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Hypoelliptic and spectral estimates for the linearized Landau operator

MOHAMAD RACHID*

Abstract

We are interested in the inhomogeneous Landau equation which describes the evolution of a particle density $f = f(t, x, v)$ representing at time $t \geq 0$, the density of particles at position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The study is motivated by the linearization of the Landau equation near Maxwellian distribution. In this article, we establish hypoelliptic estimates, a localization of the spectrum and estimates of the resolvent of the the linearized Landau operator with hard potentials and Maxwellian molecules. The proof is based on a multiplier method and requires refined pseudo-differential calculus tools.

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1 Introduction

1.1 The model.

In this paper, we study hypoellipticity and spectral properties associated to the spatially inhomogeneous Landau equation. This equation is a kinetic model in plasma physics that describes the evolution of the density function $F = F(t, x, v)$ representing at time $t \in \mathbb{R}^+$, the density of particles at position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. This equation is given by

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F) \\ F|_{t=0} = F_0, \end{cases} \quad (1)$$

where Q is the so-called Landau collision operator which acts on the variable v and which contains diffusion in velocity. More precisely, the Landau operator is defined by

$$Q(G, F) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) [G_* \partial_j F - F \partial_j G_*] dv_*, \quad (2)$$

where we use the convention of summation of repeated indices, and the derivatives are in the velocity variable v i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $G_* = G(v_*)$, $F = F(v)$, $\partial_j G_* = \partial_{v_* j} G(v_*)$, $\partial_j F = \partial_{v_j} F(v)$, etc. The matrix $A(v) = (a_{ij}(v))_{1 \leq i, j \leq 3}$ is symmetric, positive, definite, depends on the interaction between particles and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3, 1].$$

We recall the standard classification: we call hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules, i.e. $\gamma \in [0, 1]$. We denote by

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

the normalized Maxwellian which is a global equilibrium. We linearize the Landau equation around μ with the perturbation

$$F = \mu + \mu^{1/2} f.$$

The Landau equation [\(1\)](#) for $f = f(t, x, v)$ takes the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \mu^{-1/2} Q(\mu^{1/2} f, \mu) - \mu^{-1/2} Q(\mu, \mu^{1/2} f) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} f) \\ f|_{t=0} = f_0 = \mu^{-1/2} (F_0 - \mu), \end{cases} \quad (3)$$

since $Q(\mu, \mu) = 0$. Using the notation

$$\Gamma(f, g) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g),$$

we may rewrite the above equation as

$$\begin{cases} \partial_t f + \mathcal{P}f = \Gamma(f, f) \\ f|_{t=0} = f_0, \end{cases} \quad (4)$$

where the linearized Landau operator \mathcal{P} takes the form

$$\mathcal{P} = v \cdot \nabla_x - \mathcal{L} \quad (5)$$

with

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{L}_1 = \Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 = \Gamma(f, \sqrt{\mu}).$$

Operator \mathcal{P} acts only in variables (x, v) , is non selfadjoint, and consists of a transport part which is skew-adjoint, a diffusion part acting only in the v variable and a compact part (see for example Proposition 2.1 in [5]). Using for example [22], [21], the diffusion part \mathcal{L}_1 is written as follows

$$\mathcal{L}_1 f = \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] - \left(\mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2} \right) f + \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] f,$$

where $\mathbf{A}(v) = (\bar{a}_{ij}(v))_{1 \leq i, j \leq 3}$ is a symmetric matrix with

$$\bar{a}_{ij} = a_{ij} *_v \mu,$$

and the compact part \mathcal{L}_2 is given by

$$\mathcal{L}_2 f = -\mu^{-1/2} \partial_i \left\{ \mu \left[a_{ij} *_v \left\{ \mu^{1/2} \left[\partial_j f + \frac{v_j}{2} f \right] \right\} \right] \right\}.$$

Remark 1.1. Here we do not follow the same convention as the one in [8] for operators \mathcal{L}_1 and \mathcal{L}_2 .

1.2 Notations.

Throughout the paper we shall adopt the following notations: we work in dimension $n = 3$ and denote by $(x, v) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3$ the space-velocity variables. For $v \in \mathbb{R}^3$ we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$, where we recall that $|v|$ is the canonical Euclidian norm of v in \mathbb{R}^3 . The gradient in velocity (resp. space) will be denoted by ∂_v (resp. ∂_x). We shall also denote $D_v = \frac{1}{i} \partial_v$ (resp. $D_x = \frac{1}{i} \partial_x$), and denote ξ the dual variable of x , η the dual variable of v . For simplicity of notations, $a \sim b$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 b \leq a \leq c_2 b$; we abbreviate “ $\leq C$ ” to “ \lesssim ”, where C is a positive constant depending only on fixed number. Finally, the space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is an open set.

1.3 Main results and comments.

In this article, we will show a localization property of the pseudospectrum of the Landau operator \mathcal{P} , that is to say the region of the complex plane where its resolvent is a priori large (in fact we more precisely give a description of a large region where the resolvent *is* controlled, which is included in the complementary of the pseudospectrum). This result is given by the following theorem.

Theorem 1.2. *Let \mathcal{P} be the Landau operator on $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ defined in (5) with $\gamma \in [0, 1]$. Then there are two constants $C_{\mathcal{P}} > 0$ and $Q_{\mathcal{P}} > 0$ so that:*

a) The spectrum of \mathcal{P} verifies

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\},$$

with

$$S_{\mathcal{P}} = \left\{ z \in \mathbb{C}, |z+1|^{1/3} \leq C_{\mathcal{P}}(\Re z + 1), \Re z \geq -\frac{1}{2} \right\}. \quad (6)$$

b) For any $z \notin S_{\mathcal{P}}$ with $\Re z \geq -\frac{1}{2}$, the resolvent is estimated by

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq Q_{\mathcal{P}}|z+1|^{-1/3}. \quad (7)$$

Notice that if $\Re z \leq -\frac{1}{2}$ then

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq |\Re z|^{-1}. \quad (8)$$

The results of Theorem [1.2](#) can be illustrated by the following figure:

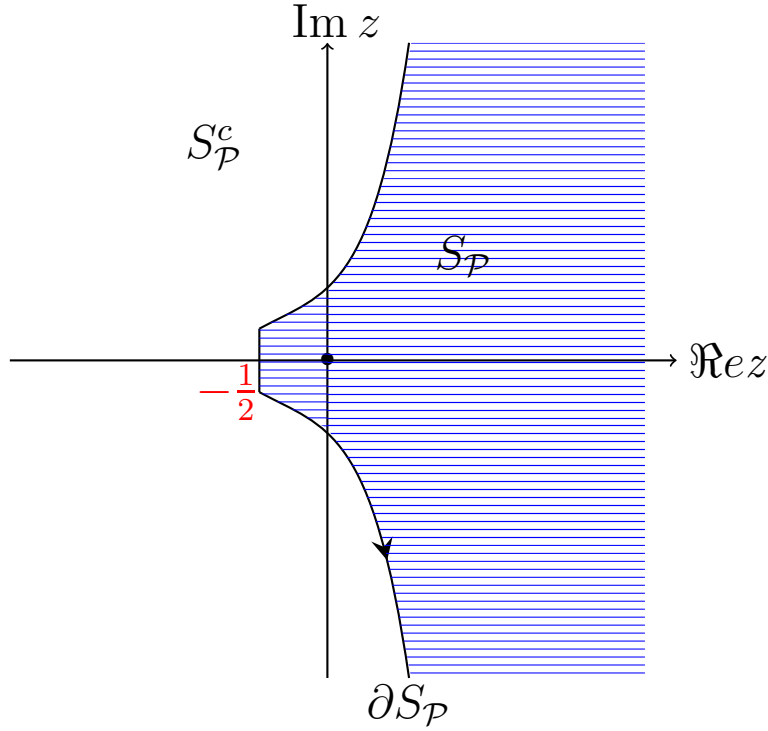


Figure 1: Localization of the pseudospectrum of the Landau operator \mathcal{P} .

In this figure $\partial S_{\mathcal{P}}$ oriented from $+i\infty$ to $-i\infty$. The hatched part is where the spectrum is localized and the non hatched part is the zone where we have good resolvent estimates (see [\(7\)](#) and [\(8\)](#)).

Let us give now some comments and motivations of this result. The cuspidal form of the pseudospectrum of linear or linearized kinetic operators was first shown in [\[9\]](#), [\[11\]](#) for the Fokker-Planck operator, and then extended to operators appearing in statistical mechanics by [\[6\]](#). The main motivation for this type of study is to be able to

understand the so-called pseudospectrum properties and derive possible trend to the equilibrium or regularization properties for the related evolution equation thanks to Cauchy formulae (see e.g. [11] for a complete result in the Fokker-Planck case). Anyway in this article, as a first step of a complete study of properties of this type, we only focus on the pseudospectral localization, but in the case of the much more complicated linearized Landau operator.

The aim of this article is double. First we show that the linearized Landau operator has indeed a pseudospectrum of cuspidal form, which is a good clue for thinking that this is a very general property of general linear or linearized kinetic operators. Second we propose a very robust and self contained pseudo-differential framework (see Section 3.1) in order to show this type of result. Our hope is that it can be used for many other kinetic models. These tools are greatly inspired by previous works (see [1], [12], [13]) concerning kinetic equations, following fundamental ideas in [18] on Wick and Weyl pseudodifferential calculus. Indeed the main remark done in all these works, following preliminary works by Alexandre and Villani [2] is that the linearized Landau operator L is, up to controlled/bounded operators, a pseudodifferential operator, namely that there exists a symbol, in a Hörmander-type class of symbol (see Section 3.1) such that

$$L \equiv a^w + \text{controlled terms},$$

where w stands for the Weyl quantization. The equivalence above can be rigorously stated and we will do it in section 2 below. The understanding of Landau type operators has been greatly improved in the past twenty years, following in particular works by Guo (see [8]).

Using these tools, the proof of the result given in Theorem 1.2 will follow the lines of [12] (see also [1], [11]), where a multiplier method is used for proving regularization (hypoelliptic) properties of the linearized Landau operator. Let us emphasize here that no regularization property is shown in this article, since we only focus on spectral estimates. Mention anyway - and this is a remarkable feature of all these hypoelliptic/hypocoercive techniques - that the same (in spirit) methods give very strong and precise results as the one given in the main theorem.

Organization of the article. In Section 2 we give some properties of the Landau operator. In Section 3, we prove hypoelliptic estimates with respect to the velocity variable for a parametric operator. In Section 4 we give hypoelliptic estimates for the linearized Landau operator. Section 5 is devoted to the proof of Theorem 1.2. An appendix is devoted to a short review of some tools used in this work (Weyl-Hörmander quantization, Wick quantization and the proof of the Theorem A.6 (Basic Theorem)).

2 Properties of the Landau operator

In this section, we first present the decomposition of the linearized Landau operator \mathcal{P} then we exhibit a simpler form of this operator. To end up we show that \mathcal{P} is a generator of a strongly continuous semigroup. Throughout this section, we work with $\gamma \in [-3, 1]$.

2.1 Splitting of the linearized operator.

Consider a smooth positive function $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $0 \leq \chi(v) \leq 1$, $\chi(v) = 1$ for $|v| \leq 1$ and $\chi(v) = 0$ for $|v| > 2$. For any $R \geq 1$ we define $\chi_R(v) = \chi(\frac{v}{R})$ and in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$. Then, we introduce the decomposition of the operator \mathcal{P} as $\mathcal{P} = \mathcal{A} + \mathcal{K}$ with

$$\mathcal{A} = -\mathcal{L}_1 + v \cdot \nabla_x + M\chi_R, \quad \mathcal{K} = -\mathcal{L}_2 - M\chi_R, \quad (9)$$

where $M > 0$ and $R > 0$ will be chosen later.

We define the function $F(v)$ as

$$F(v) = \left(\mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2} \right) - \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] + M\chi_R. \quad (10)$$

Then, we can rewrite \mathcal{A} as follows

$$\mathcal{A}f = v \cdot \nabla_x f - \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] + F(v)f. \quad (11)$$

2.2 Preliminaries.

We have the following results concerning the matrix $\mathbf{A}(v)$.

Lemma 2.1. *The following properties hold:*

- a) For $v \in \mathbb{R}^3 \setminus \{0\}$, the matrix $\mathbf{A}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace v^\perp . Moreover, when $|v| \rightarrow +\infty$ we have

$$\ell_1(v) \sim 2\langle v \rangle^\gamma \quad \text{and} \quad \ell_2(v) \sim \langle v \rangle^{\gamma+2}.$$

- b) The function \bar{a}_{ij} is smooth, for any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha \bar{a}_{ij}(v)| + |\partial_v^\alpha (\bar{a}_{ij}(v)v_j)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|},$$

- c) For $v \in \mathbb{R}^3 \setminus \{0\}$, we have

$$\bar{a}_{ij}(v)v_i v_j = \ell_1(v)|v|^2,$$

$$\bar{a}_{ii}(v) = \text{tr}(\bar{\mathbf{a}}(v)) = \ell_1(v) + 2\ell_2(v),$$

$$\bar{a}_{ij}(v)\eta_i \eta_j = \ell_1(v)|P_v \eta|^2 + \ell_2(v)|(I - P_v)\eta|^2,$$

with $\eta \in \mathbb{R}^3$ and P_v is the projection on v , i.e. $P_v \eta = \left(\eta \cdot \frac{v}{|v|} \right) \frac{v}{|v|}$.

- d) For $|v| > 1$, we have

$$|\partial_v^\alpha \ell_1(v)| \leq C_\alpha \langle v \rangle^{\gamma-|\alpha|} \quad \text{and} \quad |\partial_v^\alpha \ell_2(v)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

Proof. See for example [4, Lemma 2.4], [8, Lemma 3] and [22, Proposition 1]. \square

Lemma 2.2. For all $v \in \mathbb{R}^3$ with $|v| > 1$, we have

$$\ell_1(v) \gtrsim \langle v \rangle^\gamma \quad \text{and} \quad \ell_2(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

Proof. Using (a) in Lemma 2.1, when $|v| \rightarrow +\infty$, we have $\ell_1(v) \sim 2\langle v \rangle^\gamma$. In particular, there is a constant $N > 0$ such that for all $|v| > N$, we have

$$\ell_1(v) \geq \langle v \rangle^\gamma.$$

We have that $\ell_1(v)$ is continuous since $\mathbf{A}(v)$ is a positive definite symmetric matrix and continuous (due to convolution with μ), hence the existence of a constant $C > 0$ such that for $1 \leq |v| \leq N$

$$\ell_1(v) \geq C \langle v \rangle^\gamma,$$

and then for all $v \in \mathbb{R}^3$, perhaps with changing C ,

$$\ell_1(v) \geq C \langle v \rangle^\gamma.$$

The proof will be the same for $\ell_2(v)$. □

Lemma 2.3. Let $F(v)$ be defined in (10). Then, we can choose M and R big enough such that for all $v \in \mathbb{R}^3$, we have

$$F(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

Proof. Since

$$F(v) \geq \frac{1}{4} \ell_1(v) |v|^2 - \left| \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right| + M \chi_R,$$

according to Lemma 2.1, we have

$$\left| \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right| \lesssim \langle v \rangle^{\gamma+1}. \quad (12)$$

So, using Lemma 2.2 and (12), there exist two positive constants C_1, C_2 such that

$$F(v) \geq C_1 \langle v \rangle^{\gamma+2} - C_2 \langle v \rangle^{\gamma+1} + M \chi_R,$$

then there exist M and R such that for all $v \in \mathbb{R}^3$

$$F(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

□

Lemma 2.4. For any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha F(v)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

Proof. For $|v| > 2R$, using Leibniz's formula, we have

$$\partial_v^\alpha F(v) = \frac{1}{4} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \ell_1(v) \partial^\beta \langle v \rangle^2 - \frac{1}{4} \partial^\alpha \ell_1(v) - \partial^\alpha \left(\nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right),$$

Then, using Lemma 2.1 and Lemma 2.2 we obtain

$$|\partial_v^\alpha F(v)| \leq c_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

The function $F(v)$ being C^∞ on $|v| \leq 2R$, the estimates for $|v| \leq 2R$ are immediate. \square

Lemma 2.5. *i) $\mathbf{A}(v)$ is written as follows*

$$\mathbf{A}(v) = B^T(v)B(v), \quad (13)$$

where $B(v) = (b_{ij}(v))_{1 \leq i, j \leq 3}$ is a matrix with real-valued smooth entries.

ii) For any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha b_{ij}(v)| \leq C_\alpha \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}. \quad (14)$$

iii) There exists $c, C > 0$ such that for all $v \in \mathbb{R}^3$, for all $\eta \in \mathbb{R}^3$ we have

$$c \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2) \leq \mathbf{A}(v) \eta \cdot \eta = |B(v) \eta|^2 \leq C \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2) \quad (15)$$

Proof. i) As $\mathbf{A}(v)$ is a positive definite symmetric matrix (denoted $\mathbf{S}_3^{++}(\mathbb{R})$) and according to the spectral theorem, there exists \mathbf{Q} an orthogonal matrix such that

$$\mathbf{A}(v) = \mathbf{Q}^T(v) \mathbf{D}(v) \mathbf{Q}(v).$$

On $\{|v| > 1\}$, $v \mapsto \mathbf{Q}(v)$ can be calculated explicitly and can be chosen to be smooth. If we set

$$B(v) = \mathbf{Q}^T(v) \sqrt{\mathbf{D}(v)} \mathbf{Q}(v)$$

with $\sqrt{\mathbf{D}(v)} = \text{diag}(\sqrt{\ell_1(v)}, \sqrt{\ell_2(v)}, \sqrt{\ell_2(v)})$. We have that $v \mapsto \sqrt{\mathbf{D}(v)}$ is of class C^∞ for $|v| > 1$ (because $\ell_1(v), \ell_2(v)$ are of class C^∞ for $|v| > 1$). Then the application $v \mapsto B(v)$ is of class C^∞ on $\{|v| > 1\}$. Regarding the case where $|v| \leq 1$, we consider the following two applications:

$$\begin{aligned} \phi &: \mathbb{R}^3 \rightarrow \mathbf{S}_3^{++}(\mathbb{R}) \\ v &\mapsto \mathbf{A}(v) \end{aligned} ,$$

$$\begin{aligned} \psi &: \mathbf{S}_3^{++}(\mathbb{R}) \rightarrow \mathbf{S}_3^{++}(\mathbb{R}) \\ M &\mapsto \sqrt{M} \end{aligned} ,$$

we note that ϕ, ψ are of class C^∞ , moreover $B(v) = \psi \circ \phi(v)$. Then we have that the application $v \mapsto B(v)$ is of class C^∞ .

ii) For $|v| > 1$, we have $B(v) = \sqrt{\ell_1(v)} P_v + \sqrt{\ell_2(v)} (I - P_v)$, moreover using Lemma 2.1, we have

$$|\partial_v^\alpha \sqrt{\ell_1(v)}| \lesssim \langle v \rangle^{\frac{\gamma}{2}-|\alpha|}, \quad |\partial_v^\alpha \sqrt{\ell_2(v)}| \lesssim \langle v \rangle^{\frac{\gamma+2}{2}-|\alpha|} \quad (16)$$

and the fact that $|\partial_v^\alpha P_v| \lesssim 1$ (P_v and all its derivatives are bounded), so we get that for $|v| > 1$,

$$|\partial_v^\alpha b_{ij}(v)| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}, \quad (17)$$

where the constants in (16), (17) depend on α .

The function $b_{ij}(v)$ being C^∞ on $|v| \leq 1$, the estimates for $|v| \leq 1$ are immediate. Then, for all $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha b_{ij}(v)| \leq C_\alpha \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}. \quad (18)$$

iii) The estimate is immediate on $|v| \leq 1$ because $\mathbf{A}(v)$ is a positive definite symmetric matrix and $\mathbf{A} = B^T B$. For $|v| > 1$, using Lemma 2.1, we have

$$\begin{aligned} \bar{a}_{ij}(v)\eta_i\eta_j &= \ell_1(v)|P_v\eta|^2 + \ell_2(v)|(I - P_v)\eta|^2 \\ &\gtrsim \langle v \rangle^\gamma |\eta|^2 \cos^2(v, \eta) + \langle v \rangle^{\gamma+2} \frac{|v \wedge \eta|^2}{|v|^2} \\ &\gtrsim \langle v \rangle^\gamma |\eta|^2 \cos^2(v, \eta) + \langle v \rangle^\gamma |v \wedge \eta|^2 + \langle v \rangle^\gamma |\eta|^2 \sin^2(v, \eta) \\ &\gtrsim \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2), \end{aligned}$$

on the other hand, we have

$$\begin{aligned} \bar{a}_{ij}(v)\eta_i\eta_j &= \ell_1(v)|P_v\eta|^2 + \ell_2(v)|(I - P_v)\eta|^2 \\ &\lesssim \langle v \rangle^\gamma |\eta|^2 + \langle v \rangle^{\gamma+2} \frac{|v \wedge \eta|^2}{|v|^2} \\ &\lesssim \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2). \end{aligned}$$

Hence the proof of (iii). \square

Using Lemma 2.5, we can rewrite \mathcal{A} in the form

$$\mathcal{A} = v \cdot \nabla_x + (B(v)\nabla_v)^* \cdot B(v)\nabla_v + F(v), \quad (19)$$

where $(B(v)D_v)^* = D_v B(v)^T$, is the formal adjoint of $B(v)D_v$.

2.3 Study of the operator \mathcal{P} .

In this part, we will study the following problem:

$$\begin{cases} \partial_t f + \mathcal{P}f = 0 \\ f|_{t=0} = f_0, \end{cases} \quad (20)$$

we show that the above problem is well-posed in the space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ in the sense of semi-groups. By Hille-Yosida Theorem, it is sufficient to show that \mathcal{A} is maximal accretive in the space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$, then using the Bounded Perturbation Theorem in [7, Theorem 1.3], we get that the operator \mathcal{P} is a generator of a strongly continuous semigroup (for more details on the semi-group theory see also [19]). First, we start by recalling the basic definition of hypoellipticity.

Definition 2.6. Let P be a differential operator with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$. We say that P is a hypoelliptic operator on Ω , if, for any open $\omega \subset \Omega$, any $u \in \mathcal{D}'(\Omega)$, such that $Pu \in C^\infty(\omega)$ belongs to $C^\infty(\omega)$.

Lemma 2.7. Let \mathcal{A} be the operator defined in (19). Then, \mathcal{A} is a hypoelliptic operator.

Proof. Using formula (19), we can rewrite \mathcal{A} in the following form:

$$\begin{aligned} \mathcal{A} &= \underbrace{v \cdot \partial_x}_{X_0} + \underbrace{\left(\sum_{j=1}^3 b_{1,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{1,j}(v) \partial_{v_j} \right)}_{X_1^* X_1} + \underbrace{\left(\sum_{j=1}^3 b_{2,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{2,j}(v) \partial_{v_j} \right)}_{X_2^* X_2} \\ &+ \underbrace{\left(\sum_{j=1}^3 b_{3,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{3,j}(v) \partial_{v_j} \right)}_{X_3^* X_3} + F(v) \\ &= X_0 + \sum_{i=1}^3 X_i^* X_i + F(v). \end{aligned}$$

In addition, the coefficients $b_{ij}(v)$ and $F(v)$ are of class C^∞ , taking the vector field square brackets we get

$$Y_k = [X_k, X_0] = \sum_{j=1}^3 b_{k,j}(v) \partial_{x_j}, \quad \text{for } k \in \{1, 2, 3\}.$$

From the above, \mathcal{A} is a “type II Hörmander’s operators” (see for example [9], [14]). Moreover, the vector fields $\{X_i, Y_i, i = 1, \dots, 3\}$ generate all the space tangent to $\mathbb{R}_{x,v}^6$ so \mathcal{A} is hypoelliptic operator. \square

Theorem 2.8. Let $\gamma \in [-3, 1]$ and \mathcal{A} be the operator defined in (19). Then, its closure $\overline{\mathcal{A}}$ on the space $\mathcal{S}(\mathbb{R}_{x,v}^6)$ is maximally accretive.

Proof. We adapt here the proof given in [9, page 44]. We apply the abstract criterion by taking $\mathcal{H} = L^2(\mathbb{R}_{x,v}^6)$ and the domain of \mathcal{A} defined by $D(\mathcal{A}) = \mathcal{S}(\mathbb{R}_{x,v}^6)$. First, we show the accretivity of the operator \mathcal{A} . We want to prove that $\Re(\mathcal{A}u, u)_{\mathcal{H}} \geq 0$ for $u \in D(\mathcal{A})$. Indeed, from (19)

$$\begin{aligned} \Re(\mathcal{A}u, u)_{\mathcal{H}} &= \underbrace{\Re(v \cdot \nabla_x u, u)_{\mathcal{H}}}_{=0 \text{ since } v \cdot \nabla_x \text{ is skew-adjoint}} - \Re(\nabla_v \cdot (\mathbf{A}(v) \nabla_v u), u)_{\mathcal{H}} + \Re(F(v)u, u)_{\mathcal{H}} \\ &= \|B(v) \nabla_v u\|_{\mathcal{H}}^2 + \|\sqrt{F(v)}u\|_{\mathcal{H}}^2 \\ &\geq 0. \end{aligned}$$

Since \mathcal{A} is an accretive operator then its closure $\overline{\mathcal{A}}$ exists and it is accretive (see [9, Proposition 5.3]). Let us now show that there exists $\lambda_1 > 0$ such that the operator

$$T = \mathcal{A} + \lambda_1 \text{Id}$$

has dense image in \mathcal{H} . We take $\lambda_1 = 1$. Let $f \in \mathcal{H}$ satisfy

$$(f, Tu)_{\mathcal{H}} = 0, \quad \forall u \in D(\mathcal{A}). \quad (21)$$

we want's to prove that $f = 0$,

Since T is a differential operator then his formal adjoint T^\sharp exists (in the sense of distributions). According to (21) we obtain

$$T^\sharp f = \left(-\nabla_v \cdot \mathbf{A}(v)\nabla_v + F(v) + 1 - X_0 \right) f = 0, \text{ in } \mathcal{D}'(\mathbb{R}^6). \quad (22)$$

Using Lemma 2.7, we have $-\nabla_v \cdot \mathbf{A}(v)\nabla_v + F(v) + 1 - X_0$ is a hypoelliptic operator, so $f \in C^\infty(\mathbb{R}^6)$ (see [9, Chapter 2]).

Now we introduce the family of truncation functions ζ_k defined by

$$\zeta_k(x, v) = \zeta\left(\frac{x}{k_1}\right)\zeta\left(\frac{v}{k_2}\right), \quad \forall k = (k_1, k_2) \in (\mathbb{N}^*)^2,$$

where ζ is a C^∞ function satisfying the following conditions:

$$\begin{cases} 0 \leq \zeta \leq 1, \\ \zeta = 1 \text{ on } B(0, 1), \\ \text{supp } \zeta \subset B(0, 2), \\ \zeta \text{ is a radial function.} \end{cases}$$

The expression of $T^\sharp(\zeta_k f)$ is

$$\begin{aligned} T^\sharp(\zeta_k f) &= -\nabla_v \cdot \left(\mathbf{A}(v)\nabla_v(\zeta_k f) \right) + \left(F(v) + 1 \right) \zeta_k f - X_0(\zeta_k f) \\ &= -\nabla_v \cdot \left(\mathbf{A}(v)([\nabla_v \zeta_k]f) \right) - \nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f - X_0(\zeta_k)f - \zeta_k T^\sharp f, \end{aligned}$$

by using (22), we obtain

$$T^\sharp(\zeta_k f) = -\nabla_v \cdot \left(\mathbf{A}(v)([\nabla_v \zeta_k]f) \right) - \nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f - X_0(\zeta_k)f. \quad (23)$$

We note that $T^\sharp(\zeta_k f) \in \mathcal{H}$, taking the scalar product with $\zeta_k f$ we obtain

$$\begin{aligned} (T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} &= - \iint \nabla_v \cdot \left(\mathbf{A}(v)([\nabla_v \zeta_k]f) \right) \zeta_k f \, dx dv - \iint X_0(\zeta_k) \zeta_k |f|^2 \, dx dv \\ &\quad - \iint \left(\nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f \right) \zeta_k f \, dx dv, \end{aligned}$$

By doing an integration by parts, we obtain

$$(T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} = \iint |B\nabla_v(\zeta_k)f|^2 \, dx dv - \iint X_0(\zeta_k) \zeta_k |f|^2 \, dx dv.$$

On the other hand, using (22), we obtain

$$\begin{aligned} (T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} &= \iint |B\nabla_v(\zeta_k f)|^2 \, dx dv + \iint (F(v) + 1) |\zeta_k f|^2 \, dx dv \\ &\quad - \underbrace{\iint X_0(\zeta_k f) \zeta_k f \, dx dv}_{=0 \text{ since } X_0 \text{ is skew-adjoint}} \\ &= \iint |B\nabla_v(\zeta_k f)|^2 \, dx dv + \iint (F(v) + 1) |\zeta_k f|^2 \, dx dv. \end{aligned}$$

Using the fact that

$$\iint |B\nabla_v(\zeta_k f)|^2 \, dx dv \geq 0,$$

we obtain

$$\iint (F(v) + 1) |\zeta_k f|^2 dx dv \leq \underbrace{\iint |B\nabla_v(\zeta_k) f|^2 dx dv}_{(i)} - \underbrace{\iint X_0(\zeta_k) \zeta_k |f|^2 dx dv}_{(ii)}. \quad (24)$$

Estimate of (i): Using Lemma 2.1(b)-(d) and taking into account that the function ζ is a radial function we obtain

$$\begin{aligned} \iint |B\nabla_v(\zeta_k) f|^2 dx dv &= \iint \ell_1(v) |P_v \nabla_v \zeta_k|^2 |f|^2 dx dv + \underbrace{\iint \ell_2(v) |(I - P_v) \nabla_v \zeta_k|^2 |f|^2 dx dv}_{=0 \text{ since } v \text{ is parallel to } \nabla_v \zeta_k} \\ &\leq \frac{C_0}{k_2^2} \iint \langle v \rangle^\gamma \Phi_k |f|^2 dx dv, \end{aligned}$$

where $C_0 > 0$ and $\Phi_k = \zeta(\frac{x}{k_1}) \zeta'(\frac{v}{k_2})$. Using the fact that $\gamma \in [-3, 1]$ and the fact that Φ_k is a bounded function we have the existence of a constant $C_1 > 0$ such that,

$$\iint |B\nabla_v(\zeta_k) f|^2 dx dv \leq C_1 \left(\frac{1}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2.$$

Estimate of (ii): we have

$$\left| \iint X_0(\zeta_k) \zeta_k |f|^2 dx dv \right| \leq \frac{1}{k_1} \iint |v| \tilde{\Phi}_k \zeta_k |f|^2 dx dv,$$

where $\tilde{\Phi}_k = \zeta'(\frac{x}{k_1}) \zeta(\frac{v}{k_2})$.

Now, taking into account that the functions $\tilde{\Phi}_k$ and ζ_k are bounded, we have the existence of a constant $C_2 > 0$ such that,

$$\left| \iint X_0(\zeta_k) \zeta_k |f|^2 dx dv \right| \leq C_2 \frac{k_2}{k_1} \|f\|^2.$$

Finally, coming back to (24) we obtained the existence of a constant $C > 0$ such that,

$$\iint |\zeta_k f|^2 dx dv \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_2} + \frac{k_2}{k_1} \right) \|f\|^2, \quad \forall k. \quad (25)$$

Taking $k_1 \rightarrow +\infty$ in (25) we obtain

$$\iint |\zeta(\frac{v}{k_2}) f|^2 dx dv \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2, \quad (26)$$

and taking $k_2 \rightarrow +\infty$ in (26) we obtain

$$\iint |f|^2 dx dv = 0,$$

then $f = 0$. □

From now on, we write \mathcal{A} for the closure of the operator \mathcal{A} .

Corollary 2.9. *Let $\gamma \in [-3, 1]$ and \mathcal{P} be the operator defined in (5). Then, $-\mathcal{P}$ is a generator of a semi-group $(S(t))_{t \geq 0}$ strongly continuous on $\mathcal{H} = L^2(\mathbb{R}_{x,v}^6)$ verifying*

$$\|S(t)\| \leq e^{\|\mathcal{K}\|t} \text{ for } t \geq 0, \quad (27)$$

where \mathcal{K} the operator defined in (9).

Proof. Using (9), the operator $-\mathcal{P}$ is written as follows

$$-\mathcal{P} = -\mathcal{A} - \mathcal{K}.$$

According to Theorem 2.8, \mathcal{A} is a maximally accretive operator. According to the Hille-Yosida Theorem, $-\mathcal{A}$ is a generator of a strongly continuous semi-group of contraction. On the other hand, $-\mathcal{K}$ is a bounded operator in \mathcal{H} (we have \mathcal{L}_2 a compact operator and $M\chi_R$ is bounded). Using the Bounded Perturbation Theorem in [7, Theorem 1.3], we have that $-\mathcal{P}$ is a generator of a semi-group $(S(t))_{t \geq 0}$ strongly continuous on \mathcal{H} , moreover we have

$$\|S(t)\| \leq e^{\|\mathcal{K}\|t} \text{ for } t \geq 0. \quad (28)$$

□

3 Hypoelliptic estimates for the operator with parameters

In the following discussion, we work with $\gamma \in [0, 1]$. In this section, we will study the operator acting on the velocity variable v :

$$\mathcal{A}_\xi = iv \cdot \xi + (B(v)\nabla_v)^* \cdot B(v)\nabla_v + F(v), \quad (29)$$

where ξ is the parameter in \mathbb{R}^3 . The operator \mathcal{A}_ξ is obtained by the partial Fourier transformation in x . The goal of studying the operator \mathcal{A}_ξ and considering ξ as a parameter, is to obtain estimates of the velocity variables v uniformly with respect to ξ . Then by using the inverse Fourier transform with respect to x , we can obtain global estimates in all variables. We note that the operator \mathcal{A}_ξ verifies for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 \leq \Re e(\mathcal{A}_\xi u, u)_{L^2}. \quad (30)$$

Notations. Throughout this section, we will use $\|\cdot\|_{L^2}$ to denote the norm in the space $L^2(\mathbb{R}_v^3)$ and ξ is a parameter. We use p^{Wick} to denote the Wick quantization of p in the variables (v, η) (for more details on Wick quantization see [17] and Appendix A.3).

The main result in this section is Proposition 3.19 and Proposition 3.20.

3.1 Pseudo-differential parts

In this part, we will show several lemmas concerning pseudo-differential symbols. We need to build these symbols, who verify assumptions of Theorem A.6, such that the pseudo-differential operator associated to these symbols has good properties. These operators play an important role in hypoelliptic estimates. The standard concepts on pseudo-differential calculus are explained in Appendix A. We define for $(v, \eta) \in \mathbb{R}^6$

the following symbols, they depend on the parameter ξ but we do not mention it in our notations since ξ is seen as a parameter.

$$\lambda(v, \eta) = \left[\langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2 \right) \right]^{1/2}, \quad (31)$$

$$\mathbf{a}(v, \eta) = 1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2, \quad (32)$$

$$g_1(v, \eta) = 1 + \langle v \rangle + \langle \eta \rangle, \quad (33)$$

$$g_2(v, \eta) = 1 + \langle \xi \rangle^{1/3} + \langle \eta \rangle, \quad (34)$$

$$g_3(v, \eta) = 1 + \langle \xi \rangle^{2/3} + \langle \eta \rangle^2. \quad (35)$$

Lemma 3.1. *The above symbols are admissible weights in the sense of Definition [A.1](#) uniformly with respect to the parameter ξ in \mathbb{R}^3 .*

Proof. We have to check λ is an admissible weight. It is sufficient to verify that there exists two constants N and C , both depending only on γ , such that for all $Y = (v, \eta)$, $Y' = (v', \eta')$, we have

$$\lambda(Y) \leq C \lambda(Y') (1 + \Gamma(Y - Y'))^N,$$

where Γ is the metric defined by $\Gamma = dv^2 + d\eta^2$. We have

$$\frac{\lambda^2(v, \eta)}{\lambda^2(v', \eta')} = \frac{\langle v \rangle^\gamma}{\langle v' \rangle^\gamma} \left(\frac{1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \right). \quad (36)$$

We now use Peetre's inequality

$$\langle y \rangle^\tau \leq 2^{\frac{|\tau|}{2}} \langle y' \rangle^\tau \langle y - y' \rangle^{|\tau|}, \quad \tau \in \mathbb{R}, \quad (37)$$

to get

$$\frac{\langle v \rangle^\gamma}{\langle v' \rangle^\gamma} \leq 2^\gamma \langle v - v' \rangle^\gamma.$$

Using [\(37\)](#), we obtain

$$\frac{1 + |v|^2 + |\eta|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq \frac{\langle v \rangle^2}{\langle v' \rangle^2} + \frac{\langle \eta \rangle^2}{\langle \eta' \rangle^2} \leq 4 (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2$$

and

$$\frac{|\xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq 1 \leq (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2.$$

Using the relation

$$v \wedge \xi = (v - v') \wedge \xi + v' \wedge \xi,$$

we obtain

$$\frac{|v \wedge \xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq 1 + \frac{\langle v - v' \rangle^2 \langle \xi \rangle^2}{\langle \xi \rangle^2} \leq 2 (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2.$$

Moreover using the relation

$$v \wedge \eta = (v - v') \wedge (\eta - \eta') + (v - v') \wedge \eta' + v' \wedge (\eta - \eta') + v' \wedge \eta',$$

we compute

$$\begin{aligned}
& \frac{|v \wedge \eta|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \\
& \leq \frac{4|v - v'|^2|\eta - \eta'|^2 + 4|v - v'|^2|\eta'|^2 + 4|v'|^2|\eta - \eta'|^2 + 4|v' \wedge \eta'|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \\
& \leq 4|v - v'|^2|\eta - \eta'|^2 + 4|v - v'|^2 + 4|\eta - \eta'|^2 + 4 \\
& \leq 10(\langle v - v' \rangle + \langle \eta - \eta' \rangle)^4.
\end{aligned}$$

Combining the above inequalities, we get

$$\frac{\lambda(Y)}{\lambda(Y')} \leq C_\gamma (1 + \Gamma(Y - Y'))^{\frac{4+\gamma}{2}},$$

so λ is an admissible weight. The proof will be the same for a, g_1, g_2 and g_3 . \square

Lemma 3.2. For $m \in \mathbb{R}$,

$$\lambda^m \in S(\lambda^m, \Gamma),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 , where $S(\lambda^m, \Gamma)$ is defined in Definition [A.2](#) in Appendix [A.1](#).

Proof. We can rewrite λ as follows

$$\lambda(v, \eta) = \langle v \rangle^{\frac{\gamma}{2}} \mathbf{a}^{\frac{1}{2}}(v, \eta).$$

To prove the wanted result, we use induction on $|\alpha + \beta|$ to prove that for any $k \in \mathbb{R}$ and any $|\alpha + \beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \mathbf{a}^k(v, \eta)| \lesssim \mathbf{a}^k(v, \eta), \quad (38)$$

which obviously holds for $|\alpha + \beta| = 0$. Now suppose $|\alpha + \beta| \geq 1$, then we have either $|\alpha| \geq 1$ or $|\beta| \geq 1$, and suppose $|\beta| \geq 1$ without loss of generality. So we can write $\partial_\eta^\beta = \partial_\eta^{\tilde{\beta}} \partial_{\eta_j}$ with $|\tilde{\beta}| = |\beta| - 1$ and thus

$$\partial_v^\alpha \partial_\eta^\beta \mathbf{a}^k(v, \eta) = \partial_v^\alpha \partial_\eta^{\tilde{\beta}} \left[k \mathbf{a}^{k-1} (2\eta_j + 2(v \wedge \eta) \partial_{\eta_j}(v \wedge \eta)) \right],$$

which along with Leibniz' formula and the induction assumption yields

$$\begin{aligned}
|\partial_v^\alpha \partial_\eta^\beta \mathbf{a}^k(v, \eta)| & \lesssim \mathbf{a}^{k-1} (1 + |\eta| + |v||\eta| + |v||v \wedge \eta| + |v|^2) \\
& \lesssim \mathbf{a}^k(v, \eta),
\end{aligned}$$

on the other hand, we have for all $k \in \mathbb{R}$,

$$\langle v \rangle^k \in S(\langle v \rangle^k, \Gamma). \quad (39)$$

Finally, using [\(38\)](#), [\(39\)](#) and Leibniz' formula we conclude for all $|\alpha + \beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \lambda^m(v, \eta)| \leq C_{\alpha, \beta} \lambda^m(v, \eta). \quad (40)$$

\square

Lemma 3.3. For $m \in \mathbb{R}$,

$$|\xi \cdot \partial_\eta \lambda^m| \lesssim \lambda^m,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Lemma 3.3 follows directly from (31) and the fact that

$$|\xi \cdot \partial_\eta \lambda^2| = \lambda \langle v \rangle^{\frac{2}{3}} \mathbf{a}^{-1/2} |\xi \cdot \eta + v \wedge \xi \cdot v \wedge \eta| \lesssim \lambda^2,$$

we can conclude that for all $m \in \mathbb{R}$,

$$|\xi \cdot \partial_\eta \lambda^m| \lesssim \lambda^m,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 3.4. We have

$$g_i \in S(g_i, \Gamma), \text{ for } i = 1, \dots, 3,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Using the fact that for all $m \in \mathbb{R}$, $\langle v \rangle^m \in S(\langle v \rangle^m, \Gamma)$, $\langle \eta \rangle^m \in S(\langle \eta \rangle^m, \Gamma)$ we obtain

$$\forall \alpha, \beta \in \mathbb{N}^3, \quad |\partial_v^\alpha \partial_\eta^\beta g_i(v, \eta)| \leq C_{\alpha, \beta} g_i,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 3.5. For all $\varepsilon > 0$,

- i) $\partial_\eta \lambda \in S(\varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{7}{2}+1}, \Gamma)$, ii) $\partial_\eta g_1 \in S(\varepsilon g_1 + \varepsilon^{-1}, \Gamma)$,
- iii) $\partial_\eta g_2 \in S(\varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{6}+1}, \Gamma)$, iv) $\partial_\eta g_3 \in S(\varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{3}+1}, \Gamma)$;

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We have

$$\begin{aligned} |\partial_\eta g_1(v, \eta)| &= |\partial_\eta (1 + \langle v \rangle + \langle \eta \rangle)| \lesssim |\eta| \langle \eta \rangle^{-1} \lesssim g_1^{1/2}, \\ |\partial_\eta g_2(v, \eta)| &= |\partial_\eta (1 + \langle \xi \rangle^{1/3} + \langle \eta \rangle)| \lesssim |\eta| \langle \eta \rangle^{-1} \lesssim g_2^{1/2} \langle v \rangle^{\frac{7}{12} + \frac{1}{2}}, \\ |\partial_\eta g_3(v, \eta)| &= |\partial_\eta (1 + \langle \xi \rangle^{2/3} + \langle \eta \rangle^2)| \lesssim \langle \eta \rangle \lesssim g_3^{1/2} \langle v \rangle^{\frac{7}{6} + \frac{1}{2}}. \end{aligned}$$

Using Young's inequality we get for all $\varepsilon > 0$,

$$|\partial_\eta g_1(v, \eta)| \lesssim \varepsilon g_1 + \varepsilon^{-1}, \quad |\partial_\eta g_2(v, \eta)| \lesssim \varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{6}+1}, \quad |\partial_\eta g_3(v, \eta)| \lesssim \varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{3}+1},$$

then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_1(v, \eta)| \lesssim \varepsilon g_1 + \varepsilon^{-1}, \quad |\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_2(v, \eta)| \lesssim \varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{6}+1},$$

$$\text{and } |\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_3(v, \eta)| \lesssim \varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{3}+1}.$$

Regarding the symbol λ , we have

$$|\partial_\eta \lambda(v, \eta)| = \langle v \rangle^{\frac{2}{3}} \mathbf{a}^{-1/2} |\eta + (v \wedge \eta) \partial_\eta (v \wedge \eta)| \lesssim \langle v \rangle^{\frac{2}{3}} \mathbf{a}^{-1/2} (|\eta| + |v| |v \wedge \eta|) \lesssim \langle v \rangle^{\frac{7}{4} + \frac{1}{2}} \lambda^{1/2}.$$

Using Young's inequality we get for all $\varepsilon > 0$,

$$|\partial_\eta \lambda(v, \eta)| \lesssim \varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{7}{2}+1},$$

then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta \lambda(v, \eta)| \lesssim \varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{7}{2}+1}.$$

□

From the above, we have shown that the symbols g_1 , g_2 , g_3 and λ verify the hypotheses of Theorem [A.6](#), so we can apply the results of Theorem [A.6](#) to the following operators

$$g_{1,K}^w = (g_1 + K)^w, \quad (41)$$

$$g_{2,K}^w = (g_2 + K \langle v \rangle^{\frac{7}{6}+1})^w, \quad (42)$$

$$g_{3,K}^w = (g_3 + K \langle v \rangle^{\frac{7}{3}+1})^w, \quad (43)$$

$$\lambda_K^w = (\lambda + K \langle v \rangle^{\frac{7}{2}+1})^w, \quad (44)$$

where K the fixed constant given by Theorem [A.6](#). In Section 3.2, we will apply the results of Theorem [A.6](#) on the operators above.

Let ψ be a $C_0^\infty(\mathbb{R}, [0, 1])$ function such that

$$\psi = 1 \text{ on } [-1, 1], \text{ supp } \psi \subset [-2, 2]. \quad (45)$$

Definition 3.6. Define the real-valued symbol

$$g = -\frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}} \psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right), \quad (46)$$

where λ is the symbol defined in [\(31\)](#).

Lemma 3.7 (Lemma 3.3 in [\[12\]](#)). *We have*

$$\psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right) \in S(1, \Gamma),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Lemma 3.8. *The symbol g belongs to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 .*

Proof. Notice from [\(45\)](#) that

$$|B(v)\eta|^2 + F(v) \leq 2\lambda^{2/3}, \quad (47)$$

on the support of the function

$$\psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right). \quad (48)$$

By recalling (31) and using (15), we obtain

$$|B(v)\xi| \lesssim \langle v \rangle^{\frac{7}{2}} |\xi| + \langle v \rangle^{\frac{7}{2}} |v \wedge \xi| \lesssim \lambda. \quad (49)$$

We deduce from the Cauchy-Schwarz inequality that one can estimate

$$|B(v)\xi \cdot B(v)\eta| \lesssim |B(v)\xi| |B(v)\eta| \lesssim \lambda^{4/3}, \quad (50)$$

on the support of ψ . The symbol g is therefore a bounded function uniformly with respect to the parameter ξ in \mathbb{R}^3 . Using (14) and (15), we can estimate

$$|\partial_v^\alpha B(v)\xi| \lesssim \langle v \rangle^{\frac{7}{2}} |\xi| \lesssim |B(v)\xi| \lesssim \lambda \quad (51)$$

where $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 1$. Using again (14) and (15), we can estimate

$$|\partial_v^\alpha B(v)\eta| \lesssim \langle v \rangle^{\frac{7}{2}} |\eta| \lesssim |B(v)\eta| \lesssim \lambda^{1/3} \quad (52)$$

on the support of ψ . Moreover, one can estimate from above the modulus of all the derivatives of the term $B(v)\eta$ by a constant times $\lambda^{1/3}$ on the support of the function

$$\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right).$$

Using Leibniz's formula, Cauchy-Schwarz inequality (51), (52) one can estimate from above the modulus of all the derivatives of the term $B(v)\xi \cdot B(v)\eta$ by a constant times $\lambda^{4/3}$ on the support of the function ψ . According to Lemma 3.2 and Lemma 3.7, this proves that the symbol g belongs to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 3.9. *We have*

$$\left| \xi \cdot \partial_\eta \left[\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \right| \lesssim 1 + |B(v)\eta|^2 + F(v),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Let $\omega = \frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}$. We may write

$$\xi \cdot \partial_\eta [\psi(\omega)] = \psi'(\omega) \left[\frac{2B(v)\xi \cdot B(v)\eta}{\lambda^{2/3}} + (|B(v)\eta|^2 + F(v)) (\xi \cdot \partial_\eta) (\lambda^{-2/3}) \right].$$

Notice from (31) and (45)

$$\left| \frac{2B(v)\xi \cdot B(v)\eta}{\lambda^{2/3}} \right| \lesssim \frac{|B(v)\xi| |B(v)\eta|}{\lambda^{2/3}} \lesssim \frac{\lambda \lambda^{1/3}}{\lambda^{2/3}} \lesssim \lambda^{2/3} \lesssim |B(v)\eta|^2 + F(v),$$

on the support of the function $\psi'(\omega)$. One can then deduce Lemma 3.9 from Lemma 3.3. \square

3.2 Hypoelliptic estimates

We shall consider the multiplier $G = g^{\text{Wick}}$ defined by the Wick quantization of the symbol g . We refer the reader to Appendix [A.3](#) on Wick calculus. We begin by noticing from [\(133\)](#) that there exists a real-valued symbol \tilde{g} belonging to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 such that

$$G = g^{\text{Wick}} = \tilde{g}^w; \quad (53)$$

where \tilde{g}^w denotes the operator obtained by the Weyl quantization of the symbol \tilde{g} given by

$$(\tilde{g}^w u)(v) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} e^{i(v-v') \cdot \eta} \tilde{g}\left(\frac{v+v'}{2}, \eta\right) u(v') dv' d\eta. \quad (54)$$

We shall sometimes closely follow [\[12\]](#) and refer to appendix (Section [A](#)) for the main features of the Wick and the Weyl quantizations.

We begin by a series of Lemmas whose proof is exactly the same as the one in [\[12\]](#).

Lemma 3.10. *There exists $c_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$|(F(v)u, Gu)_{L^2}| + |(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2}| \leq c_1 \Re(\mathcal{A}_\xi u, u)_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.7 in [\[12\]](#). □

Lemma 3.11. *There exists $c_2 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\left| \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \left([4\pi^2|B(v)\eta|^2 + F(v)]^{\text{Wick}} u, u \right)_{L^2} \right| \leq c_2 \Re(\mathcal{A}_\xi u, u)_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.11 in [\[12\]](#). □

Let δ be a positive parameter such that $0 < \delta \leq 1$. We use a multiplier method and write that

$$\begin{aligned} \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} &= \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re(iv \cdot \xi u, Gu)_{L^2} \\ &\quad + \delta \Re(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} - \delta \Re(F(v)u, Gu)_{L^2}. \end{aligned} \quad (55)$$

Lemma 3.12. *We have for any $s \in \mathbb{R}$*

$$\|\langle B(v)\xi \rangle^s (1 - \delta G)u\|_{L^2} \lesssim \|\langle B(v)\xi \rangle^s u\|_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.8 in [\[12\]](#). □

Proposition 3.13. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$((\lambda^{2/3})^{\text{Wick}} u, u)_{L^2} \leq C \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Using (55), we have for $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re (iv \cdot \xi u, Gu)_{L^2} \\ &= \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} - \delta \Re (\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} + \delta \Re (F(v)u, Gu)_{L^2}, \end{aligned} \quad (56)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . Recalling (53) and noticing from (133) and (134) that $v^{\text{Wick}} = v$, we may rewrite (56) as

$$\begin{aligned} & \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re (i\xi \cdot v^{\text{Wick}}u, Gu)_{L^2} \\ &= \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} - \delta \Re (\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} + \delta \Re (F(v)u, Gu)_{L^2}. \end{aligned} \quad (57)$$

Using Lemma 3.10, we deduce that there is a constant $c_3 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re (i\xi \cdot v^{\text{Wick}}u, Gu)_{L^2} \\ & \leq c_3 \left(\Re (\mathcal{A}_\xi u, u)_{L^2} + \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right), \end{aligned} \quad (58)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . We deduce from Lemma 3.8 and (135) that

$$\begin{aligned} -\delta \Re (i\xi \cdot v^{\text{Wick}}u, Gu)_{L^2} &= -\delta \Re (g^{\text{Wick}}(i\xi \cdot v)^{\text{Wick}}u, u)_{L^2} \\ &= \delta \frac{1}{4\pi} \left(\{\xi \cdot v, g\}^{\text{Wick}}u, u \right)_{L^2}. \end{aligned} \quad (59)$$

Using (46) with a direct computation of the Poisson bracket gives that

$$\begin{aligned} \{\xi \cdot v, g\} &= (B(v)\xi \cdot B(v)\eta)(\xi \cdot \partial_\eta(\lambda^{-4/3}))\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \\ &+ \frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + \frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}}\xi \cdot \partial_\eta \left[\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]. \end{aligned} \quad (60)$$

We notice from Lemma 3.3, Lemma 3.9, (46) and (50) that

$$\left| \{\xi \cdot v, g\} - \frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right| \lesssim 1 + |B(v)\eta|^2 + F(v), \quad (61)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . It follows from (59), (58) and the fact that the Wick quantization is a positive quantization (132) that there exists constant $c_4 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \frac{\delta}{4\pi} \left(\left[\frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]^{\text{Wick}} u, u \right)_{L^2} + \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 \\ & \leq c_3 \left(\Re (\mathcal{A}_\xi u, u)_{L^2} + \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right) \\ & \quad + \delta c_4 \left(\left[1 + |B(v)\eta|^2 + F(v) \right]^{\text{Wick}} u, u \right)_{L^2}, \end{aligned} \quad (62)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . It follows from Lemma [3.11](#) that there exists $c_5 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$

$$\begin{aligned} & \delta \left(\left[\frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]^{Wick} u, u \right)_{L^2} + (4\pi^2 [|B(v)\eta|^2 + F(v)]^{Wick} u, u)_{L^2} \\ & \quad + \|u\|_{L^2}^2 \leq c_5 \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right), \end{aligned} \quad (63)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Notice from [\(31\)](#), [\(45\)](#) and [\(15\)](#) that

$$\begin{aligned} & \delta \frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + 4\pi^2 |B(v)\eta|^2 + F(v) + 1 \geq \\ & \delta \frac{|B(v)\xi|^2 + |B(v)\eta|^2 + F(v) + 1}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \\ & \quad + (|B(v)\eta|^2 + F(v) + 1) \left[1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \gtrsim \\ & \delta \frac{\lambda^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + \delta \lambda^{2/3} \left[1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \gtrsim \delta \lambda^{2/3}, \end{aligned}$$

when $0 < \frac{\delta}{\lambda^{4/3}} \leq 1$; since

$$|B(v)\eta|^2 + F(v) \geq \lambda^{2/3},$$

on the support of the function

$$1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right).$$

By using again that the Wick quantization is a positive quantization ([132](#)), we deduce that there exists $C > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$((\lambda^{2/3})^{Wick} u, u)_{L^2} \leq C \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . □

Lemma 3.14. *There exists $C > 0$ such that for all $v \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$,*

$$\langle B(v)\xi \rangle^{1/3} \leq Cm(v, \xi),$$

with

$$m(v, \xi) = \left(\int_{\mathbb{R}^3} \langle B(v + \tilde{v})\xi \rangle^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \right)^{1/2}.$$

Proof. We have that

$$m(v, \xi)^2 \gtrsim \int_{\mathbb{R}^3} (1 + |B(v + \tilde{v})\xi|^{2/3}) \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v},$$

therefore by using ([15](#)), we obtain that

$$m(v, \xi)^2 \gtrsim \int_{\mathbb{R}^3} (1 + \langle v + \tilde{v} \rangle^{\gamma/3} |\xi|^{2/3} + \langle v + \tilde{v} \rangle^{\gamma/3} |(v + \tilde{v}) \wedge \xi|^{2/3}) \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v},$$

and using Peetre's inequality (37), we have

$$\frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} \lesssim \langle v + \tilde{v} \rangle^{\gamma/3},$$

so we get

$$\begin{aligned} m(v, \xi)^2 &\gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \int_{\mathbb{R}^3} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \\ &\gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \int_{\overline{B(0,1)}} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \end{aligned}$$

where $\overline{B(0,1)}$ stands for the closed unit ball in \mathbb{R}^3 . By noticing that we have

$$|(v + \tilde{v}) \wedge \xi| \geq |v \wedge \xi| - |\tilde{v} \wedge \xi| \geq |v \wedge \xi| - |\xi| \geq \frac{1}{2} |v \wedge \xi|,$$

when $|\tilde{v}| \leq 1$ and $2|\xi| \leq |v \wedge \xi|$, it follows that

$$\int_{\mathbb{R}^3} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \gtrsim \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3},$$

when $2|\xi| \leq |v \wedge \xi|$. Since

$$\langle v \rangle^{\gamma/3} |\xi|^{2/3} \gtrsim \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3}$$

when $2|\xi| \geq |v \wedge \xi|$. Based on the above and using (15), we obtain

$$m(v, \xi)^2 \gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3} \gtrsim 1 + |B(v)\xi|^{2/3} \gtrsim \langle B(v)\xi \rangle^{2/3}.$$

□

Lemma 3.15. For all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\frac{5\gamma}{6}+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}+1} B(v)D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (64)$$

Proof. See Lemma 3.3 in [10].

□

For the rest, we need to improve this estimate. We have the following Lemma.

Lemma 3.16. For all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v)D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (65)$$

Proof. We will start by estimating the term

$$\left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2}.$$

Let $u \in \mathcal{S}(\mathbb{R}_v^3)$, we have

$$\begin{aligned} \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\leq \left| \left(B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u, B(v) D_v \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) D_v u, B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \\ &\leq \left| \left(B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u, B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) D_v u, B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|, \end{aligned}$$

using the fact that the symbol of the operator $[D_v, \langle v \rangle^{\frac{\gamma}{2}+1}]$ belongs to the class $S(\langle v \rangle^{\frac{\gamma}{2}}, \Gamma)$ uniformly with respect to the parameter ξ and using the following notations

$$H_1 = \underbrace{B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)},$$

$$H_2 = \underbrace{\langle v \rangle^{-\frac{\gamma}{2}-1} B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{-\frac{\gamma}{2}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\leq \left| \left(H_1 \langle v \rangle^{\gamma+1} u, H_1 \langle v \rangle^{\gamma+1} u \right)_{L^2} \right| \\ &\quad + \left| \left(H_1 \langle v \rangle^{\gamma+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \right)_{L^2} \right| \\ &\quad + \left| \left(\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u, H_2 \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|, \end{aligned}$$

using the fact that $\gamma + 1 \leq 2 + \frac{5\gamma}{6}$ and Lemma [3.15](#), we obtain for all $\varepsilon > 0$

$$\begin{aligned} \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\lesssim \varepsilon (\| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2) \\ &\quad + C_\varepsilon (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2). \end{aligned}$$

On the other hand, we have

$$\left| \left(\mathcal{A}_\xi \langle v \rangle^{\frac{\gamma}{2}+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \leq \left| \left(\mathcal{A}_\xi u, \langle v \rangle^{\gamma+2} u \right)_{L^2} \right| + \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|$$

and

$$\left| \left(\mathcal{A}_\xi u, \langle v \rangle^{\gamma+2} u \right)_{L^2} \right| \leq \varepsilon \| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + C_\varepsilon \| \mathcal{A}_\xi u \|_{L^2}^2.$$

Now using [\(30\)](#), we can write

$$\begin{aligned} &\| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2 \\ &\leq \| \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}+1} u \|_{L^2}^2 + \| B(v) D_v \langle v \rangle^{\frac{\gamma}{2}+1} u \|_{L^2}^2 + \| B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \|_{L^2}^2 \\ &\leq \left| \left(\mathcal{A}_\xi \langle v \rangle^{\frac{\gamma}{2}+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| + \| B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \|_{L^2}^2. \end{aligned}$$

For the last term, we have

$$\| B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \|_{L^2}^2 \leq \| H_1 \langle v \rangle^{\gamma+1} u \|_{L^2}^2 \lesssim \varepsilon \| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + C_\varepsilon (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2).$$

Finally, taking ε small enough, we obtain for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2 \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2. \quad (66)$$

□

Proposition 3.17. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\| \langle v \rangle^{\frac{\gamma}{3}} |\xi|^{2/3} u \|_{L^2}^2 \leq C (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We deduce from Proposition [3.13](#), Lemma [3.14](#) and Lemma [3.12](#) that

$$\|\langle B(v)\xi \rangle^{1/3} u\|_{L^2} \lesssim \|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi u\|_{L^2}^2 \|\langle B(v)\xi \rangle^{1/3} u\|_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . By substituting $\langle B(v)\xi \rangle^{1/3} u$ to u in this estimate, we obtain that

$$\|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}^2 \lesssim \|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi \langle B(v)\xi \rangle^{1/3} u\|_{L^2} \|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}, \quad (67)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . First, we have

$$\begin{aligned} & \langle B(v)\xi \rangle^{-1/3} \left[D_v \cdot \mathbf{A}(v) D_v, \langle B(v)\xi \rangle^{1/3} \right] \\ &= \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 D_{v_j} \cdot \bar{a}_{jk}(v) \left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \left[D_{v_j}, \langle B(v)\xi \rangle^{1/3} \right] \cdot \bar{a}_{jk}(v) D_{v_k} \\ &= \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 D_{v_j} \bar{a}_{jk}(v) \left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \bar{a}_{jk}(v) \left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] D_{v_j} \\ &+ \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \bar{a}_{jk}(v) D_{v_j} \left(\left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] \right) \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \left[D_{v_j}, \langle B(v)\xi \rangle^{1/3} \right] \cdot \bar{a}_{jk}(v) D_{v_k}. \end{aligned}$$

Symbolic calculus shows that

$$\left[D_v, \langle B(v)\xi \rangle^{1/3} \right] = \frac{1}{i} \nabla_v (\langle B(v)\xi \rangle^{1/3}),$$

and using the following notations

$$H_3 = \underbrace{\langle B(v)\xi \rangle^{-1/3} \partial_{v_j} \bar{a}_{jk}(v) \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)},$$

$$H_4 = \underbrace{\langle B(v)\xi \rangle^{-1/3} \bar{a}_{jk}(v) \partial_{v_j} \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

$$H_5 = \underbrace{\langle B(v)\xi \rangle^{-1/3} b_{ik}(v) \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)},$$

$$H_6 = \underbrace{\langle B(v)\xi \rangle^{-1/3} b_{ij}(v) \partial_{v_j} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)},$$

where we used the fact that $\mathbf{A}(v) = B^T(v)B(v)$. Then, going back to (67), we have

$$\begin{aligned}
& \|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}^2 \\
& \lesssim \|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi \langle B(v)\xi \rangle^{1/3} u\|_{L^2} \|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}, \\
& \lesssim \|\langle B(v)\xi \rangle^{-1/3} [\mathcal{A}_\xi, \langle B(v)\xi \rangle^{1/3}] u\|_{L^2}^2 + \|\mathcal{A}_\xi u\|_{L^2}^2, \\
& \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|H_3 \langle v \rangle^{\gamma+1} u\|_{L^2}^2 + \|H_4 \langle v \rangle^{\gamma+2} u\|_{L^2}^2 \\
& + \sum_{i=1}^3 \|H_5 \left(\sum_{j=1}^3 \langle v \rangle^{\frac{\gamma}{2}+1} b_{ij}(v) D_{v_j} u \right)\|_{L^2}^2 + \sum_{i=1}^3 \|H_6 \left(\sum_{k=1}^3 \langle v \rangle^{\frac{\gamma}{2}+1} b_{ik}(v) D_{v_k} u \right)\|_{L^2}^2, \\
& \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|\langle v \rangle^{\gamma+1} u\|_{L^2}^2 + \|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u\|_{L^2}^2
\end{aligned}$$

We finally conclude from Lemma 3.16 that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

This ends the proof of Proposition 3.17. \square

Lemma 3.18. *Let λ_K be the symbol defined in (44). Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C_{\tilde{\varepsilon}}$, such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\begin{aligned}
& \Re \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
& \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).
\end{aligned} \tag{68}$$

Proof. As a preliminary step we firstly show that for any $\varepsilon, \tilde{\varepsilon} > 0$ there exists a constant $C_{\varepsilon, \tilde{\varepsilon}}$, such that

$$\begin{aligned}
& \Re \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
& \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\varepsilon, \tilde{\varepsilon}} \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\
& + \varepsilon \left\{ \Re \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right\},
\end{aligned} \tag{69}$$

where a is an arbitrary symbol belonging to $S(1, \Gamma)$ uniformly with respect to the parameter ξ . Using the notation

$$\mathcal{Z}_1 = \left(\left[D_v \cdot \mathbf{A}(v) D_v, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.$$

We have

$$\begin{aligned}
\mathcal{Z}_1 &= \sum_{i,j,k=1}^3 \left(\left[b_{ik}(v) D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] u, b_{ij}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
&+ \sum_{i,j,k=1}^3 \left(b_{ik} D_{v_k} u, \left[\left(\lambda_K^{1/3} \right)^w, b_{ij}(v) D_{v_j} \right] a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.
\end{aligned}$$

Using the fact that

$$\left[B(v) D_v, \left(\lambda_K^{1/3} \right)^w \right] u = B(v) \left[D_v, \left(\lambda_K^{1/3} \right)^w \right] u - \left[\left(\lambda_K^{1/3} \right)^w, B(v) \right] D_v u,$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_1| &\leq \left| \sum_{i,j,k=1}^3 \left(b_{ik}(v) \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] u, b_{ij}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(\left[\left(\lambda_K^{1/3} \right)^w, b_{ik}(v) \right] D_{v_k} u, b_{ij}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(b_{ik}(v) D_{v_k} u, b_{ij}(v) \left[D_{v_j}, \left(\lambda_K^{1/3} \right)^w \right] a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(b_{ik}(v) D_{v_k} u, \left[\left(\lambda_K^{1/3} \right)^w, b_{ij}(v) \right] D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&\leq |\mathcal{Z}_{1,1}| + |\mathcal{Z}_{1,2}| + |\mathcal{Z}_{1,3}| + |\mathcal{Z}_{1,4}|.
\end{aligned}$$

Estimate of $\mathcal{Z}_{1,1}$: Observing $a \in S(1, \Gamma)$, $\partial_v \lambda_K^{1/3} \in S(\langle v \rangle^{\gamma/6} g_{2,K}, \Gamma)$ and using (14) with symbolic calculus shows that

$$[D_{v_j}, a^w] \in \Psi(1, \Gamma), [b_{ij}(v), a^w] \in \Psi(\langle v \rangle^{\frac{\gamma}{2}}, \Gamma) \text{ and } [D_{v_k}, \left(\lambda_K^{1/3} \right)^w] \in \Psi(\langle v \rangle^{\frac{\gamma}{6}} g_{2,K}, \Gamma)$$

uniformly with respect to the parameter ξ , where $g_{2,K}^w$ the operator defined in (42). Now using the following notation

$$H_7 = \underbrace{b_{ik}(v) \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] \left(g_{2,K}^w \right)^{-1} \langle v \rangle^{-\gamma/6} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,1}| &\leq \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, b_{ij}(v) \left[D_{v_j}, a^w \right] \langle v \rangle^{-\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}+1} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, [b_{ij}(v), a^w] \langle v \rangle^{-\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}} D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, a^w b_{ij}(v) D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&\lesssim C_\varepsilon \left(\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\gamma+1} D_v u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}} |\xi|^{2/3} u\|_{L^2}^2 \right) \\
&+ \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.
\end{aligned}$$

Moreover using Lemma 3.16 and Proposition 3.17, we obtain

$$|\mathcal{Z}_{1,1}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Estimate of $\mathcal{Z}_{1,2}$: Observing $\partial_\eta \lambda_K^{1/3} \in S(\langle v \rangle^{\frac{\gamma}{6} + \frac{1}{3}}, \Gamma)$ and using (14) with symbolic calculus shows that the symbol of the commutator $\left[\left(\lambda_K^{1/3} \right)^w, b_{ik}(v) \right]$ belongs to $S(\langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_8 = \underbrace{\left[\left(\lambda_K^{1/3} \right)^w, b_{ik}(v) \right] \langle v \rangle^{-\frac{2\gamma}{3} - \frac{1}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,2}| &\leq \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}} D_{v_k} u, b_{ij}(v) [D_{v_j}, a^w] \langle v \rangle^{-\frac{\gamma}{2} - 1} \langle v \rangle^{\frac{\gamma}{2} + 1} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}} D_{v_k} u, [b_{ij}(v), a^w] \langle v \rangle^{-\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}} D_{v_j} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}} D_{v_k} u, a^w b_{ij}(v) D_{v_j} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim C_\varepsilon \|\langle v \rangle^{\gamma+1} D_v u\|_{L^2}^2 + \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2},
\end{aligned}$$

moreover using Proposition [3.17](#), we obtain

$$|\mathcal{Z}_{1,2}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon (\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2).$$

Estimate of $\mathcal{Z}_{1,3}$: Using the following notation

$$H_9 = \underbrace{\langle v \rangle^{-\frac{\gamma}{2} - 1} b_{ij}(v) [D_{v_j}, (\lambda_K^{1/3})^w] a^w (g_{2,K}^w)^{-1} \langle v \rangle^{-\gamma/6}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,3}| &\leq \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{2} + 1} b_{ik} D_{v_k} u, H_9 \langle v \rangle^{\frac{\gamma}{6}} g_{2,K}^w (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{2} + 1} b_{ik} D_{v_k} u, \langle v \rangle^{\frac{\gamma}{6}} g_{2,K}^w (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim \varepsilon \left(\|\langle v \rangle^{\frac{\gamma}{6}} (\lambda_K^{1/3})^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{6}} D_v (\lambda_K^{1/3})^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{6}} |\xi|^{1/3} (\lambda_K^{1/3})^w u\|_{L^2}^2 \right) \\
&+ C_\varepsilon \|\langle v \rangle^{\frac{\gamma}{2} + 1} B(v) D_v u\|_{L^2}^2,
\end{aligned}$$

moreover using Lemma [3.16](#) and Proposition [3.13](#), we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,3}| &\lesssim \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (1 - \delta G) (\lambda_K^{1/3})^w u \right)_{L^2} \\
&+ C_\varepsilon (\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2).
\end{aligned}$$

Estimate of $\mathcal{Z}_{1,4}$: Using the following notations

$$H_{10} = \underbrace{\langle v \rangle^{-\frac{\gamma}{3} - 1} [(\lambda_K^{1/3})^w, b_{ij}(v)] [D_{v_j}, a^w] \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

$$H_{11} = \underbrace{\langle v \rangle^{-\frac{\gamma}{3} - 1} [(\lambda_K^{1/3})^w, b_{ij}(v)] a^w \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,4}| &\leq \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{3}+1} b_{ik} D_{v_k} u, H_{10} \langle v \rangle^{\frac{\gamma}{3}} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{3}+1} b_{ik} D_{v_k} u, H_{11} \langle v \rangle^{\frac{\gamma}{3}} D_{v_j} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim \varepsilon \left(\|\langle v \rangle^{\frac{\gamma}{3}} (\lambda_K^{1/3})^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}} D_v (\lambda_K^{1/3})^w u\|_{L^2}^2 \right) \\
&+ C_\varepsilon \|\langle v \rangle^{\frac{\gamma}{3}+1} B(v) D_v u\|_{L^2}^2,
\end{aligned}$$

Moreover using Lemma [3.15](#), we obtain

$$|\mathcal{Z}_{1,4}| \lesssim \varepsilon \Re \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right),$$

so using the estimates of $\mathcal{Z}_{1,n}$ for $n = 1, \dots, 4$, we obtain

$$|\mathcal{Z}_1| \lesssim \varepsilon \Re \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Let's look now

$$\mathcal{Z}_2 = \left(\left[F(v), (\lambda_K^{1/3})^w \right] u, a^w (\lambda_K^{1/3})^w u \right)_{L^2},$$

observing the symbol $\partial_\eta \lambda_K^{1/3} \in S(\langle v \rangle^{\frac{\gamma}{6}+\frac{1}{3}}, \Gamma)$, and using Lemma [2.4](#) with symbolic calculus shows that the symbol of the commutator $\left[F(v), (\lambda_K^{1/3})^w \right]$ belongs to $S(\langle v \rangle^{\frac{3\gamma}{2}+2}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_{12} = \underbrace{\langle v \rangle^{-\frac{\gamma}{2}-1} \left[F(v), (\lambda_K^{1/3})^w \right] \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_2| &\leq \left| \left(\langle v \rangle^{\frac{\gamma}{2}+1} H_{12} \langle v \rangle^{-\gamma-1} \langle v \rangle^{\gamma+1} u, a^w (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\leq \left| \left(H_{12} \langle v \rangle^{\gamma+1} u, a^w \langle v \rangle^{\frac{\gamma}{2}+1} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&+ \left| \left(H_{12} \langle v \rangle^{\gamma+1} u, \left[\langle v \rangle^{\frac{\gamma}{2}+1}, a^w \right] \langle v \rangle^{-\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}+1} (\lambda_K^{1/3})^w u \right)_{L^2} \right|,
\end{aligned}$$

in addition, we have that the symbol of the commutator $\left[\langle v \rangle^{\frac{\gamma}{2}+1}, a^w \right]$ belongs to $S(\langle v \rangle^{\frac{\gamma}{2}+1}, \Gamma)$ and using Lemma [3.16](#), we obtain

$$|\mathcal{Z}_2| \lesssim \varepsilon \Re \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (70)$$

Let's look now

$$\mathcal{Z}_3 = \left(\left[iv \cdot \xi, (\lambda_K^{1/3})^w \right] u, a^w (\lambda_K^{1/3})^w u \right)_{L^2},$$

using Lemma [3.3](#) with symbolic calculus shows that the symbol of the commutator $\left[iv \cdot \xi, (\lambda_K^{1/3})^w \right]$ belongs to $S(\lambda_K^{1/3}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_{13} = \underbrace{(\lambda_K^{1/3})^w (a^w)^* \left[iv \cdot \xi, (\lambda_K^{1/3})^w \right] \left((\lambda_K^{2/3})^w \right)^{-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$|\mathcal{Z}_3| \leq \left| \left(H_{13} \left(\lambda_K^{2/3} \right)^w u, u \right)_{L^2} \right| \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_{\xi} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (71)$$

From the above, using the estimates of \mathcal{Z}_l for $l = 1, \dots, 3$, we obtain (69). Next we prove (68), we have the following relation

$$\begin{aligned} & \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &= \Re e \left(\mathcal{A}_{\xi} u, \left(\lambda_K^{1/3} \right)^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_{\xi}, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &+ \Re e \left(\mathcal{A}_{\xi} u, \left(\lambda_K^{1/3} \right)^w (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_{\xi}, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &= \Re e \left(\mathcal{A}_{\xi} u, \left(\lambda_K^{1/3} \right)^w \left(\lambda_K^{1/3} \right)^w \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1} \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_{\xi}, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &+ \Re e \left(\mathcal{A}_{\xi} u, \left(\lambda_K^{1/3} \right)^w (1 - \delta G) \left(\lambda_K^{1/3} \right)^w \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1} \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \\ &+ \Re e \left(\left[\mathcal{A}_{\xi}, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \end{aligned}$$

gives, with $\tilde{\varepsilon} > 0$ arbitrary,

$$\begin{aligned} & \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_{\xi} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right) + \Re e \left(\left[\mathcal{A}_{\xi}, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & + \Re e \left(\left[\mathcal{A}_{\xi}, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}. \end{aligned}$$

We could apply (69) with $a = 1$ and $a = 1 - \delta \tilde{g}$ to control the last term in the above inequality; this gives, with $\varepsilon, \tilde{\varepsilon} > 0$ arbitrarily small,

$$\begin{aligned} & \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + \varepsilon \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_{\varepsilon, \tilde{\varepsilon}} \left(\left\| \mathcal{A}_{\xi} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ & + \varepsilon \Re e \left(\mathcal{A}_{\xi} \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}. \end{aligned}$$

Letting ε small enough yields the desired estimate (68). \square

Proposition 3.19. *Let λ_K be the symbol defined in (44). Then there exists $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 \leq C_0 \left(\left\| \mathcal{A}_{\xi} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (72)$$

Proof. Using Proposition 3.13, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\left(\left(\lambda_K^{2/3} \right)^{\text{Wick}} u, u \right)_{L^2} \lesssim \Re e \left(\mathcal{A}_{\xi} u, u \right)_{L^2} + \Re e \left(\mathcal{A}_{\xi} u, (1 - \delta G) u \right)_{L^2}, \quad (73)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . By substituting $\left(\lambda_K^{1/3} \right)^w u$ to u in the above estimate, we obtain that for all $\tilde{\varepsilon} > 0$,

$$\left(\left(\lambda_K^{2/3} \right)^{\text{Wick}} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_{\xi} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (74)$$

Notice from (133) that we may write

$$(\lambda_K^{2/3})^{\text{Wick}} = (\lambda_K^{2/3})^w + r^w, \quad (75)$$

with

$$r(v, \eta) = \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) (\lambda_K^{2/3})''(Y + \theta Y_1) Y_1 \cdot Y_1 e^{-|Y_1|^2} dY_1 d\theta,$$

where $Y, Y_1 \in \mathbb{R}^6$ and $(\lambda_K^{2/3})''(Y)$ is the Hessian of $\lambda_K^{2/3}$ at the point Y . Define

$$r_1 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) \nabla_\eta^2 (\lambda_K^{2/3})(Y + \theta Y_1) \eta_1 \cdot \eta_1 e^{-|Y_1|^2} dY_1 d\theta,$$

$$r_2 = \pi^{-3} \sum_{j,k=1}^3 \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) \partial_{v_j} \partial_{\eta_k} (\lambda_K^{2/3})(Y + \theta Y_1) (v_{1j} \eta_{1k} + v_{1k} \eta_{1j}) e^{-|Y_1|^2} dY_1 d\theta,$$

and

$$r_3 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) \nabla_v^2 (\lambda_K^{2/3})(Y + \theta Y_1) v_1 \cdot v_1 e^{-|Y_1|^2} dY_1 d\theta.$$

Using Lemma 3.5 with symbolic calculus shows that the symbol

$$\nabla_\eta^2 (\lambda_K^{2/3}), \nabla_v \nabla_\eta (\lambda_K^{2/3}) \in S(\langle v \rangle^{\gamma+2}, \Gamma),$$

uniformly with respect to the parameter ξ , then r_1, r_2 belong to $S(\langle v \rangle^{\gamma+2}, \Gamma)$. Using the following notations

$$H_{14} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_1^w \left(\lambda_K^{1/3} \right)^w \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

$$H_{15} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_2^w \left(\lambda_K^{1/3} \right)^w \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

we obtain for all $\tilde{\varepsilon} > 0$,

$$\begin{aligned} & \left| \left(r_1^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| + \left| \left(r_2^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ & \leq \left| \left(H_{14} \langle v \rangle^{\gamma+2} u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| + \left| \left(H_{15} \langle v \rangle^{\gamma+2} u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Taking into account that the symbol $\nabla_v^2 (\lambda_K^{2/3})$, belongs to $S(\langle v \rangle^{\frac{2}{3}} g_{3,K}, \Gamma)$ uniformly with respect to the parameter ξ , where $g_{3,K}$ the operator defined in (43), then the symbol r_3 belongs to $S(\langle v \rangle^{\frac{2}{3}} g_{3,K}, \Gamma)$. Using the following notation

$$H_{16} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_3^w \left(\lambda_K^{1/3} \right)^w \left(g_{3,K}^w \right)^{-1} \langle v \rangle^{-\frac{2}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} \left| \left(r_3^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| & \leq \left| \left(H_{16} \langle v \rangle^{\frac{2}{3}} g_{3,K}^w u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

From the above, using the estimates of r_l for $l = 1, \dots, 3$, we obtain

$$\left| \left(r^w (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} \right| \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2). \quad (76)$$

By applying Theorem [A.6](#) with $p = \lambda$, we obtain

$$\left((\lambda_K^{2/3})^w (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} \sim \left\| \left[(\lambda_K^{1/3})^w \right]^2 u \right\|_{L^2}^2 \sim \| (\lambda_K^{2/3})^w u \|_{L^2}^2, \quad (77)$$

uniformly with respect to the parameter ξ . Using [\(76\)](#), then taking $\tilde{\varepsilon}$ small enough, we get for all $\tilde{\varepsilon} > 0$,

$$\| (\lambda_K^{2/3})^w u \|_{L^2}^2 \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + \tilde{C}_{\tilde{\varepsilon}} (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2),$$

now taking $\tilde{\varepsilon}$ small enough, we obtain that there is a constant $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\| (\lambda_K^{2/3})^w u \|_{L^2}^2 \leq C_0 (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2). \quad (78)$$

□

Proposition 3.20. *Let λ_K be the symbol defined in [\(44\)](#). Then there exists $\mathbf{C}_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\| \mathcal{A}_\xi u \|_{L^2}^2 \leq \mathbf{C}_0 \| (\lambda_K^2)^w u \|_{L^2}^2. \quad (79)$$

Proof. We denote by $\tilde{\sigma}$ the symbol of the operator \mathcal{A}_ξ . We will show that $\tilde{\sigma}$ belongs to $S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ .

Using Lemma 3.11 in [\[12\]](#), we can write $\tilde{\sigma}$ as follows

$$\tilde{\sigma} = iv \cdot \xi + |B(v)\eta|^2 + F(v) + R_1 + R_2,$$

where R_1 (resp R_2) is a symbol belongs to $S(\langle v \rangle^{\gamma+1} \langle \eta \rangle, \Gamma)$ (resp $S(\langle v \rangle^\gamma, \Gamma)$) uniformly with respect to the parameter ξ .

For $iv \cdot \xi$: Taking into account the fact that $\gamma \geq 0$, we have

$$\begin{aligned} \text{For } |\alpha| = 0, \quad |iv \cdot \xi| &\lesssim |v|^2 + |\xi|^2 \lesssim \langle v \rangle^{\gamma+2} + \langle v \rangle^\gamma |\xi|^2 \lesssim \lambda^2, \\ \text{for } |\alpha| = 1, \quad |\partial_v^\alpha (iv \cdot \xi)| &= |\xi| \lesssim \langle v \rangle^{\gamma/2} |\xi| \lesssim \lambda^2, \\ \text{for } |\alpha| \geq 2, \quad |\partial_v^\alpha (iv \cdot \xi)| &= 0. \end{aligned}$$

For $|B(v)\eta|^2$: Using [\(14\)](#), we have

$$\text{for } |\alpha| \geq 1, \quad |\partial_v^\alpha |B(v)\eta|^2| \lesssim \langle v \rangle^\gamma |\eta|^2 \lesssim |B(v)\eta|^2, \quad (80)$$

then, using Cauchy-Schwarz we get

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_v^\alpha (|B(v)\eta|^2)| \lesssim |B(v)\eta|^2 \lesssim \lambda^2. \quad (81)$$

On the other hand, also using [\(14\)](#), we have

$$\begin{aligned} \text{pour } |\alpha| = 0, \quad |B(v)\eta|^2 &\lesssim \lambda^2, \\ \text{pour } |\alpha| = 1, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| &\lesssim |B(v)| |B(v)\eta| \lesssim \lambda^2, \\ \text{pour } |\alpha| = 2, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| &\lesssim |B^T(v)B(v)| \lesssim \lambda^2, \\ \text{pour } |\alpha| \geq 3, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| &= 0, \end{aligned}$$

so we get

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| \lesssim \lambda^2. \quad (82)$$

Moreover, one can estimate from above the modulus of all the derivatives of the term $|B(v)\eta|^2$ by a constant times λ^2 .

For $F(v)$: Using Lemma [2.4](#), we have

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_v^\alpha F(v)| \lesssim \langle v \rangle^{\gamma+2-|\alpha|} \lesssim \lambda^2, \quad (83)$$

which gives that $F(v) \in S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ .

In addition, we have that $R_1, R_2 \in S(\lambda^2, \Gamma)$. From the above, we can deduce that $\tilde{\sigma}$ belongs to $S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ . Using the following notation

$$H_{17} = \underbrace{\mathcal{A}_\xi \left((\lambda_K^2)^w \right)^{-1}}_{\in \mathcal{B}(L^2)},$$

Then there exists $\mathbf{C}_0 > 0$ such that

$$\|H_{17}\varphi\|_{L^2}^2 \leq \mathbf{C}_0 \|\varphi\|_{L^2}^2 \quad \forall \varphi \in L^2(\mathbb{R}_v^3), \quad (84)$$

uniformly with respect to the parameter ξ , which implies for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\mathcal{A}_\xi u\|_{L^2}^2 \leq \mathbf{C}_0 \left((\lambda_K^2)^w u \right)_{L^2}^2. \quad (85)$$

□

4 Hypocoelliptic estimates for the whole linearized Landau operator

In this section, we show hypoelliptic estimates with respect to the velocity and position variables for the Landau operator \mathcal{P} . These estimates allow us to locate the spectrum and estimate the resolvent of the Landau operator. We denote by Λ_K the operator associated to the symbol λ_K by considering the inverse Fourier transform with respect to the variable x .

Theorem 4.1. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (86)$$

Proof. Using Theorem [A.6](#), we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^{2/3})^w u\|_{L_v^2}^2 \sim \|(\lambda_K^w)^{2/3} u\|_{L_v^2}^2,$$

so using the Proposition [3.19](#), there exists a constant $C_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^w)^{2/3} u\|_{L_v^2}^2 \leq C_1 \left(\|\mathcal{A}_\xi u\|_{L_v^2}^2 + \|u\|_{L_v^2}^2 \right), \quad (87)$$

uniformly with respect to the parameter ξ . By integrating the previous inequality with respect to the parameter ξ in \mathbb{R}^3 and considering the inverse Fourier transform with respect to the variable x , we obtain for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C_1 \left(\|\mathcal{A}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (88)$$

Using (9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K}.$$

Consequently, using (88), we have

$$\begin{aligned} \|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 &\leq C_1 \left(\|(\mathcal{A} + \mathcal{K} - \mathcal{K})u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right) \\ &\leq C_1 \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|\mathcal{K}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right) \end{aligned}$$

and using the fact that \mathcal{K} is a bounded operator, we obtain that there exists a constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (89)$$

□

By adding a term $i\kappa$ to \mathcal{P} with $\kappa \in \mathbb{R}$, the proof doesn't change due to the never changing of the real part of \mathcal{P} as mentioned crucially in Remark 2.2 in [11]. So Theorem 4.1 admits the following extension.

Theorem 4.2. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\forall \kappa \in \mathbb{R}, \quad \|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|(\mathcal{P} - i\kappa)u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (90)$$

Theorem 4.3. *There exists $\mathbf{C} > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\|\mathcal{P}u\|_{L_{x,v}^2}^2 \leq \mathbf{C} \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (91)$$

Proof. Using Theorem A.6, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^2)^w u\|_{L_v^2}^2 \sim \|(\lambda_K^w)^2 u\|_{L_v^2}^2,$$

so using the Proposition 3.20, there exists a constant $\mathbf{C}_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\mathcal{A}_\xi u\|_{L_v^2}^2 \leq \mathbf{C}_1 \|(\lambda_K^w)^2 u\|_{L_v^2}^2, \quad (92)$$

uniformly with respect to the parameter ξ . By integrating the previous inequality with respect to the parameter ξ in \mathbb{R}^3 and considering the inverse Fourier transform with respect to the variable x , we obtain for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\mathcal{A}u\|_{L_{x,v}^2}^2 \leq \mathbf{C}_1 \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (93)$$

Using (9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K}.$$

Consequently, using (93), we have

$$\begin{aligned} \|\mathcal{P}u\|_{L^2_{x,v}}^2 &\leq \|\mathcal{A}u\|_{L^2_{x,v}}^2 + \|\mathcal{K}u\|_{L^2_{x,v}}^2 \\ &\leq \mathbf{C}_1 \|\Lambda_K^2 u\|_{L^2_{x,v}}^2 + \mathbf{C}_2 \|u\|_{L^2_{x,v}}^2, \end{aligned}$$

with $\mathbf{C}_2 > 0$. Using the fact that the operator $\Lambda_K^2 \geq \text{Id}$ (see for instance Theorem A.6 in the Appendix A.2), we obtain that there exists a constant $\mathbf{C} > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\mathcal{P}u\|_{L^2_{x,v}}^2 \leq \mathbf{C} \|\Lambda_K^2 u\|_{L^2_{x,v}}^2. \quad (94)$$

□

We strongly think the estimate in (86) is optimal in term of the index 2/3 appearing in the left hand side, although we don't prove that. This index 2/3 is classical for kinetic model. We refer for example to Proposition 5.22 in [9], partial results in this direction in the Fokker-Planck case.

5 Localisation of the spectrum for the Landau operator

In all that follows, we denote by $\sigma(\mathcal{P})$ the spectrum of the operator \mathcal{P} , $\rho(\mathcal{P})$ The resolvent set of \mathcal{P} and $\|\cdot\|_{L^2}$ to denote the norm in the space $L^2(\mathbb{R}_{x,v}^6)$. The following lemma holds for any maximally accretive operator.

Lemma 5.1. *Let $(\mathcal{A}, D(\mathcal{A}))$ be a maximally accretive operator in the Hilbert space \mathcal{H} . For any $\eta \in]0, 1[$, the estimate*

$$|z + 1|^{2\eta} \|u\|^2 \leq 4 \left(((\mathcal{A} + 1)^*(\mathcal{A} + 1))^\eta u, u \right)_{\mathcal{H}} + 4 \|(\mathcal{A} - z)u\|^2$$

holds for all $u \in D(\mathcal{A})$ and $z \in \mathbb{C}$ with $\Re z \geq -1$.

Proof. See Proposition B.1 in [11].

□

Proof of Theorem 1.2.

Proof. The proof will be divided into two steps.

First step:

Using (90) and the triangle inequality we therefore get for all $z = \nu + i\kappa \in \mathbb{C}$, with $\nu = \Re z \geq -1/2$ and $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\begin{aligned} \|\Lambda_K^{2/3} u\|_{L^2}^2 &\leq C (\|(\mathcal{P} - i\kappa)u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\leq C (\|(\mathcal{P} - i\kappa - \nu + \nu)u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\leq C (2\|(\mathcal{P} - z)u\|_{L^2}^2 + (2\nu^2 + 1)\|u\|_{L^2}^2). \end{aligned}$$

But we have $\nu \geq -1/2$ implies $2\nu + 2 \geq 1$, so we get

$$\begin{aligned}\|\Lambda_K^{2/3} u\|_{L^2}^2 &\leq C(6\|(\mathcal{P} - z)u\|_{L^2}^2 + 6(\nu + 1)^2\|u\|_{L^2}^2) \\ &\leq 6C(\|(\mathcal{P} - z)u\|_{L^2}^2 + (\nu + 1)^2\|u\|_{L^2}^2),\end{aligned}$$

By taking $\tilde{C} = 6C$, we finally obtain the following estimate

$$\|\Lambda_K^{2/3} u\|_{L^2}^2 \leq \tilde{C}(\|(\mathcal{P} - z)u\|_{L^2}^2 + (\Re z + 1)^2\|u\|_{L^2}^2), \quad (95)$$

for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$ and $z \in \mathbb{C}$ with $\Re z \geq -\frac{1}{2}$.

Second step:

First, we will show that there exists a constant \tilde{C}_1 such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we have

$$0 \leq ((\mathcal{A} + 1)^*(\mathcal{A} + 1)u, u)_{L^2} \leq \tilde{C}_1(\Lambda_K^4 u, u)_{L^2}, \quad (96)$$

where \mathcal{A} the operator defined in (9). Indeed, let $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we have

$$((1 + \mathcal{A})^*(1 + \mathcal{A})u, u)_{L^2} = ((1 + \mathcal{A})u, (1 + \mathcal{A})u)_{L^2} = \|(1 + \mathcal{A})u\|_{L^2}^2 \geq 0.$$

On the other hand, we have

$$((1 + \mathcal{A})^*(1 + \mathcal{A})u, u)_{L^2} = \|(1 + \mathcal{A})u\|_{L^2}^2 \leq \|\mathcal{A}u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Using (9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K},$$

we obtain

$$((1 + \mathcal{A})^*(1 + \mathcal{A})u, u)_{L^2} \leq \|\mathcal{P}u\|_{L^2}^2 + \|\mathcal{K}u\|_{L^2}^2 + \|u\|_{L^2}^2,$$

finally, using the fact that \mathcal{K} is a bounded operator, $\Lambda_K \geq \text{Id}$ and the estimate (91), we obtain

$$\begin{aligned}((1 + \mathcal{A})^*(1 + \mathcal{A})u, u)_{L^2} &\leq (1 + \nu_0)\|\Lambda_K^2 u\|_{L^2}^2 + \mathbf{C}\|\Lambda_K^2 u\|_{L^2}^2 \\ &\leq (1 + \nu_0 + \mathbf{C})\|\Lambda_K^2 u\|_{L^2}^2,\end{aligned}$$

where $\nu_0 > 0$. By taking $\tilde{C}_1 = (1 + \nu_0 + \mathbf{C})$, we obtain the estimate (96).

According to the monotonicity of the operator functional $\mathcal{A} \rightarrow \mathcal{A}^\alpha$ for $\alpha \in [0, 1]$, in particular with $\alpha = \frac{1}{3}$, we obtain

$$0 \leq (((1 + \mathcal{A})^*(1 + \mathcal{A}))^{1/3}u, u)_{L^2} \leq \tilde{C}_1^{2/3}(\Lambda_K^{4/3}u, u)_{L^2}.$$

According to Theorem 2.8, \mathcal{A} is maximally accretive, then by applying Lemma 5.1 with $\eta = \frac{1}{3}$, for $\Re z \geq -1/2$ and $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we obtain

$$\begin{aligned}|z + 1|^{2/3}\|u\|_{L^2}^2 &\leq 4(((1 + \mathcal{A})^*(1 + \mathcal{A}))^{1/3}u, u)_{L^2} + 4\|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4\tilde{C}_1^{2/3}(\Lambda_K^{4/3}u, u)_{L^2} + 4\|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4\tilde{C}_1^{2/3}\|\Lambda_K^{2/3}u\|_{L^2}^2 + 4\|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4\tilde{C}_1^{2/3}\|\Lambda_K^{2/3}u\|_{L^2}^2 + 4\|(\mathcal{P} - z)u\|_{L^2}^2 + 4\nu_1\|u\|_{L^2}^2.\end{aligned}$$

With the inequality (95), we obtain

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq (4\tilde{C}_1^{2/3}\tilde{C} + 4) \|(\mathcal{P} - z)u\|_{L^2}^2 + (4\tilde{C}_1^{2/3}\tilde{C} + 16\nu_1)(\Re z + 1)^2 \|u\|_{L^2}^2.$$

By taking $C_{\mathcal{P}} = \sqrt{8(4\nu_1 + \tilde{C}_1^{2/3}\tilde{C})}$, $Q_{\mathcal{P}} = \sqrt{8(1 + \tilde{C}_1^{2/3}\tilde{C})}$ we finally get $\forall z \in \mathbb{C}$, $\Re z \geq -1/2$, $\forall u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq \frac{Q_{\mathcal{P}}^2}{2} \|(\mathcal{P} - z)u\|_{L^2}^2 + \frac{C_{\mathcal{P}}^2}{2} (\Re z + 1)^2 \|u\|_{L^2}^2. \quad (97)$$

Now let $z \notin S_{\mathcal{P}}$ such that $\Re z \geq -1/2$, so according to the definition of $S_{\mathcal{P}}$ given in (6) we have

$$(\Re z + 1)^2 \leq \frac{1}{C_{\mathcal{P}}^2} |z + 1|^{2/3},$$

then for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, inequality (97) implies

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq Q_{\mathcal{P}}^2 \|(\mathcal{P} - z)u\|_{L^2}^2, \quad (98)$$

we deduce that $\mathcal{P} - z$ is injective, moreover we can replace $\mathcal{P} - z$ by $(\mathcal{P} - z)^*$ in (98), which gives $(\mathcal{P} - z)^*$ is injective and consequently $\mathcal{P} - z$ is bijective with dense image in L^2 , therefore $z \in \rho(\mathcal{P})$. By taking $v = (\mathcal{P} - z)u$ in the estimate (98) we get

$$\|(\mathcal{P} - z)^{-1}v\|_{L^2}^2 \leq Q_{\mathcal{P}}^2 |z + 1|^{-2/3} \|v\|_{L^2}^2 \quad \forall v \in L^2,$$

then we obtain the resolvent estimate (7).

On the other hand, we have that if $z \in \sigma(\mathcal{P})$ then $z \in S_{\mathcal{P}}$ and taking into account that the numerical range of the operator \mathcal{P} is the half plan $\{\Re z \geq 0\}$, we deduce that the spectrum $\sigma(\mathcal{P})$ satisfies

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\}.$$

Concerning estimate (8), we have

$$\Re((\mathcal{P} - z)u, u)_{L^2} \geq -\Re z \|u\|_{L^2}^2 \quad \text{with } \Re z \leq -\frac{1}{2} < 0,$$

which implies that

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq |\Re z|^{-1}.$$

The proof is then complete. □

A Appendix

A.1 Weyl-Hörmander calculus

We recall here some notations and basic facts of symbolic calculus, and refer to [18] and [15] for detailed discussions on the pseudo-differential calculus.

We introduce on \mathbb{R}^{2n} the following metric

$$\Gamma = dv^2 + d\eta^2.$$

Definition A.1. Let $m \geq 1$ be a C^∞ function on \mathbb{R}^{2n} . We say that m is an admissible weight for Γ if there exist two constants $C > 0$ and $N > 0$ such that

$$\forall X, Y \in \mathbb{R}^{2n}, \quad m(X) \leq C \langle X - Y \rangle^N m(Y). \quad (99)$$

Definition A.2. Let m be an admissible function. We denote by $S(m, \Gamma)$ the symbol class of all smooth functions $p(v, \eta)$ (possibly depending on parameter ξ) satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha, \beta} > 0; \forall (v, \eta) \in \mathbb{R}^{2n}, |\partial_v^\alpha \partial_\eta^\beta p(v, \eta)| \leq C_{\alpha, \beta} m(v, \eta).$$

The space of symbols $S(m, \Gamma)$ endowed with the semi-norms

$$\|p\|_{k; S(m, \Gamma)} = \sup_{|\alpha + \beta| \leq k} \sup_{(v, \eta) \in \mathbb{R}^{2n}} |m(v, \eta)^{-1} \partial_v^\alpha \partial_\eta^\beta p(v, \eta)|; \quad k \in \mathbb{N} \quad (100)$$

becomes a Fréchet space.

For such a symbol p in $S(m, \Gamma)$ we may define its Weyl quantization p^w by

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p^w u)(v) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v') \cdot \eta} p\left(\frac{v+v'}{2}, \eta\right) u(v') dv' d\eta. \quad (101)$$

The Weyl quantization of $S(m, \Gamma)$ is denoted by $\Psi(m, \Gamma)$.

Theorem A.3. (Calderon-Vaillancourt)

Let p^w be an operator in $\Psi(1, \Gamma)$. We have p^w a continuous operator on $L^2(\mathbb{R}^n)$ and

$$\forall u \in L^2(\mathbb{R}^n), \quad \|p^w u\|_{L^2} \leq C \|p\|_{N; S(1, \Gamma)} \|u\|_{L^2}, \quad (102)$$

where $C > 0$ and a positive integer N depending only on the dimension.

Proof. See Section 18 in [15]. □

Theorem A.4. Let p^w be an invertible operator in $\Psi(m, \Gamma)$, then its inverse $[p^w]^{-1}$ belongs to $\Psi(m^{-1}, \Gamma)$.

Proof. See Lemma A.2 in [11]. □

Definition A.5. Let $p^w \in \Psi(m, \Gamma)$. We say that p^w is an elliptic operator if there exists $C > 0$ such that

$$|p| \geq Cm.$$

Let us also recall here the composition formula of Weyl quantization. Let $a \in S(m_1, \Gamma)$ and $b \in S(m_2, \Gamma)$, the compositions of the pseudo-differential operators a^w and b^w are pseudo-differential operators whose symbol, denoted $a \# b$, belongs to $S(m_1 m_2, \Gamma)$ and has the following development:

$$a \# b = ab + \int_0^1 \iint e^{-i\sigma(Y-Y_1, Y-Y_2)/(2\theta)} \frac{i}{2} \sigma(\partial_{Y_1}, \partial_{Y_2}) a(Y_1) b(Y_2) dY_1 dY_2 d\theta / (\theta)^{2n}, \quad (103)$$

where σ is the symplectic form in $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ given by

$$\sigma(Z, Z') = \sum_{j=0}^n \zeta_j z'_j - z_j \zeta'_j. \quad (104)$$

As we work with pseudo-differential operators which belong to classes associated to the metric Γ , all operators will be defined as continuous operators of $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ or from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

A.2 Basic theorem

This theorem aims at giving a uniform statement for Weyl Hörmander tools with a large parameter K . This theorem gives very important results which can use these results to establish estimates which requires pseudo-differential operators. Part (I) in the theorem below has been shown in [1] with $\tau = 1$, but here we have improved this result for all $\tau \in \mathbb{R}$. In general, these results give a general and robust framework to techniques already used as well for work that requires these kinds of properties.

Theorem A.6. *Let $p, q \geq 1$ be two symbols which verify the following hypotheses:*

- i) p is an admissible weight.*
- ii) $p \sim q$.*
- iii) $p, q \in S(p, \Gamma)$.*
- iv) There exists $M \in \mathbb{R}$ such that for all $\varepsilon > 0$ we have $\partial_\eta p, \partial_\eta q \in S(\varepsilon p + \varepsilon^{-1} \langle v \rangle^M, \Gamma)$.*

We define the symbols p_K, q_K as $p_K = p + K \langle v \rangle^M$ and $q_K = q + K \langle v \rangle^M$. Then there exists K_0 such that for all $K \geq K_0$:

- I) For all $\tau \in \mathbb{R}$, $(p_K^\tau)^w$ and $(q_K^\tau)^w$ are invertible.*
- II) For all $\tau \in \mathbb{R}$, $[(p_K^\tau)^w]^{-1}$ and $[(q_K^\tau)^w]^{-1}$ are pseudo-differential operators that belong to $\Psi(p_K^{-\tau}, \Gamma)$ uniformly in K .*
- III) For all $\tau \in \mathbb{R}$ and for all $\kappa \geq 0$, $[(p_K^\kappa)^w]^\tau$ and $[(q_K^\kappa)^w]^\tau$ are pseudo-differential operators belong to $\Psi(p_K^{\kappa\tau}, \Gamma)$ uniformly in K .*
- IV) For all $\tau \in \mathbb{R}$ and for all $\kappa \geq 0$, $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|[(p_K^\kappa)^w]^\tau u\|_{L^2}^2 \sim \|(p_K^{\kappa\tau})^w u\|_{L^2}^2,$$

uniformly in K .

- V) For all $\tau \in \mathbb{R}$, $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|(p_K^w)^\tau u\|_{L^2}^2 \sim \|(p_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^w)^\tau u\|_{L^2}^2,$$

uniformly in K .

- VI) $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$(p_K^w u, u)_{L^2} \sim (q_K^w u, u)_{L^2},$$

uniformly in K .

- VII) If $M \geq 0$, we have $p_K^w, q_K^w \geq Id$.*

Proof. Let $K > 0$ and let p and q two symbols and $M \in \mathbb{R}$ such that the hypotheses from (i) to (iv) are verified. For simplification, we will prove (I, II, III) just for the operator $(p_K^\tau)^w$, and we reason in the same way for $(q_K^\tau)^w$.

I. We will show that $(p_K^\tau)^w$ is invertible for all $\tau \in \mathbb{R}$. We note that p_K is an admissible weight (p is an admissible weight). We have $p_K \in S(p_K, \Gamma)$ uniformly in K . Indeed, using (iii) we have for all $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta p_K| &\leq |\partial_v^\alpha \partial_\eta^\beta p| + |\partial_v^\alpha \partial_\eta^\beta K \langle v \rangle^M| \\ &\leq C_{\alpha, \beta} p + K C_{\alpha, \beta, M} \langle v \rangle^M \\ &\leq \tilde{C}_{\alpha, \beta, M} p_K. \end{aligned}$$

More generally we can show, by induction on $|\alpha|$ and Leibnitz's formula that for $\tau \in \mathbb{R}$,

$$\forall \alpha \in \mathbb{N}^{2n}, \quad |\partial_{v, \eta}^\alpha p_K^\tau| \leq C_{\alpha, \beta, \tau} p_K^\tau,$$

which gives $p_K^\tau \in S(p_K^\tau, \Gamma)$ uniformly in K .

Using formula (103), we may write

$$(p_K^\tau)^w (p_K^{-\tau})^w = \text{Id} - R_K^w, \quad (105)$$

where

$$R_K = - \int_0^1 (\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau})) d\theta + \int_0^1 (\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau})) d\theta$$

with $g \sharp_\theta h$ defined by

$$g \sharp_\theta h(Y) = \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} g(Y_1) h(Y_2) dY_1 dY_2 / (\pi\theta)^{2n}, \quad (106)$$

with $Y, Y_1, Y_2 \in \mathbb{R}^{2n}$ and σ a symplectic form defined in 104.

Let now N be the integer which is given in (102). By [3, Proposition 1.1], we can find a constant C_N and a positive integer l_N , both depending only on N but independent of K and θ , such that

$$\|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \|\partial_\eta p_K^\tau\|_{l_N; S(p_K^\tau, \Gamma)} \|\partial_v (p_K^{-\tau})\|_{l_N; S(p_K^{-\tau}, \Gamma)},$$

where the semi-norm $\|\cdot\|_{k; S(M, \Gamma)}$ is defined by (100).

Moreover, using (iv), we have $\partial_\eta p \in S(\varepsilon p + \varepsilon^{-1} \langle v \rangle^M, \Gamma)$, by taking $\varepsilon = K^{-1/2}$, we obtain $\partial_\eta p \in S(K^{-1/2} p_K, \Gamma)$. By writing $\partial_\eta p_K^\tau = \tau p_K^{\tau-1} \partial_\eta p$, we obtain

$$|\partial_\eta p_K^\tau| \leq \tilde{C}_N K^{-1/2} p_K^\tau.$$

Then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta p_K^\tau| \leq \tilde{C}_N K^{-1/2} p_K^\tau,$$

which gives $\partial_\eta p_K^\tau \in S(K^{-1/2} p_K^\tau, \Gamma)$ uniformly in K , moreover we have

$$\|(\partial_\eta p_K^\tau)\|_{l_N; S(p_K^\tau, \Gamma)} \leq \tilde{C}_N K^{-1/2}.$$

On the other hand we have $p_K^{-\tau} \in S(p_K^{-\tau}, \Gamma)$, and thus

$$\|\partial_v (p_K^{-\tau})\|_{l_N; S(p_K^{-\tau}, \Gamma)} \leq \tilde{C}_N,$$

with \tilde{C}_N a constant depending only on N but independent of K . As a result,

$$\|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \tilde{C}_N^2 K^{-1/2}. \quad (107)$$

Similarly,

$$\|(\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau}))\|_{N;S(1,\Gamma)} \leq C_N \tilde{C}_N^2 K^{-1/2}. \quad (108)$$

Using the estimates (107), (108) we will estimate the semi-norm $\|R_K\|_{N;S(1,\Gamma)}$. Indeed,

$$\|R_K\|_{N;S(1,\Gamma)} \leq \int_0^1 \|(\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau}))\|_{N;S(1,\Gamma)} d\theta + \int_0^1 \|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N;S(1,\Gamma)} d\theta.$$

Then

$$\|R_K\|_{N;S(1,\Gamma)} \leq 2C_N \tilde{C}_N^2 K^{-1/2},$$

and thus by (102)

$$\|R_K^w\|_{\mathcal{B}(L^2)} \leq 2CC_N \tilde{C}_N^2 K^{-1/2}$$

with C a constant depending only on the dimension. Taking $K_1 = (4CC_N \tilde{C}_N^2)^2$, so we get for all $K \geq K_1$

$$\|R_K^w\|_{\mathcal{B}(L^2)} \leq \frac{1}{2} < 1,$$

this implies $\text{Id} - R_K^w$ is invertible in the space $\mathcal{B}(L^2)$. In addition, its inverse is given by

$$(\text{Id} - R_K^w)^{-1} = \sum_{i=0}^{\infty} (R_K^w)^i \in \mathcal{B}(L^2). \quad (109)$$

Based on (105) we obtain

$$(p_K^\tau)^w \left((p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1} \right) = \text{Id}.$$

Similarly we can find a $\hat{R}_K \in S(1, \Gamma)$ such that

$$\left((\text{Id} - \hat{R}_K^w)^{-1} (p_K^{-\tau})^w \right) (p_K^\tau)^w = \text{Id}.$$

Based on the above $(p_K^\tau)^w$ is invertible and its inverse $[(p_K^\tau)^w]^{-1}$ is written in the form

$$[(p_K^\tau)^w]^{-1} = (p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1} = (\text{Id} - \hat{R}_K^w)^{-1} (p_K^{-\tau})^w.$$

We have proved the conclusion in (I).

II. According to (I), we have $[(p_K^\tau)^w]^{-1}$ has the form

$$[(p_K^\tau)^w]^{-1} = (p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1}.$$

By taking $H_K = (\text{Id} - R_K^w)^{-1}$ and using (109), we have H_K is a continuous operator in L^2 uniformly in K . So based on Theorem A.4, we have H_K is a pseudo-differential operator. We note by $\delta(H_K)$ its symbol. We have $\delta(H_K)$ belongs to $S(1, \Gamma)$ uniformly in K . From the above, we have $[(p_K^\tau)^w]^{-1}$ is a pseudo-differential operator (compositions of pseudo-differential operators). We denote h_K its symbol, h_K has the form $h_K = p_K^{-\tau} \sharp \delta(H_K)$ and belongs to $S(p_K^{-\tau}, \Gamma)$ uniformly in K .

We have proved the conclusion in (II).

III. Using the Theorem [A.4](#), we have $[(p_K^\kappa)^w]^{-1}$ is a pseudo-differential operator. Taking into account that the composition of the pseudo-differential operators is a pseudo-differential operator, we obtain

$$\forall n \in \mathbb{Z}, [(p_K^\kappa)^w]^n \in \Psi(p_K^{\kappa n}, \Gamma). \quad (110)$$

Now consider the case of the exponents $\tau \in \mathbb{R}$. By pseudo-differential calculus the problem will be reduced to

$$\forall \tau \in I, [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa \tau}, \Gamma),$$

where I an open interval of \mathbb{R} . Note that the operator $(p_K^\kappa)^w$ is self-adjoint because its symbol p_K^κ is real. Now we will show that $(p_K^\kappa)^w$ is a positive operator for K sufficiently large, which is equivalent to show that

$$\left((p_K^\kappa)^w u, u \right)_{L^2} \geq 0, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \quad (111)$$

Using again the formula [\(I03\)](#), we can write

$$(p_K^{\kappa/2})^w (p_K^{\kappa/2})^w = (p_K^\kappa)^w - R_K^w, \quad (112)$$

where

$$R_K = - \int_0^1 (\partial_\eta p_K^{\kappa/2}) \#_\theta (\partial_v (p_K^{\kappa/2})) \, d\theta + \int_0^1 (\partial_v p_K^{\kappa/2}) \#_\theta (\partial_\eta (p_K^{\kappa/2})) \, d\theta$$

with $g \#_\theta h$ defined in [\(I06\)](#). Based on the proof of (I), we have

$$\partial_\eta p_K^{\kappa/2} \in S(K^{-1/2} p_K^{\kappa/2}, \Gamma)$$

uniformly in K . Moreover we have $\partial_v p_K^{\kappa/2} \in S(p_K^{\kappa/2}, \Gamma)$, so we get

$$(\partial_\eta p_K^{\kappa/2}) \#_\theta (\partial_v (p_K^{\kappa/2})), (\partial_v p_K^{\kappa/2}) \#_\theta (\partial_\eta (p_K^{\kappa/2})) \text{ and } R_K \in S(K^{-1/2} p_K^\kappa, \Gamma)$$

uniformly in K . Writing R_K^w in the following form

$$R_K^w = K^{-1/2} (p_K^{\kappa/2})^w \underbrace{K^{1/2} [(p_K^{\kappa/2})^w]^{-1} R_K^w [(p_K^{\kappa/2})^w]^{-1} (p_K^{\kappa/2})^w}_{\in \mathcal{B}(L^2) \text{ uniformly in } K}$$

we obtain

$$|(R_K^w u, u)_{L^2}| \leq C_0 K^{-1/2} \|(p_K^{\kappa/2})^w u\|_{L^2}^2,$$

with C_0 some constant independent of K . Let $K_2 = 16C_0^2$, then using the relation [\(I12\)](#) we get for all $K \geq K_2$

$$((p_K^\kappa)^w u, u)_{L^2} \geq \frac{3}{4} \|(p_K^{\kappa/2})^w u\|_{L^2}^2 \geq 0.$$

Then, using the following formula (see for example [\[23\]](#)) we can write

$$[(p_K^\kappa)^w]^\tau = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau (s + (p_K^\kappa)^w)^{-1} \, ds, \quad \tau \in (-1, 0). \quad (113)$$

First, $s + (p_K^\kappa)^w$ is a pseudo-differential operator and its symbol $a_{K,s}$ verifies

$$\forall s \in (0, 1), \quad a_{K,s} \in S(p_K^\kappa, \Gamma)$$

and

$$\forall s \geq 1, a_{K,s} \in S(s(p_K^\kappa), \Gamma) \text{ uniformly in } K \text{ and } s.$$

So using the Theorem [A.4](#), $(s + (p_K^\kappa)^w)^{-1}$ is a pseudo-differential operator and its symbol $b_{K,s}$ verifies

$$\forall s \in (0, 1), b_{K,s} \in S((p_K^\kappa)^{-1}, \Gamma)$$

and

$$\forall s \geq 1, b_{K,s} \in S(s^{-1}(p_K^\kappa)^{-1}, \Gamma) \text{ uniformly in } K \text{ and } s.$$

Then, taking $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$[(p_K^\kappa)^w]^\tau u = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau (s + (p_K^\kappa)^w)^{-1} u \, ds,$$

using the formula ([101](#)), we get

$$\begin{aligned} [(p_K^\kappa)^w]^\tau u &= -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v')\cdot\eta} b_{K,s}\left(\frac{v+v'}{2}, \eta\right) u(v') \, dv' d\eta \right) ds, \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v')\cdot\eta} \left(-\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau b_{K,s}\left(\frac{v+v'}{2}, \eta\right) ds \right) u(v') \, dv' d\eta. \end{aligned}$$

So $[(p_K^\kappa)^w]^\tau$ is a pseudo-differential operator and its symbol d_K given by

$$d_K = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau b_{K,s} \, ds.$$

Then we will show that $d_K \in S(p_K^{\kappa\tau}, \Gamma)$. Indeed, let $\alpha, \beta \in \mathbb{N}^n, (v, \eta) \in \mathbb{R}^{2n}$,

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta d_K(v, \eta)| &\leq C \left(\int_0^1 s^\tau |\partial_v^\alpha \partial_\eta^\beta b_{K,s}| \, ds + \int_1^\infty s^\tau |\partial_v^\alpha \partial_\eta^\beta b_{K,s}| \, ds \right), \\ &\leq C_{\alpha,\beta} \left(\int_0^1 s^\tau (p_K^\kappa)^{-1} \, ds + \int_1^\infty s^{\tau-1} (p_K^\kappa)^{-1} \, ds \right), \\ &\leq \tilde{C}_{\alpha,\beta} (p_K^\kappa)^{-1}, \end{aligned}$$

which gives that $d_K \in S(p_K^{\kappa\tau}, \Gamma)$. Take $I = (-1, 0)$, then we get

$$\forall \tau \in I, [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa\tau}, \Gamma). \quad (114)$$

In addition, $[(p_K^\kappa)^w]^\tau$ is a bounded operator in L^2 uniformly in K . Indeed, $(s + (p_K^\kappa)^w)^{-1} \in \Psi(1, \Gamma)$ uniformly in K , then there exists a constant $C_1 > 0$ such that

$$\forall s \in (0, 1), \|(s + (p_K^\kappa)^w)^{-1}\|_{\mathcal{B}(L^2)} \leq C_1. \quad (115)$$

On the other hand we have

$$\forall u \in \mathcal{S}(\mathbb{R}^n), ((s + (p_K^\kappa)^w)u, u)_{L^2} \geq s\|u\|_{L^2}^2,$$

which gives

$$\forall s \geq 1, \|(s + (p_K^\kappa)^w)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s}. \quad (116)$$

Using ([115](#)), ([116](#)) and the formula ([113](#)), we obtain

$$\|[(p_K^\kappa)^w]^\tau\|_{\mathcal{B}(L^2)} \leq C_2, \quad (117)$$

with C_2 some constant independent of K .

Then based on (110), (114) we obtain for all $\tau \in I$,

$$\forall n \in \mathbb{Z}, \quad [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa n \tau}, \Gamma), \quad (118)$$

so we have

$$\forall \tau \in \mathbb{R}, \quad [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa \tau}, \Gamma). \quad (119)$$

IV. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \| [(p_K^\kappa)^w]^\tau u \|_{L^2}^2 \sim \| (p_K^{\kappa \tau})^w u \|_{L^2}^2, \quad (120)$$

otherwise we will show that there are two constants $c, C > 0$ independent of K such that for all $u \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$\| (p_K^{\kappa \tau})^w u \|_{L^2}^2 \leq c \| [(p_K^\kappa)^w]^\tau u \|_{L^2}^2, \quad (121)$$

and

$$\| [(p_K^\kappa)^w]^\tau u \|_{L^2}^2 \leq C \| (p_K^{\kappa \tau})^w u \|_{L^2}^2. \quad (122)$$

We will start with inequality (121), using the conclusion (III), we have that $[(p_K^\kappa)^w]^\tau$ is a pseudo-differential operator belongs to $\Psi(p_K^{\kappa \tau}, \Gamma)$ uniformly in K . Taking $v = [(p_K^\kappa)^w]^{-\tau} u$, show inequality (121) is equivalent to showing that

$$\| (p_K^{\kappa \tau})^w [(p_K^\kappa)^w]^{-\tau} v \|_{L^2}^2 \leq C_3 \| v \|_{L^2}^2 \quad \forall v \in L^2. \quad (123)$$

Using the Theorem A.3, we have $(p_K^{\kappa \tau})^w [(p_K^\kappa)^w]^{-\tau} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , hence the inequality (121).

Similarly, using the fact that $[(p_K^\kappa)^w]^\tau [(p_K^{\kappa \tau})^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , then taking $v = [(p_K^{\kappa \tau})^w]^{-1} u$, we get the inequality (122), so the estimate (120) is true.

V. We will first show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \| (p_K^\tau)^w u \|_{L^2}^2 \sim \| (q_K^\tau)^w u \|_{L^2}^2, \quad (124)$$

otherwise we will show that there are two constants $C_1, C_2 > 0$ independent of K such that for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\| (p_K^\tau)^w u \|_{L^2}^2 \leq C_1 \| (q_K^\tau)^w u \|_{L^2}^2, \quad (125)$$

and

$$\| (q_K^\tau)^w u \|_{L^2}^2 \leq C_2 \| (p_K^\tau)^w u \|_{L^2}^2. \quad (126)$$

We will start with inequality (125), using hypothesis (iii) we have that $(q_K^\tau)^w \in \Psi(p_K^\tau, \Gamma)$ uniformly in K . Using the conclusion (II), we have $(q_K^\tau)^w$ is an invertible operator, moreover its inverse $[(q_K^\tau)^w]^{-1}$ belongs to $\Psi(p_K^{-\tau}, \Gamma)$ uniformly in K .

Taking $v = [(q_K^\tau)^w]^{-1} u$, show inequality (125) is equivalent to showing that

$$\| (p_K^\tau)^w [(q_K^\tau)^w]^{-1} v \|_{L^2}^2 \leq C_1 \| v \|_{L^2}^2 \quad \forall v \in L^2. \quad (127)$$

Using the Theorem A.3, we have $(p_K^\tau)^w [(q_K^\tau)^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , hence the inequality (127).

Similarly, using the fact that $(q_K^\tau)^w [(p_K^\tau)^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , then taking $v = [(p_K^\tau)^w]^{-1}u$, we get the inequality (126), so the estimate (124) is true. In the same way we have

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(q_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^w)^\tau u\|_{L^2}^2. \quad (128)$$

Finally, using conclusion (IV) with $\kappa = 1$, we get that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(p_K^w)^\tau u\|_{L^2}^2 \sim \|(p_K^\tau)^w u\|_{L^2}^2. \quad (129)$$

VI. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p_K^w u, u)_{L^2} \sim (q_K^w u, u)_{L^2}. \quad (130)$$

Indeed, let $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(p_K^w u, u)_{L^2} = ((p_K^w)^{1/2} u, (p_K^w)^{1/2} u)_{L^2} = \|(p_K^w)^{1/2} u\|_{L^2}^2,$$

using the result (IV) with $\tau = \frac{1}{2}$, we obtain that

$$\|(p_K^w)^{1/2} u\|_{L^2}^2 \sim \|(p_K^{1/2})^w u\|_{L^2}^2 \sim \|(q_K^{1/2})^w u\|_{L^2}^2 \sim \|(q_K^w)^{1/2} u\|_{L^2}^2 \sim (q_K^w u, u)_{L^2}.$$

So the estimate (130) is true.

VII. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p_K^w u, u)_{L^2} \geq \|u\|_{L^2}^2.$$

Indeed, let $u \in \mathcal{S}(\mathbb{R}^n)$

$$(p_K^w u, u)_{L^2} = (p_{K-1}^w u, u)_{L^2} + (\langle v \rangle^M u, u)_{L^2},$$

based on the proof of (111), we can show that there exists a positive constant K_3 such that for all $K \geq K_3$ we have

$$(p_{K-1}^w u, u)_{L^2} \geq 0,$$

Which gives

$$(p_K^w u, u)_{L^2} = \underbrace{(p_{K-1}^w u, u)_{L^2}}_{\geq 0} + (\langle v \rangle^M u, u)_{L^2} \geq \|\langle v \rangle^{M/2} u\|_{L^2}^2.$$

With the condition $M \geq 0$ we finally obtain

$$(p_K^w u, u)_{L^2} \geq \|u\|_{L^2}^2.$$

Finally, for a symbol p which verifies the hypotheses of Theorem A.6, we can fix

$$K \geq \underbrace{\max(K_1, K_2, K_3)}_{K_0} \quad (131)$$

and we apply the results of Theorem A.6 to the operator p_K^w . \square

A.3 Wick quantization

Finally let us recall some basic properties of the Wick quantization, which is also called anti-Wick in [20]. The importance in studying the Wick quantization lies in the facts that positive symbols give rise to positive operators.

Let $Y = (v, \eta)$ be a point in \mathbb{R}^6 . The Wick quantization of a symbol q is given by

$$q^{\text{Wick}} = (2\pi)^{-3} \int_{\mathbb{R}^6} q(Y) \Pi_Y \, dY,$$

where Π_Y is the projector associated to the Gaussian φ_Y which is defined by

$$\varphi_Y(z) = \pi^{-3/4} e^{-\frac{1}{2}|z-v|^2} e^{iz \cdot \eta/2}, \quad \forall z \in \mathbb{R}^3.$$

The main property of the Wick quantization is its positivity, i.e.,

$$q(v, \eta) \geq 0, \quad \forall (v, \eta) \in \mathbb{R}^6 \text{ implies } q^{\text{Wick}} \geq 0. \quad (132)$$

According to Theorem 24.1 in [20], the Wick and Weyl quantizations of a symbol q are linked by the following identities

$$q^{\text{Wick}} = \left(q * \pi^{-3} e^{-|\cdot|^2} \right)^w = q^w + r^w \quad (133)$$

with

$$r(Y) = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1-\theta) q''(Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 \, d\theta, \quad (134)$$

where $q''(Y)$ is the Hessian of q at the point Y . Therefore, according to (102), if $q \in S(1, \Gamma)$ then q^{Wick} is a bounded operator in L^2 .

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [16]

$$q_1^{\text{Wick}} q_2^{\text{Wick}} = [q_1 q_2 - q'_1 \cdot q'_2 + \frac{1}{i} \{q_1, q_2\}]^{\text{Wick}} + T, \quad (135)$$

with T a bounded operator in $L^2(\mathbb{R}^6)$ and q' is the gradient of q with respect to Y , when $q_1 \in L^\infty(\mathbb{R}^6)$ and q_2 is a smooth symbol

whose derivatives of order ≥ 2 are bounded on \mathbb{R}^6 . The notation $\{q_1, q_2\}$ denotes by the Poisson bracket defined by

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta}. \quad (136)$$

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