AN EXISTENCE RESULT OF \((\omega, c)\)-PERIODIC MILD SOLUTIONS TO SOME FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. We first investigate in this paper further properties of the new concept of \((\omega, c)\)-periodic functions; then we apply the results to study the existence of \((\omega, c)\)-periodic mild solutions of the fractional differential equations

\[ D^\alpha_t (u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D^{\alpha-1}_t F_2(t, u(t)), \quad t \in \mathbb{R}, \]

where \(1 < \alpha < 2\), \(A : D(A) \subseteq X \to X\) is a linear densely defined operator of sectorial type on a complex Banach space \(X\), \(F_1, F_2 : \mathbb{R} \times X \to X\) are two \((\omega, c)\)-periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the sense of Riemann-Liouville.

1. Introduction

Recently, Alvarez et al. [1] introduced the concept of \((\omega, c)\)-periodic functions which contains the classes periodic, antiperiodic and Bloch periodic functions among others. This concept is motivated by the Mathieu’s equation

\[ y''(t) + [a - 2q\cos(2t)]y(t) = 0 \]

which is used in the modelling of several phenomena including the stability of railroads as trains drive over them, or seasonally forced population dynamics.

In their paper [2], the authors generalize this concept to the one of \((\omega, c)\)-pseudo periodic functions which are functions with ergodic parts. Both papers [1, 2] generalize several results in [7, 10, 11, 12].

The aim of this work is to investigate further properties of \((\omega, c)\)-periodic functions in Section 2 and their applications to some ordinary differential equations (Section 3) and fractional differential equations of the form

\[ D^\alpha_t (u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D^{\alpha-1}_t F_2(t, u(t)), \quad t \in \mathbb{R}, \]

where \(1 < \alpha < 2\) (Section 4) in a complex Banach space \((X, \| \cdot \|)\). \(D^\alpha_t\) denotes the time fractional derivative of order \(\alpha\) in the Riemann-Liouville sense.

2. \((\omega, c)\)-PERIODIC FUNCTIONS AND THEIR PROPERTIES

In this paper we will consider \((X, \| \cdot \|)\) a complex Banach space, \(\omega > 0\), and \(c \in \mathbb{C} \setminus \{0\}\).

Definition 2.1. [1, 2] A function \(f \in C(\mathbb{R}, X)\) is said to be \((\omega, c)\)-periodic if

\[ f(t + \omega) = cf(t), \quad \forall t \in \mathbb{R}. \]

In this case, \(\omega\) is called the \(c\)-period of \(f\). The set of all \((\omega, c)\)-periodic functions \(\mathbb{R} \to X\) is denoted \(P_{\omega c}(\mathbb{R}, X)\). Examples of \((\omega, c)\)-periodic include \((\omega, c)\)-periodic \((P_{\omega}(\mathbb{R}, X))\), antiperiodic \((P_{\omega a}(\mathbb{R}, X))\) and Bloch periodic \((B_{p,k}(\mathbb{R}, X))\) functions (cf. for instance [7, 10, 11, 12] for more information on these functions).
Proposition 2.2. [1] A function \( f \in C(\mathbb{R}, X) \) is \((\omega, c)\)-periodic if and only if \( f(t) = c^\omega(t)u(t), \ c^\omega(t) := c^{\omega t}, \ u \in P_{\omega}(\mathbb{R}, X) \).

Theorem 2.3. [1] Equipped with the norm \( \|f\|_{\omega c} := \sup_{t \in [0, \omega]} ||c^\omega(-t)|f(t)||, P_{\omega c}(\mathbb{R}, X) \) turns out to be a Banach space

Let’s state and prove

Theorem 2.4. Let \( f \in P_{\omega c}(\mathbb{R}, X) \). Then the following are true:

(i) If \( f(t) \neq 0 \ \forall t \in \mathbb{R}, \) then the function \( g(t) := \frac{1}{f(t)}, \ t \in \mathbb{R} \) belongs to \( P_{\omega c}(\mathbb{R}, X) \)

(ii) The function \( f_a(t) := f(t + a), \ a \in \mathbb{R} \) belongs to \( P_{\omega c}(\mathbb{R}, X) \).

(iii) If \( \lambda \in P_{\omega c}(\mathbb{R}, \mathbb{C}) \), then \( \lambda f \in P_{\omega c}(\mathbb{R}, X) \)

(iv) If \( A : X \to X \) is a bounded linear operator, then \( Af \in P_{\omega c}(\mathbb{R}, X) \).

Proof. It comes straightforward from the definition of \((\omega, c)\)-periodic functions. \( \square \)

Lemma 2.5. For all \((s, u) \in \mathbb{R} \times X, \) there exists \( \varphi \in P_{\omega c}(\mathbb{R}, X) \) such that \( \varphi(s) = u \)

Proof. Let \( \Psi : \mathbb{R} \to X \) be define by \( \Psi(t) = c^\omega(t). \) Then obviously \( \Psi \in P_{\omega c}(\mathbb{R}, X) \) with \( \Psi(0) = 1. \) This implies that the function \( \varphi(t) := \Psi(t - s)u \) satisfies the property. \( \square \)

For a given function \( F \in BC(\mathbb{R}, X), \) define the Nemytskii’s superposition operator

\[
N_F(\varphi)(\cdot) := F(\cdot, \varphi(\cdot)), \ \varphi \in P_{\omega c}(\mathbb{R}, X).
\]

In [7], the authors proved the following result in the framework of Bloch-periodic functions. This result was generalized in the case of \((\omega, c)\)-periodic functions [1]. Here we give a more rigourous proof using the lemma above.

Theorem 2.6. Let \( F \in BC(\mathbb{R}, X). \) The following are equivalent

(i) For every \( \varphi \in P_{\omega c}(\mathbb{R}, X), \ N_F(\varphi) \in P_{\omega c}(\mathbb{R}, X). \)

(ii) For all \((t, x) \in \mathbb{R} \times X, \ F(t + \omega, cx) = cF(t, x). \)

Proof. Assume (i) and let \((t, x) \in \mathbb{R} \times X. \) Then by Lemma (2.5) there exists \( \varphi \in P_{\omega c}(\mathbb{R}, X) \) such that \( \varphi(t) = x. \) Let \( \Phi(\cdot) := N(\cdot, \varphi(\cdot)). \) Then \( \Phi \in P_{\omega c}(\mathbb{R}, X) \) and using (2.1) we have

\[
F(t + \omega, cx) = F(t + \omega, c\varphi(t)) = F(t + \omega, \varphi(t + \omega)) = \Phi(t + \omega) = c\Phi(t) = cN_F(\varphi)(t)
\]

which means \( F(t + \omega, cx) = cF(t, x) \) because \( \varphi(t) = x. \)

Conversely assume that \( F(t + \omega, cx) = cF(t, x) \) and let \( \varphi \in P_{\omega c}(\mathbb{R}, X). \) Then using (2.1), we have

\[
N_F(\varphi)(t + w) = F(t + \omega, \varphi(t + \omega)) = F(t + \omega, c\varphi(t)) = cF(t, \varphi(t)) = cN_F(\varphi)(t).
\]
The proof is complete. □

**Theorem 2.7.** Suppose $f \in C^1(\mathbb{R}, X) \cap P_{\omega,c}(\mathbb{R}, X)$. Then $f' \in P_{\omega,c}(\mathbb{R}, X)$

**Proof.** We have

$$f(t+h) - f(t) = f'(t)h + o(h)$$

where $o(h) = |h|\varepsilon(h)$ with $\lim_{h \to 0} \varepsilon(h) = 0$. But

$$f(t + \omega + h) - f(t + \omega) = c[f(t+h) - f(t)] = cf'(t)h + o(h).$$

Therefore

$$f'(t + \omega) = cf'(t)$$

which means

$$f' \in P_{\omega,c}(\mathbb{R}, X).$$

□

Theorem 2.7 leads to the following corollary:

**Corollary 2.8.** Let $F \in C^1(\mathbb{R} \times X, X)$ satisfies one of the equivalent properties (i) and (ii) of Theorem 2.6. Then for every function $\varphi \in C^1(\mathbb{R}, X) \cap P_{\omega,c}(\mathbb{R}, X)$, the function $\Phi : \mathbb{R} \to X$ defined by $\Phi(\cdot) = F(\cdot, \varphi(\cdot))$ is $(\omega, c)$-periodic, differentiable and $\Phi'$ is $(\omega, c)$-periodic.

**Proposition 2.9.** Let $f \in P_{\omega,c}(\mathbb{R}, X)$ and suppose $(T(t))_{t \geq 0}$ be a $C_0$-semigroup of linear operators. Then the function $\Upsilon(t)$ defined by

$$\Upsilon(t) := \int_{-\infty}^{t} T(t-s)f(s) \, ds$$

is also in $P_{\omega,c}(\mathbb{R}, X)$.

**Proof.** It is straightforward. Indeed we have.

$$\Upsilon(t + \omega) = \int_{-\infty}^{t+\omega} T(t+\omega-s)f(s) \, ds$$

$$= \int_{-\infty}^{t} T(t-\sigma)f(\sigma + \omega) \, d\sigma$$

$$= c \int_{-\infty}^{t} T(t-\sigma)f(\sigma) \, d\sigma$$

$$= c\Upsilon(t).$$

This completes the proof. □

**Theorem 2.10.** [1] Let $f \in P_{\omega,c}(\mathbb{R}, X)$ with $f(t) = c^\lambda(t)u(t)$, $u \in P_{\omega}(\mathbb{R}, X)$. If $k^\sim(t) := c^\lambda(-t)k(t) \in L^1(\mathbb{R})$, then $(k \ast f) \in P_{\omega,c}(\mathbb{R}, X)$, where

$$(k \ast f)(t) = \int_{\mathbb{R}} k(t-\sigma)f(\sigma) \, d\sigma$$

Let $\varphi \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{C}$. Consider the operator $A_{\lambda,\varphi}$ defined by

$$A_{\lambda,\varphi}x = \lambda x + \varphi \ast x.$$

Then it obvious that $A_{\lambda,\varphi}(P_{\omega}(\mathbb{R}, X)) \subset P_{\omega}(\mathbb{R}, X)$. Moreover $A_{\lambda,\varphi}$ acts continuously on $P_{\omega,c}(\mathbb{R}, X)$, that means there exists a constant $\gamma > 0$ such that

$$\|A_{\lambda,\varphi}x\| \leq \gamma \|x\|, \quad \forall x \in P_{\omega,c}(\mathbb{R}, X).$$

Consider now

$$a(\xi) := \lambda + \mathcal{F}(\xi)$$
where $\hat{\varphi}(\xi)$ is the Fourier transform of the function $\varphi$, $a(\xi)$ is the symbol of the operator $A_{\lambda,\varphi}$, where $\varphi \in L^1(\mathbb{R})$. Now since $\lim_{\xi \to \infty} \hat{\varphi}(\xi) = 0$, the symbol $a(\xi)$ is a well-defined continuous function on $\mathbb{R} = \mathbb{R} \cup \{\infty\}$, and $a(\infty) = \lambda$.

Now we state and prove

**Theorem 2.11.** Suppose $\varphi \in L^1(\mathbb{R})$. Then the operator $A_{\lambda,\varphi}$ is invertible in $P_{\omega,c}(\mathbb{R},X)$ if $a(\xi) \neq 0$, $\forall \xi \in \mathbb{R}$.

**Proof.** Suppose $a(\xi) \neq 0$, $\forall \xi \in \mathbb{R}$. The function $\frac{1}{a(\xi)}$ is then well-defined on $\mathbb{R}$ and in view of the classical Wiener’s theorem, we get

$$\frac{1}{a(\xi)} = \frac{1}{\lambda} + \hat{\Psi}(\xi),$$

where $\Psi \in L^1(\mathbb{R})$. It is easy to verify that $A_{\frac{1}{a(\xi)},\Psi}$ is the inverse to the operator $A_{\lambda,\varphi}$ which acts on $P_{\omega,c}(\mathbb{R},X)$ in view of the remark above. \[\square\]

3. Ordinary differential equations

3.1. Linear case. Consider the following equation in $\mathbb{C}$

$$u'(t) = \lambda u(t) + h(t), \quad t \in \mathbb{R} \tag{3.1}$$

If $\Re \lambda \neq 0$, the we have either

$$u(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} h(s) \, ds \quad (\Re \lambda < 0)$$

or

$$u(t) = \int_{t}^{\infty} e^{\lambda(t-s)} h(s) \, ds \quad (\Re \lambda > 0)$$

is the unique solution of Eq. (3.1). We deduce in view of Proposition 2.9 that $u \in P_{\omega,c}(\mathbb{C})$. We have the following Massera type result.

**Theorem 3.1.** Let $f$ be $(\omega,c)$-periodic. Then every bounded solution of the differential equation

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}, \tag{3.2}$$

where $A(t) : \mathbb{R} \to M_k(\mathbb{C})$ is $\omega$-periodic, is in $P_{\omega,c}(\mathbb{C}^k)$

**Proof.** The proof is similar to Theorem 3.1 [8]. First let us note that by Floquet’s theory and without loss of generality we may assume that $A(t) = A$ is independent of $t$. Next we will show that the problem can be reduced to the one-dimensional case. In fact, if $A$ is independent of $t$, by a change of variable if necessary, we may assume that $A$ is of Jordan normal form. In this direction we can go further with assumption that $A$ has only one Jordan box. That is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} x_1'(t) \\
 x_2'(t) \\
 \vdots \\
 x_k'(t) \end{pmatrix} = \begin{pmatrix}
 \lambda & 1 & 0 & \cdots & 0 \\
 0 & \lambda & 1 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & 0 & \lambda \\
 \end{pmatrix} \begin{pmatrix} x_1(t) \\
 x_2(t) \\
 \vdots \\
 x_k(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\
 f_2(t) \\
 \vdots \\
 f_k(t) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{3.2}$$
Let’s assume that $\Re \lambda \neq 0$, then consider the last equation
\[ x_k(t) = \lambda x_k(t) + f_k(t), \]
Using the above result, we can show that the solution $x_k$ is in $P_{\omega,c}(\mathbb{C})$. The same way we can prove that $x_{k-1}$ is also in $P_{\omega,c}(\mathbb{C})$. We continue the process to prove that all $x_i$, $i = 1, 2, \ldots, k$ are in $P_{\omega,c}(\mathbb{C}^k)$. 

3.2. Semilinear case. Consider the following equation in $\mathbb{C}$
\[ u'(t) = \lambda u(t) + h(t, u(t)), \quad t \in \mathbb{R} \]
where $\Re \lambda \neq 0$ and $f : \mathbb{R} \times X \to X$ satisfies either condition (i) or condition (ii) of Theorem (2.6) and there exists $L > 0$ such that
\[ \|h(t, x) - h(t, y)\| \leq L\|x - y\|, \forall x, y \in X. \]
Then there exists a unique solution of Eq. (2.7) which is in $P_{\omega,c}(X)$ provided $L < |\Re \lambda|.$

Indeed by assumption the operator $\Lambda : P_{\omega,c}(X) \to P_{\omega,c}(X)$ defined by
\[ (\Lambda u)(t) := \int_{-\infty}^{t} e^{\lambda(t-s)} h(s, u(s)) \, ds \quad (\Re \lambda < 0) \]
is well-defined based on the assumption on $h$. Now if we take $u, v \in \int_{-\infty}^{t} e^{\lambda(t-s)} h(s, u(s)) \, ds$ (\Re \lambda < 0) then
\[ \|\Lambda u(t) - \Lambda v(t)\| \leq \int_{-\infty}^{t} \|e^{\lambda(t-s)}(h(s, u(s)) - h(s, v(s)))\| \, ds \]
\[ \leq L\|u - v\| \int_{-\infty}^{t} e^{\Re \lambda(t-s)} \, ds \]
\[ \leq \frac{L}{|\Re \lambda|} \|u - v\|_{\infty} \]
and we conclude using the Banach contraction principle.

4. Fractional Differential equations

We consider the fractional differential equation
\[ D_{\alpha}^\alpha (u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_{\alpha}^{\alpha - 1} F_2(t, u(t)), \quad t \in \mathbb{R}, \]
where $1 < \alpha < 2$, $A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $X$, $F_1, F_2 : \mathbb{R} \times X \to X$ are two $(\omega, c)$-periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the Riemann-Liouville sense.

4.1. Sectorial operators. We need to recall some definitions about sectorial operators.

Definition 4.1. A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in Banach space $X$ is said to be sectorial of type $\omega$ and angle $\theta$ if there are constants $\omega \in \mathbb{R}$, $\theta \in \left[0, \frac{\pi}{2}\right]$, and $M > 0$ such that its resolvent exists outside the sector
\[ \omega + \Sigma_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, \ |\arg(-\lambda)| < \theta\}, \]
\[ \|\lambda - A^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \lambda \notin \omega + \Sigma_\theta, \]

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for more details on sectorial operators we refer to [6].

We denote by $B(X)$ the space of all bounded linear operators from $X$ into $X$ endowed with the norm of operators. Let $A$ be a closed linear operator with domain $D(A)$ defined on a Banach space $X$, the notation $\rho(A)$ stands for the resolvent set of $A$.

To study the fractional differential equation
\begin{equation}
D_0^\alpha u(t) = Au(t) + D_0^{\alpha-1}f(t), \quad t \in \mathbb{R},
\end{equation}
where $f$ is an $X$-valued function. We consider the following concept.

**Definition 4.2.** We call $A$ the generator of a solution operator (or $\alpha$-resolvent family) if there are $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to B(X)$ such that $\{\lambda^\alpha : \text{Re}\lambda > \omega\} \subseteq \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\,dt$, $\text{Re}\lambda > \omega$, $x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by $A$. We observe that the power function $\lambda^\alpha$ is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i\text{arg}\lambda}$, with $-\pi < \text{arg}\lambda < \pi$.

We note that if $A$ is sectorial of type $\omega$ with $0 \leq \theta < \pi(1 - \frac{\omega}{2})$, then $A$ is the generator of a solution operator given by $S_\alpha(t) = \frac{1}{2\pi i} \int e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda$, where $\gamma$ is a suitable path lying outside the sector $\omega + \Sigma_{\theta}$.

**Lemma 4.3.** ([4, Theorem 1]) Let $A : D(A) \subseteq X \to X$ be a sectorial operator in a complex Banach space $X$, satisfying (4.2) and (4.3) for $M > 0$, $\omega < 0$ and $0 \leq \theta < \pi(1 - \frac{\omega}{2})$. Then there is $C(\theta, \alpha) > 0$ depending solely on $\theta$ and $\alpha$, such that
\begin{equation}
||S_\alpha(t)|| \leq \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha}, \quad t \geq 0.
\end{equation}

We recall here the following definition that will be essential in the sequel.

**Definition 4.4.** Assume that $A$ is sectorial of type $\omega < 0$ and angle $\theta$ satisfying $0 \leq \theta < \pi(1 - \frac{\omega}{2})$. A function $u : \mathbb{R} \to X$ is called a mild solution of (4.1) if the function $s \to S_\alpha(t-s)F_2(t, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and
\begin{equation}
u(t) = F_1(t, u(t)) + \int_{-\infty}^t S_\alpha(t-s)F_2(s, u(s))\,ds,
\end{equation}
for any $t \in \mathbb{R}$.

**Definition 4.5.** We call $A$ the generator of a solution operator (or $\alpha$-resolvent family) if there are $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to B(X)$ such that $\{\lambda^\alpha : \text{Re}\lambda > \omega\} \subseteq \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\,dt$, $\text{Re}\lambda > \omega$, $x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by $A$. We observe that the power function $\lambda^\alpha$ is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i\text{arg}\lambda}$, with $-\pi < \text{arg}\lambda < \pi$.

We note that if $A$ is of sectorial of type $\omega$ with $0 \leq \theta < \pi(1 - \frac{\omega}{2})$, then $A$ is the generator of a solution operator given by $S_\alpha(t) = \frac{1}{2\pi i} \int e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda$, where $\gamma$ is a suitable path lying outside the sector $\omega + \Sigma_{\theta}$.

**Lemma 4.6.** Let $A : D(A) \subseteq X \to X$ be a sectorial operator in a complex Banach space $X$, satisfying (4.2) and (4.3) for $M > 0$, $\omega < 0$ and $0 \leq \theta < \pi(1 - \frac{\omega}{2})$. Then there is $C(\theta, \alpha) > 0$ depending solely on $\theta$ and $\alpha$, such that
\begin{equation}
||S_\alpha(t)|| \leq \frac{C(\theta, \alpha)M}{1 + |\omega|t^\alpha}, \quad t \geq 0.
\end{equation}
Lemma 4.7. Assume that $A$ is sectorial of type $\omega < 0$ and angle $\theta$ satisfying $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$. If $f : \mathbb{R} \rightarrow X$ is a $(\omega, c)$-periodic function and $\Gamma f$ is given by $(\Gamma f)(t) = \int_{-\infty}^{t} S_{\alpha}(t - s) f(s) \, ds$, $t \in \mathbb{R}$, then $\Gamma f \in P_{\omega, c}(\mathbb{R}, X)$.

Proof. We have

$$
(\Gamma f)(t + \omega) = \int_{-\infty}^{t+\omega} S_{\alpha}(t + \omega - s) f(s) \, ds
$$

$$
= \int_{-\infty}^{t} S_{\alpha}(t - \sigma) f(\sigma + \omega) \, d\sigma
$$

$$
= c \int_{-\infty}^{t} S_{\alpha}(t - \sigma) f(\sigma) \, d\sigma
$$

$$
= c(\Gamma f)(t)
$$

which proves the result. 

Theorem 4.8. Assume that $A$ is sectorial of type $\omega < 0$ and angle $\theta$ satisfying $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$. Let $F_{1}, F_{2}$ satisfy Theorem 2.6 (i) and there are a positive constant $\mu_{1}$ and positive bounded continuous function $\mu_{2}(t)$ so that

(4.8) $\|F_{1}(t, u_{1}) - F_{1}(t, u_{2})\| \leq \mu_{1}\|u_{1} - u_{2}\|$, $\forall u_{1}, u_{2} \in X$, $\forall t \in \mathbb{R}$,

(4.9) $\|F_{2}(t, u_{1}) - F_{2}(t, u_{2})\| \leq \mu_{2}(t)\|u_{1} - u_{2}\|$, $\forall u_{1}, u_{2} \in X$, $\forall t \in \mathbb{R}$.

Let $\beta_{2}(t) = \int_{-\infty}^{t} \frac{\mu_{2}(s)}{1 + \|\omega(t - s)\|} \, ds$. If there is a constant $\gamma > 0$ such that

(4.10) $\mu_{1} + C(\theta, \alpha) M \beta_{2}(t) \leq \gamma < 1$,

for all $t \in \mathbb{R}$, where $C(\theta, \alpha)$ and $M$ are the constants in (4.7). Then Eq. (4.1) has a unique $(\omega, c)$-periodic mild solution.

Proof. Note that the operator $F : P_{\omega, c}(\mathbb{R}, X) \rightarrow P_{\omega, c}(\mathbb{R}, X)$ given by

(4.11) $(F\varphi)(t) = F_{1}(t, \varphi(t)) + \int_{-\infty}^{t} S_{\alpha}(t - s) F_{2}(s, \varphi(s)) \, ds,$

is well defined. Indeed, let $\varphi \in P_{\omega, c}(\mathbb{R}, X)$, then in view of Theorem 2.6, the functions $t \rightarrow F_{1}(t, \varphi(t))$ and $t \rightarrow F_{2}(t, \varphi(t))$ belong to $P_{\omega, c}(\mathbb{R}, X)$. By Lemma 4.7 we obtain that $F\varphi \in P_{\omega, c}(\mathbb{R}, X)$.

On the other hand, if $\varphi_{1}, \varphi_{2} \in P_{\omega, c}(\mathbb{R}, X)$, we have

$$
\|F\varphi_{1}(t) - F\varphi_{2}(t)\| \leq \|F_{1}(t, \varphi_{1}(t)) - F_{1}(t, \varphi_{2}(t))\|
$$

$$
+ \int_{-\infty}^{t} \|S_{\alpha}(t - s)\| \|F_{2}(s, \varphi_{1}(s)) - F_{2}(s, \varphi_{2}(s))\| \, ds
$$

$$
\leq \mu_{1}\|\varphi_{1} - \varphi_{2}\|_{\infty} + C(\theta, \alpha) M \beta_{2}(t)\|\varphi_{1} - \varphi_{2}\|_{\infty}
$$

$$
\leq \gamma\|\varphi_{1} - \varphi_{2}\|_{\infty},
$$

hence $F$ is a $\gamma$-contraction. Now, the assertion is a consequence of the contraction mapping principle. The proof is now completed. \qed
References


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