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► **To cite this version:**

Alix Munier-Kordon. Minimizing the makespan for a UET bipartite graph on a single processor with an integer precedence delay. [Research Report] lip6.2001.016, LIP6. 2001. hal-02545572

**HAL Id: hal-02545572**

**<https://hal.science/hal-02545572>**

Submitted on 17 Apr 2020

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# Minimizing the makespan for a UET bipartite graph on a single processor with an integer precedence delay

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## Abstract

We consider a set of tasks of unit execution times and a bipartite precedence delays graph with a positive precedence delay  $d$  : an arc  $(i, j)$  of this graph means that  $j$  can be executed at least  $d$  time units after the completion time of  $i$ . The problem is to sequence the tasks in order to minimize the makespan.

Firstly, we prove that the associated decision problem is NP-complete. Then, we provide a non trivial polynomial time algorithm if the degree of every tasks from one of the two sets is 2. Lastly, we give an approximation algorithm with ratio  $\frac{3}{2}$ .

## 1 Introduction

Single and multiprocessors scheduling problems have been extensively studied in the literature [16]. Scheduling problems with precedence delays arise independently in several important applications and many theoretical studies were devoted to these problems : this class of problems was considered for resource-constrained scheduling problem [3, 13]. It was also studied as a relaxation for the job-shop problem [1, 8]. For computer systems, it corresponds to the basic pipelines scheduling problems [15, 20].

An instance of a scheduling problem with precedence delays is usually defined by a set of tasks  $T = \{1, \dots, n\}$  with durations  $p_i, i \in T$ , an oriented precedence graph  $G = (T, E)$  and integer delays  $d_{ij} \geq 0, (i, j) \in E$ . For

Table 1: Complexity results

NP-Hard Problem	Reference
1 chains, $d_{ij} = d C_{\max}$	Wikum et al.[23]
1 prec, $d_{ij} = d, p_i = 1 C_{\max}$	Leung et al.[18]

every arc  $(i, j) \in E$ , task  $j$  can be executed at least  $d_{ij}$  time units after the completion time of  $i$ . The number of processors is limited. The problem is to find a schedule minimizing the makespan, or other regular criteria. Using standard notations [16], the minimization of the makespan is denoted by  $P|prec, d_{ij}|C_{\max}$ .

In this paper, we suppose that the graph  $G$  is bipartite :  $T$  is split into two sets  $X$  and  $Y$  and every arc  $(i, j) \in E$  verifies  $i \in X$  and  $j \in Y$ . We also consider that there is only one processor, the duration of tasks is one and that the delay is the same for every arc. This problem is noted  $1|bipartite, d_{ij} = d, p_i = 1|C_{max}$ . The decision problem associated is called SEQUENCING WITH DELAYS and is defined as :

- Instance : A bipartite oriented graph  $G = (X \cup Y, E)$ , a positive delay  $d$  and a deadline  $D$ .
- Question : is there a solution to the sequencing problem with a delay  $d$  and a makespan smaller than or equal to  $D$ ?

We prove in section 2 that  $1|bipartite, d_{ij} = d, p_i = 1|C_{max}$  is NP-Hard. The complexity of this problem was a challenging question since several authors proved the NP-Hardness of more general instances of this problem as shown in the table 1. In section 3, we prove that if the degree of every task in  $X$  is 2, then the problem is polynomial and we provide a greedy algorithm to solve it.

Several authors have adapted the classical polynomial algorithms for  $m$  processors and particular graphs structures to a sequencing problem with a unique delay as shown in the table 2. Note that Bampis [2] proved that  $P|bipartite, p_i = 1|C_{max}$  is NP-Hard, but his transformation doesn't seem to be easily extended to our problem.

Wikum et al. [23] also proved several complexity results, polynomial special cases and approximation algorithms for unusual particular classes of graphs (in fact, subclasses of trees). Munier and Sourd proved that  $1|chains, d_{ij} = d, p_i = p|C_{\max}$  is polynomial. Lastly, Engels et al.[9] have

Table 2: Polynomial special cases

Polynomial Problem	Reference	Comments
$1 \text{tree}, d_{ij} = d, p_i = 1 C_{max}$	Bruno et al.[6]	Based on [14]
$1 \text{prec}, d_{ij} = 1, p_i = 1 C_{max}$	Leung et al. [18]	Based on [7]
$1 \text{interval orders}, d_{ij} = d, p_i = 1 C_{max}$	Leung et al.[18]	Based on [21]

developed a polynomial algorithm for  $P|\text{tree}, d_{ij} \leq D, p_i = 1|C_{max}$  if  $D$  is a constant value.

At last, there are some approximation algorithms for problems with delays : Graham's list scheduling algorithm [11] was extended to  $P|\text{prec. delays}, d_{ij} = k, p_j = 1|C_{max}$  to give a worst-case performance ratio of  $2 - 1/(m(k + 1))$  [15, 20]. This result was extended by Munier et al. [19] to  $P|\text{prec. delays}, d_{ij}|C_{max}$ . Bernstein and Gerner [5] study the performance ratio of the Coffman-Graham algorithm for  $P|\text{prec. delays}, d_{ij} = d, p_i = 1|C_{max}$  and slightly improve it in [4]. Schuurman [22] developed a polynomial approximation scheme for a particular class of precedence constraints. We prove in section 4 that the bound 2 of Graham's list algorithm may be achieved in the worst case for  $1|\text{bipartite}, d_{ij} = d, p_i = 1|C_{max}$  and we develop a simple algorithm with worst case performance ratio equal to  $3/2$  for this problem.

## 2 Complexity of the problem

Let us consider a non oriented graph  $G = (V, E)$  and an ordering  $L$  of the vertices of  $G$  (ie, a one-to-one function  $L : V \rightarrow \{1, \dots, |V|\}$ ). For all integer  $i \in \{1, \dots, |V|\}$ , the set  $V_L(i) \subset V$  is :

$$V_L(i) = \{v \in V, L(v) \leq i \text{ and } \exists u \in V, \{v, u\} \in E \text{ and } L(u) > i\}$$

VERTEX SEPARATION is then defined as :

- Instance : A non oriented graph  $G = (V, E)$  and a positive integer  $K$ .
- Question : Is there an ordering  $L$  of the vertices of  $G$  such that, for all  $i \in \{1, \dots, |V|\}$ ,  $|V_L(i)| \leq K$ ?

This problem is proved to be NP-complete in [17]. For the following, our proofs will be more elegant if we consider the converse ordering of the

tasks. Let  $n = |V|$ . If we set,  $\forall v \in V, L'(v) = n - L(v), j = n - i + 1$  and  $B_{L'}(j) = V_L(i)$ , we get for every value  $j \in \{1, \dots, n\}$  :

$$B_{L'}(j) = \{v \in V, L'(v) > j \text{ and } \exists u \in V, \{v, u\} \in E \text{ and } L'(u) \leq j\}$$

So, the equivalent INVERSE VERTEX SEPARATION problem may be defined as :

- Instance : A non oriented graph  $G = (V, E)$  and a positive integer  $K$ .
- Question : Is there an ordering  $L$  of the vertices of  $G$  such that, for all  $i \in \{1, \dots, |V|\}, |B_L(i)| \leq K$  ?

We prove the following theorem :

**Theorem 2.1.** *There exists a polynomial transformation  $f$  from INVERSE VERTEX SEPARATION to SEQUENCING WITH DELAYS.*

*Proof.* Let  $I$  be an instance of INVERSE VERTEX SEPARATION. The associated instance  $f(I)$  is given by a bipartite graph  $G' = (X \cup Y, E')$ , a delay  $d$  and a deadline  $D$  defined as :

1. To any vertex  $v \in V$  is associated two elements  $x_v \in X$  and  $y_v \in Y$  and an arc  $(x_v, y_v) \in E'$ .
2. To any edge  $\{u, v\} \in E$  is associated the arcs  $(x_u, y_v)$  and  $(x_v, y_u)$  in  $E'$ .
3. The delay is  $d = n - 1 - K$  and the deadline  $D = 2n$ .

$f$  can be clearly computed in polynomial time (see an example figure 1).

Let us suppose that  $L$  is a solution to the instance  $I$ . Then, we build a solution to  $f(I)$  as follows :

1. Tasks from  $Y$  are executed between time  $n$  and  $2n$  following  $L$  : they are executed from  $y_{L^{-1}(1)}$  to  $y_{L^{-1}(n)}$ .
2. Let us define the partition  $P_i, i = 1 \dots n$  of  $X$  as :

$$P_i = \{x_{L^{-1}(i)}\} \cup \{x_u, u \in B_L(i)\} - \bigcup_{j=1}^{i-1} P_j$$

Tasks from  $X$  are executed between 0 and  $n$  following  $P_1 \dots P_n$ .

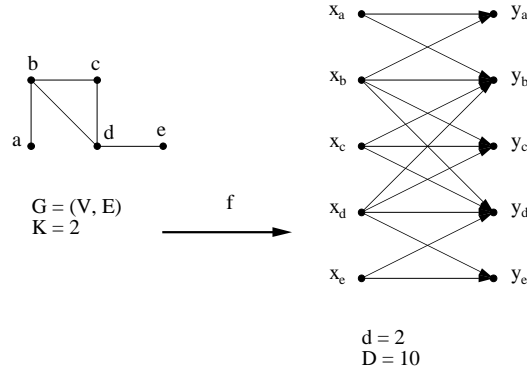


Figure 1: Example of transformation  $f$

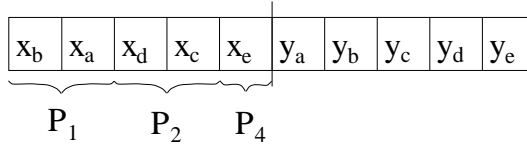


Figure 2: The schedule associated with  $L$

For example, if we consider the order defined by  $L(a) = 1$ ,  $L(b) = 2$ ,  $L(c) = 3$ ,  $L(d) = 4$  and  $L(e) = 5$ , the sets  $P_i$ ,  $i = 1 \dots 5$ , are defined by  $P_1 = \{x_a, x_b\}$ ,  $P_2 = \{x_c, x_d\}$ ,  $P_3 = \emptyset$ ,  $P_4 = \{x_e\}$  and  $P_5 = \emptyset$ . Figure 2 shows the corresponding solution for  $f(I)$  for our example.

We have to prove now that this schedule fulfill all the precedence delays of  $G'$ . Let us consider the task  $y_{L^{-1}(i)}$ ,  $i = 1 \dots n$ . We must show that all its predecessors in  $G'$  are completed at time  $(n + i - 1) - d = K + i$ .

1. We claim that all the predecessors of  $y_{L^{-1}(i)}$  in  $G'$  are in  $\bigcup_{j=1}^i P_j$ . Indeed,  $x_{L^{-1}(i)} \in P_j$ ,  $j \leq i$  by construction.

The other predecessors of  $y_{L^{-1}(i)}$  are vertices  $x_v$  with  $v$  adjacent to  $u = L^{-1}(i)$  in  $G$ . Now, if  $L(v) < L(u)$ , then  $x_v \in P_k$  with  $k \leq L(v)$ . Otherwise,  $v \in B_L(i)$  so  $x_v \in P_k$  with  $k \leq L(u)$ .

2. We show that  $|\bigcup_{j=1}^i P_j| \leq K + i$ . Indeed, this set is composed by : [1]  $i$  tasks  $x_{L^{-1}(j)}$ ,  $j = 1 \dots i$ , and [2] tasks  $x_u$  with  $L(u) > i$ , so  $u \in B_L(i)$ .

So, we built a solution to the instance  $f(I)$ .

Now, let us consider that we have a solution to  $f(I)$ . Since the graph  $G'$  is bipartite, we can exchange the tasks such that tasks from  $X$  are all completed before the first task from  $Y$ . We build an order  $L$  from tasks in  $Y$  such that,  $\forall i \in \{1, \dots, n\}$ ,  $L^{-1}(i)$  is the task  $u \in V$  such that  $y_u$  is executed at time  $n + i - 1$ . Then, we must prove that,  $\forall i \in \{1, \dots, n\}$ ,  $|B_L(i)| \leq K$ .

Let consider  $i \in \{1, \dots, n\}$ . Tasks executed during the interval  $[0, K + i)$  can be decomposed into [1]  $x_{L^{-1}(1)} \dots x_{L^{-1}(i)}$  and [2] A set  $Q_i$  of  $K$  other tasks from  $X \cup Y$ .

Let be  $v \in B_L(i)$ . We claim that  $x_v \in Q_i$ . Indeed, we get that  $L(v) > i$  and there exists  $u \in V$  with  $L(u) \leq i$  and  $\{u, v\} \in E$ . By definition of  $G'$ , we have then  $(x_v, y_u) \in E$ , so  $x_v \in Q_i$ .

We deduce that  $|B_L(i)| \leq |Q_i| = K$ . □

**Corollary 2.2.**  $1|bipartite, d_{ij} = d, p_i = 1|C_{max}$  is NP-Hard.

### 3 A polynomial special case

Let us consider a non oriented connected graph  $G = (V, E)$  without loops (*i.e.* without edges  $\{u, u\}$ ,  $u \in V$ ) and an ordering  $L$  of the vertices. We set  $|V| = n$ .  $\forall i \in \{1, \dots, n\}$ , we define the sequences  $E_L(i)$  by :

$$E_L(i) = \{\{u, v\} \in E, L(u) \leq i\}$$

$E_L(i)$  is the set of edges adjacent to at least one vertices in  $\{L^{-1}(1), \dots, L^{-1}(i)\}$ .

We define the problem MIN ADJACENT SET LINEAR ORDERING by :

- Instance : A non oriented graph  $G = (V, E)$  without loops and a positive integer  $K$ .
- Question : Is there an ordering  $L$  of the vertices of  $G$  such that, for all  $i \in \{1, \dots, |V|\}$ ,  $|E_L(i)| \leq K + i$  ?

Notice that the formulation of this problem is quite similar to MIN-CUT LINEAR ARRANGEMENT [10], which is NP-complete. In the following, we consider the subproblem  $\Pi$  of SEQUENCING WITH DELAYS with the restriction that the degree of every vertex from  $X$  is exactly 2.

**Theorem 3.1.** *There exists a polynomial transformation from  $\Pi$  to MIN ADJACENT SET LINEAR ORDERING*

*Proof.* Let us consider an instance  $I$  of  $\Pi$  given by a bipartite graph  $G = (X \cup Y, E)$ , a delay  $d$  and a deadline  $D$ . We build an instance  $f(I)$  of MIN ADJACENT SET LINEAR ORDERING as follows :

- $G' = (Y, E')$ . For every  $x \in X$  with  $(x, y_1)$  and  $(x, y_2) \in E$  is associated an edge  $e_x = \{y_1, y_2\}$  in  $E'$ .
- the value  $K = D - d - |Y| - 1$ .

$f$  can be computed in polynomial time. We prove now that  $f$  is a polynomial transformation (see figure 3 for an example)

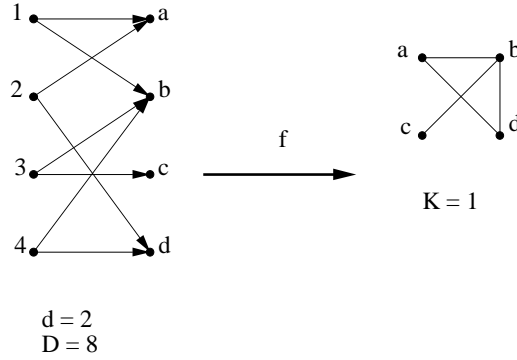


Figure 3: Example of transformation  $f$

Let us suppose that a solution to  $I$  is given. Then, without losing generality, we can suppose that the tasks from  $X$  are performed during  $[0, \dots, |X|)$  and tasks from  $Y$  during  $[D - |Y|, \dots, D)$ . We build a linear ordering  $L$  following the sequencing order of tasks  $Y : \forall i \in \{1, \dots, |Y|\}$ ,  $L(i)$  is the  $i$ th task of  $Y$  in the schedule.

$\forall i \in \{1, \dots, |Y|\}$ , let be  $t = D - |Y| + (i - 1) = K + i + d$  the starting time of the task  $L^{-1}(i)$  from  $Y$ . At time  $t - d = K + i$ , all the predecessors of  $L^{-1}(1), \dots, L^{-1}(i)$  must be completed. Now, for every edge  $e_x \in E_L(i)$  is associated exactly one of those predecessors. So,  $|E_L(i)| \leq K + i$ .

Conversely, let us suppose that a solution to  $f(\Pi)$  is given. Then, we perform tasks from  $Y$  following  $L$  during the interval  $[D - |Y|, \dots, D)$ . We define then the following sequence  $X_i \subset X$  :

1.  $X_1 = \{x \in X, e_x \in E_L(1)\}$ ,
2.  $\forall i = 2, \dots, n, X_i = \{x \in X, e_x \in E_L(i)\} - \bigcup_{j=1}^i X_j$ .



Notice that, by construction that,  $\forall i \in \{1, \dots, n\}, \bigcup_{j=1}^i X_j = \{e_x \in E_L(i)\}$ . Tasks of  $X$  are performed during  $[0, \dots, |X|)$  following  $X_1, X_2 \dots X_n$ . Every task from  $\bigcup_{j=1}^i X_j$  is then completed at time  $K + i$  (see figure 4 for the corresponding schedule).

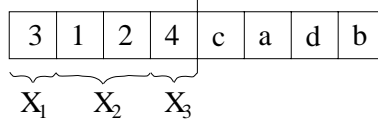


Figure 4: A corresponding schedule

We must prove that the delays constraints are fulfilled : let us consider the task  $y = (L^{-1}(i))$ . For every task  $x \in \Gamma^{-1}(y)$  is associated  $e_x \in E_L(i)$ . So,  $x \in \bigcup_{j=1}^i X_j$  and is completed at time  $K + i$ . Since  $y$  is performed at time  $t = D - |Y| + i - 1$ , we get :

$$t - (K + i) = D - |Y| + i - 1 - (K + i) = d$$

So, the delays are fulfilled.  $\square$

**Theorem 3.2.** *Let us consider an instance  $I$  of MIN ADJACENT SET LINEAR ORDERING given by a graph  $G = (V, E)$  and an integer  $K > 0$ . A necessary and sufficient condition for the existence of a solution is that*

$$|E| \leq K + |V| - 1$$

*Proof.* The condition is necessary : since the graph  $G$  is connected without loops, every linear ordering  $L$  verifies  $E_L(n - 1) = E$ . So, if  $L$  verifies the condition, we get the condition of the theorem.

The condition is sufficient : let us consider a linear ordering  $L$  and a family of graph  $G_i, i = 0, \dots, n$  defined such that,

- $G_0 = G$ ,
- $\forall i = 1, \dots, n$ , we choose a vertex  $u$  in the subgraph  $G_{i-1} = (V - \{L^{-1}(1), \dots, L^{-1}(i - 1)\}, E)$  with a minimum degree in  $G_{i-1}$  and we set  $L(u) = i$ .
- $G_n = \emptyset$ .

We note  $E_i$  the edges of  $G_i$ . Notice that,  $\forall i = 1, \dots, n$ , the two sets  $E_L(i)$  and  $E_i$  are a partition of  $E$ .

We prove by contradiction that the linear ordering  $L$  is a solution to MIN ADJACENT SET LINEAR ORDERING.

- Let us suppose that  $|E_L(1)| \geq K + 2$ , then the degree of any vertex in  $G$  is greater than or equal to  $K + 2$ . So,  $2|E| \geq |V|(K + 2)$ . By hypothesis, we get  $2K + 2|V| - 2 \geq K|V| + 2|V|$ , so  $K(2 - |V|) \geq 2$ . Since  $K > 0$ , we get that  $|V| < 2$ , so  $|V| = 1$ . In this case, we get  $|E_L(1)| = |E| = 0$ , which contradicts  $|E| \geq K + 2$ .
- Now, let us suppose that, for  $i < n - 2, \forall j \in \{1, \dots, i\}, |E_L(j)| \leq K + j$  and that  $|E_L(i + 1)| \geq (i + 1) + K + 1$ . For every vertex  $u \in G_i$ , we set  $d_{G_i}(u)$  the degree of  $u$  in  $G_i$ .

The total number of edges verifies  $|E| = |E_L(i + 1)| + |E_{i+1}|$ .

1. By hypothesis,  $|E_L(i + 1)| \geq (i + 1) + K + 1$ .
2. By definition of the sequences  $G_i, |E_{i+1}| = |E_i| - d_{G_i}(L^{-1}(i + 1))$ . Since  $u = L^{-1}(i + 1)$  is the vertex of  $G_i$  with a minimum degree, the number of arcs of  $G_i$  verifies

$$2|E_i| \geq (n - i)d_{G_i}(L^{-1}(i + 1))$$

So,

$$|E_{i+1}| \geq \frac{1}{2}(n - i)d_{G_i}(L^{-1}(i + 1)) - d_{G_i}(L^{-1}(i + 1))$$

We show that  $d_{G_i}(L^{-1}(i + 1)) \geq 2$ . Indeed, let us denote by  $e(k) = \{L^{-1}(i + 1), L^{-1}(k)\}$  an edge of  $G$  adjacent to  $L^{-1}(i + 1)$ . Then, we get easily that  $E_L(i + 1) - E_L(i) = \{e(k) \in G_i\}$ , so

$$d_{G_i}(L^{-1}(i + 1)) = |E_L(i + 1)| - |E_L(i)| \geq (i + 1) + K + 1 - (K + i) = 2$$

We deduce that

$$|E_{i+1}| \geq \frac{n - i - 2}{2}d_{G_i}(L^{-1}(i + 1)) \geq n - i - 2$$

So, the total number of edges of  $G$  verifies :

$$|E| = |E_L(i + 1)| + |E_{i+1}| \geq (i + 1) + K + 1 + n - i - 2 = |V| + K$$

which contradicts the hypothesis of the theorem.  $\square$

Notice that this proof is constructive : if the condition of the theorem is fulfilled, one can easily implements a greedy polynomial algorithm to build a linear ordering.

**Corollary 3.3.**  $\Pi$  is polynomial.

If we heavily sort the the vertices at each step of the algorithm, the complexity of the algorithm will be bounded by  $O(n^2 \log n + m)$ .

## 4 An Approximation algorithm

In this section, we consider the analysis of the performances of two approximation algorithms.

The first one is the classical Graham list scheduling algorithm [12]. At each time  $t$ , a schedulable task is chosen to be performed without any priority rule. For the bipartite graph  $G = (X \cup Y, E)$ , it consists on performing tasks from  $X$  in any order and tasks from  $Y$  as soon as possible. Several authors show that the performance ratio of this algorithm is upper bounded asymptotically by 2 [15, 20, 19]. We prove here that this bound is reached for bipartite graphs :

**Theorem 4.1.** *The performance ratio of a list scheduling for a bipartite graph tends asymptotically to 2.*

*Proof.* Let us consider a value  $d > 0$  and a bipartite graph  $G = (X \cup Y, E)$  with  $X = \{a_1, \dots, a_d\} \cup \{b\}$ ,  $Y = \{c\}$  and  $E = \{(b, c)\}$ . In the worst case for the Graham list scheduling algorithm, tasks  $\{a_1, \dots, a_d\}$  are performed first. We get then a schedule of length  $l_1 = 2d + 2$ .

Now, we can get a schedule without idle slots if we perform  $b$  first. The length of this second schedule is then  $l_2 = d + 2$ .

The performance ratio is then bounded by :  $r = \frac{2d+2}{d+2} = 2 - \frac{2}{d+2} \rightarrow_{d \rightarrow \infty} 2$ .  $\square$

We present now a slightly better approximation algorithm : let us suppose that  $G = (X \cup Y, E)$  with  $|X| = n$ ,  $|Y| = m$  and  $n \geq m$ . In the opposite, we modify the orientation of the edges and we consider the graph  $G' = (Y \cup X, E')$ . We can get a feasible schedule for  $G$  by considering the inverse order of a schedule for  $G'$ .

Let us consider the set  $X_1$  of tasks from  $X$  with a strictly positive out-degree (*i.e.*,  $X_1$  is the set of  $X$  with at least one successor in  $Y$ ). The idea is to apply a list scheduling algorithm which performs tasks from  $X_1$  before those from  $X_2 = X - X_1$ .

We denote by  $C_{opt}$  (resp.  $C_H$ ) the makespan of an optimal schedule (resp. a schedule obtained using this algorithm). We set  $|X_i| = n_i, i = 1, 2$  and  $p = \max(0, d + 1 - n_2 - m)$ . We prove the following upper bound on  $C_{opt}$  :

**Lemma 4.2.**  $C_{opt} \geq n + m + p$ .

*Proof.* The last task of  $X_1$  is performed at time  $t \geq n_1$  and has at least one successor in  $Y$ , so  $C_{opt} \geq n_1 + d + 1$ . Now, if  $p = d + 1 - n_2 - m$ ,

$n + m + p = n + m + d + 1 - n_2 - m = n_1 + d + 1$  and the inequality is true. Otherwise,  $p = 0$  and we get obviously  $C_{opt} \geq n + m$ .  $\square$

**Theorem 4.3.** *The performance ratio of this algorithm is bounded by  $\frac{3}{2}$ .*

*Proof.* We denote by  $\mathcal{I}$  the idle slots of the schedule obtained by our algorithm. We get, using the previous lemma :

$$C_H = n + m + |\mathcal{I}| \leq C_{opt} + (|\mathcal{I}| - p)$$

1. If  $|\mathcal{I}| \leq p$ , we get the theorem.
2. Let us assume now that  $|\mathcal{I}| > p$ . We build a subset  $\mathcal{I}_p \subset \mathcal{I}$  by removing from  $\mathcal{I}$  the  $p$ th first idle slots in our schedule. Let be an element  $k \in \mathcal{I}_p$  and  $t(k)$  the time of this idle slot.

Clearly, by definition of  $\mathcal{I}_p$ ,  $t(k) \geq p + n$ . Moreover, there is at least one task from  $y \in Y$  performed after  $t(k)$  such that  $y$  is not ready at time  $t(k)$ , so  $t(k) \leq n_1 + d$ . We get

$$|\mathcal{I}| - p = |\mathcal{I}_p| \leq n_1 + d - (p + n)$$

Then,

$$|\mathcal{I}| - p = |\mathcal{I}_p| \leq d - n_2 - \max(0, d + 1 - n_2 - m)$$

We deduce that

$$|\mathcal{I}_p| \leq \min(d - n_2, m - 1)$$

So,  $|\mathcal{I}_p| \leq |Y|$ .

Now, the inequality between  $C_H$  and  $C_{opt}$  becomes :

$$C_H \leq C_{opt} + |\mathcal{I}_p| \leq C_{opt} + |Y|$$

Since  $|Y| \leq |X|$ , we get that  $|Y| \leq \frac{1}{2}(|X| + |Y|) \leq \frac{1}{2}C_{opt}$  and we get the theorem.  $\square$

We can prove that the bound  $\frac{3}{2}$  is asymptotically tight : indeed, let us consider an integer  $n > 0$  and the bipartite graph  $G = (X \cup Y, E)$  with  $X = \{x_1, \dots, x_n\}$ ,  $Y = \{y_1, \dots, y_n\}$  and the arcs  $E = \{(x_i, y_j), 1 \leq j \leq i \leq n\}$ . We set  $d = n - 1$ . Note that  $|X| = n = |Y|$ .

If we perform task from  $X$  such that  $t(x_i) = i - 1, i = 1, \dots, n$ , then tasks from  $Y$  can't be performed before  $n + d - 1$ . So, we get a makespan  $L_1 = 3n - 2$ .

Now, if we perform task from from  $X$  such that  $t(x_i) = n - i, i = 1, \dots, n$ , then we get a schedule without idle slots with makespan  $L_2 = 2n$ .

So, we get  $\frac{L_1}{L_2} \rightarrow_{n \rightarrow +\infty} \frac{3}{2}$ .

## 5 Conclusions

Several new questions arise from the results presented here :

- In order to study the borderline between NP-complete and polynomial problems, the complexity of the problem with a bipartite graph where the degree of vertices from  $X$  does not exceed 3 is an interesting problem.
- The existence of better approximation algorithms is also an interesting question.

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