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Abstract

In this article, we study the long-time behaviour of a system describing the coupled motion of a rigid body and of a viscous incompressible fluid in which the rigid body is contained. We assume that the system formed by the rigid body and the fluid fills the entire space $\mathbb{R}^3$. In the case in which the rigid body is a ball and the initial data are small, we prove the global existence of solutions for this system and we describe their large time behavior. Our main result asserts, in particular, that if the initial datum is small enough in suitable norms then the position of the center of the rigid ball converges to some $h_\infty \in \mathbb{R}^3$ as time goes to infinity. This result contrasts with those known for the analogues of our system in 2 or 1 space dimensions, where it has been proved that the body quits any bounded set, provided that we wait long enough. To achieve this result we use a "monolythic" type approach, which means that we consider a linearized problem in which the equations of the solid and of the fluid are still coupled. An essential role is played by the properties of the semigroup, called *fluid-structure semigroup*, associated to this coupled linearized problem. The generator of this semigroup is called *the fluid-structure operator*. Our main tools are new $L^p - L^q$ estimates for the fluid-structure semigroup. Note that these estimates are proved for bodies of arbitrary shape. The main ingredients used to study the fluid-structure semigroup and its generator are resolvent estimates which provide both the analyticity of the fluid-structure semigroup (in the spirit of a classical work of Borchers and Sohr) and $L^p - L^q$ decay estimates (by adapting a strategy due to Iwashita).

Key words. Fluid-structure interaction, Incompressible Navier-Stokes system, Large time behaviour.

AMS subject classifications. 35Q35, 35B40, 76D03, 76D05.

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1 Introduction

We consider a homogeneous rigid body which occupies at instant $t = 0$ a ball B of radius $R > 0$ and centered at the origin and we study the motion of this body in a viscous incompressible fluid which fills the remaining part of $\mathbb{R}^3$. We denote by $h(t)$, $\mathcal{S}(t)$, $\mathcal{F}(t)$ the position of the centre of the ball, the domain occupied by the solid, which coincides with the ball of radius R centered at h, and the domain filled by the fluid, respectively, at instant $t > 0$. Moreover, the velocity and pressure fields in the fluid are denoted by u and p, respectively. With the above notation, the system describing the motion of the rigid ball in the fluid is

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu\Delta u + \nabla p = 0 & (t > 0,\ y \in \mathcal{F}(t)), \\ \operatorname{div} u = 0, & (t > 0,\ y \in \mathcal{F}(t)), \\ u(t,y) = h(t) + \omega(t) \times y & (t > 0,\ y \in \partial\mathcal{F}(t)), \\ m\ddot{h}(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} \sigma(u,p)\nu \,\mathrm{d}s & (t > 0), \\ J\dot{\omega}(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} (y - h(t)) \times \sigma(u,p)\nu \,\mathrm{d}s & (t > 0), \\ u(0,y) = u_0(y) & (y \in \mathcal{S}(0)), \\ h(0) = 0,\ \dot{h}(0) = \ell_0,\ \omega(0) = \omega_0. \end{cases} \tag{1.1}$$

In the above equations, $\omega(t)$ represents the angular velocity of the ball (with respect to its centre) and the fluid is supposed to be homogeneous with density equal to 1 and of constant viscosity $\mu > 0$. Moreover, the unit vector field normal to $\partial S(t)$ and directed towards the interior of $\mathcal{S}(t)$ is denoted by $\nu(t,\cdot)$. The constant $m > 0$ and the matrix J stand for the mass and the inertia tensor of the rigid body. Since in the above equations the rigid body is a homogeneous ball of radius R, we have

$$J = \frac{2mR^2}{5}\mathbb{I}_3.$$

Finally, the Cauchy stress tensor field in the fluid is given by the constitutive law

$$\sigma(u,p)_{k\ell} = -p\delta_{k\ell} + \mu\left(\frac{\partial u_k}{\partial y_\ell} + \frac{\partial u_\ell}{\partial y_k}\right) \qquad (1 \leqslant k, \ell \leqslant 3),$$

where $\delta_{k\ell}$ stands for the Kronecker symbol.

The system (1.1) can be easily transformed into a system in which the fluid equation is set in a fixed spatial domain. Indeed, using the change of frame $x \mapsto y(t,x) := x + h(t)$ and setting

$$v_0(x) = u_0(x), \quad v(t,x) = u(t, x+h(t)), \quad \pi(t,x) = p(t, x+h(t)), \quad \ell(t) = \dot{h}(t) \qquad (t > 0, x \in \mathcal{F}(0)),$$

and $E := \mathcal{F}(0) = \mathbb{R}^3 \setminus B$, equations (1.1) can be written in the form of the following system of unknowns v, π, ℓ and ω:

$$\begin{cases} \partial_t v + [(v - \ell) \cdot \nabla]\, v - \mu\Delta v + \nabla\pi = 0 & (t > 0,\ x \in E), \\ \operatorname{div} v = 0, & (t > 0,\ x \in E), \\ v = \ell + \omega \times x & (t > 0,\ x \in \partial E), \\ m\dot{\ell}(t) = -\displaystyle\int_{\partial E} \sigma(v,\pi)\nu \,\mathrm{d}s, & (t > 0), \\ J\dot{\omega}(t) = -\displaystyle\int_{\partial E} x \times \sigma(v,\pi)\nu \,\mathrm{d}s, & (t > 0), \\ v(0,x) = v_0(x) & (x \in E), \\ \ell(0) = \ell_0,\ \omega(0) = \omega_0. \end{cases} \tag{1.2}$$

As far as we know, the initial and boundary value problem (1.2) has been first studied in Serre [20], where it is proved, in particular, that (1.2) admits global in time weak solutions (of Leray type). The existence and uniqueness of strong solutions, with initial velocity supposed to be small (in the Sobolev space $W^{1,2}$) has been first established in Cumsille and Takahashi [4]. For the L^p theory for the local in time existence and uniqueness of strong solutions of (1.2) we refer to Geissert, Götze and Hieber [7]. Let us also mention that the analogue of (1.2) when the fluid-rigid body system fills a bounded cavity Ω (instead of the whole $\mathbb{R}^3$) has also been studied in a quite important number of papers (see, for instance, Maity and Tucsnak [18] and references therein). A natural question when considering (1.2) is the large time behaviour of the position of the mass centre of the ball, i.e., of the function h defined by

$$h(t) = \int_0^t \ell(s)\,\mathrm{d}s \qquad\qquad (t > 0).$$

It is in particular important to establish whether the centre of the rigid ball stabilizes around some position in $\mathbb{R}^3$ or its distance to the origin tends to infinity when $t \to \infty$. As far as we know, this question is open in the three dimensional context of (1.2). However, if one replaces the rigid ball by an infinite cylinder (so that the fluid can be modeled by the Navier-Stokes equations in two space dimensions) the question is studied in Ervedoza, Hillairet and Lacave [6], where it is established that the norm of $\ell(t)$ behaves like $\frac{1}{t}$ when $t \to \infty$, thus not excluding the possibility of an unbounded trajectory of the rigid ball. Other results in the same spirit concern Burgers type models for the fluid, like Vázquez and Zuazua [25], or one dimensional viscous compressible fluids, like Koike [15].

The main novelty brought in by our work is twofold. Firstly, we prove that (1.2) is well-posed (globally in time) for initial data which are small in appropriate L^q type spaces. Secondly, by appropriately choosing q, we prove that there exists $h_\infty \in \mathbb{R}^3$ such that $\lim_{t\to\infty} h(t) = h_\infty$, i.e., that the rigid body "stops" as $t \to \infty$.

To state our main result we first recall that if $G \subset \mathbb{R}^3$ is an open set, $q > 1$ and $s \in \mathbb{R}$, the notation $L^q(G)$ and $W^{s,q}(G)$ stands for the standard Lebesgue and Sobolev-Slobodeckij spaces, respectively. Our main result can be stated as follows:

Theorem 1.1. *With the above notation for the set E, let $q \in (1, 3/2]$. Then there exists $\varepsilon_0 > 0$ such that for every $v_0 \in [L^q(E)]^3 \cap \left[L^3(E)\right]^3$ and $\ell_0,\ \omega_0 \in \mathbb{R}^3$ with*

$$v_0 \in \left[W^{2,3/2}(E)\right]^3, \tag{1.3}$$

$$\operatorname{div} v_0 = 0 \text{ in } \ E, \tag{1.4}$$

$$v_0(x) = \ell_0 + \omega_0 \times x \qquad (x \in \partial B), \tag{1.5}$$

$$\|v_0\|_{[L^q(E)]^3} + \|v_0\|_{[L^3(E)]^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} \leqslant \varepsilon_0, \tag{1.6}$$

there exists a solution (v, ℓ, ω) of (1.2) in $C^0([0,\infty); \left[L^3(E)\right]^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$ such that

$$\sup_{t>0} \{t^{3/2(1/q-2/3)}(\|v(t)\|_{[L^{3/2}(E)]^3} + \|\ell(t)\|_{\mathbb{R}^3} + \|\omega(t)\|_{\mathbb{R}^3})\} < \infty. \tag{1.7}$$

Moreover, for every $q_1 \in [3, \infty)$ this solution satisfies

$$\sup_{t>0}\{t^{3/2(1/q-1/q_1)}(\|v(t)\|_{[L^{q_1}(E)]^3} + \|\ell(t)\|_{\mathbb{R}^3} + \|\omega(t)\|_{\mathbb{R}^3})\} < \infty, \tag{1.8}$$

and

$$\sup_{t>0} \|\nabla v(t)\|_{\left[L^{3/2}(E)\right]^9} < \infty.$$

In particular, if $q < 3/2$, taking q_1 sufficiently large in (1.8) we have that $\ell \in L^1([0,\infty); \mathbb{R}^3)$, hence that the position of the centre of the moving rigid ball converges to some point at finite distance $h_\infty \in \mathbb{R}^3$ as $t \to \infty$.

The proof of Theorem 1.1 is based on decay estimates for the solutions of the linearized version of (1.2). Therefore, an important part of this work is devoted to the study of the semigroup associated to the linearized problem. As shown in the forthcoming sections, this semigroup, called the *fluid-structure* semigroup, and its generator (called the *fluid-structure operator*) share several important properties of the Stokes semigroup and Stokes operator in an exterior domain. To establish this fact an essential step consists in proving that the resolvent estimates derived in Iwashita [13] and Giga [8] for the Stokes operator also hold for the fluid-structure operator (see also the corresponding estimates for the non autonomous system describing the Navier-Stokes flow around a rotating obstacle, which have been obtained in Hishida [11] and [12]). Our results on the linearized problem will be derived for a solid of arbitrary shape, opening the way to a generalization of Theorem 1.1 to situations in which the solid is not necessarily a ball. However, the fixed point methodology used in the present paper to pass from the linearized equations to the full nonlinear problem requires that the rigid body is a ball (see the comments in Section 9 below concerning some tracks towards the modification of this procedure for tackling rigid bodies of arbitrary shape).

Note that Theorem 1.1 refers to mild solutions of (1.2), i.e., satisfying the integral equation

$$\begin{bmatrix} v(t,\cdot) \\ \ell(t) \\ \omega(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} v_0 \\ \ell_0 \\ \omega_0 \end{bmatrix} + \int_0^t \mathbb{T}_{t-s}\mathbb{P} f(s)\,\mathrm{d}s \qquad (t \geqslant 0), \tag{1.9}$$

where

$$f(s,x) = -\mathbb{1}_E(x)\ \left[(v(s,x) - \ell(s))\cdot\nabla\right] v(s,x)) \qquad (x \in E,\ s \geqslant 0),$$

$\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is the *fluid-structure semigroup* and $\mathbb{P}$ is a Leray type projector on the space of free divergence vector fields on $\mathbb{R}^3$ which coincide with a rigid velocity field on B. A precise definition of these objects requires some preparation and notation, so it is postponed to Section 3. However, we mention here that the roles of the projector $\mathbb{P}$ and of the fluid-structure semigroup in this paper are very close to those played by the Leray projector and the Stokes semigroup in the analysis of the Navier-Stokes equations. Consequently, the construction and study of the fluid-structure semigroup and of its generator are essential steps of our analysis, which are detailed in Sections 4, 5, 6 and 7.

As shown in Section 8, Theorem 1.1 can be extended to weaker notions of solutions, in particular to initial data $v_0 \in [L^q(E)]^3 \cap \left[L^3(E)\right]^3$ and $v_0,\ \omega_0 \in \mathbb{R}^3$ which satisfy (1.6) and

$$\operatorname{div} v_0 = 0 \text{ in } \ E, \qquad v_0 \cdot \nu = (\ell_0 + \omega_0 \times x)\cdot\nu \text{ on } \partial B. \tag{1.10}$$

In this case, the solution of (1.2) is given in a much weaker sense which will be made precise in Section 8 Theorem 8.10. Section 9 will then provide some further comments.

The outline of the paper is as follows. In Section 2, we introduce the notation (in particular several function spaces) that will be used throughout the article and we recall several results on the Stokes system exterior domains. In Section 3 we introduce the *fluid-structure operator* and we give some of its basic properties. Section 4 is devoted to resolvent estimates for the *fluid-structure operator*. We use existing results on the Stokes system exterior domain to derive our results. In Sections 5 - 6 we show that the *fluid-structure operator* generates a bounded analytic semigroup on a suitable Banach space. We prove, in particular, $L^p - L^q$ decay estimates for the fluid-structure semigroup in Section 7. Section 8 is devoted to the proof of main result, Theorem 1.1 and its generalization. In Section 9, we formulate some open problems. Some technical results are collected in Appendix A and in Appendix B.

2 Notation and preliminaries

Throughout this paper, the notation

$$\mathbb{N},\ \mathbb{Z},\ \mathbb{R},\ \mathbb{C}$$

stands for the sets of natural numbers (starting with 1), integers, real numbers and complex numbers, respectively. For $n \in \mathbb{N}$, the euclidian norm on $\mathbb{C}^n$ will be simply denoted by $|\cdot|$. For $\theta \in (0, \pi)$ we define the sector Σ_θ in the complex plane by

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} \quad | \quad |\arg\lambda| < \theta\}. \tag{2.1}$$

Moreover, $\mathbb{Z}_+$ stands for $\mathbb{N} \cup \{0\}$. For $n,\ m \in \mathbb{N}$, $u : \mathbb{R}^n \to \mathbb{R}^m$ and $\alpha \in \mathbb{Z}_+^n$ we set $|\alpha| = \sum_{k=1}^n \alpha_k$ and we use the notation $\partial^\alpha u$ for the partial derivative $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}$.

If $G \subset \mathbb{R}^3$ is an open set, $q > 1$ and $k \in \mathbb{N}$, we denote the standard Lebesgue and Sobolev spaces by $L^q(G)$ and by $W^{k,q}(G)$, respectively. For $s \in \mathbb{R}$, $W^{s,q}(G)$ denotes the Sobolev-Slobodeckij spaces. The norms on $[L^q(G)]^n$ and $\left[W^{k,q}(G)\right]^n$ with $n \in \mathbb{N}$, will be denoted by $\|\cdot\|_{q,G}$ and $\|\cdot\|_{k,q,G}$, respectively. When $G = \mathbb{R}^3$, these norms will be simply denoted by $\|\cdot\|_q$ and $\|\cdot\|_{k,q}$, respectively. Moreover, the space $W_0^{k,q}(G)$ is the completion of $C_0^\infty(G)$ with respect to the $W^{k,q}(G)$ norm. We also introduce the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(G) := \left\{ f \in L^q_{\mathrm{loc}}(\overline{G}) \mid \nabla f \in L^q(G)^3 \right\},$$

with the norm

$$\|f\|_{\widehat{W}^{1,q}(G)} := \|\nabla f\|_{q,G},$$

where we identify to elements differing by a constant. If ∂G is locally Lipschitz, then we have

$$\overline{C_0^\infty(\overline{G})}^{\|\nabla\cdot\|_{L^q}} = \widehat{W}^{1,q}(G).$$

Furthermore, if G is bounded then we can fix a representative $f \in \widehat{W}^{1,q}(G)$ by $\displaystyle\int_G f \ \mathrm{d}x = 0$. Thus by setting

$$L_0^q(G) = \left\{ f \in L^q(G) \quad \middle| \quad \int_G f \ \mathrm{d}x = 0 \right\},$$

we can write

$$\widehat{W}^{1,q}(G) = W^{1,q}(G) \cap L_0^q(G). \tag{2.2}$$

We also introduce the space

$$L^q_\sigma(G) = \overline{\left\{ \varphi \in [C_0^\infty(G)]^3 \mid \mathrm{div}\ \varphi = 0 \right\}}^{\|\cdot\|_{q,G}}.$$

For $k \in \mathbb{N}$, and $s, q \in \mathbb{R}$ with $1 < q < \infty$, we define the weighted Sobolev spaces $W^{k,q,s}(G)$ by

$$W^{k,q,s}(G) = \left\{ f \ \middle|\ (1 + |x|^2)^{s/2}\, \partial^\alpha f \in L^q(G),\ |\alpha| \leqslant k \right\}, \tag{2.3}$$

and we set $L^{q,s}(G) = W^{0,q,s}(G)$. For $\varphi \in W^{1,q}(G)$ we denote by $D(\varphi)$ the associated strain field defined by

$$D(\varphi)_{ij} = \frac{1}{2}\left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right) \qquad (i,\ j \in \{1, 2, 3\}). \tag{2.4}$$

We use repeatedly below the following well known result due to Bogovskii ([1]):

Lemma 2.1. *Let G be a smooth bounded domain in $\mathbb{R}^3$, $q \in (1,\infty)$ and $k \in \mathbb{Z}_+$. Then there exists a linear bounded operator $\mathbb{B}_G$ from $\left[W_0^{k,q}(G)\right]^3 \cap [L_0^q(G)]^3$ to $[W_0^{k+1,q}(G)]^3$ such that*

$$\operatorname{div}\ (\mathbb{B}_G f) = f \qquad \left(f \in \left[W_0^{k,q}(G)\right]^3 \cap [L_0^q(G)]^3\right). \tag{2.5}$$

To end this section we remind, following Borchers and Sohr [2] and Iwashita [13], several results on the Stokes system in the exterior domain $E = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$, where $\mathcal{O} \subset \mathbb{R}^3$ is an open bounded set with $\partial\mathcal{O}$ of class C^2. More precisely, we consider the stationary Stokes problem:

$$\begin{cases} \lambda v - \mu\Delta v + \nabla p = f & (x \in E), \\ \operatorname{div} v = 0 & (x \in E), \\ v = 0 & (x \in \partial\mathcal{O}). \end{cases} \tag{2.6}$$

By combining Theorem 1.2 in [2] and Corollary 3.2 in [13] we have:

Theorem 2.2. *Let $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and let Σ_θ be the set defined in* (2.1). *Then*

1. *Then there exist two families of operators $(R(\lambda))_{\lambda\in\Sigma_\theta}$ and $(P(\lambda))_{\lambda\in\Sigma_\theta}$ such that for every $\lambda \in \Sigma_\theta$ we have*

$$R(\lambda) \in \mathcal{L}\left([L^q(E)]^3, \left[W^{2,q}(E)\right]^3\right), \qquad P(\lambda) \in \mathcal{L}\left([L^q(E)]^3, \widehat{W}^{1,q}(E)\right), \qquad (q>1),$$

and the functions $v = R(\lambda)f$ and $p = P(\lambda)f$ satisfy (2.6). *Moreover, there exists a positive constant M, depending only on $\mathcal{O}$, q and θ such that for every $\lambda \in \Sigma_\theta$ we have*

$$|\lambda|\|R(\lambda)f\|_{q,E} + \|\mu\Delta R(\lambda)f - \nabla P(\lambda)f\|_{q,E} \leqslant M\|f\|_{q,E} \qquad \left(q>1,\ f \in [L^q(E)]^3\right). \tag{2.7}$$

2. *For every $q > 1$, $\lambda \in \Sigma_\theta$, $m \in \mathbb{Z}_+$, $s > 3\left(1 - \frac{1}{q}\right)$ and $s' < -\frac{3}{q}$, we have*

$$R(\lambda) \in \mathcal{L}\left([W^{m,q,s}(E)]^3, \left[W^{m+2,q,s'}(E)\right]^3\right), \qquad P(\lambda) \in \mathcal{L}\left([W^{m,q,s}(E)]^3, W^{m+1,q,s'}(E)\right).$$

Moreover, the functions $\lambda \mapsto R(\lambda)$ and $\lambda \mapsto P(\lambda)$ are holomorphic from Σ_δ to $\mathcal{L}\left([W^{m,q,s}(E)]^3, \left[W^{m+2,q,s'}(E)\right]^3\right)$ and $\mathcal{L}\left([W^{m,q,s}(E)]^3, W^{m+1,q,s'}(E)\right)$, respectively. Finally, there exist

$$R_0 \in \mathcal{L}\left([W^{m,q,s}(E)]^3, \left[W^{m+2,q,s'}(E)\right]^3\right), \qquad P_0 \in \mathcal{L}\left([W^{m,q,s}(E)]^3, W^{m+1,q,s'}(E)\right)$$

such that

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0} |\lambda|^{-\frac{1}{2}}\ \|R(\lambda) - R_0\|_{\mathcal{L}\left([W^{m,q,s}(E)]^3, \left[W^{m+2,q,s'}(E)\right]^3\right)} < \infty, \tag{2.8}$$

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0} |\lambda|^{-\frac{1}{2}}\ \|P(\lambda) - P_0\|_{\mathcal{L}\left([W^{m,q,s}(E)]^3, W^{m+1,q,s'}(E)\right)} < \infty. \tag{2.9}$$

Remark 2.3. Setting $R(0) := R_0$ and $P(0) := P_0$, estimates (2.8) and (2.9) imply that the functions $\lambda \mapsto R(\lambda)$ and $\lambda \mapsto P(\lambda)$ extend to continuous functions from $\Sigma_\theta \cup \{0\}$ to $\mathcal{L}\left([W^{m,q,s}(E)]^3, \left[W^{m+2,q,s'}(E)\right]^3\right)$ and $\mathcal{L}\left([W^{m,q,s}(E)]^3, W^{m+1,q,s'}(E)\right)$, respectively.

3 Some background on the fluid-structure operator

3.1 Definition and first properties

In this section we introduce the fluid-structure operator and the fluid-structure semigroup and we remind some of their properties, as established in the existing literature. For the remaining part of this section the notation Ω designs either an open, connected and bounded subset of $\mathbb{R}^3$, with $\partial\Omega$ of class C^2, or we have $\Omega = \mathbb{R}^3$. Let $\mathcal{O}$ be an open bounded set with smooth boundary such that $\overline{\mathcal{O}} \subset \Omega$. We denote $E_\Omega = \Omega \setminus \overline{\mathcal{O}}$ and we set $E_{\mathbb{R}^3} := E$. Moreover, we denote by ν the unit normal vector on $\partial\mathcal{O}$ oriented towards the interior of $\mathcal{O}$.

Reminding notation (2.4) for the tensor field D, we introduce the function space

$$\mathbb{X}^q(\Omega) = \left\{ \Phi \in [L^q_\sigma(\Omega)]^3 \quad \middle| \quad D(\Phi) = 0 \text{ in } \mathcal{O} \right\}, \tag{3.1}$$

associated to the sets Ω and $\mathcal{O}$, which plays an important role in this work. Note that, for every $q \in (1,\infty)$ the dual $(\mathbb{X}^q(\Omega))^*$ of $\mathbb{X}^q(\Omega)$ can be identified with $\mathbb{X}^{q'}(\Omega)$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, with the duality pairing

$$\langle f, g \rangle_{\mathbb{X}^{q'}(\Omega),\mathbb{X}^q(\Omega)} = \int_{\mathcal{O}} \rho f \cdot g \, \mathrm{d}x + \int_{E_\Omega} f \cdot g \, \mathrm{d}x \qquad (f \in \mathbb{X}^{q'}(\Omega),\ g \in \mathbb{X}^q(\Omega)),$$

where ρ is the constant density of the rigid body. Our notation is making explicit only the dependence of $\mathbb{X}^q$ on Ω since these spaces will be used later on for various Ω and with fixed $\mathcal{O}$. For $\Omega = \mathbb{R}^3$, we simply set

$$\mathbb{X}^q := \mathbb{X}^q(\mathbb{R}^3). \tag{3.2}$$

Since every Φ in $\mathbb{X}^q(\Omega)$ satisfies $D(\Phi) = 0$ in $\mathcal{O}$, there exists a unique couple $\begin{bmatrix} \ell \\ \omega \end{bmatrix} \in \mathbb{C}^3 \times \mathbb{C}^3$ such that

$$\Phi(x) = \varphi(x)\mathbb{1}_{E_\Omega}(x) + (\ell + \omega \times x)\mathbb{1}_{\mathcal{O}}(x) \qquad (x \in \Omega),$$

where $\mathbb{1}_U$ stands for the characteristic function of the set U (see for instance [23, Lemma 1.1]). We can thus use the identification:

$$\mathbb{X}^q(\Omega) \simeq \left\{ \begin{bmatrix} \varphi \\ \ell \\ \omega \end{bmatrix} \in [L^q(E_\Omega)]^3 \times \mathbb{C}^3 \times \mathbb{C}^3, \text{ with } \operatorname{div}(\varphi) = 0 \text{ in } E_\Omega, \right.$$
$$\left. \partial\varphi(x)\cdot\nu(x) = (\ell + \omega \times x)\cdot\nu \text{ for } x \in \partial\mathcal{O} \text{ and } \partial\varphi(x)\cdot\nu(x) = 0 \text{ for } x \in \partial\Omega \right\}. \tag{3.3}$$

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with $\partial\Omega$ of class C^2 or let $\Omega = \mathbb{R}^3$. Let $\mathbb{P}_{q,\Omega}$ be the projection operator from $[L^q(\Omega)]^3$ onto $\mathbb{X}^q(\Omega)$. When $\Omega = \mathbb{R}^3$, $\mathbb{P}_{q,\mathbb{R}^3}$ is simply denoted by $\mathbb{P}_q$. In the following two propositions, we recall the existence of such projection operator for $\Omega = \mathbb{R}^3$ as well as for Ω bounded domain.

Proposition 3.1. *Let $\mathcal{O}$ be an open bounded set of $\mathbb{R}^3$ with $\partial\mathcal{O}$ of class C^2.For $q > 1$ let G^q_1 and G^q_2 be the spaces*

$$G^q_1 = \left\{ u \in \left[L^q(\mathbb{R}^3)\right]^3 \ \middle| \ u = \nabla q_1 \text{ for some } q_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \right\},$$

$$G^q_2 = \left\{ u \in \left[L^q(\mathbb{R}^3)\right]^3 \ \middle| \ \begin{array}{c} \operatorname{div} u = 0 \text{ in } \mathbb{R}^3, \quad u = \nabla q_2 \text{ in } E, \quad q_2 \in L^1_{loc}(\mathbb{R}^3), \\ u = \varphi \text{ in } \mathcal{O} \text{ with } \varphi \in [L^q(\mathcal{O})]^3, \\ \displaystyle\int_{\mathcal{O}} \varphi \, \mathrm{d}y = -\int_{\partial\mathcal{O}} q_2 \nu \, \mathrm{d}\gamma, \quad \int_{\mathcal{O}} \varphi \times y \, \mathrm{d}y = -\int_{\partial\mathcal{O}} q_2 \nu \times y \, \mathrm{d}\gamma \end{array} \right\}.$$

Then for every $u \in \left[L^q(\mathbb{R}^3)\right]^3$ *there exists a unique triple* $\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix} \in \mathbb{X}^q \times G_1^q \times G_2^q$ *with*

$$u = v + w_1 + w_2. \tag{3.4}$$

The map $u \mapsto v$*, denoted* $\mathbb{P}_q$*, is a projection operator form* $[L^q(\Omega)]^3$ *onto* $\mathbb{X}^q(\Omega)$*. Moreover, the dual of the operator* $\mathbb{P}_q$ *is* $\mathbb{P}_{q'}$*, where* $\dfrac{1}{q} + \dfrac{1}{q'} = 1$*.*

For the proof of Proposition 3.1 we refer to Wang and Xin [26, Theorem 2.2].

Proposition 3.2. *Let* $\Omega \subset \mathbb{R}^3$ *be an open bounded set with* $\partial\Omega$ *of class* C^2 *. Let* $\mathcal{O}$ *be an open bounded set with* $\partial\mathcal{O}$ *of class* C^2 *such that* $\overline{\mathcal{O}} \subset \Omega$*. For* $q > 1$ *let* $G_1^q(\Omega)$ *and* $G_2^q(\Omega)$ *be the spaces*

$$G_1^q(\Omega) = \left\{ u \in [L^q(\Omega)]^3 \quad | \quad u = \nabla q_1 \text{ for some } q_1 \in W_0^{1,q}(\Omega) \right\},$$

$$G_2^q(\Omega) = \left\{ u \in [L^q(\Omega)]^3 \quad \middle| \quad \begin{array}{c} \operatorname{div} u = 0 \text{ in } \Omega, \quad u = \nabla q_2 \text{ in } E_\Omega, \quad q_2 \in W^{1,q}(E_\Omega), \\ u = \varphi \text{ in } \mathcal{O} \text{ with } \varphi \in [L^q(\mathcal{O})]^3, \\ \displaystyle\int_{\mathcal{O}} \varphi \, \mathrm{d}y = -\int_{\partial\mathcal{O}} q_2 \nu \, \mathrm{d}\gamma, \quad \int_{\mathcal{O}} \varphi \times y \, \mathrm{d}y = -\int_{\partial\mathcal{O}} q_2 \nu \times y \, \mathrm{d}\gamma \end{array} \right\}.$$

Then for every $u \in [L^q(\Omega)]^3$ *there exists a unique triple* $(v, w_1, w_2) \in \mathbb{X}^q(\Omega) \times G_1^q(\Omega) \times G_2^q(\Omega)$ *with*

$$u = v + w_1 + w_2. \tag{3.5}$$

The map $u \mapsto v$*, denoted* $\mathbb{P}_{q,\Omega}$*, is a projection operator form* $[L^q(\Omega)]^3$ *onto* $\mathbb{X}^q(\Omega)$*. Furthermore, the dual of the operator* $\mathbb{P}_{q,\Omega}$ *is* $\mathbb{P}_{q',\Omega}$*, where* $\dfrac{1}{q} + \dfrac{1}{q'} = 1$*.*

The proof of Proposition 3.2 is similar to the proof of [26, Theorem 2.2]. However, for the sake of completeness we provide a short proof in Appendix A.

The *fluid-structure operator on* Ω is the operator $\mathbb{A}_{q,\Omega} : \mathcal{D}(\mathbb{A}_{q,\Omega}) \to \mathbb{X}^q(\Omega)$ defined, for every $q > 1$, by

$$\mathcal{D}(\mathbb{A}_{q,\Omega}) = \left\{ \varphi \in \left[W_0^{1,q}(\Omega)\right]^3 \cap \mathbb{X}^q(\Omega) \quad \middle| \quad \varphi_{|E_\Omega} \in \left[W^{2,q}(E_\Omega)\right]^3 \right\}, \tag{3.6}$$

$$\mathbb{A}_{q,\Omega}\varphi = \mathbb{P}_{q,\Omega}\mathcal{A}_{q,\Omega}\varphi \qquad (\varphi \in \mathcal{D}(\mathbb{A}_{q,\Omega})), \tag{3.7}$$

where $\mathbb{P}_{q,\Omega}$ is the projector introduced in Proposition 3.1, and the operator $\mathcal{A}_{q,\Omega} : \mathcal{D}(\mathcal{A}_{q,\Omega}) \to [L^q(\Omega)]^3$ is defined by $\mathcal{D}(\mathcal{A}_{q,\Omega}) = \mathcal{D}(\mathbb{A}_{q,\Omega})$ and for every $\varphi \in \mathcal{D}(\mathcal{A}_{q,\Omega})$,

$$\mathcal{A}_{q,\Omega}\varphi = \begin{cases} \mu\Delta\varphi & \text{in } E_\Omega, \\ -2\mu m^{-1} \displaystyle\int_{\partial\mathcal{O}} D(\varphi)\nu \, \mathrm{d}\gamma - \left(2\mu\mathcal{J}^{-1} \int_{\partial\mathcal{O}} y \times D(\varphi)\nu \, d\gamma \right) \times y & \text{in } \mathcal{O}. \end{cases} \tag{3.8}$$

where m and J are given in terms of the constant density ρ of the body by

$$m = \int_{\mathcal{O}} \rho \, dx, \qquad \mathcal{J} = (\mathcal{J}_{k,\ell})_{k,\ell\in\{1,2,3\}} \text{ with } \mathcal{J}_{k,\ell} = \int_{\mathcal{O}} \rho \left(\delta_{k,\ell} \, |x|^2 - x_k x_\ell \right) dx.$$

In the case $\Omega = \mathbb{R}^3$, the operators $\mathbb{P}_{q,\Omega}, \mathcal{A}_{q,\Omega}$ and $\mathbb{A}_{q,\Omega}$ are denoted by $\mathbb{P}_q, \mathcal{A}_q$ and $\mathbb{A}_q$, respectively and $\mathbb{A}_q : \mathcal{D}(\mathbb{A}_q) \to \mathbb{X}^q$ is defined, for every $q > 1$, by

$$\mathcal{D}(\mathbb{A}_q) = \left\{ \varphi \in \left[W^{1,q}(\mathbb{R}^3)\right]^3 \cap \mathbb{X}^q \quad \middle| \quad \varphi_{|E} \in \left[W^{2,q}(E)\right]^3 \right\}, \tag{3.9}$$

$$\mathbb{A}_q\varphi = \mathbb{P}_q\mathcal{A}_q\varphi \qquad (\varphi \in \mathcal{D}(\mathbb{A}_q)). \tag{3.10}$$

In the case $q = 2$ and when $\mathcal{O}$ is a ball, the fluid-structure operator A_q has been introduced in Takahashi and Tucsnak [22], where it has been proven that this operator generates an analytic semigroup on $\mathbb{X}^2$. Later, Wang and Xin [26] proved that the operator $\mathbb{A}_q$ generates an analytic semigroup on $\mathbb{H}^{6/5}(\mathbb{R}^3) \cap \mathbb{H}^q(\mathbb{R}^3)$ if $q \geqslant 2$ and that if the solid is a ball in $\mathbb{R}^3$ the operator $\mathbb{A}_{q,\mathbb{R}^3}$ generates an analytic semigroup (not necessarily bounded) on $\mathbb{H}^2(\mathbb{R}^3) \cap \mathbb{H}^q(\mathbb{R}^3)$ if $q \geqslant 6$.

It is important for future use to rephrase the resolvent equation for $\mathbb{A}_{q,\Omega}$ in a form involving only PDEs and algebraic constraints. To this aim, for $\lambda \in \mathbb{C}$, we consider the system

$$\begin{cases} \lambda v - \mu\Delta v + \nabla\pi = f & (x \in E_\Omega), \\ \operatorname{div} v = 0 & (x \in E_\Omega), \\ v = 0 & (x \in \partial\Omega), \\ v = \ell + \omega \times x & (x \in \partial\mathcal{O}), \\ m\lambda\ell = -\displaystyle\int_{\partial\mathcal{O}} \sigma(v,\pi)\nu \,\mathrm{d}s + f_\ell, \\ \mathcal{J}\lambda\omega = -\displaystyle\int_{\partial\mathcal{O}} x \times \sigma(v,\pi)\nu \,\mathrm{d}s + f_\omega. \end{cases} \tag{3.11}$$

In the above system the unknowns are $v,\ \pi,\ \ell$ and ω, whereas

$$\sigma(v,\pi) := -\pi I + 2\nu D(v).$$

By slightly adapting the methodology used in [21, 22] for the case $q = 2$, it can be checked that we have the following equivalence:

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^3$ be an open, connected and bounded set with $\partial\Omega$ of class C^2 or $\Omega = \mathbb{R}^3$. Let $1 < q < \infty$ and $\lambda \in \mathbb{C}$. Assume that $f \in [L^q(E_\Omega)]^3$ and $f_\ell, f_\omega \in \mathbb{C}^3$. Then $(v,\pi,\ell,\omega) \in \left[W^{2,q}(E_\Omega)\right]^3 \times \widehat{W}^{1,q}(E_\Omega) \times \mathbb{C}^3 \times \mathbb{C}^3$ satisfies (3.11) if*

$$(\lambda I - \mathbb{A}_{q,\Omega})V = \mathbb{P}_{q,\Omega}F, \tag{3.12}$$

where

$$V = v\mathbb{1}_{E_\Omega} + (\ell + \omega \times x)\mathbb{1}_{\mathcal{O}}, \quad F = \mathbb{P}_{q,\Omega}\left(f\mathbb{1}_{E_\Omega} + \left(m^{-1}f_\ell + \mathcal{J}^{-1}x \times f_\omega\right)\mathbb{1}_{\mathcal{O}}\right).$$

Conversely, assume that $F \in \mathbb{X}^q(\Omega)$ and $V \in \mathcal{D}(\mathbb{A}_{q,\Omega})$ satisfy (3.12). Then there exists $\pi \in \widehat{W}^{1,q}(E_\Omega)$ such that $(v,\ell,\omega) \in \left[W^{2,q}(E_\Omega)\right]^3 \times \mathbb{C}^3 \times \mathbb{C}^3$ satisfies (3.11) where

$$v = V|_{E_\Omega}, \quad \ell = \frac{1}{m}\int_{\mathcal{O}} V \,\mathrm{d}x, \quad \omega = -\mathcal{J}^{-1}\int_{\mathcal{O}} V \times x \,\mathrm{d}x,$$

and

$$f = F|_{E_\Omega}, \quad f_\ell = \frac{1}{m}\int_{\mathcal{O}} F \,\mathrm{d}x, \quad f_\omega = -\mathcal{J}^{-1}\int_{\mathcal{O}} F \times x \,\mathrm{d}x.$$

3.2 The fluid-structure semigroup on bounded domains

In this subsection we assume that Ω is an open bounded set in $\mathbb{R}^3$ with boundary of class C^2. In this case the operator $\mathbb{A}_{q,\Omega}$ has been extensively studied in Maity and Tucsnak [18]. In particular, by combining Theorem 1.3 and Theorem 4.1 from [18] we have:

Theorem 3.4. *With the above notation, let $q > 1$ and assume that $\Omega \subset \mathbb{R}^3$ is bounded, with $\partial\Omega$ of class C^2. Then the operator $\mathbb{A}_{q,\Omega}$, defined in (3.6)-(3.7), generates an analytic and exponentially stable C^0-semigroup, denoted $\mathbb{T}^{q,\Omega} = \left(\mathbb{T}^{q,\Omega}_t\right)_{t\geqslant 0}$, on $\mathbb{X}^q(\Omega)$.*

The above result has the following consequence, which follows by standard analytic semigroups theory:

Corollary 3.5. *With the notation and under the assumptions in Theorem 3.4, for every $\theta \in \left(\frac{\pi}{2}, \pi\right)$ the exists a constant M, possibly depending on q, θ, $\mathcal{O}$ and Ω, such that*

$$(1+|\lambda|)\left\|(\lambda I - \mathbb{A}_{q,\Omega})^{-1}\right\|_{\mathcal{L}(\mathbb{X}_q(\Omega))} + \left\|\mathbb{A}_{q,\Omega}(\lambda I - \mathbb{A}_{q,\Omega})^{-1}\right\|_{\mathcal{L}(\mathbb{X}_q(\Omega))} \leqslant M \qquad (\lambda \in \Sigma_\theta \cup \{0\}).$$

By combining Corollary 3.5 and Proposition 3.3 we obtain the following result:

Proposition 3.6. *Let $\theta \in (\pi/2, \pi)$, $q \in (1, \infty)$ and assume that $\Omega \subset \mathbb{R}^3$ is bounded, with $\partial\Omega$ of class C^2. Then there exists a constant $C > 0$, possibly depending on θ, q, Ω and $\mathcal{O}$, such that for all $\lambda \in \Sigma_\theta$, $f \in [L^q(E_\Omega)]^3$ and f_ℓ, $f_\omega \in \mathbb{C}^3$, there exists a unique solution $(v, \pi, \ell, \omega) \in \left[W^{2,q}(E_\Omega)\right]^3 \times \widehat{W}^{1,q}(E_\Omega) \times \mathbb{C}^3 \times \mathbb{C}^3$ of (3.11) satisfying*

$$(1+|\lambda|)\left(\|v\|_{q,E_\Omega} + |\ell| + |\omega|\right) + \|v\|_{2,q,E_\Omega} + \|\nabla\pi\|_{q,E_\Omega} \leqslant C\left(\|f\|_{q,E_\Omega} + |f_\ell| + |f_\omega|\right). \tag{3.13}$$

We need below the following slight generalization of Proposition 3.6:

Corollary 3.7. *With the notation in Proposition 3.6, let $v \in \left[W^{2,q}(E_\Omega)\right]^3$, $\pi \in \widehat{W}^{1,q}(E_\Omega)$, ℓ, $\omega \in \mathbb{C}^3$ be such that*

$$\begin{aligned} &v(x) = 0 && (x \in \partial\Omega)\\ &v = \ell + \omega \times x && (x \in \partial\mathcal{O}),\\ &\operatorname{div} v \in W_0^{1,q}(E_\Omega), \quad \int_{E_\Omega} \operatorname{div} v \, \mathrm{d}x = 0. \end{aligned}$$

Then for every $\lambda_0 > 0$ there exists a constant $c = c(\Omega, p, \lambda_0, \theta)$ such that

$$\begin{aligned} |\lambda|\left(\|v\|_{q,E_\Omega} + |\ell| + |\omega|\right) + \|D^2 v\|_{q,E_\Omega} + \|\nabla\pi\|_{q,E_\Omega}&\\ \leqslant c\Big(\|\lambda v - \Delta v + \nabla\pi\|_{q,E_\Omega} + \|\nabla \operatorname{div} v\|_{q,E_\Omega}&\\ + \left|m\lambda\ell + \int_{\partial\mathcal{O}} \sigma(v,\pi)\nu \, \mathrm{d}s\right| + \left|\mathcal{J}\lambda\omega + \int_{\partial\mathcal{O}} x \times \sigma(v,\pi)\nu \, \mathrm{d}s\right|\Big), \end{aligned} \tag{3.14}$$

for every $\lambda \in \Sigma_\theta$ with $|\lambda| \leqslant \lambda_0$.

Proof. According to Lemma 2.1 there exists $\tilde{v} \in \left[W_0^{2,q}(E_\Omega)\right]^3$ such that $\operatorname{div}\tilde{v} = \operatorname{div} v$ on E_Ω and

$$\|\tilde{v}\|_{2,q,E_\Omega} \leqslant c\|\operatorname{div} v\|_{1,q,E_\Omega}, \tag{3.15}$$

where c is a constant depending only on Ω and on q. Setting $u = v - \tilde{v}$ we see that $u \in \left[W^{2,p}(E_\Omega)\right]^3$ and

$$\begin{aligned} &u(x) = 0 && (x \in \partial\Omega),\\ &u(x) = \ell + \omega \times x && (x \in \partial\mathcal{O}),\\ &\sigma(u,\pi) = \sigma(v,\pi) && (x \in \partial\mathcal{O}),\\ &\operatorname{div} u = 0 && (x \in E_\Omega). \end{aligned}$$

By applying Proposition 3.6 and elementary inequalities, it follows that

$$\begin{aligned} |\lambda|\left(\|u\|_{q,E_\Omega} + |\ell| + |\omega|\right) + \|D^2 u\|_{q,E_\Omega} + \|\nabla\pi\|_{q,E_\Omega} &\leqslant C\|m\lambda u - \mu\Delta u + \nabla\pi\|_{q,E_\Omega}\\ &+ C\left|\lambda\ell + \int_{\partial\mathcal{O}} \sigma(v,\pi)\nu \, \mathrm{d}s\right| + C\left|\mathcal{J}\lambda\omega + \int_{\partial\mathcal{O}} x \times \sigma(v,\pi)\nu \, \mathrm{d}s\right| \qquad (\lambda \in \Sigma_\theta). \end{aligned}$$

The above estimate and (3.15) imply the conclusion (3.14). □

4 From the Stokes operator in exterior domains to the fluid-structure operator in the whole space

In this section we study the fluid structure operator $\mathbb{A}_{q,\Omega}$, defined in (3.9)-(3.10), in the case $\Omega = \mathbb{R}^3$. As mentioned in Sections 2 and 3, in this case the space $\mathbb{X}^q(\Omega)$, defined in (3.1), and the operators $\mathbb{P}_{q,\Omega}$, $\mathbb{A}_{q,\Omega}$ are simply denoted by $\mathbb{X}^q$, $\mathbb{P}_q$ and $\mathbb{A}_q$, respectively. The main idea developed in this section is that the resolvent of the fluid-structure operator can be expressed in terms of the resolvent of the Stokes operator with homogeneous Dirichlet conditions on the boundary of an obstacle of arbitrary shape $\mathcal{O}$. The connection between these two families of resolvents is then used to study the behaviour of the of $(\lambda I - \mathbb{A}_q)^{-1}$ for λ close to zero, in the spirit of the similar results for the Stokes operator in exterior domains obtained by Iwashita [13].

Let $\mathcal{O}$ be an open, bounded subset of $\mathbb{R}^3$ with $\partial\mathcal{O}$ of class C^2 and let $E = \mathbb{R}^3 \setminus \mathcal{O}$. We consider the system

$$\begin{cases} \lambda u - \mu\Delta u + \nabla\pi = f & (x \in E), \\ \operatorname{div} u = 0 & (x \in E), \\ u = \ell + \omega \times x & (x \in \partial\mathcal{O}), \\ m\lambda\ell = -\displaystyle\int_{\partial\mathcal{O}} \sigma(u,\pi)\nu \, \mathrm{d}s + f_\ell, \\ \mathcal{J}\lambda\omega = -\displaystyle\int_{\partial\mathcal{O}} x \times \sigma(u,\pi)\nu \, \mathrm{d}s + f_\omega, \end{cases} \tag{4.1}$$

where $f \in [L^q(E)]^3$, $f_\ell, f_\omega \in \mathbb{C}^3$ and $\lambda \in \mathbb{C}$. In the above system the unknowns are $u,\ \pi,\ \ell$ and ω, whereas

$$\sigma(u,\pi) := -\pi I + 2\mu D(u).$$

To study the solvability of (4.1) we introduce several auxiliary operators.

Firstly, given $\lambda \in \mathbb{C}$ and $\ell,\ \omega \in \mathbb{C}^3$, we consider the boundary value problem:

$$\begin{cases} \lambda w - \mu\Delta w + \nabla\eta = 0, \quad \operatorname{div} w = 0 & (x \in E), \\ w(x) = \ell + \omega \times x & (x \in \partial\mathcal{O}), \end{cases} \tag{4.2}$$

and we remind the notation (2.3) (and more generally the notation in Section 2) for the possibly weighted Sobolev spaces in unbounded domains.

Proposition 4.1. *Assume that $\theta \in (0,\pi)$. Then for all $q > 1$, for every $\lambda \in \Sigma_\theta$ and $\ell, \omega \in \mathbb{C}^3$, the system (4.2) admits a unique solution $(w,\eta) \in \left[W^{2,q}(E)\right]^3 \times \widehat{W}^{1,q}(E)$. Moreover, let $(D_\lambda)_{\lambda\in\Sigma_\theta}$ be the family of operators defined by*

$$D_\lambda \begin{bmatrix} \ell \\ \omega \end{bmatrix} = \begin{bmatrix} w \\ \eta \end{bmatrix} \qquad (\lambda \in \Sigma_\theta,\ \ell,\omega \in \mathbb{C}^3), \tag{4.3}$$

where $(w,\eta) \in \left[W^{2,q}(E)\right]^3 \times \widehat{W}^{1,q}(E)$ is the solution of (4.2). Then for every $\lambda \in \Sigma_\theta$ and $m \in \mathbb{N}$, we have

$$D_\lambda \in \mathcal{L}(\mathbb{C}^6, \left[W^{m+1,q}(E)\right]^3 \times \widehat{W}^{m,q}(E)), \tag{4.4}$$

$$D_\lambda \in \mathcal{L}\left(\mathbb{C}^6, \left[W^{m+1,q,s'}(E)\right]^3 \times W^{m,q,s'}(E)\right) \qquad \left(s' < -\frac{3}{q}\right). \tag{4.5}$$

Finally, there exists

$$D_0 \in \bigcap_{m\in\mathbb{N}, q>1, s'<-\frac{3}{q}} \mathcal{L}\left(\mathbb{C}^6, \left[W^{m+1,q,s'}(E)\right]^3 \times W^{m,q,s'}(E)\right)$$

such that

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0} |\lambda|^{-\frac{1}{2}}\|D(\lambda)-D_0\|_{\mathcal{L}\left(\mathbb{C}^6,\left[W^{m+1,q,s'}(E)\right]^3\times W^{m,q,s'}(E)\right)}<\infty, \tag{4.6}$$

for every $m\in\mathbb{N}$, $q>1$ *and* $s'<-\frac{3}{q}$.

Proof. We choose two balls B_1 and B_2 in $\mathbb{R}^3$ such that $\overline{\mathcal{O}}\subset B_1\subset\overline{B}_1\subset B_2$. We define a cut-off function $\chi\in C^\infty(\mathbb{R}^3)$ such that $\chi(x)\in[0,1]$ for every $x\in\mathbb{R}^3$ and

$$\chi(x)=\begin{cases}1 & \text{if } x\in\overline{B}_1,\\ 0 & \text{if } x\in E\setminus B_2.\end{cases}$$

We set

$$\overline{w}(x)=\chi(x)(\ell+\omega\times x)-\mathbb{B}_{B_2\setminus\overline{B}_1}(\nabla\chi\cdot(\ell+\omega\times x)),$$

where $\mathbb{B}_{B_2\setminus\overline{B}_1}$ is the Bogovskii operator as introduced in Lemma 2.1. It is easy to see that, div $\overline{w}=0$ in E, $\overline{w}(x)=\ell+\omega\times x$ for $x\in\partial E$ and $\overline{w}\in W^{k,q}(E)$, for any $k\in\mathbb{N}$. Since $w=\widetilde{w}+\overline{w}$, where $\widetilde{w}$ satisfies

$$\begin{cases}\lambda\widetilde{w}-\mu\Delta\widetilde{w}+\nabla\eta=-\lambda\overline{w}+\nu\Delta\overline{w}, \quad \text{div } \widetilde{w}=0 & (x\in E),\\ \widetilde{w}=0 & (x\in\partial\mathcal{O}).\end{cases}$$

We can apply classical regularity results for Stokes (e.g. [13, Proposition 2.7(i)]) to get (4.4) and Theorem 2.2 to obtain (4.5)-(4.6). □

The above result allows us to introduce the family of operators $(\mathcal{T}_\lambda)_{\lambda\in\Sigma_\theta}\subset\mathcal{L}(\mathbb{C}^6)$ defined by

$$\mathcal{T}_\lambda\begin{bmatrix}\ell\\ \omega\end{bmatrix}=\begin{bmatrix}\displaystyle\int_{\partial\mathcal{O}}\sigma(w,\eta)\nu\,\mathrm{d}s\\ \displaystyle\int_{\partial\mathcal{O}}x\times\sigma(w,\eta)\nu\,\mathrm{d}s\end{bmatrix}\qquad(\lambda\in\Sigma_\theta,\ \ell,\omega\in\mathbb{C}^3), \tag{4.7}$$

where (w,η) is the solution of (4.2), given by D_λ according to (4.3).

Proposition 4.2. *Let* $\theta\in(0,\pi)$. *For every* $\lambda\in\Sigma_\theta$ *let* $(\mathcal{T}_\lambda)_{\lambda\in\Sigma_\theta}$ *be the operators defined in* (4.7) *and let* $(\mathcal{K}_\lambda)_{\lambda\in\Sigma_\theta}$ *be the family of operators defined by*

$$\mathcal{K}_\lambda=\begin{bmatrix}\lambda mI_3 & 0\\ 0 & \lambda\mathcal{J}\end{bmatrix}+\mathcal{T}_\lambda\qquad(\lambda\in\Sigma_\theta). \tag{4.8}$$

Then there exists $\mathcal{K}_0\in\mathcal{L}(\mathbb{C}^6)$ *invertible such that*

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0}|\lambda|^{-\frac{1}{2}}\,\|\mathcal{K}_\lambda-\mathcal{K}_0\|_{\mathcal{L}(\mathbb{C}^6)}<\infty. \tag{4.9}$$

Moreover, $\mathcal{K}_\lambda$ *is invertible for every* $\lambda\in\Sigma_\theta$ *and*

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0}|\lambda|^{-\frac{1}{2}}\ \left\|\mathcal{K}_\lambda^{-1}-\mathcal{K}_0^{-1}\right\|_{\mathcal{L}(\mathbb{C}^6)}<\infty. \tag{4.10}$$

Proof. For $\ell,\ \omega\in\mathbb{C}^3$ we set

$$\begin{bmatrix}w_0\\ \eta_0\end{bmatrix}=D_0\begin{bmatrix}\ell\\ \omega\end{bmatrix},\qquad \mathcal{K}_0\begin{bmatrix}\ell\\ \omega\end{bmatrix}=\begin{bmatrix}\displaystyle\int_{\partial\mathcal{O}}\sigma(w_0,\eta_0)\nu\,\mathrm{d}s\\ \displaystyle\int_{\partial\mathcal{O}}x\times\sigma(w_0,\eta_0)\nu\,\mathrm{d}s\end{bmatrix},$$

where D_0 is the operator introduced in Proposition 4.1. Applying Proposition 4.1 and a standard trace theorem it follows that (4.9) holds. The fact that $\mathcal{K}_0$ (which is called the *resistance matrix of*

$\mathcal{O}$) is invertible is a classical result (see, for instance, Happel and Brenner [9, Section 5.4], where it is shown that this matrix is strictly positive).

On the other hand, taking the inner product in $\left[L^2(E)\right]^3$ of the first equation in (4.2) by w, integrating by parts and using the second equation in (4.2) it follows that

$$\left\langle \mathcal{T}_\lambda \begin{bmatrix} \ell \\ \omega \end{bmatrix}, \begin{bmatrix} \ell \\ \omega \end{bmatrix} \right\rangle_{\mathbb{C}^6} = \lambda \int_E |w|^2 \,\mathrm{d}x + \mu \int_E |D(w)|^2 \,\mathrm{d}x \qquad (\ell,\ \omega \in \mathbb{C}^3, \lambda \in \Sigma_\theta). \tag{4.11}$$

Assume now that $\ell,\ \omega \in \mathbb{C}^3$ and $\lambda \in \Sigma_\theta$ are such that

$$\mathcal{K}_\lambda \begin{bmatrix} \ell \\ \omega \end{bmatrix} = \begin{bmatrix} \lambda m I_3 & 0 \\ 0 & \lambda \mathcal{J} \end{bmatrix} \begin{bmatrix} \ell \\ \omega \end{bmatrix} + \mathcal{T}_\lambda \begin{bmatrix} \ell \\ \omega \end{bmatrix} = 0.$$

Taking the inner product in $\mathbb{C}^6$ of the two sides of the above formula by $\begin{bmatrix} \ell \\ \omega \end{bmatrix}$ and using (4.11) it follows that

$$\lambda m |\ell|^2 + \lambda \langle \mathcal{J}\omega, \omega \rangle_{\mathbb{C}^3} + \lambda \int_E |w|^2 \,\mathrm{d}x + \mu \int_E |D(w)|^2 = 0.$$

If $\lambda \in \Sigma_\theta$ with $\operatorname{Im}\lambda \neq 0$ it follows that $\ell = 0$ and $\omega = 0$. On the other hand, if $\lambda \in \Sigma_\theta$ and $\operatorname{Im}\lambda = 0$ we have $\operatorname{Re}\lambda > 0$. In this case, we obtain $w = 0$ and consequently $\ell = \omega = 0$. We have thus shown that the operator in (4.8) is invertible for every $\lambda \in \Sigma_\theta$. This fact, (4.9) and the fact that $\mathcal{K}_0$ is invertible finally imply (4.10). □

We are now in a position to state the main result in this section.

Theorem 4.3. *Let $q \in (1, \infty)$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$. Then*

1. *For every $\lambda \in \Sigma_\theta$ there exist operators*

$$\mathcal{R}(\lambda) \in \mathcal{L}\left([L^q(E)]^3 \times \mathbb{C}^6, \left[W^{2,q}(E)\right]^3 \times \mathbb{C}^6 \right), \qquad \mathcal{P}(\lambda) \in \mathcal{L}\left([L^q(E)]^3 \times \mathbb{C}^6, \widehat{W}^{1,q}(E) \right),$$

such that, for $f \in [L^q(E)]^3$, $f_\ell,\ f_\omega \in \mathbb{C}^3$, setting

$$\begin{bmatrix} u \\ \ell \\ \omega \end{bmatrix} = \mathcal{R}(\lambda) \begin{bmatrix} f \\ f_\ell \\ f_\omega \end{bmatrix}, \qquad \pi = \mathcal{P}(\lambda) \begin{bmatrix} f \\ f_\ell \\ f_\omega \end{bmatrix}, \tag{4.12}$$

then $u,\ \ell,\ \omega$ and π satisfy (4.1).

2. *For $\lambda \in \Sigma_\theta$, $m \in \mathbb{Z}_+$, $s > 3\left(1 - \frac{1}{q}\right)$ and $s' < -\frac{3}{q}$, we have*

$$\mathcal{R}(\lambda) \in \mathcal{L}\left([W^{m,q,s}(E)]^3 \times \mathbb{C}^6, \left[W^{m+2,q,s'}(E)\right]^3 \times \mathbb{C}^6 \right), \tag{4.13}$$

$$\mathcal{P}(\lambda) \in \mathcal{L}\left([W^{m,q,s}(E)]^3 \times \mathbb{C}^6, W^{m+1,q,s'}(E) \right). \tag{4.14}$$

Moreover, the functions $\lambda \mapsto \mathcal{R}(\lambda)$ and $\lambda \mapsto \mathcal{P}(\lambda)$ are holomorphic from Σ_δ to $\mathcal{L}\left([W^{m,q,s}(E)]^3 \times \mathbb{C}^6, \left[W^{m+2,q,s'}(E)\right]^3 \times \mathbb{C}^6 \right)$ and $\mathcal{L}\left([W^{m,q,s}(E)]^3 \times \mathbb{C}^6, W^{m+1,q,s'}(E) \right)$, respectively. Finally, there exist

$$\mathcal{R}_0 \in \mathcal{L}\left([W^{m,q,s}(E)]^3 \times \mathbb{C}^6, \left[W^{m+2,q,s'}(E)\right]^3 \times \mathbb{C}^6 \right),$$
$$\mathcal{P}_0 \in \mathcal{L}\left([W^{m,q,s}(E)]^3 \times \mathbb{C}^6, W^{m+1,q,s'}(E) \right),$$

such that

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0} \ |\lambda|^{-\frac{1}{2}} \ \|\mathcal{R}(\lambda)-\mathcal{R}_0\|_{\mathcal{L}\left([W^{m,q,s}(E)]^3\times\mathbb{C}^6,\left[W^{m+2,q,s'}(E)\times\mathbb{C}^6\right]^3\right)}<\infty, \tag{4.15}$$

$$\limsup_{\lambda\in\Sigma_\theta,\lambda\to 0} \ |\lambda|^{-\frac{1}{2}} \ \|\mathcal{P}(\lambda)-\mathcal{P}_0\|_{\mathcal{L}\left([W^{m,q,s}(E)]^3\times\mathbb{C}^6,W^{m+1,q,s'}(E)\right)}<\infty. \tag{4.16}$$

Proof. Let $m\in\mathbb{Z}_+$, $q>1$, $s>0$, $f\in[W^{m,q,s}(E)]^3$ and $f_\ell,\ f_\omega\in\mathbb{C}^3$. For $\lambda\in\Sigma_\theta\cup\{0\}$ we remind from Proposition 4.2 that the matrix $\mathcal{K}_\lambda$, defined in (4.8), is invertible and we set

$$\begin{bmatrix}\ell_\lambda\\ \omega_\lambda\end{bmatrix}=\mathcal{K}_\lambda^{-1}\begin{bmatrix} f_\ell-\displaystyle\int_{\partial\mathcal{O}}\sigma(R(\lambda)f,P(\lambda)f)\nu\,\mathrm{d}s\\ f_\omega-\displaystyle\int_{\partial\mathcal{O}}x\times\sigma(R(\lambda)f,P(\lambda)f)\nu\,\mathrm{d}s\end{bmatrix}\qquad(\lambda\in\Sigma_\theta\cup\{0\}), \tag{4.17}$$

where $(R(\lambda))$ and $(P(\lambda))$ are the families of operators introduced in Theorem 2.2 and Remark 2.3. The last formula implies, according to Proposition 4.2 and Theorem 2.2, that there exist $\delta,c_\delta>0$ such that

$$|\ell_\lambda|+|\omega_\lambda|\leqslant c_\delta\left(|f_\ell|+|f_\omega|+\|f\|_{[W^{m,q,s}(E)]^3}\right)\qquad(\lambda\in\Sigma_\theta\cup\{0\},\ |\lambda|\leqslant\delta). \tag{4.18}$$

For $\lambda\in\Sigma_\theta\cup\{0\}$ we set $\begin{bmatrix}v_\lambda\\ \eta_\lambda\end{bmatrix}=D_\lambda\begin{bmatrix}\ell_\lambda\\ \omega_\lambda\end{bmatrix}$, where $(D_\lambda)_{\lambda\in\Sigma_\theta\cup\{0\}}$ is the family of operators introduced in Proposition 4.1, and we define

$$\begin{bmatrix}u_\lambda\\ \pi_\lambda\end{bmatrix}=\begin{bmatrix}R(\lambda)f\\ P(\lambda)f\end{bmatrix}+\begin{bmatrix}v_\lambda\\ \eta_\lambda\end{bmatrix}\qquad(\lambda\in\Sigma_\theta\cup\{0\}), \tag{4.19}$$

where the operators $(R(\lambda))_{\lambda\in\Sigma_\theta\cup\{0\}}$, $(P(\lambda))_{\lambda\in\Sigma_\theta\cup\{0\}}$ have been introduced in Theorem 2.2 and Remark 2.3. By combining Theorem 2.2, Proposition 4.1 and (4.18) it follows that for every $s>3\left(1-\frac{1}{q}\right)$, $s'<-\frac{3}{q}$ and $\delta>0$ there exists $d>0$ (possibly depending on s, s' and δ) such that

$$\begin{aligned}\|u_\lambda\|_{\left[W^{m+2,q,s'}(E)\right]^3}+\|\pi_\lambda\|_{W^{m+1,q,s'}(E)}&\leqslant d\left(|f_\ell|+|f_\omega|+\|f\|_{[W^{m,q,s}(E)]^3}\right)\\ &(\lambda\in\Sigma_\theta\cup\{0\},\ |\lambda|\leqslant\delta,\ f\in[W^{m,q,s}(E)]^3,\ f_\ell,f_\omega\in\mathbb{C}^3).\end{aligned} \tag{4.20}$$

By combining (4.17) and (4.19) it follows that for every $\lambda\in\Sigma_\theta$ we have that $u=u_\lambda$, $\ell=\ell_\lambda$, $\omega=\omega_\lambda$ and $\pi=\pi_\lambda$ satisfy (4.1). Consequently, if we set

$$\mathcal{R}(\lambda)\begin{bmatrix}f\\ f_\ell\\ f_\omega\end{bmatrix}=\begin{bmatrix}R(\lambda)f+v_\lambda\\ \ell_\lambda\\ \omega_\lambda\end{bmatrix}\qquad(\lambda\in\Sigma_\theta\cup\{0\}), \tag{4.21}$$

$$\mathcal{P}(\lambda)\begin{bmatrix}f\\ f_\ell\\ f_\omega\end{bmatrix}=P(\lambda)f+\eta_\lambda,\qquad(\lambda\in\Sigma_\theta\cup\{0\}), \tag{4.22}$$

then for every $\lambda\in\Sigma_\theta$ the operators $\mathcal{R}(\lambda)$, $\mathcal{P}(\lambda)$ satisfy (4.13), (4.14) and $u,\ \ell,\ \omega$ and π defined by (4.12) is indeed a solution of (4.1).

Finally the properties (4.15) and (4.16), with $\mathcal{R}_0:=\mathcal{R}(0)$, follow now from (4.21), (4.22), together with Proposition 4.2 and the corresponding properties of P and R in Theorem 2.2. □

5 Further properties of the fluid-structure semigroup in $\mathbb{R}^3$

In this section we study the fluid structure operator $\mathbb{A}_{q,\Omega}$, defined in (3.9)-(3.10), in the case $\Omega = \mathbb{R}^3$. More precisely, we give several results opening the way to the proofs of the facts that $\mathbb{A}_q$ generates a bounded analytic semigroup and of the decay estimates for the fluid-structure operator by collecting several results which follow quite easily from the existing literature. The first one is:

Proposition 5.1. *Let $1 < q < \infty$ and let $\theta \in \left(\frac{\pi}{2}, \pi\right)$. Then there exists $\gamma > 0$ such that*

$$\left\| \lambda \left(\lambda I - \mathbb{A}_q\right)^{-1} \right\|_{\mathcal{L}(\mathbb{X}^q)} \leqslant m_{\theta,\gamma} \qquad (\lambda \in \Sigma_\theta,\ |\lambda| \geqslant \gamma). \tag{5.1}$$

Consequently, $\mathbb{A}_q$ generates an analytic semigroup on $\mathbb{X}^q$.

The proof of the above result can be obtained by a perturbation argument. Since this argument can be obtained by a slight variation of the proof of Theorem 3.1 in [18], where the similar estimate is detailed for the case of fluid-structure system confined in a bounded domain, we omit the proof. We also note that by combining Proposition 3.3 and the first statement of Theorem 4.3, we have

Proposition 5.2. *For every $\lambda \in \Sigma_\theta$ and $F \in \mathbb{X}^q$, setting*

$$\mathcal{R}(\lambda) \begin{bmatrix} F|_E \\ \ell_F \\ \omega_F \end{bmatrix} = \begin{bmatrix} u_{\lambda,F} \\ \ell_{\lambda,F} \\ \omega_{\lambda,F} \end{bmatrix}, \tag{5.2}$$

where the family $(\mathcal{R}(\lambda))$ has been introduced in (4.3) and

$$\ell_F = \frac{1}{m} \int_{\mathcal{O}} F \, \mathrm{d}x, \quad \omega_F = -\mathcal{J}^{-1} \int_{\mathcal{O}} F \times x \, \mathrm{d}x, \tag{5.3}$$

we have

$$\left(\lambda I - \mathbb{A}_q\right)^{-1} F = u_{\lambda,F} \mathbb{1}_E + (\ell_{\lambda,F} + \omega_{\lambda,F} \times x) \mathbb{1}_{\mathcal{O}}. \tag{5.4}$$

The result below provides some simple but important properties of the fluid-structure operator $\mathbb{A}_q$.

Proposition 5.3. *For every $1 < q < \infty$, the dual $\mathbb{A}_q^*$ of $\mathbb{A}_q$ is given by $\mathbb{A}_q^* = \mathbb{A}_{q'}$, with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.*

Proof. For $G \in \mathbb{X}^{q'}$, we set

$$\ell_G = \frac{1}{m} \int_{\mathcal{O}} G \, \mathrm{d}x, \quad \omega_G = -\mathcal{J}^{-1} \int_{\mathcal{O}} G \times x \, \mathrm{d}x.$$

We consider the equation

$$(\lambda I - \mathbb{A}_{q'}) W = G, \tag{5.5}$$

which according to Proposition 3.3 is equivalent to the system

$$\begin{cases} \lambda\varphi - \operatorname{div} \sigma(\varphi, \pi_\varphi) = G|_E, \quad \operatorname{div}\varphi = 0 & (x \in E), \\ \varphi = \psi + \kappa \times x, & (x \in \partial\mathcal{O}), \\ \lambda m \psi = -\displaystyle\int_\Gamma \sigma(\varphi, \pi_\varphi)\nu \, \mathrm{d}s + \ell_G, \\ \lambda \mathcal{J} \kappa = -\displaystyle\int_\Gamma y \times \sigma(\varphi, \pi_\varphi)\nu \, \mathrm{d}s + \omega_G, \end{cases} \tag{5.6}$$

where

$$\varphi = W|_E, \quad \psi = \frac{1}{m} \int_{\mathcal{O}} W \, \mathrm{d}x, \quad \kappa = -\mathcal{J}^{-1} \int_{\mathcal{O}} W \times x \, \mathrm{d}x.$$

Assume that $u \in \left[W^{2,q}(E)\right]^3, \pi \in \widehat{W}^{1,q}(E), \ell \in \mathbb{C}^3$ and $\omega \in \mathbb{C}^3$ satisfy the system (4.1). Taking the inner product in $\mathbb{C}^3$, of $(5.6)_1$ by u and of (4.1) by φ, integrating by parts and summing up the two formulas we obtain

$$\int_E \langle f, \varphi \rangle_{\mathbb{C}^3} \, \mathrm{d}x + \int_{\mathcal{O}} \sigma(u, \pi)\nu \cdot \varphi \, \mathrm{d}s. = \int_E \langle u, G \rangle_{\mathbb{C}^3} \, \mathrm{d}x + \int_{\mathcal{O}} \sigma(\varphi, \pi_\varphi)\nu \cdot u \, \mathrm{d}s. \tag{5.7}$$

Using the boundary conditions, the above relation can be written as

$$\int_E \langle f, \varphi \rangle_{\mathbb{C}^3} \, \mathrm{d}x + \langle f_\ell, \psi \rangle_{\mathbb{C}^3} + \langle f_\omega, \kappa \rangle_{\mathbb{C}^3} = \int_E \langle u, G \rangle_{\mathbb{C}^3} \, \mathrm{d}x + \langle \ell, \ell_G \rangle_{\mathbb{C}^3} + \langle \omega, \omega_G \rangle_{\mathbb{C}^3}. \tag{5.8}$$

In terms of the operator $\mathbb{A}_q$ and $\mathbb{A}_{q'}$, the above equality reads as

$$\langle (\lambda I - \mathbb{A}_q)U, W \rangle_{\mathbb{X}^q, \mathbb{X}^{q'}} = \langle U, (\lambda I - \mathbb{A}_{q'})W \rangle_{\mathbb{X}^q, \mathbb{X}^{q'}}, \qquad (U \in \mathcal{D}(\mathbb{A}_q), W \in \mathcal{D}(\mathbb{A}_{q'})),$$

with $U = u\mathbb{1}_E + (\ell + \omega \times y)\mathbb{1}_{\mathcal{O}}$. Therefore from the above identity we deduce $\mathcal{D}(\mathbb{A}_{q'}) \subset \mathcal{D}(\mathbb{A}_q^*)$. In order to prove the reverse inclusion, we first note that, for $\lambda_0 > 0$ large enough the operator $(\lambda_0 I - \mathbb{A}_{q'})$ is invertible (see Proposition 5.1). Take λ_0 as above and $W \in \mathcal{D}((\lambda_0 I - \mathbb{A}_q)^*)$. Since $\mathbb{X}_q^* = \mathbb{X}_{q'}$, there exists $\widetilde{U} \in \mathcal{D}(\mathbb{A}_{q'})$ such that

$$(\lambda_0 I - \mathbb{A}_{q'})\, \widetilde{U} = \left(\lambda_0 I - \mathbb{A}_q^*\right) W.$$

Let $U \in \mathcal{D}(\mathbb{A}_q)$. Then using the last two formulas, we obtain

$$\begin{aligned} \langle (\lambda_0 I - \mathbb{A}_q)U, W \rangle_{\mathbb{X}^q, \mathbb{X}^{q'}} = \left\langle U, (\lambda_0 I - \mathbb{A}_q^*)W \right\rangle_{\mathbb{X}^q, \mathbb{X}^{q'}} &= \left\langle U, (\lambda_0 I - \mathbb{A}_{q'})\widetilde{U} \right\rangle_{\mathbb{X}^q, \mathbb{X}^{q'}} \\ &= \left\langle (\lambda I - \mathbb{A}_q)U, \widetilde{U} \right\rangle_{\mathbb{X}^q, \mathbb{X}^{q'}}. \end{aligned}$$

In particular, we have

$$\left\langle (\lambda_0 I - \mathbb{A}_q)U, W - \widetilde{U} \right\rangle_{\mathbb{X}^q, \mathbb{X}^{q'}} = 0 \text{ for all } U \in \mathcal{D}(\mathbb{A}_q). \tag{5.9}$$

Therefore $W = \widetilde{U}$ and this completes the proof. □

The last result in this section provides some information on the resolvent equation associated to $\mathbb{A}_q$.

Proposition 5.4. *Let $\lambda \in \mathbb{C}$, such that $\lambda \notin (-\infty, 0)$. Then for every $q \in (1, \infty)$ we have*

(i) $\mathrm{Ker}\,(\lambda I - \mathbb{A}_q) = \{0\}$.

(ii) $\overline{\mathrm{Range}\,(\lambda I - \mathbb{A}_q)} = \mathbb{X}^q$.

Proof. Due to Proposition 3.3, it is enough to show that if $(u, \pi, \ell, \omega) \in \left[W^{2,q}(E)\right]^3 \times \widehat{W}^{1,q}(E) \times \mathbb{C}^3 \times \mathbb{C}^3$ satisfies the system (4.1) with $(f, f_\ell, f_\omega) = 0$, then $u = \pi = \ell = \omega = 0$.

We first consider the case $q = 2$. Multiplying, $(4.1)_1$ by u, $(4.1)_4$ by ℓ and $(4.1)_5$ by ω, we obtain after integration by parts:

$$\lambda \int_E |u|^2 + 2\mu \int_E |D(u)|^2 + \lambda m|\ell|^2 + \lambda \left\langle \mathcal{J}\omega, \omega \right\rangle_{\mathbb{C}^3} = 0. \tag{5.10}$$

If $\mathrm{Im}\lambda \neq 0$, we take the imaginary part of this identity and obtain that $u = \pi = \ell = \omega = 0$. If $\mathrm{Im}\lambda = 0$, then $\mathrm{Re}\lambda \geqslant 0$, hence using the above identity and the boundary conditions we also obtain $u = \pi = \ell = \omega = 0$.

Let us then consider the case $q > 2$ and $\lambda \neq 0$. Let B_1 and B_2 be two open balls in $\mathbb{R}^3$ such that

$$\overline{\mathcal{O}} \subset B_1, \qquad \overline{B_1} \subset B_2,$$

and let $\varphi_1,\ \varphi_2 \in C^\infty(\mathbb{R}^3)$ be such that $\varphi_1(x) \geqslant 0$, $\varphi_2(x) \geqslant 0$, $\varphi_1(x) + \varphi_2(x) = 1$ for every $x \in \mathbb{R}^3$, $\varphi_1 = 1$ on $\overline{B_1}$, $\varphi_1 = 0$ on $\mathbb{R}^3 \setminus B_2$, $\varphi_2 = 1$ on $\mathbb{R}^3 \setminus B_2$ and $\varphi_2 = 0$ on some open neighbourhood of $\overline{B_1}$. Then $\varphi_1 u$ satisfies the following system

$$\begin{cases} \lambda(\varphi_1 u) - \Delta(\varphi_1 u) + \nabla(\varphi_1 \pi) = -2(\nabla u)(\nabla \varphi_1) - (\Delta \varphi_1)u + \pi \nabla \varphi_1 & (x \in B_2 \setminus \overline{\mathscr{O}}), \\ \operatorname{div}(\varphi_1 u) = (\nabla \varphi_1) \cdot u & (x \in B_2 \setminus \overline{\mathscr{O}}), \\ \varphi_1 u = 0 & (x \in \partial B_2), \\ \varphi_1 u = \ell + \omega \times x & (x \in \partial \mathcal{O}), \\ m\lambda \ell + \displaystyle\int_{\partial \mathcal{O}} \sigma(\varphi_1 u, \varphi_1 \pi)\nu \, \mathrm{d}s = 0, \\ \mathcal{J}\lambda\omega + \displaystyle\int_{\partial \mathcal{O}} x \times \sigma(\varphi_1 u, \pi)\nu = 0. \end{cases} \tag{5.11}$$

Note that $-2(\nabla u)(\nabla \varphi_1) - (\Delta \varphi_1)u + \pi \nabla \varphi_1 \in \left[L^2(B_2 \setminus \overline{\mathscr{O}})\right]^3$. Therefore, by using Corollary 3.7 we obtain $(\varphi_1 u, \varphi_1 \pi) \in \left[W^{2,2}(B_2 \setminus \overline{\mathscr{O}})\right]^3 \times W^{1,2}(B_2 \setminus \overline{\mathscr{O}})$. Similarly, $(\varphi_2 u, \varphi_2 \pi)$ satisfies the following system

$$\begin{cases} \lambda(\varphi_2 u) - \Delta(\varphi_2 u) + \nabla(\varphi_2 \pi) = -2(\nabla u)(\nabla \varphi_2) - (\Delta \varphi_2)u + \pi \nabla \varphi_2 & (x \in \mathbb{R}^3), \\ \operatorname{div}(\varphi_2 u) = (\nabla \varphi_2) \cdot u & (x \in \mathbb{R}^3). \end{cases} \tag{5.12}$$

We also have $2(\nabla u)(\nabla \varphi_2) - (\Delta \varphi_2)u + \pi \nabla \varphi_2 \in \left[L^2(\mathbb{R}^3)\right]^3$. By standard results on Stokes operator in the whole space, we also get $(\varphi_2 u, \varphi_2 \pi) \in \left[W^{2,2}(\mathbb{R}^3)\right]^3 \times \widehat{W}^{1,2}(\mathbb{R}^3)$. Combining the above results we obtain $u \in \left[W^{2,2}(E)\right]^3$ and $\pi \in \widehat{W}^{1,2}(E)$.

Let us consider the case $1 < q < 2$ and $\lambda \neq 0$. We use a bootstrap argument here. Let us set $\bar{f}_i = -2(\nabla u)(\nabla \varphi_i) - (\Delta \varphi_i)u + \pi \nabla \varphi_i$. By Sobolev imbedding theorem we obtain $\bar{f}_1,\ \bar{f}_2 \in \left[L^r(B_2 \setminus \overline{\mathscr{O}})\right]^3$, for $r > q$, with $\dfrac{1}{3} + \dfrac{1}{r} = \dfrac{1}{q}$. This implies that $\varphi_1 u \in \left[W^{2,r}(B_2 \setminus \overline{\mathscr{O}})\right]^3$ and $\varphi_2 u \in \left[W^{2,r}(\mathbb{R}^3)\right]^3$, hence $u \in \left[W^{2,r}(E)\right]^3$. If $r \geqslant 2$, we are reduced to the previous case. Otherwise, we continue the process until we get $u \in \left[W^{2,2}(E)\right]^3$.

Now we consider the case $\lambda = 0$. By looking at the identity (5.10), we just need to show that $\nabla u \in \left[L^2(E)\right]^9$. By [3, Theorem 2.1], we know that

$$\|\nabla u\|_{2,E} \leqslant C \|D^2 u\|_{6/5,E}.$$

Therefore the conclusion follows as soon as we prove that $D^2 u \in \left[L^{6/5}(E)\right]^{27}$. The argument is similar to the case $\lambda \neq 0$. If $q \geqslant 6/5$, then it follows by a similar argument as above. Indeed, we have $\bar{f}_1 \in \left[L^{6/5}(B_2 \setminus \overline{\mathscr{O}})\right]^3$ and $\bar{f}_2 \in \left[L^{6/5}(\mathbb{R}^3)\right]^3$. Then $D^2(\varphi_1 u) \in \left[L^{6/5}(B_2 \setminus \overline{\mathscr{O}})\right]^{27}$ and $D^2(\varphi_2 u) \in \left[L^{6/5}(\mathbb{R}^3)\right]^{27}$. So we consider the case $1 < q < 6/5$. As before, using embedding theorem we first obtain $\bar{f}_1 \in \left[L^s(B_2 \setminus \overline{\mathscr{O}})\right]^3$ and $\bar{f}_2 \in \left[L^s(\mathbb{R}^3)\right]^3$ for some $s > q$ with $\dfrac{1}{n} + \dfrac{1}{s} = \dfrac{1}{q}$. Consequently, we have $D^2(\varphi_1 u) \in \left[L^s(B_2 \setminus \overline{\mathscr{O}})\right]^{27}$ and $D^2(\varphi_2 u) \in \left[L^s(\mathbb{R}^3)\right]^{27}$. Observe that $s \geqslant 6/5$, thus we are reduced to the previous case. □

6 Analyticity of the Fluid-Structure semigroup

The main result in this section is:

Theorem 6.1. *For every $1 < q < \infty$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ there exists $M_{q,\theta} > 0$ such that the operator $\mathbb{A}_q$ satisfies*

$$\left\|\lambda(\lambda I - \mathbb{A}_q)^{-1}\right\|_{\mathcal{L}(\mathbb{X}^q)} \leqslant M_{q,\theta} \qquad (\lambda \in \Sigma_\theta). \tag{6.1}$$

Consequently, $\mathbb{A}_q$ generates a bounded analytic semigroup $\mathbb{T}^q = (\mathbb{T}^q_t)_{t\geqslant 0}$ on $\mathbb{X}^q$.

The guiding idea in proving the above result is borrowed from Borchers and Sohr [2] and it consists in using a contradiction argument and appropriate cut-off functions, combined with Proposition 3.6 and classical results for the Stokes operator in the whole space.

A first step towards the proof of Theorem 6.1 is the following result, concerning the case $q \in (1, 3/2)$:

Proposition 6.2. *Let $q \in \left(1, \frac{3}{2}\right)$ and $\theta \in (\frac{\pi}{2}, \pi)$. Let $(\mathcal{R}(\lambda))$ and $(P(\lambda))$ be the family of the operators introduced in Theorem 4.3. For $(f, f_\ell, f_\omega) \in [L^q(E)]^3 \times \mathbb{C}^3 \times \mathbb{C}^3$, we set*

$$\begin{bmatrix} u \\ \ell \\ \omega \end{bmatrix} = \mathcal{R}(\lambda) \begin{bmatrix} f \\ f_\ell \\ f_\omega \end{bmatrix}, \quad \pi = \mathcal{P}(\lambda) \begin{bmatrix} f \\ f_\ell \\ f_\omega \end{bmatrix} \qquad (\lambda \in \Sigma_\theta). \tag{6.2}$$

Then there exists a constant $M_{q,\theta} > 0$ such that, for every $(f, f_\ell, f_\omega) \in [L^q(E)]^3 \times \mathbb{C}^3 \times \mathbb{C}^3$ and for every $\lambda \in \Sigma_\theta$,

$$|\lambda|(\|u\|_{q,E} + |\ell| + |\omega|) + \left\|D^2 u\right\|_{q,E} + \|\nabla\pi\|_{q,E} + |\ell| + |\omega| \leqslant M_{q,\theta}(\|f\|_{q,E} + |f_\ell| + |f_\omega|). \tag{6.3}$$

Proof. First remark that Proposition 5.1 easily implies (6.3) for $\lambda \in \Sigma_\theta$ with $|\lambda| \geqslant \gamma$. We thus focus on the proof of the estimate (6.3) for $\lambda \in \Sigma_\theta$ with $|\lambda| \leqslant \gamma$. Assume that (6.3) is false for some $q \in \left(1, \frac{3}{2}\right)$ for $\lambda \in \Sigma_\theta$ with $|\lambda| \leqslant \gamma$. Then there exists a sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$, together with a sequence (u_n, ℓ_n, ω_n) in $\left[W^{2,q}(E)\right]^3 \times \mathbb{C}^3 \times \mathbb{C}^3$ and (π_n) in $\widehat{W}^{1,q}(E)$ such that

$$0 < |\lambda_n| \leqslant \gamma, \qquad |\arg\lambda_n| \leqslant \theta \qquad (n \in \mathbb{N}), \tag{6.4}$$

$$|\lambda_n|(\|u_n\|_{q,E} + |\ell_n| + |\omega_n|) + \left\|D^2 u_n\right\|_{q,E} + \|\nabla\pi_n\|_{q,E} + |\ell_n| + |\omega_n| = 1 \qquad (n \in \mathbb{N}), \tag{6.5}$$

$$\|\lambda_n u_n - \mu\Delta u_n + \nabla\pi_n\|_{q,E} \to 0, \quad \text{as } n \to \infty, \tag{6.6}$$

$$m\lambda_n \ell_n + \int_{\partial\mathcal{O}} \sigma(u_n, \pi_n)\nu \, \mathrm{d}s \to 0, \quad \text{as } n \to \infty, \tag{6.7}$$

$$\mathcal{J}\lambda_n\omega_n + \int_{\partial\mathcal{O}} x \times \sigma(u_n, \pi_n)\nu \, \mathrm{d}s \to 0, \qquad \text{as } n \to \infty. \tag{6.8}$$

To obtain the desired contradiction we proceed, following [2], in several steps.

Step 1: Localization.

Let B_1 and B_2 be two open balls in $\mathbb{R}^3$ such that

$$\overline{\mathcal{O}} \subset B_1, \qquad \overline{B_1} \subset B_2,$$

and let $\varphi_1,\ \varphi_2 \in C^\infty(\mathbb{R}^3)$ be such that $\varphi_1(x) \geqslant 0$, $\varphi_2(x) \geqslant 0$, $\varphi_1(x) + \varphi_2(x) = 1$ for every $x \in \mathbb{R}^3$, $\varphi_1 = 1$ on $\overline{B_1}$, $\varphi_1 = 0$ on $\mathbb{R}^3 \setminus B_2$, $\varphi_2 = 1$ on $\mathbb{R}^3 \setminus B_2$ and $\varphi_2 = 0$ on some open neighbourhood of

$\overline{B_1}$. After some calculations, we see that for each $n \in \mathbb{N}$ we have

$$\begin{cases} \lambda_n(\varphi_1 u_n) - \Delta(\varphi_1 u_n) + \nabla(\varphi_1 \pi_n) = \varphi_1 \left(\lambda_n u_n - \Delta u_n + \nabla \pi_n\right) \\ \qquad\qquad -2(\nabla u_n)(\nabla \varphi_1) - (\Delta \varphi_1) u_n + \pi_n \nabla \varphi_1 & (x \in B_2 \setminus \overline{\mathscr{O}}), \\ \operatorname{div}(\varphi_1 u_n) = (\nabla \varphi_1) \cdot u_n & (x \in B_2 \setminus \overline{\mathscr{O}}), \\ \varphi_1 u_n = 0 & (x \in \partial B_2), \\ \varphi_1 u_n = \ell_n + \omega_n \times x & (x \in \partial \mathcal{O}), \\ m\lambda_n \ell_n + \displaystyle\int_{\partial \mathcal{O}} \sigma(\varphi_1 u_n, \varphi_1 \pi_n)\nu \, \mathrm{d}s = m\lambda_n \ell_n + \int_{\partial \mathcal{O}} \sigma(u_n, \pi_n)\nu, \\ \mathcal{J}\lambda_n \omega_n + \displaystyle\int_{\partial \mathcal{O}} x \times \sigma(\varphi_1 u_n, \pi_n)\nu = \mathcal{J}\lambda_n \omega_n + \int_{\partial \mathcal{O}} x \times \sigma(u_n, \pi_n)\nu \, \mathrm{d}s. \end{cases} \tag{6.9}$$

By applying Corollary 3.7 and using the fact that φ_1 vanishes outside B_2, it follows that there exists $c > 0$ such that for every $n \in \mathbb{N}$ we have

$$\begin{aligned} &|\lambda_n| \left(\|\varphi_1 u_n\|_{q,E} + |\ell_n| + |\omega_n|\right) + \|D^2(\varphi_1 u_n)\|_{q,E} + \|\nabla(\varphi_1 \pi_n)\|_{q,E} + |\ell_n| + |\omega_n| \\ &\qquad \leqslant c \left(\|\varphi_1(\lambda u_n - \Delta u_n + \nabla \pi_n)\|_{q,E} + \|\nabla(\nabla \varphi_1 \cdot u_n)\|_{q,E} + 2\|\nabla u_n \cdot \nabla \varphi_1\|_{q,E}\right) \\ &\qquad\qquad + c \left(\|(\Delta \varphi_1) u_n\|_{q,E} + \|\pi_n \nabla \varphi_1\|_{q,E}\right) \\ &\qquad\qquad + c \left(\left| m\lambda_n \ell_n + \int_{\partial \mathcal{O}} \sigma(u_n, \pi_n)\nu \right| + \left| \mathcal{J}\lambda_n \omega_n + \int_{\partial \mathcal{O}} x \times \sigma(u_n, \pi_n)\nu \, \mathrm{d}s \right| \right). \end{aligned} \tag{6.10}$$

On the other hand, using the fact that $\varphi_2 = 0$ on some open neighbourhood of $\overline{B_1}$, for each $n \in \mathbb{N}$ we have:

$$\begin{cases} \lambda_n(\varphi_2 u_n) - \Delta(\varphi_2 u_n) + \nabla(\varphi_2 \pi_n) = \varphi_2 \left(\lambda_n u_n - \Delta u_n + \nabla \pi_n\right) \\ \qquad\qquad -2(\nabla u_n)(\nabla \varphi_2) - (\Delta \varphi_2) u_n + \pi_n \nabla \varphi_2 & (x \in \mathbb{R}^3), \\ \operatorname{div}(\varphi_2 u_n) = (\nabla \varphi_2) \cdot u_n & (x \in \mathbb{R}^3). \end{cases} \tag{6.11}$$

Using classical results for the Stokes operator in $\mathbb{R}^3$ (see, for instance, McCracken [19]), it follows that, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} &|\lambda_n| \|\varphi_2 u_n\|_{q,E} + \|D^2(\varphi_2 u_n)\|_{q,E} + \|\nabla(\varphi_2 \pi_n)\|_{q,E} \\ &\qquad \leqslant c \left(\|\varphi_2(\lambda u_n - \Delta u_n + \nabla \pi_n)\|_{q,E} + \|\nabla(\nabla \varphi_2 \cdot u_n)\|_{q,E} + 2\|\nabla u_n \cdot \nabla \varphi_2\|_{q,E}\right) \\ &\qquad\qquad + c \left(\|(\Delta \varphi_2) u_n\|_{q,E} + \|\pi_n \nabla \varphi_2\|_{q,E}\right). \end{aligned} \tag{6.12}$$

By combining (6.10) and (6.12) it follows that for every $n \in \mathbb{N}$ we have

$$\begin{aligned} &|\lambda_n| \left(\|u_n\|_{q,E} + |\ell_n| + |\omega_n|\right) + \|D^2 u_n\|_{q,E} + \|\nabla \pi_n\|_{q,E} + |\ell_n| + |\omega_n| \\ &\qquad \leqslant |\lambda_n| \left(\|\varphi_1 u_n\|_{q,E} + \|\varphi_2 u_n\|_{q,E} + |\ell_n| + |\omega_n|\right) + \|D^2(\varphi_1 u_n)\|_{q,E} + \|D^2(\varphi_2 u_n)\|_{q,E} \\ &\qquad\qquad + \|\nabla(\pi_n \nabla \varphi_1)\|_{q,E} + \|\nabla(\pi_n \nabla \varphi_2)\|_{q,E} \leqslant c\|\lambda u_n - \Delta u_n + \nabla \pi_n\|_{q,E} \\ &+ c \left(\left| m\lambda_n \ell_n + \int_{\partial \mathcal{O}} \sigma(u_n, \pi_n)\nu \right| + \left| \mathcal{J}\lambda_n \omega_n + \int_{\partial \mathcal{O}} x \times \sigma(u_n, \pi_n)\nu \, \mathrm{d}s \right| \right) + W(u_n, \nabla u_n, \pi_n), \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} W(u_n, \nabla u_n, \pi_n) = c \sum_{j=1}^{2} &\left(\|\nabla(\nabla \varphi_j \cdot u_n)\|_{q,E} + 2\|\nabla u_n \cdot \nabla \varphi_j\|_{q,E}\right) \\ &+ c \sum_{j=1}^{2} \left(\|(\Delta \varphi_j) u_n\|_{q,E} + \|\pi_n \nabla \varphi_j\|_{q,E}\right) \qquad (n \in \mathbb{N}). \end{aligned} \tag{6.14}$$

Step 2. Passage to the limit.
Let $r, s > 1$ be defined by

$$\frac{1}{3} + \frac{1}{s} = \frac{1}{q} \quad \text{and} \quad \frac{1}{3} + \frac{1}{r} = \frac{1}{s},$$

so that

$$\frac{2}{3} + \frac{1}{r} = \frac{1}{q} \quad \text{and } 1 < s < 3. \tag{6.15}$$

By Theorem 2.1 and Lemma 3.1 in Crispo and Maremonti [3] and (6.5), we have

$$\|u_n\|_{r,E} \leqslant C\|\nabla u_n\|_{s,E} \leqslant C\|D^2 u_n\|_{q,E} \leqslant C, \tag{6.16}$$

$$\|\pi_n\|_{s,E} \leqslant C\|\nabla \pi_n\|_{q,E} \leqslant C. \tag{6.17}$$

Thus, there exist a subsequence, still denoted by (u_n), (π_n), (ℓ_n), (ω_n) and $u \in [L^r(E)]^3$, $\pi \in L^s(E)$, $(\ell, \omega) \in \mathbb{C}^3 \times \mathbb{C}^3$ and $\lambda \in \overline{\Sigma_\theta}$ such that

$$u_n \rightharpoonup_{[L^r(E)]^3} u, \quad \pi_n \rightharpoonup_{L^s(E)} \pi, \quad \ell_n \to \ell, \quad \omega_n \to \omega, \quad \lambda_n \to \lambda, \quad \text{as } n \to \infty, \tag{6.18}$$

where $\rightharpoonup_X$ stands for the weak convergence in a Banach space X. Let us set

$$U_n = u_n \mathbb{1}_E + (\ell_n + \omega_n \times x)\mathbb{1}_{\mathcal{O}} \quad (n \in \mathbb{N}), \quad U = u\mathbb{1}_E + (\ell + \omega \times x)\mathbb{1}_{\mathcal{O}}. \tag{6.19}$$

Then $U_n \in \mathbb{X}^r$ and the sequence (U_n) weakly converges to U in $\mathbb{X}^r$. According to (6.6)–(6.8) and by the definition of the operator $\mathbb{A}_q$, we have that

$$U_n \in \mathcal{D}(\mathbb{A}_q) \text{ for all } n \in \mathbb{N}, \quad \text{and } \ (\lambda_n - \mathbb{A}_q)U_n \to_{\mathbb{X}^q(E)} 0 \text{ as } n \to \infty.$$

Let $W \in \mathcal{D}(\mathbb{A}_{q'}) \cap \mathcal{D}(\mathbb{A}_{r'})$. By Proposition 5.3,

$$0 = \lim_{n\to\infty} \langle (\lambda_n I - \mathbb{A}_q)U_n, W\rangle_{\mathbb{X}^q, \mathbb{X}^{q'}} = \langle U, (\lambda I - \mathbb{A}_{r'})W\rangle_{\mathbb{X}^r, \mathbb{X}^{r'}}.$$

Since the set $\{(\lambda I - \mathbb{A}_{r'})W \mid W \in \mathcal{D}(\mathbb{A}_{q'}) \cap \mathcal{D}(\mathbb{A}_{r'})\} \subseteq \mathbb{X}^{q'} \cap \mathbb{X}^{r'}$ is dense in $\mathbb{X}^{r'}$ (see Proposition 5.4), the last formula implies that $U = 0$. Consequently, using (6.6),

$$\lambda_n u_n \rightharpoonup_{[L^q(E)]^3} 0, \quad \Delta u_n \rightharpoonup_{[L^q(E)]^3} 0, \quad \nabla \pi_n \rightharpoonup_{[L^q(E)]^3} 0, \quad \text{as } n \to \infty.$$

Next using the fact that $\sup_n \|\pi_n\|_{L^s(\Omega)} < \infty$ (see (6.17)) we deduce that $\pi = 0$.

Now we consider the expression $W(u_n, \nabla u_n, \pi_n)$ defined in (6.14). We claim that

$$\lim_{n\to\infty} W(u_n, \nabla u_n, \pi_n) = 0. \tag{6.20}$$

To shorten the proof, since all the terms in $W(u_n, \nabla u_n, \pi_n)$ are the same as in [2], we consider only one term of $W(u_n, \nabla u_n, \pi_n)$, say $f_{j,n} = \nabla(\nabla\varphi_j \cdot u_n)$ for $j \in \{1, 2\}$, since the other terms can be estimated in a similar manner. Note that, $f_{j,n} \in \left[W_0^{1,q}(B_2 \setminus \overline{B}_1)\right]^3$ for every $n \in \mathbb{N}$ and using (6.16), (6.18) and the fact that $u = 0$ we also have $(f_{j,n})$ converges weakly to 0 in $\left[L^q(B_2 \setminus \overline{B}_1)\right]^3$. Moreover, using (6.17)

$$\begin{aligned} \|f_{j,n}\|_{1,q,B_2\setminus\overline{B}_1} &\leqslant C(\|u_n\|_{q,B_2\setminus\overline{B}_1} + \|\nabla u_n\|_{q,B_2\setminus\overline{B}_1}) \\ &\leqslant C(\|u_n\|_{r,B_2\setminus\overline{B}_1} + \|\nabla u_n\|_{s,B_2\setminus\overline{B}_1}) \qquad (\text{ since } r, s > q) \\ &\leqslant C(\|u_n\|_{r,E} + \|\nabla u_n\|_{s,E}) \leqslant C. \end{aligned} \tag{6.21}$$

Thus, $f_{j,n}$ converges strongly to 0 in $\left[L^q(B_2 \setminus \overline{B}_1)\right]^3$ as $n \to \infty$. Consequently, we obtain (6.20). This, together with (6.5), contradicts the estimate (6.13), which ends the proof. □

We are now in position to prove the main result in this section.

Proof of Theorem 6.1 . We first note that from Proposition 3.3, Theorem 4.3 and Proposition 6.2, we obtain (6.1) for $1 < q < \frac{3}{2}$. In the case $\frac{3}{2} \leqslant q \leqslant 2$ we take $q_0 \in (1, \frac{3}{2})$. We define $0 \leqslant s \leqslant 1$ by

$$\frac{1}{q} = \frac{s}{q_0} + \frac{1-s}{2}.$$

Since (6.1) holds for q_0, there exists a constant $M_{\theta,q_0} > 0$ such that

$$\left\|\lambda(\lambda I - \mathbb{A}_{q_0})^{-1}\right\|_{\mathcal{L}(\mathbb{X}^{q_0})} \leqslant M_{\theta,q_0} \qquad (\lambda \in \Sigma_\theta).$$

On the other hand, $\mathbb{A}_2$ is a self-adjoint operator on $\mathbb{X}^2$ (see [22]). Therefore, we also have

$$\left\|\lambda(\lambda I - \mathbb{A}_2)^{-1}\right\|_{\mathcal{L}(\mathbb{X}^2)} \leqslant M_{\theta,2} \qquad (\lambda \in \Sigma_\theta),$$

for some $M_{\theta,2}$ depending only on θ. Then by Riesz-Thorin interpolation theorem (see for instance [24, Theorem 1, Section 1.18.7]), we obtain

$$\left\|\lambda(\lambda I - \mathbb{A}_q)^{-1}\right\|_{\mathcal{L}(\mathbb{X}^q)} \leqslant M_{\theta,q_0}^s M_{\theta,2}^{1-s} \qquad (\lambda \in \Sigma_\theta). \tag{6.22}$$

This ends the proof of (6.1) for $\dfrac{3}{2} \leqslant q \leqslant 2$.

In the case $2 < q < \infty$, we take $1 < q' \leqslant 2$ such that $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. By Proposition 5.3, we have $\lambda(\lambda I - \mathbb{A}_q)^{-1} = [\lambda(\lambda I - \mathbb{A}_{q'})^{-1}]^*$, so that

$$\left\|\lambda(\lambda I - \mathbb{A}_q)^{-1}\right\|_{\mathcal{L}(\mathbb{X}^q)} = \left\|\lambda(\lambda I - \mathbb{A}_{q'})^{-1}\right\|_{\mathcal{L}(\mathbb{X}^{q'})} \qquad (\lambda \in \Sigma_\theta).$$

We have already seen that (6.1) holds for $1 < q \leqslant 2$. Thus from the above identity we infer that, (6.1) holds for any $2 < q < \infty$, which ends the proof. □

We end this section with the result below, whose proof can be easily obtained by combining Theorem 6.1 and the results from Lunardi [17, Chapter 3]:

Corollary 6.3. *With the assumptions and notations of Theorem 6.1, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that*

$$\left\|\mathbb{A}_q^{k-1} U\right\|_{\mathbb{X}^q} \leqslant \varepsilon \left\|\mathbb{A}_q^k U\right\|_{\mathbb{X}^q} + C_\varepsilon \left\|U\right\|_{\mathbb{X}^q} \qquad (U \in \mathcal{D}(\mathbb{A}_q^k), k \in \mathbb{N}). \tag{6.23}$$

7 Decay estimates for the fluid-structure semigroup

Based on Theorem 6.1, we consider the *fluid-structure semigroup* which is, for each $q \in (1, \infty)$, the bounded analytic semigroup $\mathbb{T}^q$ introduced in Theorem 6.1. Our main result in this section is:

Theorem 7.1. *(i) Let $1 < q < \infty$. Let $R_0 > 0$ be such that $\overline{\mathcal{O}} \subset B_{R_0}$. Then for any $R > R_0$, there exists a constant $C > 0$, depending on q and R, such that*

$$\left\|\mathbb{T}_t^q U\right\|_{q,B_R} \leqslant C t^{-\frac{3}{2q}} \left\|U\right\|_{\mathbb{X}^q} \qquad (t > 1,\ U \in \mathbb{X}^q). \tag{7.1}$$

(ii) Let $1 < q \leqslant r < \infty$ and $\sigma = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$. Then there exists a constant $C > 0$, depending on q and r, such that

$$\left\|\mathbb{T}_t^q U\right\|_{\mathbb{X}^r} \leqslant C t^{-\sigma} \left\|U\right\|_{\mathbb{X}^q} \qquad (t > 0,\ U \in \mathbb{X}^q). \tag{7.2}$$

(iii) Let $1 < q \leqslant r \leqslant 3$. Then there exists a constant $C > 0$, depending on q and r, such that

$$\|\nabla \mathbb{T}_t^q U\|_{r,E} \leqslant C t^{-\sigma-1/2} \|U\|_{\mathbb{X}^q} \qquad (t > 0,\ U \in \mathbb{X}^q). \tag{7.3}$$

Our methodology to prove the above result is inspired by [13] and it consists in using the resolvent estimates developed in Section 4 - Section 6. However, applying the strategy proposed in [13] requires several adaptations which are described below.

To start with, we state the following regularity result of the projection operator $\mathbb{P}_q$.

Proposition 7.2. *Let $k \in \mathbb{N}$. Assume that $1 < r \leqslant q < \infty$. Let $u \in \left[L^q(\mathbb{R}^3)\right]^3$ be such that* div $u = 0$ *in $\mathcal{D}'(\mathbb{R}^3)$ and $\partial^\alpha u \in [L^r(E)]^3$ for every multi-index $\alpha \in \mathbb{Z}_+^3$ with $|\alpha| = k$. Then $\partial^\alpha(\mathbb{P}_q u) \in [L^r(E)]^3$ for every multi-index $\alpha \in \mathbb{Z}_+^3$ with $|\alpha| = k$. Moreover, there exists a constant C independent of the choice of u with the above properties, such that*

$$\sum_{|\alpha|=k} \|\partial^\alpha \mathbb{P}_q u\|_{r,E} \leqslant C \left(\sum_{|\alpha|=k} \|\partial^\alpha u\|_{r,E} + \|u\|_q \right). \tag{7.4}$$

Proof. Let $v = \mathbb{P}_q u$. Then

$$v(x) = \ell_v + \omega_v \times x \qquad (x \in \mathcal{O}), \tag{7.5}$$

where

$$\ell_v = \frac{1}{m} \int_{\mathcal{O}} v \, \mathrm{d}x, \qquad \omega_v = -\frac{1}{\mathcal{J}} \int_{\mathcal{O}} v \times x \, \mathrm{d}x. \tag{7.6}$$

Moreover, there exists a positive constant C, depending only on q and on $\mathcal{O}$, such that (see for instance [26, Proof of Theorem 2.2, Eq. (3.14)])

$$|\ell_v| + |\omega_v| \leqslant C \|u\|_q. \tag{7.7}$$

Since div $u = 0$, we have that w_1 from the decomposition (3.4) of u vanishes and, according to [26, Proof of Theorem 2.2, Eq. (3.15)], w_2 from the same decomposition satisfies $w_2 = \nabla \pi_2$, with π_2 satisfying

$$\Delta \pi_2 = 0 \text{ in } E, \quad \frac{\partial \pi_2}{\partial \nu} = u \cdot \nu - (\ell_v + \omega_v \times x) \cdot \nu \text{ on } \partial\mathcal{O}. \tag{7.8}$$

Then estimate (7.4) follows from (7.7) and from Giga and Sohr [8, proof of Lemma 2.3]. □

We next provide two results characterising the graph norm of $\mathbb{A}_q^m$ in terms of Sobolev spaces.

Proposition 7.3. *Let $1 < q < \infty$.*

(i) Assume that $U \in \mathcal{D}(\mathbb{A}_q)$ and $\mathbb{A}_q U_{|E} \in [W^{m,q}(E)]^3$ for some $m \in \mathbb{Z}_+$. Then $U_{|E} \in \left[W^{m+2,q}(E)\right]^3$ and there exists a constant $C_m > 0$ such that

$$\|U\|_{m+2,q,E} \leqslant C_m \left(\|\mathbb{A}_q U\|_{m,q,E} + \|U\|_{\mathbb{X}^q} \right). \tag{7.9}$$

(ii) For every $m \in \mathbb{N}$, if $U \in \mathcal{D}(\mathbb{A}_q^m)$, then $U_{|E} \in W^{2m,q}(E)$ and

$$\|U\|_{2m,q,E} \leqslant C \left(\left\|\mathbb{A}_q^m U\right\|_{\mathbb{X}^q} + \|U\|_{\mathbb{X}^q} \right) \qquad (U \in \mathcal{D}(\mathbb{A}_q^m)). \tag{7.10}$$

Proof. Let us set $\mathbb{A}_q U = -F$, so that $F_{|E} \in [W^{m,q}(E)]^3$. Moreover, we denote

$$u = U_{|E}, \qquad \ell = \frac{1}{m} \int_{\mathcal{O}} U \, \mathrm{d}x, \quad \omega = -\mathcal{J}^{-1} \int_{\mathcal{O}} U \times x \, \mathrm{d}x.$$

Then according to Proposition 5.2 there exists $\pi \in \widehat{W}^{1,q}(E)$ such that u, π, ℓ and ω satisfy

$$\begin{cases} -\mu\Delta u + \nabla\pi = F & (x \in E), \\ \operatorname{div} u = 0 & (x \in E), \\ u = \ell + \omega \times x & (x \in \partial\mathcal{O}). \end{cases}$$

Let $\begin{bmatrix} w_1 \\ \eta_1 \end{bmatrix} = D_1 \begin{bmatrix} \ell \\ \omega \end{bmatrix}$, where D_1 is the Dirichlet map introduced in Proposition 4.1. According to Proposition 4.1, for every $k \in \mathbb{N}$ there exists positive constants $C_{1,k}$, $C_{2,k}$ such that

$$\|w_1\|_{k+1,q,E} + \|\pi_1\|_{\widehat{W}^{k,q}(E)} \leqslant C_{1,k}\left(|\ell| + |\omega|\right) \leqslant C_{2,k}\|U\|_{\mathbb{X}^q}. \tag{7.11}$$

We denote $\widetilde{u} = u - w_1$ and $\widetilde{\pi} = \pi - \eta_1$. Then $\widetilde{u}$ and $\widetilde{\pi}$ satisfy

$$\begin{cases} -\mu\Delta\widetilde{u} + \nabla\widetilde{\pi} = F + w_1 & (x \in E), \\ \operatorname{div} \widetilde{u} = 0 & (x \in E), \\ \widetilde{u} = 0 & (x \in \partial\mathcal{O}). \end{cases}$$

According to [13, Proposition 2.7(i)], for every $m \in \mathbb{N}$ there exists a positive constant $C_{3,m}$ such that

$$\|\widetilde{u}\|_{m+2,q,E} \leqslant C_{3,m}\left(\|F\|_{m,q,E} + \|w_1\|_{m,q,E} + \|\widetilde{u}\|_{q,E}\right).$$

The above estimate together with (7.11) implies the estimate (7.9).

To prove (7.10), we use an induction argument. We first note that (7.10) is true for $m = 1$, since it is nothing else but the estimate (7.9) for $m = 0$. Let us assume that (7.10) is true for some $m \in \mathbb{N}$ and $U \in \mathcal{D}(\mathbb{A}_q^{m+1})$. Then by (7.9) and induction hypothesis, there exists a positive constant $C_m > 0$ such that

$$\|U\|_{2m+2,q,E} \leqslant C_m\left(\|\mathbb{A}_q U\|_{2m,q,E} + \|U\|_{\mathbb{X}^q}\right) \leqslant C\left(\left\|\mathbb{A}_q^{m+1}U\right\|_{\mathbb{X}^q} + \|\mathbb{A}_q U\|_{\mathbb{X}^q} + \|U\|_{\mathbb{X}^q}\right). \tag{7.12}$$

Then the assertion (7.10) holds for m replaced by $m+1$ by applying Corollary 6.3 repeatedly and (7.12). This completes the proof of the proposition. □

Proposition 7.4. *Let $q \in (1, \infty)$. Then:*

(i) For any $m \in \mathbb{N}$, there exists a positive constant $C_m > 0$ such that

$$\left\|\mathbb{A}_q^m U\right\|_{\mathbb{X}^q} \leqslant C_m\left(\|U\|_{2m,q,E} + \|U\|_{\mathbb{X}^q}\right) \qquad \left(U \in \mathcal{D}(\mathbb{A}_q^m)\right). \tag{7.13}$$

(ii) Let $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $m \in \mathbb{N}$. Then there exists a positive constant $C_m > 0$ such that

$$\left\|(\lambda I - \mathbb{A}_q)^{-1}F\right\|_{2m+2,q,E} \leqslant C_m\left(\|F\|_{2m,q,E} + \|F\|_{\mathbb{X}^q}\right), \qquad \left(F \in \mathcal{D}(\mathbb{A}_q^m), \lambda \in \Sigma_\theta, |\lambda| \geqslant 1\right). \tag{7.14}$$

Proof. We use an induction argument to prove (7.13). Using Proposition 3.1, (3.7) and (3.8) we first note that the estimate (7.9) is true for $m = 1$. Assume that (7.13) holds for some $m \in \mathbb{N}$ and $U \in \mathcal{D}(\mathbb{A}_q^{m+1})$. By the induction hypothesis, we have

$$\left\|\mathbb{A}_q^{m+1}U\right\|_{\mathbb{X}^q} \leqslant C_m\left(\|\mathbb{A}_q U\|_{2m,q,E} + \|\mathbb{A}_q U\|_{\mathbb{X}^q}\right), \qquad \left(U \in \mathcal{D}(\mathbb{A}_q^{m+1})\right). \tag{7.15}$$

By applying Proposition 7.2 and Corollary 6.3, the above estimate implies that

$$\left\|\mathbb{A}_q^{m+1}U\right\|_{\mathbb{X}^q} \leqslant C_m\left(\|U\|_{2m+2,q,E}+\|U\|_{\mathbb{X}^q}\right) \qquad \left(U\in\mathcal{D}(\mathbb{A}_q^{m+1})\right). \tag{7.16}$$

Thus (7.13) also holds when m is replaced by $m+1$.

Finally (7.14) follows from the facts that

$$\left\|\lambda(\lambda I-\mathbb{A}_q)^{-1}\right\|_{\mathcal{L}(\mathcal{D}(\mathbb{A}_q^m))}\leqslant M, \qquad (\lambda\in\Sigma_\theta) \tag{7.17}$$

together with the estimates (7.10) and (7.13). □

Remark 7.5. Putting together (7.10) and (7.13), it follows that, for every $m\in\mathbb{N}$, the graph norm of $\mathbb{A}_q^m$ is equivalent to $\|\cdot\|_{2m,q,E}+\|\cdot\|_{\mathbb{X}^q}$. We also note that this equivalence also holds for the bounded domain version of the fluid-structure operator, i.e., $\Omega\subset\mathbb{R}^3$ open and bounded, and the operator $\mathbb{A}_{q,\Omega}$ defined in (3.7). Moreover, elements φ of $\mathcal{D}(\mathbb{A}_q^m)$ are such that $\varphi\in\left[W^{1,q}(\mathbb{R}^3)\right]^3\cap\mathbb{X}^q(\mathbb{R}^3)$ and $\varphi_{|E}\in\left[W^{2m,q}(E)\right]^3$.

To state the next results, which yield decay estimates for the fluid-structure semigroup in weighted L^p spaces, we remind from Section 2 the notation $L^{q,s}$ for the weighted Lebesgue spaces introduced in (2.3).

Theorem 7.6. *Let $1<q<\infty$. Let s and s' be real numbers such that $s>3(1-1/q)$ and $s'<-3/q$. Then there exists a positive constant C, depending only on q,s and s', such that*

$$\|\mathbb{T}_t^qU\|_{\left[L^{q,s'}(\mathbb{R}^3)\right]^3}\leqslant C(1+t)^{-\frac{3}{2}}\|U\|_{[L^{q,s}(\mathbb{R}^3)]^3} \qquad U\in\mathbb{X}^q\cap\left[L^{q,s}(\mathbb{R}^3)\right]^3. \tag{7.18}$$

Proof. We first note that Theorem 4.3 is a complete analogue of Corollary 3.2 in [13], and Theorem 6.1 is the analogue of the main result in [2]. We can thus complete the proof following line by line the proof of Theorem 1.1 in [13]. □

Remark 7.7. For $U_0\in\mathcal{D}(\mathbb{A}_q)$, we denote by

$$u_0=U_0|_E,\quad \ell_0=\frac{1}{m}\int_{\mathcal{O}}U_0\,\mathrm{d}x,\quad \omega_0=-\frac{1}{\mathcal{J}}\int_{\mathcal{O}}U_0\times x\,\mathrm{d}x.$$

Moreover for every $t\geqslant 0$, we set $U(t)=\mathbb{T}_t^qU_0$ and

$$u(t)=U(t)|_E,\quad \ell(t)=\frac{1}{m}\int_{\mathcal{O}}U(t)\,\mathrm{d}x,\quad \omega(t)=-\frac{1}{\mathcal{J}}\int_{\mathcal{O}}U(t)\times x\,\mathrm{d}x.$$

Then there exists $\pi\in C([0,\infty);\widehat{W}^{1,q}(E))$ such that (u,π,ℓ,ω) satisfies the following system

$$\begin{cases}\partial_t u-\mu\Delta u+\nabla\pi=0,\quad \mathrm{div}u=0 & (t\geqslant 0,x\in E),\\ u=\ell+\omega\times x & (t\geqslant 0,x\in\partial\mathcal{O}),\\ m\dot{\ell}+\displaystyle\int_{\partial\mathcal{O}}\sigma(u,\pi)\nu\,\mathrm{d}s=0 & (t\geqslant 0),\\ \mathcal{J}\dot{\omega}+\displaystyle\int_{\partial\mathcal{O}}x\times\sigma(u,\pi)\nu\,\mathrm{d}s=0, & (t\geqslant 0),\\ u(0)=u_0 & (x\in E),\\ \ell(0)=\ell_0,\qquad \omega(0)=\omega_0.\end{cases} \tag{7.19}$$

Our next result in this section provides L^q-L^r decay estimates for the fluid-structure semigroup $\mathbb{T}^q$ for small time:

Theorem 7.8. *Let $1 < q \leqslant r < \infty$ and $\sigma = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$. Then for each $\tau \in (0, \infty)$, there exists a constant $C > 0$, depending on τ, q and r, such that*

$$\|\mathbb{T}_t^q U\|_{\mathbb{X}^r} \leqslant C t^{-\sigma} \|U\|_{\mathbb{X}^q} \qquad (t \leqslant \tau,\ U \in \mathbb{X}^q), \tag{7.20}$$

$$\|\nabla \mathbb{T}_t^q U\|_{r,E} \leqslant C t^{-\sigma-\frac{1}{2}} \|U\|_{\mathbb{X}^q} \qquad (t \leqslant \tau,\ U \in \mathbb{X}^q). \tag{7.21}$$

Proof. Let $N = [2\sigma]$, where $[\cdot]$ denotes the integer part function. Let us assume that N is even. Then by (7.10), there exists a constant $C > 0$ depending on τ, q and r, such that

$$\begin{aligned}\|\mathbb{T}_t^q U\|_{N,q,E} + |\ell(t)| + |\omega(t)| &\leqslant C\left(\left\|\mathbb{A}_q^{N/2}\mathbb{T}_t^q U\right\|_{\mathbb{X}^q} + \|\mathbb{T}_t^q U\|_{\mathbb{X}^q}\right)\\ &\leqslant C\left(t^{-\frac{N}{2}}\|U\|_{\mathbb{X}^q} + \tau^{\frac{N}{2}} t^{-\frac{N}{2}}\|U\|_{\mathbb{X}^q}\right) \leqslant C t^{-\frac{N}{2}}\|U\|_{\mathbb{X}^q} \qquad (t \in (0,\tau]).\end{aligned} \tag{7.22}$$

In a similar manner, we also obtain

$$\|\mathbb{T}_t^q U\|_{N+2,q,E} + |\ell(t)| + |\omega(t)| \leqslant C t^{-\frac{N+2}{2}}\|U\|_{\mathbb{X}^q} \qquad (t \in (0,\tau]). \tag{7.23}$$

Thus by Sobolev embedding, interpolation and using (7.22)-(7.23), we obtain

$$\begin{aligned}\|\mathbb{T}_t^q U\|_{\mathbb{X}^r} &\leqslant C\left(\|\mathbb{T}_t^q U\|_{r,E} + |\ell(t)| + |\omega(t)|\right)\\ &\leqslant C\left(\|\mathbb{T}_t^q U\|_{2\sigma,q,E} + |\ell(t)| + |\omega(t)|\right)\\ &\leqslant C\left(\|\mathbb{T}_t^q U\|_{N+2,q,E}^{\frac{2\sigma-N}{2}}\|\mathbb{T}_t^q U\|_{N,q,E}^{\frac{N+2-2\sigma}{2}} + |\ell(t)| + |\omega(t)|\right) \leqslant C t^{-\sigma}\|U\|_{\mathbb{X}^q},\quad t \in (0,\tau].\end{aligned}$$

If N is odd then we replace N by $N-1$. This completes the proof of (7.20). The proof of (7.21) is completely, thus omitted here. □

The next step towards the proof of Theorem 7.1 is the following result:

Lemma 7.9. *With the notations and assumptions of Theorem 7.1, let $d > R_0$ and let $m \in \mathbb{N}$. Moreover, denote $E_d := \{x \in E \mid |x| < d\}$. Then*

(i) There exists a constant $C > 0$ depending on d and m such that for all $t > 0$,

$$\|\mathbb{T}_t^q U\|_{q,B_d} + \|\mathbb{T}_t^q U\|_{2m,q,E_d} \leqslant C(1+t)^{-\frac{3}{2}}\left(\|U\|_{2m,q,E_d} + \|U\|_{\mathbb{X}^q}\right), \tag{7.24}$$

for every $U \in \mathcal{D}(\mathbb{A}_q^m)$ with $U = 0$ for $|x| > d$.

(ii) There exists a constant $C > 0$ depending on d and m such that for all $t > 0$,

$$\|\partial_t \mathbb{T}_t^q U\|_{q,B_d} + \|\partial_t \mathbb{T}_t^q U\|_{2m,q,E_d} \leqslant C(1+t)^{-\frac{5}{2}}\left(\|U\|_{2m+2,q,E_d} + \|U\|_{\mathbb{X}^q}\right), \tag{7.25}$$

for every $U \in \mathcal{D}(\mathbb{A}_q^{m+1})$ with $U = 0$ for $|x| > d$.

Proof. The proof can be obtained following line by line the proof Lemma 5.2 from Iwashita [13]. More precisely, it suffices to use instead of Proposition 2.7 and Lemma 2.8 in [13] our results in Proposition 7.3 and Proposition 7.4 above, respectively, and to replace expansion (3.2) in [13] by (4.15) above. □

Proposition 7.10. *With the notation and assumptions of Remark 7.7 and Theorem 7.1, let $d > R_0 + 5$ and $m \in \mathbb{N}$. Moreover, assume that $U_0 \in \mathrm{Ran}(\mathbb{T}_1^q)$. Then there exists a positive constant C, depending only on d, m and q, such that, for every $t \geqslant 0$ we have*

$$\|u(t,\cdot)\|_{2m,q,E_d} + |\ell(t)| + |\omega(t)| \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{q,[3/q]+2m+2,E} + |\ell_0| + |\omega_0| \right), \tag{7.26}$$

$$\|\partial_t u(t,\cdot)\|_{2m,q,E_d} + |\dot{\ell}(t)| + |\dot{\omega}(t)| \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{q,[3/q]+2m+4,E} + |\ell_0| + |\omega_0| \right), \tag{7.27}$$

$$\|\pi(t,\cdot)\|_{2m+1,q,E_d} \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{q,[3/q]+2m+4,E} + |\ell_0| + |\omega_0| \right), \tag{7.28}$$

where $[s]$ denotes the integer part of $s \in \mathbb{R}$.

Proof. We follow with minor modifications the steps of the proof of Lemma 5.3 in [13].

Step 1. Since $U_0 \in \mathrm{Ran}(\mathbb{T}_1^q)$, we have $U_0 \in \mathcal{D}(\mathbb{A}_q^k)$ for all $k \in \mathbb{N}$. Let $\widetilde{u}_0$ be an extension of u_0 to $\mathbb{R}^3$ such that $\widetilde{u}_0 \in \left[W^{2m,q}(\mathbb{R}^3)\right]^3$ and $\|\widetilde{u}_0\|_{2m,q} \leqslant C \|U_0\|_{\mathcal{D}(\mathbb{A}_q^m)}$, where C is a constant independent of U_0. Then $\mathrm{div}\, \widetilde{u}_0 \in W_0^{2m-1,q}(\mathcal{O})$ and $\displaystyle\int_{\mathcal{O}} \mathrm{div}\, \widetilde{u}_0 = \int_{\partial\mathcal{O}} (\ell_0 + \omega_0 \times x) \cdot \nu \, \mathrm{d}s = 0$. Then by Lemma 2.1 we have that $\mathbb{B}_{\mathcal{O}}(\mathrm{div}\widetilde{u}_0) \in W_0^{2m,q}(\mathcal{O})$. Let us set

$$\psi = \widetilde{u}_0 - \mathbb{B}_{\mathcal{O}}(\mathrm{div}\widetilde{u}_0),$$

where $\mathbb{B}_{\mathcal{O}}(\mathrm{div}\widetilde{u}_0)$ is seen as a function in $\left[W^{2m,q}(\mathbb{R}^3)\right]^3$ after its extension by 0 in E. Then $\psi \in \left[W^{2m,q}(\mathbb{R}^3)\right]^3$ has the following properties

$$\psi(x) = U_0(x) = u_0(x) \text{ for all } x \in \mathbb{E}, \quad \mathrm{div}\, \psi(x) = 0 \text{ in } \mathbb{R}^3, \quad \|\psi\|_{2m,q} \leqslant C \|U_0\|_{\mathcal{D}(\mathbb{A}_q^m)}. \tag{7.29}$$

Step 2. We consider the following Stokes system in $\mathbb{R}^3$

$$\begin{cases} \partial_t v_0(t,x) - \mu \Delta v_0(t,x) = 0, \ \mathrm{div}\, v_0(t,x) = 0 & (t \geqslant 0, x \in \mathbb{R}^3), \\ v_0(0,x) = \psi(x) & (x \in \mathbb{R}^3). \end{cases} \tag{7.30}$$

Let q and r be such that $1 < q \leqslant r \leqslant \infty$ and define $\sigma = \frac{3}{2}(\frac{1}{q} - \frac{1}{r})$. According to classical estimates (see, for instance, [13, Lemma 5.1]) for the heat kernel, for every $m \in \mathbb{Z}_+$, there exists a constant $C_m > 0$, depending on q and r, with

$$\|\nabla^m v_0(t,\cdot)\|_r \leqslant C_m t^{-\sigma - \frac{m}{2}} \|\psi\|_q \qquad (t \geqslant 0), \tag{7.31}$$

$$\|\nabla^m v_0(t,\cdot)\|_r \leqslant C_m (1+t)^{-\sigma - \frac{m}{2}} \|\psi\|_{[2\sigma]+m+1,q} \qquad (t \geqslant 0), \tag{7.32}$$

$$\|\partial_t v_0(t,\cdot)\|_r \leqslant C_m t^{-\sigma-1} \|\psi\|_q \qquad (t \geqslant 0), \tag{7.33}$$

$$\|\nabla^m \partial_t v_0(t,\cdot)\|_r \leqslant C_m (1+t)^{-\sigma-1-\frac{m}{2}} \|\psi\|_{[2\sigma]+m+3,q} \qquad (t \geqslant 0). \tag{7.34}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be such that $\varphi(x) = 1$ for $|x| \leqslant d-2$ and $\varphi(x) = 0$ for $|x| > d-1$. Denote $\Omega_d = \left\{x \in \mathbb{R}^3 \mid d-2 \leqslant |x| \leqslant d-1\right\}$ and let $\mathbb{B}_d : \mathcal{D}(\Omega_d) \to [\mathcal{D}(\Omega_d)]^3$ be the Bogovskii operator such that $\mathrm{div}(\mathbb{B}_d h) = h$ if $\displaystyle\int_{\Omega_d} h = 0$. We define

$$v_1(t,\cdot) = \mathbb{B}_d(-\nabla\varphi \cdot v_0(t,\cdot)) \qquad (t \geqslant 0). \tag{7.35}$$

By applying (7.32) and (7.34), it follows that there exists a constant $C_m > 0$, depending on q, such that

$$\|v_1(t,\cdot)\|_{m,q,E} \leqslant C_m \|\nabla\varphi \cdot v_0(t,\cdot)\|_{m-1,q,E} \leqslant C_m (1+t)^{-3/2q} \|\psi\|_{[3/q]+m,q} \qquad (t \geqslant 0), \tag{7.36}$$

$$\|\partial_t v_1(t,\cdot)\|_{m,q,E} \leqslant C_m \|\nabla\varphi\cdot\partial_t v_0(t,\cdot)\|_{m-1,q,E} \leqslant C_m(1+t)^{-3/2q}\|\psi\|_{[3/q]+m+2,q} \qquad (t\geqslant 0). \tag{7.37}$$

Step 3. We now set

$$v_2(t,x) = u(t,x) - (1-\varphi)v_0(t,x) + v_1(t,x) \qquad (t\geqslant 0,\ x\in E). \tag{7.38}$$

Then v_2, π, ℓ and ω satisfy

$$\begin{cases} \partial_t v_2 - \mu\Delta v_2 + \nabla\pi = f, \quad \operatorname{div} v_2 = 0 & (t>0,\ x\in E),\\ v_2 = \ell + \omega\times x & (t>0, x\in\partial\mathcal{O}),\\ m\dot\ell + \displaystyle\int_{\partial\mathcal{O}} \sigma(v_2,\pi)\nu\,\mathrm{d}s = 0 & (t>0),\\ \mathcal{J}\dot\omega + \displaystyle\int_{\partial\mathcal{O}} x\times\sigma(v_2,\pi)\nu\,\mathrm{d}s = 0 & (t>0),\\ v_2(0,x) = \zeta(x) & (x\in E),\\ \ell(0)=\ell_0, \qquad \omega(0)=\omega_0, \end{cases} \tag{7.39}$$

where

$$f = -2(\nabla\varphi\cdot\nabla)v_0 - \mu(\Delta\varphi)v_0 + \partial_t v_1 - \mu\Delta v_1, \tag{7.40}$$

$$\zeta(x) = \varphi(x)\psi(x) + v_1(0,x) \qquad (x\in E). \tag{7.41}$$

Moreover, we have

$$\operatorname{div} f(t,x) = 0, \qquad \operatorname{div}\zeta(x) = 0 \qquad (t>0, x\in E), \tag{7.42}$$

$$\operatorname{supp} f(t,\cdot)\subset\{d-2\leqslant|x|\leqslant d-1\} \qquad (t>0).$$

Denote

$$V_{20} = \zeta\mathbb{1}_E + (\ell_0+\omega_0\times x)\mathbb{1}_{\mathcal{O}},$$

and

$$V_2(t,x) = v_2(t,x)\mathbb{1}_E(x) + (\ell(t)+\omega(t)\times x)\mathbb{1}_{\mathcal{O}}(x) \qquad (t>0,\ x\in\mathbb{R}^3).$$

Recall that $U_0\in\mathrm{Ran}(\mathbb{T}_1^q)$, in particular $U_0\in\mathcal{D}(\mathbb{A}_q^m)$ for every $m\in\mathbb{N}$. Therefore $V_{20}\in\mathcal{D}(\mathbb{A}_q^m)$ for every $m\in\mathbb{N}$ and there exists a constant $C>0$, depending on m and q, such that

$$\|V_{20}\|_{\mathcal{D}(\mathbb{A}_q^m)} \leqslant C_m\|U_0\|_{\mathcal{D}(\mathbb{A}_q^m)}. \tag{7.43}$$

Using (7.31), (7.32), (7.36) and (7.37), we infer that there exists a constant $C>0$, depending only on m and q, such that

$$\|f(t,\cdot)\|_{m,q,E} \leqslant C_m(1+t)^{-3/2q}\|\psi\|_{[3/q]+m+2,q} \qquad (t>0). \tag{7.44}$$

On the other hand, applying the variation of the constants formula to (7.39), we have

$$V_2(t,\cdot) = \mathbb{T}_t^q V_{20} + \int_0^t \mathbb{T}_{t-s}^q f(s,\cdot)\,\mathrm{d}s \qquad (t\geqslant 0). \tag{7.45}$$

The last estimate, combined with Lemma 7.9, can be used, following line by line the end of the third step of the proof of Lemma 5.3 in [13], to obtain the existence of a constant C (depending only on d, m and q), such that for every $t>0$ we have

$$\|V_2(t,\cdot)\|_{\mathcal{D}(\mathbb{A}_q^m)} \leqslant C_m(1+t)^{-3/2q}\left(\|u_0\|_{[3/q]+2m+2,q,E} + |\ell_0| + |\omega_0|\right). \tag{7.46}$$

$$\|\partial_t V_2(t,\cdot)\|_{\mathcal{D}(\mathbb{A}_q^m)} \leqslant C_m(1+t)^{-3/2q}\left(\|u_0\|_{[3/q]+2m+4,q,E} + |\ell_0| + |\omega_0|\right). \tag{7.47}$$

Final step. Estimates (7.26) easily follow by combining (7.38) with the estimates (7.31) - (7.33), (7.36), (7.37), (7.46) and (7.47). The estimate (7.27) can be obtained similarly. Putting together (7.26) and (7.27), from $(7.19)_1$ we obtain

$$\begin{aligned}\|\nabla\pi(t,\cdot)\|_{2m,q,E_d} &\leqslant \|\partial_t u(t,\cdot)\|_{2m,q,E_d} + \|\Delta u(t,\cdot)\|_{2m,q,E_d}\\ &\leqslant C(1+t)^{-3/2q}\left(\|u_0\|_{q,[3/q]+2m+4,E} + |\ell_0| + |\omega_0|\right) \qquad (t\geqslant 0).\end{aligned}$$

Then the estimate (7.28) follows from the above estimate after redefining π as $\pi - \displaystyle\int_{E_d}\pi\,\mathrm{d}x$ and applying Poincaré type inequalities. □

The results in Lemma 7.9 and Proposition 7.10 provide decay estimates for the restrictions to bounded sets of the solution u of the linearized problem. The result below provides decay estimates for the restriction of $u(t,\cdot)$ to the exterior of the bounded set E_d introduced in Lemma 7.9.

Proposition 7.11. *With the notation and assumptions of Remark 7.7 and Theorem 7.1, let $d > R_0 + 5$. Moreover, assume that $U_0 \in \mathrm{Ran}(\mathbb{T}_1^q)$. Then there exists a positive constant C, depending only on d and q, such that, for every $t \geqslant 0$ we have*

$$\|u(t,\cdot)\|_{r,\{|x|>d\}} \leqslant C(1+t)^{-\sigma}\left(\|u_0\|_{q,[3/q]+[2\sigma]+7,E} + |\ell_0| + |\omega_0|\right) \quad (1<q\leqslant r<\infty), \tag{7.48}$$

$$\|\nabla u(t,\cdot)\|_{r,\{|x|>d\}} \leqslant C(1+t)^{-\sigma}\left(\|u_0\|_{q,[3/q]+[2\sigma]+7,E} + |\ell_0| + |\omega_0|\right) \quad (1<q\leqslant r\leqslant 3), \tag{7.49}$$

where $[s]$ denotes the integer part of $s\in\mathbb{R}$.

Proof. Let $\chi\in C^\infty(\mathbb{R}^3)$ be such that $\chi(x)=1$ for $|x|>d$ and $\chi(x)=0$ for $|x|<d-1$. It follows that for every $t\geqslant 0$ we have that $\mathrm{supp}\,\mathrm{div}(\chi u(t,\cdot))\subset\{d-1<|x|<d\}$. Then there exists $v_3(t,\cdot)$ such that $\mathrm{div}\,v_3 = \mathrm{div}(\chi u)$, $\mathrm{supp}\,v_3(t,\cdot)\subset\{d-1<|x|<d\}$ and for every $m\in\mathbb{N}$, we have

$$\|v_3(t,\cdot)\|_{m,q,E} \leqslant C(1+t)^{-\frac{3}{2q}}\left(\|u_0\|_{q,[3/q]+m+2,E} + |\ell_0| + |\omega_0|\right), \tag{7.50}$$

$$\|\partial_t v_3(t,\cdot)\|_{m,q,E} \leqslant C(1+t)^{-\frac{3}{2q}}\left(\|u_0\|_{q,[3/q]+m+4,E} + |\ell_0| + |\omega_0|\right), \tag{7.51}$$

for some constant $C>0$ depending on m and q. To derive the last two estimates we have used Bogovskii Lemma, (7.26) and (7.27). We now define

$$v_4(t,x) = \chi(x)u(t,x) - v_3(t,x) \qquad (t>0,\ x\in\mathbb{R}^3).$$

Note that $\mathrm{div}\,v_4 = 0$ so that v_4 satisfies

$$\begin{cases}\partial_t v_4 - \mu\Delta v_4 + \nabla(\chi\pi) = h, \qquad \mathrm{div} v_4 = 0, & (t>0, x\in\mathbb{R}^3),\\ v_4(0,x) = v_{40}(x) & (x\in\mathbb{R}^3),\end{cases} \tag{7.52}$$

where

$$h = -2(\nabla\chi\cdot\nabla)u(t) - \mu(\Delta\chi)u + \partial_t v_3 - \mu\Delta v_3 + \pi\nabla\psi, \tag{7.53}$$

and

$$v_{40}(x) = \chi(x)U_0(x) - v_3(0,x) \qquad (x\in\mathbb{R}^3). \tag{7.54}$$

Note that, since all the functions appearing in (7.52) are supported away from $\mathcal{O}$, the function v_4 shares all the properties derived in Lemma 5.5 and proofs of Theorems, 1.2 and 1.3 of [13]. In particular,

$$\|v_4(t,\cdot)\|_r \leqslant C(1+t)^{-\sigma}\left(\|u_0\|_{q,[3/q]+[2\sigma]+7,E} + |\ell_0| + |\omega_0|\right) \qquad (1<q\leqslant r<\infty), \tag{7.55}$$

and

$$\|\nabla v_4(t,\cdot)\|_r \leqslant C(1+t)^{-\sigma-1/2}\left(\|u_0\|_{q,[3/q]+[2\sigma]+7,E}+|\ell_0|+|\omega_0|\right) \qquad (1<q\leqslant r\leqslant 3). \tag{7.56}$$

By combining (7.55), (7.50) (with $m=[2\sigma]+1$) and Sobolev's embedding theorem we conclude that

$$\begin{aligned}
&\|u(t)\|_{r,\{|x|>d\}} \leqslant \|v_3(t)\|_{r,E}+\|v_4(t)\|_r\\
&\leqslant C\,\|v_3(t)\|_{[2\sigma]+1,q,E}+\|v_4(t)\|_r\\
&\leqslant C(1+t)^{-\frac{3}{2q}}\left(\|u_0\|_{q,[3/q]+m+3,E}+|\ell_0|+|\omega_0|\right)+C(1+t)^{-\sigma}\left(\|u_0\|_{q,[3/q]+m+7,E}+|\ell_0|+|\omega_0|\right)\\
&\leqslant C(1+t)^{-\sigma}\left(\|u_0\|_{q,[3/q]+m+7,E}+|\ell_0|+|\omega_0|\right).
\end{aligned}$$

This completes the proof of (7.48).

Finally, the proof of (7.49) is obtained similarly from (7.56), together with (7.50) (with $m=[2\sigma]+2$). □

We are now in a position to prove the main result in this section.

Proof of Theorem 7.1. For small times, Theorem 7.1 simply is Theorem 7.8, and we thus only focus on the estimates of Theorem 7.1 for times larger than 1.

To prove (7.1), it suffices to note that, for every $U\in\mathbb{X}^q$ we have $\mathbb{T}_1^q U\in\mathcal{D}(\mathbb{A}_q^k)$ for any $k\in\mathbb{N}$, so that applying (7.26) with $m=0$ we obtain

$$\left\|\mathbb{T}_{t+1}^q U\right\|_{q,B_R}\leqslant C(1+t)^{-3/2q}\|U\|_{q,\mathbb{R}^3} \qquad (t>0, U\in\mathbb{X}^q). \tag{7.57}$$

Concerning (7.2), we first note that this estimate holds for $t\in(0,1]$ (see (7.20)). Again applying (7.26) with $m=[2\sigma]+1$, we get

$$\left\|\mathbb{T}_{t+1}^q U\right\|_{r,E_d}\leqslant C\left\|\mathbb{T}_{t+1}^q U\right\|_{q,[2\sigma]+1,E_d}\leqslant C(1+t)^{-3/2q}\|U\|_{q,\mathbb{R}^3} \qquad (t>0, U\in\mathbb{X}^q),$$

and by (7.48)

$$\left\|\mathbb{T}_{t+1}^q U\right\|_{r,\{|x|>d\}}\leqslant C(1+t)^{-\sigma}\|U\|_{q,\mathbb{R}^3} \qquad (t>0, U\in\mathbb{X}^q).$$

The above two estimates give (7.2) for $t\geqslant 1$.

The proof of (7.3) follows analogously by combining (7.21), (7.26) and (7.49). □

8 Proof of the main results

In this section, we focus on the analysis of the non-linear fluid-structure model when the initial datum is small and the rigid body is a ball. More precisely, our main goal here is to prove Theorem 1.1. Moreover, in Theorem 8.10, we consider a weaker concept of solutions for which we also provide a global in time existence result and we describe their large time behaviour.

The arguments we are using are close to those in Kato [14], with several adaptations necessary to tackle the extra term coming from the motion of the rigid body, in a spirit close to [6], and with the extensive use of the results obtained in the previous sections on the fluid-structure semigroup, and in particular Theorem 7.1.

We rely, in particular, on the following lemma, which is a rather straightforward consequence of Theorem 7.1:

Lemma 8.1. *Let p_0 and q_0 be such that $3/2\leqslant q_0\leqslant p_0$. Then there exists $C>0$ such that for every $F\in L^{q_0}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ satisfying $F=0$ in B and $\operatorname{div} F\in\left[L^r(\mathbb{R}^3)\right]^3$ for some $r\in(1,p_0]$ we have*

$$\|\mathbb{T}_t^r\mathbb{P}_r\operatorname{div} F\|_{p_0}\leqslant Ct^{-3/2(1/q_0-1/p_0)-1/2}\|F\|_{q_0,E} \qquad (t>0). \tag{8.1}$$

Proof. The proof follows the same steps as those appearing in [6, Proof of Corollary 3.10]. More precisely, for $F \in L^{q_0}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ satisfying $F=0$ in B and $\operatorname{div} F \in \left[L^r(\mathbb{R}^3)\right]^3$ for some $r \in (1,p_0]$, we necessarily have $F\cdot\nu = 0$ on ∂B and $\mathbb{T}_t^r\mathbb{P}_r \operatorname{div} F$ is a well-defined element of $\mathbb{X}^{p_0}$ (see Theorem 7.1). Let p_0' and r' be the conjugates of p_0 and r respectively. We thus have, using the density of $\mathbb{X}^{r'}$ in $\mathbb{X}^{p_0'} \cap \mathbb{X}^{r'}$, that

$$
\begin{aligned}
&\|\mathbb{T}_t^r\mathbb{P}_r \operatorname{div}(F)\|_{p_0} \\
&= \sup_{\varphi\in\mathbb{X}^{p_0'},\, \|\varphi\|_{\mathbb{X}^{p_0'}}\leqslant 1} \left\{ \langle \mathbb{T}_t^r\mathbb{P}_r \operatorname{div}(F), \varphi\rangle_{\mathbb{X}^{p_0},\mathbb{X}^{p_0'}} \right\} \\
&= \sup_{\varphi\in\mathbb{X}^{p_0'}\cap\mathbb{X}^{r'},\, \|\varphi\|_{\mathbb{X}^{p_0'}}\leqslant 1} \left\{ \langle \operatorname{div} F, \mathbb{T}_t^{r'}\varphi\rangle_{\mathbb{X}^r,\mathbb{X}^{r'}} \right\} \quad (\text{ by } \textit{Proposition } 3.1 \text{ and } \textit{Proposition } 5.3) \\
&= \sup_{\varphi\in\mathbb{X}^{p_0'}\cap\mathbb{X}^{r'},\, \|\varphi\|_{\mathbb{X}^{p_0'}}\leqslant 1} \left\{ -\int_E F\cdot\nabla\mathbb{T}_t^{p_0'}\varphi \,\mathrm{d}x \right\} \quad (\text{ as } F=0 \text{ in } B) \\
&\leqslant \|F\|_{q_0,E} \sup_{\varphi\in\mathbb{X}^{p_0'}\cap\mathbb{X}^{r'},\, \|\varphi\|_{\mathbb{X}^{p_0'}}\leqslant 1} \|\nabla\mathbb{T}_t^{p_0'}\varphi\|_{q_0',E}.
\end{aligned}
$$

Finally, using (7.3) we obtain the estimate (8.1). □

In the remaining part of this section, when there is no risk of confusion, the fluid-structure semigroup will be simply denoted by $\mathbb{T}$ (instead of $\mathbb{T}^q$). Similarly, if the appropriate value of q is clear from the context, the projector $\mathbb{P}_q$ is simply denoted by $\mathbb{P}$.

Remark 8.2. Note that from Theorem 7.1, there exists $C>0$ such that for every $V_0 \in \mathcal{D}(\mathbb{A}_{3/2})$ we have

$$\|\ \mathbb{T}_t V_0\|_{3/2} \leqslant C\|V_0\|_{3/2} \leqslant C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})} \qquad (t\geqslant 0),$$

and

$$\|\ \mathbb{A}_{3/2}\mathbb{T}_t V_0\|_{3/2} \leqslant C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})} \qquad (t\geqslant 0),$$

so that we easily deduce that

$$\|\ \nabla\mathbb{T}_t V_0\|_{3/2,E} \leqslant C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})} \qquad (V_0\in\mathcal{D}(\mathbb{A}_{3/2}), t\geqslant 0). \tag{8.2}$$

Note that, according to (7.10) and [3], if $V_0 \in \mathcal{D}(\mathbb{A}_{3/2})$, V_0 also belongs to $\mathbb{X}^p$ for any $p\ \in [3/2,\infty)$, and thus for all $p\in[3/2,\infty)$, there exists C such that

$$\|\ \mathbb{T}_t V_0\|_p \leqslant C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})} \qquad (V_0\in\mathcal{D}(\mathbb{A}_{3/2}), t\geqslant 0). \tag{8.3}$$

Before going further, let us mention the following local existence result, whose functional setting corresponds to the global existence result in Theorem 1.1.

Theorem 8.3. *Let $V_0 \in \mathcal{D}(\mathbb{A}_{3/2})$. Then, for all $p\in(3,\infty)$, there exists $T_0>0$ depending on $\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})}$ such that there exists a unique solution $V = \begin{bmatrix} v \\ \ell \\ \omega \end{bmatrix} \in C^0([0,T_0];\mathbb{X}^p)$ with $\nabla v \in C^0([0,T_0];\left[L^{3/2}(E)\right]^9)$ of (1.2).*

Furthermore, for $p\in(3,\infty)$ and $T>0$, two solutions of (1.2) with the same intial datum which both belong to $C^0([0,T];\mathbb{X}^p)\cap C^0([0,T];\left[W^{1,3/2}(E)\right]^3)$ coincide on $(0,T)$.

We postpone the proof of Theorem 8.3 to the appendix, since it is not the main result of our article and other (weaker) functional settings can be found in the literature for local existence results, see, for instance, [7].

Note that Theorem 8.3 corresponds to an initial datum $V_0 \in \mathcal{D}(\mathbb{A}_{3/2})$, which is characterized, through the identification described in (3.3), to data (v_0, ℓ_0, ω_0) in $\left[L^3(E)\right]^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying (1.3)–(1.4)–(1.5).

We are now in a position to prove our main result.

Proof of Theorem 1.1. The proof is divided into several steps.
Step 1: Reformulation of the non-linear problem. As mentioned in the introduction, we are looking for mild solutions V of the non-linear problem (1.2), i.e. solutions of the equation (1.9). For each $t > 0$ we identify $V(t,\cdot)$ with a triple $\begin{bmatrix} v(t,\cdot) \\ \ell(t) \\ \omega(t) \end{bmatrix}$, where $v(t,\cdot) : E \to \mathbb{R}^3$ and $\ell(t),\ \omega(t) \in \mathbb{R}^3$, as described in (3.3).

We first remark that, since $(v-h)\cdot\nu = 0$ on ∂B, we have

$$-\left[(v-\ell)\cdot\nabla\right] v = \operatorname{div} F, \tag{8.4}$$

where

$$F(s,x) = -\mathbb{1}_E(x)(v(s,x)-\ell(s)) \otimes v(s,x) \qquad (s>0,\ x\in\mathbb{R}^3). \tag{8.5}$$

In particular the triple $V = \begin{bmatrix} v \\ \ell \\ \omega \end{bmatrix}$ is a mild solution of (1.2) iff it satisfies

$$V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s}\,\mathbb{P}\operatorname{div} F(s)\,\mathrm{d}s \qquad (t \geqslant 0), \tag{8.6}$$

where F is defined in (8.5). The above formulation will be intensively used in the remaining part of the section, in conjonction with Lemma 8.1.
Step 2: Construction of a mild solution through a fixed point process. Let $q \in (1, 3/2]$ and let

$$\begin{aligned}\mathscr{C} = \{V \in C^0([0,\infty);\mathbb{X}^3) \ \mid\ & t^{3/2(1/q-2/3)}V(t) \in C_b^0((0,\infty);\mathbb{X}^{3/2}),\\ & (1+t^{3/2(1/q-1/3)})V(t) \in C_b^0([0,\infty);\mathbb{X}^3),\ (t^{3/2(1/3-1/6)}+t^{3/2(1/q-1/6)})V(t) \in C_b^0([0,\infty);\mathbb{X}^6)\},\end{aligned} \tag{8.7}$$

where $C_b^0(I;U)$ stands for the set of bounded continuous functions from the interval I to the Banach space U. Note that $\mathscr{C}$ is a Banach space when endowed with the norm

$$\begin{aligned}\|V\|_{\mathscr{C}} = \sup_{t>0}\{t^{3/2(1/q-2/3)}\|V(t)\|_{3/2} + (1+t^{3/2(1/q-1/3)})\|V(t)\|_3 \\ + (t^{3/2(1/3-1/6)}+t^{3/2(1/q-1/6)})\|V(t)\|_6\}.\end{aligned} \tag{8.8}$$

For $K > 0$, we denote by $\mathscr{C}(K)$ the ball

$$\mathscr{C}(K) = \{V \in \mathscr{C},\ \text{ with } \|V\|_{\mathscr{C}} \leqslant K\}. \tag{8.9}$$

We also introduce the space $\mathcal{C}$ defined by

$$\mathcal{C} = \left\{V \in L^\infty(0,\infty;\mathbb{X}^3),\ \text{with } \nabla V \in \left[L^\infty(0,\infty;L^{3/2}(E))\right]^9\right\}. \tag{8.10}$$

and, for $K' > 0$, we define

$$\mathcal{C}(K') = \{V \in \mathcal{C},\ \text{ with } \sup_{t>0}\|\nabla V(t,\cdot)\|_{3/2,E} \leqslant K'\}. \tag{8.11}$$

Moreover, for $V_0 \in \mathbb{X}^q \cap \mathbb{X}^3 \cap \mathcal{D}(\mathbb{A}_{3/2})$ with $\|V_0\|_{\mathbb{X}^q \cap \mathbb{X}^3} \leqslant \varepsilon_0$ we set

$$(\Lambda V)(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s} \mathbb{P} \operatorname{div} F(s)\, \mathrm{d}s \qquad (V \in \mathscr{C} \cap \mathcal{C},\ t > 0), \tag{8.12}$$

where F has been defined in (8.5). An essential step in the proof of Theorem 1.1 is the following lemma:

Lemma 8.4. *There exists $C > 0$ such that for every $K, K' > 0$ the function Λ defined in* (8.12) *maps $\mathscr{C}(K) \cap \mathcal{C}(K')$ into $\mathscr{C}(K_1) \cap \mathcal{C}(K_1')$, where*

$$K_1 = C\|V_0\|_{\mathbb{X}^q \cap \mathbb{X}^3} + CK^2, \quad K_1' = C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})} + CKK'. \tag{8.13}$$

Moreover, the constant C introduced above is such that:

$$\|\Lambda(V^a) - \Lambda(V^b)\|_{\mathscr{C}} \leqslant CK\|V^a - V^b\|_{\mathscr{C}} \qquad (V^a,\ V^b \in \mathscr{C}(K) \cap \mathcal{C}(K')), \tag{8.14}$$

and

$$\begin{aligned} \sup_{t>0} \|(\nabla(\Lambda(V^a)) - \nabla(\Lambda(V^b)))(t,\cdot)\|_{3/2,E} \ &\leqslant CK'\|V^a - V^b\|_{\mathscr{C}} \\ &+ CK \sup_{t>0} \|(\nabla(\Lambda(V^a)) - \nabla(\Lambda(V^b)))(t,\cdot)\|_{3/2,E}, \qquad (V^a,\ V^b \in \mathscr{C}(K) \cap \mathcal{C}(K')). \end{aligned} \tag{8.15}$$

Proof. According to Theorem 7.1, since $V_0 \in \mathbb{X}^q \cap \mathbb{X}^3$ and it satisfies (1.6), we have $\mathbb{T}_t V_0 \in \mathscr{C}$ and

$$\begin{aligned} \sup_{t>0} \{ t^{3/2(1/q-2/3)} \|\mathbb{T}_t V_0\|_{3/2} + (1 + t^{3/2(1/q-1/3)}) \|\mathbb{T}_t V_0\|_3 \\ + (t^{3/2(1/3-1/6)} + t^{3/2(1/q-1/6)}) \|\mathbb{T}_t V_0\|_6 \} \leqslant C\|V_0\|_{\mathbb{X}^q \cap \mathbb{X}^3}. \end{aligned} \tag{8.16}$$

Moreover, since $V_0 \in \mathcal{D}(\mathbb{A}_{3/2})$, we also have that $\mathbb{T}_t V_0 \in L^\infty\left([0,\infty); \left[W^{1,3/2}(\mathbb{R}^3)\right]^3\right)$, see (8.2). In particular, we have that

$$\mathbb{T}_t V_0 \in \mathcal{C} \quad \text{for all } t > 0, \qquad \sup_{t>0} \|\nabla \mathbb{T}_t V_0\|_{3/2,E} \leqslant C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})}. \tag{8.17}$$

We now want to show that second term in the right hand side of (8.12) belongs to $\mathscr{C}$. To this aim, we first notice that for $V \in \mathscr{C}(K)$ we have

$$|\ell(s)| \leqslant \frac{K}{s^{1/2}} \qquad (s > 0). \tag{8.18}$$

Reminding to the reader that F is given by (8.5), we can combine (8.18) with the fact that

$$\sup_{t>0} \{\|V(t)\|_3 + t^{3/2(1/3-1/6)} \|V(t)\|_6\} \leqslant K \qquad (V \in \mathscr{C}(K)),$$

to obtain that

$$\begin{aligned} \|F(s)\|_{3,E} &\leqslant \|v(s)\|_{6,E}^2 + |\ell(s)|\ \|v(s)\|_{3,E} \\ &\leqslant K^2 \left[\left(\frac{1}{s^{3/2(1/3-1/6)}} \right)^2 + \frac{1}{s^{1/2}} \right] \leqslant \frac{2K^2}{s^{1/2}}, \qquad (s > 0, V \in \mathscr{C}(K)). \end{aligned} \tag{8.19}$$

Moreover, using the fact that

$$\sup_{t>0} \{ t^{3/2(1/q-1/3)} \|V(t)\|_3 + t^{3/2(1/q-1/6)}) \|V(t)\|_6 \} \leqslant K \qquad (V \in \mathscr{C}(K)),$$

it follows that

$$\begin{aligned}\|F(s)\|_{3,E} &\leqslant \|v(s)\|_{6,E}^2 + |\ell(s)|\ \|v(s)\|_{3,E}\\ &\leqslant K^2\left(\frac{1}{s^{3/2(1/q-1/6)}}\frac{1}{s^{3/2(1/3-1/6)}} + \frac{1}{s^{1/2}}\frac{1}{s^{3/2(1/q-1/3)}}\right)\\ &\leqslant \frac{2K^2}{s^{3/(2q)}}, \qquad\qquad (s>0, V\in\mathscr{C}(K)).\end{aligned} \tag{8.20}$$

Similarly, using (8.18) and the estimate

$$\sup_{t>0}\{t^{3/2(1/q-2/3)}\|V(t)\|_{3/2,E} + t^{3/2(1/q-1/3)}\|V(t)\|_3\} \leqslant K \qquad\qquad (V\in\mathscr{C}(K)),$$

we get

$$\begin{aligned}\|F(s)\|_{3/2,E} &\leqslant \|v(s)\|_{3,E}^2 + |\ell(s)|\ \|v(s)\|_{3/2,E}\\ &\leqslant K^2\left[\left(\frac{1}{s^{3/2(1/q-1/3)}}\right) + \frac{1}{s^{1/2}}\frac{1}{s^{3/2(1/q-2/3)}}\right]\\ &\leqslant \frac{2K^2}{s^{3/2(1/q-1/3)}}, \qquad\qquad (s>0, V\in\mathscr{C}(K)).\end{aligned} \tag{8.21}$$

We are now in a position to estimate the second term in the right hand side of (8.12). Note that, using (8.5), for $V\in\mathscr{C}\cap\mathcal{C}$, we have

$$\operatorname{div} F(s,\cdot)\in\left[L^{3/2}(\mathbb{R}^3)\right]^3 + \left[L^{6/5}(\mathbb{R}^3)\right]^3 \qquad\qquad (s>0).$$

It thus follows from the estimate (8.21) and Lemma 8.1 with $p_0=q_0=3/2$ that for $V\in\mathscr{C}(K)\cap\mathcal{C}$ and $t>0$ we have

$$\begin{aligned}\left\|\int_0^t \mathbb{T}_{t-s}\mathbb{P}\operatorname{div} F(s)\,\mathrm{d}s\right\|_{3/2} &\leqslant \int_0^t \frac{1}{(t-s)^{1/2}}\|F(s)\|_{3/2,E}\,\mathrm{d}s\\ &\leqslant C\int_0^t \frac{1}{(t-s)^{1/2}}\frac{K^2}{s^{3/2(1/q-1/3)}}\,\mathrm{d}s \leqslant \frac{CK^2}{t^{3/2(1/q-2/3)}}.\end{aligned} \tag{8.22}$$

Lemma 8.1 with $p_0=q_0=3$ and the estimate (8.19) yields:

$$\left\|\int_0^t \mathbb{T}_{t-s}\mathbb{P}\operatorname{div} F(s)\,\mathrm{d}s\right\|_3 \leqslant \int_0^t \frac{1}{(t-s)^{1/2}}\|F(s)\|_{3,E}\,\mathrm{d}s \leqslant C\int_0^t \frac{1}{(t-s)^{1/2}}\frac{K^2}{s^{1/2}}\,\mathrm{d}s \leqslant CK^2. \tag{8.23}$$

Applying Lemma 8.1 with $p_0=3$, $q_0=3/2$ for $s\in(0,t/2)$, and with $p_0=q_0=3$ for $s\in(t/2,t)$ and the estimates (8.20),(8.21), we get

$$\begin{aligned}&\left\|\int_0^t \mathbb{T}_{t-s}\mathbb{P}\operatorname{div} F(s)\,\mathrm{d}s\right\|_3\\ &\leqslant \int_0^{t/2} \frac{1}{(t-s)^{3/2(2/3-1/3)+1/2}}\|F(s)\|_{3/2,E}\,\mathrm{d}s + \int_{t/2}^t \frac{1}{(t-s)^{1/2}}\|F(s)\|_{3,E}\,\mathrm{d}s\\ &\leqslant C\int_0^{t/2} \frac{1}{(t-s)^{3/2(2/3-1/3)+1/2}}\frac{K^2}{s^{3/2(1/q-1/3)}}\,\mathrm{d}s + \int_{t/2}^t \frac{1}{(t-s)^{1/2}}\frac{K^2}{s^{3/(2q)}}\,\mathrm{d}s\\ &\leqslant \frac{CK^2}{t^{3/2(1/q-1/3)}}.\end{aligned} \tag{8.24}$$

According to Lemma 8.1 with $p_0 = 6, q_0 = 3$ and (8.19)

$$\left\| \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\, F(s)\,\mathrm{d}s \right\|_6 \leqslant \int_0^t \frac{1}{(t-s)^{3/2(1/3-1/6)+1/2}} \|F(s)\|_{3,E}\,\mathrm{d}s$$
$$\leqslant C\int_0^t \frac{1}{(t-s)^{3/2(1/3-1/6)+1/2}} \frac{K^2}{s^{1/2}}\,\mathrm{d}s \leqslant \frac{CK^2}{t^{3/2(1/3-1/6)}}. \tag{8.25}$$

Finally, using Lemma 8.1 with $p_0 = 6, q_0 = 3/2$ for $s \in (0, t/2)$ and $p_0 = 6, q_0 = 3$ for $s \in (t/2, t)$ and using the estimates (8.20)- (8.21), we deduce

$$\left\| \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\, F(s)\,\mathrm{d}s \right\|_6$$
$$\leqslant \int_0^{t/2} \frac{1}{(t-s)^{3/2(2/3-1/6)+1/2}} \|F(s)\|_{3/2,E}\,\mathrm{d}s + \int_{t/2}^t \frac{1}{(t-s)^{3/2(1/3-1/6)+1/2}} \|F(s)\|_{3,E}\,\mathrm{d}s$$
$$\leqslant C\int_0^{t/2} \frac{1}{(t-s)^{3/2(2/3-1/6)+1/2}} \frac{K^2}{s^{3/2(1/q-1/3)}}\,\mathrm{d}s + \int_{t/2}^t \frac{1}{(t-s)^{3/2(1/3-1/6)+1/2}} \frac{K^2}{s^{3/(2q)}}\,\mathrm{d}s$$
$$\leqslant \frac{CK^2}{t^{3/2(1/q-1/6)}}. \tag{8.26}$$

Therefore, combining the estimates (8.16) and (8.22)-(8.26), we have proved that if $V \in \mathscr{C}(K) \cap \mathcal{C}$,

$$\|\Lambda(V)\|_{\mathscr{C}} \leqslant C\|V_0\|_{\mathbb{X}^q \cap \mathbb{X}^3} + CK^2,$$

i.e. that $\Lambda(V) \in \mathscr{C}(K_1)$ with K_1 as in (8.13).

In order to show the second estimate (8.13), we use (8.17) for the first term of (8.6) and (7.3) to estimate the second term, to obtain that for all $V \in \mathscr{C}(K) \cap \mathcal{C}(K')$, for every $t > 0$ we have:

$$\left\| \nabla \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\,(F(s))\,\mathrm{d}s \right\|_{3/2,E}$$
$$\leqslant \left\| \int_0^t \nabla\mathbb{T}_{t-s}\mathbb{P}\left\{ [(\ell(s) - v(s)) \cdot \nabla]\, v(s) \right\} \mathrm{d}s \right\|_{3/2,E}$$
$$\leqslant \int_0^t \frac{C}{(t-s)^{1/2}} |\ell(s)| \|\nabla v(s)\|_{3/2,E}\,ds + \int_0^t \frac{C}{(t-s)^{3/4}} \|v(s)\|_{6,E} \|\nabla v(s)\|_{3/2,E}\,\mathrm{d}s$$
$$\leqslant K' \left(\int_0^t \frac{C}{(t-s)^{1/2}} \frac{K}{s^{1/2}}\,\mathrm{d}s + \int_0^t \frac{C}{(t-s)^{3/4}} \frac{K}{s^{1/4}}\,\mathrm{d}s \right)$$
$$\leqslant CKK'. \tag{8.27}$$

To study the Lipschitz character of the map Λ, we simply remark that, for V^a and V^b in $\mathscr{C}(K) \cap \mathcal{C}(K')$,

$$(\Lambda(V^a) - \Lambda(V^b))(t) = \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\,(F^a - F^b)(s)\,\mathrm{d}s, \qquad (t \geqslant 0), \tag{8.28}$$

and that

$$F^a - F^b = -\mathbb{1}_E(v^a - \ell^a) \otimes v^a + \mathbb{1}_E(v^b - \ell^b) \otimes v^b$$
$$= -\mathbb{1}_E((v^a - v^b) - (\ell^a - \ell^b)) \otimes v^a + \mathbb{1}_E(v^b - \ell^b) \otimes (v^b - v^a).$$

We can thus estimate $F^a - F^b$ similarly as what has been done for F, starting from the estimate

$$|(\ell^a - \ell^b)(s)| \leqslant \frac{K}{s^{1/2}}, \quad (s > 0,\ V^a, V^b \in \mathscr{C}(K)),$$

to obtain that there exists $C>0$ such that for all $s>0$,

$$\|(F^a-F^b)(s)\|_{3,E} \leqslant \frac{C(\|\ V^a\|_{\mathscr{C}}+\|\ V^b\|_{\mathscr{C}})\|V^a-V^b\|_{\mathscr{C}}}{s^{1/2}},$$
$$\|(F^a-F^b)(s)\|_{3,E} \leqslant \frac{C(\|\ V^a\|_{\mathscr{C}}+\|\ V^b\|_{\mathscr{C}})\|V^a-V^b\|_{\mathscr{C}}}{s^{3/(2q)}}, \tag{8.29}$$
$$\|(F^a-F^b)(s)\|_{3/2,E} \leqslant \frac{C(\|\ V^a\|_{\mathscr{C}}+\|\ V^b\|_{\mathscr{C}})\|V^a-V^b\|_{\mathscr{C}}}{s^{3/2(1/q-1/3)}}. \tag{8.30}$$

The above inequalities allow us to estimate (8.28) in the norm of $\mathscr{C}$ by using arguments similar to those used to estimate the second term in (8.12) and thus further to prove (8.14).

Similarly, it is not difficult to show that for every $s>0$ we have

$$\begin{aligned}\|(\ell^a\cdot\nabla v^a-\ell^b\cdot\nabla v^b)(s)\|_{3/2,E} &\leqslant \frac{C}{s^{1/2}}(\|\ V^a\|_{\mathscr{C}}+\|\ V^b\|_{\mathscr{C}})\|\nabla V^a-\nabla V^b\|_{L^\infty(0,\infty;L^{3/2}(E))}\\ &\quad+\frac{C}{s^{1/2}}(\|\nabla V^a\|_{L^\infty(0,\infty;L^{3/2}(E))}+\|\nabla V^b\|_{L^\infty(0,\infty;L^{3/2}(E))})\|V^a-V^b\|_{\mathscr{C}}.\\ \|(v^a\cdot\nabla v^a-v^b\cdot\nabla v^b)(s)\|_{6/5,E} &\leqslant \frac{C}{s^{1/4}}(\|\ V^a\|_{\mathscr{C}}+\|\ V^b\|_{\mathscr{C}})\|\nabla V^a-\nabla V^b\|_{L^\infty(0,\infty;L^{3/2}(E))}\\ &\quad+\frac{C}{s^{1/4}}(\|\nabla V^a\|_{L^\infty(0,\infty;L^{3/2}(E))}+\|\nabla V^b\|_{L^\infty(0,\infty;L^{3/2}(E))})\|V^a-V^b\|_{\mathscr{C}}.\end{aligned}$$

Mimicking the proof of (8.27), we easily deduce (8.15). This concludes the proof of Lemma 8.4. □

Setting

$$K=1/(2C),\quad \text{and } K'=2C\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})}, \tag{8.31}$$

Lemma 8.4 implies that for V_0 such that $\|V_0\|_{\mathbb{X}^q\cap\mathbb{X}^3}\leqslant\frac{1}{4C^2}$, the map Λ maps $\mathscr{C}(K)\cap\mathcal{C}(K')$ to itself and it is strictly contractive for the norm

$$\|V\|_{\mathscr{C}}+\frac{K}{2K'}\|\nabla V\|_{3/2,E}\,.$$

It thus admits a unique fixed point V in that class, for which we have in particular that $V\in\mathscr{C}(K)\cap\mathscr{C}(K')$.

Step 3: Decay in $\mathbb{X}^{q_1}$ norm for $q_1\geqslant 3$. The fixed point V of Λ given by Lemma 8.4 for small enough initial datum is the solution we are looking for. It remains to prove the decay estimate (1.8). Let us only check the decay of the $\mathbb{X}^{q_1}$ norm (this is only needed if $q_1>6$, since otherwise a simple interpolation argument gives the result, but in fact our argument works for any $q_1\geqslant 3$). In order to do that, we proceed as for the estimate of the $\mathbb{X}^6$ norm (see (8.25)- (8.26)): for all $t>0$,

$$\begin{aligned}\left\|\int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\,F(s)\,\mathrm{d}s\right\|_{q_1} &\leqslant \int_0^{t/2}\frac{1}{(t-s)^{3/2(2/3-1/q_1)+1/2}}\|F(s)\|_{3/2,E}\,ds\\ &\quad+\int_{t/2}^t\frac{1}{(t-s)^{3/2(1/3-1/q_1)+1/2}}\|F(s)\|_{3,E}\,ds\\ &\leqslant C\int_0^{t/2}\frac{1}{(t-s)^{3/2(2/3-1/q_1)+1/2}}\frac{K^2}{s^{3/2(1/q-1/3)}}\,ds\\ &\quad+\int_{t/2}^t\frac{1}{(t-s)^{3/2(1/3-1/q_1)+1/2}}\frac{K^2}{s^{3/(2q)}}\,ds\\ &\leqslant \frac{CK^2}{t^{3/2(1/q-1/q_1)}}.\end{aligned} \tag{8.32}$$

Since V satisfies (8.6), we thus obtain that for $t > 0$,

$$t^{3/2(1/q-1/q_1)} \, \|V(t)\|_{q_1} \leqslant C\|V_0\|_{\mathbb{X}^q\cap\mathbb{X}^3} + CK^2 \leqslant C\|V_0\|_{\mathbb{X}^q\cap\mathbb{X}^3}.$$

This concludes the proof of Theorem 1.1. □

Remark 8.5 (Uniqueness of the solution provided by Theorem 1.1)**.** We claim that the solution V of (1.2) provided by Theorem 1.1 in fact belongs to $C^0_b([0,\infty);\mathbb{X}^p)$ for some $p > 3$. Accordingly, by Theorem 8.3, the maximal solution of (1.2) is defined globally in time.

Let us explain why $V \in C^0_b([0,\infty);\mathbb{X}^p)$ for some $p > 3$. Fix $p \in (3,\infty)$. Since we already have that $t^{3/2(1/q-1/p)}V(t) \in C^0_b([0,\infty);\mathbb{X}^p)$, we only have to focus on the behaviour of the norm $\mathbb{X}^p$ for small times. Now, since V is constructed as the fixed point of the map Λ which satisfies the assumptions of Banach fixed point theorem, V is the limit of the sequence $(V_n)_{n\in\mathbb{N}}$ given by $V_0 = 0$ and for $n \in \mathbb{N}$, $V_{n+1} = \Lambda(V_n)$. Thus, it is clear that the sequence $(V_n|_{(0,T_0)})_{n\in\mathbb{N}}$ satisfies $V_{n+1}|_{(0,T_0)} = \Lambda_{T_0}(V_n|_{(0,T_0)})$, where T_0 is the time given by Theorem 7.8 and Λ_{T_0} is the map introduced in the proof of Theorem 8.3 in (B.1). Arguing as in the proof of Theorem 8.3, it follows that$(V_n|_{(0,T)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T];\mathbb{X}^p)$, which strongly converges to the unique solution of (1.2) given in Theorem 8.3.

We have thus proved that $V \in C^0_b([0,\infty);\mathbb{X}^p)$, while V also satisfies $\nabla V \in C^0_b([0,\infty);\left[L^{3/2}(E)\right]^9)$, so that V is the solution of (1.2) within this class according to Theorem 8.3.

Remark 8.6. In fact, to run the above proof, the only assumption needed on V_0 in order to construct a solution is that

$$\sup_{t>0}\,\{t^{3/2(1/q-2/3)}\|\mathbb{T}_tV_0\|_{3/2} + (1+t^{3/2(1/q-1/3)})\|\mathbb{T}_tV_0\|_3 \\ + (t^{3/2(1/3-1/6)} + t^{3/2(1/q-1/6)})\|\mathbb{T}_tV_0\|_6 + \|\nabla\mathbb{T}_tV_0\|_{3/2,E}\} < \infty. \quad (8.33)$$

The assumptions in Theorem 1.1 imply that $V_0 \in \mathcal{D}(\mathbb{A}_{3/2})$, which obviously implies (8.33). However, it semms likely that assuming $V_0 \in \mathcal{D}(\mathbb{A}^{1/2}_{3/2})$ is sufficient to guarantee (8.33).

For the uniqueness of solutions we only use that the map $t \mapsto \mathbb{T}_tV_0$ is also in $C^0_b([0,\infty);\mathbb{X}^6)$, which can also be guaranteed by assuming the less demanding assumption $V_0 \in \mathbb{X}^6$.

Therefore, the class of initial data proposed in Theorem 1.1 is by no means sharp. As we shall see afterwards in Theorem 8.10, we will anyway be able to extend this theorem to a much larger class of initial data, for which the solutions will be considered in a weaker sense.

Remark 8.7. The usual smallness condition for global existence of solution (see [14]) simply is the smallness of V_0 in $\mathbb{X}^3$. The difficulty here is that the above estimates strongly require the estimate (8.18), which we do not know how to prove for initial data small only in $\mathbb{X}^3$.

Therefore, in our approach, we used that the initial datum is small in a set of the form $\mathbb{X}^{q_0}\cap\mathbb{X}^3$ for some $q_0 < 3$. In fact, Theorem 1.1 focus on the case $q_0 \in (1,3/2)$, because that will allow to conclude that the body stops as $t \to \infty$, but the following result can also be proved along the same lines:

Theorem 8.8. *Let $q \in (3/2,3)$ and $q_1 \in (3,\infty)$ such that $3/2(1/q-1/q_1) \geqslant 1/2$. Then there exists $\varepsilon_0 > 0$ such that for all $V_0 = \begin{bmatrix} v_0 \\ \ell_0 \\ \omega_0 \end{bmatrix} \in \mathbb{X}^q\cap\mathbb{X}^3$ satisfying (1.3)–(1.4)–(1.5)–(1.6), there is a solution $V = \begin{bmatrix} v(t,\cdot) \\ \ell(t) \\ \omega(t) \end{bmatrix}$ of (1.2) such that*

$$\sup_{t>0}\,\{(1+t^{3/2(1/q-1/3)})\|V(t)\|_3\} < \infty, \quad (8.34)$$

and for all $q_1 \in (3,\infty)$,

$$\sup_{t>0} \{(t^{3/2(1/3-1/q_1)} + t^{3/2(1/q-1/q_1)})\|V(t)\|_{q_1}\} < \infty.$$

The detailed proof is left to the reader, since it mainly follows the one of Theorem 1.1, with the important change that the class $\mathscr{C}$ and its norm should be redefined as

$$\mathscr{C} = \{V \in C^0(0,\infty;\mathbb{X}^3), \text{ such that } (1+t^{3/2(1/q-1/3)})V(t) \in C^0_b(0,\infty;\mathbb{X}^3), \\ (t^{3/2(1/3-1/q_1)} + t^{3/2(1/q-1/q_1)})V(t) \in C^0_b(0,\infty;\mathbb{X}^{q_1})\},$$

for some q_1 with $3/2(1/q-1/q_1) \geqslant 1/2$, which we endow with the norm

$$\|V\|_{\mathscr{C}} = \sup_{t>0}\{(1+t^{3/2(1/q-1/3)})\|V(t)\|_3 + (t^{3/2(1/3-1/q_1)} + t^{3/2(1/q-1/q_1)})\|V(t)\|_{q_1}\}. \tag{8.35}$$

The critical point in the proof of the existence of a fixed point to the map Λ will again be that since $3/2(1/q-1/q_1) \geqslant 1/2$, for $V \in \mathscr{C}$, we again have, similarly as in (8.18),

$$\sup_{s>0} s^{1/2}\ |\ell(s)| \leqslant \|V\|_{\mathscr{C}}.$$

We end this section by an extension of Theorem 1.1 to the case $V_0 = \begin{bmatrix} v_0 \\ \ell_0 \\ \omega_0 \end{bmatrix} \in [L^q(E)]^3 \cap \big[L^3(E)\big]^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying (1.6) and (1.10). In other words, we obtain here a global in time existence result for a weaker notion of solution.

In order to do that, we will rely on the following lemma:

Lemma 8.9. *Let* $\mathcal{O} = B$ *and* $E = \mathbb{R}^3 \setminus B$, *and* $q \in (1,3/2]$. *Choosing* $\varepsilon_0 > 0$ *as in Theorem 1.1, for all* $V_0^a = \begin{bmatrix} v_0^a \\ \ell_0^a \\ \omega_0^a \end{bmatrix}, V_0^b = \begin{bmatrix} v_0^b \\ \ell_0^b \\ \omega_0^b \end{bmatrix}$ *in* $\mathbb{X}^q \cap \mathbb{X}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ *satisfying* (1.3)–(1.4)–(1.5)–(1.6), *the respective solutions* $V^a = \begin{bmatrix} v^a \\ \ell^a \\ \omega^a \end{bmatrix}$ *and* $V^b = \begin{bmatrix} v^b \\ \ell^b \\ \omega^b \end{bmatrix}$ *of* (1.2) *with initial datum* V_0^a *and* V_0^b *given by Theorem 1.1 satisfy*

$$\|V^a - V^b\|_{\mathscr{C}} \leqslant C\|V_0^a - V_0^b\|_{\mathbb{X}^q\cap\mathbb{X}^3}. \tag{8.36}$$

where $\|\cdot\|_{\mathscr{C}}$ *is defined by* (8.8). *Besides, for all* $q_1 \geqslant 3$, *we have*

$$\sup_{t>0}\{t^{3/2(1/q-1/q_1)}(\|(v^a-v^b)(t)\|_{q_1,E} + \|(\ell^a-\ell^b)(t)\|_{\mathbb{R}^3} \\ + \|(\omega^a-\omega^b)(t)\|_{\mathbb{R}^3})\} \leqslant C\|V_0^a - V_0^b\|_{\mathbb{X}^q\cap\mathbb{X}^3}. \tag{8.37}$$

Proof. The proof of Lemma 8.9 is in fact almost contained in the proof of Theorem 1.1. Indeed, for V_0^a and V_0^b as in the assumption, we have

$$V^a(t) = \mathbb{T}_t V_0^a + \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\, F^a(s)\,\mathrm{d}s, \text{ where } F^a(s) = -1_E(v^a(s)-\ell^a(s))\otimes v^a(s).$$
$$V^b(t) = \mathbb{T}_t V_0^b + \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\, F^b(s)\,\mathrm{d}s, \text{ where } F^b(s) = -1_E(v^b(s)-\ell^b(s))\otimes v^b(s).$$

Thus taking the difference of the last two forrmulas, we get

$$V^a(t) - V^b(t) = \mathbb{T}_t(V_0^a - V_0^b) + \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\,(F^a - F^b(s))\,\mathrm{d}s,$$

so that

$$\|V^a - V^b\|_{\mathscr{C}} \leqslant \|\mathbb{T}_t(V_0^a - V_0^b)\|_{\mathscr{C}} + \left\| \int_0^t \mathbb{T}_{t-s}\mathbb{P}\mathrm{div}\,(F^a - F^b(s))\,\mathrm{d}s \right\|_{\mathscr{C}}$$
$$\leqslant C\|V_0^a - V_0^b\|_{\mathbb{X}^q\cap\mathbb{X}^3} + \frac{1}{2}\|V^a - V^b\|_{\mathscr{C}},$$

where the first estimate comes from Theorem 7.6, while the second follows from estimate (8.14) with the choice (8.31) done in the proof of Theorem 1.1.

In order to prove (8.37), we do as in (8.32) using the bounds (8.29) and (8.30). Details are left to the reader. □

We thus have the following consequence of Theorem 1.1 and Lemma 8.9:

Theorem 8.10. *Let $\mathcal{O} = B$ and $E = \mathbb{R}^3 \setminus B$, and $q \in (1, 3/2]$. Then there exists $\varepsilon_0 > 0$ such that for all initial datum $V_0 = \begin{bmatrix} v_0 \\ \ell_0 \\ \omega_0 \end{bmatrix} \in \mathbb{X}^q \cap \mathbb{X}^3$ satisfying (1.6) and (1.10) and any sequence $(V_{0,n})_{n\in\mathbb{N}}$ converging to V_0 in $\mathbb{X}^q \cap \mathbb{X}^3$ and such that for all $n \in \mathbb{N}$, $V_{0,n}$ satisfies (1.3)–(1.4)–(1.5)–(1.6), the corresponding solutions V_n of (1.2) form a Cauchy sequence in $\mathscr{C}$ (defined in (8.7)), which converges to a function $V \in \mathscr{C}$ satisfying the decay estimates (1.7)–(1.8). Besides, this limit V is independent of the sequence of initial data considered to approximate V_0.*

The proof of Theorem 8.10 is a straightforward consequence of Lemma 8.9, so it is omitted here. Note that Theorem 8.10 gives a very weak meaning to solutions of (1.2) for initial datum in $\mathbb{X}^q \cap \mathbb{X}^3$ with small norms as the unique limit of solutions of (1.2) for more regular initial data converging in $\mathbb{X}^q \cap \mathbb{X}^3$.

9 Concluding remarks and open questions

The main result in this paper, namely Theorem 1.1, concerns the wellposednes of the system modelling the motion of a rigid ball in a viscous incompressible fluid filling the remaining part of $\mathbb{R}^3$ and asserts that the position of the centre of the ball tends, when $t \to \infty$, to some position $h_\infty \in \mathbb{R}^3$. This result differs from those previously obtained in two space dimensions in [6], or for a simplified $1D$ model in [25], where it has been shown that the distance of the centre of the ball to the origin tends to $+\infty$ when $t \to \infty$. Several open questions seem natural in view of our results.

One of the most challenging ones, for which we have no track at this stage, is determining h_∞ from the initial data.

Another natural question is the generalization of Theorem 1.1 for a body of arbitrary shape. When the rigid body is not a ball, writing the equations in a fixed domain requires the use of more delicate change of variables, since it has to include the rotation of the body. There are basically two ways of doing that: one consists in setting $v(t,x) = Q^*(t)u(t, h(t) + Q(t)x)$, where $Q(t)$ is the rotation matrix of the body, that is the solution of $\dot{Q}(t)Q^*(t)x = \omega(t) \times x$ starting from $Q(0) = Id$. The problem is that such change of frame would induce in the fixed frame a term of the form $(\omega \times x)\cdot\nabla v$ which our estimate does not allow to handle since the identity mapping does not belong to $L^\infty(E)$. The alternative approach proposed in [4], which consists in constructing a change of variable which follows the structure in a neighbourhood of it and equals the identity far from the body, seems more suitable to deal with the non-linear terms. However, this change of variable introduces a lot of delicate terms which we do not know how to handle in the above setting so far. In fact, even in two space dimension, the existing results (see [6]) provide an analysis of the motion of a rigid body in a viscous incompressible fluid in $\mathbb{R}^2$ only in the case when the rigid body is a disk.

The smallness conditions in Theorem 1.1 also raises some natural issues. The usual smallness condition for global existence of solutions of Navier-Stokes equations (see, for instance, [14]) is simply given in terms of the norm of V_0 in $\mathbb{X}^3$, so that it might seem surprising to further impose a smallness condition on $\|V_0\|_q$ with $q < 3/2$. In fact, an easy adaptation of the proof of Theorem 1.1 shows that a similar global existence result holds when imposing a smallness condition on the $L^q \cap L^3$ norm of V_0 with $q \in (1,3)$ (see Remark 8.7). However, it is not clear if we can get a similar global existence result by simply assuming that the L^3 norm of V_0 is small. This property is very likely strongly linked with the following question. For $U_0 \in \mathbb{X}^3$, can we guarantee that, with the notation of Remark 7.7, there exists a constant $C > 0$ such that for all $t \geqslant 0$,

$$|\ell(t)| \leqslant \frac{C}{t^{1/2}} \ ?$$

Indeed, this is precisely this estimate (recall (8.18)) that was used in our proof of Theorem 1.1.

Finally, let us mention that the counterparts in two space dimensions of some of our results in Sections 4–7 have been used in Takahashi and Lacave [16] to study the behaviour of solutions of (1.2) when the radius of the rigid ball tends to zero (see also He and Iftimie [10] and references therein). We believe that the approach in [16] can be adapted to the three dimensional case by using our results on the fluid structure-semigroup and its generator, but this deserves further work.

A Proof of Proposition 3.2

We first show that every $u \in [L^q(\Omega)]^3$ can be written in the form $u = v + w_1 + w_2$, with $v \in \mathbb{X}^q(\Omega)$, $w_1 \in G_1^q(\Omega)$ and $w_2 \in G_2^q(\Omega)$. To this aim, let q_1 be the solution of the problem

$$\Delta q_1 = \operatorname{div} u \text{ in } \Omega, \quad q_1 = 0 \text{ on } \partial\Omega.$$

Thus $q_1 \in W_0^{1,q}(\Omega)$ and setting $w_1 = \nabla q_1$, we clearly have $w_1 \in G_1^q(\Omega)$. Since we are looking for $v \in \mathbb{X}^q(\Omega)$, we know that

$$v = \ell_v + \omega_v \times x \text{ for } x \in \mathcal{O}, \tag{A.1}$$

for some $\ell_v \in \mathbb{R}^3$ and $\omega_v \in \mathbb{R}^3$. We set

$$\varphi(x) = u(x) - \nabla q_1(x) - \ell_v - \omega_v \times x \qquad x \in \mathcal{O}, \tag{A.2}$$

and

$$w_2 = \begin{cases} \nabla q_2 & \text{in } E_\Omega \\ \varphi & \text{in } \mathcal{O}. \end{cases} \tag{A.3}$$

Since we are looking for $w_2 \in G_2^q(\Omega)$ we require

$$\ell_v = \frac{1}{m}\int_{\mathcal{O}} (u - \nabla q_1 - \varphi)\, \mathrm{d}x = \frac{1}{m}\left[\int_{\mathcal{O}} (u - \nabla q_1)\, \mathrm{d}x + \int_{\partial\mathcal{O}} q_2 \nu \, \mathrm{d}\gamma\right], \tag{A.4}$$

and

$$\omega_v = -\frac{1}{\mathcal{J}}\int_{\mathcal{O}} (u - \nabla q_1 - \varphi) \times x \, \mathrm{d}x = -\frac{1}{\mathcal{J}}\left[\int_{\mathcal{O}} (u - \nabla q_1) \times x \, \mathrm{d}x + \int_{\partial\mathcal{O}} q_2 \nu \times x \, \mathrm{d}\gamma\right]. \tag{A.5}$$

Now we define q_2 as the solution of the Neumann problem

$$\begin{cases} \Delta q_2 = 0 & \text{in } E_\Omega, \\ \dfrac{\partial q_2}{\partial n} = (u - \nabla q_1)\cdot n & \text{on } \partial\Omega, \\ \dfrac{\partial q_2}{\partial n} = (u - \nabla q_1)\cdot n - (\ell_v + \omega_v \times x)\cdot n & \text{on } \partial\mathcal{O}, \end{cases} \tag{A.6}$$

where ℓ_v and ω_v are defined in (A.4) and (A.5), respectively. Note that, q_2 solves a Laplace equation with non-local boundary condition. As shown below, we have that $q_2 \in W^{1,q}(E_\Omega)$ and there exists a constant C depending on q, Ω and $\mathcal{O}$ such that

$$\|q_2\|_{1,q,E_\Omega} \leqslant C \|u\|_{q,\Omega}\,. \tag{A.7}$$

In this case we can determine ℓ_v and ω_v from (A.4) and (A.5) respectively. Consequently, we obtain φ and w_2 from (A.2) and (A.3) respectively. In particular, we have that $w_2 \in G_2^q(\Omega)$ and by setting $v = u - w_1 - w_2$ we can verify that $v \in \mathbb{X}^q(\Omega)$.

We still have to prove that $q_2 \in W^{1,q}(\Omega)$ and (A.7) holds. If $q = 2$, this is a consequence of Lax-Milgram Theorem (see for instance [5, Lemma 1]). If $q \neq 2$, we employ a density argument. Assume that $u \in [C_0^\infty(\Omega)]^3$ and q_2 solves (A.6). Then there exists a constant C depending only on q, Ω and $\mathcal{O}$ such that

$$\|q_2\|_{1,q,E_\Omega} \leqslant C \left(\|u\|_{q,\Omega} + |\ell_v| + |\omega_v| \right). \tag{A.8}$$

Next, by following the arguments of the proof of Theorem 2.2 in [26], we have

$$|\ell_v| + |\omega_v| \leqslant C \|u\|_{q,\Omega}\,, \tag{A.9}$$

where C is a positive constant depending only on q, Ω and $\mathcal{O}$. The above two estimates yield that there exists a positive constant C, depending only on q, Ω and $\mathcal{O}$, such that estimate (A.7) holds. Thus the conclusion follows by a density argument. This completes the proof of the existence of a decomposition with the required properties. The proof of uniqueness of the decomposition is similar to that of [26, Theorem 2.2].

B Proof of Theorem 8.3

As in the proof of Theorem 1.1, we start by remarking that we are looking for a solution V of (1.9) or equivalently (8.6).

Let $p \in (3, \infty)$. For $T > 0$, we introduce the class:

$$\mathscr{C}(T) = \left\{ V = \begin{bmatrix} v \\ \ell \\ \omega \end{bmatrix} \in C^0([0,T]; \mathbb{X}^p) \text{ with } \nabla v \in \left[C^0([0,T]; L^{3/2}(E)) \right]^9 \right\},$$

which we endow with the norm

$$\|V\|_{\mathscr{C}(T)} = \|V\|_{L^\infty(0,T;\mathbb{X}^p)} + \|\nabla v\|_{L^\infty(0,T;L^{3/2}(E))}.$$

We then define the map $\Lambda_T : V \in \mathscr{C}(T) \mapsto \Lambda_T V$ defined for $t \in [0,T]$ by

$$\Lambda_T V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s} \mathbb{P}((\ell(s) - v(s)) \cdot \nabla v(s))\, ds, \tag{B.1}$$

or, equivalently,

$$\Lambda_T V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s} \mathbb{P} \mathrm{div}\, ((\ell(s) - v(s)) \otimes v(s))\, ds,$$

For V^a and $V^b \in \mathscr{C}(T)$, we estimate $\Lambda_T V^a - \Lambda_T V^b$ in the norm $C^0([0,T]; \mathbb{X}^p)$, for which we use

Lemma 8.1 with $q = p$ and $q = p/2$: for all $t \in [0,T]$,

$$\begin{aligned}
\big\|\Lambda_T V^a - \Lambda_T V^b(t)\big\|_p &\leqslant C\int_0^t \frac{1}{(t-s)^{1/2}}\|(\ell^a(s)\otimes V^a(s) - \ell^b(s)\otimes V^b(s))\|_p\,ds \\
&\quad + C\int_0^t \frac{1}{(t-s)^{1/2+3/2(2/p-1/p)}}\|V^a(s)\otimes V^a(s) - V^b(s)\otimes V^b(s)\|_{p/2}\,ds \\
&\leqslant C\int_0^t \frac{1}{(t-s)^{1/2}}\|V^a - V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\left(\|V^a\|_{L^\infty(0,T;\mathbb{X}^p)} + \|V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\right)\,ds \\
&\quad + C\int_0^t \frac{1}{(t-s)^{1/2+3/2(2/p-1/p)}}\|V^a - V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\left(\|V^a\|_{L^\infty(0,T;\mathbb{X}^p)} + \|V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\right)\,ds \\
&\leqslant C(T^{1/2} + T^{1/2-3/2p})\|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right),
\end{aligned}$$

so that we have, for some constant C independent of $T > 0$,

$$\|\Lambda_T(V^a) - \Lambda_T(V^b)\|_{L^\infty(0,T;\mathbb{X}^p)} \leqslant C(T + T^{1/2-3/2p})\|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right).$$

Then, for V^a and $V^b \in \mathscr{C}(T)$, we estimate $\nabla(\Lambda_T V^a - \Lambda_T V^b)$ in $C^0([0,T];\big[L^{3/2}(E)\big]^9)$ using the gradient estimate (7.3) with $r = 3/2$ and $q = 3/2$ and $r = 3/2$ and q_p defined by $1/q_p = 2/3 + 1/p$: for all $t \in [0,T]$,

$$\begin{aligned}
\big\|\nabla(\Lambda_T V^a - \Lambda_T V^b)(t)\big\|_{p,E} &\leqslant C\int_0^t \frac{1}{(t-s)^{1/2}}\|\ell^a(s)\cdot\nabla V^a(s) - \ell^b(s)\cdot\nabla V^b(s))\|_{3/2,E}\,ds \\
&\quad + C\int_0^t \frac{1}{(t-s)^{1/2+3/2(2/3+1/p-2/3)}}\|V^a(s)\cdot\nabla V^a(s) - V^b(s)\cdot\nabla V^b(s)\|_{q_p}\,ds \\
&\leqslant C\int_0^t \frac{1}{(t-s)^{1/2}}\|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right)\,ds \\
&\quad + C\int_0^t \frac{1}{(t-s)^{1/2+3/(2p)}}\|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right)\,ds \\
&\leqslant C(T^{1/2} + T^{1/2-3/2p})\|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right),
\end{aligned}$$

where we have used that, for all $s \in (0,T)$,

$$\begin{aligned}
&\|\ell^a(s)\cdot\nabla V^a(s) - \ell^b(s)\cdot\nabla V^b(s))\|_{3/2,E} \\
&\leqslant \|V^a - V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\|\nabla V^a\|_{L^\infty(0,T;L^{3/2}(E))} + \|V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\|\nabla V^a - \nabla V^b\|_{L^\infty(0,T;L^{3/2}(E))} \\
&\leqslant \|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|V^a(s)\cdot\nabla V^a(s) - V^b(s)\cdot\nabla V^b(s)\|_{q_p} \\
&\leqslant \|V^a - V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\|\nabla V^a\|_{L^\infty(0,T;L^{3/2}(E))} + \|V^b\|_{L^\infty(0,T;\mathbb{X}^p)}\|\nabla V^a - \nabla V^b\|_{L^\infty(0,T;L^{3/2}(E))} \\
&\leqslant \|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right).
\end{aligned}$$

We have thus proved that there exists a constant $C > 0$ such that for all $T > 0$, for all V^a and V^b in $\mathscr{C}(T)$,

$$\|\Lambda_T(V^a) - \Lambda_T(V^b)\|_{\mathscr{C}(T)} \leqslant C(T^{1/2} + T^{1/2-3/2p})\|V^a - V^b\|_{\mathscr{C}(T)}\left(\|V^a\|_{\mathscr{C}(T)} + \|V^b\|_{\mathscr{C}(T)}\right). \quad \text{(B.2)}$$

Since we have from (8.2)–(8.3) that

$$\|\Lambda_T(0)\|_{\mathscr{C}(T)} \leqslant C_0\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})},$$

for some C_0 independent of T, routine arguments show that, setting $K = 2C_0\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})}$ and $T_0 > 0$ such that

$$4CC_0\|V_0\|_{\mathcal{D}(\mathbb{A}_{3/2})}(T_0^{1/2} + T_0^{1/2-3/2p}) = 1/2,$$

we have, for all $V \in \mathscr{C}(T_0)$ with $\|V\|_{\mathscr{C}(T_0)} \leqslant K$, $\|\Lambda_{T_0}V\|_{\mathscr{C}(T_0)} \leqslant K$, and for all V^a and V^b in $\mathscr{C}(T_0)$ with $\|V^a\|_{\mathscr{C}(T_0)} \leqslant K$ and $\|V^b\|_{\mathscr{C}(T_0)} \leqslant K$, $\|\Lambda_{T_0}(V^a) - \Lambda_{T_0}(V^b)\|_{\mathscr{C}(T_0)} \leqslant \|V^a - V^b\|_{\mathscr{C}(T_0)}/2$. Thus, Banach fixed point theorem shows the existence of a fixed point V to Λ_{T_0} in the class $\mathscr{C}(T_0)$.

To get the uniqueness part in the proof of Theorem 8.3, let V^a and V^b be two solutions of (1.2) with the same initial datum in the class $\mathscr{C}(T)$. Then the above computations show that, setting

$$e(t) = \big\|(V^a - V^b)(t)\big\|_{\mathbb{X}^p} + \big\|(\nabla V^a - \nabla V^b)(t)\big\|_{3/2,E} \qquad (t \in (0,T)),$$

we have

$$e(t) \leqslant C\int_0^t \left(\frac{1}{(t-s)^{1/2}} + \frac{1}{(t-s)^{1/2+3/(2p)}}\right) e(s)\,\mathrm{d}s \qquad (t \in (0,T)),$$

while $e(0) = 0$. We can thus conclude that e vanish identically on $(0,T)$ by Gronwall's Lemma.

This concludes the proof of Theorem 8.3.

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