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GISÈLE MOPHOU AND GASTON M. N'GUÉRÉKATA

Abstract. We first investigate in this paper further properties of the new concept of (ω, c)-periodic functions; then we apply the results to study the existence of (ω, c)-periodic mild solutions of the fractional differential equations $D_t^\alpha(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1}F_2(t, u(t)),\ t \in \mathbb{R}$, where $1 < \alpha < 2,\ A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, $F_1, F_2 : \mathbb{R} \times X \to X$ are two (ω, c)-periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the sense of Riemann-Liouville.

1. Introduction

Recently, Alvarez et al. [1] introduced the concept of (ω,c)-periodic functions which contains the classes periodic, antiperiodic and Bloch periodic functions among others. This concept is motivated by the Mathieu's equation

$$y''(t) + [a - 2qcos(2t)]y(t) = 0$$

which is used in the modelling of several phenomena including the stability of railroads as trains drive over them, or seasonally forced population dynamics.

In their paper [2], the authors generalize this concept to the one of (ω,c)-pseudo periodic functions which are functions with ergodic parts. Both papers [1, 2] generalize several results in [7, 10, 11, 12].

The aim of this work is to investigate further properties of (ω,c)-periodic functions in Section 2 and their applications to some ordinary differential equations (Section 3) and fractional differential equations of the form $D_t^\alpha(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1}F_2(t, u(t)),\ t \in \mathbb{R}$, where $1 < \alpha < 2$ (Section 4) in a complex Banach space $(X, \|\cdot\|)$. D_t^α denotes the time fractional derivative of order α in the Riemann-Liouville sense.

2. (ω, c)-periodic functions and their properties

In this paper we will consider $(X, \|\cdot\|)$ a complex Banach space, $\omega > 0$, and $c \in \mathbb{C} \setminus \{0\}$.

Definition 2.1. [1, 2] *A function $f \in C(\mathbb{R}, X)$ is said to be (ω,c)-periodic if*

$$f(t + \omega) = cf(t),\ \ \forall\, t \in \mathbb{R}.$$

In this case, ω is called the c-period of f. The set of all (ω,c)-periodic functions $\mathbb{R} \to X$ is denoted $P_{\omega c}(\mathbb{R}, X)$. Examples of ($\omega$, c)-periodic include ω- periodic ($P_\omega(\mathbb{R}, X)$), antiperiodic ($P_{a\omega}(\mathbb{R}, X)$) and Bloch periodic ($B_{p,k}(\mathbb{R}, X)$) functions (cf. for instance [7, 10, 11, 12] for more information on these functions).

Proposition 2.2. [1] *A function $f \in C(\mathbb{R}, X)$ is (ω,c)-periodic if and only if*

$$f(t) = c^{\wedge}(t)u(t), \quad c^{\wedge}(t) := c^{\frac{t}{\omega}}, \quad u \in P_{\omega}(\mathbb{R}, X).$$

Theorem 2.3. [1] *Equipped with the norm $\|f\|_{\omega c} := \sup_{t\in[0,\omega]} \|\|c^{\wedge}(-t)|f(t)\|$, $P_{\omega c}(\mathbb{R}, X)$ turns out to be a Banach space*

Let's state and prove

Theorem 2.4. *Let $f \in P_{\omega,c}(\mathbb{R}, X)$. Then the following are true:*

(i) If $f(t) \neq 0 \quad \forall t \in \mathbb{R}$, then the function $g(t) := \frac{1}{f(t)}, \quad t \in \mathbb{R}$ belongs to $P_{\omega,\frac{1}{c}}(\mathbb{R}, X)$

(ii) The function $f_a(t) := f(t + a), \quad a \in \mathbb{R}$ belongs to $P_{\omega,c}(\mathbb{R}, X)$.

(iii) If $\lambda \in P_{\omega,c}(\mathbb{R}, \mathbb{C})$, then $\lambda f \in P_{\omega,c^2}(\mathbb{R}, X)$

(iv) If $A : X \to X$ is a bounded linear operator, then $Af \in P_{\omega,c}(\mathbb{R}, X)$.

Proof. It comes straightforward from the definition of (ω,c)-periodic functions. □

Lemma 2.5. *For all $(s, u) \in \mathbb{R}\times X$, there exists $\varphi \in P_{\omega,c}(\mathbb{R}, X)$ such that $\varphi(s) = u$*

Proof. Let $\Psi : \mathbb{R} \to X$ be define by $\Psi(t) = c^{\wedge}(t)$. Then obviously $\Psi \in P_{\omega,c}(\mathbb{R}, X)$ with $\Psi(0) = 1$. This implies that the function $\varphi(t) := \Psi(t - s)u$ satisfies the property. □

For a given function $F \in BC(\mathbb{R}, X)$, define the Nemytskii's superposition operator

$$\mathcal{N}_F(\varphi)(\cdot) := F(\cdot, \varphi(\cdot)), \quad \varphi \in P_{\omega,c}(\mathbb{R}, X). \tag{2.1}$$

In [7], the authors proved the following result in the framework of Bloch-periodic functions.This result was generalized in the case of (ω, c)-periodic functions [1]. Here we give a more rigourous proof using the lemma above.

Theorem 2.6. *Let $F \in BC(\mathbb{R}, X)$. The following are equivalent*

(i) For every $\varphi \in P_{\omega,c}(\mathbb{R}, X)$, $\mathcal{N}_F(\varphi) \in P_{\omega,c}(\mathbb{R}, X)$.

(ii) For all $(t, x) \in \mathbb{R} \times X, \quad F(t + \omega, cx) = cF(t, x)$.

Proof. Assume (i) and let $(t, x) \in \mathbb{R} \times X$. Then by Lemma (2.5) there exists $\varphi \in P_{\omega,c}(\mathbb{R}, X)$ such that $\varphi(t) = x$. Let $\Phi(.) := \mathcal{N}(\cdot, \varphi(\cdot))$. Then $\Phi \in P_{\omega,c}(\mathbb{R}, X)$ and using (2.1) we have

$$\begin{aligned} F(t + \omega, cx) &= F(t + \omega, c\varphi(t)) \\ &= F(t + \omega, \varphi(t + \omega)) \\ &= \Phi(t + \omega) \\ &= c\Phi(t) \\ &= c\mathcal{N}_F(\varphi)(t) \\ &= cF(t, \varphi(t)) \end{aligned}$$

which means $F(t + \omega, cx) = cF(t, x)$ because $\varphi(t) = x$.

Conversely assume that $F(t + \omega, cx) = cF(t, x)$ and let $\varphi \in P_{\omega,c}(\mathbb{R}, X)$. Then using (2.1), we have

$$\begin{aligned} \mathcal{N}_F(\varphi)(t + w) &= F(t + \omega, \varphi(t + \omega)) \\ &= F(t + \omega, c\varphi(t)) \\ &= cF(t, \varphi(t)) \\ &= c\mathcal{N}_F(\varphi)(t). \end{aligned}$$

The proof is complete. □

Theorem 2.7. *Suppose $f \in C^1(\mathbb{R}, X) \cap P_{\omega,c}(\mathbb{R}, X)$. Then $f' \in P_{\omega,c}(\mathbb{R}, X)$*

Proof. We have

$$f(t+h) - f(t) = f'(t)h + o(h)$$

where $o(h) = |h|\varepsilon(h)$ with $\lim_{h\to 0} \varepsilon(h) = 0$. But

$$f(t+\omega+h) - f(t+\omega) = c[f(t+h) - f(t)] = cf'(t)h + o(h).$$

Therefore

$$f'(t+\omega) = cf'(t)$$

which means

$$f' \in P_{\omega,c}(\mathbb{R}, X).$$

□

Theorem 2.7 leads to the following corollary:

Corollary 2.8. *Let $F \in C^1(\mathbb{R} \times X, X)$ satisfies one of the equivalent properties (i) and (ii) of Theorem 2.6. Then for every function $\varphi \in C^1(\mathbb{R}, X) \cap P_{\omega,c}(\mathbb{R}, X)$, the function $\Phi : \mathbb{R} \longrightarrow X$ defined by $\Phi(\cdot) = F(\cdot, \varphi(\cdot))$ is (ω, c)-periodic), differentiable and Φ' is (ω, c)-periodic.*

Proposition 2.9. *Let $f \in P_{\omega,c}(\mathbb{R}, X)$ and suppose $(T(t))_{t\geq 0}$ be a C_0-semigroup of linear operators. Then the function $\Upsilon(t)$ defined by*

$$\Upsilon(t) := \int_{-\infty}^{t} T(t-s)f(s)\,\mathrm{d}s$$

is also in $P_{\omega,c}(\mathbb{R}, X)$.

Proof. It is straightforward. Indeed we have.

$$\begin{aligned} \Upsilon(t+\omega) &= \textstyle\int_{-\infty}^{t+\omega} T(t+\omega-s)f(s)\,\mathrm{d}s \\ &= \textstyle\int_{-\infty}^{t} T(t-\sigma)f(\sigma+\omega)\,\mathrm{d}\sigma \\ &= c\textstyle\int_{-\infty}^{t} T(t-\sigma)f(\sigma)\,\mathrm{d}\sigma \\ &= c\Upsilon(t). \end{aligned}$$

This completes the proof.

□

Theorem 2.10. [1] *Let $f \in P_{\omega,c}(\mathbb{R}, X)$ with $f(t) = c^{\wedge}(t)u(t)$, $u \in P_{\omega}(\mathbb{R}, X)$. If $k^{\sim}(t) := c^{\wedge}(-t)k(t) \in L^1(\mathbb{R})$, then $(k \star f) \in P_{\omega c}(\mathbb{R}, X)$, where*

$$(k \star f)(t) = \int_{\mathbb{R}} k(t-\sigma)f(\sigma)\,\mathrm{d}\sigma$$

Let $\varphi \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{C}$. Consider the operator $A_{\lambda,\varphi}$ defined by

$$A_{\lambda,\varphi}x = \lambda x + \varphi \star x.$$

Then it obvious that $A_{\lambda,\varphi}(P_{\omega}(\mathbb{R}, X)) \subset P_{\omega}(\mathbb{R}, X)$. Moreover $A_{\lambda,\varphi}$ acts continuously on $P_{\omega c}(\mathbb{R}, X)$, that means there exists a constant $\gamma > 0$ such that

$$\|A_{\lambda,\varphi}x\| \leq \gamma\|x\|, \quad \forall x \in P_{\omega c}(\mathbb{R}, X).$$

Consider now

$$a(\xi) := \lambda + \hat{\varphi}(\xi)$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of the function φ, $a(\xi)$ is the *symbol* of the operator $A_{\lambda,\varphi}$, where $\varphi \in L^1(\mathbb{R})$. Now since $\lim_{\xi\to\infty} \hat{\varphi}(\xi) = 0$, the symbol $a(\xi)$ is a well-defined continuous function on $\overline{\mathbb{R}} = \mathbb{R} \cup (\infty)$, and $a(\infty) = \lambda$.

Now we state and prove

Theorem 2.11. *Suppose $\varphi \in L^1(\mathbb{R})$. Then the operator $A_{\lambda,\varphi}$ is invertible in $P_{\omega c}(\mathbb{R}, X)$ if $a(\xi) \neq 0$, $\forall \xi \in \overline{\mathbb{R}}$.*

Proof. Suppose $a(\xi) \neq 0$, $\forall \xi \in \bar{\mathbb{R}}$. The function $\frac{1}{a(\xi)}$ is then well-defined on $\overline{\mathbb{R}}$ and in view of the classical Wiener's theorem, we get

$$\frac{1}{a(\xi)} = \frac{1}{\lambda} + \hat{\Psi}(\xi),$$

where $\Psi \in L^1(\mathbb{R})$. It is easy to verify that $A_{\frac{1}{\lambda},\Psi}$ is the inverse to the operator $A_{\lambda,\varphi}$ which acts on $P_{\omega c}(\mathbb{R}, X)$ in view of the remark above.

□

3. Ordinary differential equations

3.1. **Linear case.** Consider the following equation in $\mathbb{C}$

$$u'(t) = \lambda u(t) + h(t), \quad t \in \mathbb{R} \tag{3.1}$$

If $\mathcal{R}e\lambda \neq 0$, the we have either

$$u(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} h(s)\, \mathrm{d}s \ \ (\mathcal{R}e\lambda < 0)$$

or

$$u(t) = \int_{t}^{\infty} e^{\lambda(t-s)} h(s)\, \mathrm{d}s \ \ (\mathcal{R}e\lambda > 0)$$

is the unique solution of Eq. (3.1). We deduce in view of Proposition 2.9 that $u \in P_{\omega,c}(\mathbb{C})$. We have the following Massera type result.

Theorem 3.1. *Let f be (ω, c)-periodic. Then every bounded solution of the differential equation*

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}, \tag{3.2}$$

where $A(t) : \mathbb{R} \to \mathcal{M}_k(\mathbb{C})$ is ω-periodic, is in $P_{\omega,c}(\mathbb{C}^k)$

Proof. The proof is similar to Theorem 3.1 [8]. First let us note that by Floquet's theory and without loss of generality we may assume that $A(t) = A$ is independent of t. Next we will show that the problem can be reduced to the one-dimensional case. In fact, if A is independent of t, by a change of variable if necessary, we may assume that A is of Jordan normal form. In this direction we can go further with assumption that A has only one Jordan box. That is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_k'(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_k(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Let's assume that $\mathcal{R}e\lambda \neq 0$, then consider the last equation

$$x_k'(t) = \lambda x_k(t) + f_k(t),$$

Using the above result, we can show that the solution x_k is in $P_{\omega,c}(\mathbb{C})$. The same way we can prove that x_{k-1} is also in $P_{\omega,c}(\mathbb{C})$ We continue the process to prove that all $x_i,\ i = 1, 2, ...k$ are in $P_{\omega,c}(\mathbb{C})$, which shows that $x \in P_{\omega,c}(\mathbb{C}^k)$

□

3.2. **Semilinear case.** Consider the following equation in $\mathbb{C}$

$$u'(t) = \lambda u(t) + h(t, u(t)),\ \ t \in \mathbb{R} \tag{3.3}$$

where $\mathcal{R}e\lambda \neq 0$ and $f : \mathbb{R} \times X \to X$ satisfies either condition (i) or condition (ii) of Theorem (2.6) and there exists $L > 0$ such that

$$\|h(t, x) - h(t, y)\| \leq L\|x - y\|,\ \ \forall x, y \in X.$$

Then there exists a unique solution of Eq. (2.7) which is in $P_{\omega,c}(X)$ provided $L < |\mathcal{R}e\lambda|$.

Indeed by assumption the operator $\Lambda : P_{\omega,c}(X) \to P_{\omega,c}(X)$ defined by

$$(\Lambda u)(t) := \int_{-\infty}^{t} e^{\lambda(t-s)} h(s, u(s))\, \mathrm{d}s\ \ (\mathcal{R}e\lambda < 0)$$

is well-defined based on the assumption on h. Now if we take $u, b \in \int_{-\infty}^{t} e^{\lambda(t-s)} h(s)\, \mathrm{d}s\ \ (\mathcal{R}e\lambda < 0)$ then

$$\begin{aligned}(\|\Lambda u)(t) - \Lambda v)(t)\| &\leq \int_{-\infty}^{t} \|e^{\lambda(t-s)}(h(s, u(s)) - h(s, v(s)\| \,\mathrm{d}s \\ &\leq L\|u - v\|_\infty \int_{-\infty}^{t} e^{\mathcal{R}e\lambda(t-s)}\, \mathrm{d}s \\ &\leq \frac{L}{|\mathcal{R}e\lambda|}\|u - v\|_\infty\end{aligned}$$

and we conclude using the Banach contraction principle.

4. Fractional Differential equations

We consider the fractional differential equation

$$D_t^\alpha(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1} F_2(t, u(t)),\ t \in \mathbb{R}, \tag{4.1}$$

where $1 < \alpha < 2,\ A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, $F_1, F_2 : \mathbb{R} \times X \to X$ are two (ω, c)-periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the Riemann-Liouville sense.

4.1. **Sectorial operators.** We need to recall some definitions about sectorial operators.

Definition 4.1. *A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in Banach space X is said to be sectorial of type ω and angle θ if there are constants $\omega \in \mathbb{R},\ \theta \in]0, \frac{\pi}{2}[$, and $M > 0$ such that its resolvent exists outside the sector*

$$\omega + \Sigma_\theta := \{\omega + \lambda : \lambda \in \mathbb{C},\ |arg(-\lambda)| < \theta\}, \tag{4.2}$$

$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda - \omega|},\ \lambda \notin \omega + \Sigma_\theta, \tag{4.3}$$

for more details on sectorial operators we refer to [6].

We denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into X endowed with the norm of operators. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X, the notation $\rho(A)$ stands for the resolvent set of A.

To study the fractional differential equation

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t),\ t \in \mathbb{R}, \tag{4.4}$$

where f is an $X-$valued function. We consider the following concept.

Definition 4.2. *We call A the generator of a solution operator (or α-resolvent family) if there are $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subseteq \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,\mathrm{d}t$, $Re\lambda > \omega$, $x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by A. We observe that the power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{iarg\lambda}$, with $-\pi < arg(\lambda) < \pi$.*

We note that if A is sectorial of type ω with $0 \le \theta < \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by $S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\,\mathrm{d}\lambda$, where γ is a suitable path lying outside the sector $\omega + \Sigma_\theta$.

Lemma 4.3. *([4, Theorem 1]) Let $A : D(A) \subseteq X \to X$ be a sectorial operator in a complex Banach space X, satisfying (4.2) and (4.3) for $M > 0$, $\omega < 0$ and $0 \le \theta < \pi(1 - \frac{\alpha}{2})$. Then there is $C(\theta, \alpha) > 0$ depending solely on θ and α, such that*

$$||S_\alpha(t)|| \le \frac{C(\theta,\alpha)M}{1 + |\omega|t^\alpha},\ t \ge 0. \tag{4.5}$$

We recall here the following definition that will be essential in the sequel.

Definition 4.4. *Assume that A is sectorial of type $\omega < 0$ and angle θ satisfying $0 \le \theta < \pi(1 - \frac{\alpha}{2})$. A function $u : \mathbb{R} \to X$ is called a mild solution of (4.1) if the function $s \to S_\alpha(t-s)F_2(s, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = F_1(t, u(t)) + \int_{-\infty}^t S_\alpha(t-s)F_2(s, u(s))\,\mathrm{d}s, \tag{4.6}$$

for any $t \in \mathbb{R}$.

Definition 4.5. *We call A the generator of a solution operator (or α-resolvent family) if there are $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subseteq \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,\mathrm{d}t$, $Re\lambda > \omega$, $x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by A. We observe that the power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{iarg\lambda}$, with $-\pi < arg(\lambda) < \pi$.*

We note that if A is of sectorial of type ω with $0 \le \theta < \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by $S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\,\mathrm{d}\lambda$, where γ is a suitable path lying outside the sector $\omega + \Sigma_\theta$.

Lemma 4.6. *Let $A : D(A) \subseteq X \to X$ be a sectorial operator in a complex Banach space X, satisfying (4.2) and (4.3) for $M > 0$, $\omega < 0$ and $0 \le \theta < \pi(1 - \frac{\alpha}{2})$. Then there is $C(\theta, \alpha) > 0$ depending solely on θ and α, such that*

$$||S_\alpha(t)|| \le \frac{C(\theta,\alpha)M}{1 + |\omega|t^\alpha},\ t \ge 0. \tag{4.7}$$

Lemma 4.7. *Assume that A is sectorial of type $\omega < 0$ and angle θ satisfying $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$. If $f : \mathbb{R} \to X$ is a (ω, c)-periodic function and Γf is given by $(\Gamma f)(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, then $\Gamma f \in P_{\omega,c}(\mathbb{R}, X)$.*

Proof. We have

$$\begin{aligned}(\Gamma f)(t+\omega) &= \int_{-\infty}^{t+\omega} S_\alpha(t+\omega-s)f(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma+\omega)\,\mathrm{d}\sigma \\ &= c\int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma)\,\mathrm{d}\sigma \\ &= c(\Gamma f)(t)\end{aligned}$$

which proves the result. □

Theorem 4.8. *Assume that A is sectorial of type $\omega < 0$ and angle θ satisfying $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$. Let F_1, F_2 satisfy Theorem 2.6 (i) and there are a positive constant μ_1 and positive bounded continuous function $\mu_2(t)$ so that*

$$||F_1(t, u_1) - F_1(t, u_2)|| \leq \mu_1||u_1 - u_2||,\ \forall\ u_1, u_2 \in X,\ \forall\ t \in \mathbb{R}, \tag{4.8}$$

$$||F_2(t, u_1) - F_2(t, u_2)|| \leq \mu_2(t)||u_1 - u_2||,\ \forall\ u_1, u_2 \in X,\ \forall\ t \in \mathbb{R}. \tag{4.9}$$

Let $\beta_2(t) = \int_{-\infty}^{t} \frac{\mu_2(s)}{1+|\omega|(t-s)^\alpha}\,\mathrm{d}s$. If there is a constant $\gamma > 0$ such that

$$\mu_1 + C(\theta, \alpha)M\beta_2(t) \leq \gamma < 1, \tag{4.10}$$

for all $t \in \mathbb{R}$, where $C(\theta, \alpha)$ and M are the constants in (4.7). Then Eq. (4.1) has a unique (ω,c)-periodic mild solution.

Proof. Note that the operator $F : P_{\omega,c}(\mathbb{R}, X) \to P_{\omega,c}(\mathbb{R}, X)$ given by

$$(F\varphi)(t) = F_1(t, \varphi(t)) + \int_{-\infty}^{t} S_\alpha(t-s)F_2(s, \varphi(s))\,\mathrm{d}s, \tag{4.11}$$

is well defined. Indeed, let $\varphi \in P_{\omega,c}(\mathbb{R}, X)$, then in view of Theorem 2.6, the functions $t \to F_1(t, \varphi(t))$ and $t \to F_2(t, \varphi(t))$ belong to $P_{\omega,c}(\mathbb{R}, X)$. By Lemma 4.7 we obtain that $F\varphi \in P_{\omega,c}(\mathbb{R}, X)$.

On the other hand, if $\varphi_1,\ \varphi_2 \in P_{\omega,c}(\mathbb{R}, X)$, we have

$$\begin{aligned}||F\varphi_1(t) - F\varphi_2(t)|| &\leq ||F_1(t, \varphi_1(t)) - F_1(t, \varphi_2(t))|| \\ &+ \int_{-\infty}^{t} ||S_\alpha(t-s)||\,||F_2(s, \varphi_1(s)) - F_2(s, \varphi_2(s))||\,\mathrm{d}s \\ &\leq \mu_1||\varphi_1 - \varphi_2||_\infty + C(\theta, \alpha)M\beta_2(t)||\varphi_1 - \varphi_2||_\infty \\ &\leq \gamma||\varphi_1 - \varphi_2||_\infty,\end{aligned}$$

hence F is a γ-contraction. Now, the assertion is a consequence of the contraction mapping principle. The proof is now completed. □

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