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Hypoelliptic and spectral estimates for the linearized Landau operator

MOHAMAD RACHID*

Abstract

We are interested in the inhomogeneous Landau equation which describes the evolution of a particle density $f = f(t, x, v)$ representing at time $t \geq 0$, the density of particles at position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The study is motivated by the linearization of the Landau equation near Maxwellian distribution. In this article, we establish hypoelliptic estimates, a localization of the spectrum and estimates of the resolvent of the linear Landau operator with hard potentials and Maxwellian molecules. The proof is based on a multiplier method and requires fine pseudo-differential calculus tools.

Contents

1	Introduction	3
1.1	The model.	3
1.2	Notations.	4
1.3	Main results.	4
2	Properties of the Landau operator	6
2.1	Splitting of the linearized operator.	7
2.2	Preliminaries.	7
2.3	Study of the operator \mathcal{P}	10
3	Hypoelliptic estimates for the operator with parameters	14
3.1	Pseudo-differential parts	14
3.2	Hypoelliptic estimates	20
4	Hypoelliptic estimates for the whole linearized Landau operator	33

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5 Localisation of the spectrum for the Landau operator **35**

A Appendix **37**

A.1 Weyl-Hörmander calculus 37

A.2 Basic theorem 39

A.3 Wick quantization 46

Acknowledgement **46**

1 Introduction

1.1 The model.

In this paper, we study the hypoellipticity and the spectral properties associated to the spatially inhomogeneous Landau equation. This equation is a kinetic model in plasma physics that describes the evolution of the density function $F = F(t, x, v)$ representing at time $t \in \mathbb{R}^+$, the density of particles at position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. This equation is given by

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F) \\ F|_{t=0} = F_0, \end{cases} \quad (1)$$

where Q is the so-called Landau collision operator which acts on the variable v and which contains diffusion in velocity. More precisely, the Landau operator is defined by

$$Q(G, F) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) [G_* \partial_j F - F \partial_j G_*] dv_*, \quad (2)$$

and we use the convention of summation of repeated indices, and the derivatives are in the velocity variable v i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $G_* = G(v_*)$, $F = F(v)$, $\partial_j G_* = \partial_{v_* j} G(v_*)$, $\partial_j F = \partial_{v_j} F(v)$, etc. The matrix $A(v) = (a_{i,j}(v))_{1 \leq i, j \leq 3}$ is symmetric, positive, definite, depends on the interaction between particles and is given by

$$a_{i,j}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3, 1].$$

We have the following classification: we call hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules, i.e. $\gamma \in [0, 1]$. We denote by

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

the normalized Maxwellian which is a global equilibrium. We linearize the Landau equation around μ with the perturbation

$$F = \mu + \mu^{1/2} f.$$

The Landau equation [\(1\)](#) for $f = f(t, x, v)$ takes the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \mu^{-1/2} Q(\mu^{1/2} f, \mu) - \mu^{-1/2} Q(\mu, \mu^{1/2} f) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} f) \\ f|_{t=0} = f_0 = \mu^{-1/2} (F_0 - \mu), \end{cases} \quad (3)$$

since $Q(\mu, \mu) = 0$. Using the notation

$$\Gamma(f, g) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g),$$

we may rewrite the above equation as

$$\begin{cases} \partial_t f + \mathcal{P}f = \Gamma(f, f) \\ f|_{t=0} = f_0, \end{cases} \quad (4)$$

where the linearized Landau operator \mathcal{P} takes the form

$$\mathcal{P} = v \cdot \nabla_x - \mathcal{L} \quad (5)$$

with

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{L}_1 = \Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 = \Gamma(f, \sqrt{\mu}).$$

The operator \mathcal{P} acts only in variables (x, v) , is non selfadjoint, and consists of a transport part which is skew-adjoint, a diffusion part acting only in the v variable and a compact part. Using for example [18], [5], we show that the diffusion part \mathcal{L}_1 is written as follows

$$\mathcal{L}_1 f = \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] - (\mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2}) f + \nabla_v \cdot [\mathbf{A}(v) \frac{v}{2}] f,$$

with $\mathbf{A}(v) = (\bar{a}_{i,j}(v))_{1 \leq i,j \leq 3}$ is a symmetric matrix where

$$\bar{a}_{i,j} = a_{i,j} *_v \mu,$$

and the compact part \mathcal{L}_2 is given by

$$\mathcal{L}_2 f = -\mu^{-1/2} \partial_i \left\{ \mu \left[a_{ij} *_v \left\{ \mu^{1/2} \left[\partial_j f + \frac{v_j}{2} f \right] \right\} \right] \right\}.$$

1.2 Notations.

Throughout the paper we shall adopt the following notations: we work in dimension $n = 3$ and denote by $(x, v) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3$ the space-velocity variables. For $v \in \mathbb{R}^3$ we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$, where we recall that $|v|$ is the canonical Euclidian norm of v in \mathbb{R}^3 . The gradient in velocity (resp. space) will be denoted by ∂_v (resp. ∂_x). We shall also denote $D_v = \frac{1}{i} \partial_v$ (resp. $D_x = \frac{1}{i} \partial_x$), and denote ξ the dual variable of x , η the dual variable of v . For simplicity of notations, $a \sim b$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 b \leq a \leq c_2 b$; we abbreviate “ $\leq C$ ” to “ \lesssim ”, where C is a positive constant depending only on fixed number. Finally, the space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is an open set.

1.3 Main results.

In this article, we will show a localization property of the spectrum of the Landau operator \mathcal{P} . This result is given by the following theorem.

Theoreme 1.1. *Let \mathcal{P} be the Landau operator defined in (5) with $\gamma \in [0, 1]$. Then there are two constants $C_{\mathcal{P}} > 0$ and $Q_{\mathcal{P}} > 0$ so that:*

a) *The spectrum of \mathcal{P} verifies*

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\},$$

with

$$S_{\mathcal{P}} = \left\{ z \in \mathbb{C}, |z + 1|^{1/3} \leq C_{\mathcal{P}} (\Re z + 1), \Re z \geq -\frac{1}{2} \right\}. \quad (6)$$

b) For any $z \notin S_{\mathcal{P}}$ with $\Re z \geq -\frac{1}{2}$, the resolvent is estimated by

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L^2_{x,v})} \leq Q_{\mathcal{P}}|z + 1|^{-1/3}. \quad (7)$$

Notice that if $\Re z \leq -\frac{1}{2}$ then

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L^2_{x,v})} \leq |\Re z|^{-1}. \quad (8)$$

To prove this theorem we need subelliptic estimates. To establish those estimates, an adapted pseudo-differential calculation is used. Standard notions concerning symbolic estimates and the pseudo-differential calculus are explained at the beginning of Section A.

The results of Theorem [1.1](#) allow to obtain the following figure:

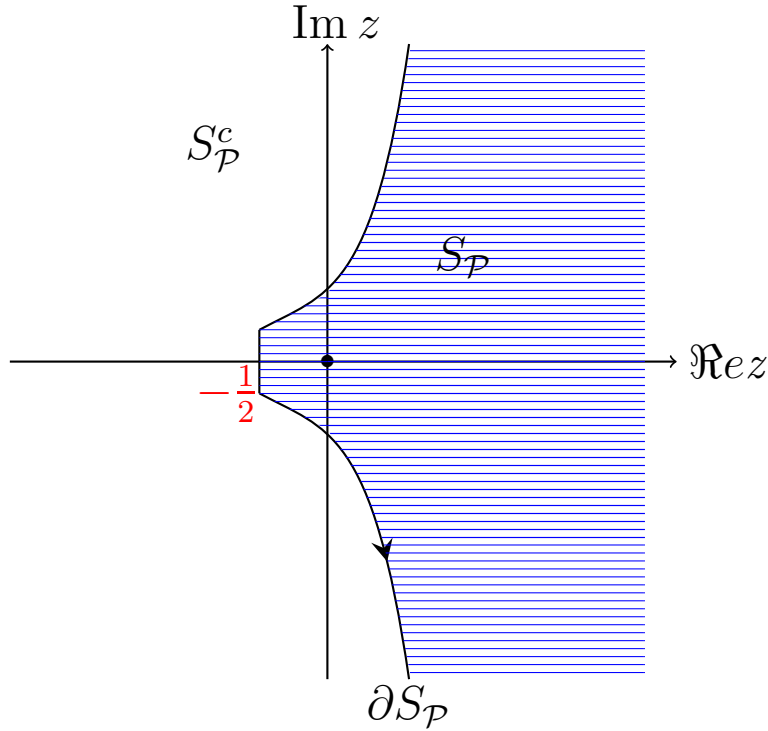


Figure 1: Localization of the spectrum of the Landau operator \mathcal{P} .

In this figure $\partial S_{\mathcal{P}}$ oriented from $+i\infty$ to $-i\infty$. The hatched part is where the spectrum is localized and the non hatched part is the zone where we have good resolvent estimates (see [\(7\)](#) and [\(8\)](#)).

Our strategy to prove the Theorem [1.1](#), is to adapt the method used in [\[6\]](#) and [\[8\]](#) to localize the spectrum of Fokker-Planck operator. This method is based on hypoelliptic estimates. The difference between the technique used in [\[6\]](#), [\[8\]](#) and our technique is that, in our work, we are working on the Landau collision operator Q (defined in [\(2\)](#)) which is more complicated than the Fokker-Planck operator. We need more technicality and fine tools for pseudo-differential calculus. In addition, we need to build symbols such that the pseudo-differential operator associated to these symbols has good properties (see Section [3.1](#)). These symbols require some assumptions, hence

the idea of introducing Theorem [A.6](#). We need Theorem [A.6](#) which gives very important properties on pseudo-differential calculus for a very large constant which will be noted K and it is very important to establish hypoelliptic estimates. This Theorem gives a general and robust framework to techniques already used in [\[10\]](#) and thus for work which requires pseudodifferential operators having good properties. The main idea for the first part (I) of the proof of Theorem [A.6](#), is based on the techniques as in the Lemma 4.3 in the article [\[1\]](#) and the others part requires other techniques. In addition, to establish hypoelliptic estimates, we make use of multiplier method to explore the intrinsic hypoelliptic structure of operator $\mathcal{P} = v \cdot \nabla_x - \mathcal{L}$ defined in [\(5\)](#). By multiplier method we mean finding a bounded selfadjoint operator G , such that on one side the commutator between the transport part and G gives some “elliptic” properties in spatial variables, and on the other side we can control the upper bound for the term $|(\mathcal{A}u, Gu)_{L^2}|$ (see Section [3.2](#)). This method was introduced for the first time by F. Hérau and F. Nier in [\[8\]](#). The multiplier method makes a strong use of pseudo-differential and Wick calculus, widely studied in particular by Lerner (see [\[14\]](#) and [\[15\]](#)). We use the same multiplier method as in articles [\[7\]](#), [\[9\]](#) which study and prove hypoellipticity with optimal loss of $4/3$ derivatives for the linear Landau-type operators with and without external potential. Linear Landau-type equations are a class of inhomogeneous kinetic equations whose study is motivated by the linearization of the Landau equation. Details about the Landau equation may be found in the works by Y. Guo [\[5\]](#) and K.-C. Wu [\[18\]](#). F. Hérau and W.-X. Li in [\[7\]](#), F. Hérau and K. Pravda-Starov in [\[9\]](#), studied part of the Landau operator which can be noted here in our work by \mathcal{A} and they took several hypotheses on this operator to obtain a more simplified form. But in our work, we prove these hypotheses to obtain a more simplified form of the operator \mathcal{A} (see [\(19\)](#)) and we work on the original Linearized Landau operator which is the most physically interesting. In this article, we used the decomposition of the Landau operator also introduced in [\[18\]](#), [\[5\]](#) to get a simpler form of this operator and some properties of this operator obtained in [\[3\]](#) to obtain more robust properties for this operator (see Section [1.1](#)). In addition, we used some results obtained by [\[7\]](#), [\[9\]](#) for linear Landau-type operator. On the other hand, we improve these results to establish our results on the Landau operator which are based on the same technique as in [\[7\]](#) and [\[9\]](#) where preliminary subelliptic were given.

Organization of the article. In Section 2 we give some properties of the Landau operator. In Section 3, we prove hypoelliptic estimates with respect to the velocity variable for a parametric operator. In Section 4 we give hypoelliptic estimates for the linearized Landau operator. Section 5 is devoted to the proof of Theorem [1.1](#). An appendix is devoted to a short review of some tools used in this work (Weyl-Hörmander quantization, Wick quantization and the proof of the Theorem [A.6](#) (Basic Theorem)).

2 Properties of the Landau operator

In this section, first we present the decomposition of the linearized Landau operator \mathcal{P} then we show that it verifies important properties which gives us a simpler form of this operator to end up showing that \mathcal{P} is a generator of a strongly continuous family of semi-group. Throughout this section, we work with $\gamma \in [-3, 1]$.

2.1 Splitting of the linearized operator.

Consider a smooth positive function $\chi \in C_c^\infty(\mathbb{R}_v^3)$ such that $0 \leq \chi(v) \leq 1$, $\chi(v) = 1$ for $|v| \leq 1$ and $\chi(v) = 0$ for $|v| > 2$. For any $R \geq 1$ we define $\chi_R(v) = \chi(\frac{v}{R})$ and in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$. Then, we introduce the decomposition of the operator \mathcal{P} as $\mathcal{P} = \mathcal{A} + \mathcal{K}$ with

$$\mathcal{A} = -\mathcal{L}_1 + v \cdot \nabla_x + M\chi_R, \quad \mathcal{K} = -\mathcal{L}_2 - M\chi_R, \quad (9)$$

where $M > 0$ and $R > 0$ will be chosen later.

We define the function $F(v)$ as

$$F(v) = (\mathbf{A}(v)\frac{v}{2} \cdot \frac{v}{2}) - \nabla_v \cdot [\mathbf{A}(v)\frac{v}{2}] + M\chi_R. \quad (10)$$

Then, we can rewrite \mathcal{A} as follows

$$\mathcal{A}f = v \cdot \nabla_x f - \nabla_v \cdot [\mathbf{A}(v)\nabla_v f] + F(v)f. \quad (11)$$

2.2 Preliminaries.

We have the following results concerning the matrix $\mathbf{A}(v)$.

Lemma 2.1. *The following properties hold:*

- a) For $v \in \mathbb{R}^3 \setminus \{0\}$, the matrix $\mathbf{A}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace v^\perp . Moreover, when $|v| \rightarrow +\infty$ we have

$$\ell_1(v) \sim 2\langle v \rangle^\gamma \quad \text{and} \quad \ell_2(v) \sim \langle v \rangle^{\gamma+2}.$$

- b) The function $\bar{a}_{i,j}$ is smooth, for any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha \bar{a}_{i,j}(v)| + |\partial_v^\alpha (\bar{a}_{i,j}(v)v_j)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|},$$

- c) For $v \in \mathbb{R}^3 \setminus \{0\}$, we have

$$\bar{a}_{i,j}(v)v_i v_j = \ell_1(v)|v|^2,$$

$$\bar{a}_{i,i}(v) = \text{tr}(\bar{a}(v)) = \ell_1(v) + 2\ell_2(v),$$

$$\bar{a}_{i,j}(v)\eta_i \eta_j = \ell_1(v)|P_v \eta|^2 + \ell_2(v)|(I - P_v)\eta|^2,$$

with $\eta \in \mathbb{R}^3$ and P_v is the projection on v , i.e. $P_v \eta = \left(\eta \cdot \frac{v}{|v|}\right) \frac{v}{|v|}$.

- d) For $|v| > 1$, we have

$$|\partial_v^\alpha \ell_1(v)| \leq C_\alpha \langle v \rangle^{\gamma-|\alpha|} \quad \text{and} \quad |\partial_v^\alpha \ell_2(v)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

Proof. See for example [3, Lemma 2.4], [5, Lemma 3] et [18, Proposition 1]. □

Lemma 2.2. For all $v \in \mathbb{R}^3$ with $|v| > 1$, we have

$$\ell_1(v) \gtrsim \langle v \rangle^\gamma \quad \text{et} \quad \ell_2(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

Proof. Using (a) in Lemma 2.1, when $|v| \rightarrow +\infty$, we have $\ell_1(v) \sim 2\langle v \rangle^\gamma$. In particular, there is a constant $N > 0$ such that for all $|v| > N$, we have

$$\ell_1(v) \geq \langle v \rangle^\gamma.$$

We have that $\ell_1(v)$ is continuous since $\mathbf{A}(v)$ is a positive definite symmetric matrix and continuous (due to convolution with μ), hence the existence of a constant $C > 0$ such that for $1 \leq |v| \leq N$

$$\ell_1(v) \geq C \langle v \rangle^\gamma,$$

and then for all $v \in \mathbb{R}^3$, perhaps with changing C ,

$$\ell_1(v) \geq C \langle v \rangle^\gamma.$$

The proof will be the same for $\ell_2(v)$. □

Lemma 2.3. Let $F(v)$ be defined in (10). Then, we can choose M and R big enough such that for all $v \in \mathbb{R}^3$, we have

$$F(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

Proof. Since

$$F(v) \geq \frac{1}{4} \ell_1(v) |v|^2 - \left| \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right| + M \chi_R,$$

according to Lemma 2.1, we have

$$\left| \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right| \lesssim \langle v \rangle^{\gamma+1}. \quad (12)$$

So, using Lemma 2.2 and (12), there exist two positive constants C_1, C_2 such that

$$F(v) \geq C_1 \langle v \rangle^{\gamma+2} - C_2 \langle v \rangle^{\gamma+1} + M \chi_R,$$

then there exist M and R such that for all $v \in \mathbb{R}^3$

$$F(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

□

Lemma 2.4. For any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha F(v)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

Proof. For $|v| > 2R$, using Leibniz's formula, we have

$$\partial_v^\alpha F(v) = \frac{1}{4} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \ell_1(v) \partial^\beta \langle v \rangle^2 - \frac{1}{4} \partial^\alpha \ell_1(v) - \partial^\alpha \left(\nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right),$$

Then, using Lemma 2.1 and Lemma 2.2 we obtain

$$|\partial_v^\alpha F(v)| \leq c_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

The function $F(v)$ being C^∞ on $|v| \leq 2R$, the estimates for $|v| \leq 2R$ are immediate. □

Lemma 2.5. *i) $\mathbf{A}(v)$ is written as follows*

$$\mathbf{A}(v) = B^T(v)B(v), \quad (13)$$

avec $B(v) = (b_{i,j}(v))_{1 \leq i,j \leq 3}$ is a matrix with real-valued smooth entries.

ii) For any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha b_{i,j}(v)| \leq C_\alpha \langle v \rangle^{\frac{7}{2}+1-|\alpha|}. \quad (14)$$

iii) There exists $c, C > 0$ such that for all $v \in \mathbb{R}^3$, for all $\eta \in \mathbb{R}^3$ we have

$$c \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2) \leq \mathbf{A}(v) \eta \cdot \eta = |B(v) \eta|^2 \leq C \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2) \quad (15)$$

Proof. i) As $\mathbf{A}(v)$ is a positive definite symmetric matrix (denoted $\mathbf{S}_3^{++}(\mathbb{R})$) and according to the spectral theorem, there exists \mathbf{Q} an orthogonal matrix such that

$$\mathbf{A}(v) = \mathbf{Q}^T(v) \mathbf{D}(v) \mathbf{Q}(v).$$

On $\{|v| > 1\}$, $v \mapsto \mathbf{Q}(v)$ can be calculated explicitly and can be chosen to be smooth. If we set

$$B(v) = \mathbf{Q}^T(v) \sqrt{\mathbf{D}(v)} \mathbf{Q}(v)$$

with $\sqrt{\mathbf{D}(v)} = \text{diag}(\sqrt{\ell_1(v)}, \sqrt{\ell_2(v)}, \sqrt{\ell_2(v)})$. We have that $v \mapsto \sqrt{\mathbf{D}(v)}$ is of class C^∞ for $|v| > 1$ (because $\ell_1(v), \ell_2(v)$ are of class C^∞ for $|v| > 1$). Then the application $v \mapsto B(v)$ is of class C^∞ on $\{|v| > 1\}$. Regarding the case where $|v| \leq 1$, we consider the following two applications:

$$\begin{aligned} \phi &: \mathbb{R}^3 \rightarrow \mathbf{S}_3^{++}(\mathbb{R}) \\ v &\mapsto \mathbf{A}(v) \end{aligned} ,$$

$$\begin{aligned} \psi &: \mathbf{S}_3^{++}(\mathbb{R}) \rightarrow \mathbf{S}_3^{++}(\mathbb{R}) \\ M &\mapsto \sqrt{M} \end{aligned} ,$$

we note that ϕ, ψ are of class C^∞ , moreover $B(v) = \psi \circ \phi(v)$. Then we have that the application $v \mapsto B(v)$ is of class C^∞ .

ii) For $|v| > 1$, we have $B(v) = \sqrt{\ell_1(v)} P_v + \sqrt{\ell_2(v)} (I - P_v)$, moreover using Lemma [2.1](#), we have

$$|\partial_v^\alpha \sqrt{\ell_1(v)}| \lesssim \langle v \rangle^{\frac{7}{2}-|\alpha|}, \quad |\partial_v^\alpha \sqrt{\ell_2(v)}| \lesssim \langle v \rangle^{\frac{7+2}{2}-|\alpha|} \quad (16)$$

and the fact that $|\partial_v^\alpha P_v| \lesssim 1$ (P_v and all its derivatives are bounded), so we get that for $|v| > 1$,

$$|\partial_v^\alpha b_{i,j}(v)| \lesssim \langle v \rangle^{\frac{7}{2}+1-|\alpha|}, \quad (17)$$

where the constants in [\(16\)](#), [\(17\)](#) depend on α .

The function $b_{i,j}(v)$ being C^∞ on $|v| \leq 1$, the estimates for $|v| \leq 1$ are immediate. Then, for all $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha b_{i,j}(v)| \leq C_\alpha \langle v \rangle^{\frac{7}{2}+1-|\alpha|}. \quad (18)$$

iii) The estimate is immediate on $|v| \leq 1$ because $\mathbf{A}(v)$ is a positive definite symmetric matrix and $\mathbf{A} = B^T B$. For $|v| > 1$, using Lemma [2.1](#), we have

$$\begin{aligned} \bar{a}_{i,j}(v)\eta_i\eta_j &= \ell_1(v)|P_v\eta|^2 + \ell_2(v)|(I - P_v)\eta|^2 \\ &\gtrsim \langle v \rangle^\gamma |\eta|^2 \cos^2(v, \eta) + \langle v \rangle^{\gamma+2} \frac{|v \wedge \eta|^2}{|v|^2} \\ &\gtrsim \langle v \rangle^\gamma |\eta|^2 \cos^2(v, \eta) + \langle v \rangle^\gamma |v \wedge \eta|^2 + \langle v \rangle^\gamma |\eta|^2 \sin^2(v, \eta) \\ &\gtrsim \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2), \end{aligned}$$

on the other hand, we have

$$\begin{aligned} \bar{a}_{i,j}(v)\eta_i\eta_j &= \ell_1(v)|P_v\eta|^2 + \ell_2(v)|(I - P_v)\eta|^2 \\ &\lesssim \langle v \rangle^\gamma |\eta|^2 + \langle v \rangle^{\gamma+2} \frac{|v \wedge \eta|^2}{|v|^2} \\ &\lesssim \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2). \end{aligned}$$

Hence the proof of (iii). □

Using Lemma [2.5](#), we can rewrite \mathcal{A} in the form

$$\mathcal{A} = v \cdot \nabla_x + (B(v)\nabla_v)^* \cdot B(v)\nabla_v + F(v), \quad (19)$$

where $(B(v)D_v)^* = D_v B(v)^T$, is the formal adjoint of $B(v)D_v$.

2.3 Study of the operator \mathcal{P} .

In this part, we will study the following problem:

$$\begin{cases} \partial_t f + \mathcal{P}f = 0 \\ f|_{t=0} = f_0, \end{cases} \quad (20)$$

we show that the above problem is well-posed in the space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ in the sense of semi-groups. By Hille-Yosida Theorem, it is sufficient to show that \mathcal{A} is maximal accretive in the space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$, then using the Bounded Perturbation Theorem in [[4](#), Theorem 1.3], we get that the operator \mathcal{P} is a generator of a strongly continuous family of semi-group (for more details on the semi-group theory see also [[16](#)]). First, we start by recalling the basic definition of hypoellipticity.

Definition 2.6. Let P be a differential operator with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$. We say that P is a hypoelliptic operator on Ω , if, for any $\omega \subset \Omega$, any $u \in \mathcal{D}'(\Omega)$, such that $Pu \in C^\infty(\omega)$ belongs to $C^\infty(\omega)$.

Lemma 2.7. Let \mathcal{A} be the operator defined in ([19](#)). Then, \mathcal{A} is a hypoelliptic operator.

Proof. Using formula (19), we can rewrite \mathcal{A} in the following form:

$$\begin{aligned}\mathcal{A} &= \underbrace{v \cdot \partial_x}_{X_0} + \underbrace{\left(\sum_{j=1}^3 b_{1,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{1,j}(v) \partial_{v_j} \right)}_{X_1^* X_1} + \underbrace{\left(\sum_{j=1}^3 b_{2,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{2,j}(v) \partial_{v_j} \right)}_{X_2^* X_2} \\ &\quad + \underbrace{\left(\sum_{j=1}^3 b_{3,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{3,j}(v) \partial_{v_j} \right)}_{X_3^* X_3} + F(v) \\ &= X_0 + \sum_{i=1}^3 X_i^* X_i + F(v).\end{aligned}$$

In addition, the coefficients $b_{i,j}(v)$ and $F(v)$ are of class C^∞ , taking the vector field square brackets we get

$$Y_k = [X_k, X_0] = \sum_{j=1}^3 b_{k,j}(v) \partial_{x_j}, \quad \text{for } k \in \{1, 2, 3\}.$$

From the above, \mathcal{A} which is a “type II Hörmander’s operators” (see for example [6], [11]). Moreover, the vector fields $\{X_i, Y_i, i = 1, \dots, 3\}$ generate all the space tangent to $\mathbb{R}_{x,v}^6$ so \mathcal{A} is hypoelliptic operator. \square

Theorem 2.8. *Let $\gamma \in [-3, 1]$ and \mathcal{A} be the operator defined in (9). Then, its closure $\overline{\mathcal{A}}$ on the space $\mathcal{S}(\mathbb{R}_{x,v}^6)$ is maximally accretive.*

Proof. We adapt here the proof given in [6, page 44]. We apply the abstract criterion by taking $\mathcal{H} = L^2(\mathbb{R}_{x,v}^6)$ and the domain of \mathcal{A} defined by $D(\mathcal{A}) = \mathcal{S}(\mathbb{R}_{x,v}^6)$. First, we show the accretivity of the operator \mathcal{A} . We want to prove that $\Re(\mathcal{A}u, u)_{\mathcal{H}} \geq 0$ for $u \in D(\mathcal{A})$. Indeed, from (19)

$$\begin{aligned}\Re(\mathcal{A}u, u)_{\mathcal{H}} &= \underbrace{\Re(v \cdot \nabla_x u, u)_{\mathcal{H}}}_{=0 \text{ since } v \cdot \nabla_x \text{ is skew-adjoint}} - \Re(\nabla_v \cdot (\mathbf{A}(v) \nabla_v u), u)_{\mathcal{H}} + \Re(F(v)u, u)_{\mathcal{H}} \\ &= \|B(v) \nabla_v u\|_{\mathcal{H}}^2 + \|\sqrt{F(v)}u\|_{\mathcal{H}}^2 \\ &\geq 0.\end{aligned}$$

Since \mathcal{A} is an accretive operator then its closure $\overline{\mathcal{A}}$ exists and it is accretive (see [6, Proposition 5.3]). Let us now show that there exists $\lambda_1 > 0$ such that the operator

$$T = \mathcal{A} + \lambda_1 \text{Id}$$

has dense image in \mathcal{H} . We take $\lambda_1 = 1$. Let $f \in \mathcal{H}$ satisfy

$$(f, Tu)_{\mathcal{H}} = 0, \quad \forall u \in D(\mathcal{A}). \quad (21)$$

we want to prove that $f = 0$.

Since T is a differential operator then his formal adjoint T^\sharp exists (in the sense of distributions). According to (21) we obtain

$$T^\sharp f = \left(-\nabla_v \cdot \mathbf{A}(v) \nabla_v + F(v) + 1 - X_0 \right) f = 0, \quad \text{in } D'(\mathbb{R}^6). \quad (22)$$

Using Lemma 2.7, we have $-\nabla_v \cdot \mathbf{A}(v)\nabla_v + F(v) + 1 - X_0$ is a hypoelliptic operator, so $f \in C^\infty(\mathbb{R}^6)$ (see [6, Chapter 2]).

Now we introduce the family of truncation functions ζ_k defined by

$$\zeta_k(x, v) = \zeta\left(\frac{x}{k_1}\right)\zeta\left(\frac{v}{k_2}\right), \quad \forall k = (k_1, k_2) \in (\mathbb{N}^*)^2,$$

where ζ is a C^∞ function satisfying the following conditions:

$$\begin{cases} 0 \leq \zeta \leq 1, \\ \zeta = 1 \text{ on } B(0, 1), \\ \text{supp } \zeta \subset B(0, 2), \\ \zeta \text{ is a radial function.} \end{cases}$$

The expression of $T^\sharp(\zeta_k f)$ is

$$\begin{aligned} T^\sharp(\zeta_k f) &= -\nabla_v \cdot (\mathbf{A}(v)\nabla_v(\zeta_k f)) + (F(v) + 1)\zeta_k f - X_0(\zeta_k f) \\ &= -\nabla_v \cdot (\mathbf{A}(v)([\nabla_v \zeta_k]f)) - \nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f - X_0(\zeta_k)f - \zeta_k T^\sharp f, \end{aligned}$$

by using (22), we obtain

$$T^\sharp(\zeta_k f) = -\nabla_v \cdot (\mathbf{A}(v)([\nabla_v \zeta_k]f)) - \nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f - X_0(\zeta_k)f. \quad (23)$$

We note that $T^\sharp(\zeta_k f) \in \mathcal{H}$, taking the scalar product with $\zeta_k f$ we obtain

$$\begin{aligned} (T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} &= -\iint \nabla_v \cdot (\mathbf{A}(v)([\nabla_v \zeta_k]f))\zeta_k f \, dx dv - \iint X_0(\zeta_k)\zeta_k |f|^2 \, dx dv \\ &\quad - \iint (\nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f)\zeta_k f \, dx dv, \end{aligned}$$

By doing an integration by parts, we obtain

$$(T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} = \iint |B\nabla_v(\zeta_k)f|^2 \, dx dv - \iint X_0(\zeta_k)\zeta_k |f|^2 \, dx dv.$$

On the other hand, using (22), we obtain

$$\begin{aligned} (T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} &= \iint |B\nabla_v(\zeta_k f)|^2 \, dx dv + \iint (F(v) + 1)|\zeta_k f|^2 \, dx dv \\ &\quad - \underbrace{\iint X_0(\zeta_k f)\zeta_k f \, dx dv}_{=0 \text{ since } X_0 \text{ is skew-adjoint}} \\ &= \iint |B\nabla_v(\zeta_k f)|^2 \, dx dv + \iint (F(v) + 1)|\zeta_k f|^2 \, dx dv. \end{aligned}$$

Using the fact that

$$\iint |B\nabla_v(\zeta_k f)|^2 \, dx dv \geq 0,$$

we obtain

$$\iint (F(v) + 1)|\zeta_k f|^2 \, dx dv \leq \underbrace{\iint |B\nabla_v(\zeta_k)f|^2 \, dx dv}_{(i)} - \underbrace{\iint X_0(\zeta_k)\zeta_k |f|^2 \, dx dv}_{(ii)}. \quad (24)$$

Estimate of (i): Using Lemma [2.1](#)-(b)-(d) and taking into account that the function ζ is a radial function we obtain

$$\begin{aligned} \iint |B\nabla_v(\zeta_k)f|^2 dx dv &= \iint \ell_1(v) |P_v \nabla_v \zeta_k|^2 |f|^2 dx dv + \underbrace{\iint \ell_2(v) |(I - P_v) \nabla_v \zeta_k|^2 |f|^2 dx dv}_{=0 \text{ since } v \text{ is parallel to } \nabla_v \zeta_k} \\ &\leq \frac{C_0}{k_2^2} \iint \langle v \rangle^\gamma \Phi_k |f|^2 dx dv, \end{aligned}$$

where $C_0 > 0$ and $\Phi_k = \zeta(\frac{x}{k_1})\zeta'(\frac{v}{k_2})$. Using the fact that $\gamma \in [-3, 1]$ and the fact that Φ_k is a bounded function we have the existence of a constant $C_1 > 0$ such that,

$$\iint |B\nabla_v(\zeta_k)f|^2 dx dv \leq C_1 \left(\frac{1}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2.$$

Estimate of (ii): we have

$$\left| \iint X_0(\zeta_k) \zeta_k |f|^2 dx dv \right| \leq \frac{1}{k_1} \iint |v| \tilde{\Phi}_k \zeta_k |f|^2 dx dv,$$

where $\tilde{\Phi}_k = \zeta'(\frac{x}{k_1})\zeta(\frac{v}{k_2})$.

Now, taking into account that the functions $\tilde{\Phi}_k$ and ζ_k are bounded, we have the existence of a constant $C_2 > 0$ such that,

$$\left| \iint X_0(\zeta_k) \zeta_k |f|^2 dx dv \right| \leq C_2 \frac{k_2}{k_1} \|f\|^2.$$

Finally, coming back to [\(24\)](#) we obtained the existence of a constant $C > 0$ such that,

$$\iint |\zeta_k f|^2 dx dv \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_2} + \frac{k_2}{k_1} \right) \|f\|^2, \quad \forall k. \quad (25)$$

Taking $k_1 \rightarrow +\infty$ in [\(25\)](#) we obtain

$$\iint |\zeta(\frac{v}{k_2})f|^2 dx dv \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2, \quad (26)$$

and taking $k_2 \rightarrow +\infty$ in [\(26\)](#) we obtain

$$\iint |f|^2 dx dv = 0,$$

then $f = 0$. □

From now on, we write \mathcal{A} for the closure of the operator \mathcal{A} .

Corollary 2.9. *Let $\gamma \in [-3, 1]$ and \mathcal{P} be the operator defined in [\(5\)](#). Then, $-\mathcal{P}$ is a generator of a semi-group $(S(t))_{t \geq 0}$ strongly continuous on $\mathcal{H} = L^2(\mathbb{R}_{x,v}^6)$ verifying*

$$\|S(t)\| \leq e^{\|\mathcal{K}\|t} \text{ for } t \geq 0, \quad (27)$$

where \mathcal{K} the operator defined in [\(9\)](#).

Proof. Using (9), the operator $-\mathcal{P}$ is written as follows

$$-\mathcal{P} = -\mathcal{A} - \mathcal{K}.$$

According to Theorem 2.8, \mathcal{A} is a maximally accretive operator. According to the Hille-Yosida Theorem, $-\mathcal{A}$ is a generator of a strongly continuous semi-group of contraction. On the other hand, $-\mathcal{K}$ is a bounded operator in \mathcal{H} (we have K a compact operator and $M\chi_R$ is bounded). Using the Bounded Perturbation Theorem in [4, Theorem 1.3], we have that $-\mathcal{P}$ is a generator of a semi-group $(S(t))_{t \geq 0}$ strongly continuous on \mathcal{H} , moreover we have

$$\|S(t)\| \leq e^{\|\mathcal{K}\|t} \text{ for } t \geq 0. \quad (28)$$

□

3 Hypoelliptic estimates for the operator with parameters

In the following discussion, we work with $\gamma \in [0, 1]$. In this section, we will study the operator acting on the velocity variable v :

$$\mathcal{A}_\xi = iv \cdot \xi + (B(v)\nabla_v)^* \cdot B(v)\nabla_v + F(v), \quad (29)$$

where ξ is the parameter in \mathbb{R}^3 . The operator \mathcal{A}_ξ is obtained by the partial Fourier transformation in x . The goal of studying the operator \mathcal{A}_ξ and considering ξ as a parameter, is to obtain estimates of the velocity variables v uniformly with respect to ξ . Then by using the inverse Fourier transform with respect to x , we can obtain global estimates in all variables. We note that the operator \mathcal{A}_ξ verifies for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 \leq \Re e(\mathcal{A}_\xi u, u)_{L^2}. \quad (30)$$

Notations. Throughout this section, we will use $\|\cdot\|_{L^2}$ to denote the norm in the space $L^2(\mathbb{R}_v^3)$ and ξ is a parameter. We use p^{Wick} to denote the Wick quantization of p in the variables (v, η) (for more details on Wick quantization see [14] and Section A.3).

The main result in this section is Proposition 3.19 and Proposition 3.20.

3.1 Pseudo-differential parts

In this part, we will show several lemmas concerning pseudo-differential symbols. We need to build these symbols, who verify assumptions of Theorem A.6, such that the pseudo-differential operator associated to these symbols has good properties. This operator play an important role in hypoelliptic estimates. The standard concepts on pseudo-differential calculus are explained in Appendix A. We define for $(v, \eta) \in \mathbb{R}^6$ the

following symbols, they depend on the parameter ξ but we do not mention it in our notations since ξ is seen as a parameter.

$$\lambda(v, \eta) = \left[\langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2 \right) \right]^{1/2}, \quad (31)$$

$$a(v, \eta) = 1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2, \quad (32)$$

$$g_1(v, \eta) = 1 + \langle v \rangle + \langle \eta \rangle, \quad (33)$$

$$g_2(v, \eta) = 1 + \langle \xi \rangle^{1/3} + \langle \eta \rangle, \quad (34)$$

$$g_3(v, \eta) = 1 + \langle \xi \rangle^{2/3} + \langle \eta \rangle^2. \quad (35)$$

Lemma 3.1. *The above symbols are admissible weights uniformly with respect to the parameter ξ in \mathbb{R}^3 .*

Proof. We have to check λ is an admissible weight. It is sufficient to verify that there exists two constants N and C , both depending only on γ , such that for all $Y = (v, \eta)$, $Y' = (v', \eta')$, we have

$$\lambda(Y) \leq C \lambda(Y') (1 + \Gamma(Y - Y'))^N.$$

We have

$$\frac{\lambda^2(v, \eta)}{\lambda^2(v', \eta')} = \frac{\langle v \rangle^\gamma}{\langle v' \rangle^\gamma} \left(\frac{1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \right). \quad (36)$$

We now use Peetre's inequality

$$\langle y \rangle^\tau \leq 2^{\frac{|\tau|}{2}} \langle y' \rangle^\tau \langle y - y' \rangle^{|\tau|}, \quad \tau \in \mathbb{R}, \quad (37)$$

to get

$$\frac{\langle v \rangle^\gamma}{\langle v' \rangle^\gamma} \leq 2^\gamma \langle v - v' \rangle^\gamma.$$

Using (37), we obtain

$$\frac{1 + |v|^2 + |\eta|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq \frac{\langle v \rangle^2}{\langle v' \rangle^2} + \frac{\langle \eta \rangle^2}{\langle \eta' \rangle^2} \leq 4 (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2$$

and

$$\frac{|\xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq 1 \leq (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2.$$

Using the relation

$$v \wedge \xi = (v - v') \wedge \xi + v' \wedge \xi,$$

we obtain

$$\frac{|v \wedge \xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq 1 + \frac{\langle v - v' \rangle^2 \langle \xi \rangle^2}{\langle \xi \rangle^2} \leq 2 (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2.$$

Moreover using the relation

$$v \wedge \eta = (v - v') \wedge (\eta - \eta') + (v - v') \wedge \eta' + v' \wedge (\eta - \eta') + v' \wedge \eta',$$

we compute

$$\begin{aligned}
& \frac{|v \wedge \eta|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \\
& \leq \frac{4|v - v'|^2|\eta - \eta'|^2 + 4|v - v'|^2|\eta'|^2 + 4|v'|^2|\eta - \eta'|^2 + 4|v' \wedge \eta'|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \\
& \leq 4|v - v'|^2|\eta - \eta'|^2 + 4|v - v'|^2 + 4|\eta - \eta'|^2 + 4 \\
& \leq 10(\langle v - v' \rangle + \langle \eta - \eta' \rangle)^4.
\end{aligned}$$

Combining the above inequalities, we get

$$\frac{\lambda(Y)}{\lambda(Y')} \leq C_\gamma (1 + \Gamma(Y - Y'))^{\frac{4+\gamma}{2}},$$

so λ is an admissible weight. The proof will be the same for a, g_1, g_2 and g_3 . \square

Lemma 3.2. For $m \in \mathbb{R}$,

$$\lambda^m \in S(\lambda^m, \Gamma),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We can rewrite λ as follows

$$\lambda(v, \eta) = \langle v \rangle^{\frac{\gamma}{2}} a^{\frac{1}{2}}(v, \eta).$$

To prove the wanted result, this we use induction on $|\alpha + \beta|$ to prove that for any $k \in \mathbb{R}$ and any $|\alpha + \beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta a^k(v, \eta)| \lesssim a^k(v, \eta), \quad (38)$$

which obviously holds for $|\alpha + \beta| = 0$. Now suppose $|\alpha + \beta| \geq 1$, then we have either $|\alpha| \geq 1$ or $|\beta| \geq 1$, and suppose $|\beta| \geq 1$ without loss of generality. So we can write $\partial_\eta^\beta = \partial_\eta^{\tilde{\beta}} \partial_{\eta_j}$ with $|\tilde{\beta}| = |\beta| - 1$ and thus

$$\partial_v^\alpha \partial_\eta^\beta a^k(v, \eta) = \partial_v^\alpha \partial_\eta^{\tilde{\beta}} \left[k a^{k-1} (2\eta_j + 2(v \wedge \eta) \partial_{\eta_j} (v \wedge \eta)) \right],$$

which along with Leibniz' formula and the induction assumption yields

$$\begin{aligned}
|\partial_v^\alpha \partial_\eta^\beta a^k(v, \eta)| & \lesssim a^{k-1} (1 + |\eta| + |v||\eta| + |v||v \wedge \eta| + |v|^2) \\
& \lesssim a^k(v, \eta),
\end{aligned}$$

on the other hand, we have for all $k \in \mathbb{R}$,

$$\langle v \rangle^k \in S(\langle v \rangle^k, \Gamma). \quad (39)$$

Finally, using (38), (39) and Leibniz' formula we conclude for all $|\alpha + \beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \lambda^m(v, \eta)| \leq C_{\alpha, \beta} \lambda^m(v, \eta). \quad (40)$$

\square

Lemma 3.3. For $m \in \mathbb{R}$,

$$|\xi \cdot \partial_\eta \lambda^m| \lesssim \lambda^m,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Lemma 3.3 follows directly from (31) and the fact that

$$|\xi \cdot \partial_\eta \lambda^2| = \lambda \langle v \rangle^{\frac{2}{3}} a^{-1/2} |\xi \cdot \eta + v \wedge \xi \cdot v \wedge \eta| \lesssim \lambda^2,$$

we can conclude that for all $m \in \mathbb{R}$,

$$|\xi \cdot \partial_\eta \lambda^m| \lesssim \lambda^m,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 3.4. We have

$$g_i \in S(g_i, \Gamma), \text{ for } i = 1, \dots, 3,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Using the fact that for all $m \in \mathbb{R}$, $\langle v \rangle^m \in S(\langle v \rangle^m, \Gamma)$, $\langle \eta \rangle^m \in S(\langle \eta \rangle^m, \Gamma)$ we obtain

$$\forall \alpha, \beta \in \mathbb{N}^3, \quad |\partial_v^\alpha \partial_\eta^\beta g_i(v, \eta)| \leq C_{\alpha, \beta} g_i,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 3.5. For all $\varepsilon > 0$,

- i) $\partial_\eta \lambda \in S(\varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{7}{2}+1}, \Gamma)$,
- ii) $\partial_\eta g_1 \in S(\varepsilon g_1 + \varepsilon^{-1}, \Gamma)$,
- iii) $\partial_\eta g_2 \in S(\varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{6}+1}, \Gamma)$,
- iv) $\partial_\eta g_3 \in S(\varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{3}+1}, \Gamma)$;

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We have

$$\begin{aligned} |\partial_\eta g_1(v, \eta)| &= |\partial_\eta (1 + \langle v \rangle + \langle \eta \rangle)| \lesssim |\eta| \langle \eta \rangle^{-1} \lesssim g_1^{1/2}, \\ |\partial_\eta g_2(v, \eta)| &= |\partial_\eta (1 + \langle \xi \rangle^{1/3} + \langle \eta \rangle)| \lesssim |\eta| \langle \eta \rangle^{-1} \lesssim g_2^{1/2} \langle v \rangle^{\frac{7}{12} + \frac{1}{2}}, \\ |\partial_\eta g_3(v, \eta)| &= |\partial_\eta (1 + \langle \xi \rangle^{2/3} + \langle \eta \rangle^2)| \lesssim \langle \eta \rangle \lesssim g_3^{1/2} \langle v \rangle^{\frac{7}{6} + \frac{1}{2}}. \end{aligned}$$

Using Young's inequality we get for all $\varepsilon > 0$,

$$|\partial_\eta g_1(v, \eta)| \lesssim \varepsilon g_1 + \varepsilon^{-1}, \quad |\partial_\eta g_2(v, \eta)| \lesssim \varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{6}+1}, \quad |\partial_\eta g_3(v, \eta)| \lesssim \varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{3}+1},$$

then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_1(v, \eta)| \lesssim \varepsilon g_1 + \varepsilon^{-1}, \quad |\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_2(v, \eta)| \lesssim \varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{6}+1},$$

$$\text{and } |\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_3(v, \eta)| \lesssim \varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{7}{3}+1}.$$

Regarding the symbol λ , we have

$$|\partial_\eta \lambda(v, \eta)| = \langle v \rangle^{\frac{2}{3}} a^{-1/2} |\eta + (v \wedge \eta) \partial_\eta (v \wedge \eta)| \lesssim \langle v \rangle^{\frac{2}{3}} a^{-1/2} (|\eta| + |v| |v \wedge \eta|) \lesssim \langle v \rangle^{\frac{7}{4} + \frac{1}{2}} \lambda^{1/2},$$

Using Young's inequality we get for all $\varepsilon > 0$,

$$|\partial_\eta \lambda(v, \eta)| \lesssim \varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{7}{2}+1},$$

then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta \lambda(v, \eta)| \lesssim \varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{7}{2}+1}.$$

□

From the above, we have shown that the symbols g_1 , g_2 , g_3 and λ verify the hypotheses of Theorem [A.6](#), so we can apply the results of Theorem [A.6](#) to the following operators

$$g_{1,K}^w = (g_1 + K)^w, \quad (41)$$

$$g_{2,K}^w = (g_2 + K \langle v \rangle^{\frac{7}{6}+1})^w, \quad (42)$$

$$g_{3,K}^w = (g_3 + K \langle v \rangle^{\frac{7}{3}+1})^w, \quad (43)$$

$$\lambda_K^w = (\lambda + K \langle v \rangle^{\frac{7}{2}+1})^w, \quad (44)$$

where K the fixed constant given by Theorem [A.6](#). In Section [3.2](#), we will apply the results of Theorem [A.6](#) on the operators above.

Let ψ be a $C_0^\infty(\mathbb{R}, [0, 1])$ function such that

$$\psi = 1 \text{ on } [-1, 1], \quad \text{supp } \psi \subset [-2, 2]. \quad (45)$$

Definition 3.6. Define the real-valued symbol

$$g = -\frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right), \quad (46)$$

where λ is the symbol defined in [\(31\)](#).

Lemma 3.7 (Lemme 3.3 dans [9]). *We have*

$$\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \in S(1, \Gamma),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Lemma 3.8. *The symbol g belongs to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 .*

Proof. Notice from [\(45\)](#) that

$$|B(v)\eta|^2 + F(v) \leq 2\lambda^{2/3}, \quad (47)$$

on the support of the function

$$\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right). \quad (48)$$

By recalling (31) and using (15), we obtain

$$|B(v)\xi| \lesssim \langle v \rangle^{\frac{7}{2}} |\xi| + \langle v \rangle^{\frac{7}{2}} \|v \wedge \xi\| \lesssim \lambda. \quad (49)$$

We deduce from the Cauchy-Schwarz inequality that one can estimate

$$|B(v)\xi \cdot B(v)\eta| \lesssim |B(v)\xi| |B(v)\eta| \lesssim \lambda^{4/3}, \quad (50)$$

on the support of ψ . The symbol g is therefore a bounded function uniformly with respect to the parameter ξ in \mathbb{R}^3 . Using (14) and (15), we can estimate

$$|\partial_v^\alpha B(v)\xi| \lesssim \langle v \rangle^{\frac{7}{2}} |\xi| \lesssim |B(v)\xi| \lesssim \lambda \quad (51)$$

where $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 1$. Using again (14) and (15), we can estimate

$$|\partial_v^\alpha B(v)\eta| \lesssim \langle v \rangle^{\frac{7}{2}} |\eta| \lesssim |B(v)\eta| \lesssim \lambda^{1/3} \quad (52)$$

on the support of ψ . Moreover, one can estimate from above the modulus of all the derivatives of the term $B(v)\eta$ by a constant times $\lambda^{1/3}$ on the support of the function

$$\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right).$$

Using Leibniz's formula, Cauchy-Schwarz inequality (51), (52) one can estimate from above the modulus of all the derivatives of the term $B(v)\xi \cdot B(v)\eta$ by a constant times $\lambda^{4/3}$ on the support of the function ψ . According to Lemma 3.2 and Lemma 3.7, this proves that the symbol g belongs to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 3.9. *We have*

$$\left| \xi \cdot \partial_\eta \left[\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \right| \lesssim 1 + |B(v)\eta|^2 + F(v),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Let $\omega = \frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}$. We may write

$$\xi \cdot \partial_\eta [\psi(\omega)] = \psi'(\omega) \left[\frac{2B(v)\xi \cdot B(v)\eta}{\lambda^{2/3}} + (|B(v)\eta|^2 + F(v)) (\xi \cdot \partial_\eta) (\lambda^{-2/3}) \right].$$

Notice from (31) and (45)

$$\left| \frac{2B(v)\xi \cdot B(v)\eta}{\lambda^{2/3}} \right| \lesssim \frac{|B(v)\xi| |B(v)\eta|}{\lambda^{2/3}} \lesssim \frac{\lambda \lambda^{1/3}}{\lambda^{2/3}} \lesssim \lambda^{2/3} \lesssim |B(v)\eta|^2 + F(v),$$

on the support of the function $\psi'(\omega)$. One can then deduce Lemma 3.9 from Lemma 3.3. \square

3.2 Hypoelliptic estimates

We shall consider the multiplier $G = g^{\text{Wick}}$ defined by the Wick quantization of the symbol g . We refer the reader to Appendix [A](#) on Wick calculus. We begin by noticing from [\(133\)](#) that there exists a real-valued symbol \tilde{g} belonging to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 such that

$$G = g^{\text{Wick}} = \tilde{g}^w; \quad (53)$$

where \tilde{g}^w denotes the operator obtained by the Weyl quantization of the symbol \tilde{g} given by

$$(\tilde{g}^w u)(v) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} e^{i(v-v') \cdot \eta} \tilde{g}\left(\frac{v+v'}{2}, \eta\right) u(v') dv' d\eta. \quad (54)$$

We shall sometimes closely follow [\[9\]](#) and refer to appendix (Section A) for the main features of the Wick and the Weyl quantizations.

We begin by a series of Lemmas whose proof is exactly the same as the one in [\[9\]](#).

Lemma 3.10. *There exists $c_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$|(F(v)u, Gu)_{L^2}| + |(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2}| \leq c_1 \Re(\mathcal{A}_\xi u, u)_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.7 in [\[9\]](#). □

Lemma 3.11. *There exists $c_2 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\left| \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \left([4\pi^2|B(v)\eta|^2 + F(v)]^{\text{Wick}} u, u \right)_{L^2} \right| \leq c_2 \Re(\mathcal{A}_\xi u, u)_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.11 in [\[9\]](#). □

Let δ be a positive parameter such that $0 < \delta \leq 1$. We use a multiplier method and write that

$$\begin{aligned} \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} &= \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re(iv \cdot \xi u, Gu)_{L^2} \\ &\quad + \delta \Re(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} - \delta \Re(F(v)u, Gu)_{L^2}. \end{aligned} \quad (55)$$

Lemma 3.12. *We have for any $s \in \mathbb{R}$*

$$\|\langle B(v)\xi \rangle^s (1 - \delta G)u\|_{L^2} \lesssim \|\langle B(v)\xi \rangle^s u\|_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.8 in [\[9\]](#). □

Proposition 3.13. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$((\lambda^{2/3})^{\text{Wick}} u, u)_{L^2} \leq C \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Using (55), we have for $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re (iv \cdot \xi u, Gu)_{L^2} \\ &= \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} - \delta \Re (\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} + \delta \Re (F(v)u, Gu)_{L^2}, \end{aligned} \quad (56)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . Recalling (53) and noticing from (133) and (134) that $v^{\text{Wick}} = v$, we may rewrite (56) as

$$\begin{aligned} & \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re (i\xi \cdot v^{\text{Wick}}u, Gu)_{L^2} \\ &= \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} - \delta \Re (\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} + \delta \Re (F(v)u, Gu)_{L^2}. \end{aligned} \quad (57)$$

Using Lemma 3.10, we deduce that there is a constant $c_3 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re (i\xi \cdot v^{\text{Wick}}u, Gu)_{L^2} \\ & \leq c_3 \left(\Re (\mathcal{A}_\xi u, u)_{L^2} + \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right), \end{aligned} \quad (58)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . We deduce from Lemma 3.8 and (135) that

$$\begin{aligned} -\delta \Re (i\xi \cdot v^{\text{Wick}}u, Gu)_{L^2} &= -\delta \Re (g^{\text{Wick}}(i\xi \cdot v)^{\text{Wick}}u, u)_{L^2} \\ &= \delta \frac{1}{4\pi} \left(\{\xi \cdot v, g\}^{\text{Wick}}u, u \right)_{L^2}. \end{aligned} \quad (59)$$

Using (46) with a direct computation of the Poisson bracket gives that

$$\begin{aligned} \{\xi \cdot v, g\} &= (B(v)\xi \cdot B(v)\eta)(\xi \cdot \partial_\eta(\lambda^{-4/3}))\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \\ &+ \frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + \frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}}\xi \cdot \partial_\eta \left[\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]. \end{aligned} \quad (60)$$

We notice from Lemma 3.3, Lemma 3.9, (46) and (50) that

$$\left| \{\xi \cdot v, g\} - \frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right| \lesssim 1 + |B(v)\eta|^2 + F(v), \quad (61)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . It follows from (59), (58) and the fact that the Wick quantization is a positive quantization (132) that there exists constant $c_4 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \frac{\delta}{4\pi} \left(\left[\frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]^{\text{Wick}} u, u \right)_{L^2} + \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 \\ & \leq c_3 \left(\Re (\mathcal{A}_\xi u, u)_{L^2} + \Re (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right) \\ & \quad + \delta c_4 \left(\left[1 + |B(v)\eta|^2 + F(v) \right]^{\text{Wick}} u, u \right)_{L^2}, \end{aligned} \quad (62)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . It follows from Lemma [3.11](#) that there exists $c_5 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$

$$\begin{aligned} & \delta \left(\left[\frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]^{Wick} u, u \right)_{L^2} + (4\pi^2 [|B(v)\eta|^2 + F(v)]^{Wick} u, u)_{L^2} \\ & \quad + \|u\|_{L^2}^2 \leq c_5 \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right), \end{aligned} \quad (63)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Notice from [\(31\)](#), [\(45\)](#) and [\(15\)](#) that

$$\begin{aligned} & \delta \frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + 4\pi^2 |B(v)\eta|^2 + F(v) + 1 \geq \\ & \delta \frac{|B(v)\xi|^2 + |B(v)\eta|^2 + F(v) + 1}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \\ & \quad + (|B(v)\eta|^2 + F(v) + 1) \left[1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \gtrsim \\ & \delta \frac{\lambda^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + \delta \lambda^{2/3} \left[1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \gtrsim \delta \lambda^{2/3}, \end{aligned}$$

when $0 < \frac{\delta}{\lambda^{4/3}} \leq 1$; since

$$|B(v)\eta|^2 + F(v) \geq \lambda^{2/3},$$

on the support of the function

$$1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right).$$

By using again that the Wick quantization is a positive quantization ([132](#)), we deduce that there exists $C > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$((\lambda^{2/3})^{Wick} u, u)_{L^2} \leq C \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . □

Lemma 3.14. *There exists $C > 0$ such that for all $v \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$,*

$$\langle B(v)\xi \rangle^{1/3} \leq Cm(v, \xi),$$

with

$$m(v, \xi) = \left(\int_{\mathbb{R}^3} \langle B(v + \tilde{v})\xi \rangle^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \right)^{1/2}.$$

Proof. We have that

$$m(v, \xi)^2 \gtrsim \int_{\mathbb{R}^3} (1 + |B(v + \tilde{v})\xi|^{2/3}) \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v},$$

therefore by using ([15](#)), we obtain that

$$m(v, \xi)^2 \gtrsim \int_{\mathbb{R}^3} (1 + \langle v + \tilde{v} \rangle^{\gamma/3} |\xi|^{2/3} + \langle v + \tilde{v} \rangle^{\gamma/3} |(v + \tilde{v}) \wedge \xi|^{2/3}) \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v},$$

and using Peetre's inequality (37), we have

$$\frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} \lesssim \langle v + \tilde{v} \rangle^{\gamma/3},$$

so we get

$$\begin{aligned} m(v, \xi)^2 &\gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \int_{\mathbb{R}^3} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \\ &\gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \int_{\overline{B(0,1)}} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \end{aligned}$$

where $\overline{B(0,1)}$ stands for the closed unit ball in \mathbb{R}^3 . By noticing that we have

$$|(v + \tilde{v}) \wedge \xi| \geq |v \wedge \xi| - |\tilde{v} \wedge \xi| \geq |v \wedge \xi| - |\xi| \geq \frac{1}{2} |v \wedge \xi|,$$

when $|\tilde{v}| \leq 1$ and $2|\xi| \leq |v \wedge \xi|$, it follows that

$$\int_{\mathbb{R}^3} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \gtrsim \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3},$$

when $2|\xi| \leq |v \wedge \xi|$. Since

$$\langle v \rangle^{\gamma/3} |\xi|^{2/3} \gtrsim \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3}$$

when $2|\xi| \geq |v \wedge \xi|$. Based on the above and using (15), we obtain

$$m(v, \xi)^2 \gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3} \gtrsim 1 + |B(v)\xi|^{2/3} \gtrsim \langle B(v)\xi \rangle^{2/3}.$$

□

Lemma 3.15. For all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\frac{5\gamma}{6}+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}+1} B(v)D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (64)$$

Proof. See Lemma 3.3 in [7]. □

For the rest, we need to improve this estimate. We have the following Lemma.

Lemma 3.16. For all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v)D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (65)$$

Proof. We will start by estimating the term

$$\left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2}.$$

Let $u \in \mathcal{S}(\mathbb{R}_v^3)$, we have

$$\begin{aligned} \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\leq \left| \left(B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u, B(v) D_v \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) D_v u, B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \\ &\leq \left| \left(B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u, B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) D_v u, B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|, \end{aligned}$$

using the fact that the symbol of the operator $[D_v, \langle v \rangle^{\frac{\gamma}{2}+1}]$ belongs to the class $S(\langle v \rangle^{\frac{\gamma}{2}}, \Gamma)$ uniformly with respect to the parameter ξ and using the following notations

$$H_1 = \underbrace{B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)},$$

$$H_2 = \underbrace{\langle v \rangle^{-\frac{\gamma}{2}-1} B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] \langle v \rangle^{-\frac{\gamma}{2}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\leq \left| \left(H_1 \langle v \rangle^{\gamma+1} u, H_1 \langle v \rangle^{\gamma+1} u \right)_{L^2} \right| \\ &\quad + \left| \left(H_1 \langle v \rangle^{\gamma+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \right)_{L^2} \right| \\ &\quad + \left| \left(\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u, H_2 \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|, \end{aligned}$$

using the fact that $\gamma + 1 \leq 2 + \frac{5\gamma}{6}$ and Lemma [3.15](#), we obtain for all $\varepsilon > 0$

$$\begin{aligned} \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\lesssim \varepsilon (\| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2) \\ &\quad + C_\varepsilon (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2). \end{aligned}$$

On the other hand, we have

$$\left| \left(\mathcal{A}_\xi \langle v \rangle^{\frac{\gamma}{2}+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \leq \left| \left(\mathcal{A}_\xi u, \langle v \rangle^{\gamma+2} u \right)_{L^2} \right| + \left| \left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|$$

and

$$\left| \left(\mathcal{A}_\xi u, \langle v \rangle^{\gamma+2} u \right)_{L^2} \right| \leq \varepsilon \| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + C_\varepsilon \| \mathcal{A}_\xi u \|_{L^2}^2.$$

Now using [\(30\)](#), we can write

$$\begin{aligned} &\| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2 \\ &\leq \| \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}+1} u \|_{L^2}^2 + \| B(v) D_v \langle v \rangle^{\frac{\gamma}{2}+1} u \|_{L^2}^2 + \| B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \|_{L^2}^2 \\ &\leq \left| \left(\mathcal{A}_\xi \langle v \rangle^{\frac{\gamma}{2}+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| + \| B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \|_{L^2}^2. \end{aligned}$$

For the last term, we have

$$\| B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u \|_{L^2}^2 \leq \| H_1 \langle v \rangle^{\gamma+1} u \|_{L^2}^2 \lesssim \varepsilon \| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + C_\varepsilon (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2).$$

Finally, taking ε small enough, we obtain for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2 \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2. \quad (66)$$

□

Proposition 3.17. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\| \langle v \rangle^{\frac{\gamma}{3}} |\xi|^{2/3} u \|_{L^2}^2 \leq C (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We deduce from Proposition [3.13](#), Lemma [3.14](#) and Lemma [3.12](#) that

$$\|\langle B(v)\xi \rangle^{1/3} u\|_{L^2}^2 \lesssim \left(\|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi u\|_{L^2}^2 \|\langle B(v)\xi \rangle^{1/3} u\|_{L^2}^2 \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . By substituting $\langle B(v)\xi \rangle^{1/3} u$ to u in this estimate, we obtain that

$$\|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}^2 \lesssim \|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi \langle B(v)\xi \rangle^{1/3} u\|_{L^2}^2 \|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}^2, \quad (67)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . First, we have

$$\begin{aligned} & \langle B(v)\xi \rangle^{-1/3} \left[D_v \cdot \mathbf{A}(v) D_v, \langle B(v)\xi \rangle^{1/3} \right] \\ &= \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 D_{v_j} \cdot \bar{a}_{j,k}(v) \left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \left[D_{v_j}, \langle B(v)\xi \rangle^{1/3} \right] \cdot \bar{a}_{j,k}(v) D_{v_k} \\ &= \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 D_{v_j} \bar{a}_{j,k}(v) \left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \bar{a}_{j,k}(v) \left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] D_{v_j} \\ &+ \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \bar{a}_{j,k}(v) D_{v_j} \left(\left[D_{v_k}, \langle B(v)\xi \rangle^{1/3} \right] \right) \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \left[D_{v_j}, \langle B(v)\xi \rangle^{1/3} \right] \cdot \bar{a}_{j,k}(v) D_{v_k}. \end{aligned}$$

Symbolic calculus shows that

$$\left[D_v, \langle B(v)\xi \rangle^{1/3} \right] = \frac{1}{i} \nabla_v (\langle B(v)\xi \rangle^{1/3}),$$

and using the following notations

$$\begin{aligned} H_3 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} \partial_{v_j} \bar{a}_{j,k}(v) \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)}, \\ H_4 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} \bar{a}_{j,k}(v) \partial_{v_j} \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)}, \\ H_5 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} b_{i,k}(v) \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)}, \\ H_6 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} b_{i,j}(v) \partial_{v_j} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)}, \end{aligned}$$

where we used the fact that $\mathbf{A}(v) = B^T(v)B(v)$. Then, going back to (67), we have

$$\begin{aligned}
& \| \langle B(v)\xi \rangle^{2/3} u \|_{L^2}^2 \\
& \lesssim \| \langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi \langle B(v)\xi \rangle^{1/3} u \|_{L^2} \| \langle B(v)\xi \rangle^{2/3} u \|_{L^2}, \\
& \lesssim \| \langle B(v)\xi \rangle^{-1/3} [\mathcal{A}_\xi, \langle B(v)\xi \rangle^{1/3}] u \|_{L^2}^2 + \| \mathcal{A}_\xi u \|_{L^2}^2, \\
& \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| H_3 \langle v \rangle^{\gamma+1} u \|_{L^2}^2 + \| H_4 \langle v \rangle^{\gamma+2} u \|_{L^2}^2 \\
& + \sum_{i=1}^3 \| H_5 \left(\sum_{j=1}^3 \langle v \rangle^{\frac{\gamma}{2}+1} b_{i,j}(v) D_{v_j} u \right) \|_{L^2}^2 + \sum_{i=1}^3 \| H_6 \left(\sum_{k=1}^3 \langle v \rangle^{\frac{\gamma}{2}+1} b_{i,k}(v) D_{v_k} u \right) \|_{L^2}^2, \\
& \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| \langle v \rangle^{\gamma+1} u \|_{L^2}^2 + \| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2
\end{aligned}$$

We finally conclude from Lemma 3.16 that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\| \langle B(v)\xi \rangle^{2/3} u \|_{L^2}^2 \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2.$$

This ends the proof of Proposition 3.17. \square

Lemma 3.18. *Let λ_K be the symbol defined in (44). Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C_{\tilde{\varepsilon}}$, such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\begin{aligned}
& \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
& \lesssim \tilde{\varepsilon} \| \left(\lambda_K^{2/3} \right)^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right).
\end{aligned} \tag{68}$$

Proof. As a preliminary step we firstly show that for any $\varepsilon, \tilde{\varepsilon} > 0$ there exists a constant $C_{\varepsilon, \tilde{\varepsilon}}$, such that

$$\begin{aligned}
& \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
& \lesssim \tilde{\varepsilon} \| \left(\lambda_K^{2/3} \right)^w u \|_{L^2}^2 + C_{\varepsilon, \tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right) \\
& + \varepsilon \left\{ \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right\},
\end{aligned} \tag{69}$$

where a is an arbitrary symbol belonging to $S(1, \Gamma)$ uniformly with respect to the parameter ξ . Using the notation

$$\mathcal{Z}_1 = \left(\left[D_v \cdot \mathbf{A}(v) D_v, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.$$

We have

$$\begin{aligned}
\mathcal{Z}_1 & = \sum_{i,j,k=1}^3 \left(\left[b_{i,k}(v) D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] u, b_{i,j}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
& + \sum_{i,j,k=1}^3 \left(b_{i,k} D_{v_k} u, \left[\left(\lambda_K^{1/3} \right)^w, b_{i,j}(v) D_{v_j} \right] a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.
\end{aligned}$$

Using the fact that

$$\left[B(v) D_v, \left(\lambda_K^{1/3} \right)^w \right] u = B(v) \left[D_v, \left(\lambda_K^{1/3} \right)^w \right] u - \left[\left(\lambda_K^{1/3} \right)^w, B(v) \right] D_v u,$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_1| &\leq \left| \sum_{i,j,k=1}^3 \left(b_{i,k}(v) \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] u, b_{i,j}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(\left[\left(\lambda_K^{1/3} \right)^w, b_{i,k}(v) \right] D_{v_k} u, b_{i,j}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(b_{i,k}(v) D_{v_k} u, b_{i,j}(v) \left[D_{v_j}, \left(\lambda_K^{1/3} \right)^w \right] a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(b_{i,k}(v) D_{v_k} u, \left[\left(\lambda_K^{1/3} \right)^w, b_{i,j}(v) \right] D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&\leq |\mathcal{Z}_{1,1}| + |\mathcal{Z}_{1,2}| + |\mathcal{Z}_{1,3}| + |\mathcal{Z}_{1,4}|.
\end{aligned}$$

Estimate of $\mathcal{Z}_{1,1}$: Observing $a \in S(1, \Gamma)$, $\partial_v \lambda_K^{1/3} \in S(\langle v \rangle^{\gamma/6} g_{2,K}^w, \Gamma)$ and using (14) with symbolic calculus shows that

$$\left[D_{v_j}, a^w \right] \in \Psi(1, \Gamma), \quad [b_{i,j}(v), a^w] \in \Psi(\langle v \rangle^{\frac{\gamma}{2}}, \Gamma) \quad \text{and} \quad \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] \in \Psi(\langle v \rangle^{\frac{\gamma}{6}} g_{2,K}, \Gamma)$$

uniformly with respect to the parameter ξ , where $g_{2,K}^w$ the operator defined in (42). Now using the following notation

$$H_7 = \underbrace{b_{i,k}(v) \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] \left(g_{2,K}^w \right)^{-1} \langle v \rangle^{-\gamma/6} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,1}| &\leq \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, b_{i,j}(v) \left[D_{v_j}, a^w \right] \langle v \rangle^{-\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}+1} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, [b_{i,j}(v), a^w] \langle v \rangle^{-\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}} D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, a^w b_{i,j}(v) D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\
&\lesssim C_\varepsilon \left(\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\gamma+1} D_v u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}} |\xi|^{2/3} u\|_{L^2}^2 \right) \\
&+ \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.
\end{aligned}$$

Moreover using Lemma 3.16 and Proposition 3.17, we obtain

$$|\mathcal{Z}_{1,1}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\hat{\mathcal{A}}u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Estimate of $\mathcal{Z}_{1,2}$: Observing $\partial_\eta \lambda_K^{1/3} \in S(\langle v \rangle^{\frac{\gamma}{6}+\frac{1}{3}}, \Gamma)$ and using (14) with symbolic calculus shows that the symbol of the commutator $\left[\left(\lambda_K^{1/3} \right)^w, b_{i,k}(v) \right]$ belongs to $S(\langle v \rangle^{\frac{2\gamma}{3}+\frac{1}{3}}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_8 = \underbrace{\left[\left(\lambda_K^{1/3} \right)^w, b_{i,k}(v) \right] \langle v \rangle^{-\frac{2\gamma}{3}-\frac{1}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,2}| &\leq \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}} D_{v_k} u, b_{i,j}(v) [D_{v_j}, a^w] \langle v \rangle^{-\frac{\gamma}{2} - 1} \langle v \rangle^{\frac{\gamma}{2} + 1} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}} D_{v_k} u, [b_{i,j}(v), a^w] \langle v \rangle^{-\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}} D_{v_j} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&+ \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3} + \frac{1}{3}} D_{v_k} u, a^w b_{i,j}(v) D_{v_j} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim C_\varepsilon \|\langle v \rangle^{\gamma+1} D_v u\|_{L^2}^2 + \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2},
\end{aligned}$$

moreover using Proposition [3.17](#), we obtain

$$|\mathcal{Z}_{1,2}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon (\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2).$$

Estimate of $\mathcal{Z}_{1,3}$: Using the following notation

$$H_9 = \underbrace{\langle v \rangle^{-\frac{\gamma}{2} - 1} b_{i,j}(v) [D_{v_j}, (\lambda_K^{1/3})^w] a^w (g_{2,K}^w)^{-1} \langle v \rangle^{-\gamma/6}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,3}| &\leq \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{2} + 1} b_{i,k} D_{v_k} u, H_9 \langle v \rangle^{\frac{\gamma}{6}} g_{2,K}^w (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{2} + 1} b_{i,k} D_{v_k} u, \langle v \rangle^{\frac{\gamma}{6}} g_{2,K}^w (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim \varepsilon \left(\|\langle v \rangle^{\frac{\gamma}{6}} (\lambda_K^{1/3})^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{6}} D_v (\lambda_K^{1/3})^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{6}} |\xi|^{1/3} (\lambda_K^{1/3})^w u\|_{L^2}^2 \right) \\
&+ C_\varepsilon \|\langle v \rangle^{\frac{\gamma}{2} + 1} B(v) D_v u\|_{L^2}^2,
\end{aligned}$$

moreover using Lemma [3.16](#) and Proposition [3.13](#), we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,3}| &\lesssim \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + \varepsilon \Re e \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (1 - \delta G) (\lambda_K^{1/3})^w u \right)_{L^2} \\
&+ C_\varepsilon (\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2).
\end{aligned}$$

Estimate of $\mathcal{Z}_{1,4}$: Using the following notations

$$H_{10} = \underbrace{\langle v \rangle^{-\frac{\gamma}{3} - 1} \left[(\lambda_K^{1/3})^w, b_{i,j}(v) \right] [D_{v_j}, a^w] \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

$$H_{11} = \underbrace{\langle v \rangle^{-\frac{\gamma}{3} - 1} \left[(\lambda_K^{1/3})^w, b_{i,j}(v) \right] a^w \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_{1,4}| &\leq \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{3}+1} b_{i,k} D_{v_k} u, H_{10} \langle v \rangle^{\frac{\gamma}{3}} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\quad + \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{3}+1} b_{i,k} D_{v_k} u, H_{11} \langle v \rangle^{\frac{\gamma}{3}} D_{v_j} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\lesssim \varepsilon \left(\|\langle v \rangle^{\frac{\gamma}{3}} (\lambda_K^{1/3})^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}} D_v (\lambda_K^{1/3})^w u\|_{L^2} \right) \\
&\quad + C_\varepsilon \|\langle v \rangle^{\frac{\gamma}{3}+1} B(v) D_v u\|_{L^2}^2,
\end{aligned}$$

Moreover using Lemma [3.15](#), we obtain

$$|\mathcal{Z}_{1,4}| \lesssim \varepsilon \Re \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right),$$

so using the estimates of $\mathcal{Z}_{1,n}$ for $n = 1, \dots, 4$, we obtain

$$|\mathcal{Z}_1| \lesssim \varepsilon \Re \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Let's look now

$$\mathcal{Z}_2 = \left([F(v), (\lambda_K^{1/3})^w] u, a^w (\lambda_K^{1/3})^w u \right)_{L^2},$$

observing the symbol $\partial_\eta \lambda_K^{1/3} \in S(\langle v \rangle^{\frac{2}{6}+\frac{1}{3}}, \Gamma)$, and using Lemma [2.4](#) with symbolic calculus shows that the symbol of the commutator $[F(v), (\lambda_K^{1/3})^w]$ belongs to $S(\langle v \rangle^{\frac{3\gamma}{2}+2}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_{12} = \underbrace{\langle v \rangle^{-\frac{\gamma}{2}-1} [F(v), (\lambda_K^{1/3})^w] \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{Z}_2| &\leq \left| \left(\langle v \rangle^{\frac{\gamma}{2}+1} H_{12} \langle v \rangle^{-\gamma-1} \langle v \rangle^{\gamma+1} u, a^w (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\leq \left| \left(H_{12} \langle v \rangle^{\gamma+1} u, a^w \langle v \rangle^{\frac{\gamma}{2}+1} (\lambda_K^{1/3})^w u \right)_{L^2} \right| \\
&\quad + \left| \left(H_{12} \langle v \rangle^{\gamma+1} u, [\langle v \rangle^{\frac{\gamma}{2}+1}, a^w] \langle v \rangle^{-\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}+1} (\lambda_K^{1/3})^w u \right)_{L^2} \right|,
\end{aligned}$$

in addition, we have that the symbol of the commutator $[\langle v \rangle^{\frac{\gamma}{2}+1}, a^w]$ belongs to $S(\langle v \rangle^{\frac{\gamma}{2}+1}, \Gamma)$ and using Lemma [3.16](#), we obtain

$$|\mathcal{Z}_2| \lesssim \varepsilon \Re \left(\mathcal{A}_\xi (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (70)$$

Let's look now

$$\mathcal{Z}_3 = \left([iv \cdot \xi, (\lambda_K^{1/3})^w] u, a^w (\lambda_K^{1/3})^w u \right)_{L^2},$$

using Lemma [3.3](#) with symbolic calculus shows that the symbol of the commutator $[iv \cdot \xi, (\lambda_K^{1/3})^w]$ belongs to $S(\lambda_K^{1/3}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_{13} = \underbrace{(\lambda_K^{1/3})^w (a^w)^* [iv \cdot \xi, (\lambda_K^{1/3})^w] ((\lambda_K^{2/3})^w)^{-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$|\mathcal{Z}_3| \leq \left| \left(H_{13} \left(\lambda_K^{2/3} \right)^w u, u \right)_{L^2} \right| \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right). \quad (71)$$

From the above, using the estimates of \mathcal{Z}_l for $l = 1, \dots, 3$, we obtain (69). Next we prove (68), we have the following relation

$$\begin{aligned} & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &= \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &+ \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &= \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w \left(\lambda_K^{1/3} \right)^w \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1} \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &+ \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w (1 - \delta G) \left(\lambda_K^{1/3} \right)^w \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1} \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \\ &+ \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \end{aligned}$$

gives, with $\tilde{\varepsilon} > 0$ arbitrary,

$$\begin{aligned} & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right) + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}. \end{aligned}$$

We could apply (69) with $a = 1$ and $a = 1 - \delta \tilde{g}$ to control the last term in the above inequality; this gives, with $\varepsilon, \tilde{\varepsilon} > 0$ arbitrarily small,

$$\begin{aligned} & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_{\varepsilon, \tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right) \\ & + \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}. \end{aligned}$$

Letting ε small enough yields the desired estimate (68). \square

Proposition 3.19. *Let λ_K be the symbol defined in (44). Then there exists $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\| (\lambda_K^{2/3})^w u \|_{L^2}^2 \leq C_0 \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right). \quad (72)$$

Proof. Using Proposition 3.13, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\left((\lambda_K^{2/3})^{\text{Wick}} u, u \right)_{L^2} \lesssim \Re e \left(\mathcal{A}_\xi u, u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi u, (1 - \delta G) u \right)_{L^2}, \quad (73)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . By substituting $\left(\lambda_K^{1/3} \right)^w u$ to u in the above estimate, we obtain that for all $\tilde{\varepsilon} > 0$,

$$\left((\lambda_K^{2/3})^{\text{Wick}} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right). \quad (74)$$

Notice from (133) that we may write

$$(\lambda_K^{2/3})^{\text{Wick}} = (\lambda_K^{2/3})^w + r^w, \quad (75)$$

with

$$r(v, \eta) = \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) (\lambda_K^{2/3})'' (Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 d\theta,$$

où $Y, Y_1 \in \mathbb{R}^6$. Define

$$r_1 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) \nabla_\eta^2 (\lambda_K^{2/3}) (Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 d\theta,$$

$$r_2 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) \nabla_v \nabla_\eta (\lambda_K^{2/3}) (Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 d\theta,$$

and

$$r_3 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) \nabla_v^2 (\lambda_K^{2/3}) (Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 d\theta.$$

Using Lemma 3.5 with symbolic calculus shows that the symbol

$$\nabla_\eta^2 (\lambda_K^{2/3}), \nabla_v \nabla_\eta (\lambda_K^{2/3}) \in S(\langle v \rangle^{\gamma+2}, \Gamma),$$

uniformly with respect to the parameter ξ , then r_1, r_2 belong to $S(\langle v \rangle^{\gamma+2}, \Gamma)$. Using the following notations

$$H_{14} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_1^w \left(\lambda_K^{1/3} \right)^w \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

$$H_{15} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_2^w \left(\lambda_K^{1/3} \right)^w \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

we obtain for all $\tilde{\varepsilon} > 0$,

$$\begin{aligned} & \left| \left(r_1^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| + \left| \left(r_2^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ & \leq \left| \left(H_{14} \langle v \rangle^{\gamma+2} u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| + \left| \left(H_{15} \langle v \rangle^{\gamma+2} u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Taking into account that the symbol $\nabla_v^2 (\lambda_K^{2/3})$, belongs to $S(\langle v \rangle^{\frac{2}{3}} g_{3,K}, \Gamma)$ uniformly with respect to the parameter ξ , where $g_{3,K}$ the operator defined in (43), then the symbol r_3 belongs to $S(\langle v \rangle^{\frac{2}{3}} g_{3,K}, \Gamma)$. Using the following notation

$$H_{16} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_3^w \left(\lambda_K^{1/3} \right)^w \left(g_{3,K}^w \right)^{-1} \langle v \rangle^{-\frac{2}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} \left| \left(r_3^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| & \leq \left| \left(H_{16} \langle v \rangle^{\frac{2}{3}} g_{3,K}^w u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

From the above, using the estimates of r_l for $l = 1, \dots, 3$, we obtain

$$\left| \left(r^w (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} \right| \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2). \quad (76)$$

By applying Theorem [A.6](#) with $p = \lambda$, we obtain

$$\left((\lambda_K^{2/3})^w (\lambda_K^{1/3})^w u, (\lambda_K^{1/3})^w u \right)_{L^2} \sim \left\| \left[(\lambda_K^{1/3})^w \right]^2 u \right\|_{L^2}^2 \sim \| (\lambda_K^{2/3})^w u \|_{L^2}^2, \quad (77)$$

uniformly with respect to the parameter ξ . Using [\(76\)](#), then taking $\tilde{\varepsilon}$ small enough, we get for all $\tilde{\varepsilon} > 0$,

$$\| (\lambda_K^{2/3})^w u \|_{L^2}^2 \lesssim \tilde{\varepsilon} \| (\lambda_K^{2/3})^w u \|_{L^2}^2 + \tilde{C}_{\tilde{\varepsilon}} (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2),$$

now taking $\tilde{\varepsilon}$ small enough, we obtain that there is a constant $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\| (\lambda_K^{2/3})^w u \|_{L^2}^2 \leq C_0 (\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2). \quad (78)$$

□

Proposition 3.20. *Let λ_K be the symbol defined in [\(44\)](#). Then there exists $\mathbf{C}_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\| \mathcal{A}_\xi u \|_{L^2}^2 \leq \mathbf{C}_0 \| (\lambda_K^2)^w u \|_{L^2}^2. \quad (79)$$

Proof. We denote by $\tilde{\sigma}$ the symbol of the operator \mathcal{A}_ξ . We will show that $\tilde{\sigma}$ belongs to $S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ .

Using Lemma 3.11 in [\[9\]](#), we can write $\tilde{\sigma}$ as follows

$$\tilde{\sigma} = iv \cdot \xi + |B(v)\eta|^2 + F(v) + R_1 + R_2,$$

where R_1 (resp R_2) is a symbol belongs to $S(\langle v \rangle^{\gamma+1} \langle \eta \rangle, \Gamma)$ (resp $S(\langle v \rangle^\gamma, \Gamma)$) uniformly with respect to the parameter ξ .

For $iv \cdot \xi$: Taking into account the fact that $\gamma \geq 0$, we have

$$\begin{aligned} \text{For } |\alpha| = 0, \quad |iv \cdot \xi| &\lesssim |v|^2 + |\xi|^2 \lesssim \langle v \rangle^{\gamma+2} + \langle v \rangle^\gamma |\xi|^2 \lesssim \lambda^2, \\ \text{for } |\alpha| = 1, \quad |\partial_v^\alpha (iv \cdot \xi)| &= |\xi| \lesssim \langle v \rangle^{\gamma/2} |\xi| \lesssim \lambda^2, \\ \text{for } |\alpha| \geq 2, \quad |\partial_v^\alpha (iv \cdot \xi)| &= 0. \end{aligned}$$

For $|B(v)\eta|^2$: Using [\(14\)](#), we have

$$\text{for } |\alpha| \geq 1, \quad |\partial_v^\alpha |B(v)\eta|^2| \lesssim \langle v \rangle^\gamma |\eta|^2 \lesssim |B(v)\eta|^2, \quad (80)$$

then, using Cauchy-Schwarz we get

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_v^\alpha (|B(v)\eta|^2)| \lesssim |B(v)\eta|^2 \lesssim \lambda^2. \quad (81)$$

On the other hand, also using [\(14\)](#), we have

$$\begin{aligned} \text{pour } |\alpha| = 0, \quad |B(v)\eta|^2 &\lesssim \lambda^2, \\ \text{pour } |\alpha| = 1, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| &\lesssim |B(v)| |B(v)\eta| \lesssim \lambda^2, \\ \text{pour } |\alpha| = 2, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| &\lesssim |B^T(v)B(v)| \lesssim \lambda^2, \\ \text{pour } |\alpha| \geq 3, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| &= 0, \end{aligned}$$

so we get

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| \lesssim \lambda^2. \quad (82)$$

Moreover, one can estimate from above the modulus of all the derivatives of the term $|B(v)\eta|^2$ by a constant times λ^2 .

For $F(v)$: Using Lemma [2.4](#), we have

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_v^\alpha F(v)| \lesssim \langle v \rangle^{\gamma+2-|\alpha|} \lesssim \lambda^2, \quad (83)$$

which gives that $F(v) \in S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ .

In addition, we have that $R_1, R_2 \in S(\lambda^2, \Gamma)$. From the above, we can deduce that $\tilde{\sigma}$ belongs to $S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ . Using the following notation

$$H_{17} = \underbrace{\mathcal{A}_\xi \left((\lambda_K^2)^w \right)^{-1}}_{\in \mathcal{B}(L^2)},$$

Then there exists $\mathbf{C}_0 > 0$ such that

$$\|H_{17}\varphi\|_{L^2}^2 \leq \mathbf{C}_0 \|\varphi\|_{L^2}^2 \quad \forall \varphi \in L^2(\mathbb{R}_v^3), \quad (84)$$

uniformly with respect to the parameter ξ , which implies for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\mathcal{A}_\xi u\|_{L^2}^2 \leq \mathbf{C}_0 \left((\lambda_K^2)^w u \right)_{L^2}^2. \quad (85)$$

□

4 Hypocoelliptic estimates for the whole linearized Landau operator

In this section, we show hypoelliptic estimates with respect to the velocity and position variables for the Landau operator \mathcal{P} . These estimates allow us to locate the spectrum and estimate the resolvent of the Landau operator. We denote by Λ_K the operator associated to the symbol λ_K by considering the inverse Fourier transform with respect to the variable x .

Theoreme 4.1. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (86)$$

Proof. Using Theorem [A.6](#), we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\left\| (\lambda_K^{2/3})^w u \right\|_{L_v^2}^2 \sim \left\| (\lambda_K^w)^{2/3} u \right\|_{L_v^2}^2,$$

so using the Proposition [3.19](#), there exists a constant $C_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\left\| (\lambda_K^w)^{2/3} u \right\|_{L_v^2}^2 \leq C_1 \left(\|\mathcal{A}_\xi u\|_{L_v^2}^2 + \|u\|_{L_v^2}^2 \right), \quad (87)$$

uniformly with respect to the parameter ξ . By integrating the previous inequality with respect to the parameter ξ in \mathbb{R}^3 and considering the inverse Fourier transform with respect to the variable x , we obtain for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C_1 \left(\|\mathcal{A}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (88)$$

Using (9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K}.$$

Consequently, using (88), we have

$$\begin{aligned} \|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 &\leq C_1 \left(\|(\mathcal{A} + \mathcal{K} - \mathcal{K})u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right) \\ &\leq C_1 \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|\mathcal{K}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right) \end{aligned}$$

and using the fact that \mathcal{K} is a bounded operator, we obtain that there exists a constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (89)$$

□

By adding a term $i\kappa$ to \mathcal{P} with $\kappa \in \mathbb{R}$, the proof doesn't change due to the never changing of the real part of \mathcal{P} . So Theorem 4.1 admits the following extension.

Theoreme 4.2. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\forall \kappa \in \mathbb{R}, \|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|(\mathcal{P} - i\kappa)u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (90)$$

Theoreme 4.3. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\|\mathcal{P}u\|_{L_{x,v}^2}^2 \leq C \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (91)$$

Proof. Using Theorem A.6, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^2)^w u\|_{L_v^2}^2 \sim \|(\lambda_K^w)^2 u\|_{L_v^2}^2,$$

so using the Proposition 3.20, there exists a constant $C_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\mathcal{A}_\xi u\|_{L_v^2}^2 \leq C_1 \|(\lambda_K^w)^2 u\|_{L_v^2}^2, \quad (92)$$

uniformly with respect to the parameter ξ . By integrating the previous inequality with respect to the parameter ξ in \mathbb{R}^3 and considering the inverse Fourier transform with respect to the variable x , we obtain for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\mathcal{A}u\|_{L_{x,v}^2}^2 \leq C_1 \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (93)$$

Using (9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K}.$$

Consequently, using (93), we have

$$\begin{aligned}\|\mathcal{P}u\|_{L^2_{x,v}}^2 &\leq \|\mathcal{A}u\|_{L^2_{x,v}}^2 + \|\mathcal{K}u\|_{L^2_{x,v}}^2 \\ &\leq \mathbf{C}_1 \|\Lambda_K^2 u\|_{L^2_{x,v}}^2 + \mathbf{C}_2 \|u\|_{L^2_{x,v}}^2,\end{aligned}$$

with $\mathbf{C}_2 > 0$. Using the fact that the operator $\Lambda_K^2 \geq \text{Id}$ (see for instance Theorem A.6 in the Appendix), we obtain that there exists a constant $\mathbf{C} > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^6_{x,v})$,

$$\|\mathcal{P}u\|_{L^2_{x,v}}^2 \leq \mathbf{C} \|\Lambda_K^2 u\|_{L^2_{x,v}}^2. \quad (94)$$

□

The estimate in (86) is optimal in term of the index 2/3 appearing in the left hand side and the estimate in (91) is optimal in term of the index 2 appearing in the right hand side.

5 Localisation of the spectrum for the Landau operator

In all that follows, we denote by $\sigma(\mathcal{P})$ the spectrum of the operator \mathcal{P} , $\rho(\mathcal{P})$ The resolvent set of \mathcal{P} and $\|\cdot\|_{L^2}$ to denote the norm in the space $L^2(\mathbb{R}^6_{x,v})$. The following lemma holds for any maximally accretive operator.

Lemma 5.1. *Let $(\mathcal{A}, D(\mathcal{A}))$ be a maximally accretive operator in the Hilbert space \mathcal{H} . For any $\eta \in]0, 1[$, the estimate*

$$|z + 1|^{2\eta} \|u\|^2 \leq 4 \left(((\mathcal{A} + 1)^*(\mathcal{A} + 1))^\eta u, u \right)_{\mathcal{H}} + 4 \|(\mathcal{A} - z)u\|^2$$

holds for all $u \in D(\mathcal{A})$ and $z \in \mathbb{C}$ with $\Re z \geq -1$.

Proof. See Proposition B.1 in [8].

□

Proof of Theorem 1.1.

Proof. The proof will be divided into two steps.

First step:

Using (90) and the triangle inequality we therefore get for all $z = \nu + i\kappa \in \mathbb{C}$, with $\nu = \Re z \geq -1/2$ and $u \in \mathcal{S}(\mathbb{R}^6_{x,v})$,

$$\begin{aligned}\|\Lambda_K^{2/3} u\|_{L^2}^2 &\leq C (\|(\mathcal{P} - i\kappa)u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\leq C (\|(\mathcal{P} - i\kappa - \nu + \nu)u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\leq C (2\|(\mathcal{P} - z)u\|_{L^2}^2 + (2\nu^2 + 1)\|u\|_{L^2}^2).\end{aligned}$$

But we have $\nu \geq -1/2$ implies $2\nu + 2 \geq 1$, so we get

$$\begin{aligned}\|\Lambda_K^{2/3} u\|_{L^2}^2 &\leq C (6\|(\mathcal{P} - z)u\|_{L^2}^2 + 6(\nu + 1)^2 \|u\|_{L^2}^2) \\ &\leq 6C (\|(\mathcal{P} - z)u\|_{L^2}^2 + (\nu + 1)^2 \|u\|_{L^2}^2),\end{aligned}$$

By taking $\tilde{C} = 6C$, we finally obtain the following estimate

$$\|\Lambda_K^{2/3} u\|_{L^2}^2 \leq \tilde{C} \left(\|(\mathcal{P} - z)u\|_{L^2}^2 + (\Re z + 1)^2 \|u\|_{L^2}^2 \right), \quad (95)$$

for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$ and $z \in \mathbb{C}$ with $\Re z \geq -\frac{1}{2}$.

Second step:

First, we will show that there exists a constant \tilde{C}_1 such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we have

$$0 \leq \left((\mathcal{A} + 1)^*(\mathcal{A} + 1)u, u \right)_{L^2} \leq \tilde{C}_1 \left(\Lambda_K^4 u, u \right)_{L^2}, \quad (96)$$

where \mathcal{A} the operator defined in (9). Indeed, let $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we have

$$\left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} = \left((1 + \mathcal{A})u, (1 + \mathcal{A})u \right)_{L^2} = \|(1 + \mathcal{A})u\|_{L^2}^2 \geq 0.$$

On the other hand, we have

$$\left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} = \|(1 + \mathcal{A})u\|_{L^2}^2 \leq \|\mathcal{A}u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Using (9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K},$$

we obtain

$$\left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} \leq \|\mathcal{P}u\|_{L^2}^2 + \|\mathcal{K}u\|_{L^2}^2 + \|u\|_{L^2}^2,$$

finally, using the fact that \mathcal{K} is a bounded operator, $\Lambda_K \geq \text{Id}$ and the estimate (91), we obtain

$$\begin{aligned} \left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} &\leq (1 + \nu_0) \|\Lambda_K^2 u\|_{L^2}^2 + \mathbf{C} \|\Lambda_K^2 u\|_{L^2}^2 \\ &\leq (1 + \nu_0 + \mathbf{C}) \|\Lambda_K^2 u\|_{L^2}^2, \end{aligned}$$

where $\nu_0 > 0$. By taking $\tilde{C}_1 = (1 + \nu_0 + \mathbf{C})$, we obtain the estimate (96).

According to the monotonicity of the operator functional $\mathcal{A} \rightarrow \mathcal{A}^\alpha$ for $\alpha \in [0, 1]$, in particular with $\alpha = \frac{1}{3}$, we obtain

$$0 \leq \left(((1 + \mathcal{A})^*(1 + \mathcal{A}))^{1/3} u, u \right)_{L^2} \leq \tilde{C}_1^{2/3} \left(\Lambda_K^{4/3} u, u \right)_{L^2}.$$

According to Theorem 2.8, \mathcal{A} is maximally accretive, then by applying Lemma 5.1 with $\eta = \frac{1}{3}$, for $\Re z \geq -1/2$ and $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we obtain

$$\begin{aligned} |z + 1|^{2/3} \|u\|_{L^2}^2 &\leq 4 \left(((1 + \mathcal{A})^*(1 + \mathcal{A}))^{1/3} u, u \right)_{L^2} + 4 \|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4 \tilde{C}_1^{2/3} \left(\Lambda_K^{4/3} u, u \right)_{L^2} + 4 \|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4 \tilde{C}_1^{2/3} \|\Lambda_K^{2/3} u\|_{L^2}^2 + 4 \|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4 \tilde{C}_1^{2/3} \|\Lambda_K^{2/3} u\|_{L^2}^2 + 4 \|(\mathcal{P} - z)u\|_{L^2}^2 + 4\nu_1 \|u\|_{L^2}^2. \end{aligned}$$

With the inequality (95), we obtain

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq (4\tilde{C}_1^{2/3}\tilde{C} + 4) \|(\mathcal{P} - z)u\|_{L^2}^2 + (4\tilde{C}_1^{2/3}\tilde{C} + 16\nu_1)(\Re z + 1)^2 \|u\|_{L^2}^2.$$

By taking $C_{\mathcal{P}} = \sqrt{8(4\nu_1 + \tilde{C}_1^{2/3}\tilde{C})}$, $Q_{\mathcal{P}} = \sqrt{8(1 + \tilde{C}_1^{2/3}\tilde{C})}$ we finally get $\forall z \in \mathbb{C}$, $\Re z \geq -1/2$, $\forall u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq \frac{Q_{\mathcal{P}}^2}{2} \|(\mathcal{P} - z)u\|_{L^2}^2 + \frac{C_{\mathcal{P}}^2}{2} (\Re z + 1)^2 \|u\|_{L^2}^2. \quad (97)$$

Now let $z \notin S_{\mathcal{P}}$ such that $\Re z \geq -1/2$, so according to the definition of $S_{\mathcal{P}}$ given in (6) we have

$$(\Re z + 1)^2 \leq \frac{1}{C_{\mathcal{P}}^2} |z + 1|^{2/3},$$

then for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, inequality (97) implies

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq Q_{\mathcal{P}} \|(\mathcal{P} - z)u\|_{L^2}^2, \quad (98)$$

we deduce that $\mathcal{P} - z$ is injective, moreover we can replace $\mathcal{P} - z$ by $(\mathcal{P} - z)^*$ in (98), which gives $(\mathcal{P} - z)^*$ is injective and consequently $\mathcal{P} - z$ is bijective with dense image in L^2 , therefore $z \in \rho(\mathcal{P})$. By taking $v = (\mathcal{P} - z)u$ in the estimate (98) we get

$$\|(\mathcal{P} - z)^{-1}v\|_{L^2}^2 \leq Q_{\mathcal{P}}^2 |z + 1|^{-2/3} \|v\|_{L^2}^2 \quad \forall v \in L^2,$$

Then we obtain the resolvent estimate (7).

On the other hand, we have that if $z \in \sigma(\mathcal{P})$ then $z \in S_{\mathcal{P}}$ and taking into account that the numerical range of the operator \mathcal{P} is the half plan $\{\Re z \geq 0\}$, we deduce that the spectrum $\sigma(\mathcal{P})$ satisfies

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\},$$

the proof is then complete. \square

A Appendix

A.1 Weyl-Hörmander calculus

We recall here some notations and basic facts of symbolic calculus, and refer to [15] and [12] for detailed discussions on the pseudo-differential calculus.

We introduce on \mathbb{R}^{2n} the following metric

$$\Gamma = dv^2 + d\eta^2.$$

Definition A.1. Let $m \geq 1$ be a C^∞ function on \mathbb{R}^{2n} . We say that m is an admissible weight for Γ if there exist two constants $C > 0$ and $N > 0$ such that

$$\forall X, Y \in \mathbb{R}^{2n}, \quad m(X) \leq C \langle X - Y \rangle^N m(Y). \quad (99)$$

Definition A.2. Let m be an admissible function. We denote by $S(m, \Gamma)$ the symbol class of all smooth functions $p(v, \eta)$ (possibly depending on parameter ξ) satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha, \beta} > 0; \forall (v, \eta) \in \mathbb{R}^{2n}, |\partial_v^\alpha \partial_\eta^\beta p(v, \eta)| \leq C_{\alpha, \beta} m(v, \eta).$$

The space of symbols $S(m, \Gamma)$ endowed with the semi-norms

$$\|p\|_{k; S(m, \Gamma)} = \sup_{|\alpha+\beta| \leq k} \sup_{(v, \eta) \in \mathbb{R}^{2n}} |m(v, \eta)^{-1} \partial_v^\alpha \partial_\eta^\beta p(v, \eta)|; \quad k \in \mathbb{N} \quad (100)$$

becomes a Fréchet space.

For such a symbol p in $S(m, \Gamma)$ we may define its Weyl quantization p^w by

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p^w u)(v) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v') \cdot \eta} p\left(\frac{v+v'}{2}, \eta\right) u(v') \, dv' d\eta. \quad (101)$$

The Weyl quantization of $S(m, \Gamma)$ is denoted by $\Psi(m, \Gamma)$.

Theoreme A.3. (Calderon-Vaillancourt)

Let p^w be an operator in $\Psi(1, \Gamma)$. We have p^w a continuous operator on $L^2(\mathbb{R}^n)$ and

$$\forall u \in L^2(\mathbb{R}^n), \quad \|p^w u\|_{L^2} \leq C \|p\|_{N; S(1, \Gamma)} \|u\|_{L^2}, \quad (102)$$

where $C > 0$ and a positive integer N depending only on the dimension.

Proof. See Section 18 in [12]. □

Theoreme A.4. Let p^w be an invertible operator in $\Psi(m, \Gamma)$, then its inverse $[p^w]^{-1}$ belongs to $\Psi(m^{-1}, \Gamma)$.

Proof. See Lemma A.2 in [8]. □

Definition A.5. Let $p^w \in \Psi(m, \Gamma)$. We say that p^w is an elliptic operator if there exists $C > 0$ such that

$$|p| \geq Cm.$$

Let us also recall here the composition formula of Weyl quantization. Let $a \in S(m_1, \Gamma)$ and $b \in S(m_2, \Gamma)$, the compositions of the pseudo-differential operators a^w and b^w are pseudo-differential operators whose symbol, denoted $a \# b$, belongs to $S(m_1 m_2, \Gamma)$ and has the following development:

$$a \# b = ab + \int_0^1 \iint e^{-i\sigma(Y-Y_1, Y-Y_2)/(2\theta)} \frac{i}{2} \sigma(\partial_{Y_1}, \partial_{Y_2}) a(Y_1) b(Y_2) dY_1 dY_2 d\theta / (\theta)^{2n}, \quad (103)$$

where σ is the symplectic form in $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ given by

$$\sigma(Z, Z') = \sum_{j=0}^n \zeta_j z'_j - z_j \zeta'_j. \quad (104)$$

As we work with pseudo-differential operators which belong to classes associated to the metric Γ , all operators will be defined as continuous operators of $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ or from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

A.2 Basic theorem

This theorem aims at giving a uniform statement for Weyl Hörmander tools with a large parameter K . This theorem gives very important results which can use these results to establish estimates which requires pseudo-differential operators. part (I) in the theorem below has been shown in [1] with $\tau = 1$, but here we have improved this result for all $\tau \in \mathbb{R}$. In general, these results give a general and robust framework to techniques already used as well for work that requires these kinds of properties.

Theoreme A.6. *Let $p, q \geq 1$ be two symbols which verify the following hypotheses:*

- i) p is an admissible weight.*
- ii) $p \sim q$.*
- iii) $p, q \in S(p, \Gamma)$.*
- iv) There exists $M \in \mathbb{R}$ such that for all $\varepsilon > 0$ we have $\partial_\eta p, \partial_\eta q \in S(\varepsilon p + \varepsilon^{-1} \langle v \rangle^M, \Gamma)$.*

We define the symbols p_K, q_K as $p_K = p + K \langle v \rangle^M$ and $q_K = q + K \langle v \rangle^M$. Then there exists K_0 such that for all $K \geq K_0$:

- I) For all $\tau \in \mathbb{R}$, $(p_K^\tau)^w$ and $(q_K^\tau)^w$ are invertible.*
- II) For all $\tau \in \mathbb{R}$, $[(p_K^\tau)^w]^{-1}$ and $[(q_K^\tau)^w]^{-1}$ are pseudo-differential operators belong to $\Psi(p_K^{-\tau}, \Gamma)$ uniformly in K .*
- III) For all $\tau \in \mathbb{R}$ and for all $\kappa \geq 0$, $[(p_K^\kappa)^w]^\tau$ and $[(q_K^\kappa)^w]^\tau$ are pseudo-differential operators belong to $\Psi(p_K^{\kappa\tau}, \Gamma)$ uniformly in K .*
- IV) For all $\tau \in \mathbb{R}$ and for all $\kappa \geq 0$, $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|[(p_K^\kappa)^w]^\tau u\|_{L^2}^2 \sim \|(p_K^{\kappa\tau})^w u\|_{L^2}^2,$$

uniformly in K .

- V) For all $\tau \in \mathbb{R}$, $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|(p_K^w)^\tau u\|_{L^2}^2 \sim \|(p_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^w)^\tau u\|_{L^2}^2,$$

uniformly in K .

- VI) $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$(p_K^w u, u)_{L^2} \sim (q_K^w u, u)_{L^2},$$

uniformly in K .

- VII) If $M \geq 0$, we have $p_K^w, q_K^w \geq Id$.*

Proof. Let $K > 0$ and let p and q two symbols and $M \in \mathbb{R}$ such that the hypotheses from (i) to (iv) are verified. For simplification, we will prove (I, II, III) just for the operator $(p_K^\tau)^w$, and we reason in the same way for $(q_K^\tau)^w$.

I. We will show that $(p_K^\tau)^w$ is invertible for all $\tau \in \mathbb{R}$. We note that p_K is an admissible weight (p is an admissible weight). We have $p_K \in S(p_K, \Gamma)$ uniformly in K . Indeed, using (iii) we have for all $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta p_K| &\leq |\partial_v^\alpha \partial_\eta^\beta p| + |\partial_v^\alpha \partial_\eta^\beta K \langle v \rangle^M| \\ &\leq C_{\alpha, \beta} p + K C_{\alpha, \beta, M} \langle v \rangle^M \\ &\leq \tilde{C}_{\alpha, \beta, M} p_K. \end{aligned}$$

More generally we can show, by induction on $|\alpha|$ and Leibnitz's formula that for $\tau \in \mathbb{R}$,

$$\forall \alpha \in \mathbb{N}^{2n}, \quad |\partial_{v, \eta}^\alpha p_K^\tau| \leq C_{\alpha, \beta, \tau} p_K^\tau,$$

which gives $p_K^\tau \in S(p_K^\tau, \Gamma)$ uniformly in K .

Using formula (103), we may write

$$(p_K^\tau)^w (p_K^{-\tau})^w = \text{Id} - R_K^w, \quad (105)$$

where

$$R_K = - \int_0^1 (\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau})) d\theta + \int_0^1 (\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau})) d\theta$$

with $g \sharp_\theta h$ defined by

$$g \sharp_\theta h(Y) = \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} g(Y_1) h(Y_2) dY_1 dY_2 / (\pi\theta)^{2n}, \quad (106)$$

with $Y, Y_1, Y_2 \in \mathbb{R}^{2n}$ and σ a symplectic form defined in (104).

Let now N be the integer which is given in (102). By [2, Proposition 1.1], we can find a constant C_N and a positive integer l_N , both depending only on N but independent of K and θ , such that

$$\|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \|\partial_\eta p_K^\tau\|_{l_N; S(p_K^\tau, \Gamma)} \|(\partial_v (p_K^{-\tau}))\|_{l_N; S(p_K^{-\tau}, \Gamma)},$$

where the semi-norm $\|\cdot\|_{k; S(M, \Gamma)}$ is defined by (100).

Moreover, using (iv), we have $\partial_\eta p \in S(\varepsilon p + \varepsilon^{-1} \langle v \rangle^M, \Gamma)$, by taking $\varepsilon = K^{-1/2}$, we obtain $\partial_\eta p \in S(K^{-1/2} p_K, \Gamma)$. By writing $\partial_\eta p_K^\tau = \tau p_K^{\tau-1} \partial_\eta p$, we obtain

$$|\partial_\eta p_K^\tau| \leq \tilde{C}_N K^{-1/2} p_K^\tau.$$

Then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta p_K^\tau| \leq \tilde{C}_N K^{-1/2} p_K^\tau,$$

which gives $\partial_\eta p_K^\tau \in S(K^{-1/2} p_K^\tau, \Gamma)$ uniformly in K , moreover we have

$$\|(\partial_\eta p_K^\tau)\|_{l_N; S(p_K^\tau, \Gamma)} \leq \tilde{C}_N K^{-1/2}.$$

On the other hand we have $p_K^{-\tau} \in S(p_K^{-\tau}, \Gamma)$, and thus

$$\|(\partial_v (p_K^{-\tau}))\|_{l_N; S(p_K^{-\tau}, \Gamma)} \leq \tilde{C}_N,$$

with \tilde{C}_N a constant depending only on N but independent of K . As a result,

$$\|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \tilde{C}_N^2 K^{-1/2}. \quad (107)$$

Similarly,

$$\|(\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau}))\|_{N;S(1,\Gamma)} \leq C_N \tilde{C}_N^2 K^{-1/2}. \quad (108)$$

Using the estimates (107), (108) we will estimate the semi-norm $\|R_K\|_{N;S(1,\Gamma)}$. Indeed,

$$\|R_K\|_{N;S(1,\Gamma)} \leq \int_0^1 \|(\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau}))\|_{N;S(1,\Gamma)} d\theta + \int_0^1 \|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N;S(1,\Gamma)} d\theta.$$

Then

$$\|R_K\|_{N;S(1,\Gamma)} \leq 2C_N \tilde{C}_N^2 K^{-1/2},$$

and thus by (102)

$$\|R_K^w\|_{\mathcal{B}(L^2)} \leq 2CC_N \tilde{C}_N^2 K^{-1/2}$$

with C a constant depending only on the dimension. Taking $K_1 = (4CC_N \tilde{C}_N^2)^2$, so we get for all $K \geq K_1$

$$\|R_K^w\|_{\mathcal{B}(L^2)} \leq \frac{1}{2} < 1,$$

this implies $\text{Id} - R_K^w$ is invertible in the space $\mathcal{B}(L^2)$. In addition, its inverse is given by

$$(\text{Id} - R_K^w)^{-1} = \sum_{i=0}^{\infty} (R_K^w)^i \in \mathcal{B}(L^2). \quad (109)$$

Based on (105) we obtain

$$(p_K^\tau)^w \left((p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1} \right) = \text{Id}.$$

Similarly we can find a $\hat{R}_K \in S(1, \Gamma)$ such that

$$\left((\text{Id} - \hat{R}_K^w)^{-1} (p_K^{-\tau})^w \right) (p_K^\tau)^w = \text{Id}.$$

Based on the above $(p_K^\tau)^w$ is invertible and its inverse $[(p_K^\tau)^w]^{-1}$ is written in the form

$$[(p_K^\tau)^w]^{-1} = (p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1} = (\text{Id} - \hat{R}_K^w)^{-1} (p_K^{-\tau})^w.$$

We have proved the conclusion in (I).

II. According to (I), we have $[(p_K^\tau)^w]^{-1}$ has the form

$$[(p_K^\tau)^w]^{-1} = (p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1}.$$

By taking $H_K = (\text{Id} - R_K^w)^{-1}$ and using (109), we have H_K is a continuous operator in L^2 uniformly in K . So based on Theorem A.4, we have H_K is a pseudo-differential operator. We note by $\delta(H_K)$ its symbol. We have $\delta(H_K)$ belongs to $S(1, \Gamma)$ uniformly in K . From the above, we have $[(p_K^\tau)^w]^{-1}$ is a pseudo-differential operator (compositions of pseudo-differential operators). We denote h_K its symbol, h_K has the form $h_K = p_K^{-\tau} \sharp \delta(H_K)$ and belongs to $S(p_K^{-\tau}, \Gamma)$ uniformly in K .

We have proved the conclusion in (II).

III. Using the Theorem [A.4](#), we have $[(p_K^\kappa)^w]^{-1}$ is a pseudo-differential operator. Taking into account that the composition of the pseudo-differential operators is a pseudo-differential operator, we obtain

$$\forall n \in \mathbb{Z}, [(p_K^\kappa)^w]^n \in \Psi(p_K^{\kappa n}, \Gamma). \quad (110)$$

Now consider the case of the exponents $\tau \in \mathbb{R}$. By pseudo-differential calculus the problem will be reduced to

$$\forall \tau \in I, [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa \tau}, \Gamma),$$

where I an open interval of \mathbb{R} . Note that the operator $(p_K^\kappa)^w$ is self-adjoint because its symbol p_K^κ is real. Now we will show that $(p_K^\kappa)^w$ is a positive operator for K sufficiently large, which is equivalent to show that

$$\left((p_K^\kappa)^w u, u \right)_{L^2} \geq 0, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \quad (111)$$

Using again the formula [\(I03\)](#), we can write

$$(p_K^{\kappa/2})^w (p_K^{\kappa/2})^w = (p_K^\kappa)^w - R_K^w, \quad (112)$$

where

$$R_K = - \int_0^1 (\partial_\eta p_K^{\kappa/2}) \#_\theta (\partial_v (p_K^{\kappa/2})) \, d\theta + \int_0^1 (\partial_v p_K^{\kappa/2}) \#_\theta (\partial_\eta (p_K^{\kappa/2})) \, d\theta$$

with $g \#_\theta h$ defined in [\(I06\)](#). Based on the proof of (I), we have

$$\partial_\eta p_K^{\kappa/2} \in S(K^{-1/2} p_K^{\kappa/2}, \Gamma)$$

uniformly in K . Moreover we have $\partial_v p_K^{\kappa/2} \in S(p_K^{\kappa/2}, \Gamma)$, so we get

$$(\partial_\eta p_K^{\kappa/2}) \#_\theta (\partial_v (p_K^{\kappa/2})), (\partial_v p_K^{\kappa/2}) \#_\theta (\partial_\eta (p_K^{\kappa/2})) \text{ and } R_K \in S(K^{-1/2} p_K^\kappa, \Gamma)$$

uniformly in K . Writing R_K^w in the following form

$$R_K^w = K^{-1/2} (p_K^{\kappa/2})^w \underbrace{K^{1/2} [(p_K^{\kappa/2})^w]^{-1} R_K^w [(p_K^{\kappa/2})^w]^{-1}}_{\in \mathcal{B}(L^2) \text{ uniformément en } K} (p_K^{\kappa/2})^w,$$

we obtain

$$|(R_K^w u, u)_{L^2}| \leq C_0 K^{-1/2} \|(p_K^{\kappa/2})^w\|_{L^2}^2,$$

with C_0 some constant independent of K . Let $K_2 = 16C_0^2$, then using the relation [\(I12\)](#) we get for all $K \geq K_2$

$$\left((p_K^\kappa)^w u, u \right)_{L^2} \geq \frac{3}{4} \|(p_K^{\kappa/2})^w\|_{L^2}^2 \geq 0.$$

Then, using the following formula (see for example [\[19\]](#)) we can write

$$[(p_K^\kappa)^w]^\tau = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau (s + (p_K^\kappa)^w)^{-1} \, ds, \quad \tau \in (-1, 0). \quad (113)$$

First, $s + (p_K^\kappa)^w$ is a pseudo-differential operator and its symbol $a_{K,s}$ verifies

$$\forall s \in (0, 1), \quad a_{K,s} \in S(p_K^\kappa, \Gamma)$$

and

$$\forall s \geq 1, a_{K,s} \in S(s(p_K^\kappa), \Gamma) \text{ uniformly in } K \text{ and } s.$$

So using the Theorem [A.4](#), $(s + (p_K^\kappa)^w)^{-1}$ is a pseudo-differential operator and its symbol $b_{K,s}$ verifies

$$\forall s \in (0, 1), b_{K,s} \in S((p_K^\kappa)^{-1}, \Gamma)$$

and

$$\forall s \geq 1, b_{K,s} \in S(s^{-1}(p_K^\kappa)^{-1}, \Gamma) \text{ uniformly in } K \text{ et } s.$$

Then, taking $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$[(p_K^\kappa)^w]^\tau u = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau (s + (p_K^\kappa)^w)^{-1} u \, ds,$$

using the formula ([101](#)), we get

$$\begin{aligned} [(p_K^\kappa)^w]^\tau u &= -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v')\cdot\eta} b_{K,s}\left(\frac{v+v'}{2}, \eta\right) u(v') \, dv' d\eta \right) ds, \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v')\cdot\eta} \left(-\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau b_{K,s}\left(\frac{v+v'}{2}, \eta\right) ds \right) u(v') \, dv' d\eta. \end{aligned}$$

So $[(p_K^\kappa)^w]^\tau$ is a pseudo-differential operator and its symbol d_K given by

$$d_K = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau b_{K,s} \, ds.$$

Then we will show that $d_K \in S(p_K^{\kappa\tau}, \Gamma)$. Indeed, let $\alpha, \beta \in \mathbb{N}^n, (v, \eta) \in \mathbb{R}^{2n}$,

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta d_K(v, \eta)| &\leq C \left(\int_0^1 s^\tau |\partial_v^\alpha \partial_\eta^\beta b_{K,s}| \, ds + \int_1^\infty s^\tau |\partial_v^\alpha \partial_\eta^\beta b_{K,s}| \, ds \right), \\ &\leq C_{\alpha,\beta} \left(\int_0^1 s^\tau (p_K^\kappa)^{-1} \, ds + \int_1^\infty s^{\tau-1} (p_K^\kappa)^{-1} \, ds \right), \\ &\leq \tilde{C}_{\alpha,\beta} (p_K^\kappa)^{-1}, \end{aligned}$$

which gives that $d_K \in S(p_K^{\kappa\tau}, \Gamma)$. Take $I = (-1, 0)$, then we get

$$\forall \tau \in I, [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa\tau}, \Gamma). \quad (114)$$

In addition, $[(p_K^\kappa)^w]^\tau$ is a bounded operator in L^2 uniformly in K . Indeed, $(s + (p_K^\kappa)^w)^{-1} \in \Psi(1, \Gamma)$ uniformly in K , then there exists a constant $C_1 > 0$ such that

$$\forall s \in (0, 1), \|(s + (p_K^\kappa)^w)^{-1}\|_{\mathcal{B}(L^2)} \leq C_1. \quad (115)$$

On the other hand we have

$$\forall u \in \mathcal{S}(\mathbb{R}^n), ((s + (p_K^\kappa)^w)u, u)_{L^2} \geq s\|u\|_{L^2}^2,$$

which gives

$$\forall s \geq 1, \|(s + (p_K^\kappa)^w)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s}. \quad (116)$$

Using ([115](#)), ([116](#)) and the formula ([113](#)), we obtain

$$\|[(p_K^\kappa)^w]^\tau\|_{\mathcal{B}(L^2)} \leq C_2, \quad (117)$$

with C_2 some constant independent of K .

Then based on (110), (114) we obtain for all $\tau \in I$,

$$\forall n \in \mathbb{Z}, \quad [|(p_K^\kappa)^w]^\tau]^n \in \Psi(p_K^{\kappa n \tau}, \Gamma), \quad (118)$$

so we have

$$\forall \tau \in \mathbb{R}, \quad [|(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa \tau}, \Gamma). \quad (119)$$

IV. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|[|(p_K^\kappa)^w]^\tau u\|_{L^2}^2 \sim \|(p_K^{\kappa \tau})^w u\|_{L^2}^2, \quad (120)$$

otherwise we will show that there are two constants $c, C > 0$ independent of K such that for all $u \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$\|(p_K^{\kappa \tau})^w u\|_{L^2}^2 \leq c \|[|(p_K^\kappa)^w]^\tau u\|_{L^2}^2, \quad (121)$$

and

$$\|[|(p_K^\kappa)^w]^\tau u\|_{L^2}^2 \leq C \|(p_K^{\kappa \tau})^w u\|_{L^2}^2. \quad (122)$$

We will start with inequality (121), using the conclusion (III), we have that $[|(p_K^\kappa)^w]^\tau$ is a pseudo-differential operator belongs to $\Psi(p_K^{\kappa \tau}, \Gamma)$ uniformly in K . Taking $v = [|(p_K^\kappa)^w]^{-\tau} u$, show inequality (121) is equivalent to showing that

$$\|(p_K^{\kappa \tau})^w [|(p_K^\kappa)^w]^{-\tau} v\|_{L^2}^2 \leq C_3 \|v\|_{L^2}^2 \quad \forall v \in L^2. \quad (123)$$

Using the Theorem A.3, we have $(p_K^{\kappa \tau})^w [|(p_K^\kappa)^w]^{-\tau} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , hence the inequality (121).

Similarly, using the fact that $[|(p_K^\kappa)^w]^\tau [(p_K^{\kappa \tau})^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , then taking $v = [(p_K^{\kappa \tau})^w]^{-1} u$, we get the inequality (122), so the estimate (120) is true.

V. We will first show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(p_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^\tau)^w\|_{L^2}^2, \quad (124)$$

otherwise we will show that there are two constants $C_1, C_2 > 0$ independent of K such that for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\|(p_K^\tau)^w u\|_{L^2}^2 \leq C_1 \|(q_K^\tau)^w u\|_{L^2}^2, \quad (125)$$

and

$$\|(q_K^\tau)^w u\|_{L^2}^2 \leq C_2 \|(p_K^\tau)^w u\|_{L^2}^2. \quad (126)$$

We will start with inequality (125), using hypothesis (iii) we have that $(q_K^\tau)^w \in \Psi(p_K^\tau, \Gamma)$ uniformly in K . Using the conclusion (II), we have $(q_K^\tau)^w$ is an invertible operator, moreover its inverse $[(q_K^\tau)^w]^{-1}$ belongs to $\Psi(p_K^{-\tau}, \Gamma)$ uniformly in K .

Taking $v = [(q_K^\tau)^w]^{-1} u$, show inequality (125) is equivalent to showing that

$$\|(p_K^\tau)^w [(q_K^\tau)^w]^{-1} v\|_{L^2}^2 \leq C_1 \|v\|_{L^2}^2 \quad \forall v \in L^2. \quad (127)$$

Using the Theorem A.3, we have $(p_K^\tau)^w [(q_K^\tau)^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , hence the inequality (127).

Similarly, using the fact that $(q_K^\tau)^w [(p_K^\tau)^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , then taking $v = [(p_K^\tau)^w]^{-1}u$, we get the inequality (126), so the estimate (124) is true. In the same way we have

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(q_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^w)^\tau u\|_{L^2}^2. \quad (128)$$

Finally, using conclusion (IV) with $\kappa = 1$, we get that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(p_K^w)^\tau u\|_{L^2}^2 \sim \|(p_K^\tau)^w u\|_{L^2}^2. \quad (129)$$

VI. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p_K^w u, u)_{L^2} \sim (q_K^w u, u)_{L^2}. \quad (130)$$

Indeed, let $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(p_K^w u, u)_{L^2} = ((p_K^w)^{1/2} u, (p_K^w)^{1/2} u)_{L^2} = \|(p_K^w)^{1/2} u\|_{L^2}^2,$$

using the result (IV) with $\tau = \frac{1}{2}$, we obtain that

$$\|(p_K^w)^{1/2} u\|_{L^2}^2 \sim \|(p_K^{1/2})^w u\|_{L^2}^2 \sim \|(q_K^{1/2})^w u\|_{L^2}^2 \sim \|(q_K^w)^{1/2} u\|_{L^2}^2 \sim (q_K^w u, u)_{L^2}.$$

So the estimate (130) is true.

VII. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p_K^w u, u)_{L^2} \geq \|u\|_{L^2}^2.$$

Indeed, let $u \in \mathcal{S}(\mathbb{R}^n)$

$$(p_K^w u, u)_{L^2} = (p_{K-1}^w u, u)_{L^2} + (\langle v \rangle^M u, u)_{L^2},$$

based on the proof of (111), we can show that there exists a positive constant K_3 such that for all $K \geq K_3$ we have

$$(p_{K-1}^w u, u)_{L^2} \geq 0,$$

Which gives

$$(p_K^w u, u)_{L^2} = \underbrace{(p_{K-1}^w u, u)_{L^2}}_{\geq 0} + (\langle v \rangle^M u, u)_{L^2} \geq \|\langle v \rangle^{M/2} u\|_{L^2}^2.$$

With the condition $M \geq 0$ we finally obtain

$$(p_K^w u, u)_{L^2} \geq \|u\|_{L^2}^2.$$

Finally, for a symbol p which verifies the hypotheses of Theorem A.6, we can fix

$$K \geq \underbrace{\max(K_1, K_2, K_3)}_{K_0} \quad (131)$$

and we apply the results of Theorem A.6 to the operator p_K^w . \square

A.3 Wick quantization

Finally let us recall some basic properties of the Wick quantization, which is also called anti-Wick in [17]. The importance in studying the Wick quantization lies in the facts that positive symbols give rise to positive operators.

Let $Y = (v, \eta)$ be a point in \mathbb{R}^6 . The Wick quantization of a symbol q is given by

$$q^{\text{Wick}} = (2\pi)^{-3} \int_{\mathbb{R}^6} q(Y) \Pi_Y \, dY,$$

where Π_Y is the projector associated to the Gaussian φ_Y which is defined by

$$\varphi_Y(z) = \pi^{-3/4} e^{-\frac{1}{2}|z-v|^2} e^{iz \cdot \eta/2}, \quad \forall z \in \mathbb{R}^3.$$

The main property of the Wick quantization is its positivity, i.e.,

$$q(v, \eta) \geq 0, \quad \forall (v, \eta) \in \mathbb{R}^6 \text{ implies } q^{\text{Wick}} \geq 0. \quad (132)$$

According to Theorem 24.1 in [17], the Wick and Weyl quantizations of a symbol q are linked by the following identities

$$q^{\text{Wick}} = \left(q * \pi^{-3} e^{-|\cdot|^2} \right)^w = q^w + r^w \quad (133)$$

with

$$r(Y) = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1-\theta) r''(Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 \, d\theta. \quad (134)$$

Therefore, according to (102), if $q \in S(1, \Gamma)$ then q^{Wick} is a bounded operator in L^2 . We also recall the following composition formula obtained in the proof of Proposition 3.4 in [13]

$$q_1^{\text{Wick}} q_2^{\text{Wick}} = [q_1 q_2 - q_1' \cdot q_2' + \frac{1}{i} \{q_1, q_2\}]^{\text{Wick}} + T, \quad (135)$$

with T a bounded operator in $L^2(\mathbb{R}^6)$, when $q_1 \in L^\infty(\mathbb{R}^6)$ and q_2 is a smooth symbol whose derivatives of order ≥ 2 are bounded on \mathbb{R}^6 . The notation $\{q_1, q_2\}$ denotes by the Poisson bracket defined by

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta}. \quad (136)$$

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