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# Stackelberg control with constraints on the state for a linear backward heat equation

L. L. DJOMEGNE NJOUKOU \*G. MOPHOU † G. DEUGOUE ‡

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## Abstract

We are interested in the hierarchic control problem for a linear backward heat equation. We assume that we can act on the system via two distributed controls. One control called follower solves an optimal control problem which consist in bringing the flux of the system to zero. The other control called leader solves a null controllability problem. The results are achieved by means of inequality of observability associated to a non-homogeneous Dirichlet boundary condition.

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**Key-words** : linear heat equation, optimal control, Carleman inequality, Controllability, Euler-Lagrange formula.

## 1 Introduction

Let  $n \in \mathbb{N} \setminus \{0\}$  and  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  with boundary  $\Gamma$  of class  $\mathcal{C}^2$ . Let  $\mathcal{O}$ ,  $\omega_1$  and  $\omega_2$  be three non-empty subsets of  $\Omega$  such that  $\omega_1 \subset \omega_2$ . For a time  $T > 0$ , we set  $Q = (0, T) \times \Omega$ ,  $\mathcal{O}_T = (0, T) \times \mathcal{O}$ ,  $\omega_1^T = (0, T) \times \omega_1$ ,  $\omega_2^T = (0, T) \times \omega_2$ ,  $\Sigma = (0, T) \times \Gamma$  and  $\Sigma_1 = (0, T) \times \Gamma_1$ . Then, we consider the following linear heat equation:

$$\begin{cases} -\frac{\partial q}{\partial t} - \Delta q + a_0 q &= h_0 \mathbf{1}_{\mathcal{O}} + v \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2} & \text{in } Q, \\ q &= 0 & \text{on } \Sigma, \\ q(T) &= 0 & \text{in } \Omega, \end{cases} \quad (1)$$

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where  $a_0 \in L^\infty(Q)$ ,  $h_0 \in L^2(\mathcal{O}_T)$ , the control  $v$  and  $k$  belong respectively to  $L^2(\omega_1^T)$  and  $L^2(\omega_2^T)$ . The function  $\mathbf{1}_{\mathbb{X}}$  denotes the characteristic function on the open set  $\mathbb{X}$ . System (1) is a backward heat equation. Under the above assumptions on the data, it is well known that system (1) has a unique solution  $q(v, k) := q(t, x; v, k) \in H^{2,1}(Q) \cap L^2((0, T); H_0^1(\Omega))$  where from now on,

$$H^{r,s}((0, T) \times \mathbb{X}) = L^2((0, T); H^r(\mathbb{X})) \cap H^s((0, T); L^2(\mathbb{X})).$$

Moreover, there exists a positive constant  $C = C(\|a_0\|_{L^\infty(Q)}, T)$  such that

$$\|q\|_{H^{2,1}(Q)} \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|v\|_{L^2(\omega_1^T)} + \|k\|_{L^2(\omega_2^T)} \right), \quad (2)$$

and it follows from the continuity of the trace that,

$$\left\| \frac{\partial q}{\partial \nu}(v, k) \right\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma)}^2 \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|v\|_{L^2(\omega_1^T)} + \|k\|_{L^2(\omega_2^T)} \right). \quad (3)$$

We refer to [1] for more literature on the regularity results on the parabolic equations.

In this paper we are concerned by the Stackelberg control which is a hierarchical optimization problem with here,  $k$  which is the "Leader" and  $v$  the "Follower". The Stackelberg problem consists in the following: given  $k$ , determine firstly the control  $v$  solution of an optimal control problem;  $v$  being founded as a function of  $k$ , determine secondly the control  $k$  solution of a null controllability problem. More precisely, we are interesting in a two following problems:

**Problem 1** (*Optimal control problem*)

Let  $\omega_1$  and  $\omega_2$  be two non-empty subsets of  $\Omega$ . Given  $k \in L^2(\omega_2^T)$ , find the control  $u := u(k) \in L^2(\omega_1^T)$  such that

$$J(u) = \inf_{v \in L^2(\omega_1^T)} J(v) \quad (4)$$

with

$$J(v) = \left\| \frac{\partial q}{\partial \nu}(v, k) \right\|_{L^2(\Sigma_1)}^2 + N \|v\|_{L^2(\omega_1^T)}^2, \quad (5)$$

where  $N > 0$ ,  $\nu$  is the unit exterior normal vector of  $\Gamma$ ,  $\frac{\partial q}{\partial \nu}$  is the normal derivative of  $q$  with respect to  $\nu$  and  $q(v, k)$  is solution of system (1).

**Problem 2** (*Null controllability problem*)

Let  $\omega_1$  and  $\omega_2$  be two non-empty subsets of  $\Omega$  such that  $\omega_1 \subset \omega_2$ . Let also  $u(k)$  be the optimal control obtain in the Problem 1. Then, find a control  $k \in L^2(\omega_2^T)$  such that if  $q = q(t, x; u(k), k)$  is solution of (1), then

$$q(0) = q(0, x; u(k), k) = 0 \text{ for } x \in \Omega. \quad (6)$$

**Remark 1** Note that from (3), the cost function defined by (5) is well defined.

The idea of the Stackelberg model was initiated by H. Von Stackelberg in [9]. This model states that we have two companies that share the production in the market. A company called a leader is better known and therefore is better placed to decide first that it sells quantity. The other company the follower observes and decides on its quantity to sell according to the strategy of the leader. There are more works on Stackelberg's control of partial differential equations. The first author used this concept in distributed systems is J.L. Lions. In [8], this author uses this concept for a linear parabolic equation with two controls; the one called follower aims to bring the state of the system to a desired state, while the other control the leader solves a problem of approximate controllability. In [10], this same author uses the strategy of Stackelberg given in [8] for a linear wave equation with boundary control. In [11] O. Nakoulima uses the concept of Stackelberg for a linear and backward parabolic equation with two controls to be determined: one called follower solves a null controllability problem with constraints on control while the other control called the leader solves an optimal control problem.

Problem 1 is an optimal control problem associated to a linear parabolic equation. This problem has been studied by J. L. Loins in [7]. The purpose of this kind of problem is to bring the state of the system to a desired state by minimizing some functional. Problem 2 is a null controllability problem for a linear parabolic equation. In [4] D. Russel proved that the linear heat equation is null controllable in any time  $T$  provided the wave equation is exactly controllable for some time  $T$ . In [5] Lebeau and Robbiano solved the linear problem of null controllability using observability inequalities deriving from Carleman inequalities. In [6] Nakoulima gives a result of null controllability for the linear heat equation with constraint on the distributed control. His result is based on an observability inequality adapted to the constraint. In [12] Mophou et al solves a linear null controllability problem with constraint on the control deriving from boundary sentinels problem. The authors used an observability inequality adapted to the constraint to prove the results.

In this paper, we propose the Stackelberg strategy using the notion of optimal control for the follower and appropriate Carleman inequality for the leader. More precisely, we prove the following result:

**Theorem 1.1** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \in \mathbb{N}^*$  be a bounded subset with boundary  $\Gamma$  of class  $\mathcal{C}^2$ . Let  $\omega_1$  and  $\omega_2$  be two non empty subsets of  $\Omega$ . Let also  $k \in L^2(\omega_2^T)$ . Then, there exists  $p \in L^2(Q)$  such that the optimal control problem (4) has a unique solution  $u \in L^2(\omega_1^T)$  which is characterized by the following optimality system:*

$$\begin{cases} -\frac{\partial q}{\partial t} - \Delta q + a_0 q &= h_0 \mathbf{1}_O + u \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2} & \text{in } Q, \\ q &= 0 & \text{on } \Sigma, \\ q(T) &= 0 & \text{in } \Omega, \end{cases} \quad (7)$$

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p + a_0 p = 0 & \text{in } Q, \\ p = \frac{\partial q}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \\ p(0) = 0 & \text{in } \Omega, \end{cases} \quad (8)$$

and

$$u = \frac{p}{N} \text{ in } \omega_1^T. \quad (9)$$

Moreover there exists a constants  $C = C(\|a\|_{L^\infty(Q)}, T, N) > 0$  such that

$$\|u\|_{L^2(\omega_1^T)} \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|k\|_{L^2(\omega_2^T)} \right). \quad (10)$$

**Theorem 1.2** *Assume that the assumptions of Theorem 1.1 hold. Assume also that  $\omega_1$  and  $\omega_2$  are such that  $\omega_1 \subset \omega_2$ . Then, there exists a positive weight function  $\theta$  (a precise definition of  $\theta$  will be given later on) such that for any  $h_0 \in L^2(\mathcal{O}_T)$  with  $\theta h_0 \in L^2(\mathcal{O}_T)$ , there exists a unique control  $\hat{k} \in L^2(\omega_2^T)$  such that  $(\hat{k}, q, p)$  is solution of the null controllability problem (7)-(9). Moreover*

$$\hat{k} = \hat{\rho} \text{ in } \omega_2^T \quad (11)$$

where  $\rho$  satisfies

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial t} - \Delta \hat{\rho} + a_0 \hat{\rho} = 0 & \text{in } Q, \\ \hat{\rho} = \frac{\partial \hat{\Psi}}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \end{cases} \quad (12)$$

with  $\hat{\Psi}$  solution of

$$\begin{cases} -\frac{\partial \hat{\Psi}}{\partial t} - \Delta \hat{\Psi} + a_0 \hat{\Psi} = \frac{\hat{\rho}}{N} \mathbf{1}_{\omega_1} & \text{in } Q, \\ \hat{\Psi} = 0 & \text{on } \Sigma, \\ \hat{\Psi}(T) = 0 & \text{in } \Omega. \end{cases} \quad (13)$$

In addition, there exists a constant  $C = C(\|a_0\|_{L^\infty(Q)}, T, N)$  such that

$$\|\hat{k}\|_{L^2(\omega_2^T)} \leq \sqrt{C} \|\theta h_0\|_{L^2(\mathcal{O}_T)}. \quad (14)$$

The rest of this paper is organized as follows: In Section 2 we prove the Theorem 1.1. In Section 3, we establish an appropriate inequality of observability. Section 4 is devoted to study a null controllability of the linear system and give the proof of the Theorem 1.2.

From now on, we will denote by  $C(X)$  a generic positive constant whose value varies from a line to another but depending on  $X$ .

## 2 Proof of Theorem 1.1

In this section, we are interested in the resolution of Problem 1.

**Proposition 2.1** *For any  $k \in L^2(\omega_2^T)$ , there exists a unique optimal control  $u := u(k) \in L^2(\omega_1^T)$  such that (4) holds true.*

**Proof.** Since  $J(v) \geq 0$  for all  $v \in L^2(\omega_1^T)$ , we deduce that  $\inf_{v \in L^2(\omega_1^T)} J(v)$  exists. Let  $(v_n) \subset L^2(\omega_1^T)$  be a minimizing sequence such that

$$\lim_{n \rightarrow +\infty} J(v_n) = \inf_{v \in L^2(\omega_1^T)} J(v). \quad (15)$$

Then,  $q_n := q(v_n)$  is solution of system (1). This means that  $q_n$  satisfies

$$\begin{cases} -\frac{\partial q_n}{\partial t} - \Delta q_n + a_0 q_n = h_0 \mathbf{1}_{\mathcal{O}} + v_n \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2} & \text{in } Q, \\ q_n = 0 & \text{on } \Sigma, \\ q_n(T) = 0 & \text{in } \Omega. \end{cases} \quad (16)$$

Moreover, in view of (15), there exists a constant  $C > 0$  independent of  $n$  such that

$$J(v_n) \leq C. \quad (17)$$

Therefore, from the structure of  $J$  we get

$$\|v_n\|_{L^2(\omega_1^T)} \leq C(N), \quad (18)$$

$$\left\| \frac{\partial q_n}{\partial \nu} \right\|_{L^2(\Sigma_1)} \leq C. \quad (19)$$

In view of (16),(18) and (2), we deduce that

$$\|q_n\|_{H^{2,1}(Q)} \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|k\|_{L^2(\omega_2^T)} \right), \quad (20)$$

where  $C = C(\|a_0\|_{L^\infty(Q)}, T, N) > 0$ . Hence, there exist  $u \in L^2(\omega_1^T)$ ,  $\beta \in L^2(\Sigma_1)$ ,  $q \in H^{2,1}(Q)$  and subsequences extracted from  $(v_n)$ ,  $(\frac{\partial q_n}{\partial \nu})$  and  $(q_n)$  (still called  $(v_n)$ ,  $(\frac{\partial q_n}{\partial \nu})$  and  $(q_n)$ ) such that

$$v_n \rightharpoonup u \text{ weakly in } L^2(\omega_1^T), \quad (21a)$$

$$q_n \rightharpoonup q \text{ weakly in } H^{2,1}(Q), \quad (21b)$$

$$\frac{\partial q_n}{\partial \nu} \rightharpoonup \beta \text{ weakly in } L^2(\Sigma_1). \quad (21c)$$

Let  $\mathbb{D}(Q)$  the set of functions of class  $\mathcal{C}^\infty$  on  $Q$  with compact support. If we multiply the first equation of (16) by  $\varphi \in \mathbb{D}(Q)$  and integrate by parts over  $Q$ , we obtain

$$\int_Q q_n \left( \frac{\partial \varphi}{\partial t} - \Delta \varphi + a_0 \varphi \right) dxdt = \int_Q (h_0 \mathbf{1}_{\mathcal{O}} + v_n \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2}) \varphi dxdt, \quad \forall \varphi \in \mathbb{D}(Q). \quad (22)$$

Passing to the limit in (22) while using (21a) and (21b), we obtain

$$\int_Q q \left( \frac{\partial \varphi}{\partial t} - \Delta \varphi + a_0 \varphi \right) dxdt = \int_Q (h_0 \mathbf{1}_O + u \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2}) \varphi dxdt, \quad \forall \varphi \in \mathbb{D}(Q),$$

which after an integration by parts gives

$$\int_Q \left( -\frac{\partial q}{\partial t} - \Delta q + a_0 q \right) \varphi dxdt = \int_Q (h_0 \mathbf{1}_O + u \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2}) \varphi dxdt, \quad \forall \varphi \in \mathbb{D}(Q).$$

Thus, we obtain

$$-\frac{\partial q}{\partial t} - \Delta q + a_0 q = h_0 \mathbf{1}_O + u \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2} \quad \text{in } Q. \quad (23)$$

As  $q \in H^{2,1}(Q)$ , we have the existence of the traces  $q|_{\Sigma}, \frac{\partial q}{\partial \nu}|_{\Sigma} \in L^2(\Sigma)$  and  $q(T) \in L^2(\Omega)$ . So if we multiplying the first equation of (16) by  $\varphi \in C^\infty(\overline{Q})$  such that  $\varphi = 0$  on  $\Sigma \setminus \Sigma_1$  and  $\varphi(0) = 0$  in  $\Omega$  and integrate by parts over  $Q$ , we obtain

$$\begin{aligned} & - \int_{\Sigma_1} \frac{\partial q_n}{\partial \nu} \varphi d\sigma dt + \int_Q q_n \left( \frac{\partial \varphi}{\partial t} - \Delta \varphi + a_0 \varphi \right) dxdt \\ & = \int_Q (h_0 \mathbf{1}_O + v_n \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2}) \varphi dxdt. \end{aligned} \quad (24)$$

Passing to the limit in (24) when  $n \rightarrow +\infty$  while using (21a)-(21c), we obtain

$$\begin{aligned} & - \int_{\Sigma_1} \beta \varphi dxdt + \int_Q q \left( \frac{\partial \varphi}{\partial t} - \Delta \varphi + a_0 \varphi \right) dxdt \\ & = \int_Q (h_0 \mathbf{1}_O + u \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2}) \varphi dxdt, \end{aligned} \quad (25)$$

$\forall \varphi \in C^\infty(\overline{Q})$  with  $\varphi = 0$  on  $\Sigma \setminus \Sigma_1$ ,  $\varphi(0) = 0$  in  $\Omega$ .

Integrating by parts this latter identity, we obtain

$$\begin{aligned} & - \int_{\Sigma_1} \beta \varphi dxdt + \int_\Omega q(T) \varphi(T) dx - \int_\Sigma \frac{\partial \varphi}{\partial \nu} q d\sigma dt \\ & + \int_{\Sigma_1} \varphi \frac{\partial q}{\partial \nu} d\sigma dt + \int_Q \left( -\frac{\partial q}{\partial t} - \Delta q + a_0 q \right) \varphi dxdt \\ & = \int_Q (h_0 \mathbf{1}_O + u \mathbf{1}_{\omega_1} + k \mathbf{1}_{\omega_2}) \varphi dxdt, \end{aligned} \quad (26)$$

$\forall \varphi \in C^\infty(\overline{Q})$  with  $\varphi = 0$  on  $\Sigma \setminus \Sigma_1$ ,  $\varphi(0) = 0$  in  $\Omega$ ,

which in view of (23) yields

$$\int_{\Sigma_1} \left( \frac{\partial q}{\partial \nu} - \beta \right) \varphi d\sigma dt + \int_\Omega q(T) \varphi(T) dx - \int_\Sigma q \frac{\partial \varphi}{\partial \nu} d\sigma dt = 0, \quad (27)$$

$\forall \varphi \in C^\infty(\overline{Q})$  with  $\varphi = 0$  on  $\Sigma \setminus \Sigma_1$ ,  $\varphi(0) = 0$  in  $\Omega$ .

Choosing successively in (27),  $\varphi(T) = 0$  in  $\Omega$  and  $\varphi = 0$  on  $\Sigma_1$ , then  $\varphi(T) = 0$  in  $\Omega$ , we successively get

$$q = 0 \text{ in } \Sigma \quad (28)$$

and

$$\beta = \frac{\partial q}{\partial \nu} \text{ on } \Sigma_1. \quad (29)$$

Finally, it follows from (27) that

$$q(T) = 0 \text{ in } \Omega. \quad (30)$$

From (21c) and (29), we have

$$\frac{\partial q_n}{\partial \nu} \rightharpoonup \frac{\partial q}{\partial \nu} \text{ weakly in } L^2(\Sigma_1). \quad (31)$$

In view of (23), (28) and (30), we deduce that  $q = q(u(k))$  is solution of (1). Using the weak lower semi-continuity of the function  $v \mapsto J(v)$ , (31) and (21a), we obtain that

$$J(u) \leq \liminf_{n \rightarrow +\infty} J(v_n),$$

which according to (15) implies that

$$J(u) \leq \inf_{v \in L^2(\omega_1^T)} J(v).$$

Thus

$$J(u) = \inf_{v \in L^2(\omega_1^T)} J(v).$$

The uniqueness of  $u$  comes from the fact that the functional  $J$  is strictly convex.

■

In order to characterize the optimal control  $u$ , we express the Euler-Lagrange optimality conditions:

$$\lim_{\lambda \rightarrow 0} \frac{J(u + \lambda v) - J(u)}{\lambda} = 0, \text{ for all } v \in L^2(\omega_1^T). \quad (32)$$

After some calculations, (32) gives

$$\int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial q}{\partial \nu} d\sigma dt + N \int_0^T \int_{\omega_1} uv dx dt = 0, \forall v \in L^2(\omega_1^T), \quad (33)$$

where  $z = z(v) \in H^{2,1}(Q)$  is solution of

$$\begin{cases} -\frac{\partial z}{\partial t} - \Delta z + a_0 z = v \mathbf{1}_{\omega_1} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega. \end{cases} \quad (34)$$



To interpret (33), we consider the adjoint state solution of system (8):

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p + a_0 p = 0 & \text{in } Q, \\ p = \frac{\partial q}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \\ p(0) = 0 & \text{in } \Omega. \end{cases}$$

Since  $q \in H^{2,1}(Q)$ , we have that  $\frac{\partial q}{\partial \nu} \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma) \subset L^2(\Sigma)$  (see Theorem 2.1, [1], Vol. 2). Therefore we can prove by transposition that there exists a unique  $p \in L^2(Q)$  such that

$$\int_{\Sigma_1} \frac{\partial q}{\partial \nu} \frac{\partial \Phi}{\partial \nu} d\sigma dt + \int_Q p \left( -\frac{\partial \Phi}{\partial t} - \Delta \Phi + a_0 \Phi \right) dx dt = 0,$$

$$\forall \Phi \in H^{2,1}(Q) \text{ such that } \Phi|_{\Sigma} = 0, \Phi(T) = 0 \text{ in } \Omega.$$

Thus, multiplying the first equation of (34) by  $p$  and integrating by parts over  $Q$ , we obtain:

$$\int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial q}{\partial \nu} d\sigma dt = - \int_0^T \int_{\omega_1} v p dx dt. \quad (35)$$

Hence, combining (33) and (35), we deduce that

$$\int_0^T \int_{\omega_1} (-p + Nu)v dx dt = 0, \quad \forall v \in L^2(\omega_1^T).$$

Consequently,

$$u = \frac{p}{N} \text{ in } \omega_1^T.$$

**Proposition 2.2** *Let  $u \in L^2(\omega_1^T)$  be the solution of (4). Then we have the following estimate:*

$$\|u\|_{L^2(\omega_1^T)} \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|k\|_{L^2(\omega_2^T)} \right), \quad (36)$$

where  $C = C(\|a\|_{L^\infty(Q)}, T, N) > 0$ .

**Proof.** Let  $z := z(v)$  and  $l := l(k)$  be respectively solutions of

$$\begin{cases} -\frac{\partial z}{\partial t} - \Delta z + a_0 z = v \mathbf{1}_{\omega_1} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega, \end{cases} \quad (37)$$

and

$$\begin{cases} -\frac{\partial l}{\partial t} - \Delta l + a_0 l = h_0 \mathbf{1}_{\mathcal{O}} + k \mathbf{1}_{\omega_2} & \text{in } Q, \\ l = 0 & \text{on } \Sigma, \\ l(T) = 0 & \text{in } \Omega. \end{cases} \quad (38)$$

Then  $q = z + l$  and it follows from the data that  $z, l \in H^{2,1}(Q)$  and there exists  $C = C(\|a\|_{L^\infty(Q)}, T) > 0$  such that

$$\|z\|_{H^{2,1}(Q)} \leq C\|v\|_{L^2(\omega_1^T)} \quad (39)$$

and

$$\|l\|_{H^{2,1}(Q)} \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|k\|_{L^2(\omega_2^T)} \right). \quad (40)$$

Define on  $L^2(\omega_1^T) \times L^2(\omega_1^T)$  the bilinear form  $a(., .)$  by

$$a(v, w) = \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial z(w)}{\partial \nu} d\sigma dt + N \int_0^T \int_{\omega_1} v w dx dt \quad \forall v, w \in L^2(\omega_1^T), \quad (41)$$

and on  $L^2(\omega_1^T)$  the linear form

$$\mathcal{L}(v) = \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial l(k)}{\partial \nu} d\sigma dt \quad \forall v \in L^2(\omega_1^T). \quad (42)$$

Then, for any  $v \in L^2(\omega_1^T)$ , the bilinear form  $a(., .)$  is coercive because

$$a(u, u) = \left\| \frac{\partial z(u)}{\partial \nu} \right\|_{L^2(\Sigma_1)}^2 + N \|u\|_{L^2(\omega_1^T)}^2 \geq N \|u\|_{L^2(\omega_1^T)}^2.$$

In view of (33), we have

$$\begin{aligned} 0 &= \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial q}{\partial \nu} d\sigma dt + N \int_0^T \int_{\omega_1} u v dx dt \\ &= \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial z(u)}{\partial \nu} d\sigma dt + \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial l(k)}{\partial \nu} d\sigma dt + N \int_0^T \int_{\omega_1} u v dx dt \\ &= a(v, u) + \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu} \frac{\partial l(k)}{\partial \nu} d\sigma dt, \forall v \in L^2(\omega_1^T), \end{aligned}$$

where the bilinear form  $a(., .)$  is given by (41) and  $l = l(k)$  is solution to (38). Hence, taking  $v = u$  in this latter identity and using the coercivity of  $a(., .)$ , we deduce that

$$\begin{aligned} N \|u\|_{L^2(\omega_1^T)}^2 &\leq \left\| \frac{\partial z(u)}{\partial \nu} \right\|_{L^2(\Sigma_1)} \left\| \frac{\partial l(k)}{\partial \nu} \right\|_{L^2(\Sigma_1)}, \\ &\leq C \|z(u)\|_{H^{2,1}(Q)} \|l(k)\|_{H^{2,1}(Q)}, \end{aligned}$$

which in view of (39) and (40) gives

$$\|u\|_{L^2(\omega_1^T)} \leq C \left( \|h_0\|_{L^2(\mathcal{O}_T)} + \|k\|_{L^2(\omega_2^T)} \right),$$

where  $C = C(\|a\|_{L^\infty(Q)}, T, N) > 0$ . ■

### 3 Carleman inequality

In order to solve problem (2), we need an appropriate inequality associated to adjoint of systems (7)-(9).

**Lemma 3.1** [2] *Let  $\omega_2$  be an arbitrary non empty open set of  $\Omega$ . Then there exists  $\psi \in C^2(\bar{\Omega})$  such that*

$$\psi(x) > 0 \quad \forall x \in \Omega, \quad \psi(x) = 0 \quad \forall x \in \Gamma, \quad (43)$$

$$|\nabla\psi(x)| > 0 \quad \forall x \in \bar{\Omega} - \omega_2. \quad (44)$$

Let  $\lambda \geq 1$  a real number and  $m_1$  and  $m_2$  be two positive constants such that  $m_1 \leq m_2$ . For any  $(t, x) \in Q$ , we set

$$\varphi(t, x) = \frac{e^{\lambda(\psi(x)+m_1)}}{t^2(T-t)^2}, \quad (45)$$

$$\eta(t, x) = \frac{e^{\lambda(\psi(x)+m_1)} - e^{\lambda(\|\psi\|_{L^\infty(\Omega)}+m_2)}}{t^2(T-t)^2}. \quad (46)$$

**Remark 2** *For  $x \in \Gamma$ , we have  $\psi(x) = 0$ ; therefore, the function  $\varphi$  and  $\eta$  defined respectively by (45) and (46) depends only on  $t$  and we have*

$$\varphi^{-\frac{1}{4}} \leq C(T).$$

We consider the following system

$$\begin{cases} -\frac{\partial z}{\partial t} - \Delta z + a_0 z = 0 & \text{in } Q, \\ z = h & \text{on } \Sigma, \end{cases} \quad (47)$$

where  $a_0 \in L^\infty(Q)$  and  $h \in H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)$ .

Set

$$W(Q) = \left\{ z | z, \frac{\partial z}{\partial x_i} \in L^2(Q), i = 1, \dots, n; \frac{\partial z}{\partial t} \in L^2((0, T); H^{-1}(\Omega)) \right\}.$$

We give now the global Carleman inequality for system (47).

**Proposition 3.1** [3] *Let  $\psi$ ,  $\varphi$  and  $\eta$  be respectively defined by (43)-(46). Let  $\omega_2$  be a non empty open subset of  $\Omega$ . Then there exists a positive constants  $\lambda_0 \geq 1$  and  $s_0 \geq 1$  and there exists a constant  $C_1 > 0$  independent of  $s \geq s_0$  and  $\lambda \geq \lambda_0$  such that for any  $s \geq s_0$  and  $\lambda \geq \lambda_0$ , and for all  $z \in W(Q)$  solution of (47), we have*

$$\begin{aligned} & \int_Q \frac{1}{s\varphi} e^{-2s\eta} |\nabla z|^2 dxdt + s\lambda^2 \int_Q \varphi e^{-2s\eta} |z|^2 dxdt \leq \\ & C_1 \left( s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} h e^{-s\eta}\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 + s\lambda^2 \int_0^T \int_{\omega_2} \varphi e^{-2s\eta} |z|^2 dxdt \right). \end{aligned} \quad (48)$$

For any suitable function  $z$ , we adopt the following notation

$$I(z) = \int_Q \frac{1}{s\varphi} e^{-2s\eta} |\nabla z|^2 dxdt + s\lambda^2 \int_Q e^{-2s\eta} \varphi |z|^2 dxdt. \quad (49)$$

For any  $\rho^0 \in L^2(\Omega)$ , we consider the following systems:

$$\begin{cases} \frac{\partial \rho}{\partial t} - \Delta \rho + a_0 \rho = 0 & \text{in } Q, \\ \rho = \frac{\partial \Psi}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \\ \rho(0) = \rho^0 & \text{in } \Omega, \end{cases} \quad (50)$$

$$\begin{cases} -\frac{\partial \Psi}{\partial t} - \Delta \Psi + a_0 \Psi = \frac{\rho}{N} \mathbf{1}_{\omega_1} & \text{in } Q, \\ \Psi = 0 & \text{on } \Sigma, \\ \Psi(T) = 0 & \text{in } \Omega. \end{cases} \quad (51)$$

Using Proposition , we have the following results

**Proposition 3.2** *Let  $\omega_1$  and  $\omega_2$  be two non empty open subsets of  $\Omega$  such that  $\omega_1 \subset \omega_2$ . Then, there exists a positive constant  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N)$  and a weight function  $\theta$  define by (58) such that the following observability inequality holds true for any  $\rho$  solution of (50)-(51):*

$$\int_Q e^{-2s\eta} \varphi |\rho|^2 dxdt \leq C \int_0^T \int_{\omega_2} |\rho|^2 dxdt. \quad (52)$$

**Proof.** Applying (48) to  $\rho$  solution of (50)-(51), then using the notation (49), we have that

$$I(\rho) \leq C_1 \left( \left\| s^{-\frac{1}{4}} \varphi^{-\frac{1}{4}} e^{-s\eta} \frac{\partial \Psi}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma_1)}^2 + s\lambda^2 \int_0^T \int_{\omega_2} \varphi e^{-2s\eta} |\rho|^2 dxdt \right). \quad (53)$$

In view of the definition of  $\psi$ ,  $\varphi$   $\eta$  given respectively by (43), (45) and (46), we have that there exists  $C(T)$  such that

$$\left\| s^{-\frac{1}{4}} \varphi^{-\frac{1}{4}} e^{-s\eta} \frac{\partial \Psi}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma_1)}^2 \leq C(T) \left\| \frac{\partial \Psi}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma_1)}^2. \quad (54)$$

Using on the one hand the fact that

$$\left\| \frac{\partial \Psi}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma_1)}^2 \leq C \|\Psi\|_{H^{2,1}(Q)}^2, \quad (55)$$

and, on the other hand that  $\Psi$  satisfies (51), we deduce that

$$\|\Psi\|_{H^{2,1}(Q)}^2 \leq C(\|a_0\|_{L^\infty(Q)}, T, N) \|\rho\|_{L^2(\omega_T^T)}^2. \quad (56)$$

Combining (54)-(56) and using the fact that  $\omega_1 \subset \omega_2$ , we have that

$$\left\| s^{-\frac{1}{4}} \varphi^{-\frac{1}{4}} e^{-s\eta} \frac{\partial \Psi}{\partial \nu} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma_1)}^2 \leq C(\|a_0\|_{L^\infty(Q)}, T, N) \|\rho\|_{L^2(\omega_2^T)}^2. \quad (57)$$

In view of (57) and the fact that  $s, \lambda \geq 1$ , it follows from (53) that

$$\int_Q e^{-2s\eta} \varphi |\rho|^2 dx dt \leq C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) \|\rho\|_{L^2(\omega_2^T)}^2 + C_1 \int_0^T \int_{\omega_2} \varphi e^{-2s\eta} |\rho|^2 dx dt.$$

Since  $e^{-2s\eta} \varphi \in L^\infty(Q)$ , we deduce from this latter inequality (52):

$$\int_Q e^{-2s\eta} \varphi |\rho|^2 dx dt \leq C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) \|\rho\|_{L^2(\omega_2^T)}^2.$$

Set

$$\frac{1}{\theta^2} = e^{-2s\eta} \varphi. \quad (58)$$

Then according to the definition of  $\eta$  and  $\varphi$ , we have that  $\frac{1}{\theta^2} \in L^\infty(Q)$  and (52) can be rewritten as

$$\int_Q \frac{1}{\theta^2} |\rho|^2 dx dt \leq C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) \|\rho\|_{L^2(\omega_2^T)}^2. \quad (59)$$

■

## 4 Proof of Theorem 1.2

In this section, we are concerned with the Problem 2. More precisely, we are interested in the following null controllability problem: Find a control  $k \in L^2(\omega_2^T)$  such that if  $q = q(t, x; u(k), k) \in H^{2,1}(Q)$  is solution of (7)-(9), then (6) holds true, i.e.:

$$q(0, \cdot; u(k), k) = 0 \text{ in } \Omega.$$

If we multiply the first line in (7) by  $\rho$  solution of (50) and the first line in (8) by  $\Psi$  solution of (51) and integrate by parts over  $Q$ , we obtain the following equations:

$$\int_\Omega q(0) \rho^0 dx - \int_{\Sigma_1} \frac{\partial q}{\partial \nu} \frac{\partial \Psi}{\partial \nu} d\sigma dt = \int_{\mathcal{O}_T} h_0 \rho dx dt + \frac{1}{N} \int_{\omega_1^T} p \rho dx dt + \int_{\omega_2^T} k \rho dx dt \quad (60)$$

and

$$\int_{\Sigma_1} \frac{\partial q}{\partial \nu} \frac{\partial \Psi}{\partial \nu} d\sigma dt + \frac{1}{N} \int_{\omega_1^T} p \rho dx dt = 0. \quad (61)$$

Combining (60) and (61), we obtain

$$\int_\Omega q(0) \rho^0 dx - \int_{\mathcal{O}_T} h_0 \rho dx dt - \int_{\omega_2^T} k \rho dx dt = 0.$$

The null controllability property is equivalent to find a control  $k \in L^2(\omega_2^T)$  such that for any  $\rho^0 \in L^2(\Omega)$ , we have

$$\int_{\omega_2^T} k\rho \, dxdt + \int_{\mathcal{O}_T} h_0\rho \, dxdt = 0.$$

To find such a control, we consider for any  $\varepsilon > 0$  and for any  $\rho^0 \in L^2(\Omega)$  the following functional:

$$J_\varepsilon(\rho^T) = \frac{1}{2} \int_0^T \int_{\omega_2} |\rho|^2 dxdt + \int_{\mathcal{O}_T} h_0\rho \, dxdt + \varepsilon \|\rho^0\|_{L^2(\Omega)}, \quad (62)$$

where  $\rho$  and  $\Psi$  are solutions of (50) and (51). We need to prove that the functional has a minimum in  $L^2(\Omega)$  and then the controllability of (7)-(??).

**Proposition 4.1** *Assume that  $a_0 \in L^\infty(Q)$  and that  $\omega_1 \subset \omega_2$ . Let  $\theta$  be defined by (58) and  $h_0 \in L^2(\mathcal{O}_T)$  be such that  $\theta h_0 \in L^2(\mathcal{O}_T)$ . Then there exists a unique point  $\rho_\varepsilon^0 \in L^2(\Omega)$  such that*

$$J_\varepsilon(\rho_\varepsilon^0) = \inf_{\rho^0 \in L^2(\Omega)} J_\varepsilon(\rho^0). \quad (63)$$

Moreover, when  $\rho_\varepsilon^0 \neq 0$ , we have the following optimality condition

$$\begin{aligned} 0 &= \int_0^T \int_{\omega_2} \rho_\varepsilon \rho dxdt + \int_{\mathcal{O}_T} h_0 \rho dxdt \\ &\quad + \varepsilon \frac{1}{\|\rho_\varepsilon^0\|_{L^2(\Omega)}} \int_{\Omega} \rho_\varepsilon^0 \rho^0 dx, \quad \forall \rho^0 \in L^2(\Omega), \end{aligned} \quad (64)$$

where  $\rho_\varepsilon$  and  $\Psi_\varepsilon$  are solutions to (50) and (51) corresponding to  $\rho^0 = \rho_\varepsilon^0$ . In addition, there exists a positive constant  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N)$  independent of  $\varepsilon$  such that  $k_\varepsilon = \rho_\varepsilon \mathbf{1}_{\omega_2}$  satisfies:

$$\|k_\varepsilon\|_{L^2(\omega_2^T)} \leq \sqrt{C} \|\theta h_0\|_{L^2(\mathcal{O}_T)}. \quad (65)$$

**Proof.** It is clear that  $J_\varepsilon$  is continuous and strictly convex on  $L^2(\Omega)$ . It remains to prove that  $J_\varepsilon$  is coercive to obtain that there exists a unique point  $\rho_\varepsilon^0 \in L^2(\Omega)$  where the functional  $J_\varepsilon$  reaches its minimum. So, using Cauchy Schwarz inequality and the definition of  $\theta$  given by (58), we have that

$$\int_Q (-h_0)\rho \, dxdt \leq \|\theta h_0\|_{L^2(\mathcal{O}_T)} \left\| \frac{1}{\theta} \rho \right\|_{L^2(Q)},$$

which in view of (52) gives

$$\int_Q (-h_0)\rho \, dxdt \leq \sqrt{C} \|\theta h_0\|_{L^2(\mathcal{O}_T)} \|\rho\|_{L^2(\omega_2^T)}$$

where  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) > 0$ . Using Young inequality, we obtain that

$$\int_Q (-h_0)\rho \, dxdt \leq C\|\theta h_0\|_{L^2(\mathcal{O}_T)}^2 + \frac{1}{4}\|\rho\|_{L^2(\omega_2^T)}^2,$$

which combining with definition of  $J_\varepsilon$  give by (62) yields

$$J_\varepsilon(\rho^0) \geq \frac{1}{4} \int_0^T \int_{\omega_2} |\rho|^2 dxdt + \varepsilon\|\rho^0\|_{L^2(\Omega)} - C\|\theta h_0\|_{L^2(\mathcal{O}_T)}^2$$

where  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) > 0$ . Hence we deduce that  $J_\varepsilon$  is coercive on  $L^2(\Omega)$ . Consequently, there exists a unique point  $\rho_\varepsilon^0 \in L^2(\Omega)$  where the functional  $J_\varepsilon$  reaches its minimum.

Now, assume that  $\rho_\varepsilon^0 \neq 0$ . In order to prove (64), we write the Euler-Lagrange condition which characterize the minimizer  $\rho_\varepsilon^0 \in L^2(\Omega)$ :

$$\lim_{\lambda \rightarrow 0} \frac{J_\varepsilon(\rho_\varepsilon^0 + \lambda\rho^0) - J_\varepsilon(\rho_\varepsilon^0)}{\lambda} = 0, \quad \forall \rho^0 \in L^2(\Omega). \quad (66)$$

After some calculations (66) yields (64).

Let  $\rho_\varepsilon^0$  be the solution of (63) and  $\rho_\varepsilon$  be the solution of (50)-(51) associated to  $\rho_\varepsilon^0$ . Let also  $q_\varepsilon, p_\varepsilon$  be the solution associated to  $k = k_\varepsilon = \rho_\varepsilon$  of systems (7) and (8) respectively:

$$\begin{cases} -\frac{\partial q_\varepsilon}{\partial t} - \Delta q_\varepsilon + a_0 q_\varepsilon &= h_0 \mathbf{1}_\mathcal{O} + u_\varepsilon \mathbf{1}_{\omega_1} + k_\varepsilon \mathbf{1}_{\omega_2} & \text{in } Q, \\ q_\varepsilon &= 0 & \text{on } \Sigma, \\ q_\varepsilon(T) &= 0 & \text{in } \Omega, \end{cases} \quad (67)$$

and

$$\begin{cases} \frac{\partial p_\varepsilon}{\partial t} - \Delta p_\varepsilon + a_0 p_\varepsilon &= 0 & \text{in } Q, \\ p_\varepsilon &= \frac{\partial q_\varepsilon}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \\ p_\varepsilon(0) &= 0 & \text{in } \Omega, \end{cases} \quad (68)$$

with

$$u_\varepsilon = \frac{p_\varepsilon}{N} \text{ in } \omega_1^T, \quad (69)$$

and

$$k_\varepsilon = \rho_\varepsilon \mathbf{1}_{\omega_2} \text{ in } Q. \quad (70)$$

Multiplying the first line of (67) by  $\rho$  and the first line of (68) by  $\Psi$ , where  $\rho$  and  $\Psi$  are respectively solutions of (50) and (51) and integrating by parts over  $Q$ , we have successively

$$\int_\Omega q_\varepsilon(0)\rho^0 \, dx - \int_{\Sigma_1} \frac{\partial q_\varepsilon}{\partial \nu} \frac{\partial \Psi}{\partial \nu} \, d\sigma dt = \int_{\mathcal{O}_T} h_0 \rho \, dxdt + \frac{1}{N} \int_{\omega_1^T} p_\varepsilon \rho \, dxdt + \int_{\omega_2^T} \rho_\varepsilon \rho \, dxdt \quad (71)$$

and

$$\int_{\Sigma_1} \frac{\partial q_\varepsilon}{\partial \nu} \frac{\partial \Psi}{\partial \nu} d\sigma dt + \frac{1}{N} \int_{\omega_1^T} p_\varepsilon \rho dx dt = 0. \quad (72)$$

Combining (71) and (72) with (64), we obtain

$$0 = \int_{\Omega} \left( q_\varepsilon(0, x) + \varepsilon \frac{1}{\|\rho_\varepsilon^0\|_{L^2(\Omega)}} \rho_\varepsilon^0(x) \right) \rho^0(x) dx, \quad \forall \rho^0 \in L^2(\Omega).$$

Hence,

$$q_\varepsilon(0, x) = -\varepsilon \frac{1}{\|\rho_\varepsilon^0\|_{L^2(\Omega)}} \rho_\varepsilon^0(x).$$

Consequently,

$$\|q_\varepsilon(0, x)\|_{L^2(\Omega)} = \varepsilon. \quad (73)$$

Now, if we take  $\rho^0 = \rho_\varepsilon^0$  in (64), we obtain that

$$\begin{aligned} \|k_\varepsilon\|_{L^2(\omega_2^T)}^2 &= \int_0^T \int_{\omega} |\rho_\varepsilon|^2 dx dt = - \int_{\mathcal{O}_T} h_0 \rho_\varepsilon dx dt - \varepsilon \|\rho_\varepsilon^0\|_{L^2(\Omega)} \\ &\leq \|\theta h_0\|_{L^2(\mathcal{O}_T)} \left\| \frac{1}{\theta} \rho_\varepsilon \right\|_{L^2(Q)}. \end{aligned}$$

It then follows from (52) that

$$\|k_\varepsilon\|_{L^2(\omega_2^T)} \leq \sqrt{C} \|\theta h_0\|_{L^2(\mathcal{O}_T)},$$

where  $C = C(\|a_0\|_{L^\infty(Q)}, T, N) > 0$ . ■

#### 4.1 Proof of Theorem 1.2

We proceed in three steps.

**Step1.** We give some a priori estimates on  $u_\varepsilon$ ,  $q_\varepsilon$  and  $p_\varepsilon$ .

In view of (36) and (65), we have that

$$\|u_\varepsilon\|_{L^2(\omega_2^T)} \leq C (\|h_0\|_{L^2(\mathcal{O}_T)} + \|\theta h_0\|_{L^2(\mathcal{O}_T)}), \quad (74)$$

where  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) > 0$ . Since  $q_\varepsilon$ ,  $p_\varepsilon$  satisfy (67)-(68), using (65) and (74), we prove that

$$\|q_\varepsilon\|_{H^{2,1}(Q)} \leq C (\|h_0\|_{L^2(\mathcal{O}_T)} + \|\theta h_0\|_{L^2(\mathcal{O}_T)}), \quad (75)$$

$$\left\| \frac{\partial q_\varepsilon}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C (\|h_0\|_{L^2(\mathcal{O}_T)} + \|\theta h_0\|_{L^2(\mathcal{O}_T)}), \quad (76)$$

and

$$\|p_\varepsilon\|_{L^2(Q)} \leq C (\|h_0\|_{L^2(\mathcal{O}_T)} + \|\theta h_0\|_{L^2(\mathcal{O}_T)}), \quad (77)$$

where  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) > 0$ .



**Step2.** We study the convergence when  $\varepsilon \rightarrow 0$  to the sequences  $k_\varepsilon, u_\varepsilon, q_\varepsilon, \frac{\partial q_\varepsilon}{\partial \nu}$  and  $p_\varepsilon$ .

In view of (65), (73), (74)-(77), we can extract subsequences still denoted by  $k_\varepsilon, u_\varepsilon, q_\varepsilon, \frac{\partial q_\varepsilon}{\partial \nu}$  and  $p_\varepsilon$  such that when  $\varepsilon \rightarrow 0$ , we have

$$k_\varepsilon \rightharpoonup \hat{k} \text{ weakly in } L^2(\omega_2^T), \quad (78a)$$

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(\omega_1^T), \quad (78b)$$

$$q_\varepsilon \rightharpoonup q \text{ weakly in } H^{2,1}(Q), \quad (78c)$$

$$\frac{\partial q_\varepsilon}{\partial \nu} \rightharpoonup \beta \text{ weakly in } L^2(\Sigma), \quad (78d)$$

$$p_\varepsilon \rightharpoonup p \text{ weakly in } L^2(Q), \quad (78e)$$

$$q_\varepsilon(0) \rightarrow 0 \text{ strongly in } L^2(\Omega). \quad (78f)$$

From (69), (78b) and (78e), we obtain

$$u = \frac{p}{N} \text{ in } \omega_1^T. \quad (79)$$

Moreover using the weak lower semi-continuity of the norm, we deduce from (78b) and (74) that

$$\|u\|_{L^2(\omega_1^T)} \leq C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) (\|h_0\|_{L^2(\mathcal{O}_T)} + \|\theta h_0\|_{L^2(\mathcal{O}_T)}). \quad (80)$$

Now, proceeding as for the convergence of  $q^n$  in page 5-7, we prove using (78a)-(78d) and (78f) that  $q$  is solution of

$$\begin{cases} -\frac{\partial q}{\partial t} - \Delta q + a_0 q &= h_0 \mathbf{1}_{\mathcal{O}} + u \mathbf{1}_{\omega_1} + \hat{k} \mathbf{1}_{\omega_2} & \text{in } Q, \\ q &= 0 & \text{on } \Sigma, \\ q(T) &= 0 & \text{in } \Omega, \end{cases} \quad (81)$$

$$\frac{\partial q_\varepsilon}{\partial \nu} \rightharpoonup \beta = \frac{\partial q}{\partial \nu} \text{ weakly in } L^2(\Sigma) \quad (82)$$

and

$$q(0) = 0 \text{ in } \Omega. \quad (83)$$

Now, if we multiply the first equation in (68) by  $\xi \in \mathbb{D}(Q)$  and integrate by parts over  $Q$ , then take the limit when  $\varepsilon \rightarrow 0$  while using (78), we deduce that

$$\int_Q p \left( -\frac{\partial \xi}{\partial t} - \Delta \xi + a_0 \xi \right) dx dt = 0,$$

which after an integration by parts over  $Q$  gives

$$\int_Q \left( \frac{\partial p}{\partial t} - \Delta p + a_0 p \right) \xi dx dt = 0, \quad \forall \xi \in \mathbb{D}(Q).$$

Hence, we deduce that

$$-\frac{\partial p}{\partial t} - \Delta p + a_0 p = 0 \text{ in } Q. \quad (84)$$

Since  $p \in L^2(Q)$  and  $\Delta p \in H^{-1}((0, T); L^2(\Omega))$ , we deduce that  $p|_\Sigma$  and  $\frac{\partial p}{\partial \nu}|_\Sigma$  exist and belong to  $H^{-1}((0, T); H^{-1/2}(\Gamma))$  and  $H^{-1}((0, T); H^{-3/2}(\Gamma))$  respectively. On the other hand observing that  $p \in L^2(Q)$  and  $\frac{\partial p}{\partial t} \in L^2((0, T); H^{-2}(\Omega))$ , we deduce that  $p(0)$  and  $p(T)$  exist in  $H^{-1}(\Omega)$ .

So, if we multiply the first equation in (68) by  $\xi \in \mathcal{C}^\infty(\overline{Q})$  such that  $\xi|_\Sigma = 0$ ,  $\xi(T) = 0$  in  $\Omega$ , then integrate by parts over  $Q$ , we obtain

$$0 = \int_{\Sigma_1} \frac{\partial q_\varepsilon}{\partial \nu} \frac{\partial \xi}{\partial \nu} d\sigma dt + \int_Q p_\varepsilon \left( -\frac{\partial \xi}{\partial t} - \Delta \xi + a_0 \xi \right) dx dt, \\ \forall \xi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \xi|_\Sigma = 0, \xi(T) = 0 \text{ in } \Omega.$$

Passing this latter identity to the limit when  $\varepsilon$  tends toward zero, while using (78e) and (82), we obtain that

$$0 = \int_{\Sigma_1} \frac{\partial q}{\partial \nu} \frac{\partial \xi}{\partial \nu} d\sigma dt + \int_Q p \left( -\frac{\partial \xi}{\partial t} - \Delta \xi + a_0 \xi \right) dx dt, \\ \forall \xi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \xi|_\Sigma = 0, \xi(T) = 0 \text{ in } \Omega,$$

which after an integration by parts yields

$$0 = \langle p(0), \xi(0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Sigma_1} \frac{\partial q}{\partial \nu} \frac{\partial \xi}{\partial \nu} d\sigma dt + \\ - \left\langle p, \frac{\partial \xi}{\partial \nu} \right\rangle_{H^{-3/2}((0, T), H^{-1/2}(\Gamma)), H^{3/2}((0, T), H^{1/2}(\Gamma))} \\ + \int_Q \left( \frac{\partial p}{\partial t} - \Delta p + a_0 p \right) \xi dx dt, \\ \forall \xi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \xi|_\Sigma = 0, \xi(T) = 0 \text{ in } \Omega.$$

Using (84) in this latter identity, we get

$$0 = \langle p(0), \xi(0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Sigma_1} \frac{\partial q}{\partial \nu} \frac{\partial \xi}{\partial \nu} d\sigma dt + \\ - \left\langle p, \frac{\partial \xi}{\partial \nu} \right\rangle_{H^{-3/2}((0, T), H^{-1/2}(\Gamma)), H^{3/2}((0, T), H^{1/2}(\Gamma))} \quad (85) \\ \forall \xi \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } \xi|_\Sigma = 0, \xi(T) = 0 \text{ in } \Omega.$$

Take in (85),  $\xi$  such that  $\frac{\partial \xi}{\partial \nu} = 0$  on  $\Sigma$ , we deduce that

$$p(0) = 0 \text{ in } \Omega, \text{ and finally,} \quad (86)$$

$$p = \frac{\partial q}{\partial \nu} \text{ on } \Sigma_1. \quad (87)$$

It then follows from (84), (87) and (86) that  $p$  satisfies

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p + a_0 p = 0 & \text{in } Q, \\ p = \frac{\partial q}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \\ p(0) = 0 & \text{in } \Omega, \end{cases} \quad (88)$$

Finally, using the weak-lower semi-continuity of the norm and (78a), we deduce from (65) the estimate (14).

**Step3.** Observing that  $k_\varepsilon = \rho_\varepsilon \mathbf{1}_{\omega_2}$ , where  $\rho_\varepsilon$  satisfies:

$$\begin{cases} \frac{\partial \rho_\varepsilon}{\partial t} - \Delta \rho_\varepsilon + a_0 \rho_\varepsilon = 0 & \text{in } Q, \\ \rho_\varepsilon = \frac{\partial \Psi_\varepsilon}{\partial \nu} \mathbf{1}_{\Gamma_1} & \text{on } \Sigma, \\ \rho_\varepsilon(0) = \rho_\varepsilon^0 & \text{in } \Omega, \end{cases} \quad (89)$$

with  $\Psi_\varepsilon$  solution of

$$\begin{cases} -\frac{\partial \Psi_\varepsilon}{\partial t} - \Delta \Psi_\varepsilon + a_0 \Psi_\varepsilon = -\frac{\rho_\varepsilon}{N} \mathbf{1}_{\omega_1} & \text{in } Q, \\ \Psi_\varepsilon = 0 & \text{on } \Sigma, \\ \Psi_\varepsilon(T) = 0 & \text{in } \Omega, \end{cases} \quad (90)$$

we know that

$$\begin{aligned} \|\Psi_\varepsilon\|_{H^{2,1}(Q)} &\leq C(\|a_0\|_{L^\infty(Q)}, T, N) \|rho_\varepsilon\|_{L^2(\omega_2^T)} \\ &\leq C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) \|\theta h_0\|_{L^2(\mathcal{O}_T)}, \\ \left\| \frac{\partial \Psi_\varepsilon}{\partial \nu} \right\|_{L^2(\Sigma)} &\leq C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) \|\theta h_0\|_{L^2(\mathcal{O}_T)}, \end{aligned}$$

because  $\omega_1 \subset \omega_2$ ,  $k_\varepsilon = \rho_\varepsilon \mathbf{1}_{\omega_2}$  and (65) holds true.

Consequently, there exist  $\hat{\Psi} \in H^{2,1}(Q)$  and  $\beta \in L^2(\Sigma)$  such that

$$\Psi_\varepsilon \rightharpoonup \hat{\Psi} \text{ weakly in } H^{2,1}(Q), \quad (91a)$$

$$\frac{\partial \Psi_\varepsilon}{\partial \nu} \rightharpoonup \beta \text{ weakly in } L^2(\Sigma). \quad (91b)$$

On the other hand, from (52), (65) and (70), there exists  $C = C(C_1, \|a_0\|_{L^\infty(Q)}, T, N) > 0$  such that

$$\left\| \frac{1}{\theta} \rho_\varepsilon \right\|_{L^2(Q)}^2 \leq C \|\theta h_0\|_{L^2(\mathcal{O}_T)}^2.$$

Hence, if we set  $L^2(\frac{1}{\theta}, X) = \left\{ z \in L^2(X), \int_X \frac{1}{\theta^2} |z|^2 dX < \infty \right\}$ , we deduce from this latter inequality that  $\rho_\varepsilon$  is bounded in  $L^2(\frac{1}{\theta}, Q)$ . Consequently, there exists  $\rho \in L^2(\frac{1}{\theta}, Q)$  and a subsequence of  $(\rho_\varepsilon)$  still denoted  $(\rho_\varepsilon)$  such that

$$\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } L^2\left(\frac{1}{\theta}, Q\right). \quad (92)$$

If we refer of the definition of  $\psi$ ,  $\varphi$  and  $\eta$  given by (43)-(46) and the definition of  $\theta$  given by (58), we can see that for all  $\tau > 0$ ,

$$\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } L^2(] \tau, T - \tau[ \times \Omega),$$

This implies that

$$\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } \mathbb{D}'(Q). \quad (93)$$

where  $\mathbb{D}'(Q)$  is the dual of  $\mathbb{D}(Q)$ . Therefore, it follows from (65) and (70) that

$$\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } L^2(\omega_2^T) \quad (94)$$

and we have

$$\hat{k} = \hat{\rho} \mathbf{1}_{\omega_2}. \quad (95)$$

Now proceeding as for the convergence of  $q^n$  in page 5-7, while passing to the limit in (90), we prove using (91) and (94) that  $\hat{\Psi}$  is solution of (13) and that

$$\frac{\partial \Psi_\varepsilon}{\partial \nu} \rightharpoonup \frac{\partial \hat{\Psi}}{\partial \nu} \text{ weakly in } L^2(\Sigma). \quad (96)$$

Using (93) and inter0bb while passing to the limit in (89), we obtain that  $\hat{\rho}$  is solution of (12).

In short, In view of (81), (83) and (88), we have that  $(\hat{k}, q, p)$  is solution of the null controllability problems (7)-(??).

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