j) &= 0, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S.
\end{aligned} \tag{3.3}$$

In order to understand the limiting procedure as  $\epsilon \rightarrow 0$ , we look at the adjoint equation and multiply it by  $p\epsilon$ . We obtain

$$\begin{aligned}
\int_0^{\ell_i} \epsilon (p_i^\epsilon)^2 + \alpha_i \beta'(y_i^\epsilon) (p_i^\epsilon)^2 + (\epsilon \partial_x p_i^\epsilon + \beta'(\partial_x y_i^\epsilon) \partial_x p_i^\epsilon) \partial_x p_i^\epsilon dx \\
= \int_0^{\ell_i} |y_i^\epsilon - y_i^d|^{p-2} (y_i^\epsilon - y_i^d) p_i^\epsilon dx.
\end{aligned}$$

If now  $y_i^\epsilon \rightarrow y_i$  strongly in  $W$ , then

$$\epsilon + \alpha_i \beta'(y_i^\epsilon) \rightharpoonup \alpha_i \beta'(y_i), \quad \epsilon + \alpha_i \beta'(\partial_x y_i^\epsilon) \rightharpoonup \alpha_i \beta'(\partial_x y_i).$$

Moreover, for bounded  $(y_i^\epsilon)_\epsilon, (p_i^\epsilon)_\epsilon$ , the terms in the equation stay bounded and if  $p_i^\epsilon \rightarrow p_i$  (even weakly), the limiting equation is

$$\int_0^{\ell_i} \alpha_i \beta'(y_i) (p_i)^2 + (\beta'(\partial_x y_i) \partial_x p_i) \partial_x p_i dx = \int_0^{\ell_i} |y_i - y_i^d|^{p-2} (y_i - y_i^d) p_i dx.$$

We are now in the position to show the following limiting result.

**Theorem 3.3.** *There exists  $\bar{p} \in V$  satisfying together with the optimal pair  $(\bar{y}, \bar{u})$  the first order optimality condition.*

$$\begin{aligned}
\alpha_i \beta(\bar{y}_i) - \partial_x(\beta(\partial_x \bar{y}_i)) &= f_i, \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
\alpha_i \beta'(\bar{y}_i) \bar{p}_i - \partial_x(\beta'(\partial_x \bar{y}_i) \partial_x \bar{p}_i) &= \kappa(|\bar{y}_i - y_i^d|^{p-2}(\bar{y}_i - y_i^d)), \quad i \in \mathcal{I}, \quad x \in (0, \ell_i) \\
\bar{y}_i(n_j) &= \bar{y}_k(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad \bar{p}_i(n_j) = \bar{p}_k(n_j), \quad \forall i, k \in \mathcal{I}_j, \quad j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij}(\beta(\partial_x \bar{y}_i)(n_j)) &= 0, \quad j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij}(\beta'(\partial_x \bar{y}_i)(n_j) \partial_x \bar{p}_i(n_j)) &= 0, \quad j \in \mathcal{J}^M \\
\beta(\partial_x \bar{y}_i)(n_j) &= \bar{u}_j, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
\beta'(\partial_x \bar{y}_i)(n_j) \partial_x \bar{p}_i(n_j) &= 0, \quad d_j = 1, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S \\
\bar{y}_i(n_j) &= 0, \quad \bar{p}_i(n_j) = 0 \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_D^S \\
\bar{p}_i^\epsilon(n_j) + \nu \bar{u}_j &= 0, \quad i \in \mathcal{I}_j, \quad j \in \mathcal{J}_N^S.
\end{aligned} \tag{3.4}$$

*Example.* We consider the tripod above, where we use  $\alpha_i = \alpha = 10$ ,  $f_i = f = 10$  and demonstrate the effect of the second order nonlinear operator by choosing coefficients  $\gamma_i = \gamma = 1$  or 100. The penalty for  $\kappa_i = \kappa$  is chosen equal to 1000. For the calculations, we used a regularization, as in the text as  $\epsilon = 0.01$ . The numerical studies reveal that the system behaviour does not largely depend on  $\epsilon$  as long this number is in order of magnitude smaller than the resolution tolerance which was set here to  $1e - 4$ . The optimality system is solved using the MATLAB routine *bvp4c*. See Fig. 3.1 and Fig. 3.2.

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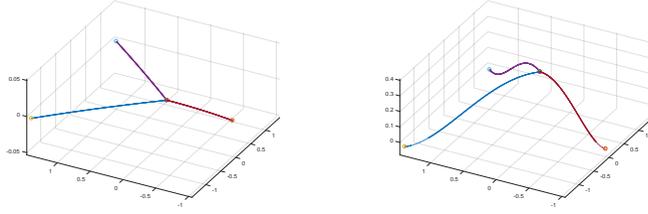


FIGURE 3. left:  $\alpha = 10$ ,  $\gamma = 1$ ,  $f_i = 10$ ; right:  $\alpha = 10$ ,  $\gamma = 100$ ,  $f_i = 10$

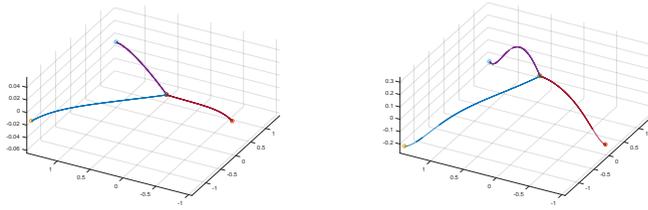


FIGURE 4. left:  $\alpha = 100$ ,  $\gamma = 1$ ,  $f_i = 10$ ; right:  $\alpha = 1$ ,  $\gamma = 0.1$ ,  $f_i = 10$

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