



HAL
open science

Betweenness of partial orders

Bruno Courcelle

► **To cite this version:**

| Bruno Courcelle. Betweenness of partial orders. 2020. ⟨hal-02547130⟩

HAL Id: hal-02547130

<https://hal.science/hal-02547130v1>

Preprint submitted on 20 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Betweenness of partial orders

Bruno Courcelle
LaBRI, CNRS and Bordeaux University,
33405 Talence, France
courcell@labri.fr

April 19, 2020

Abstract: We construct a monadic second-order sentence that characterizes the ternary relations that are the betweenness relations of finite or infinite partial orders. We prove that no first-order sentence can do that. We characterize the partial orders that can be reconstructed from their betweenness relations. We propose a polynomial time algorithm that tests if a finite relation is the betweenness of a partial order.

Keywords : Betweenness, partial order, axiomatization, monadic second-order logic, comparability graph

Introduction

Betweenness is a standard notion in the study of structures such as trees, partial orders and graphs. It is defined as the ternary relation $B(x, y, z)$ expressing that an element y is *between* x and z , in a sense that depends on the considered structure. This relation is easy to understand and axiomatize in first-order (FO) logic for *linear orders*. In particular, a linear order can be uniquely described, up to reversal, from its betweenness relation. However, the notion of *partial betweenness*¹ raises some difficult algorithmic and logical problems ([8], Chapter 9).

Betweenness in *partial orders* is axiomatized in [9] by an infinite set of FO sentences that cannot be replaced by a finite one, as we will prove. In this article, we axiomatize betweenness in partial orders by a *single monadic second-order* (MSO) sentence. We characterize the partial orders that are uniquely reconstructible, up to reversal, from their betweenness relations. We show that an MSO formula can describe *some* partial order P such that $B_P = B_Q$ from

¹The question is whether a given ternary relation B is included in the betweenness relation of a linear order. The corresponding problem is NP-complete. This question is not related with the betweenness of partial orders.

the betweenness relation B_Q of a partial order Q : this definition yields a partial order that may be a proper suborder of Q . We give a polynomial time algorithm to test if a finite ternary structure is the betweenness relation of some partial order and to produce relevant partial orders if this is possible.

Several notions of betweenness in *graphs* have also been investigated and axiomatized. We only refer to the survey [1] that contains a rich bibliography. Another reference is [2]. In previous articles we have studied betweenness in finite or infinite trees, and also in *generalized trees*, defined as the partial orders such that the set of elements larger than any one is linearly ordered [4, 5, 6, 7]. The corresponding betweenness relations are defined from partial orders, but not as in [9] and in the present article.

This work contributes to the understanding of the expressive power of monadic second-order logic in finite and infinite graphs and related relational structures. We refer to [3, 8] for monadic second-order logic.

1 Definitions and known results

All partial orders, graphs and relational structures are finite or countably infinite.

Definitions 1 : *Betweenness.*

To shorten writings $\neq (x_1, x_2, \dots, x_n)$ means that x_1, x_2, \dots, x_n are pairwise distinct.

(a) *Betweenness in linear orders*

Let $L = (V, \leq)$ be a linear order. Its *betweenness relation* B_L is the ternary relation on V defined by :

$$B_L(x, y, z) :\iff x < y < z \text{ or } z < y < x.$$

The following properties hold for $B = B_L$ and all $x, y, z, u \in V$:

$$B1 : B(x, y, z) \Rightarrow \neq (x, y, z).$$

$$B2 : B(x, y, z) \Rightarrow B(z, y, x).$$

$$B3 : B(x, y, z) \Rightarrow \neg B(x, z, y).$$

$$B4 : B(x, y, z) \wedge B(y, z, u) \Rightarrow B(x, y, u) \wedge B(x, z, u).$$

$$B5 : B(x, y, z) \wedge B(x, u, y) \Rightarrow B(x, u, z) \wedge B(u, y, z).$$

$$B6 : \neq (x, y, z) \Rightarrow B(x, y, z) \vee B(x, z, y) \vee B(y, x, z).$$

We get an axiomatization by finitely many universal first-order sentences: if a ternary structure $S = (V, B)$ satisfies these properties, then $B = B_L$ for a linear order $L = (V, \leq)$. We will say that the class of betweenness relations of linear orders is *first-order (FO) definable*. The order $L = (V, \leq)$ and its reversal $L^{rev} := (V, \geq)$ are the only ones whose betweenness relation is B_L , see [4, 5, 8]. We will say that \leq is *uniquely defined, up to reversal* (written *u.t.r.*), or *reconstructible* from its betweenness relation.

(b) *Betweenness in partial orders*

The *betweenness relation* B_P of a partial order $P = (V, \leq)$ – we will also say a *poset*² – is the ternary relation on V defined, as in (a), by :

$$B_P(x, y, z) : \iff x < y < z \text{ or } z < y < x.$$

We denote by $Bet(P)$ the ternary structure (V, B_P) . For all $x, y, z, u, v \in V$, the relation $B = B_P$ satisfies Properties B1 to B5 together with:

$$X : B(x, y, z) \wedge B(u, y, v) \Rightarrow B(x, y, u) \vee B(x, y, v),$$

$$F : B(x, y, z) \wedge B(y, u, v) \Rightarrow B(x, y, u) \vee B(z, y, u),$$

and an infinite set \mathcal{O} of properties expressed by universal first-order sentences. The notation is borrowed to the article by Lihova [9] who proved, conversely, that if a ternary structure $S = (V, B)$ satisfies these properties, then $B = B_P$ for a poset $P = (V, \leq)$ and, of course for its reversal $P^{rev} := (V, \geq)$. We will prove that no finite set of first-order sentences can characterize betweenness in posets. Our proof will use the following examples.

(c) A *B-cycle* is a ternary structure (V, B) such that $V = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$, $n \geq 2$, and B consists of the triples $(a_1, b_1, a_2), (a_2, b_2, a_3), \dots, (a_{n-1}, b_{n-1}, a_n), (a_n, b_n, a_1)$ and their inverse ones, $(a_2, b_1, a_1), (a_3, b_2, a_2), \dots$ so that B2 is satisfied. This structure satisfies Properties B1-B5. If n is even, then $B = B_P$ where $P = (V, \leq)$ is the poset such that :

$$a_1 < b_1 < a_2 > b_2 > a_3 < b_3 < a_4 > \dots a_{n-1} < b_{n-1} < a_n > b_n > a_1,$$

and no other inequality holds except by transitivity (*e.g.* $a_1 < a_2$). If n is odd, no such partial order does exist (cf. Lemma 14). Consider for example the case $n = 3$. A partial order P such that $a_1 < b_1$ and $B_P = B$ must verify $b_1 < a_2 > b_2 > a_3 < b_3 < a_1$ but then, we would have (b_3, a_1, b_1) in B , which is not assumed. The set \mathcal{O} excludes these odd *B-cycles*.

We will prove that the class of betweenness relations of partial orders is monadic second-order (MSO) definable without using the set \mathcal{O} . We will also identify the partial orders that can be reconstructed *u.t.r.* from their betweenness relations, independently of any logical description. We refer to [3, 8] for first-order and monadic second-order logic.

Definitions 2 : *Ternary structures and their Gaifman graphs.*

(a) A *ternary structure* is a pair $S = (V, B)$ such that $B \subseteq V^3$. Its *Gaifman graph*³ is $Gf(S) := (V, E)$ where there is an edge $u - v$ in E if and only if $u \neq v$ and u and v belong to a same triple in B .

We say that S is *connected* if $Gf(S)$ is. If $Gf(S)$ is not connected, then S is the union of the pairwise disjoint induced structures $S[X] := (X, B \cap X^3)$,

²This is an inelegant but short terminology for *partial order*, *partial ordering* or *partially ordered set*.

³This graph is defined similarly for arbitrary relational structures, not only for ternary ones.

called the *connected components* of S , where the sets X are the vertex sets of the connected components of $Gf(S)$.

(b) If $P = (V, \leq)$ is a partial order, its *comparability graph* $Comp(P)$ having vertex set V and an edge $u-v$ if and only if u and v are different and *comparable*, i.e., $u < v$ or $v < u$. It is the Gaifman graph of the binary structure P .

We say that P is *connected* if $Comp(P)$ is. If $Comp(P)$ is not connected, then P is the union of the pairwise disjoint posets $P[X] := (X, \leq \cap X^2)$ where the sets X are the vertex sets of the connected components of $Comp(P)$.

We have $Gf(Bet(P)) \subseteq Comp(P)$. The inclusion may be proper.

If $Gf(Bet(P))$ is connected, then so is $Comp(P)$, but not necessarily conversely, because $Gf(Bet(P))$ has no edge if P has no chain $x < y < z$.

Example 3 : Here is an example where $Gf(Bet(P)) \subset Comp(P)$. Let $P = (V, \leq)$ where $V = \{a, b, c, d, e, f\}$ and \leq is generated by $a < b < c < d$, $e < c, e < f, b < f$, reflexivity and transitivity. The edge $e-f$ of $Comp(P)$ is not in $Gf(Bet(P))$ because e and f do not belong to any chain of size 3. If we remove the clause $e < f$, the resulting partial order has the same betweenness relation as P and $Gf(Bet(P)) = Comp(P)$. We will generalize this observation in Proposition 6.

2 Betweenness in partial orders

Definitions 4 : *Chains, antichains and B-minimality.*

Let $P = (V, \leq)$ be a partial order.

(a) A *chain* (resp. an *antichain*) is a subset X of V that is linearly ordered (resp. where any two elements are incomparable.) Its *size* is $|X| \in \mathbb{N} \cup \{\omega\}$. Maximality of chains and antichains is understood for set inclusion.

We use $A_P(x, y, z)$ to abbreviate $B_P(x, y, z) \vee B_P(x, z, y) \vee B_P(y, x, z)$, meaning that x, y and z belong to a chain of size at least 3.

(b) We say that P is *B-minimal* if $Gf(Bet(P)) = Comp(P)$, equivalently, if every two comparable elements belong to a chain of size at least 3, or, as we will see, that it is the unique minimal poset P such that $Gf(Bet(P)) = Gf(Bet(Q))$ for some poset Q , where posets are related by inclusion of the defining binary relations.

(c) We define $Min(P)$ and $Max(P)$ as the sets of minimal and maximal elements respectively of a poset P . They are its *extremal elements*. An element is *isolated* if it is so in $Comp(P)$, equivalently, if it belongs to $Min(P) \cap Max(P)$.

(d) In a ternary structure $S = (V, B)$ that satisfies Properties B1, B2 and B3 (in order to avoid uninteresting cases), we say that an element x is *extremal* if $B(y, x, z)$ does not hold for any y, z . The extremal elements of a structure $Bet(P)$ are the extremal elements of P . \square

Example 5: Let $P = (V, \leq)$ where $V = \{a, b, c, d, e, f\}$ and \leq is generated by $a < b < c, d < e < f, d < c$, reflexivity and transitivity. Then $Comp(P)$ is connected but $Gf(Bet(P))$ is not. The only orderings on V that yield $Comp(P)$

as comparability graph are P and P^{rev} . The graph $Gf(Bet(P))$ has two connected components with vertex sets $\{a, b, c\}$ and $\{d, e, f\}$. From it, one obtains 4 orderings on V that yield the betweenness structure $Bet(P)$. We will develop this observation. \square

Proposition 6 : Let $P = (V, \leq)$ be a poset and \leq' be defined by :

$x \leq' y$ if and only if $x = y$ or
 $x < y$ and, if $x \in Min(P)$ and $y \in Max(P)$, then $x < z < y$ for
some z .

The poset $\tilde{P} := (V, \leq') \subseteq P$ is B-minimal and $Bet(\tilde{P}) = Bet(P)$. It is the unique minimal poset Q such that $Bet(Q) = Bet(P)$ and $Q \subseteq P$.

Proof : We have $\leq' \subseteq \leq$, hence, $\tilde{P} \subseteq P$. If $x <' y <' z$, then $x < y < z$, hence $x < z$, and if $x \in Min(P)$ and $z \in Max(P)$, we have y between them, hence $x <' z$. Reflexivity and antisymmetry are clear and so we have a partial order.

We have $Bet(\tilde{P}) \subseteq Bet(P)$. However, if $x < y < z$, we have $x <' y <' z$ by the definitions. Hence, \tilde{P} and P have the same chains of size at least 3. In particular, $Bet(\tilde{P}) = Bet(P)$.

If \tilde{P} is not B-minimal, there are x, y such that $x <' y$ and x and y do not belong to any chain of size 3 in \tilde{P} , whence, in P . As $x < y$ this implies that $x \in Min(P)$ and $y \in Max(P)$, but we have $x < z < y$ for some z , hence we have $x <' z <' y$, which contradicts the assumption that x and y do not belong to a chain of size 3 in \tilde{P} .

Assume that $Q = (V, \leq_Q) \subseteq P$ and $Bet(Q) = Bet(P)$. If $x <' y$, then, we have $x <_Q y$: to prove this, we observe that the definitions yield $x < y < z$ or $z < x < y$ or $x < z < y$ of some z . In the first case, $(x, y, z) \in B_P = B_Q$, hence $x <_Q y <_Q z$ because $Q \subseteq P$, and so $x <_Q y$. The proofs are similar for the two other cases. Hence $\tilde{P} \subseteq Q$. Hence, \tilde{P} is the unique minimal poset Q such that $Bet(Q) = Bet(P)$ and $Q \subseteq P$. \square

A partial order P is *B-reconstructible* (that is *reconstructible from its betweenness relation*) if P and P^{rev} are the only ones whose betweenness structure is $Bet(P)$.

Theorem 7 : A partial order is B-reconstructible if and only if it is B-minimal and, either it is connected or it has exactly two connected components that are one without extremal elements and an isolated element. A finite B-reconstructible partial order is B-minimal and connected.

Proof : We first recall the case of a linear order $L = (V, \leq)$ from [5] that is connected and B-minimal. If $a < b$ in V , then \leq is the unique linear order⁴ such that $a < b$ and whose betweenness relation is B_L .

"If" Let $P = (V, \leq)$ be connected and B-minimal, and let B be its betweenness relation. Let C be a maximal chain (it has size at least 3) and a, b in C

⁴This order is FO definable in the structure (V, B_L, a, b) .

such that $a < b$. The linear order (C, \leq) is uniquely determined by $B \cap C^3$ and the pair (a, b) .

Consider another maximal chain D . If $|C \cap D| \geq 2$, we say that C and D *merge*. Then, the order on D is uniquely determined from that on C and the relation $B \cap (C \cup D)^3$.

If $C \cap D = \{c\}$, then either $Min(C) = Min(D) = c$ or $Max(C) = Max(D) = c$, and we say that C and D *join*. The order \leq on D is uniquely determined from that on C and $B \cap (C \cup D)^3$ because we have, for any $d \in D$:

$$c < d \text{ if } c < a \text{ and } \neg B(a, c, d) \text{ and, dually,}$$

$$d < c \text{ if } a < c \text{ and } \neg B(a, c, d).$$

Then, the ordering on D between c and d determines in a unique way the ordering on D . Hence, if there is a sequence of maximal chains $C = C_1, C_2, \dots, C_n$ such that C_i and C_{i+1} merge or join for each i , then the ordering on all of them is determined from the ordering on C .

Finally, we prove that any two comparable elements of V are related in a unique way, provided the ordering of a maximal chain C is fixed. Let $a \in C$ and $a = b_0 - b_1 - b_2 - \dots - b_n$ be a path in the comparability graph. There is a sequence of maximal chains $C = C_1, C_2, \dots, C_n$ such that, for each i , C_i and C_{i+1} merge or join, and b_i and b_{i+1} are in C_{i+1} . The proof is by induction on n . We only consider the first step.

Let C be a maximal chain containing a , and $b = b_1 \notin C$ be such that $a < b$. There is a maximal chain D that contains a and b . If C and D merge, we are done. Otherwise, $C \cap D = \{a\}$. If there is $c \in C$ such that $c < a$, then we can replace D by a maximal chain D' containing c, a and b , and it merges with C ; otherwise, $a = Min(C)$ and by the maximality of D , we have $a = Min(D)$, hence C and D join. The proof is similar if $b < a$. We let C_1 be D or D' in the second case.

It follows that the partial order on V is uniquely determined from B and the linear order on C , because if x, y are adjacent in $Comp(P)$, then there is a sequence $C = C_1, C_2, \dots, C_n$ such that C_i and C_{i+1} merge or join, and $x, y \in C_n$. The relation $x < y$ or $y < x$, is thus determined in a unique way.

As there are on C exactly two linear orders compatible with B , there are exactly two partial orders on V , P and P^{rev} whose betweenness relation is the given B_P .

If P is B-minimal and has one connected component X without extremal elements and one isolated element u , then, the set X can be ordered in exactly two ways, and u must be isolated in any poset Q such that $Bet(Q) = Bet(P)$, otherwise, any ordering $u < v$ or $v < u$ for $v \in X$ would add triples to B_P .

"Only if" Let $P = (V, \leq)$ be not B-minimal. There are $x, y \in V$ that do not belong to any chain of size 3, and such that $x < y$. Hence, $x \in Min(P)$ and $y \in Max(P)$. If we remove from \leq the pair (x, y) , as in the definition of \tilde{P} , we obtain a poset with same betweenness relation as P and that is not the reversal of P .

If P is B-minimal and has two connected components that are not singletons, then each of them can be ordered in two ways while giving the same betweenness relation as P . Hence, V can be ordered in at least 4 ways, hence, P is not B-reconstructible.

Let P be B-minimal. If it has two isolated elements u and v we can order them $u < v$, or $v < u$ or leave them incomparable, without modifying $Bet(P)$. If P has one isolated element u and a component having an extremal element w , then we can order V so that w is maximal and then, we define $u < w$ or leave u isolated. Hence, P is not B-reconstructible.

As a connected component without extremal elements must be infinite, the last assertion holds. \square

Next we prove definability results in MSO logic. We let \mathcal{B}_{PO} be the class of structures $Bet(P)$ for posets P .

Theorem 8: The class \mathcal{B}_{PO} is MSO definable. There is a pair of monadic second-order formulas that defines, for each structure S in \mathcal{B}_{PO} , some partial order P such that $S = Bet(P)$.

Definition 9 : *Cut of a partial order.*

A *cut* of a poset $P = (V, \leq)$ is a partition (L, U) of V such that :

- (i) L is downwards closed and U is upwards closed,
- (ii) Every maximal chain meets L and U .

Note that (U, L) is a cut of the reversal (V, \geq) of P . These cuts are Dedekind cuts in linear orders.

Lemma 10 : Every poset P without isolated element has a cut.

Proof: We let A be a maximal antichain of P . There exists one that is constructible from an enumeration⁵ v_1, v_2, \dots of V . We define

$$U := \{x \in V - Min(P) \mid y \leq x \text{ for some } y \in A\} \text{ and } L := V - U.$$

We prove that (L, U) is a cut. From the definition, U is upwards closed, and so, L is downwards closed. Let C be a maximal chain: it contains a unique element $a \in A$. If $a = Min(C)$, then $a \in Min(P) \subseteq L$. As a is not isolated in P , $a \neq Max(C)$, hence, we have $a < x$ for some $x \in C$, and so $x \in U$. Otherwise, we have $y < a$ for some $y \in C$, hence $a \notin Min(P)$ and so $a \in U$ and $y \notin U$. Hence, (L, U) is a cut. \square

Proof of Theorem 8 :

It follows from Proposition 6 that \mathcal{B}_{PO} is the class of structures $Bet(P)$ for B-minimal posets P .

First part: We first characterize the structures $Bet(P)$ for B-minimal posets P without isolated elements.

Let $P = (V, \leq)$ be B-minimal without isolated elements, and let (L, U) be a cut of it.

⁵We consider finite or countably infinite sets that are *effectively given*, see [8], hence, that have some explicit or implicit enumeration. The Choice Axiom can also be used to assert the existence of an enumeration.

We claim that \leq can be defined from L, U and its betweenness relation B_P by FO formulas⁶.

Claim 1 : For $x, y \in V$, we have $x < y$ if and only if one of following conditions holds, for $B := B_P$:

- (i) $x \in L, y \in U$, and $B(x, y, z) \vee B(x, z, y) \vee B(z, x, y)$ holds for some z ,
- (ii) $x, y \in L$ and $B(x, y, w)$ holds for some $w \in U$,
- (iii) $x, y \in U$ and $B(w, x, y)$ holds for some $w \in L$.

Proof : Let x, y be such that $x < y$. As P is B-minimal, x and y belong to a maximal chain C of size at least 3. This chain contains some z such that $x < y < z, x < z < y$ or $z < x < y$ and meets L and U .

If $x \in L$ and $y \in U$, then each of these three cases can hold and yields respectively $B(x, y, z), B(x, z, y)$ or $B(z, x, y)$. If $x, y \in L$, then, since C meets U , we have some $w \in U$ such that $x < y < w$ and $B(x, y, w)$ holds. If $x, y \in U$ then, since C meets L , we have some $w \in L$ such that $w < x < y$ and $B(w, x, y)$ holds. Hence, we have one of the exclusive cases (i), (ii) or (iii). \square

Let $\varphi(B, L, x, y)$ be the FO formula expressing Conditions (i),(ii) and (iii) of the claim in a ternary structure $S = (V, B)$, where $L \subseteq V$, and U is defined as $V - L$. Then, there exists an FO formula $\psi(B, L)$ expressing the following:

(a) B satisfies properties B1, B2 and B3, and every element of V belongs to some triple in B .

(b) L and its complement $U := V - L$ are not empty,

(c) the binary relation $\{(x, y) \in V \times V \mid S \models \varphi(B, L, x, y)\}$ is a strict partial order $<$ that is B-minimal⁷,

(d) B is the betweenness relation of $<$.

Finally, we let $\theta(B)$ be the MSO sentence $\exists L.\psi(B, L)$.

Claim 2 : For a ternary structure $S = (V, B)$, we have $S \models \theta(B)$ if and only if $S = \text{Bet}(P)$ for a B-minimal poset $P = (V, \leq)$ and S has no isolated elements.

Proof: "If" Let $P = (V, \leq)$ be B-minimal, $B = B_P$ be its betweenness relation, and assume that $\text{Bet}(P)$ has no isolated elements. The poset P has none either and has a cut (L, U) . Properties (a) and (b) hold by the definitions. By Claim 1, $\varphi(B, L, x, y)$ defines the strict partial order $<$ and so, (c) and (d) hold. So we have $S \models \theta(B)$.

Conversely, assume that B and L satisfy $\psi(B, L)$. Properties (c) and (d) hold hence $B = B_P$ for the strict and B-minimal partial order P defined by $\varphi(B, L, x, y)$. By Property (a), S has no isolated elements. \square

Hence, to define a B-minimal partial order P such that $\text{Bet}(P) = S$ where S satisfies $\theta(B)$, we use the following MSO formulas:

$\psi(B, L)$ intended to select in $S = (V, B)$ an appropriate set L .

$\varphi(B, L, x, y)$ that defines the partial order in terms of L assumed to satisfy $\psi(B, L)$.

⁶We allow free set variables, here L , in FO formulas. We make explicit the dependence on the relation B .

⁷I.e., whose associated partial order \leq is B-minimal.

If P is a partial order such that $Bet(P)$ has no isolated elements, then, $Bet(P) \models \theta(B)$ because, by Proposition 6, $Bet(P) = Bet(\tilde{P})$ where \tilde{P} is B-minimal. But the formula $\varphi(B, L, x, y)$ defines in the structure $Bet(P)$ partial orders Q such that $Bet(Q) = Bet(P)$.

Second part : Let be given a structure $S = (V, B)$. It is the union of its connected components $S[X]$ where the sets X are the vertex sets of the connected components of $Gf(S)$. There is nothing to verify for the components which are isolated elements. The others can identified by an MSO formula $\gamma(X)$, cf. [8]. Then, $S \in \mathcal{B}_{PO}$ if and only if each of these components satisfies $\theta(B)$. For this purpose, we translate $\theta(B)$ into a formula θ' such that, for every subset X of V ,

$$S \models \theta'(B, X) \text{ if and only if } S[X] \models \theta(B \cap X^3).$$

This is a classical construction called *relativization of quantifications* to a set X , see *e.g.* [8]. Hence, a structure S belongs to \mathcal{B}_{PO} if and only if:

$$S \models \forall X.(\gamma(X) \implies \theta'(B, X)).$$

Similarly, $\psi(B, L)$ can be transformed into ψ' such that, if $X, L \subseteq V$, then:

$$S \models \psi'(B, L \cap X, X) \text{ if and only if } S[X] \models \psi(B \cap X^3, L \cap X).$$

It follows that, from a set $L \subseteq V$ such that:

$$S \models \forall X.(\gamma(X) \implies \psi'(B, L \cap X, X)),$$

one can define partial orders for the components $S[X]$. We transform $\varphi(B, L, x, y)$ into $\varphi'(B, L, X, x, y)$ that defines a partial order on X , by using $L \cap X$. Then, from L as above, one obtains a strict partial order on V defined by :

$$x < y : \iff S \models \exists X.(x, y \in X \wedge \gamma(X) \wedge \varphi'(B, L, X, x, y))$$

whose betweenness relation is B . This completes the proof of the theorem.

□

Remark 11: A B-minimal poset $P = (V, \leq)$ has several cuts (L, U) . However, from the structure $Bet(P)$, they yield only two orders whose betweenness relation is $Bet(P)$. If $Gf(Bet(P))$ has n connected components that are not singleton, the formulas ψ' and φ' define the 2^n partial orders whose betweenness is $Bet(P)$.

Remark 12 : A partial order is reconstructible, *u.t.r.*, from its comparability graph if and only if this graph is *prime*. This notion is relative to the theory of modular decomposition. Furthermore, there is an MSO formula that defines the two transitive orientations of a prime comparability graph G , equivalently, the two partial orders P such that $G = Comp(P)$. More generally, the class of comparability graphs is MSO definable, and primality is MSO definable. These results are proved in Section 5 of [3]. They concern finite and countably infinite partial orders.

3 Finite partial orders

We give an algorithm that decides in polynomial time whether a finite ternary structure $S = (V, B)$ is $Bet(P)$ for some poset P , and produces one if possible.

Lemma 13 : Let $P = (V, \leq)$ is a finite and B-minimal partial order. For $x, y \in V$, we have:

$x < y$ if and only if
either $B_P(x, y, z)$ holds for some $z \in Max(P)$, or
 $y \in Max(P)$ and $B_P(w, x, y) \vee B_P(x, w, y)$ holds for some $w \in V$.

Proof : The "if" direction is clear, since $Max(P)$ is not empty.

For the converse, assume that $x < y$. The elements x and y belong to a chain of size at least 3 with maximal element $z \in Max(P)$. If $y \neq z$ we have $B_P(x, y, z)$, otherwise $B_P(w, x, y) \vee B_P(x, w, y)$ for some $w \in V$ (the last case is for the case where x is minimal). \square

In a ternary structure $S = (V, B)$ that satisfies Properties B1, B2 and B3, we say that an element $x \in V$ is *extremal* if $B(y, x, z)$ does not hold for any y, z . We let $Ext(S)$ be the graph whose vertex set is the set of extremal elements denoted by V_{Ext} , and $u - v$ is an edge if and only if $B(u, w, v)$ holds for some w (necessarily not in V_{Ext}). It follows that $Ext(S)$ is a subgraph of $Gf(S)$.

Lemma 14 : Let $P = (V, \leq)$ be a finite and B-minimal partial order without isolated elements. The graph $Ext(Bet(P))$ is bipartite, with bipartition $(Max(P), Min(P))$. It is connected if P is.

Proof: Each element of $Max(P) \cup Min(P)$ is extremal in $Bet(P)$. If $w \notin Max(P) \cup Min(P)$ then $u < w < v$ for some u, v and so $B_P(u, w, v)$ holds and w is not extremal in $Bet(P)$.

The set $Max(P) \cap Min(P)$ is empty. If $u - v$ is an edge of $Ext(Bet(P))$, then $B_P(u, w, v)$ holds hence $u < w < v$ or $v < w < u$ and u and v cannot be both in $Max(P)$ or in $Min(P)$. Hence, $Ext(Bet(P))$ is bipartite with bipartition $(Max(P), Min(P))$.

Assume now that P is connected. Consider a sequence of maximal chains C_1, C_2, \dots, C_n such that, for each i , C_i and C_{i+1} merge or join, as in the proof of Theorem 7. We have in $Ext(Bet(P))$ a path

$$Min(C_1) - Max(C_1) = Max(C_2) - Min(C_2) = Min(C_3) - \dots - Min(C_n)$$

or vice-versa by exchanging Min and Max . It follows that $Ext(Bet(P))$ is connected. \square

This lemma proves in particular that the odd B -cycles (cf. Definition 1(c)) are not $Bet(P)$ for any partial order, because their extremal elements are a_1, \dots, a_n forming an odd cycle.

Theorem 15 : There exists a polynomial time algorithm that decides whether a finite ternary structure $S = (V, B)$ is the betweenness structure of a partial order P , and produce one if possible.

Proof : Let be given a finite ternary structure $S = (V, B)$, $n := |V|$ and $m := |B| = O(n^3)$.

Step 1 : In order to eliminate trivial cases, the algorithm first checks Properties B1, B2 and B3, and if they hold, it constructs the graph $Gf(S)$ and determines its connected components. This step takes time $O(n + m)$.

Each connected component is then considered and still denoted by S .

Step 2: The algorithm constructs the graph $Ext(S)$, checks if it is bipartite and if so determines its bipartition (V_1, V_2) that is unique because $Ext(S)$ is connected. If $Ext(S)$ is not bipartite, the algorithm stops and returns a negative answer. This step takes time $O(m)$.

Step 3 : By taking V_1 as intended set of maximal elements, the algorithm define a binary relation \leq as follows:

- $x \leq y$ if and only if either $x = y$,
- or $B(x, y, z)$ holds for some $z \in V_1$,
- or $y \in V_1$ and $B(w, x, y) \vee B(x, w, y)$ holds for some $w \in V - V_1$.

This step take time $O(m)$. If $S = Bet(P)$ for some poset P , then \leq is a B-minimal partial order such that $Bet(Q) = S$, where $Q := (V, \leq)$

Step 4 : The algorithm verifies that $B = Bet(Q)$. This step takes time $O(n^3)$. If $B \neq Bet(Q)$, it can report a failure. \square

This algorithm can perhaps be made quicker by means of clever data-structures.

4 No finite first-order axiomatization of \mathcal{B}_{PO} .

Proposition 16 : The class \mathcal{B}_{PO} is not FO definable, *i.e.*, is not characterized by a single first-order sentence.

We recall that it is by the conjunction of an infinite set of universal first-order sentences [9].

Proof : The proof is based on a reduction to a first-order definability result for languages.

A word w of length $p > 0$ over the alphabet $\{a, b\}$ can be represented by the relational structure $S(w) := ([p], \leq, A)$ whose domain $[p]$ is the set of *positions* of letters, \leq is the standard order and $A \subseteq [p]$ is the set of positions of letter a . Hence $B := [p] - A$ is the set of positions of b . We let $suc(x, y)$ express that y is the position following x ; this relation is FO definable from \leq .

The language $K := \{(ab)^m \mid m \geq 2\}$ is FO definable, which means that there exists an FO sentence κ such that, for every nonempty word w over $\{a, b\}$, we have $w \in K$ if and only if $S(w) \models \kappa$.

We define an FO formula $\beta(x, y, z)$ relative to $S(w)$ that expresses the following property of a triple (x, y, z) of positions of a word w in K :

- $x \in A \wedge suc(x, y) \wedge suc(y, z)$ or $z \in A \wedge suc(z, y) \wedge suc(y, x)$ or
- $x \in A \wedge suc(y, z)$, x is the first position of w and z is the last one or
- $z \in A \wedge suc(y, x)$, z is the first position of w and x is the last one.

These triples and the opposite ones (to satisfy Property B2) form a set $B_w \subseteq [2m]^3$. The structure $T(w) := ([2m], B_w)$ is thus a B -cycle as in Definition 1(c). Its vertices a_1, \dots, a_m are the odd positions, those of letter a , its vertices b_1, \dots, b_m are the even positions, those of letter b .

It follows from Definition 1(c) and Lemma 14 that $T(w) \in \mathcal{B}_{PO}$ if and only if m is even.

Assume now that \mathcal{B}_{PO} is axiomatized by a first-order sentence θ , possibly not universal. This sentence can be translated into an FO sentence φ such that, for every word w over $\{a, b\}$, we have:

$$S(w) \models \kappa \wedge \varphi \text{ if and only if } w \in K \text{ and } T(w) \models \theta$$

$$\text{if and only if } T(w) \in \mathcal{B}_{PO} \text{ if and only if } w = (ab)^{2n} \text{ for some } n.$$

Then the language $L := \{(ab)^{2n} \mid n \geq 1\}$ would be first-order definable, which is not the case by a classical result due to McNaughton, Papert and Schützenberger, see [10].

We get a contradiction hence, the class \mathcal{B}_{PO} is not axiomatizable by a single first-order sentence. \square

Acknowledgement: I thank Maurice Pouzet for his comments from which I obtained the easy proof of Proposition 16.

References

- [1] M. Changat, P. Narasimha-Shenoi, and G. Seethakuttyamma, Betweenness in graphs: A short survey on shortest and induced path betweenness, *AKCE International Journal of Graphs and Combinatorics*, **16** (2019) 96-109. (Available on ScienceDirect.com, <https://doi.org/10.1016/j.akcej.2018.06.007>)
- [2] V. Chvatal, Antimatroids, betweenness, convexity, in *Research Trends in Combinatorial Optimization*, Springer, 2008, pp. 57-64.
- [3] B. Courcelle, The monadic second-order logic of graphs XV: On a conjecture by D. Seese. *J. Appl. Log.* **4** (2006) 79-114.
- [4] B. Courcelle, Several notions of rank-width for countable graphs, *J. Combinatorial Theory B* **123** (2017) 186-214
- [5] B. Courcelle, Algebraic and logical descriptions of generalized trees, *Logical Methods in Computer Science* **13** (2017) Issue 3.
- [6] B. Courcelle, Axiomatization of betweenness in order-theoretic trees, February 2019, <https://hal.archives-ouvertes.fr/hal-02205829>
- [7] B. Courcelle, Betweenness in order-theoretic trees, to appear in *Fields of Logic and Computation III, Lecture Notes in Computer Science*, Springer 2020.

- [8] B. Courcelle and J. Engelfriet, *Graph structure and monadic second-order logic, a language theoretic approach*, Cambridge University Press, 2012.
- [9] J. Lihova, Strict-order betweenness, *Acta Univ. M. Belii Ser. Math.* **8** (2000) 27-33. Available from <https://actamath.savbb.sk/acta0804.shtml>.
- [10] W. Thomas. Classifying regular events in symbolic logic. *J. Comput. Syst. Sci.*, **25** (1982)360–376