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## SPATIAL ASYMPTOTICS OF MILD SOLUTIONS TO THE TIME-DEPENDENT OSEEN SYSTEM

PAUL DEURING

Univ. du Littoral Côte d'Opale  
EA 2797 – LMPA – Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville  
F-62228 Calais, France

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**ABSTRACT.** We consider mild solutions to the 3D time-dependent Oseen system with homogeneous Dirichlet boundary conditions, under weak assumptions on the data. Such solutions are defined via the semigroup generated by the Oseen operator in  $L^q$ . They turn out to be also  $L^q$ -weak solutions to the Oseen system. On the basis of known results about spatial asymptotics of the latter type of solutions, we then derive pointwise estimates of the spatial decay of mild solutions. The rate of decay depends in particular on  $L^p$ -integrability in time of the external force.

**1. Introduction.** In this article, we consider mild solutions to the 3D time-dependent Oseen system

$$u' - \Delta_x u + \tau \partial_{x_1} u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty), \quad (1)$$

where  $\overline{\Omega}^c := \mathbb{R}^3 \setminus \overline{\Omega}$ , with  $\Omega$  an open, bounded set in  $\mathbb{R}^3$  with smooth boundary. Thus  $\overline{\Omega}^c$  is an exterior domain, which we suppose to be connected. Equation (1) is supplemented by homogeneous Dirichlet boundary conditions on  $\partial\Omega$  and an initial condition,

$$u(t)|_{\partial\Omega} = 0 \quad \text{for } t \in (0, \infty), \quad u(0) = U_0. \quad (2)$$

The Oseen system is a linearization of the time-dependent Navier-Stokes system with Oseen term,

$$u' - \Delta_x u + \tau \partial_{x_1} u + (u \cdot \nabla_x) u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty). \quad (3)$$

This latter system is usually considered as a model for the flow of a viscous incompressible fluid around a rigid body moving with constant velocity and without rotation, with the set  $\Omega$  corresponding to the rigid body. The functions  $u : \overline{\Omega}^c \times (0, \infty) \mapsto \mathbb{R}^3$  (velocity) and  $\pi : \overline{\Omega}^c \times (0, \infty) \mapsto \mathbb{R}$  (pressure) are the unknowns of problem (1), (2), whereas the functions  $f : \overline{\Omega}^c \times (0, \infty) \mapsto \mathbb{R}^3$  (volume force) and  $U_0 : \overline{\Omega}^c \mapsto \mathbb{R}^3$  (initial velocity), as well as the number  $\tau \in (0, \infty)$  (Reynolds number), are given quantities.

Mild solutions to (1), (2), which only involve the velocity among the two unknowns velocity and pressure, are introduced via the semigroup generated by the Oseen operator. We refer to (5) for the definition of this operator, and to the proof

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of Theorem 3.3 for a discussion of the associated semigroup. A precise definition of a mild solution to (1), (2) is given in Theorem 3.5.

In the work at hand, we study the spatial decay of such solutions. It turned out their decay rate is highest if  $U_0 \in L^q_\sigma(\overline{\Omega}^c)$  and  $f \in L^1(0, \infty, L^q_\sigma(\overline{\Omega}^c))$  for some  $q \in (1, 3/2)$ , and if  $|U_0(x)|$  and  $|f(x, t)|$  tend to zero sufficiently fast for  $|x| \rightarrow \infty$ . (See Section 2 for the definition of  $L^q_\sigma(\overline{\Omega}^c)$ ). In this situation we obtain that

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-(3+|\alpha|)/2} \quad (4)$$

for a. e.  $t \in (0, \infty)$ , a. e.  $x \in B_{R_0}^c := \mathbb{R}^3 \setminus B_{R_0}$ , and for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ , with  $\mathfrak{C}$  being independent of  $x$  and  $t$ . The parameter  $R_0$  is some fixed number from  $(0, \infty)$  with  $\Omega \subset B_{R_0}$ . Concerning the condition  $|\alpha| \leq 1$ , it means that  $u$  and the spatial gradient  $\nabla_x u$  are estimated in (4). The function  $\nu$  appearing in (4) is defined by  $\nu(x) := 1 + |x| - x_1$  for  $x \in \mathbb{R}^3$ . Its presence should be interpreted as a mathematical manifestation of the wake extending downstream in the flow behind a rigid body. If  $f \in L^p(0, \infty, L^q_\sigma(\overline{\Omega}^c))$  for some  $p \in (1, \infty)$  and  $q \in (1, 3/2)$ , the rate of decay of  $|\partial_x^\alpha u(x, t)|$  diminishes to  $-(3 + |\alpha|)/2 + 1/(2p')$ . So in particular this rate is linked to  $L^p$ -integrability in time of  $f$ . If the relation  $U_0 \in L^q_\sigma(\overline{\Omega}^c)$  only holds for some  $q \in [3/2, 3)$ , we also obtain a lower rate, and if  $q \geq 3$ , we have to suppose that  $U_0 = 0$  because otherwise our theory is mute. We refer to Theorem 4.6 for a detailed statement of our results. In the remark following this theorem, we explain the condition mentioned above that  $|U_0(x)|$  and  $|f(x, t)|$  are to tend to zero sufficiently fast for  $|x| \rightarrow \infty$ .

A link between the rate of spatial decay of  $\partial_x^\alpha u$  on the one hand and  $L^p$ -integrability of  $f$  with respect to time on the other already appears in [7, Theorem 6.1] and [8, Theorem 5.2], but with certain  $L^p$ -norms in time and in space of  $u$  additionally influencing the spatial asymptotics of  $u$ . The former reference specifies the spatial decay of regular solutions to (1), and the latter one extends these results to  $L^q$ -weak solutions of (1). No specific boundary conditions are imposed in [7] and [8]. A simplified version of [8, Theorem 5.2] is stated below as Theorem 2.7, which is the starting point of the work at hand. In [7] and [8], we applied [7, Theorem 6.1] and [8, Theorem 5.2], respectively, to some solutions which are known to exist; see [7, Theorem 6.2, 6.3] and [8, Theorem 6.1, 6.2, 6.3]. However, these examples are either restricted to an  $L^2$ -framework ([7, Theorem 6.2] and [8, Theorem 6.1, 6.2, 6.3]), or they deal with solutions whose lifespan  $T_0 \in (0, \infty)$  is finite, which are much more regular than required for our theory and are associated with a right-hand side  $f \in L^p(0, T_0, L^q(\overline{\Omega}^c)^3)$  required to satisfy the condition  $p = q$  ([7, Theorem 6.3]). But as the key point of [8, Theorem 6.1, 6.2] we could improve the decay rates obtained in existing literature ([1], [3]).

The work at hand with its focus on mild solutions shows that we may handle solutions with data of low regularity. In fact, these solutions exist for any  $t > 0$  if  $U_0 \in L^q_\sigma(\overline{\Omega}^c)$  and  $f \in L^1_{loc}([0, \infty), L^q_\sigma(\overline{\Omega}^c))$  for some  $q > 1$  (Theorem 3.5). In addition, we are able to express our decay bounds exclusively in terms of the data; no norms of the solution are involved. Moreover, in the assumptions  $U_0 \in L^q_\sigma(\overline{\Omega}^c)$  and  $f \in L^p(0, \infty, L^q_\sigma(\overline{\Omega}^c))$ , a large range of parameters  $q \in (1, \infty)$ ,  $p \in [1, \infty)$  is admitted, and our estimates exhibit how the choice of these parameters influences the spatial asymptotics of the solution. In this respect we recall the example given in (4): if  $q \in (1, 3/2)$ ,  $p = 1$ , then  $|\partial_x^\alpha u(x, t)|$  tends to zero as  $O(|x| \nu(x))^{-(3+|\alpha|)/2}$  for  $|x| \rightarrow \infty$ . This rate is best possible in the sense that it coincides with standard decay estimates of a fundamental solution to (1); see Lemma 2.4.

Our proof of (4) consists in verifying the assumptions of Theorem 2.7, with two main points. Firstly it must be shown that mild solutions are also  $L^q$ -weak solutions as considered in that latter theorem. This is not completely obvious due to the low regularity of  $f$ ; see the proof of Theorem 3.5. Secondly, certain  $L^p$ -norms of  $u$  with respect to space and time variables must be estimated by the data (Corollary 4.3 – 4.5). This is achieved by means of  $L^p - L^q$ -estimates of the Oseen semigroup. In our context the key feature of these estimates is the rate of temporal decay of spatial  $L^q$ -norms of this semigroup. Such rates are derived in [19], [11], [12], [16] and [17]. We will use results established in [19] and [17]; see Theorem 3.3 and 4.1.

We point out that according to [10], the velocity part  $U$  of a solution  $(U, \Pi)$  to the Oseen resolvent system  $-\Delta U + \tau \partial_1 U + \lambda U + \nabla \Pi = F$ ,  $\operatorname{div} U = 0$  in the whole space  $\mathbb{R}^3$  does not satisfy the estimate  $\|U\|_2 \leq C_0 |\lambda|^{-1} \|F\|_2$  with a single constant  $C_0 > 0$  for all  $F \in L^2(\mathbb{R}^3)^3$  and all  $\lambda \in \mathbb{C}$  with  $\Re \lambda > 0$ . As a consequence of this negative result, which arises because small values of  $|\lambda|$  are admitted, an analogous resolvent estimate cannot be expected to hold for solutions to the Oseen resolvent problem in  $\overline{\Omega}^c$ , under whatever boundary conditions. Therefore, in view of [25, Theorem 4.2, point 3.]), it is a safe guess that maximal regularity is not valid for solutions of problem (1), (2). The negative result in [10] is the reason why we discuss some properties of the Oseen operator and its associated semigroup in greater detail; see the proof of Theorem 3.3.

We mention that pointwise spatial decay of solutions to the nonlinear problem (3), (2) is considered in [18], [23], [6], [4], [6] and [9].

**2. Notation. Some auxiliary results.** The symbol  $|\cdot|$  denotes the Euclidean norm of  $\mathbb{R}^n$  for any  $n \in \mathbb{N}$ , and the length  $\alpha_1 + \alpha_2 + \alpha_3$  of a multi-index  $\alpha \in \mathbb{N}_0^3$ . For  $R \in (0, \infty)$ ,  $x \in \mathbb{R}^3$ , put  $B_R(x) := \{y \in \mathbb{R}^3 : |x - y| < R\}$ . In the case  $x = 0$ , we write  $B_R$  instead of  $B_R(0)$ .

The set  $\Omega \subset \mathbb{R}^3$  and the parameter  $\tau \in (0, \infty)$  introduced in Section 1 will be kept fixed throughout. Recall that  $\Omega$  is open and bounded, with smooth boundary and connected complement. Further recall that  $n^{(\Omega)}$  denotes the outward unit normal to  $\Omega$ . We fix a number  $R_0 \in (0, \infty)$  with  $\overline{\Omega} \subset B_{R_0}$ . For  $R \in (0, \infty)$ , we define  $\Omega_R := B_R \setminus \overline{\Omega}$  and  $Z_{R, \infty} := \Omega_R \times (0, \infty)$ .

We additionally recall that also in Section 1, we introduced the weight function  $\nu : \mathbb{R}^3 \mapsto [1, \infty)$  by setting  $\nu(x) := 1 + |x| - x_1$  for  $x \in \mathbb{R}^3$ .

For  $I \subset \mathbb{R}$ , let  $\chi_I$  stand for the characteristic function of  $I$  on  $\mathbb{R}$ . If  $A \subset \mathbb{R}^3$ , we denote by  $A^c$  the complement  $\mathbb{R}^3 \setminus A$  of  $A$  in  $\mathbb{R}^3$ . Put  $e_l := (\delta_{jl})_{1 \leq j \leq 3}$  for  $1 \leq l \leq 3$  (unit vector in  $\mathbb{R}^3$ ). If  $A$  is some nonempty set and  $\gamma : A \mapsto \mathbb{R}$  a function, we set  $|\gamma|_\infty := \sup\{|\gamma(x)| : x \in A\}$ . Let  $p \in [1, \infty)$ ,  $m \in \mathbb{N}$ . If  $A \subset \mathbb{R}^3$  is open, we write  $\|\cdot\|_p$  for the norm of the Lebesgue space  $L^p(A)$ , and  $\|\cdot\|_{m,p}$  for the usual norm of the Sobolev space  $W^{m,p}(A)$  of order  $m$  and exponent  $p$ . For an open set  $B \subset \mathbb{R}^3$ , the spaces  $L^p_{loc}(B)$  and  $W^{m,p}_{loc}(B)$  are defined as the set of all functions  $V$  from  $B$  into  $\mathbb{R}$  or  $\mathbb{C}$  such that  $V|_A \in L^p(A)$  and  $V|_A \in W^{1,p}(A)$ , respectively, for any open, bounded set  $A \subset \mathbb{R}^3$  with  $\overline{A} \subset B$ . We put  $\nabla V := (\partial_k V_j)_{1 \leq j, k \leq 3}$  for  $V \in W^{1,1}_{loc}(B)^3$ .

Let  $n \in \mathbb{N}$  and let  $\mathcal{V}$  be a normed space, with norm denoted by  $\|\cdot\|$ . Then we will use the same notation  $\|\cdot\|$  for the norm of  $\mathcal{V}^n$  defined by  $\|(f_1, \dots, f_n)\| := (\sum_{j=1}^n \|f_j\|^2)^{1/2}$  for  $(f_1, \dots, f_n) \in \mathcal{V}^n$ . The space  $\mathcal{V}^{3 \times 3}$ , as concerns its norm, is identified with  $\mathcal{V}^9$ .

Let  $A \subset \mathbb{R}^3$  be open and  $p \in (1, \infty)$ . Put  $C_{0,\sigma}^\infty(A) := \{V \in C_0^\infty(A)^3 : \operatorname{div} V = 0\}$ . We write  $L^p_\sigma(A)$  for the closure of  $C_{0,\sigma}^\infty(A)$  with respect to the norm of  $L^p(A)^3$ . This

function space  $L^p_\sigma(A)$  ("space of solenoidal  $L^p$ -functions") is equipped with the norm  $\|\cdot\|_p$ .

Let  $p \in [1, \infty]$  and  $\mathfrak{B}$  be a Banach space. For any interval  $J \subset \mathbb{R}$ , the notation  $\|\cdot\|_{L^p(J, \mathfrak{B})}$  stands for the norm of  $L^p(J, \mathfrak{B})$ . Let  $a, b \in \mathbb{R} \cup \{\infty\}$  with  $a < b$ . Then we write  $L^p(a, b, \mathfrak{B})$  instead of  $L^p((a, b), \mathfrak{B})$ . The expression  $L^p_{loc}([a, b], \mathfrak{B})$  denotes the space of all functions  $v : (a, b) \mapsto \mathfrak{B}$  such that  $v|_{(a, T)} \in L^p(a, T, \mathfrak{B})$  for any  $T \in (a, b)$ . This space is to be distinguished from the space  $L^p_{loc}(a, b, \mathfrak{B})$ , defined in the usual way. Let  $T \in (0, \infty]$ ,  $A \subset \mathbb{R}^3$  open,  $p \in [1, \infty]$ ,  $q \in (1, \infty)$  and  $n \in \{1, 3\}$ . Then we write  $\|\cdot\|_{q, p; T}$  instead of  $\|\cdot\|_{L^p(0, T, L^q(A)^n)}$ . For an interval  $J \subset \mathbb{R}$  and a function  $v : J \mapsto W^{1,1}_{loc}(A)^3$ , the notation  $\nabla_x v$  stands for the gradient of  $v$  with respect to  $x \in A$ , in the sense that

$$\nabla_x v : J \mapsto L^1_{loc}(A)^{3 \times 3}, \quad \nabla_x v(t)(x) := \left( \partial_{x_k}(v_j(t))(x) \right)_{1 \leq j, k \leq 3} \quad \text{for } t \in J, x \in A$$

(spatial gradient of  $v$ ). Similar conventions are to be valid with respect to the expressions  $\Delta_x v$ ,  $\operatorname{div}_x v$  and  $\partial_{x_j} v$ .

For the definition of the Bochner integral, we refer to [26, p. 132-133], or to [15, p. 78-80].

We write  $C$  for numerical constants and  $C(\gamma_1, \dots, \gamma_n)$  for constants depending exclusively on parameters  $\gamma_1, \dots, \gamma_n \in [0, \infty)$  for some  $n \in \mathbb{N}$ . However, such a precise bookkeeping will be possible only at some places. Mostly we will use the symbol  $\mathfrak{C}$  for constants whose dependence on parameters is not indicated. Sometimes we write  $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$  in order to indicate that the constant in question is influenced in particular but not exclusively by the quantities  $\gamma_1, \dots, \gamma_n$ . However, whenever the symbol  $\mathfrak{C}$  appears, it stands for a constant that does not depend on the quantities in a list introduced by the word "for" and preceding or following the respective inequality. In particular, such a constant never depends on the variable  $t$ .

We state an estimate involving the function  $\nu$ .

**Lemma 2.1.** *The inequality  $\nu(x) \leq C(1 + |y|)\nu(x - y)$  holds for  $x, y \in \mathbb{R}^3$ .*

*Proof.* Let  $x, y \in \mathbb{R}$ . If  $|x| - x_1 \leq 4|y|$ , we get  $\nu(x) \leq 1 + 4|y| \leq 4(1 + |y|)\nu(x - y)$ . On the other hand, if  $|x| - x_1 \geq 4|y|$ , hence  $\nu(x) \geq 4|y|$ , we have

$$\nu(x) = 1 + |x - y + y| - (x - y + y)_1 \leq \nu(x - y) + 2|y| \leq \nu(x - y) + \nu(x)/2,$$

so  $\nu(x) \leq 2\nu(x - y)(1 + |y|)$ .  $\square$

The Helmholtz-Fujita decomposition of  $L^q(\overline{\Omega}^c)^3$  will play an important role in what follows. The ensuing theorem serves to introduce this decomposition, fix the related notation, and indicate which properties of the operators in question will be used.

**Theorem 2.2.** *For  $q \in (1, \infty)$ , there is a linear bounded operator  $\mathcal{P}_q : L^q(\overline{\Omega}^c)^3 \mapsto L^q_\sigma(\overline{\Omega}^c)$  with  $\mathcal{P}_q(V) = V$  for  $V \in L^q_\sigma(\overline{\Omega}^c)$ . Moreover  $\mathcal{P}'_q = \mathcal{P}_q$  for  $q \in (1, \infty)$ .*

*Proof.* See [14, Section III.1], [5, Corollary 2.3].  $\square$

We state two well-known properties of Bochner integrals in view of clarifying some arguments further below.

**Theorem 2.3.** *Let  $\mathfrak{B}$  be a Banach space and  $f : \mathbb{R} \mapsto \mathfrak{B}$  a Bochner integrable function. Then  $\int_{\mathbb{R}} \|f(s + h) - f(s)\|_{\mathfrak{B}} ds \rightarrow 0$  for  $h \rightarrow 0$ , where  $\|\cdot\|_{\mathfrak{B}}$  denotes the norm of  $\mathfrak{B}$ .*

Let  $\mathfrak{A}$  be another Banach space,  $A : \mathfrak{B} \mapsto \mathfrak{A}$  a linear and bounded operator,  $J \subset \mathbb{R}$  an interval and  $f : J \mapsto \mathfrak{B}$  a Bochner integrable mapping. Then  $A \circ f : J \mapsto \mathfrak{A}$  is Bochner integrable, too, and  $A(\int_J f dx) = \int_J A \circ f dx$ , where the integral on the left-hand side is  $\mathfrak{B}$ -valued and the one on the right-hand side  $\mathfrak{A}$ -valued.

*Proof.* See [15, Theorem 3.8.3], [26, p. 134, Corollary 2], [15, Theorem 3.7.12].  $\square$

We define some fundamental solutions. Set  $\mathfrak{H}(z, t) := (4\pi t)^{-3/2} e^{-|z|^2/(4t)}$  for  $z \in \mathbb{R}^3$ ,  $t \in (0, \infty)$  (heat kernel),

$$\Gamma_{jk}(z, t) := \mathfrak{H}(z, t) \delta_{jk} + \int_t^\infty \partial_{z_j} \partial_{z_k} \mathfrak{H}(z, s) ds \quad \text{for } z \in \mathbb{R}^3, t \in (0, \infty), 1 \leq j, k \leq 3$$

(fundamental solution to the time-dependent Stokes system), and

$$\Lambda_{jk}^{(\tau)}(z, t) := \Gamma_{jk}(z - \tau t e_1, t) \quad \text{for } z, t, j, k \text{ as before.}$$

(fundamental solution to the time-dependent Oseen system (1)) We will need the following estimate of  $\Lambda^{(\tau)}$ .

**Lemma 2.4** ([7, Corollary 3.3]). *Let  $K > 0$ . Then for  $z \in B_K^c$ ,  $t \in (0, \infty)$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 2$ , the estimate  $|\partial_z^\alpha \Lambda^{(\tau)}(z, t)| \leq C(K, \tau) (|z| \nu(z) + t)^{-(3+|\alpha|)/2}$  holds.*

The potential functions introduced in the two ensuing lemmas are needed in order to state the decay result from [8] which we will apply later on (proof of Theorem 4.6).

**Lemma 2.5** ([7, Corollary 3.5]). *Let  $q \in [1, \infty)$  and  $V \in L^q(\mathbb{R}^3)^3$ . Then the integral  $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda^{(\tau)}(x - y, t) V(y)| dy$  is finite for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ ,  $x \in \mathbb{R}^3$ ,  $t \in (0, \infty)$ . Define the function  $\mathfrak{J}^{(\tau)}(V) : \mathbb{R}^3 \mapsto \mathbb{R}^3$  by setting*

$$\mathfrak{J}^{(\tau)}(V)(x, t) := \int_{\mathbb{R}^3} \Lambda^{(\tau)}(x - y, t) \cdot V(y) dy \quad \text{for } x \in \mathbb{R}^3, t \in (0, \infty).$$

*The derivative  $\partial_{x_i} \mathfrak{J}^{(\tau)}(V)(x, t)$  exists and equals  $\int_{\mathbb{R}^3} \partial_{x_i} \partial_t^\sigma \Lambda^{(\tau)}(x - y, t) \cdot V(y) dy$  for  $x, t$  as above and for  $l \in \{1, 2, 3\}$ . The functions  $\mathfrak{J}^{(\tau)}(V)$  and  $\partial_l \mathfrak{J}^{(\tau)}(V)$  are continuous in  $\mathbb{R}^3 \times (0, \infty)$ .*

*If  $A \subset \mathbb{R}^3$  is measurable and  $V \in L^q(A)^3$ , the term  $\mathfrak{J}^{(\tau)}(V)$  is defined in an obvious way via the zero extension of  $V$  to  $\mathbb{R}^3$ .*

**Lemma 2.6** ([7, Lemma 3.8]). *Let  $q \in [1, \infty)$  and  $f \in L_{loc}^1([0, \infty), L^q(\mathbb{R}^3)^3)$ . Then the integral  $\int_0^t \int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda^{(\tau)}(x - y, t - \sigma) \cdot f(y, \sigma)| dy d\sigma$  is finite for a. e.  $t \in (0, \infty)$ , a. e.  $x \in \mathbb{R}^3$  and for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ . Thus we may define*

$$\mathfrak{R}^{(\tau)}(f)(x, t) := \int_0^t \int_{\mathbb{R}^3} \Lambda^{(\tau)}(x - y, t - \sigma) \cdot f(y, \sigma) dy d\sigma$$

*for such  $t$  and  $x$ . The relation  $\mathfrak{R}^{(\tau)}(f)(t) \in W_{loc}^{1,1}(\mathbb{R}^3)^3$  holds for a. e.  $t \in (0, \infty)$ .*

*If  $T_0 \in (0, \infty]$ ,  $A \subset \mathbb{R}^3$  measurable and  $f \in L_{loc}^1([0, T_0), L^q(A)^3)$ , then  $\mathfrak{R}^{(\tau)}(f)$  is defined in an obvious way via the zero extension of  $f$  to  $\mathbb{R}^3 \times (0, \infty)$ .*

The next theorem states the decay result from [8] we mentioned in Section 1.

**Theorem 2.7.** *Take  $S_0 \in (0, R_0)$  with  $\bar{\Omega} \subset B_{S_0}$ . Let  $\tilde{q}, r_1, r_2, r_3 \in (1, \infty)$  and take functions  $U_0 \in L^{\tilde{q}}(\bar{\Omega}^c)^3$ ,  $f \in L_{loc}^1([0, \infty), L^{r_1}(\bar{\Omega}^c)^3)$  and  $u : (0, \infty) \mapsto W_{loc}^{1,1}(\bar{\Omega}^c)^3$*

with  $u \in C^0([0, \infty), L^{r_2}(\overline{\Omega^c})^3)$ ,  $\nabla_x u \in L^1_{loc}([0, \infty), L^{r_3}(\overline{\Omega^c})^9)$  and  $\operatorname{div}_x u = 0$ . Further suppose that  $u$  satisfies the equation

$$\int_0^{T_0} \int_{\overline{\Omega^c}} \left( -\gamma'(t) u(t) \cdot \vartheta + \gamma(t) [\nabla_x u(t) \cdot \nabla \vartheta + \tau \partial_{x_1} u(t) \cdot \vartheta - f(t) \cdot \vartheta] \right) dx dt - \gamma(0) \int_{\overline{\Omega^c}} U_0 \cdot \vartheta dx = 0 \quad \text{for } \gamma \in C_0^\infty([0, T_0]), \vartheta \in C_{0,\sigma}^\infty(\overline{\Omega^c}).$$

Assume there are numbers  $q \in (1, \infty)$  and  $v_1, v_2, v_3 \in [1, \infty]$  such that the function  $u|_{Z_{R_0, \infty}}$  belongs to  $L^\infty(0, \infty, L^q(Z_{R_0, \infty})^3)$  and to  $L^{v_1}(0, \infty, L^q(Z_{R_0, \infty})^3)$ , the restriction  $\nabla_x u|_{Z_{R_0, \infty}}$  is in  $L^{v_2}(0, \infty, L^q(Z_{R_0, \infty})^9)$ , and  $f|_{Z_{R_0, \infty}}$  belongs to  $L^{v_3}(0, \infty, L^q(Z_{R_0, \infty})^3)$ . Suppose that the zero flux condition  $\int_{\partial \Omega} u(t) \cdot n^{(\Omega)} dx = 0$  holds for  $t \in (0, \infty)$ . Then there is a zero-measure subset  $\mathfrak{S}_\infty$  of  $(0, \infty)$  such that

$$\begin{aligned} & |\partial_x^\alpha [u - \mathfrak{R}^{(\tau)}(f|_{B_{S_0}^c} \times (0, \infty)) - \mathfrak{J}^{(\tau)}(U_0|_{B_{S_0}^c})](x, t)| \\ & \leq \mathfrak{C} (\|u|_{Z_{R_0, \infty}}\|_{q, \infty; \infty} + \|u|_{Z_{R_0, \infty}}\|_{q, v_1; \infty} + \|\nabla_x u|_{Z_{R_0, \infty}}\|_{q, v_2; \infty} \\ & \quad + \|f|_{Z_{R_0, \infty}}\|_{q, v_3; \infty} + \|U_0\|_{\tilde{q}}) (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2 \min\{v'_1, v'_2, v'_3\})} \end{aligned}$$

for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ ,  $t \in (0, \infty) \setminus \mathfrak{S}_\infty$ ,  $x \in \overline{B_{R_0}}^c \setminus N_t$ , where  $N_t$  is some zero-measure subset of  $\overline{B_{R_0}}^c$ . The constant  $\mathfrak{C}$  in (5) does not depend on  $t$  in particular.

*Proof.* This theorem is a somewhat less general version of [8, Theorem 5.2], adapted to what will be needed in Section 4.  $\square$

Under suitable assumptions on  $U_0$ , the potential function  $\mathfrak{J}^{(\tau)}(U_0)(x, t)$  diminishes as  $O([\nu(x)]^{-2})$  for  $|x| \rightarrow \infty$ . Here are the details of this result, which will be interesting in the context of our decay estimate of weak solutions presented in Section 4 (Theorem 4.6). It is the highest rate of decay we could find for  $\mathfrak{J}^{(\tau)}(U_0)$ .

**Lemma 2.8.** *Let  $q \in (3, \infty)$ ,  $U_0 \in L^q_\sigma(\overline{\Omega^c}) \cap L^1_\sigma(\overline{\Omega^c})^3$  with  $\operatorname{supp}(U_0)$  compact. Take  $R \in (0, \infty)$  with  $\operatorname{supp}(U_0) \subset B_R$ . Then*

$$|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0)(x, t)| \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2} \|U_0\|_1$$

for  $x \in B_{2R}^c$ ,  $t \in (0, \infty)$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ .

*Proof.* We apply an approach used by Kozono [21, p. 724] in a different context (temporal decay). By [20, Lemma 2.2], the function  $U_0$  has mean value zero:  $\int_{\overline{\Omega^c}} U_0 dx = 0$ . Take  $x, t, \alpha$  as in the lemma. Then we get for  $y \in B_R$  that  $|x - y| \geq |x|/2 \geq R$  and  $\nu(x) \leq C(1 + |y|)\nu(x - y)$  (Lemma 2.1), so with Lemma 2.4,

$$\begin{aligned} & |\partial_x^\alpha \partial_{y_i} \Lambda^{(\tau)}(x - y, t)| \leq C(\tau, R) (|x - y| \nu(x - y) + t)^{-(4+|\alpha|)/2} \\ & \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2}. \end{aligned}$$

Now we find with Lemma 2.5 that

$$\begin{aligned}
 & \left| \partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0)(x, t) \right| = \left| \int_{B_R} [\partial_x^\alpha \Lambda^{(\tau)}(x - z, t) - \partial_x^\alpha \Lambda^{(\tau)}(x, t)] U_0(z) dz \right| \\
 & = \left| \int_{B_R} \int_0^1 \sum_{l=1}^3 \partial_x^\alpha \partial_{y_l} \Lambda^{(\tau)}(x - y, t) \Big|_{y=\vartheta z} z_l d\vartheta U_0(z) dz \right| \\
 & \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2} \int_{B_R} |z| |U_0(z)| dz \\
 & \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2} \|U_0(z)\|_1.
 \end{aligned}$$

□

**3. Mild solutions of (1), (2).** We begin by recalling some known results, occasionally discussing a proof if the result in question is slightly modified or is not stated clearly in literature.

**Lemma 3.1.** *Let  $(B, \|\cdot\|)$  be a Banach space,  $S : [0, \infty) \mapsto B$  a  $C^0$ -semigroup on  $B$  and  $f \in L^1_{loc}([0, \infty), B)$ . Then  $\int_0^t \|S(t-s)f(s)\| ds < \infty$  for  $t \in (0, \infty)$ . Define  $u(t) := \int_0^t S(t-s)f(s) ds$  for  $t \in (0, \infty)$ . Then  $u \in C^0([0, \infty), B)$ .*

*Proof.* See [15, Theorem 3.8.4] and its proof, and the first statement of Theorem 2.3. □

Next we introduce the Oseen operator  $\mathcal{O}_q : \mathcal{D}(\mathcal{O}_q) \mapsto L^q_\sigma(\overline{\Omega}^c)$  for  $q \in (1, \infty)$  by setting

$$\begin{aligned}
 \mathcal{D}(\mathcal{O}_q) & := L^q_\sigma(\overline{\Omega}^c) \cap W_0^{1,q}(\overline{\Omega}^c)^3 \cap W^{2,q}(\overline{\Omega}^c)^3, \quad \mathcal{O}_q(V) := \mathcal{P}_q(\Delta V - \tau \partial_1 V) \\
 & \text{for } V \in \mathcal{D}(\mathcal{O}_q),
 \end{aligned} \tag{5}$$

where the operator  $\mathcal{P}_q$  was introduced in Theorem 2.2. We denote the identity mapping on  $L^q_\sigma(\overline{\Omega}^c)$  by  $\mathcal{I}_q$ . The ensuing theorem gives some details on the resolvent of  $\mathcal{O}_q$ .

**Theorem 3.2.** *Let  $q \in (1, \infty)$ . Then the resolvent set  $\varrho(\mathcal{O}_q)$  of  $\mathcal{O}_q$  is given by  $\varrho(\mathcal{O}_q) = \{\lambda \in \mathbb{C} : \tau^2 \Re \lambda > -(\Im \lambda)^2\}$ . The relation  $(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F) \in \mathcal{D}(\mathcal{O}_q)$  holds for  $\lambda \in \varrho(\mathcal{O}_q)$  and for  $F \in L^q_\sigma(\overline{\Omega}^c)$ .*

*Let  $\vartheta_0 \in (\pi/2, \pi)$ . There is  $r_0 \in (0, \infty)$  such that*

$$\{\lambda \in \mathbb{C} : |\lambda| \geq r_0, |\arg \lambda| \leq \vartheta_0\} \subset \varrho(\mathcal{O}_q)$$

and

$$|\lambda| \|U\|_q + |\lambda|^{1/2} \|U\|_{1,q} + \|U\|_{2,q} \leq \mathfrak{C} \|F\|_q \quad \text{for } F \in L^q_\sigma(\overline{\Omega}^c) \tag{6}$$

and for  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq r_0$ ,  $|\arg \lambda| \leq \vartheta_0$ , where  $U := (\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F)$ .

*Proof.* The first claim of that theorem, pertaining to  $\varrho(\mathcal{O}_q)$ , holds according to [13, Theorem 3.1]. The relation  $(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F) \in \mathcal{D}(\mathcal{O}_q)$  for  $\lambda \in \varrho(\mathcal{O}_q)$ ,  $F \in L^q_\sigma(\overline{\Omega}^c)$  is obvious by the definition of the resolvent. By [19, Lemma 4.5], there is  $r_0 \in (0, \infty)$  such that  $|\lambda| \|U\|_q + \|U\|_{2,q} \leq \mathfrak{C} \|F\|_q$  for  $\lambda \in \mathbb{C}$ ,  $|\lambda| \geq r_0$ ,  $|\arg \lambda| \leq \vartheta_0$ ,  $F \in L^q_{loc}(\overline{\Omega}^c)$ , with  $U$  defined as above. It follows by interpolation that  $|\lambda|^{1/2} \|\nabla V\|_q \leq \mathfrak{C} \|F\|_q$ . □



The ensuing theorem deals with the semigroup generated by  $\mathcal{O}_q$ . In particular it presents an  $L^q$ - $L^q$ -estimate (inequality (7)), which we take from [19] and [17]. The estimate in the following theorem is “global” in the sense that it gives an upper bound of the Oseen semigroup with respect to  $L^q$ -norms on the exterior domain  $\overline{\Omega}^c$ . In Section 4, we will additionally need “local”  $L^p$ – $L^q$ -estimates, that is, upper bounds for  $L^p$ -norms on  $\Omega_{R_0}$  instead of  $\overline{\Omega}^c$ . These latter estimates yield decay rates which are not always available in the global setting. We further note that [16] and [17] deal with the case of time-dependent coefficients and rotational terms in the differential equations, a level of generality not needed here.

**Theorem 3.3.** *Let  $q \in (1, \infty)$ . The operator  $\mathcal{O}_q$  generates an analytic semigroup on  $L^q_\sigma(\overline{\Omega}^c)$ . We write  $e^{t\mathcal{O}_q}$  for its value in  $t \in [0, \infty)$ . Let  $U \in L^q_\sigma(\overline{\Omega}^c)$ , and put*

$$u_U(t) := e^{t\mathcal{O}_q}U \quad \text{for } t \in [0, \infty).$$

*Then  $u_U \in C^0([0, \infty), L^q_\sigma(\overline{\Omega}^c)) \cap C^\infty((0, \infty), L^q_\sigma(\overline{\Omega}^c))$ ,  $u_U(t) \in \mathcal{D}(\mathcal{O}_q)$ ,  $u'_U(t) = \mathcal{O}_q u_U(t)$  for  $t \in (0, \infty)$ , and  $u_U(0) = U$ . Moreover*

$$\|\partial_x^\alpha u_U(t)\|_q \leq \mathfrak{C} (\chi_{(0,1]}(t) t^{-|\alpha|/2} + \chi_{[1, \infty)}(t) t^{-\min\{1/2, 3/(2q)\}|\alpha|}) \|U\|_q \quad (7)$$

*for  $t \in (0, \infty)$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ . In particular  $\nabla_x u_U \in L^1_{loc}([0, \infty), L^q(\overline{\Omega}^c)^9)$ .*

*Proof.* According to Miyakawa [22, Theorem 4.2], the Oseen operator  $\mathcal{O}_q$  generates an analytical semigroup on  $L^q_\sigma(\overline{\Omega}^c)$ . Since Theorem 3.2 was not yet available in [22], but allows to directly reduce this semigroup property to standard results in [24], we indicate a proof based on such a reduction, for the convenience of the reader, although the argument in question is in principle well known. Take  $\vartheta_0 \in (\pi/2, \pi)$  and choose a number  $r_0$  associated to  $\vartheta_0$  as in Theorem 3.2. By that theorem  $\mathcal{S}_0 := \{\lambda \in \mathbb{C} : |\lambda| \geq r_0, |\arg \lambda| \leq \vartheta_0\} \subset \varrho(\mathcal{O}_q)$  and

$$\|(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F)\|_q \leq \mathfrak{C} |\lambda|^{-1} \|F\|_q \quad \text{for } F \in L^q_\sigma(\overline{\Omega}^c), \lambda \in \mathcal{S}_0. \quad (8)$$

We may choose  $a_0 > 0$  such that  $S_{\vartheta_0, a_0} := \{\lambda \in \mathbb{C} \setminus \{a_0\} : |\arg(\lambda - a_0)| \leq \vartheta_0\} \subset \mathcal{S}_0$ . As a consequence  $S_{\vartheta_0, a_0} \cup \{a_0\} \subset \varrho(\mathcal{O}_q)$  and inequality (8) holds for  $F \in L^q_\sigma(\overline{\Omega}^c)$ ,  $\lambda \in S_{\vartheta_0, a_0} \cup \{a_0\}$ . But for  $\lambda \in S_{\vartheta_0, a_0}$ , we have  $|\lambda| \geq |\lambda - a_0| \sin(\vartheta_0)$ , so we may conclude with (6) that  $\|(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F)\|_q \leq \mathfrak{C} |\lambda - a_0|^{-1} \|F\|_q$  for  $F$  and  $\lambda$  as before. As a consequence the sets  $\{0\}$  and  $S_{\vartheta_0, 0} := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| \leq \vartheta_0\}$  are contained in the resolvent set  $\varrho(-a_0 \mathcal{I}_q + \mathcal{O}_q)$  of the operator  $-a_0 \mathcal{I}_q + \mathcal{O}_q$ , and  $\|(\lambda - (-a_0 \mathcal{I}_q + \mathcal{O}_q))^{-1}(F)\|_q \leq \mathfrak{C} |\lambda|^{-1} \|F\|_q$  for  $F \in L^q_\sigma(\overline{\Omega}^c)$  and  $\lambda \in S_{\vartheta_0, 0}$ . Now it follows that  $-a_0 \mathcal{I}_q + \mathcal{O}_q$  generates an analytic semigroup on  $L^q_\sigma(\overline{\Omega}^c)$  ([24, Theorem 1.7.7, 2.5.2]). We write  $e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$  for its value in  $t \in [0, \infty)$ , so that the mapping  $t \mapsto e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$  ( $t \in [0, \infty)$ ) is the restriction of an analytic semigroup to  $[0, \infty)$ . Then the mapping  $t \mapsto e^{t a_0} e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$  ( $t \in [0, \infty)$ ) is also such a restriction, and the operator  $\mathcal{O}_q$  is the infinitesimal generator of this semigroup. In view of the uniqueness result in [24, Theorem 1.2.6] and the notation introduced in Theorem 3.3, we have  $e^{t\mathcal{O}_q} = e^{t a_0} e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$  for  $t \in [0, \infty)$ . The function  $u_U$  introduced in Theorem 3.3 thus belongs to  $C^0([0, \infty), L^q_\sigma(\overline{\Omega}^c))$  and to  $C^\infty((0, \infty), L^q_\sigma(\overline{\Omega}^c))$ , with  $u_U(t) \in \mathcal{D}(\mathcal{O}_q)$ ,  $u'_U(t) = \mathcal{O}_q u_U(t)$  for  $t \in (0, \infty)$ , and  $u_U(0) = U$  ([24, Corollary 1.2.3, Theorem 2.5.2 (d), Corollary 2.4.4, Lemma 2.4.2]).

Concerning inequality (7), we indicate that it is valid in the case  $t \leq 1$  according to [19, (6.38)], whereas the case  $t \geq 1$  is covered by [19, (1.2)] if  $\alpha = 0$ , [19, (1.3)] if  $|\alpha| = 1$ ,  $q \leq 3$ , and [17, (2.23)] if  $|\alpha| = 1$ ,  $q > 3$ . Actually, in the case

$t \geq 1$ ,  $|\alpha| = 1$ ,  $q > 3$ , another reference would be [19, (1.6)], but that latter inequality is proved only implicitly in [19].  $\square$

**Theorem 3.4.** *Let  $q \in (1, \infty)$  and  $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ . Define*

$$u_f(t) := \int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds \quad \text{for } t \in [0, \infty)$$

(see Lemma 3.1), with the preceding integral denoting an  $L_\sigma^q(\overline{\Omega}^c)$ -valued and an  $L^q(\overline{\Omega}^c)^3$ -valued Bochner integral. Then the integral  $\int_0^t \|\partial_{x_l}(e^{(t-s)\mathcal{O}_q} f(s))\|_q ds$  exists and  $u_f(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$  for a. e.  $t > 0$ . Moreover  $\operatorname{div}_x u_f = 0$ ,  $\nabla_x u_f \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^3)$ , and

$$\partial_{x_l} u_f(t) = \int_0^t \partial_{x_l}(e^{(t-s)\mathcal{O}_q} f(s)) ds \quad \text{for } 1 \leq l \leq 3 \text{ and a. e. } t \in (0, \infty). \quad (9)$$

*Proof.* Of course, it does not make any difference whether the integral in the definition of  $u_f$  is considered as an  $L_\sigma^q(\overline{\Omega}^c)$ -valued or an  $L^q(\overline{\Omega}^c)^3$ -valued Bochner integral, due to Theorem 2.3 and the fact that the canonical imbedding of  $L_\sigma^p(\overline{\Omega}^c)$  into  $L^q(\overline{\Omega}^c)^3$  is linear and bounded. Let  $T \in (0, \infty)$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ , and put  $Z_T := \overline{\Omega}^c \times (0, T)$ . Then by (7),

$$\begin{aligned} \int_0^T \int_0^t \|\partial_x^\alpha(e^{(t-s)\mathcal{O}_q} f(s))\|_q ds dt &\leq \mathfrak{C} \int_0^T \int_0^t (\chi_{(0,1)}(t-s)(t-s)^{-|\alpha|/2} \\ &\quad + \chi_{[1,\infty)}(t-s)) \chi_{(0,T)}(t-s) \|f(s)\|_q ds dt \\ &\leq \mathfrak{C} \int_{\mathbb{R}} (\chi_{(0,1)}(r) r^{-|\alpha|/2} + \chi_{[1,\infty)}(r)) \chi_{(0,T)}(r) dr \|f\|_{Z_T} \|f\|_{q,1;T}, \end{aligned} \quad (10)$$

where we used Young's inequality in the second estimate. Inequality (10) yields in particular that  $\int_0^t \|\partial_x^\alpha(e^{(t-s)\mathcal{O}_q} f(s))\|_q ds < \infty$  for a. e.  $t \in (0, \infty)$  and for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ .

Take any such  $t$ . Let  $l \in \{1, 2, 3\}$  and  $\psi \in C_0^\infty(\overline{\Omega})^3$ . By the second statement in Theorem 2.3 and because  $e^{(t-s)\mathcal{O}_q} f(s) \in \mathcal{D}(\mathcal{O}_q)$  for  $s \in (0, t)$ , we get that

$$\int_{\overline{\Omega}^c} \partial_l \psi(x) \cdot u_f(x, t) dx = - \int_{\overline{\Omega}^c} \psi(x) \cdot \left( \int_0^t \partial_{x_l}(e^{(t-s)\mathcal{O}_q} f(s)) ds \right)(x) dx.$$

Thus the weak derivative  $\partial_{x_l} u_f(t)$  exists and equation (9) holds. Since the integral  $\int_0^t \|\partial_x^\alpha(e^{(t-s)\mathcal{O}_q} f(s))\|_q ds$  is finite by the choice of  $t$ , we thus have  $u_f(t) \in W^{1,q}(\overline{\Omega}^c)^3$ . Equation (9) and inequality (10) imply that the function  $\nabla_x u_f$  belongs to  $L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^3)$ .

In order to show that  $u_f(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$  and  $\operatorname{div}_x u_f(t) = 0$ , again take  $t \in (0, \infty)$  with  $\int_0^t \|\partial_x^\alpha(e^{(t-s)\mathcal{O}_q} f(s))\|_q ds < \infty$  for  $\alpha \in \mathbb{N}_0^3$ ,  $|\alpha| \leq 1$ . We have  $e^{(t-s)\mathcal{O}_q} f(s) \in \mathcal{D}(\mathcal{O}_q) \subset W_0^{1,q}(\overline{\Omega}^c)^3 \cap L_\sigma^q(\overline{\Omega}^c)$  for  $s \in (0, t)$ , in particular  $\operatorname{div}_x(e^{(t-s)\mathcal{O}_q} f(s)) = 0$  by a density argument in  $L_\sigma^q(\overline{\Omega}^c)$ . It follows with (9) that  $\operatorname{div}_x u_f(t) = 0$ . By the choice of  $t$  and because  $e^{(t-s)\mathcal{O}_q} f(s) \in W_0^{1,q}(\overline{\Omega}^c)^3$  we may conclude that the integral  $\int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$  exists also as a  $W_0^{1,q}(\overline{\Omega}^c)^3$ -valued Bochner integral. For  $\psi \in C_0^\infty(\overline{\Omega}^c)^3$ , the operator  $V \mapsto \int_{\overline{\Omega}^c} \psi V dx$  is linear and bounded as a mapping on  $L_\sigma^q(\overline{\Omega}^c)$  and on  $W_0^{1,q}(\overline{\Omega}^c)^3$ . This observation and Theorem 2.3 imply that the integral

$\int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$  yields the same function both as  $L_\sigma^q(\overline{\Omega}^c)$ - and  $W_0^{1,q}(\overline{\Omega}^c)^3$ -valued Bochner integral. As a consequence we have  $u_f(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$ .  $\square$

In the ensuing theorem, we collect some of our previous results. They allow us to introduce the notion of “mild solution”. The theorem then states that such a solution satisfies (1) in the sense of an  $L^q$ -weak solutions as formulated in (5), for a right-hand side  $f$  of low regularity.

**Theorem 3.5.** *Let  $q \in (1, \infty)$ ,  $U_0 \in L_\sigma^q(\overline{\Omega}^c)$  and  $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ . Let the functions  $u_{U_0}$  and  $u_f$  be defined as in Theorem 3.3 and 3.4, respectively, and put  $u := u_{U_0} + u_f$ .*

*This function  $u$  is called a “mild solution” to (1). It satisfies the relations  $u \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ ,  $u(0) = U_0$ ,  $u(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$ ,  $\operatorname{div}_x u(t) = 0$  for  $t \in (0, \infty)$ , and  $\nabla_x u \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^9)$ . Moreover the function  $u$  fulfills equation (5).*

*Proof.* By Theorem 3.3 we know that  $u_{U_0} \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ ,  $u_{U_0}(0) = U_0$  and  $u_{U_0}(t) \in \mathcal{D}(\mathcal{O}_q)$ . Let  $\gamma \in C_0^\infty([0, \infty))$  and  $\vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c)$ . Choose some  $T \in (0, \infty)$  with  $\operatorname{supp}(\gamma) \subset [0, T]$ . The properties of  $u_{U_0}$  listed above imply that the function  $t \mapsto \int_{\overline{\Omega}^c} u_{U_0}(t) \cdot \vartheta dx$  ( $t \in [0, \infty)$ ) belongs to  $C^0([0, \infty))$  and to  $C^\infty((0, \infty))$ , with  $\partial_t(\int_{\overline{\Omega}^c} u_{U_0}(t) \cdot \vartheta dx) = \int_{\overline{\Omega}^c} (u_{U_0})'(t) \cdot \vartheta dx$  for  $t \in (0, \infty)$ , and  $\int_{\overline{\Omega}^c} u_{U_0}(t) \cdot \vartheta dx|_{t=0} = \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx$ . Thus  $\gamma(\epsilon) \int_{\overline{\Omega}^c} u_{U_0}(\epsilon) \cdot \vartheta dx \rightarrow \gamma(0) \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx$  for  $\epsilon \downarrow 0$ . From the preceding relations we get

$$\begin{aligned} \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u_{U_0}(t) \cdot \vartheta dx dt &= \lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma'(t) \int_{\overline{\Omega}^c} u_{U_0}(t) \cdot \vartheta dx dt \\ &= -\lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} (u_{U_0})'(t) \cdot \vartheta dx dt - \gamma(0) \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx. \end{aligned} \quad (11)$$

Since by (7),  $\int_0^T \int_{\overline{\Omega}^c} |\nabla_x u_{U_0}(t) \cdot \nabla \vartheta| dx dt < \infty$ , and because  $u_{U_0}(t) \in W^{2,q}(\overline{\Omega}^c)^3$  for  $t > 0$ , we have

$$\int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \nabla_x u_{U_0}(t) \cdot \nabla \vartheta dx dt = -\lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} \Delta_x u_{U_0}(t) \cdot \vartheta dx ds.$$

But  $\vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c)$ , so  $\mathcal{P}_q'(\vartheta) = \vartheta$  by Theorem 2.2. Therefore due to the equation  $\mathcal{P}_q = \mathcal{P}_q'$  (Theorem 2.2), we get

$$\int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \nabla_x u_{U_0}(t) \cdot \nabla \vartheta dx dt = -\lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} \mathcal{P}_q(\Delta_x u_{U_0}(t)) \cdot \vartheta dx ds. \quad (12)$$

By a similar reasoning we find that

$$\int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \tau \partial_{x_1} u_{U_0}(t) \cdot \vartheta dx dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} \mathcal{P}_q(\tau \partial_{x_1} u_{U_0}(t)) \cdot \vartheta dx ds. \quad (13)$$

Since  $(u_{U_0})'(t) = \mathcal{O}_q u_{U_0}(t)$  for  $t > 0$ , we may conclude from (11) – (13) that equation (5) is valid with  $f = 0$  and with  $u_{U_0}$  in the role of  $u$ . Note that in the preceding argument, the integral  $\int_0^T \gamma(t) \int_{\overline{\Omega}^c} \mathcal{P}_q(\Delta_x u_{U_0}(t)) \cdot \vartheta dx ds$ , which does not exist in general, does not arise.

By Lemma 3.1 we know that  $u_f \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ . Moreover Theorem 3.4 yields that  $u_f(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$ ,  $\operatorname{div}_x u_f(t) = 0$  for a. e.  $t \in (0, \infty)$ , and  $u_f \in$

$L^1_{loc}([0, \infty), L^q(\overline{\Omega}^c)^\vartheta)$ . Take  $\gamma, \vartheta$  and  $T$  as above. By the previous relation and the second claim in Theorem 2.3, we get

$$\begin{aligned} & \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u_f(t) \cdot \vartheta \, dx \, dt \\ &= \int_0^T \int_0^t \int_{\overline{\Omega}^c} \gamma'(t) \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, ds \, dt. \end{aligned} \quad (14)$$

Due to inequality (7), we have  $\|e^{(t-s)\mathcal{O}_q} f(s)\|_q \leq \mathfrak{C}(T) \|f(s)\|_q$  for  $t \in (0, T)$ ,  $s \in (0, t)$ . Moreover the function  $s \mapsto \|f(s)\|_q$  ( $s \in (0, T)$ ) is integrable, the function  $\gamma'$  is bounded, and  $\vartheta$  is bounded with compact support. As a consequence the integral  $\int_0^T \int_0^t \int_{\overline{\Omega}^c} |\gamma'(t) \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q} f(s))(x)| \, dx \, ds \, dt$  exists. Thus we may apply Fubini's and Lebesgue's theorem on the right-hand side of (14), to obtain

$$\begin{aligned} & \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u_f(t) \cdot \vartheta \, dx \, dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma'(t) \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds. \end{aligned} \quad (15)$$

By Theorem 3.3 we know that for  $s \in (0, \infty)$ , the function  $t \mapsto e^{(t-s)\mathcal{O}_q} f(s)$  ( $t \in [s, \infty)$ ) belongs to  $C^0([s, \infty), L^q_\sigma(\overline{\Omega}^c))$  and to  $C^\infty((s, \infty), L^q_\sigma(\overline{\Omega}^c))$ . Hence, for  $s \in (0, \infty)$ , the function  $K_{\vartheta, s}(t) := \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx$  ( $t \in [s, \infty)$ ) belongs to  $C^0([s, \infty)) \cap C^\infty((s, \infty))$ , with  $K'_{\vartheta, s}(t) := \int_{\overline{\Omega}^c} \vartheta(x) \cdot \partial_t (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx$ . Thus the right-hand side in (15) may be transformed by an integration by parts into the term

$$\lim_{\epsilon \downarrow 0} \left[ - \int_0^T \int_{s+\epsilon}^T \gamma(t) K'_{\vartheta, s}(t) \, dt \, ds - \int_0^t \gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^\epsilon \mathcal{O}_q f(s))(x) \, dx \, ds \right].$$

But with Hölder's inequality and (7),

$$\left| \gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^\epsilon \mathcal{O}_q f(s))(x) \, dx \right| \leq \mathfrak{C} |\gamma|_\infty \|\vartheta\|_{q'} \|f(s)\|_q,$$

with the function  $s \mapsto \|f(s)\|_q$  ( $s \in (0, T)$ ) being integrable, as already mentioned before. Moreover, by the continuity of the function  $r \mapsto e^r \mathcal{O}_q V$  ( $r \in [0, \infty)$ ), and because this function takes the value  $V$  if  $r = 0$ , for  $V \in L^q_\sigma(\overline{\Omega}^c)$  (Theorem 3.3), we obtain that  $\gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^\epsilon \mathcal{O}_q f(s))(x) \, dx \rightarrow \gamma(s) \int_{\overline{\Omega}^c} \vartheta(x) \cdot f(x, s) \, dx$  ( $\epsilon \downarrow 0$ ). Therefore

$$\int_0^T \gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^\epsilon \mathcal{O}_q f(s))(x) \, dx \, ds \rightarrow \int_0^T \gamma(s) \int_{\overline{\Omega}^c} \vartheta(x) \cdot f(x, s) \, dx \, ds \quad (\epsilon \downarrow 0)$$

by Lebesgue's theorem. From (15), the transformation of the right-hand side of (15) presented above and the preceding relation, we deduce that

$$\begin{aligned} & \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u_f(t) \cdot \vartheta \, dx \, dt = - \int_0^T \gamma(t) \int_{\overline{\Omega}^c} f(t) \cdot \vartheta \, dx \, dt \\ & \quad - \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \gamma(t) \int_{\overline{\Omega}^c} \vartheta(x) \cdot \partial_t (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds. \end{aligned} \quad (16)$$

Since  $\nabla_x u_f \in L^1_{loc}([0, \infty), L^q(\overline{\Omega}^c)^3)$  by Theorem 3.4, and because of the second statement in Theorem 2.3, we get

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \tau \partial_{x_1} u_f(t) \cdot \vartheta \, dx \, dt \\ &= \int_0^T \gamma(t) \int_0^t \int_{\overline{\Omega}^c} \vartheta(x) \cdot \tau \partial_{x_1} (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, ds \, dt. \end{aligned} \quad (17)$$

By (7),  $\|\nabla_x (e^{r\mathcal{O}_q} f(s))\|_q \leq \mathfrak{C}(T) r^{-1/2} \|f(s)\|_q$  for  $r, s \in (0, T)$ . Thus with Hölder's inequality,

$$\begin{aligned} & \int_0^T \int_0^t \int_{\overline{\Omega}^c} |\gamma(t) \vartheta(x) \cdot \tau \partial_{x_1} (e^{(t-s)\mathcal{O}_q} f(s))(x)| \, dx \, ds \, dt \\ & \leq \mathfrak{C}(T) |\gamma|_\infty \|\vartheta\|_{q'} \int_0^T \int_0^t (t-s)^{-1/2} \|f(s)\|_q \, ds \, dt. \end{aligned}$$

But

$$\begin{aligned} & \int_0^T \int_0^t (t-s)^{-1/2} \|f(s)\|_q \, ds \, dt = \int_0^T \int_s^T (t-s)^{-1/2} \, dt \|f(s)\|_q \, ds \\ & \leq \mathfrak{C}(T) \int_0^T \|f(s)\|_q \, ds. \end{aligned}$$

Since the function  $s \mapsto \|f(s)\|_q$  ( $s \in (0, T)$ ) is integrable, it follows that the integral  $\int_0^T \int_0^t \int_{\overline{\Omega}^c} |\gamma(t) \vartheta(x) \cdot \tau \partial_{x_1} (e^{(t-s)\mathcal{O}_q} f(s))(x)| \, dx \, ds \, dt$  is finite. Thus from (17) and Fubini's and Lebesgue's theorem,

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \tau \partial_{x_1} u_f(t) \cdot \vartheta \, dx \, dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \vartheta(x) \cdot \tau \partial_{x_1} (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds. \end{aligned} \quad (18)$$

The same reasoning yields that

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \nabla_x u_f(t) \cdot \nabla \vartheta \, dx \, dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \nabla \vartheta(x) \cdot \nabla_x (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds. \end{aligned} \quad (19)$$

On the other hand, we have  $e^{(t-s)\mathcal{O}_q} f(s) \in W^{2,q}(\overline{\Omega}^c)^3$  for  $s \in (0, \infty)$ ,  $t \in (s, \infty)$ , hence we obtain

$$\int_{\overline{\Omega}^c} \nabla \vartheta(x) \cdot \nabla_x (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx = - \int_{\overline{\Omega}^c} \vartheta(x) \cdot \Delta_x (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx$$

for  $s, t$  as before. So we may combine (18) and (19) to get

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \left( \nabla_x u_f(t) \cdot \nabla \vartheta + \tau \partial_{x_1} u_f(t) \cdot \vartheta \right) \, dx \, dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \vartheta(x) \cdot [-\Delta_x + \tau \partial_{x_1}] (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds. \end{aligned} \quad (20)$$

At this point we proceed as in the first part of the proof, using the equations  $\mathcal{P}_{q'}(\vartheta) = \vartheta$  and  $\mathcal{P}'_q = \mathcal{P}_{q'}$  provided by Theorem 2.2. Due to them, we may rewrite

(20) as

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \left( \nabla_x u_f(t) \cdot \nabla \vartheta + \tau \partial_{x_1} u_f(t) \cdot \vartheta \right) dx dt \\ &= - \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \vartheta(x) \cdot \mathcal{O}_q \left( e^{(t-s)\mathcal{O}_q} f(s) \right) (x) dx dt ds. \end{aligned} \quad (21)$$

But  $(\partial_t - \mathcal{O}_q)(e^{(t-s)\mathcal{O}_q} f(s)) = 0$  for  $s \in (0, \infty)$ ,  $t \in (s, \infty)$  by Theorem 3.3, so it follows from (16) and (21) that equation (5) holds with  $U_0 = 0$  and  $u_f$  in the role of  $u$ . Since  $u = u_{U_0} + u_f$ , equation (5) holds as stated in the theorem.  $\square$

**4. Spatial decay of mild solutions.** We use  $L^p - L^q$ -estimates of the Oseen semigroup  $e^{t\mathcal{O}_q}$  in order to deduce rates of spatial decay of mild solutions to (1), (2). Our main tools are Theorem 3.3 as well as the following theorem which reproduces results from [19] and [17]. Recall that the parameter  $R_0 \in (0, \infty)$  and the set  $Z_{R_0, \infty}$  were introduced at the beginning of Section 2, and the functions  $u_U$  and  $u_f$  in Theorem 3.3 and 3.4, respectively, for  $U \in L^q_\sigma(\overline{\Omega}^c)$  and  $f \in L^1_{loc}([0, \infty), L^q_\sigma(\overline{\Omega}^c))$ .

**Theorem 4.1** ([19, (6.18)], [17, (6.4)]). *Let  $q \in (1, \infty)$ . Then*

$$\|u_V|_{\Omega_{R_0}}\|_{1,q} \leq \mathfrak{C} t^{-3/(2q)} \|V\|_q \quad \text{for } V \in L^q_\sigma(\overline{\Omega}^c), t \in [1, \infty),$$

with the constant  $\mathfrak{C}$  being independent of  $t$  and  $V$ .

**Corollary 4.2.** *Let  $q \in (1, \infty)$ . Then*

$$\|\partial_x^\alpha u_V|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \left( \chi_{(0,1)}(t) t^{-|\alpha|/2} + \chi_{[1,\infty)}(t) t^{-3/(2q)} \right) \|V\|_q$$

for  $V \in L^q_\sigma(\overline{\Omega}^c)$ , a. e.  $t \in (0, \infty)$  and  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ .

*Proof.* Theorem 3.3 ( $t \leq 1$ ) and 4.1 ( $t > 1$ ).  $\square$

In the ensuing three corollaries, we apply Theorem 3.3 and the preceding corollary in order to estimate the functions  $u_f$  and  $u_{U_0}$ .

**Corollary 4.3.** *Let  $q \in (1, \infty)$ . If  $q < 3/2$ , take  $p_1 \in [1, \infty)$  and set  $p := p_1$ .*

*In the case  $q \geq 3/2$ , let  $p_1 \in [1, (1 - 3/(2q))^{-1})$ . Then  $1 \geq 3/(2q) > 1 - 1/p_1 \geq 0$ . Fix some  $p_2 \in (1, \infty)$  with  $3/(2q) > 1/p_2 > 1 - 1/p_1$ . For example, choose  $p_2 := 2(3/(2q) + 1 - 1/p_1)^{-1}$ . Then  $1 \geq 1/p_1 > 1/p_1 + 1/p_2 - 1 > 0$ . Define  $p := (1/p_1 + 1/p_2 - 1)^{-1}$ . Then  $p_1, p_2, p \in [1, \infty)$  with  $1/p = 1/p_1 + 1/p_2 - 1$ ,  $3p_2/(2q) > 1$  and  $p \in [p_1, \infty)$ . Moreover*

$$\|u_f|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty} \quad \text{and} \quad \|u_f(t)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$$

for  $f \in L^{p_1}(0, \infty, L^q_\sigma(\overline{\Omega}^c))$ ,  $t \in (0, \infty)$ , with  $\mathfrak{C}$  independent of  $t$  and  $f$ .

*Proof.* For any  $f \in L^1_{loc}([0, \infty), L^q_\sigma(\overline{\Omega}^c))$ ,  $t \in (0, \infty)$ , we may deduce from the second claim in Theorem 2.3 and from Corollary 4.2 that

$$\|u_f(t)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \int_0^t g(t-s) \|f(s)\|_q ds, \quad (22)$$

with  $g(r) := \chi_{(0,1)}(r) + \chi_{[1,\infty)}(r) r^{-3/(2q)}$  for  $r \in \mathbb{R}$ . In the case  $q < 3/2$ , put  $p_2 := 1$ . Then we may conclude for any choice of  $q$  that  $p_1, p_2, p \in [1, \infty)$ ,  $1/p = 1/p_1 + 1/p_2 - 1$  and  $3p_2/(2q) > 1$ . The latter inequality yields that  $g \in L^{p_2}(\mathbb{R})$ . At this point we see that inequality (22) and Young's inequality imply the estimate  $\|u_f|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$  for  $f \in L^{p_1}(0, \infty, L^q_\sigma(\overline{\Omega}^c))$ . Since  $p_1 <$

$(1 - 3/(2q))^{-1}$  in the case  $q \geq 3/2$ , and  $3/(2q) > 1$  if  $q < 3/2$ , we have in any case that  $3p'_1/(2q) > 1$ . Therefore (22) and Hölder's inequality yield the estimate of  $\|u_f(t)|\Omega_{R_0}\|_q$  stated in the corollary.  $\square$

**Corollary 4.4.** *Let the numbers  $q, p_1, p_2$  be given in one of the following three ways:*

*Either  $q \in [1, 3/2)$ ,  $p_1 \in [1, \infty)$ ,  $p := p_1$ ,  
or  $q \in [3/2, 3)$ ,  $p_1 \in [1, (1 - 3/(2q))^{-1})$ , and  $p_2$  and  $p$  are defined by  $p_2 := 2(3/(2q) + \max\{1/2, 1 - 1/p_1\})^{-1}$ ,  $p := (1/p_1 + 1/p_2 - 1)^{-1}$ ,  
or  $q \in [3, \infty)$ ,  $p_1 \in [1, (1 - 3/(2q))^{-1})$ ,  $p_2 := 2(3/(2q) + 1 - 1/p_1)^{-1}$  and again  $p := (1/p_1 + 1/p_2 - 1)^{-1}$ .*

*In any case,  $p$  belongs to  $[p_1, \infty)$ . If  $q < 3$ , the inequality*

$$\|(\nabla_x u_f)|Z_{R_0, \infty}\|_{q, p; \infty} \leq \mathfrak{C} \|f\|_{q, p_1; \infty}$$

*holds for  $f \in L^{p_1}(0, \infty, L^q_\sigma(\overline{\Omega}^c))$ . In the case  $q \geq 3$ , the estimate*

$$\|(\nabla_x u_f)|Z_{R_0, \infty}\|_{q, p; \infty} \leq \mathfrak{C} (\|f\|_{q, p_1; \infty} + \|f\|_{q, p; \infty})$$

*is valid for functions  $f$  belonging to  $L^v(0, \infty, L^q_\sigma(\overline{\Omega}^c))$  for both  $v = p_1$  and  $v = p$ .*

*Proof.* Suppose that  $q \in [3/2, 3)$ . Then  $3/(2q) \in (1/2, 1]$ . On the other hand, the assumption  $p_1 \in [1, (1 - 3/(2q))^{-1})$  implies  $3/(2q) > 1 - 1/p_1$ . Therefore we have  $3/(2q) > \max\{1/2, 1 - 1/p_1\} > 0$ , so  $3/(2q) > 1/p_2 > \max\{1/2, 1 - 1/p_1\}$ , hence  $1 > 1/p_2 > 1/2$ , that is,  $p_2 \in (1, 2)$ . Moreover  $3/(2q) > 1/p_2 > 1 - 1/p_1$ , so by Corollary 4.3 we get  $1/p_1 > 1/p_1 + 1/p_2 - 1 > 0$  and  $3p_2/(2q) > 1$ . In particular  $p$  is well defined and belongs to  $[p_1, \infty)$ .

Next assume that  $q \geq 3$ . Then the conditions on  $p_1$  and the definition of  $p_2$  imply that  $3/(2q) > 1/p_2 > 1 - 1/p_1$ , so Corollary 4.3 yields that  $p_2 \in (1, \infty)$ ,  $1/p_1 > 1/p_1 + 1/p_2 - 1 > 0$  and  $3p_2/(2q) > 1$ . In particular  $p$  is again well defined and  $p \in [p_1, \infty)$ .

If  $q < 3/2$ , choose  $p_2 = 1$ . Then  $(1/p_1 + 1/p_2 - 1)^{-1} = p$  by the choice of  $p$ , and  $3p_2/(2q) > 1$  because  $3/(2q) > 1$  in the case under consideration.

Altogether we have for any choice of  $q$  that  $p_1, p_2, p \in [1, \infty)$ ,  $3p_2/(2q) > 1$  and  $1/p = (1/p_1 + 1/p_2 - 1)^{-1}$ . If  $q < 3$ , we additionally have  $p_2 \in [1, 2)$ . Theorem 3.4 and 2.3 yield that  $(\partial_{x_l} u_f(t))|_{\Omega_{R_0}} = \int_0^t [\partial_{x_l} (e^{(t-s)\mathcal{O}_q} f(s))] |_{\Omega_{R_0}} ds$  for  $f \in L^1_{loc}([0, \infty), L^q_\sigma(\overline{\Omega}^c))$ ,  $1 \leq l \leq 3$  and a. e.  $t \in (0, \infty)$ , for any choice of  $q$ ; see the beginning of the proof of Corollary 4.3 as concerns the role of Theorem 2.3. Hence by Corollary 4.2,

$$\|(\nabla_x u_f(t))|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \int_0^t g(t-s) \|f(s)\|_q ds \quad (23)$$

for  $f, t$  as before, with  $g(r) := \chi_{(0,1)}(r) r^{-1/2} + \chi_{[1, \infty)}(r) r^{-3/(2q)}$  for  $r \in \mathbb{R}$ . Since  $p_2 \in [1, 2)$  and  $3p_2/(2q) > 1$  in the case  $q < 3$ , we obtain  $g \in L^{p_2}(\mathbb{R})$  in that case. Recalling that  $p_1, p_2, p \in [1, \infty)$  and  $p = (1/p_1 + 1/p_2 - 1)^{-1}$ , we may thus conclude from (23) and Young's inequality that in the case  $q < 3$ , the estimate  $\|(\nabla_x u_f)|Z_{R_0, \infty}\|_{q, p; \infty} \leq \mathfrak{C} \|f\|_{q, p_1; \infty}$  is valid for  $f \in L^{p_1}(0, \infty, L^q_\sigma(\overline{\Omega}^c))$ .

Now suppose that  $q \geq 3$ . Since in this case the relations  $p = (1/p_1 + 1/p_2 - 1)^{-1}$  and  $3p_2/(2q) > 1$  are valid, too, we may apply Young's inequality once more, to obtain  $(\int_0^\infty (\int_{\mathbb{R}} \chi_{[1, \infty)}(t-s) (t-s)^{-3/(2q)} \|f(s)\|_q ds)^p dt)^{1/p} \leq \mathfrak{C} \|f\|_{q, p_1; \infty}$ . Also by Young's inequality we get  $(\int_0^\infty (\int_{\mathbb{R}} \chi_{(0,1)}(t-s) (t-s)^{-1/2} \|f(s)\|_q ds)^p dt)^{1/p} \leq$

$\mathfrak{C} \|f\|_{q,p;\infty}$ . Due to (23) and the two preceding estimates, we may conclude that  $\|(\nabla_x u_f)|_{Z_{R_0,\infty}}\|_{q,p;\infty} \leq \mathfrak{C} (\|f\|_{q,p_1;\infty} + \|f\|_{q,p;\infty})$ .  $\square$

**Corollary 4.5.** *Let  $q \in (1, \infty)$ . If  $q < 3/2$ , take  $p \in [1, \infty)$ , else let  $p \in (2q/3, \infty)$ . Then the inequalities*

$$\|u_{U_0}|_{Z_{R_0,\infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|U_0\|_q \quad \text{and} \quad \|u_{U_0}(t)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \|U_0\|_q$$

hold for  $U_0 \in L_\sigma^q(\overline{\Omega}^c)$ ,  $t \in (0, \infty)$ .

Let  $q \in (1, 3)$ . If  $q < 3/2$ , take  $p \in [1, 2)$ , else let  $p \in (2q/3, 2)$ . Then

$$\|(\nabla_x u_{U_0})|_{Z_{R_0,\infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|U_0\|_q \quad \text{for } U_0 \in L_\sigma^q(\overline{\Omega}^c).$$

*Proof.* Use Corollary 4.2.  $\square$

Now we are in a position to establish our decay result for mild solutions to (1), (2).

**Theorem 4.6.** *Fix some  $S_0 \in (0, R_0)$  with  $\overline{\Omega} \subset B_{S_0}$ . Choose parameters  $q, p_0, p_1, p_2, \gamma$  in the following way:*

*Either take  $q \in (1, 3/2)$ ,  $p_1 \in [1, \infty)$  and set  $p_0 := 1$ ,  $\gamma := p_1$ ,*

*or let  $q \in [3/2, 3)$ ,  $p_0 \in (2q/3, 2)$  and  $p_1 \in [1, (1 - 3/(2q))^{-1})$ , and define*

$$p_2 := 2 \left( 3/(2q) + \max\{1/2, 1 - 1/p_1\} \right)^{-1}, \quad \gamma := (1/p_1 + 1/p_2 - 1)^{-1},$$

*or choose  $q \in [3, \infty)$  and  $p_1 \in [1, (1 - 3/(2q))^{-1})$ , and then define  $p_2 := 2 \left( 3/(2q) + 1 - 1/p_1 \right)^{-1}$  and again  $\gamma := (1/p_1 + 1/p_2 - 1)^{-1}$ .*

*According to Corollary 4.4, the parameter  $\gamma$  is well defined in all three cases and belongs to  $[p_1, \infty)$ .*

*In the case  $q < 3$ , let  $U_0 \in L_\sigma^q(\overline{\Omega}^c)$  and  $f \in L^{p_1}(0, \infty, L_\sigma^q(\overline{\Omega}^c))$ , and define  $u := u_{U_0} + u_f$ .*

*If  $q \geq 3$ , take  $f \in L^{p_1}(0, \infty, L_\sigma^q(\overline{\Omega}^c)) \cap L^\gamma(0, \infty, L_\sigma^q(\overline{\Omega}^c))$  and set  $u := u_f$ .*

*Then there is a zero-measure subset  $\mathfrak{S}_\infty$  of  $(0, \infty)$  and for any  $t \in (0, \infty) \setminus \mathfrak{S}_\infty$  a zero-measure subset  $N_t$  of  $\overline{B_{R_0}^c}$  such that*

$$\begin{aligned} & \left| \partial_x^\alpha [u - \mathfrak{R}^{(\tau)}(f|_{B_{S_0}^c \times (0, \infty)}) - \mathfrak{J}^{(\tau)}(U_0|_{B_{S_0}^c})](x, t) \right| \\ & \leq \mathfrak{C} \left[ \|U_0\|_q (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2p_0')} + (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2\gamma')} \|f\|_{q,p_1;\infty} \right] \end{aligned} \quad (24)$$

*if  $q < 3$ , and*

$$\begin{aligned} & \left| \partial_x^\alpha [u - \mathfrak{R}^{(\tau)}(f|_{B_{S_0}^c \times (0, \infty)})](x, t) \right| \\ & \leq \mathfrak{C} (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2\gamma')} (\|f\|_{q,p_1;\infty} + \|f\|_{q,\gamma;\infty}) \end{aligned} \quad (25)$$

*else, for  $t \in (0, \infty) \setminus \mathfrak{S}_\infty$ ,  $x \in \overline{B_{R_0}^c} \setminus N_t$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ . The constant in these inequalities is independent of  $U_0, f, x$  and  $t$ .*

The asymptotics of the functions  $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|_{B_{S_0}^c \times (0, \infty)})|$  and  $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|_{B_{S_0}^c})|$  (Lemma 2.6 and 2.5) are separate problems, only depending on the behaviour of  $f$  and  $U_0$ , respectively. If both these latter functions have compact support and are  $L^1$ , then the former two functions are bounded by  $\mathfrak{C} (|x| \nu(x))^{-(3+|\alpha|)/2}$  for  $x \in B_{R_0}^c$ ,  $t > 0$ ; see [7, Lemma 4.1, 4.2] In this situation the functions  $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|_{B_{S_0}^c})|$  and  $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|_{B_{S_0}^c \times (0, \infty)})|$  decrease faster than the right-hand side of (24), except if  $q < 3/2$ ,  $p_1 = 1$  in Theorem 4.6. Then the two convergence rates coincide.



Under the assumptions of Lemma 2.8, the term  $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)|$  even goes to zero as  $O([\nu(x)]^{-(4+|\alpha|)/2})$  for  $|x| \rightarrow \infty$ . For conditions on  $f$  and  $U_0$  leading to the decay bound  $\mathfrak{C}(|x|\nu(x))^{-(2+|\alpha|)/2}$  for  $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)|$  and  $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty))|$ , we refer to [3, Theorem 3.1] and [2, Theorem 1.1], respectively. An asymptotic behaviour of  $f$  and  $U_0$  entailing the bound  $\mathfrak{C}(|x|\nu(x))^{-(5/2+|\alpha|)/2}$  – with a slightly larger exponent in the case of  $\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty))$  if  $|\alpha| = 1$  – is considered in [8, Theorem 4.13, 4.14]. These remarks explain why we stated in Section 1 that inequality (4) holds if  $U_0(x)$  and  $f(x, t)$  decay sufficiently fast for  $|x| \rightarrow \infty$ .

It should be noted that the sum  $\mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty)) + \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)$  solves (1) in the whole space  $\mathbb{R}^3 \times (0, \infty)$ . So the left-hand side in (5) may be interpreted as the perturbation generated by the presence of the rigid object, in the region far from that object.

*Proof of Theorem 4.6.* Suppose that  $q < 3$ . Then we have  $u = u_{U_0} + u_f$ . Corollary 4.5 yields the estimate

$$\|u_{U_0}|Z_{R_0, \infty}\|_{q, \infty; \infty} + \|u_{U_0}|Z_{R_0, \infty}\|_{q, p_0; \infty} + \|(\nabla_x u_{U_0})|Z_{R_0, \infty}\|_{q, p_0; \infty} \leq \mathfrak{C}\|U_0\|_q.$$

Moreover, by Corollary 4.3 and 4.4,

$$\|u_f|Z_{R_0, \infty}\|_{q, \infty; \infty} + \|u_f|Z_{R_0, \infty}\|_{q, \gamma; \infty} + \|(\nabla_x u_f)|Z_{R_0, \infty}\|_{q, \gamma; \infty} \leq \mathfrak{C}\|f\|_{q, p_1; \infty}.$$

The assumptions in Theorem 4.6, the preceding inequalities and Theorem 3.5 show that the conditions on  $U_0$ ,  $f$  and  $u$  in Theorem 2.7 are fulfilled if this latter theorem is applied separately to the cases  $f = 0$  and  $U_0 = 0$ . By making use of this theorem in this way, we may conclude that inequality (24) is valid in the case  $q < 3$  under consideration. Note that since  $\gamma \geq p_1$  (Corollary 4.4), the rate of decay of  $\partial_x^\alpha u_f$  provided by Theorem 2.7 is  $-(3 + |\alpha|)/2 + 1/(2\gamma')$ .

In the case  $q \geq 3$ , we obtain (25) by the same reasoning, but only Corollary 4.3 and 4.4 are relevant because of our definition  $u := u_f$  if  $q \geq 3$ .  $\square$

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*E-mail address:* [deuring@univ-littoral.fr](mailto:deuring@univ-littoral.fr)