



**HAL**  
open science

# AN EXISTENCE RESULT OF $(\omega, C)$ -PERIODIC MILD SOLUTIONS TO SOME FRACTIONAL DIFFERENTIAL EQUATION

Gisèle Mophou, Gaston M N'Guérékata

► **To cite this version:**

Gisèle Mophou, Gaston M N'Guérékata. AN EXISTENCE RESULT OF  $(\omega, C)$ -PERIODIC MILD SOLUTIONS TO SOME FRACTIONAL DIFFERENTIAL EQUATION. *Nonlinear Studies*, In press. hal-02548014

**HAL Id: hal-02548014**

**<https://hal.science/hal-02548014>**

Submitted on 20 Apr 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# AN EXISTENCE RESULT OF $(\omega, c)$ -PERIODIC MILD SOLUTIONS TO SOME FRACTIONAL DIFFERENTIAL EQUATION

GISÈLE MOPHOU AND GASTON M. N'GUÉRÉKATA

ABSTRACT. We first investigate in this paper further properties of the new concept of  $(\omega, c)$ -periodic functions; then we apply the results to study the existence of  $(\omega, c)$ -periodic mild solutions of the fractional differential equations  $D_t^\alpha(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1}F_2(t, u(t))$ ,  $t \in \mathbb{R}$ , where  $1 < \alpha < 2$ ,  $A : D(A) \subseteq X \rightarrow X$  is a linear densely defined operator of sectorial type on a complex Banach space  $X$ ,  $F_1, F_2 : \mathbb{R} \times X \rightarrow X$  are two  $(\omega, c)$ -periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the sense of Riemann-Liouville.

## 1. INTRODUCTION

Recently, Alvarez et al. [1] introduced the concept of  $(\omega, c)$ -periodic functions which contains the classes periodic, antiperiodic and Bloch periodic functions among others. This concept is motivated by the Mathieu's equation

$$y''(t) + [a - 2q\cos(2t)]y(t) = 0$$

which is used in the modelling of several phenomena including the stability of railroads as trains drive over them, or seasonally forced population dynamics.

In their paper [2], the authors generalize this concept to the one of  $(\omega, c)$ -pseudo periodic functions which are functions with ergodic parts. Both papers [1, 2] generalize several results in [7, 10, 11, 12].

The aim of this work is to investigate further properties of  $(\omega, c)$ -periodic functions in Section 2 and their applications to some ordinary differential equations (Section 3) and fractional differential equations of the form  $D_t^\alpha(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1}F_2(t, u(t))$ ,  $t \in \mathbb{R}$ , where  $1 < \alpha < 2$  (Section 4) in a complex Banach space  $(X, \|\cdot\|)$ .  $D_t^\alpha$  denotes the time fractional derivative of order  $\alpha$  in the Riemann-Liouville sense.

## 2. $(\omega, c)$ -PERIODIC FUNCTIONS AND THEIR PROPERTIES

In this paper we will consider  $(X, \|\cdot\|)$  a complex Banach space,  $\omega > 0$ , and  $c \in \mathbb{C} \setminus \{0\}$ .

**Definition 2.1.** [1, 2] *A function  $f \in C(\mathbb{R}, X)$  is said to be  $(\omega, c)$ -periodic if*

$$f(t + \omega) = cf(t), \quad \forall t \in \mathbb{R}.$$

In this case,  $\omega$  is called the  $c$ -period of  $f$ . The set of all  $(\omega, c)$ -periodic functions  $\mathbb{R} \rightarrow X$  is denoted  $P_{\omega c}(\mathbb{R}, X)$ . Examples of  $(\omega, c)$ -periodic include  $\omega$ -periodic ( $P_\omega(\mathbb{R}, X)$ ), antiperiodic ( $P_{a\omega}(\mathbb{R}, X)$ ) and Bloch periodic ( $B_{p,k}(\mathbb{R}, X)$ ) functions (cf. for instance [7, 10, 11, 12] for more information on these functions).

**Proposition 2.2.** [1] *A function  $f \in C(\mathbb{R}, X)$  is  $(\omega, c)$ -periodic if and only if*

$$f(t) = c^\wedge(t)u(t), \quad c^\wedge(t) := c^{\frac{t}{\omega}}, \quad u \in P_\omega(\mathbb{R}, X).$$

**Theorem 2.3.** [1] *Equipped with the norm  $\|f\|_{\omega c} := \sup_{t \in [0, \omega]} \|c^\wedge(-t)|f(t)\|$ ,  $P_{\omega c}(\mathbb{R}, X)$*

*turns out to be a Banach space*

Let's state and prove

**Theorem 2.4.** *Let  $f \in P_{\omega, c}(\mathbb{R}, X)$ . Then the following are true:*

- (i) *If  $f(t) \neq 0 \quad \forall t \in \mathbb{R}$ , then the function  $g(t) := \frac{1}{f(t)}$ ,  $t \in \mathbb{R}$  belongs to  $P_{\omega, \frac{1}{c}}(\mathbb{R}, X)$*
- (ii) *The function  $f_a(t) := f(t + a)$ ,  $a \in \mathbb{R}$  belongs to  $P_{\omega, c}(\mathbb{R}, X)$ .*
- (iii) *If  $\lambda \in P_{\omega, c}(\mathbb{R}, \mathbb{C})$ , then  $\lambda f \in P_{\omega, c^2}(\mathbb{R}, X)$*
- (iv) *If  $A : X \rightarrow X$  is a bounded linear operator, then  $Af \in P_{\omega, c}(\mathbb{R}, X)$ .*

*Proof.* It comes straightforward from the definition of  $(\omega, c)$ -periodic functions.  $\square$

**Lemma 2.5.** *For all  $(s, u) \in \mathbb{R} \times X$ , there exists  $\varphi \in P_{\omega, c}(\mathbb{R}, X)$  such that  $\varphi(s) = u$*

*Proof.* Let  $\Psi : \mathbb{R} \rightarrow X$  be define by  $\Psi(t) = c^\wedge(t)$ . Then obviously  $\Psi \in P_{\omega, c}(\mathbb{R}, X)$  with  $\Psi(0) = 1$ . This implies that the function  $\varphi(t) := \Psi(t - s)u$  satisfies the property.  $\square$

For a given function  $F \in BC(\mathbb{R}, X)$ , define the Nemytskii's superposition operator

$$(2.1) \quad \mathcal{N}_F(\varphi)(\cdot) := F(\cdot, \varphi(\cdot)), \quad \varphi \in P_{\omega, c}(\mathbb{R}, X).$$

In [7], the authors proved the following result in the framework of Bloch-periodic functions. This result was generalized in the case of  $(\omega, c)$ -periodic functions [1]. Here we give a more rigourous proof using the lemma above.

**Theorem 2.6.** *Let  $F \in BC(\mathbb{R}, X)$ . The following are equivalent*

- (i) *For every  $\varphi \in P_{\omega, c}(\mathbb{R}, X)$ ,  $\mathcal{N}_F(\varphi) \in P_{\omega, c}(\mathbb{R}, X)$ .*
- (ii) *For all  $(t, x) \in \mathbb{R} \times X$ ,  $F(t + \omega, cx) = cF(t, x)$ .*

*Proof.* Assume (i) and let  $(t, x) \in \mathbb{R} \times X$ . Then by Lemma (2.5) there exists  $\varphi \in P_{\omega, c}(\mathbb{R}, X)$  such that  $\varphi(t) = x$ . Let  $\Phi(\cdot) := \mathcal{N}(\cdot, \varphi(\cdot))$ . Then  $\Phi \in P_{\omega, c}(\mathbb{R}, X)$  and using (2.1) we have

$$\begin{aligned} F(t + \omega, cx) &= F(t + \omega, c\varphi(t)) \\ &= F(t + \omega, \varphi(t + \omega)) \\ &= \Phi(t + \omega) \\ &= c\Phi(t) \\ &= c\mathcal{N}_F(\varphi)(t) \\ &= cF(t, \varphi(t)) \end{aligned}$$

which means  $F(t + \omega, cx) = cF(t, x)$  because  $\varphi(t) = x$ .

Conversely assume that  $F(t + \omega, cx) = cF(t, x)$  and let  $\varphi \in P_{\omega, c}(\mathbb{R}, X)$ . Then using (2.1), we have

$$\begin{aligned} \mathcal{N}_F(\varphi)(t + w) &= F(t + \omega, \varphi(t + \omega)) \\ &= F(t + \omega, c\varphi(t)) \\ &= cF(t, \varphi(t)) \\ &= c\mathcal{N}_F(\varphi)(t). \end{aligned}$$

The proof is complete.  $\square$

**Theorem 2.7.** *Suppose  $f \in C^1(\mathbb{R}, X) \cap P_{\omega, c}(\mathbb{R}, X)$ . Then  $f' \in P_{\omega, c}(\mathbb{R}, X)$*

*Proof.* We have

$$f(t+h) - f(t) = f'(t)h + o(h)$$

where  $o(h) = |h|\varepsilon(h)$  with  $\lim_{h \rightarrow 0} \varepsilon(h) = 0$ . But

$$f(t+\omega+h) - f(t+\omega) = c[f(t+h) - f(t)] = cf'(t)h + o(h).$$

Therefore

$$f'(t+\omega) = cf'(t)$$

which means

$$f' \in P_{\omega, c}(\mathbb{R}, X).$$

$\square$

Theorem 2.7 leads to the following corollary:

**Corollary 2.8.** *Let  $F \in C^1(\mathbb{R} \times X, X)$  satisfies one of the equivalent properties (i) and (ii) of Theorem 2.6. Then for every function  $\varphi \in C^1(\mathbb{R}, X) \cap P_{\omega, c}(\mathbb{R}, X)$ , the function  $\Phi : \mathbb{R} \rightarrow X$  defined by  $\Phi(\cdot) = F(\cdot, \varphi(\cdot))$  is  $(\omega, c)$ -periodic, differentiable and  $\Phi'$  is  $(\omega, c)$ -periodic.*

**Proposition 2.9.** *Let  $f \in P_{\omega, c}(\mathbb{R}, X)$  and suppose  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup of linear operators. Then the function  $\Upsilon(t)$  defined by*

$$\Upsilon(t) := \int_{-\infty}^t T(t-s)f(s) ds$$

*is also in  $P_{\omega, c}(\mathbb{R}, X)$ .*

*Proof.* It is straightforward. Indeed we have.

$$\begin{aligned} \Upsilon(t+\omega) &= \int_{-\infty}^{t+\omega} T(t+\omega-s)f(s) ds \\ &= \int_{-\infty}^t T(t-\sigma)f(\sigma+\omega) d\sigma \\ &= c \int_{-\infty}^t T(t-\sigma)f(\sigma) d\sigma \\ &= c\Upsilon(t). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 2.10.** [1] *Let  $f \in P_{\omega, c}(\mathbb{R}, X)$  with  $f(t) = c^\wedge(t)u(t)$ ,  $u \in P_\omega(\mathbb{R}, X)$ . If  $k^\sim(t) := c^\wedge(-t)k(t) \in L^1(\mathbb{R})$ , then  $(k \star f) \in P_{\omega, c}(\mathbb{R}, X)$ , where*

$$(k \star f)(t) = \int_{\mathbb{R}} k(t-\sigma)f(\sigma) d\sigma$$

Let  $\varphi \in L^1(\mathbb{R})$  and  $\lambda \in \mathbb{C}$ . Consider the operator  $A_{\lambda, \varphi}$  defined by

$$A_{\lambda, \varphi}x = \lambda x + \varphi \star x.$$

Then it obvious that  $A_{\lambda, \varphi}(P_\omega(\mathbb{R}, X)) \subset P_\omega(\mathbb{R}, X)$ . Moreover  $A_{\lambda, \varphi}$  acts continuously on  $P_{\omega, c}(\mathbb{R}, X)$ , that means there exists a constant  $\gamma > 0$  such that

$$\|A_{\lambda, \varphi}x\| \leq \gamma\|x\|, \quad \forall x \in P_{\omega, c}(\mathbb{R}, X).$$

Consider now

$$a(\xi) := \lambda + \hat{\varphi}(\xi)$$

where  $\hat{\varphi}(\xi)$  is the Fourier transform of the function  $\varphi$ ,  $a(\xi)$  is the *symbol* of the operator  $A_{\lambda,\varphi}$ , where  $\varphi \in L^1(\mathbb{R})$ . Now since  $\lim_{\xi \rightarrow \infty} \hat{\varphi}(\xi) = 0$ , the symbol  $a(\xi)$  is a well-defined continuous function on  $\overline{\mathbb{R}} = \mathbb{R} \cup (\infty)$ , and  $a(\infty) = \lambda$ .

Now we state and prove

**Theorem 2.11.** *Suppose  $\varphi \in L^1(\mathbb{R})$ . Then the operator  $A_{\lambda,\varphi}$  is invertible in  $P_{\omega c}(\mathbb{R}, X)$  if  $a(\xi) \neq 0, \forall \xi \in \overline{\mathbb{R}}$ .*

*Proof.* Suppose  $a(\xi) \neq 0, \forall \xi \in \overline{\mathbb{R}}$ . The function  $\frac{1}{a(\xi)}$  is then well-defined on  $\overline{\mathbb{R}}$  and in view of the classical Wiener's theorem, we get

$$\frac{1}{a(\xi)} = \frac{1}{\lambda} + \hat{\Psi}(\xi),$$

where  $\Psi \in L^1(\mathbb{R})$ . It is easy to verify that  $A_{\frac{1}{\lambda},\Psi}$  is the inverse to the operator  $A_{\lambda,\varphi}$  which acts on  $P_{\omega c}(\mathbb{R}, X)$  in view of the remark above.  $\square$

### 3. ORDINARY DIFFERENTIAL EQUATIONS

**3.1. Linear case.** Consider the following equation in  $\mathbb{C}$

$$(3.1) \quad u'(t) = \lambda u(t) + h(t), \quad t \in \mathbb{R}$$

If  $\mathcal{R}e\lambda \neq 0$ , then we have either

$$u(t) = \int_{-\infty}^t e^{\lambda(t-s)} h(s) ds \quad (\mathcal{R}e\lambda < 0)$$

or

$$u(t) = \int_t^{\infty} e^{\lambda(t-s)} h(s) ds \quad (\mathcal{R}e\lambda > 0)$$

is the unique solution of Eq. (3.1). We deduce in view of Proposition 2.9 that  $u \in P_{\omega,c}(\mathbb{C})$ . We have the following Massera type result.

**Theorem 3.1.** *Let  $f$  be  $(\omega, c)$ -periodic. Then every bounded solution of the differential equation*

$$(3.2) \quad x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R},$$

where  $A(t) : \mathbb{R} \rightarrow \mathcal{M}_k(\mathbb{C})$  is  $\omega$ -periodic, is in  $P_{\omega,c}(\mathbb{C}^k)$

*Proof.* The proof is similar to Theorem 3.1 [8]. First let us note that by Floquet's theory and without loss of generality we may assume that  $A(t) = A$  is independent of  $t$ . Next we will show that the problem can be reduced to the one-dimensional case. In fact, if  $A$  is independent of  $t$ , by a change of variable if necessary, we may assume that  $A$  is of Jordan normal form. In this direction we can go further with assumption that  $A$  has only one Jordan box. That is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_k'(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_k(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Let's assume that  $\operatorname{Re}\lambda \neq 0$ , then consider the last equation

$$x'_k(t) = \lambda x_k(t) + f_k(t),$$

Using the above result, we can show that the solution  $x_k$  is in  $P_{\omega,c}(\mathbb{C})$ . The same way we can prove that  $x_{k-1}$  is also in  $P_{\omega,c}(\mathbb{C})$ . We continue the process to prove that all  $x_i$ ,  $i = 1, 2, \dots, k$  are in  $P_{\omega,c}(\mathbb{C})$ , which shows that  $x \in P_{\omega,c}(\mathbb{C}^k)$   $\square$

**3.2. Semilinear case.** Consider the following equation in  $\mathbb{C}$

$$(3.3) \quad u'(t) = \lambda u(t) + h(t, u(t)), \quad t \in \mathbb{R}$$

where  $\operatorname{Re}\lambda \neq 0$  and  $f : \mathbb{R} \times X \rightarrow X$  satisfies either condition (i) or condition (ii) of Theorem (2.6) and there exists  $L > 0$  such that

$$\|h(t, x) - h(t, y)\| \leq L\|x - y\|, \quad \forall x, y \in X.$$

Then there exists a unique solution of Eq. (2.7) which is in  $P_{\omega,c}(X)$  provided  $L < |\operatorname{Re}\lambda|$ .

Indeed by assumption the operator  $\Lambda : P_{\omega,c}(X) \rightarrow P_{\omega,c}(X)$  defined by

$$(\Lambda u)(t) := \int_{-\infty}^t e^{\lambda(t-s)} h(s, u(s)) ds \quad (\operatorname{Re}\lambda < 0)$$

is well-defined based on the assumption on  $h$ . Now if we take  $u, v \in \int_{-\infty}^t e^{\lambda(t-s)} h(s) ds$  ( $\operatorname{Re}\lambda < 0$ ) then

$$\begin{aligned} \|(\Lambda u)(t) - \Lambda v(t)\| &\leq \int_{-\infty}^t \|e^{\lambda(t-s)}(h(s, u(s)) - h(s, v(s)))\| ds \\ &\leq L\|u - v\|_{\infty} \int_{-\infty}^t e^{\operatorname{Re}\lambda(t-s)} ds \\ &\leq \frac{L}{|\operatorname{Re}\lambda|} \|u - v\|_{\infty} \end{aligned}$$

and we conclude using the Banach contraction principle.

#### 4. FRACTIONAL DIFFERENTIAL EQUATIONS

We consider the fractional differential equation

$$(4.1) \quad D_t^{\alpha}(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1}F_2(t, u(t)), \quad t \in \mathbb{R},$$

where  $1 < \alpha < 2$ ,  $A : D(A) \subseteq X \rightarrow X$  is a linear densely defined operator of sectorial type on a complex Banach space  $X$ ,  $F_1, F_2 : \mathbb{R} \times X \rightarrow X$  are two  $(\omega, c)$ -periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the Riemann-Liouville sense.

**4.1. Sectorial operators.** We need to recall some definitions about sectorial operators.

**Definition 4.1.** A closed linear operator  $(A, D(A))$  with dense domain  $D(A)$  in Banach space  $X$  is said to be sectorial of type  $\omega$  and angle  $\theta$  if there are constants  $\omega \in \mathbb{R}$ ,  $\theta \in ]0, \frac{\pi}{2}[$ , and  $M > 0$  such that its resolvent exists outside the sector

$$(4.2) \quad \omega + \Sigma_{\theta} := \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\},$$

$$(4.3) \quad \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + \Sigma_{\theta},$$

for more details on sectorial operators we refer to [6].

We denote by  $\mathcal{B}(X)$  the space of all bounded linear operators from  $X$  into  $X$  endowed with the norm of operators. Let  $A$  be a closed linear operator with domain  $D(A)$  defined on a Banach space  $X$ , the notation  $\rho(A)$  stands for the resolvent set of  $A$ .

To study the fractional differential equation

$$(4.4) \quad D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t), \quad t \in \mathbb{R},$$

where  $f$  is an  $X$ -valued function. We consider the following concept.

**Definition 4.2.** *We call  $A$  the generator of a solution operator (or  $\alpha$ -resolvent family) if there are  $\omega \in \mathbb{R}$  and a strongly continuous function  $S_\alpha : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  such that  $\{\lambda^\alpha : \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$  and  $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x \, dt$ ,  $\operatorname{Re} \lambda > \omega$ ,  $x \in X$ . In this case,  $S_\alpha(t)$  is called the solution operator generated by  $A$ . We observe that the power function  $\lambda^\alpha$  is uniquely defined as  $\lambda^\alpha = |\lambda|^\alpha e^{i \arg \lambda}$ , with  $-\pi < \arg(\lambda) < \pi$ .*

We note that if  $A$  is sectorial of type  $\omega$  with  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ , then  $A$  is the generator of a solution operator given by  $S_\alpha(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda$ , where  $\gamma$  is a suitable path lying outside the sector  $\omega + \Sigma_\theta$ .

**Lemma 4.3.** ([4, Theorem 1]) *Let  $A : D(A) \subseteq X \rightarrow X$  be a sectorial operator in a complex Banach space  $X$ , satisfying (4.2) and (4.3) for  $M > 0$ ,  $\omega < 0$  and  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ . Then there is  $C(\theta, \alpha) > 0$  depending solely on  $\theta$  and  $\alpha$ , such that*

$$(4.5) \quad \|S_\alpha(t)\| \leq \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha}, \quad t \geq 0.$$

We recall here the following definition that will be essential in the sequel.

**Definition 4.4.** *Assume that  $A$  is sectorial of type  $\omega < 0$  and angle  $\theta$  satisfying  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ . A function  $u : \mathbb{R} \rightarrow X$  is called a mild solution of (4.1) if the function  $s \rightarrow S_\alpha(t-s)F_2(s, u(s))$  is integrable on  $(-\infty, t)$  for each  $t \in \mathbb{R}$  and*

$$(4.6) \quad u(t) = F_1(t, u(t)) + \int_{-\infty}^t S_\alpha(t-s)F_2(s, u(s)) \, ds,$$

for any  $t \in \mathbb{R}$ .

**Definition 4.5.** *We call  $A$  the generator of a solution operator (or  $\alpha$ -resolvent family) if there are  $\omega \in \mathbb{R}$  and a strongly continuous function  $S_\alpha : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  such that  $\{\lambda^\alpha : \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$  and  $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x \, dt$ ,  $\operatorname{Re} \lambda > \omega$ ,  $x \in X$ . In this case,  $S_\alpha(t)$  is called the solution operator generated by  $A$ . We observe that the power function  $\lambda^\alpha$  is uniquely defined as  $\lambda^\alpha = |\lambda|^\alpha e^{i \arg \lambda}$ , with  $-\pi < \arg(\lambda) < \pi$ .*

We note that if  $A$  is of sectorial of type  $\omega$  with  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ , then  $A$  is the generator of a solution operator given by  $S_\alpha(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda$ , where  $\gamma$  is a suitable path lying outside the sector  $\omega + \Sigma_\theta$ .

**Lemma 4.6.** *Let  $A : D(A) \subseteq X \rightarrow X$  be a sectorial operator in a complex Banach space  $X$ , satisfying (4.2) and (4.3) for  $M > 0$ ,  $\omega < 0$  and  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ . Then there is  $C(\theta, \alpha) > 0$  depending solely on  $\theta$  and  $\alpha$ , such that*

$$(4.7) \quad \|S_\alpha(t)\| \leq \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha}, \quad t \geq 0.$$

**Lemma 4.7.** *Assume that  $A$  is sectorial of type  $\omega < 0$  and angle  $\theta$  satisfying  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ . If  $f : \mathbb{R} \rightarrow X$  is a  $(\omega, c)$ -periodic function and  $\Gamma f$  is given by  $(\Gamma f)(t) = \int_{-\infty}^t S_\alpha(t-s)f(s) ds$ ,  $t \in \mathbb{R}$ , then  $\Gamma f \in P_{\omega,c}(\mathbb{R}, X)$ .*

*Proof.* We have

$$\begin{aligned} (\Gamma f)(t+\omega) &= \int_{-\infty}^{t+\omega} S_\alpha(t+\omega-s)f(s) ds \\ &= \int_{-\infty}^t S_\alpha(t-\sigma)f(\sigma+\omega) d\sigma \\ &= c \int_{-\infty}^t S_\alpha(t-\sigma)f(\sigma) d\sigma \\ &= c(\Gamma f)(t) \end{aligned}$$

which proves the result.  $\square$

**Theorem 4.8.** *Assume that  $A$  is sectorial of type  $\omega < 0$  and angle  $\theta$  satisfying  $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$ . Let  $F_1, F_2$  satisfy Theorem 2.6 (i) and there are a positive constant  $\mu_1$  and positive bounded continuous function  $\mu_2(t)$  so that*

$$(4.8) \quad \|F_1(t, u_1) - F_1(t, u_2)\| \leq \mu_1 \|u_1 - u_2\|, \quad \forall u_1, u_2 \in X, \quad \forall t \in \mathbb{R},$$

$$(4.9) \quad \|F_2(t, u_1) - F_2(t, u_2)\| \leq \mu_2(t) \|u_1 - u_2\|, \quad \forall u_1, u_2 \in X, \quad \forall t \in \mathbb{R}.$$

Let  $\beta_2(t) = \int_{-\infty}^t \frac{\mu_2(s)}{1+|\omega|(t-s)^\alpha} ds$ . If there is a constant  $\gamma > 0$  such that

$$(4.10) \quad \mu_1 + C(\theta, \alpha)M\beta_2(t) \leq \gamma < 1,$$

for all  $t \in \mathbb{R}$ , where  $C(\theta, \alpha)$  and  $M$  are the constants in (4.7). Then Eq. (4.1) has a unique  $(\omega, c)$ -periodic mild solution.

*Proof.* Note that the operator  $F : P_{\omega,c}(\mathbb{R}, X) \rightarrow P_{\omega,c}(\mathbb{R}, X)$  given by

$$(4.11) \quad (F\varphi)(t) = F_1(t, \varphi(t)) + \int_{-\infty}^t S_\alpha(t-s)F_2(s, \varphi(s)) ds,$$

is well defined. Indeed, let  $\varphi \in P_{\omega,c}(\mathbb{R}, X)$ , then in view of Theorem 2.6, the functions  $t \rightarrow F_1(t, \varphi(t))$  and  $t \rightarrow F_2(t, \varphi(t))$  belong to  $P_{\omega,c}(\mathbb{R}, X)$ . By Lemma 4.7 we obtain that  $F\varphi \in P_{\omega,c}(\mathbb{R}, X)$ .

On the other hand, if  $\varphi_1, \varphi_2 \in P_{\omega,c}(\mathbb{R}, X)$ , we have

$$\begin{aligned} \|F\varphi_1(t) - F\varphi_2(t)\| &\leq \|F_1(t, \varphi_1(t)) - F_1(t, \varphi_2(t))\| \\ &\quad + \int_{-\infty}^t \|S_\alpha(t-s)\| \|F_2(s, \varphi_1(s)) - F_2(s, \varphi_2(s))\| ds \\ &\leq \mu_1 \|\varphi_1 - \varphi_2\|_\infty + C(\theta, \alpha)M\beta_2(t) \|\varphi_1 - \varphi_2\|_\infty \\ &\leq \gamma \|\varphi_1 - \varphi_2\|_\infty, \end{aligned}$$

hence  $F$  is a  $\gamma$ -contraction. Now, the assertion is a consequence of the contraction mapping principle. The proof is now completed.  $\square$



## REFERENCES

1. A. Alvarez, A. Gómez, M. Pinto, *( $\omega, c$ )-periodic functions and mild solutions to abstract fractional integr-differential equations*, Electronic J. Qual. Theory Diff. Equ. (2018), No.16, 1-8; <http://doi.org/10.14232/ejqtde.2018.1.16>
2. A. Alvarez, S. Castillo, M. Pinto, *( $\omega, c$ )-pseudo periodic functions, first order Cauchy problem and Lasota-Ważewska model with ergodic and unbounded oscillating production of cells*, Boundary Value Problems (2019), 2019:106, 1-20; <http://doi.org/10.1186/s13661-019-1217-x>
3. Y.Q. Chen, *Anti-periodic solutions solutions for semilinear evolution equations*, J. Anal. Appl. bf 315, N0.1 (2016), 337-348.
4. E. Cuesta, *Asymptotic behavior of the solutions of fractional integro-differential equations and some time discretizations*, Discrete Contin. Dyn. Syst. (Supplement) (2007), 277-285.
5. A. Haraux, *Anti-periodic solutions of some nonlinear evolution equations*, Manuscripta Math. **63**, N0.4 (1989), 479-505
6. A. Lunardi, *Analytic Semigroup and Optimal Regularity in Parabolic Problems*, Progr. Non-linear Differential Equations Appl. Vol. **16**, Birkhäuser-Verlag, Basel, 1995.
7. M.F. Hasler, G.M. N'Guérékata, *Bloch-periodic functions and some applications*, Nonlinear Studies, **21** (1) (2014), 21-30.
8. J.H. Liu, G. N'Guérékata, Nguyen van Minh, *A Massera type theorem for almost automorphic solutions of differential equations*, J. Math. Anal. Appl. **299** (2004), 587-599.
9. G. Mophou, G.M. N'Guérékata, *Pseudo almost automorphic solutions of a neutral functional fractional differetial equation*, Intern. J. Evol. Equ. **4** (2) (2009), 129-139
10. G.M. N'Guérékata, V. Valmorin, *Antiperiodic solutions of semilinear integrodifferential equations in Banach spaces*, Appl. Math. Comput., **218** (2012), 11118-11124
11. G. Mophou, G.M. N'Guérékata, V. Valmorin, *Asymptotic behavior of mild solutions of some fractional functional integro-differential equations*, African Diaspora J. Math., **X**, N. X (2013), 1-13
12. M.F. Hasler, G.M. N'Guérékata, *Bloch-periodic functions and some applications*, Nonlinear Stud. **21**, No.1 (2014), 21-30

LABORATOIRE LAMIA, UNIVERSITÉ DES ANTILLES, CAMPUS FOUILLOLE, 97159 POINTE-À-PITRE, GUADELOUPE, (FWI)

*Email address:* [gisle.mophou@univ-ag.fr](mailto:gisle.mophou@univ-ag.fr)

UNIVERSITY DISTINGUISHED PROFESSOR, DEPARTMENT OF MATHEMATICS, MORGAN STATE UNIVERSITY, BALTIMORE, MD 21251, USA

*Email address:* [Gaston.N'Guerekata@morgan.edu](mailto:Gaston.N'Guerekata@morgan.edu)