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Asymptotics of mild solutions to the time-dependent Oseen system.

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Abstract

We consider mild solutions to the 3D time-dependent Oseen system with homogeneous Dirichlet boundary conditions, under weak assumptions on the data. Such solutions are defined via the semigroup generated by the Oseen operator in L^q . They turn out to be also L^q -weak solutions to the Oseen system. On the basis of known results about spatial asymptotics of the latter type of solutions, we then derive pointwise estimates of the spatial decay of mild solutions. The rate of decay depends in particular on L^p -integrability in time of the external force.

AMS subject classifications. 35Q35, 76D07, 35B40, 35D30.

Key words. Oseen system, mild solution, spatial decay, Oseen semigroup.

1 Introduction

In this article, we consider mild solutions to the 3D the time-dependent Oseen system

$$u' - \Delta_x u + \tau \partial_{x_1} u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty), \quad (1.1)$$

where $\overline{\Omega}^c := \mathbb{R}^3 \setminus \overline{\Omega}$, with Ω an open, bounded set in \mathbb{R}^3 with smooth boundary. Thus $\overline{\Omega}^c$ is an exterior domain, which we suppose to be connected. Equation (1.1) is supplemented by homogeneous Dirichlet boundary conditions on $\partial\Omega$ and an initial condition,

$$u(t)|_{\partial\Omega} = 0 \quad \text{for } t \in (0, \infty), \quad u(0) = U_0. \quad (1.2)$$

The Oseen system is a linearization of the time-dependent Navier-Stokes system with Oseen term,

$$u' - \Delta_x u + \tau \partial_{x_1} u + (u \cdot \nabla_x) u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty). \quad (1.3)$$

This latter system is usually considered as a model for the flow of a viscous incompressible fluid around a rigid body moving with constant velocity and without rotation, with the set Ω corresponding to the rigid body. The functions $u : \overline{\Omega}^c \times (0, \infty) \mapsto \mathbb{R}^3$ (velocity) and $\pi : \overline{\Omega}^c \times (0, \infty) \mapsto \mathbb{R}$ (pressure) are the unknowns of problem (1.1), (1.2), whereas the functions $f : \overline{\Omega}^c \times (0, \infty) \mapsto \mathbb{R}^3$ (volume force) and $U_0 : \overline{\Omega}^c \mapsto \mathbb{R}^3$ (initial velocity), as well as the number $\tau \in (0, \infty)$ (Reynolds number), are given quantities.

Mild solutions to (1.1), (1.2), which only involve the velocity among the two unknowns velocity and pressure, are introduced via the semigroup generated by the Oseen operator. We refer to (3.1) for the definition of this operator, and to the proof of Theorem 3.2 for a discussion of the associated semigroup. A precise definition of a mild solutions to (1.1), (1.2) is given in Theorem 3.4.

In the work at hand, we study the spatial decay of such solutions. It turned out their decay rate is highest if $U_0 \in L_\sigma^{q_0}(\overline{\Omega}^c)$ and $f \in L^1(0, \infty, L_\sigma^{q_1}(\overline{\Omega}^c))$ for some $q_0, q_1 \in (1, 3/2)$, and if $|U_0(x)|$ and $|f(x, t)|$ tend to zero sufficiently fast for $|x| \rightarrow \infty$. (See Section 2 for the definition of $L_\sigma^q(\overline{\Omega}^c)$). In this situation we obtain that

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-(3+|\alpha|)/2} \quad (1.4)$$

for a. e. $t \in (0, \infty)$, a. e. $x \in B_{R_0}^c := \mathbb{R}^3 \setminus B_{R_0}$, and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Here u denotes the mild solution to (1.1), (1.2) associated with U_0 and f (Theorem 3.4). The parameter R_0 is some fixed number from $(0, \infty)$ with $\overline{\Omega} \subset B_{R_0}$. Concerning the condition $|\alpha| \leq 1$, it means that u and the spatial gradient $\nabla_x u$ are estimated in (1.4). The function ν appearing in (1.4) is defined by $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$. Its presence should be interpreted as a mathematical manifestation of the wake extending downstream in the flow behind a rigid body. If $f \in L^p(0, \infty, L_\sigma^{q_1}(\overline{\Omega}^c))$ for some $p \in (1, \infty)$, the rate of decay of $|\partial_x^\alpha u(x, t)|$ diminishes to $-(3 + |\alpha|)/2 + 1/(2p')$. So in particular this rate is linked to L^p -integrability in time of f . If the relation $U_0 \in L_\sigma^{q_0}(\overline{\Omega}^c)$ only holds for some $q_0 \in [3/2, 3)$, we also obtain a lower rate, and if $q_0 \geq 3$, we have no result at all. We refer to Theorem 4.2 for a detailed statement of our results. In the remark following this theorem, we explain what exactly we mean by the condition mentioned above that $|U_0(x)|$ and $|f(x, t)|$ are to tend to zero sufficiently fast for $|x| \rightarrow \infty$.

A link between the rate of spatial decay of $\partial_x^\alpha u$ on the one hand and L^p -integrability of f with respect to time on the other already appears in [17, Theorem 6.1] and [18, Theorem 5.2], but with certain L^p -norms in time and in space of u additionally influencing the spatial asymptotics of u . The former theorem specifies the spatial decay of regular solutions to (1.1), and the latter one extends these results to L^q -weak solutions of (1.1). No specific boundary conditions are imposed in [17] and [18]. A simplified version of [18, Theorem 5.2] is stated below as Theorem 2.4, which is the starting point of the work at hand. In [17] and [18], we applied [17, Theorem 6.1] and [18, Theorem 5.2], respectively, to some solutions whose existence is known from literature; see [17, Theorem 6.2, 6.3] and [18, Theorem 6.1, 6.2, 6.3]. However, these examples are either restricted to an L^2 -framework ([17, Theorem 6.2] and [18, Theorem 6.1, 6.2, 6.3]), or they deal with solutions whose lifespan $T_0 \in (0, \infty)$ is finite, which are much more regular than required for our theory and are associated with a right-hand side $f \in L^p(0, T_0, L^q(\overline{\Omega}^c)^3)$ required to satisfy the condition $p = q$ ([17, Theorem 6.3]). But as the key point of [18, Theorem 6.1, 6.2] we could improve the decay rates obtained in [9] and [12].

The work at hand with its focus on mild solutions shows that we may handle solutions with data of low regularity. In fact, these solutions exist for any $t > 0$ if $U_0 \in L_\sigma^q(\overline{\Omega}^c)$ and $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ for some $q > 1$ (Theorem 3.4). In addition, we are able to express our decay bounds exclusively in terms of the data; no norms of the solution are involved. Moreover, in the assumptions $U_0 \in L_\sigma^{q_0}(\overline{\Omega}^c)$ and $f \in L^p(0, \infty, L_\sigma^{q_1}(\overline{\Omega}^c))$, a large range of parameters $q_0, q_1 \in (1, \infty)$, $p_1 \in [1, \infty)$ is admitted, and our estimates exhibit how the choice of these parameters influences the spatial asymptotics of the solution. In this respect we recall the example given in (1.4): if $q_1, q_2 \in (1, 3/2)$, $p = 1$, then $|\partial_x^\alpha u(x, t)|$ tends to zero as $O([|x| \nu(x)]^{-(3+|\alpha|)/2})$ for $|x| \rightarrow \infty$. This rate is best possible in the sense explained in the remark following Theorem 4.2.

Our proof of (1.4) consists in verifying the assumptions of Theorem 2.4, with two main

points. Firstly it must be shown that mild solutions are also L^q -weak solutions as considered in that latter theorem. This is not completely obvious due to the low regularity of f ; see the proof of Theorem 3.4. Secondly, certain L^p -norms of u with respect to space and time variables must be estimated by the data (Corollary 4.2 – 4.5). This is achieved by means of $L^p - L^q$ -estimates of the Oseen semigroup. In our context the key feature of these estimates is the rate of temporal decay of spatial L^q -norms of this semigroup. Such rates are derived in [34], [22], [23], [31] and [32]. We will use results established in [34] and [32]; see Theorem 3.2 and 4.1.

We point out that according to [21], the velocity part U of a solution (U, Π) to the Oseen resolvent system $-\Delta U + \tau \partial_1 U + \lambda U + \nabla \Pi = F$, $\operatorname{div} U = 0$ in the whole space \mathbb{R}^3 does not satisfy the estimate $\|U\|_2 \leq C_0 |\lambda|^{-1} \|F\|_2$ with a single constant $C_0 > 0$ for all $F \in L^2(\mathbb{R}^3)^3$ and all $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$. As a consequence of this negative result, which arises because small values of $|\lambda|$ are admitted, an analogous resolvent estimate cannot be expected to hold for solutions to the Oseen resolvent problem in $\bar{\Omega}^c$, under whatever boundary conditions. Therefore, in view of [45, Theorem 4.2, point 3.]), it is a safe guess that maximal regularity is not valid for solutions of problem (1.1), (1.2), not even in an L^2 -framework. The negative result in [21] is the reason why we discuss some properties of the Oseen operator and its associated semigroup in greater detail; see the proof of Theorem 3.2.

Let us mention some references more distantly related to the work at hand. Knightly [33] considers even the case that the velocity of the rigid body changes with time. However, his results are valid only under various smallness assumptions. Mizumachi [39, Theorem 2] identified a class of initial data U_0 and a class of solutions (u, π) to the nonlinear problem (1.3), (1.2) with $f = 0$ such that $|u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-1}$ for x, t as in (1.4). Takahashi [44] deals with the Navier-Stokes system with Oseen term in the case $\Omega = \emptyset$. In [2], [3], solutions to (1.1) and (1.3) are estimated in weighted L^p -norms, with the weights adapted to the wake in the flow field downstream to the rigid body. Reference [16] by the present author combines decay estimates in time and in space for solutions of (1.1) and of a generalization of (1.3), under Dirichlet boundary conditions and in an L^2 -framework, with lower decay rates than those obtained in (1.4) in the case $p = 2$. Predecessor papers to this article are [5] – [13]. The theory developed in [17] and [18] is applied in [19] to a generalization of (1.3), the same as in [16]. Questions of existence, regularity and stability related to (1.1) or (1.3) are addressed in [24], [28], [29], [37], [38], [41], [42].

2 Notation. Some auxiliary results.

The symbol $|\cdot|$ denotes the Euclidean norm of \mathbb{R}^n for any $n \in \mathbb{N}$, and the length $\alpha_1 + \alpha_2 + \alpha_3$ of a multi-index $\alpha \in \mathbb{N}_0^3$. For $R \in (0, \infty)$, $x \in \mathbb{R}^3$, put $B_R(x) := \{y \in \mathbb{R}^3 : |x - y| < R\}$. In the case $x = 0$, we write B_R instead of $B_R(0)$.

The set $\Omega \subset \mathbb{R}^3$ and the parameter $\tau \in (0, \infty)$ introduced in Section 1 will be kept fixed throughout. Recall that Ω is open and bounded, with smooth boundary and connected complement. Further recall that $n^{(\Omega)}$ denotes the outward unit normal to Ω . We fix a number $R_0 \in (0, \infty)$ with $\bar{\Omega} \subset B_{R_0}$. For $R \in (0, \infty)$, we define $\Omega_R := B_R \setminus \bar{\Omega}$ and $Z_{R, \infty} := \Omega_R \times (0, \infty)$.

We additionally recall that also in Section 1, we introduced the function $\nu : \mathbb{R}^3 \mapsto [1, \infty)$ by setting $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$.

For $I \subset \mathbb{R}$, let χ_I stand for the characteristic function of I on \mathbb{R} . If $A \subset \mathbb{R}^3$, we denote by A^c the complement $\mathbb{R}^3 \setminus A$ of A in \mathbb{R}^3 . Put $e_l := (\delta_{jl})_{1 \leq j \leq 3}$ for $1 \leq l \leq 3$ (unit vector in \mathbb{R}^3). If A is some nonempty set and $\gamma : A \mapsto \mathbb{R}$ a function, we set $|\gamma|_\infty := \sup\{|\gamma(x)| : x \in A\}$.

Let $p \in [1, \infty)$, $m \in \mathbb{N}$. If $A \subset \mathbb{R}^3$ is open, we write $\|\cdot\|_p$ for the norm of the Lebesgue space $L^p(A)$, and $\|\cdot\|_{m,p}$ for the usual norm of the Sobolev space $W^{m,p}(A)$ of order m and exponent p . For an open set $B \subset \mathbb{R}^3$, the spaces $L^p_{loc}(B)$ and $W^{m,p}_{loc}(B)$ are defined as the set of all functions V from B into \mathbb{R} or \mathbb{C} such that $V|_A \in L^p(A)$ and $V|_A \in W^{1,p}(A)$, respectively, for any open, bounded set $A \subset \mathbb{R}^3$ with $\bar{A} \subset B$. We put $\nabla V := (\partial_k V_j)_{1 \leq j,k \leq 3}$ for $V \in W^{1,1}_{loc}(B)$.

Let \mathcal{V} be a normed space, and let the norm of \mathcal{V} be denoted by $\|\cdot\|$. Take $n \in \mathbb{N}$. Then we will use the same notation $\|\cdot\|$ for the norm of \mathcal{V}^n defined by $\|(f_1, \dots, f_n)\| := (\sum_{j=1}^n \|f_j\|^2)^{1/2}$ for $(f_1, \dots, f_n) \in \mathcal{V}^n$. The space $\mathcal{V}^{3 \times 3}$, as concerns its norm, is identified with \mathcal{V}^9 .

Let $A \subset \mathbb{R}^3$ be open and $p \in (1, \infty)$. We define $C^\infty_{0,\sigma}(A) := \{V \in C^\infty_0(A)^3 : \operatorname{div} V = 0\}$, and we write $L^p_\sigma(A)$ for the closure of $C^\infty_{0,\sigma}(A)$ with respect to the norm of $L^p(A)^3$. This function space $L^p_\sigma(A)$ ("space of solenoidal L^p -functions") is equipped with the norm $\|\cdot\|_p$.

Let $p \in [1, \infty]$ and \mathfrak{B} be a Banach space. For any interval $J \subset \mathbb{R}$, the notation $\|\cdot\|_{L^p(J, \mathfrak{B})}$ stands for the norm of $L^p(J, \mathfrak{B})$. Let $a, b \in \mathbb{R} \cup \{\infty\}$ with $a < b$. Then we write $L^p(a, b, \mathfrak{B})$ instead of $L^p((a, b), \mathfrak{B})$. The expression $L^p_{loc}([a, b], \mathfrak{B})$ denotes the space of all functions $v : (a, b) \mapsto \mathfrak{B}$ such that $v|_{(a, T)} \in L^p(a, T, \mathfrak{B})$ for any $T \in (a, b)$. This space is to be distinguished from the space $L^p_{loc}(a, b, \mathfrak{B})$, defined in the usual way. Let $T \in (0, \infty]$, $A \subset \mathbb{R}^3$ open, $p \in [1, \infty]$, $q \in (1, \infty)$ and $n \in \{1, 3\}$. Then we write $\|\cdot\|_{q,p;T}$ instead of $\|\cdot\|_{L^p(0,T, L^q(A)^n)}$. For an interval $J \subset \mathbb{R}$ and a function $v : J \mapsto W^{1,1}_{loc}(A)^3$, the notation $\nabla_x v$ stands for the gradient of v with respect to $x \in A$, in the sense that

$$\nabla_x v : J \mapsto L^1_{loc}(A)^{3 \times 3}, \quad \nabla_x v(t)(x) := \left(\partial x_k (v_j(t))(x) \right)_{1 \leq j,k \leq 3} \quad \text{for } t \in J, x \in A$$

(spatial gradient of v). Similar conventions are to be valid with respect to the expressions $\Delta_x v$, $\operatorname{div}_x v$ and $\partial x_j v$.

For the definition of the Bochner integral, we refer to [46, p. 132-133], or to [30, p. 78-80].

We write C for numerical constants and $C(\gamma_1, \dots, \gamma_n)$ for constants depending exclusively on parameters $\gamma_1, \dots, \gamma_n \in [0, \infty)$ for some $n \in \mathbb{N}$. However, such a precise bookkeeping will be possible only at some places. Mostly we will use the symbol \mathfrak{C} for constants whose dependence on parameters must be traced from context. Sometimes we write $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$ in order to indicate that the constants in question is influenced by the quantities $\gamma_1, \dots, \gamma_n$. But in such cases, this constant depends on other parameters as well.

We state an estimate involving the function ν .

Lemma 2.1 ([20, Lemma 4.8]) *The inequality $\nu(x) \leq C(1 + |y|)\nu(x - y)$ holds for $x, y \in \mathbb{R}^3$.*

Young's inequality for integrals will be used frequently. Stated her for the convenience of

the reader, we will refer to it as “Young’s inequality”.

Lemma 2.2 ([1, Corollary 2.25]) *Let $n \in \mathbb{N}$ and $p, p_1, p_2 \in [1, \infty]$ with $1/p = 1/p_1 + 1/p_2$. Then*

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} U(x-y)V(y) dy \right|^p dx \right)^{1/p} \leq C \|U\|_{p_1} \|V\|_{p_2} \quad \text{for } U \in L^{p_1}(\mathbb{R}^n), V \in L^{p_2}(\mathbb{R}^n).$$

Functions V from $L^q_\sigma(\overline{\Omega}^c)$ with sufficient regularity satisfy the equation $\operatorname{div} V = 0$:

Lemma 2.3 *Let $q \in (1, \infty)$ and $V \in L^q_\sigma(\overline{\Omega}^c) \cap W^{1,q}(\overline{\Omega}^c)^3$. Then $\operatorname{div} V = 0$.*

Proof: Let $\psi \in C_0^\infty(\overline{\Omega}^c)$. Since $u \in L^q_\sigma(\overline{\Omega}^c)$, there is a sequence (φ_n) in $C_{0,\sigma}^\infty(\overline{\Omega}^c)$ with $\operatorname{div} \varphi_n = 0$ for $n \in \mathbb{N}$ and $\|\varphi_n - V\|_q \rightarrow 0$. Then $\int_{\overline{\Omega}^c} \operatorname{div} V \cdot \psi dx = \int_{\overline{\Omega}^c} -V \cdot \nabla \psi dx = \lim_{n \rightarrow \infty} \int_{\overline{\Omega}^c} -\varphi_n \cdot \nabla \psi dx = 0$. This implies the lemma. \square

The Helmholtz-Fujita decomposition of $L^q(\overline{\Omega}^c)^3$ will play an important role in what follows. Here are the facts we will use.

Theorem 2.1 *For $q \in (1, \infty)$, there is a linear bounded operator $\mathcal{P}_q : L^q(\overline{\Omega}^c)^3 \mapsto L^q_\sigma(\overline{\Omega}^c)$ with $\mathcal{P}_q(V) = V$ for $V \in L^q_\sigma(\overline{\Omega}^c)$. Moreover $\mathcal{P}'_q = \mathcal{P}_{q'}$ for $q \in (1, \infty)$.*

Proof: See [27, Section III.1], [15, Corollary 2.3]. \square

Next we state a basic tool for handling Bochner integrals.

Theorem 2.2 *Let B_1, B_2 be Banach spaces, $A : B_1 \mapsto B_2$ a linear and bounded operator, $n \in \mathbb{N}$, $J \subset \mathbb{R}^n$ an open set and $f : J \mapsto B_1$ a Bochner integrable mapping. Then $A \circ f : J \mapsto B_2$ is Bochner integrable, too, and $A(\int_J f dx) = \int_J A \circ f dx$, where the integral on the left-hand side is B_1 -valued and the one on the right-hand side B_2 -valued.*

Proof: See [46, p. 134, Corollary 2], [30, Theorem 3.7.12]. \square

We will need mean continuity of the Bochner integral.

Theorem 2.3 ([30, Theorem 3.8.3]) *Let B be a Banach space and $f : \mathbb{R} \mapsto B$ a Bochner integrable function. Then $\int_{\mathbb{R}} \|f(s+h) - f(s)\|_B ds \rightarrow 0$ for $h \rightarrow 0$, where $\|\cdot\|_B$ denotes the norm of B .*

We define some fundamental solutions, first the heat kernel, $\mathfrak{H}(z, t) := (4\pi t)^{-3/2} e^{-|z|^2/(4t)}$ for $z \in \mathbb{R}^3$, $t \in (0, \infty)$, then a fundamental solution to the time-dependent Stokes system, $\Gamma_{jk}(z, t) := \mathfrak{H}(z, t) \delta_{jk} + \int_t^\infty \partial z_j \partial z_k \mathfrak{H}(z, s) ds$ for $z \in \mathbb{R}^3$, $t \in (0, \infty)$, $j, k \in \{1, 2, 3\}$, and finally a fundamental solution to the time-dependent Oseen system (1.1), $\Lambda_{jk}(z, t) := \Gamma_{jk}(z - \tau t e_1, t)$ for z, t, j, k as before. We will need the following estimate of Λ .

Lemma 2.4 ([17, Corollary 3.3]) *Let $K > 0$. Then for $z \in B_K^c$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$, the estimate $|\partial_z^\alpha \Lambda(z, t)| \leq C(K, \tau) (|z| \nu(z) + t)^{-(3+|\alpha|)/2}$ holds.*

The potential functions introduced in the two ensuing lemmas are needed in order to state the decay result from [18] which we will apply later on (proof of Theorem 4.2).

Lemma 2.5 ([17, Corollary 3.5]) *Let $A \subset \mathbb{R}^3$ be measurable, $q \in [1, \infty)$, $V \in L^q(A)^3$, and let \tilde{V} denote the zero extension of V to \mathbb{R}^3 . Then $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y, t) \tilde{V}(y)| dy < \infty$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $x \in \mathbb{R}^3$, $t \in (0, \infty)$. Define the function $\mathfrak{J}^{(\tau)}(V) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by setting $\mathfrak{J}^{(\tau)}(V)(x, t) := \int_{\mathbb{R}^3} \Lambda(x-y, t) \cdot \tilde{V}(y) dy$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$.*

The derivative $\partial x_l \mathfrak{J}^{(\tau)}(V)(x, t)$ exists and equals $\int_{\mathbb{R}^3} \partial x_l \partial_t^\alpha \Lambda(x - y, t) \cdot \tilde{V}(y) dy$ for x, t as above and for $l \in \{1, 2, 3\}$. The functions $\mathfrak{J}^{(\tau)}(V)$ and $\partial x_l \mathfrak{J}^{(\tau)}(V)$ are continuous in $\mathbb{R}^3 \times (0, \infty)$.

Lemma 2.6 ([17, Lemma 3.8]) *Let $T_0 \in (0, \infty]$, $A \subset \mathbb{R}^3$ measurable, $q \in [1, \infty)$ and f a function from $L^1_{loc}([0, T_0], L^q(A)^3)$. Let \tilde{f} denote the zero extension of f to $\mathbb{R}^3 \times (0, \infty)$. Then the integral $\int_0^t \int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x - y, t - \sigma) \cdot \tilde{f}(y, \sigma)| dy d\sigma$ is finite for a. e. $t \in (0, \infty)$, a. e. $x \in \mathbb{R}^3$ and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Thus we may define*

$$\mathfrak{R}^{(\tau)}(f)(x, t) := \int_0^t \int_{\mathbb{R}^3} \Lambda(x - y, t - \sigma) \cdot \tilde{f}(y, \sigma) dy d\sigma$$

for such t and x . The relation $\mathfrak{R}^{(\tau)}(f)(t) \in W_{loc}^{1,1}(\mathbb{R}^3)^3$ holds for a. e. $t \in (0, \infty)$.

The following theorem states the decay result from [18] we mentioned in Section 1.

Theorem 2.4 *Take $S_0 \in (0, R_0)$ with $\bar{\Omega} \subset B_{S_0}$. Let $\tilde{q}, r_1, r_2, r_3 \in (1, \infty)$ and take functions $U_0 \in L^{\tilde{q}}(\bar{\Omega}^c)^3$, $f \in L^1_{loc}([0, \infty), L^{r_1}(\bar{\Omega}^c)^3)$ and $u : (0, \infty) \mapsto W_{loc}^{1,1}(\bar{\Omega}^c)^3$ with $u \in C^0([0, \infty), L^{r_2}(\bar{\Omega}^c)^3)$, $\nabla_x u \in L^1_{loc}([0, \infty), L^{r_3}(\bar{\Omega}^c)^9)$ and $\operatorname{div}_x u = 0$. Further suppose that u satisfies the equation*

$$\begin{aligned} \int_0^{T_0} \int_{\bar{\Omega}^c} \left(-\gamma'(t) u(t) \cdot \vartheta + \gamma(t) [\nabla_x u(t) \cdot \nabla \vartheta + \tau \partial x_1 u(t) \cdot \vartheta - f(t) \cdot \vartheta] \right) dx dt \quad (2.1) \\ -\gamma(0) \int_{\bar{\Omega}^c} U_0 \cdot \vartheta dx = 0 \quad \text{for } \gamma \in C_0^\infty([0, T_0]), \vartheta \in C_{0,\sigma}^\infty(\bar{\Omega}^c). \end{aligned}$$

Assume there are numbers $q \in (1, \infty)$ and $\gamma_1, \gamma_2, \gamma_3 \in [1, \infty]$ such that the function $u|_{Z_{R_0,\infty}}$ belongs to $L^\infty(0, \infty, L^q(Z_{R_0,\infty})^3)$ and to $L^{\gamma_1}(0, \infty, L^q(Z_{R_0,\infty})^3)$, the restriction $\nabla_x u|_{Z_{R_0,\infty}}$ is in $L^{\gamma_2}(0, \infty, L^q(Z_{R_0,\infty})^9)$ and $f|_{Z_{R_0,\infty}}$ in $L^{\gamma_3}(0, \infty, L^q(Z_{R_0,\infty})^3)$. Suppose that the zero flux condition $\int_{\partial\Omega} u(t) \cdot n^{(\Omega)} d\sigma_x = 0$ holds for $t \in (0, \infty)$. Then there is a zero-measure subset \mathfrak{S}_∞ of $(0, \infty)$ such that

$$\begin{aligned} |\partial_x^\alpha [u - \mathfrak{R}^{(\tau)}(f|_{B_{S_0}^c} \times (0, \infty)) - \mathfrak{J}^{(\tau)}(U_0|_{B_{S_0}^c})](x, t)| \quad (2.2) \\ \leq \mathfrak{C} (\|u|_{Z_{R_0,\infty}}\|_{q,\infty;T_0} + \|u|_{Z_{R_0,\infty}}\|_{q,\gamma_1;T_0} + \|\nabla_x u|_{Z_{R_0,\infty}}\|_{q,\gamma_2;T_0} + \|f|_{Z_{R_0,\infty}}\|_{q,\gamma_3;T_0} \\ + \|U_0\|_{\tilde{q}}) (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2\min\{\gamma'_1, \gamma'_2, \gamma'_3\})} \end{aligned}$$

for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $t \in (0, \infty) \setminus \mathfrak{S}_\infty$, $x \in \overline{B_{R_0}}^c \setminus N_t$, where N_t is some zero-measure subset of $\overline{B_{R_0}}^c$.

Proof: This theorem is a simplified but less general version of [18, Theorem 5.2], adapted to what will be needed in Section 4. \square

Under suitable assumptions on U_0 , the potential function $\mathfrak{J}^{(\tau)}(U_0)(x, t)$ diminishes as $O([\|x| \nu(x)\|^{-2})$ for $|x| \rightarrow \infty$. Here are the details of this result, which will be interesting in the context of our decay estimate of weak solutions presented in Section 4 (Theorem 4.2). It is the highest rate of decay we could find for $\mathfrak{J}^{(\tau)}(U_0)$.

Lemma 2.7 *Let $q \in (3, \infty)$, $U_0 \in L^q_\sigma(\bar{\Omega}^c) \cap L^1_\sigma(\bar{\Omega}^c)^3$ with $\operatorname{supp}(U_0)$ compact. Take $R \in (0, \infty)$ with $\operatorname{supp}(U_0) \subset B_R$. Then $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0)(x, t)| \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2} \|U_0\|_1$ for $x \in B_{2R}^c$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Proof: We apply an approach used by Kozono [36, p. 724] in a different context (temporal decay). By [35, Lemma 2.2], the function u_0 has mean value zero: $\int_{\overline{\Omega}^c} U_0 dx = 0$. Take x, t, α as in the lemma. Then we get for $y \in B_R$ that $|x - y| \geq |x|/2 \geq R$ and $\nu(x) \geq C(1 + |y|)\nu(x - y)$ (Lemma 2.1), so with Lemma 2.4,

$$\partial_x^\alpha \partial y_l \Lambda(x - y, t) \leq C(\tau, R) (|x - y| \nu(x - y) + t)^{-(4+|\alpha|)/2} \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2}.$$

Now we find

$$\begin{aligned} \left| \partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0)(x, t) \right| &\leq \left| \int_{B_R} [\partial_x^\alpha \Lambda(x - z, t) - \partial_x^\alpha \Lambda(x, t)] U_0(z) dz \right| \\ &= \left| \int_{B_R} \int_0^1 \sum_{l=1}^3 \partial_x^\alpha \partial y_l \Lambda(x - y, t)|_{y=\vartheta z} z_l d\vartheta U_0(z) dz \right| \\ &\leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2} \int_{B_R} |z| |U_0(z)| dz \leq C(\tau, R) (|x| \nu(x))^{-(4+|\alpha|)/2} \|U_0(z)\|_1. \end{aligned}$$

□

3 Mild solutions of (1.1), (1.2)

We begin by recalling some known results, occasionally discussing a proof if the result in question is slightly modified or is not stated clearly in literature.

Lemma 3.1 *Let B be a Banach space, $S : [0, \infty) \mapsto B$ a C^0 -semigroup on B , $\mathcal{D}(A)$ a dense subspace of B , $A : \mathcal{D}(A) \mapsto B$ the infinitesimal generator of S , and $f \in L_{loc}^1([0, \infty), B)$. Let the norm of B be denoted by $\| \cdot \|$. Then $\int_0^t \|S(t-s)f(s)\| ds < \infty$ for $t \in (0, \infty)$. Define $u(t) := \int_0^t S(t-s)f(s) ds$ for $t \in (0, \infty)$. Then $u \in C^0([0, \infty), B)$.*

Proof: According to [40, Theorem 1.2.2], there are constants $M \in [1, \infty)$, $\omega \in [0, \infty)$ with $\|S(t)X\| \leq M e^{\omega t} \|X\|$ for $t \in [0, \infty)$. The first claim in the lemma follows from this. Let $t \in (0, \infty)$, $h \in (0, 1]$. Then we have $u(t+h) - u(t) = \mathfrak{A}_1 + \mathfrak{A}_2$, with $\mathfrak{A}_1 := \int_0^h S(t+h-s)f(s) ds$, $\mathfrak{A}_2 := \int_h^{t+h} S(t+h-s)f(s) ds - \int_0^t S(t-s)f(s) ds$. By the estimate of $\|S(t)(X)\|$ mentioned above, we get $\|\mathfrak{A}_1\| \leq M e^{\omega(t+1)} \int_0^h \|f(s)\| ds$, and

$$\|\mathfrak{A}_2\| = \int_0^t \|S(t-s)(f(s+h) - f(s))\| ds \leq M e^{\omega t} \int_0^t \|f(s+h) - f(s)\| ds.$$

But $\int_0^h \|f(s)\| ds \rightarrow 0$ for $h \downarrow 0$ by Lebesgue's theorem, and $\int_0^t \|f(s+h) - f(s)\| ds \rightarrow 0$ also for $h \downarrow 0$ by Theorem 2.3. Thus the second claim of the theorem is proved. □

Next we introduce the Oseen operator $\mathcal{O}_q : \mathcal{D}(\mathcal{O}_q) \mapsto L_\sigma^q(\overline{\Omega}^c)$ for $q \in (1, \infty)$ by setting

$$\begin{aligned} \mathcal{D}(\mathcal{O}_q) &:= L_\sigma^q(\overline{\Omega}^c) \cap W_0^{1,q}(\overline{\Omega}^c)^3 \cap W^{2,q}((\overline{\Omega}^c)^3), \quad \mathcal{O}_q(V) := \mathcal{P}_q(\Delta V - \tau \partial_1 V) \quad (3.1) \\ &\text{for } V \in \mathcal{D}(\mathcal{O}_q), \end{aligned}$$

where the operator \mathcal{P}_q was introduced in Theorem 2.1. We denote the identity mapping on $L_\sigma^q(\overline{\Omega}^c)$ by \mathcal{I}_q . The ensuing theorem gives some details on the resolvent of \mathcal{O}_q .

Theorem 3.1 *Let $q \in (1, \infty)$. Then the resolvent set $\varrho(\mathcal{O}_q)$ of \mathcal{O}_q is given by $\varrho(\mathcal{O}_q) = \{\lambda \in \mathbb{C} : \tau^2 \Re \lambda > -(\Im \lambda)^2\}$. The relation $(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F) \in \mathcal{D}(\mathcal{O}_q)$ holds for $\lambda \in \varrho(\mathcal{O}_q)$ and for $F \in L_\sigma^q(\overline{\Omega}^c)$.*

Let $\vartheta_0 \in (\pi, \pi/2)$. There is $r_0 \in (0, \infty)$ such that $\{\lambda \in \mathbb{C} : |\lambda| \geq r_0, |\arg \lambda| \leq \vartheta_0\} \subset \varrho(\mathcal{O}_q)$ and

$$|\lambda| \|U\|_q + |\lambda|^{1/2} \|U\|_{1,q} + \|U\|_{2,q} \leq \mathfrak{C} \|F\|_q \quad \text{for } F \in L_\sigma^q(\overline{\Omega}^c) \quad (3.2)$$

and for $\lambda \in \mathbb{C}$ with $|\lambda| \geq r_0, |\arg \lambda| \leq \vartheta_0$, where $U := (\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F)$.

Proof: The first claim of that theorem, pertaining to $\varrho(\mathcal{O}_q)$, holds according to [25, Theorem 3.1]. The relation $(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F) \in \mathcal{D}(\mathcal{O}_q)$ for $\lambda \in \varrho(\mathcal{O}_q)$, $F \in L_\sigma^q(\overline{\Omega}^c)$ is obvious by the definition of the resolvent. By [34, Lemma 4.5], there is $r_0 \in (0, \infty)$ such that $|\lambda| \|U\|_q + \|U\|_{2,q} \leq \mathfrak{C} \|F\|_q$ for $\lambda \in \mathbb{C}$, $|\lambda| \geq r_0, |\arg \lambda| \leq \vartheta_0$, $F \in L_{loc}^q(\overline{\Omega}^c)$, with U defined as above. It follows by interpolation that $|\lambda|^{1/2} \|\nabla V\|_q \leq \mathfrak{C} \|F\|_q$. \square

The ensuing theorem deals with the semigroup generated by \mathcal{O}_q . In particular it presents an L^q - L^q -estimate (inequality (3.3)), which we take from [34] and [32]. The estimate in the following theorem is “global” in the sense that it gives an upper bound of the Oseen semigroup with respect to L^q -norms on the exterior domain $\overline{\Omega}^c$. In Section 4, we will additionally need “local” L^p – L^q -estimates, that is, upper bounds for L^p -norms on Ω_{R_0} instead of $\overline{\Omega}^c$. These latter estimates yield decay rates which are not always available in the global setting. We further note that [31] and [32] deal with the case of time-dependent coefficients and rotational terms in the differential equations, a level of generality not needed here.

Theorem 3.2 *Let $q \in (1, \infty)$. The operator \mathcal{O}_q generates an analytic semigroup on $L_\sigma^q(\overline{\Omega}^c)$. We write $e^{t\mathcal{O}_q}$ for its value in $t \in [0, \infty)$. Let $U \in L_\sigma^q(\overline{\Omega}^c)$, and put $u(t) := e^{t\mathcal{O}_q}U$ for $t \in [0, \infty)$. Then $u \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c)) \cap C^\infty((0, \infty), L_\sigma^q(\overline{\Omega}^c))$, $u(t) \in \mathcal{D}(\mathcal{O}_q)$, $u'(t) = \mathcal{O}_q u(t)$ for $t \in (0, \infty)$, $u(0) = U$. Moreover*

$$\|\partial_x^\alpha u(t)\|_q \leq \mathfrak{C} (\chi_{(0,1]}(t) t^{-|\alpha|/2} + \chi_{[1,\infty)}(t) t^{-\min\{1/2, 3/(2q)\}|\alpha|}) \|U\|_q \quad (3.3)$$

for $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. In particular $\nabla_x u \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^9)$.

Proof: According to Miyakawa [38, Theorem 4.2], the Oseen operator \mathcal{O}_q generates an analytical semigroup on $L_\sigma^q(\overline{\Omega}^c)$. Since Theorem 3.1 was not yet available in [38], but allows to directly reduce this semigroup property to standard results in [40], we present a proof based on such a reduction, for the convenience of the reader, although the argument in question is in principle well known. Take $\vartheta_0 \in (\pi/2, \pi)$ and choose a number r_0 associated to ϑ_0 as in Theorem 3.1. We may suppose that $r_0 \geq 1$. By that theorem $\mathcal{S}_0 := \{\lambda \in \mathbb{C} : |\lambda| \geq r_0, |\arg \lambda| \leq \vartheta_0\} \subset \varrho(\mathcal{O}_q)$ and

$$\|(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F)\|_q \leq \mathfrak{C} |\lambda|^{-1} \|F\|_q \quad \text{for } F \in L_\sigma^q(\overline{\Omega}^c), \lambda \in \mathcal{S}_0. \quad (3.4)$$

Put $a_0 := 2r_0/\sin \vartheta_0$, $\mathcal{S}_{\vartheta_0, a_0} := \{\lambda \in \mathbb{C} \setminus \{a_0\} : |\arg(\lambda - a_0)| \leq \vartheta_0\}$. Then $\mathcal{S}_{\vartheta_0, a_0} \cup \{a_0\} \subset \mathcal{S}_0$. In fact, take $\lambda \in \mathbb{C} \setminus \{a_0\}$ with $|\arg(\lambda - a_0)| \leq \vartheta_0$. We note that for $b > 0$, the function $f(x) := x(x^2 + b)^{-1/2}$ ($x \in \mathbb{R}$) is increasing. Obviously $\Re(\lambda - a_0) < \Re \lambda$, so if $\Im \lambda \neq 0$, and thus $\Im \lambda = \Im(\lambda - a_0) \neq 0$, we get with $b = (\Im \lambda)^2 = (\Im(\lambda - a_0))^2$ that

$f(\Re(\lambda - a_0)) \leq f(\Re\lambda)$, that is, $\Re(\lambda - a_0)/|\lambda - a_0| \leq \Re(\lambda)/|\lambda|$, and therefore $\arg(\lambda - a_0) = \arccos(\Re(\lambda - a_0)/|\lambda - a_0|) \geq \arccos(\Re(\lambda)/|\lambda|) = \arg(\lambda) > 0$. Since $|\arg(\lambda - a_0)| \leq \vartheta_0$, it follows that $|\arg \lambda| \leq \vartheta_0$ in the case $\Im\lambda \neq 0$. If $\Im\lambda = 0$, then $\lambda - a_0 \in \mathbb{R} \setminus \{0\}$, and so $\lambda \in \mathbb{R}$. Since $|\arg(\lambda - a_0)| \leq \vartheta_0 < \pi$, we get $\lambda - a_0 > 0$, hence $\lambda > a_0 > 0$, and thus $|\arg \lambda| = 0 < \vartheta_0$. Thus we may conclude that in any case $|\arg(\lambda)| \leq \vartheta_0$.

In order to obtain that $\lambda \in \mathcal{S}_0$, we still have to show that $|\lambda| \geq r_0$. To this end, we first suppose that $|\arg(\lambda - a_0)| \geq \pi/2$. Then $|\sin(\arg(\lambda - a_0))| \geq \sin \vartheta_0$. Thus, if we assume $|\lambda| < (a_0 \sin \vartheta_0)/2$, we get a contradiction via the estimate

$$\begin{aligned} |\lambda| &\geq |\Im\lambda| = |\Im(\lambda - a_0)| \geq |\lambda - a_0| \sin \vartheta_0 \geq (a_0 - |\lambda|) \sin \vartheta_0 \\ &\geq (a_0 - (a_0 \sin \vartheta_0)/2) \sin \vartheta_0 \geq (a_0/2) \sin \vartheta_0. \end{aligned} \quad (3.5)$$

Therefore $|\lambda| \geq (a_0 \sin \vartheta_0)/2$, hence $|\lambda| \geq r_0$ by the definition of a_0 . Altogether we get $\lambda \in \mathcal{S}_0$ if $|\arg(\lambda - a_0)| \geq \pi/2$. Suppose that $|\arg(\lambda - a_0)| \leq \pi/2$. Then $\Re(\lambda - a_0) \geq 0$, so that $|\lambda| \geq \Re\lambda \geq a_0 \geq r_0$. Since the inequality $|\arg(\lambda)| \leq \vartheta_0$ is already proved, we again get $\lambda \in \mathcal{S}_0$. Obviously $a_0 \in \mathcal{S}_0$, so we have confirmed that $S_{\vartheta_0, a_0} \cup \{a_0\} \subset \mathcal{S}_0$.

As a consequence $S_{\vartheta_0, a_0} \cup \{a_0\} \subset \varrho(\mathcal{O}_q)$ and inequality (3.4) holds for $F \in L_\sigma^q(\overline{\Omega}^c)$, $\lambda \in S_{\vartheta_0, a_0} \cup \{a_0\}$. But for $\lambda \in S_{\vartheta_0, a_0}$ with $|\arg(\lambda - a_0)| > \pi/2$, we get $|\lambda| \geq |\lambda - a_0| \sin \vartheta_0$ as in (3.5). In the case $|\arg(\lambda - a_0)| \leq \pi/2$, we have $\Re(\lambda - a_0) \geq 0$, hence $\Re\lambda > 0$, and then $|\lambda| \geq |\lambda - a_0|$ since $\Im\lambda = \Im(\lambda - a_0)$. We may conclude with (3.2) that $\|(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}(F)\|_q \leq \mathfrak{C} |\lambda - a_0|^{-1} \|F\|_q$ for F and λ as before. As a consequence the sets $\{0\}$ and $S_{\vartheta_0, 0} := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| \leq \vartheta_0\}$ are contained in the resolvent set $\varrho(-a_0 \mathcal{I}_q + \mathcal{O}_q)$ of the operator $-a_0 \mathcal{I}_q + \mathcal{O}_q$, and by the preceding estimate of $(\lambda \mathcal{I}_q - \mathcal{O}_q)^{-1}$, we have $\|(\lambda - (-a_0 \mathcal{I}_q + \mathcal{O}_q))^{-1}(F)\|_q \leq \mathfrak{C} |\lambda|^{-1} \|F\|_q$ for $F \in L_\sigma^q(\overline{\Omega}^c)$ and $\lambda \in S_{\vartheta_0, 0}$. Now it follows that $-a_0 \mathcal{I}_q + \mathcal{O}_q$ generates an analytic semigroup on $L_\sigma^q(\overline{\Omega}^c)$ ([40, Theorem 1.7.7, 2.5.2]). We write $e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$ for its value in $t \in [0, \infty)$, so that the mapping $t \mapsto e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$ ($t \in [0, \infty)$) is the restriction of an analytic semigroup to $[0, \infty)$. Therefore the mapping $t \mapsto e^{t a_0} e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$ ($t \in [0, \infty)$) also is the restriction of an analytic semigroup to $[0, \infty)$, and the operator \mathcal{O}_q is the infinitesimal generator of this semigroup. In view of the uniqueness result in [40, Theorem 1.2.6] and the notation introduced in Theorem 3.2, we have $e^{t \mathcal{O}_q} = e^{t a_0} e^{t(-a_0 \mathcal{I}_q + \mathcal{O}_q)}$ for $t \in [0, \infty)$. The function u introduced in Theorem 3.2 thus belongs to $C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ and to $C^\infty((0, \infty), L_\sigma^q(\overline{\Omega}^c))$, with $u(t) \in \mathcal{D}(\mathcal{O}_q)$, $u'(t) = \mathcal{O}_q u(t)$ for $t \in (0, \infty)$, and $u(0) = U_0$ ([40, Corollary 1.2.3, Theorem 2.5.2 (d), Corollary 2.4.4, Lemma 2.4.2]).

Concerning inequality (3.3), we indicate that (3.3) in the case $t \leq 1$ holds according to [34, (6.38)], whereas the case $t \geq 1$ is covered by [34, (1.2)] if $\alpha = 0$, [34, (1.3)] if $|\alpha| = 1$, $q \leq 3$, and [32, (2.23)] if $|\alpha| = 1$, $q > 3$. Actually, in the case $t \geq 1$, $|\alpha| = 1$, $q > 3$, another reference would be [34, (1.6)], but that latter inequality is proved only implicitly in [34].

□

Theorem 3.3 *Let $q \in (1, \infty)$, $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega}^c))$, $u(t) := \int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$ for $t \in [0, \infty)$ (see Lemma 3.1), with the preceding integral denoting an $L_\sigma^q(\overline{\Omega}^c)$ -valued and an $L^q(\overline{\Omega}^c)^3$ -valued Bochner integral. Then $\int_0^t \|\partial x_l(e^{(t-s)\mathcal{O}_q} f(s))\|_q ds < \infty$, $u(t) \in$*

$W_0^{1,q}(\overline{\Omega}^c)^3$ for a. e. $t > 0$, $\operatorname{div}_x u = 0$, $\nabla_x u \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^3)$, and

$$\partial x_l u(t) = \int_0^t \partial x_l (e^{(t-s)\mathcal{O}_q} f(s)) ds \quad \text{for } 1 \leq l \leq 3 \text{ and a. e. } t \in (0, \infty). \quad (3.6)$$

Proof: Of course, it does not make any difference whether the integral in the definition of u is considered as an $L_\sigma^q(\overline{\Omega}^c)$ -valued or an $L^q(\overline{\Omega}^c)^3$ -valued Bochner integral, due to Theorem 2.2 and the fact that the canonical imbedding of $L_\sigma^p(\overline{\Omega}^c)$ into $L^q(\overline{\Omega}^c)^3$ is linear and bounded. Let $T \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then by (3.3),

$$\begin{aligned} \int_0^T \int_0^t \|\partial_x^\alpha (e^{(t-s)\mathcal{O}_q} f(s))\|_q ds dt &\leq \mathfrak{C} \int_0^T \int_0^t (\chi_{(0,1)}(t-s) (t-s)^{-|\alpha|/2} \\ &\quad + \chi_{[1,\infty)}(t-s)) \chi_{(0,T)}(t-s) \chi_{(0,T)}(s) \|f(s)\|_q ds dt \\ &\leq \mathfrak{C} \int_{\mathbb{R}} (\chi_{(0,1)}(r) r^{-|\alpha|/2} + \chi_{[1,\infty)}(r)) \chi_{(0,T)}(r) dr \|f\|_{Z_T} \|f\|_{Z_T} \end{aligned} \quad (3.7)$$

where we used Young's inequality in the second estimate. Inequality (3.7) yields in particular that $\int_0^t \|\partial_x^\alpha (e^{(t-s)\mathcal{O}_q} f(s))\|_q ds < \infty$ for a. e. $t \in (0, \infty)$ and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Take any such t . Let $l \in \{1, 2, 3\}$ and $\psi \in C_0^\infty(\overline{\Omega})^3$. For $\sigma \in \{0, 1\}$, the operator $V \mapsto \int_{\overline{\Omega}^c} \partial_l^\sigma \psi(x) \cdot V(x) dx$ ($V \in L_\sigma^q(\overline{\Omega}^c)$) belongs to $(L_\sigma^q(\overline{\Omega}^c))'$, so we get with Theorem 2.2 and because $e^{(t-s)\mathcal{O}_q} f(s) \in \mathcal{D}(\mathcal{O}_q)$ for $s \in (0, t)$ that the equation $\int_{\overline{\Omega}^c} \partial_l \psi(x) \cdot u(x, t) dx = - \int_{\overline{\Omega}^c} \psi(x) \cdot (\int_0^t \partial x_l (e^{(t-s)\mathcal{O}_q} f(s) ds)(x) dx$ holds. Thus the weak derivative $\partial x_l u(t)$ exists and equation (3.6) holds. Since $\int_0^t \|\partial_x^\alpha (e^{(t-s)\mathcal{O}_q} f(s))\|_q ds < \infty$ by the choice of t , we thus have $u(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$. Equation (3.6) and inequality (3.7) imply that $\nabla_x u \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^3)$. In order to show that $u(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$ and $\operatorname{div}_x u(t) = 0$, again take $t \in (0, \infty)$ with $\int_0^t \|\partial_x^\alpha (e^{(t-s)\mathcal{O}_q} f(s))\|_q ds < \infty$ for $\alpha \in \mathbb{N}_0^3$, $|\alpha| \leq 1$. We have $e^{(t-s)\mathcal{O}_q} f(s) \in \mathcal{D}(\mathcal{O}_q) \subset W_0^{1,q}(\overline{\Omega}^c)^3 \cap L_\sigma^q(\overline{\Omega}^c)$ for $s \in (0, t)$, in particular $\operatorname{div}_x (e^{(t-s)\mathcal{O}_q} f(s)) = 0$ by Lemma 2.3. It follows with (3.6) that $\operatorname{div}_x u(t) = 0$. By the choice of t and because $e^{(t-s)\mathcal{O}_q} f(s) \in W_0^{1,q}(\overline{\Omega}^c)^3$ we may conclude that the integral $\int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$ exists also as a $W_0^{1,q}(\overline{\Omega}^c)^3$ -valued Bochner integral. For $\psi \in C_0^\infty(\overline{\Omega}^c)^3$, the operator $V \mapsto \int_{\overline{\Omega}^c} \psi V dx$ is linear and bounded as a mapping on $L_\sigma^q(\overline{\Omega}^c)$ and on $W_0^{1,q}(\overline{\Omega}^c)^3$. This observation and Theorem 2.2 imply that the integral $\int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$ yields the same function both as $L_\sigma^q(\overline{\Omega}^c)$ - and $W_0^{1,q}(\overline{\Omega}^c)^3$ -valued Bochner integral. As a consequence we have $u(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$. \square

In the ensuing theorem, we collect some of our previous results. They allow us to introduce the notion of ‘‘mild solution’’. The theorem then states that such a solution satisfies (1.1) in the sense of an L^q -weak solutions as formulated in (2.1), for a right-hand side f of low regularity.

Theorem 3.4 *Let $q \in (1, \infty)$, $U_0 \in L_\sigma^q(\overline{\Omega}^c)$, $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega}^c))$, and define $u(t) := e^{t\mathcal{O}_q} U_0 + \int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$ for $t \in [0, \infty)$. This function u is called a ‘‘mild solution’’ to (1.1). It satisfies the relations $u \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$, $u(0) = U_0$, $u(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$, $\operatorname{div}_x u(t) = 0$ for $t \in (0, \infty)$, and $\nabla_x u \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^9)$. Moreover the function u fulfills equation (2.1).*

Proof: Put $u^{(1)}(t) := e^{t\mathcal{O}_q}U_0$, $u^{(2)}(t) := \int_0^t e^{(t-s)\mathcal{O}_q}f(s) ds$ for $t \in [0, \infty)$. By Theorem 3.2 we know that $u^{(1)} \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$, $u^{(1)}(0) = U_0$, $u^{(1)}(t) \in \mathcal{D}(\mathcal{O}_q)$ (in particular $u^{(1)}(t) \in W^{2,q}(\overline{\Omega}^c)^3 \cap W_0^{1,q}(\overline{\Omega}^c)^3$), $\operatorname{div}_x u^{(1)}(t) = 0$ for $t \in (0, \infty)$, and $\nabla_x u^{(1)} \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^9)$. Let $\gamma \in C_0^\infty([0, \infty))$ and $\vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c)$. Choose some $T \in (0, \infty)$ with $\operatorname{supp}(\gamma) \subset [0, T]$. The operator $V \mapsto \int_{\overline{\Omega}^c} V \cdot \vartheta dx$ ($V \in L_\sigma^q(\overline{\Omega}^c)$) is linear and bounded. It follows from this and the properties of $u^{(1)}$ listed above that the function $t \mapsto \int_{\overline{\Omega}^c} u^{(1)}(t) \cdot \vartheta dx$ ($t \in [0, \infty)$) belongs to $C^0([0, \infty))$ and to $C^\infty((0, \infty))$, with $\partial_t(\int_{\overline{\Omega}^c} u^{(1)}(t) \cdot \vartheta dx) = \int_{\overline{\Omega}^c} (u^{(1)})'(t) \cdot \vartheta dx$ for $t \in (0, \infty)$, and $\int_{\overline{\Omega}^c} u^{(1)}(t) \cdot \vartheta dx|_{t=0} = \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx$. Thus $\gamma(\epsilon) \int_{\overline{\Omega}^c} u^{(1)}(\epsilon) \cdot \vartheta dx \rightarrow \gamma(0) \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx$. From the preceding relations we get

$$\begin{aligned} \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u^{(1)}(t) \cdot \vartheta dx dt &= \lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma'(t) \int_{\overline{\Omega}^c} u^{(1)}(t) \cdot \vartheta dx dt \\ &= -\lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} (u^{(1)})'(t) \cdot \vartheta dx dt - \gamma(0) \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx. \end{aligned} \quad (3.8)$$

Since by (3.3), $\int_0^T \int_{\overline{\Omega}^c} |\nabla_x u^{(1)}(t) \cdot \nabla \vartheta| dx dt < \infty$, and because $u^{(1)}(t) \in W^{2,q}(\overline{\Omega}^c)^3$ for $t > 0$, we have $\int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \nabla_x u^{(1)}(t) \cdot \nabla \vartheta dx dt = -\lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} \Delta_x u^{(1)}(t) \cdot \vartheta dx ds$. But $\vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c)$, so $\mathcal{P}_q'(\vartheta) = \vartheta$ by Theorem 2.1. Therefore due to the equation $\mathcal{P}_q = \mathcal{P}_q'$ (Theorem 2.1), we get

$$\int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \nabla_x u^{(1)}(t) \cdot \nabla \vartheta dx dt = -\lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} \mathcal{P}_q(\Delta_x u^{(1)}(t)) \cdot \vartheta dx ds. \quad (3.9)$$

By a similar reasoning we find that

$$\int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \tau \partial_{x_1} u^{(1)}(t) \cdot \vartheta dx dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^T \gamma(t) \int_{\overline{\Omega}^c} \mathcal{P}_q(\tau \partial_{x_1} u^{(1)}(t)) \cdot \vartheta dx ds. \quad (3.10)$$

Since $(u^{(1)})'(t) = \mathcal{O}_q u^{(1)}(t)$ for $t > 0$, we may conclude from (3.8) – (3.10) that equation (2.1) is valid with $f = 0$ and with $u^{(1)}$ in the role of u . Note that in the preceding argument, the integral $\int_0^T \gamma(t) \int_{\overline{\Omega}^c} \mathcal{P}_q(\Delta_x u^{(1)}(t)) \cdot \vartheta dx ds$, which does not exist in general, does not arise.

By Lemma 3.1 we know that $u^{(2)} \in C^0([0, \infty), L_\sigma^q(\overline{\Omega}^c))$. Moreover Theorem 3.3 yields that $u^{(2)}(t) \in W_0^{1,q}(\overline{\Omega}^c)^3$, $\operatorname{div}_x u^{(2)}(t) = 0$ for a. e. $t \in (0, \infty)$, and $u^{(2)} \in L_{loc}^1([0, \infty), L^q(\overline{\Omega}^c)^9)$. Take γ, ϑ and T as above. By the previous relation and Theorem 2.2, and because the operator $V \mapsto \int_{\overline{\Omega}^c} V \cdot \vartheta dx$ ($V \in L_\sigma^q(\overline{\Omega}^c)$) is linear and bounded, we get

$$\int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u^{(2)}(t) \cdot \vartheta dx dt = \int_0^T \int_0^t \int_{\overline{\Omega}^c} \gamma'(t) \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q}f(s))(x) dx ds dt. \quad (3.11)$$

Due to inequality (3.3), we have $\|e^{(t-s)\mathcal{O}_q}f(s)\|_q \leq \mathfrak{C}(T) \|f(s)\|_q$ for $t \in (0, T)$, $s \in (0, t)$. Moreover the function $s \mapsto \|f(s)\|_q$ ($s \in (0, T)$) is integrable, γ' is bounded, and ϑ is bounded with compact support, so $\int_0^T \int_0^t \int_{\overline{\Omega}^c} |\gamma'(t) \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q}f(s))(x)| dx ds dt < \infty$. Thus we may apply Fubini's and Lebesgue's theorem on the right-hand side of (3.11), to

obtain

$$\begin{aligned} & \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u^{(2)}(t) \cdot \vartheta \, dx \, dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma'(t) \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds. \end{aligned} \quad (3.12)$$

By Theorem 3.2 we know that for $s \in (0, \infty)$, the function $t \mapsto e^{(t-s)\mathcal{O}_q} f(s)$ ($t \in [s, \infty)$) belongs to $C^0([s, \infty), L^q_\sigma(\overline{\Omega}^c))$ and to $C^\infty((s, \infty), L^q_\sigma(\overline{\Omega}^c))$. Using once more that the operator $V \mapsto \int_{\overline{\Omega}^c} V \cdot \vartheta \, dx$ ($V \in L^q_\sigma(\overline{\Omega}^c)$) is a bounded functional, we see that for $s \in (0, \infty)$, the function $K_{\vartheta, s}(t) := \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx$ ($t \in [s, \infty)$) belongs to $C^0([s, \infty)) \cap C^\infty((s, \infty))$, with $K'_{\vartheta, s}(t) := \int_{\overline{\Omega}^c} \vartheta(x) \cdot \partial_t(e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx$. Thus the right-hand side in (3.12) may be transformed by an integration by parts into $\lim_{\epsilon \downarrow 0} [-\int_0^T \int_{s+\epsilon}^T \gamma(t) K'_{\vartheta, s}(t) \, dt \, ds - \int_0^T \gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^{\epsilon \mathcal{O}_q} f(s))(x) \, dx \, ds]$. But with Hölder's inequality and (3.3), $|\gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^{\epsilon \mathcal{O}_q} f(s))(x) \, dx| \leq \mathfrak{C} |\gamma|_\infty \|\vartheta\|_{q'} \|f(s)\|_q$, with the function $s \mapsto \|f(s)\|_q$ ($s \in (0, T)$) being integrable, as already mentioned before. Moreover, by the continuity of the function $r \mapsto e^{r\mathcal{O}_q} V$ ($r \in [0, \infty)$), and because this function takes the value V if $r = 0$, for $V \in L^q_\sigma(\overline{\Omega}^c)$ (Theorem 3.2), we obtain that $\gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^{\epsilon \mathcal{O}_q} f(s))(x) \, dx \rightarrow \gamma(s) \int_{\overline{\Omega}^c} \vartheta(x) \cdot f(x, s) \, dx$ ($\epsilon \downarrow 0$). Therefore $\int_0^T \gamma(s+\epsilon) \int_{\overline{\Omega}^c} \vartheta(x) \cdot (e^{\epsilon \mathcal{O}_q} f(s))(x) \, dx \, ds \rightarrow \int_0^T \gamma(s) \int_{\overline{\Omega}^c} \vartheta(x) \cdot f(x, s) \, dx \, ds$ ($\epsilon \downarrow 0$) by Lebesgue's theorem. From (3.12), the transformation of the right-hand side of (3.12) presented above and the preceding relation, we deduce that

$$\begin{aligned} & \int_0^\infty \gamma'(t) \int_{\overline{\Omega}^c} u^{(2)}(t) \cdot \vartheta \, dx \, dt \\ &= -\lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \gamma(t) \int_{\overline{\Omega}^c} \vartheta(x) \cdot \partial_t(e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, dt \, ds - \int_0^T \gamma(t) \int_{\overline{\Omega}^c} f(t) \cdot \vartheta \, dx \, dt. \end{aligned} \quad (3.13)$$

Since $\nabla_x u^{(2)} \in L^1_{loc}([0, \infty), L^q(\overline{\Omega}^c)^9)$ by Theorem 3.3, and because of Theorem 2.2 and the linearity and boundedness of the operator $V \mapsto \int_{\overline{\Omega}^c} V \cdot \vartheta \, dx$ ($V \in L^q(\overline{\Omega}^c)^3$), we get

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \tau \partial x_1 u^{(2)}(t) \cdot \vartheta \, dx \, dt \\ &= \int_0^T \gamma(t) \int_0^t \int_{\overline{\Omega}^c} \vartheta(x) \cdot \tau \partial x_1 (e^{(t-s)\mathcal{O}_q} f(s))(x) \, dx \, ds \, dt. \end{aligned} \quad (3.14)$$

By (3.3) $\|\nabla_x(e^{r\mathcal{O}_q} f(s))\|_q \leq \mathfrak{C}(T) r^{-1/2} \|f(s)\|_q$ for $r, s \in (0, T)$. Thus with Hölder's inequality,

$$\begin{aligned} & \int_0^T \int_0^t \int_{\overline{\Omega}^c} |\gamma(t) \vartheta(x) \cdot \tau \partial x_1 (e^{(t-s)\mathcal{O}_q} f(s))(x)| \, dx \, ds \, dt \\ & \leq \mathfrak{C}(T) |\gamma|_\infty \|\vartheta\|_{q'} \int_0^T \int_0^t (t-s)^{-1/2} \|f(s)\|_q \, ds \, dt. \end{aligned}$$

But $\int_0^T \int_0^t (t-s)^{-1/2} \|f(s)\|_q \, ds \, dt = \int_0^T \int_s^T (t-s)^{-1/2} \, dt \|f(s)\|_q \, ds \leq \mathfrak{C}(T) \int_0^T \|f(s)\|_q \, ds$. Since the function $s \mapsto \|f(s)\|_q$ ($s \in (0, T)$) is integrable, it follows that the integral

$\int_0^T \int_0^t \int_{\overline{\Omega}^c} |\gamma(t) \vartheta(x) \cdot \tau \partial x_1 (e^{(t-s)\mathcal{O}_q} f(s))(x)| dx ds dt$ is finite. Thus from (3.14) and Fubini's and Lebesgue's theorem,

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \tau \partial x_1 u^{(2)}(t) \cdot \vartheta dx dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \vartheta(x) \cdot \tau \partial x_1 (e^{(t-s)\mathcal{O}_q} f(s))(x) dx dt ds. \end{aligned} \quad (3.15)$$

Since for $1 \leq l \leq 3$, the operator $V \mapsto \int_{\overline{\Omega}^c} V \cdot \partial_l \vartheta dx$ ($V \in L^q(\overline{\Omega}^c)^3$) is also linear and bounded, the same reasoning yields that

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \nabla_x u^{(2)}(t) \cdot \nabla \vartheta dx dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \nabla \vartheta(x) \cdot \nabla_x (e^{(t-s)\mathcal{O}_q} f(s))(x) dx dt ds. \end{aligned} \quad (3.16)$$

On the other hand, we have $e^{(t-s)\mathcal{O}_q} f(s) \in W^{2,q}(\overline{\Omega}^c)^3$ for $s \in (0, \infty)$, $t \in (s, \infty)$, hence we obtain $\int_{\overline{\Omega}^c} \nabla \vartheta(x) \cdot \nabla_x (e^{(t-s)\mathcal{O}_q} f(s))(x) dx = - \int_{\overline{\Omega}^c} \vartheta(x) \cdot \Delta_x (e^{(t-s)\mathcal{O}_q} f(s))(x) dx$ for s, t as before. So we may combine (3.15) and (3.16) to obtain

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \left(\nabla_x u^{(2)}(t) \cdot \nabla \vartheta + \tau \partial x_1 u^{(2)}(t) \cdot \vartheta \right) dx dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \vartheta(x) \cdot [-\Delta_x + \tau \partial x_1] (e^{(t-s)\mathcal{O}_q} f(s))(x) dx dt ds. \end{aligned} \quad (3.17)$$

At this point we proceed as in the first part of the proof, using the relations $\mathcal{P}_{q'}(\vartheta) = \vartheta$ and $\mathcal{P}'_q = \mathcal{P}_{q'}$ provided by Theorem 2.1. Due to these equations, we may rewrite (3.17) as

$$\begin{aligned} & \int_0^\infty \gamma(t) \int_{\overline{\Omega}^c} \left(\nabla_x u^{(2)}(t) \cdot \nabla \vartheta + \tau \partial x_1 u^{(2)}(t) \cdot \vartheta \right) dx dt \\ &= - \lim_{\epsilon \downarrow 0} \int_0^T \int_{s+\epsilon}^T \int_{\overline{\Omega}^c} \gamma(t) \vartheta(x) \cdot \mathcal{O}_q (e^{(t-s)\mathcal{O}_q} f(s))(x) dx dt ds. \end{aligned} \quad (3.18)$$

But $(\partial_t - \mathcal{O}_q)(e^{(t-s)\mathcal{O}_q} f(s)) = 0$ for $s \in (0, \infty)$, $t \in (s, \infty)$ by Theorem 3.2, so it follows from (3.13) and (3.18) that equation (2.1) holds with $U_0 = 0$ and $u^{(2)}$ in the role of u . Since $u = u^{(1)} + u^{(2)}$, equation (2.1) holds as stated in the theorem. \square

4 Spatial decay of mild solutions.

We use $L^p - L^q$ -estimates of the Oseen semigroup $e^{t\mathcal{O}_q}$ in order to deduce rates of spatial decay of mild solutions to (1.1), (1.2). Our main tools are Theorem 3.2 as well as the following theorem which reproduces results from [34] and [32]. Recall that the parameter $R_0 \in (0, \infty)$ was fixed at the beginning of Section 2.

Theorem 4.1 ([34, (6.18)], [32, (6.4)]) *Let $q \in (1, \infty)$. Then*

$$\|e^{t\mathcal{O}_q} V|_{\Omega_{R_0}}\|_{1,q} \leq \mathfrak{C} t^{-3/(2q)} \|V\|_q \quad \text{for } V \in L^q_\sigma(\overline{\Omega}^c), t \in [1, \infty).$$

Corollary 4.1 *Let $q \in (1, \infty)$. Then*

$$\|\partial_x^\alpha (e^{t\mathcal{O}_q} V)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \left(\chi_{(0,1)}(t) t^{-|\alpha|/2} + \chi_{[1,\infty)}(t) t^{-3/(2q)} \right) \|V\|_q$$

for $V \in L_\sigma^q(\overline{\Omega^c})$, a. e. $t \in (0, \infty)$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.

Proof: Theorem 3.2 ($t \leq 1$) and 4.1 ($t > 1$). \square

In the ensuing four corollaries, we apply Theorem 3.2 and the preceding corollary in order to estimate the functions $\int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$ and $e^{t\mathcal{O}_q} U_0$. Recall that the set $Z_{R_0, \infty}$ was introduced at the beginning of Section 2.

Corollary 4.2 *Let $q \in (1, \infty)$. If $q < 3/2$, take $p_1 \in [1, \infty)$ and set $p := p_1$.*

In the case $q \geq 3/2$, let $p_1 \in [1, (1 - 3/(2q))^{-1})$. Then $1 \geq 3/(2q) > 1 - 1/p_1 \geq 0$. Fix some $p_2 \in (1, \infty)$ with $3/(2q) > 1/p_2 > 1 - 1/p_1$, for example $p_2 := 2(3/(2q) + 1 - 1/p_1)^{-1}$. Then $1 \geq 1/q_1 > 1/q_1 + 1/q_2 - 1 > 0$. Define $p := (1/q_1 + 1/q_2 - 1)^{-1}$. Then $p_1, p_2, p \in [1, \infty)$ with $1/p = 1/p_1 + 1/p_2 - 1$, $3p_2/(2q) > 1$ and $p \in [p_1, \infty)$.

For $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega^c}))$, $t \in (0, \infty)$, define $u_f(t) := \int_0^t e^{(t-s)\mathcal{O}_q} f(s) ds$. Then the inequalities $\|u_f|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$ and $\|u_f(t)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$ hold for $f \in L^{p_1}(0, \infty, L_\sigma^q(\overline{\Omega^c}))$, $t \in (0, \infty)$.

Proof: The operator $V \mapsto V|_{\Omega_{R_0}}$ from $L_\sigma^q(\overline{\Omega^c})$ into $L^q(\Omega_{R_0})^3$ is linear and bounded. Thus, for any $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega^c}))$, $t \in (0, \infty)$, we may deduce from Theorem 2.2 that $u_f(t)|_{\Omega_{R_0}} = \int_0^t [e^{(t-s)\mathcal{O}_q} f(s)]|_{\Omega_{R_0}} ds$, hence with Corollary 4.1,

$$\|u_f(t)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \int_0^t g(t-s) \|f(s)\|_q ds, \quad (4.1)$$

with $g(r) := \chi_{(0,1)}(r) + \chi_{[1,\infty)}(r) r^{-3/(2q)}$ for $r \in \mathbb{R}$. In the case $q < 3/2$, put $p_2 := 1$. Then we may conclude for any choice of q that $p_1, p_2, q \in [1, \infty)$, $1/p = 1/p_1 + 1/p_2 - 1$ and $3p_2/(2q) > 1$. The latter inequality yields that $g \in L^{p_2}(\mathbb{R})$. At this point we see that inequality (4.1) and Young's inequality imply the estimate $\|u_f|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$ for $f \in L^{p_1}(0, \infty, L_\sigma^q(\overline{\Omega^c}))$. Since $p_1 < (1 - 3/(2q))^{-1}$ in the case $q \geq 3/2$, and $3/(2q) > 1$ if $q < 3/2$, we have in any case that $3p_1/(2q) > 1$. Therefore (4.1) and Hölder's inequality yield the estimate of $\|u_f(t)|_{\Omega_{R_0}}\|_q$ stated in the corollary. \square

Corollary 4.3 *Let the numbers q, p_1, p_2 be given in one of the following three ways:*

$q \in [1, 3/2)$, $p_1 \in [1, \infty)$, $p := p_1$,

or $q \in [3/2, 3)$, $p_1 \in [1, (1 - 3/(2q))^{-1})$, $p_2 := 2(3/(2q) + \max\{1/2, 1 - 1/p_1\})^{-1}$, $p := (1/p_1 + 1/p_2 - 1)^{-1}$,

or $q \in [3, \infty)$, $p_1 \in [1, (1 - 3/(2q))^{-1})$, $p_2 := 2(3/(2q) + 1 - 1/p_1)^{-1}$ and again $p := (1/p_1 + 1/p_2 - 1)^{-1}$.

Then p is well defined and belongs to $[p_1, \infty)$. Let $l \in \{1, 2, 3\}$. Then if $q < 3$, the inequality $\|(\partial x_l u_f)|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$ holds for $f \in L^{p_1}(0, \infty, L_\sigma^q(\overline{\Omega^c}))$. In the case $q \geq 3$, the estimate $\|(\partial x_l u_f)|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} (\|f\|_{q,p_1;\infty} + \|f\|_{q,p;\infty})$ is valid for functions f belonging to $L^\gamma(0, \infty, L_\sigma^q(\overline{\Omega^c}))$ with $\gamma = p_1$ and $\gamma = p$. (The function u_f was introduced in Corollary 4.2.

Proof: Suppose that $q \in [3/2, 3)$. Then $3/(2q) \in (1/2, 1]$. On the other hand, the assumption $p_1 \in [1, (1 - 3/(2q))^{-1})$ implies $3/(2q) > 1 - 1/p_1$. Therefore we have $3/(2q) > \max\{1/2, 1 - 1/p_1\} > 0$, so $3/(2q) > 1/p_2 > \max\{1/2, 1 - 1/p_1\}$, hence $1 > 1/p_2 > 1/2$, that is, $p_2 \in (1, 2)$. Moreover $3/(2q) > 1/p_2 > 1 - 1/p_1$, so by Corollary 4.2 we get $1/p_1 > 1/p_1 + 1/p_2 - 1 > 0$ and $3p_2/(2q) > 1$. In particular p is well defined and belongs to $[p_1, \infty)$.

Next assume that $q \geq 3$. Then the conditions on p_1 and the definition of p_2 imply that $3/(2q) > 1/p_2 > 1 - 1/p_1$, so Corollary 4.2 yields that $p_2 \in (1, \infty)$, $1/p_1 > 1/p_1 + 1/p_2 - 1 > 0$ and $3p_2/(2q) > 1$. In particular p is again well defined and $p \in [p_1, \infty)$.

If $q < 3/2$, choose $p_2 = 1$. Then $(1/p_1 + 1/p_2 - 1)^{-1} = p$ by the choice of p in the corollary, and $3p_2/(2q) > 1$ because $3/(2q) > 1$ in the case under consideration. Altogether we have for any choice of q that $p_1, p_2, p \in [1, \infty)$, $1/p = (1/p_1 + 1/p_2 - 1)^{-1}$ and $3p_2/(2q) > 1$. If $q < 3$, we additionally have $p_2 \in [1, 2)$.

Theorem 3.3 and 2.2 yield that $(\partial x_l u_f(t))|_{\Omega_{R_0}} = \int_0^t [\partial x_l (e^{(t-s)\mathcal{O}_q} f(s))] |_{\Omega_{R_0}} ds$ for $f \in L_{loc}^1([0, \infty), L_\sigma^q(\overline{\Omega}^c))$ and a. e. $t \in (0, \infty)$, for any choice of q ; see the beginning of the proof of Corollary 4.2 as concerns the role of Theorem 2.2. Hence by Corollary 4.1,

$$\|(\partial x_l u_f(t))|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \int_0^t g(t-s) \|f(s)\|_q ds \quad (4.2)$$

for f, t as before, with $g(r) := \chi_{(0,1)}(r) r^{-1/2} + \chi_{[1,\infty)}(r) r^{-3/(2q)}$ for $r \in \mathbb{R}$. Since $p_2 \in [1, 2)$ and $3p_2/(2q) > 1$ in the case $q < 3$, we obtain $g \in L^{p_2}(\mathbb{R})$ in that case. Recalling that $p_1, p_2, p \in [1, \infty)$ and $1/p = (1/p_1 + 1/p_2 - 1)^{-1}$, we may thus conclude from (4.2) and Young's inequality that in the case $q < 3$, the estimate $\|(\partial x_l u_f)|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$ is valid for $f \in L^{p_1}(0, \infty, L_\sigma^q(\overline{\Omega}^c))$.

Now suppose that $q \geq 3$. Since in this case the relations $1/p = (1/p_1 + 1/p_2 - 1)^{-1}$ and $3p_2/(2q) > 1$ are valid, too, we may apply Young's inequality once more, to obtain $(\int_0^\infty (\int_{\mathbb{R}} \chi_{[1,\infty)}(t-s) (t-s)^{-3/(2q)} \|f(s)\|_q ds)^p dt)^{1/p} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}$. Also by Young's inequality we get $(\int_0^\infty (\int_{\mathbb{R}} \chi_{(0,1)}(t-s) (t-s)^{-1/2} \|f(s)\|_q ds)^p dt)^{1/p} \leq \mathfrak{C} \|f\|_{q,p;\infty}$. Due to (4.2) and the two preceding estimates, we may conclude that $\|(\partial x_l u_f)|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} (\|f\|_{q,p_1;\infty} + \|f\|_{q,p;\infty})$. \square

Corollary 4.4 For $q \in (1, \infty)$, $U_0 \in L_\sigma^q(\overline{\Omega}^c)$, $t \in (0, \infty)$, put $u_{U_0}(t) := e^{t\mathcal{O}_q} U_0$.

Let $q \in (1, \infty)$. If $q < 3/2$, take $p \in [1, \infty)$, else let $p \in (2q/3, \infty)$. Then the inequalities $\|u_{U_0}|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|U_0\|_q$ and $\|u_{U_0}(t)|_{\Omega_{R_0}}\|_q \leq \mathfrak{C} \|U_0\|_q$ hold for $U_0 \in L_\sigma^q(\overline{\Omega}^c)$, $t \in (0, \infty)$.

Proof: Use Corollary 4.1 with $\alpha = 0$. \square

Corollary 4.5 Let $q \in (1, 3)$. If $q < 3/2$, take $p \in [1, 2)$, else let $p \in (2q/3, 2)$. Take $l \in \{1, 2, 3\}$. Then $\|(\partial x_l u_{U_0})|_{Z_{R_0, \infty}}\|_{q,p;\infty} \leq \mathfrak{C} \|U_0\|_q$ for $U_0 \in L_\sigma^q(\overline{\Omega}^c)$, where u_{U_0} was defined in the preceding corollary.

Proof: Use Corollary 4.1 with $\alpha = e_l$. \square

Now we are in a position to establish our decay result for mild solutions to (1.1), (1.2).

Theorem 4.2 Fix some $S_0 \in (0, R_0)$ with $\overline{\Omega} \subset B_{S_0}$. Choose parameters q, p_0, p_1, p_2, γ

in the following way:

Take $q \in (1, 3/2)$, $p_1 \in [1, \infty)$ and set $p_0 := 1$, $\gamma := p_1$,

or let $q \in [3/2, 3)$, $p_0 \in (2q/3, 2)$ and $p_1 \in [1, (1 - 3/(2q))^{-1})$, and define $p_2 := 2(3/(2q) + \max\{1/2, 1 - 1/p_1\})^{-1}$, $\gamma := (1/p_1 + 1/p_2 - 1)^{-1}$,

or choose $q \in [3, \infty)$, $p_1 \in [1, (1 - 3/(2q))^{-1})$ and put $p_2 := 2(3/(2q) + 1 - 1/p_1)^{-1}$ and again $\gamma := (1/p_1 + 1/p_2 - 1)^{-1}$.

According to Corollary 4.3, the parameter γ is well defined in all three cases and belongs to $[p_1, \infty)$. In the case $q < 3$, let $U_0 \in L^q_\sigma(\overline{\Omega}^c)$ and $f \in L^{p_1}(0, \infty, L^q_\sigma(\overline{\Omega}^c))$, and define $u(t) := e^{t\mathcal{O}_q}U_0 + \int_0^t e^{(t-s)\mathcal{O}_q}f(s)ds$ for $t \in (0, \infty)$. If $q \geq 3$, take $f \in L^{p_1}(0, \infty, L^q_\sigma(\overline{\Omega}^c)) \cap L^\gamma(0, \infty, L^q_\sigma(\overline{\Omega}^c))$ and set $u(t) := \int_0^t e^{(t-s)\mathcal{O}_q}f(s)ds$, again for $t \in (0, \infty)$.

Then there is a zero-measure subset \mathfrak{S}_∞ of $(0, \infty)$ and for any $t \in (0, \infty) \setminus \mathfrak{S}_\infty$ a zero-measure subset N_t of $\overline{B_{R_0}^c}$ such that

$$\begin{aligned} & |\partial_x^\alpha [u - \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty)) - \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)](x, t)| \\ & \leq \mathfrak{C} [\|U_0\|_q (|x|\nu(x))^{-(3+|\alpha|)/2+1/(2p_0')} + (|x|\nu(x))^{-(3+|\alpha|)/2+1/(2\gamma')} \|f\|_{q, p_1; \infty}] \end{aligned} \quad (4.3)$$

if $q < 3$, and

$$\begin{aligned} & |\partial_x^\alpha [u - \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty))](x, t)| \\ & \leq \mathfrak{C} (|x|\nu(x))^{-(3+|\alpha|)/2+1/(2\gamma')} (\|f\|_{q, p_1; \infty} + \|f\|_{q, \gamma; \infty}) \end{aligned} \quad (4.4)$$

else, for $t \in (0, \infty) \setminus \mathfrak{S}_\infty$, $x \in \overline{B_{R_0}^c} \setminus N_t$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.

The asymptotics of the functions $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty))|$ and $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)|$ are a separate problem, only depending on the behaviour of f and U_0 , respectively. If both these latter functions have compact support and are L^1 , then the former two functions are bounded by $\mathfrak{C} (|x|\nu(x))^{-(3+|\alpha|)/2}$ for $x \in B_{R_0}^c$, $t > 0$; see [17, Lemma 4.1] as concerns $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)|$, and [17, Lemma 4.2] with respect to $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty))|$. In this situation the functions $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)|$ and $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty))|$ decrease faster than the right-hand side of (4.3), except if $q < 3/2$, $p_0 = 1$, $p_1 = 1$ in Theorem 4.2. Then the two convergence rates coincide. Under the assumptions of Lemma 2.7, $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)|$ even goes to zero as $O([|x|\nu(x)]^{-(4+|\alpha|)/2})$ for $|x| \rightarrow \infty$. For conditions on f and U_0 leading to the decay bound $\mathfrak{C} (|x|\nu(x))^{-(2+|\alpha|)/2}$, we refer to [12, Theorem 3.1] and [11, Theorem 1.1], respectively. These remarks explain why we stated in Section 1 that inequality (1.4) holds if $U_0(x)$ and $f(x, t)$ decay sufficiently fast for $|x| \rightarrow \infty$. It should be noted that the sum $\mathfrak{R}^{(\tau)}(f|B_{S_0}^c \times (0, \infty)) + \mathfrak{J}^{(\tau)}(U_0|B_{S_0}^c)$ solves (1.1) in the whole space $\mathbb{R}^3 \times (0, \infty)$. So the left-hand side in (2.2) may be interpreted as the perturbation generated by the presence of the rigid object, in the region far from that object.

Proof of Theorem 4.2: The relation $\gamma \geq p_1$ holds according to Corollary 4.3. Suppose that $q < 3$. Then we have $u = u_{U_0} + u_f$, with u_{U_0} and u_f defined as in Corollary 4.2 and 4.4, respectively. Corollary 4.4 and 4.5 yield that

$$\|u_{U_0}|Z_{R_0, \infty}\|_{q, \infty; \infty} + \|u_{U_0}|Z_{R_0, \infty}\|_{q, p_0; \infty} + \|(\nabla_x u_{U_0})|Z_{R_0, \infty}\|_{q, p_0; \infty} \leq \mathfrak{C} \|U_0\|_q.$$

Moreover, by Corollary 4.2 and 4.3,

$$\|u_f|Z_{R_0,\infty}\|_{q,\infty;\infty} + \|u_f|Z_{R_0,\infty}\|_{q,\gamma;\infty} + \|(\nabla_x u_f)|Z_{R_0,\infty}\|_{q,\gamma;\infty} \leq \mathfrak{C} \|f\|_{q,p_1;\infty}.$$

The assumptions in Theorem 4.2, the preceding inequalities and Theorem 3.4 show that the conditions on U_0 , f and u in Theorem 2.4 are fulfilled if this latter theorem is applied separately to the cases $f = 0$ and $U_0 = 0$, respectively. By making use of this theorem in this way and applying the preceding inequalities, we may conclude that inequality (4.3) is valid. Note that since $\gamma \geq p_1$, the rate of decay of $\partial_x^\alpha u_f$ obtained from Theorem 2.4 is $-(3 + |\alpha|)/2 + 1/(2\gamma')$, in view of the fact that Corollary 4.4 yields a bound for $\|u|Z_{R_0,\infty}\|_{q,\gamma;\infty}$ and $\|\nabla_x u|Z_{R_0,\infty}\|_{q,\gamma;\infty}$.

In the case $q \geq 3$, we obtain (4.4) by the same reasoning, but only Corollary 4.2 and 4.3 are relevant, whereas Corollary 4.4 is not needed and Corollary 4.5 is mute in this case. \square

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