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# Feedback stabilization of parabolic systems with input delay

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## Abstract

This work is devoted to the stabilization of parabolic systems with a finite-dimensional control subjected to a constant delay. Our main result shows that the Fattorini-Hautus criterion yields the existence of such a feedback control, as in the case of stabilization without delay. The proof consists in splitting the system into a finite dimensional unstable part and a stable infinite-dimensional part and to apply the Artstein transformation on the finite-dimensional system to remove the delay in the control. Using our abstract result, we can prove new results for the stabilization of parabolic systems with constant delay: the  $N$ -dimensional linear reaction-convection-diffusion equation with  $N \geq 1$  and the Oseen system. We end the article by showing that this theory can be used to stabilize nonlinear parabolic systems with input delay by proving the local feedback distributed stabilization of the Navier-Stokes system around a stationary state.

**Keywords:** stabilizability, delay control, parabolic systems, finite-dimensional control

**2010 Mathematics Subject Classification** 93B52, 93D15, 35Q30, 76D05, 93C20.

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## 1 Introduction

Time delay phenomena appear in many applications, for instance in biology, mechanics, automatic control or engineering and are inevitable due to the time-lag between the measurements and their exploitation. For instance in control problems, one need to take into account the analysis time or the computation time. We aim at showing that, under quite general hypotheses, one can deduce the exponential stabilization with delay of a parabolic system from its exponential stabilization without delay. One of the first article devoted to the parabolic case is [23] with a backstepping method (see [13] for a similar method for the wave equation). We can also quote [12], [28], where the approach is to construct a feedback by a predictor approach. Several works have considered different extensions to this problem: the case of non constant delay (see, for instance, [9], [29])

1 or the case of multiple delay (see, for instance, [10]). Note that in the context of stability problems for partial  
2 differential equations with delay, some particular features can appear for hyperbolic systems: a small delay in  
3 the feedback mechanism can destabilize a system (see for instance [16, 15]) and a delay term can also improve  
4 the performance of a system (see for instance [1]). It is not known if these phenomena occur also for parabolic  
5 systems.

6 This article is devoted to the feedback stabilization of the system

$$z' = Az + Bv + f, \quad z(0) = z^0, \quad (1.1)$$

7 where  $A$  is the generator of an analytic semigroup  $(e^{tA})_{t \geq 0}$  on a Hilbert space  $\mathbb{H}$ , where  $B : \mathbb{U} \rightarrow \mathcal{D}(A^*)'$  is a  
8 linear operator on a Hilbert space  $\mathbb{U}$  and where  $f$  is a given source satisfying an exponential decay at infinity.  
9 The purpose of this source term in what follows is to handle nonlinearities (see Section 5).

10 Our aim is to obtain a feedback control  $v(t)$  that depends on the values of  $z(s)$  for  $s \leq t - \tau$ , where  $\tau > 0$   
11 is a positive constant corresponding to a delay. With such a feedback control, our aim is to obtain exponential  
12 stabilization of (1.1) if we assume that it is the case without delay. A characterization of the exponential  
13 stabilization of (1.1) with rate  $\sigma > 0$  in the case without delay is the well-known Fattorini-Hautus criterion (see  
14 [19], [20] and [5]):

$$\forall \varepsilon \in \mathcal{D}(A^*), \forall \lambda \in \mathbb{C}, \operatorname{Re} \lambda \geq -\sigma \quad A^* \varepsilon = \lambda \varepsilon \quad \text{and} \quad B^* \varepsilon = 0 \quad \implies \quad \varepsilon = 0. \quad (\text{UC}_\sigma)$$

15 In the case of a system governed by partial differential equations, this criterion corresponds generally to a  
16 Unique Continuation property (UC) and in many situations it holds true for any  $\sigma$  (see the examples given in  
17 this article). Here and in what follows, we denote by  $A^* : \mathcal{D}(A^*) \rightarrow \mathbb{H}$  and by  $B^* : \mathcal{D}(A^*) \rightarrow \mathbb{U}$  the adjoint  
18 operators of  $A$  and  $B$ . Note that we have identified  $\mathbb{H}$  and  $\mathbb{U}$  with their duals. This criterion is equivalent, in  
19 the case without delay, to the exponential stabilization of (1.1) with a rate larger than  $\sigma$  provided we assume  
20 the following hypotheses:

$$\begin{aligned} \text{The spectrum of } A \text{ consists of isolated eigenvalues } (\lambda_j) \text{ with finite algebraic multiplicity } N_j \\ \text{and there is no finite cluster point in } \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\sigma\}, \end{aligned} \quad (\text{Hyp1})$$

$$B \in \mathcal{L}(\mathbb{U}, \mathbb{H}_{-\gamma}) \quad \text{for some } \gamma \in [0, 1). \quad (\text{Hyp2})$$

22 Let  $\rho(A)$  be the resolvent set of  $A$ . The spaces  $\mathbb{H}_\alpha$  are defined as follows: we fix  $\mu_0 \in \rho(A)$ , then

$$\mathbb{H}_\alpha := \begin{cases} \mathcal{D}((\mu_0 - A)^\alpha) & \text{if } \alpha \geq 0 \\ \mathcal{D}((\mu_0 - A^*)^{-\alpha})' & \text{if } \alpha < 0 \end{cases} \quad \text{and} \quad \mathbb{H}_\alpha^* := \begin{cases} \mathcal{D}((\mu_0 - A^*)^\alpha) & \text{if } \alpha \geq 0 \\ \mathcal{D}((\mu_0 - A)^{-\alpha})' & \text{if } \alpha < 0. \end{cases} \quad (1.2)$$

We recall that if  $\alpha > 0$ , a norm for  $\mathbb{H}_{-\alpha}$  is

$$\|f\|_{\mathbb{H}_{-\alpha}} := \|(\mu_0 - A)^{-\alpha} f\|_{\mathbb{H}},$$

23 see, for instance, [37, Proposition 2.10.2]. To deal with the source  $f$ , we also assume the following hypothesis

$$\mathbb{H}_\alpha = [\mathbb{H}, \mathcal{D}(A)]_\alpha \quad (\alpha \in [0, 1]), \quad (\text{Hyp3})$$

24 where  $[\cdot, \cdot]_\alpha$  denotes the complex interpolation method (see, for instance, [36, Section 1.9, pp.55-61]). Using  
25 [36, p.143, Remarks 3 and 4], we have that  $\mathbb{H}_\alpha = (\mathbb{H}, \mathcal{D}(A))_{\alpha, 2}$  for  $\alpha \in [0, 1]$ , where  $(\cdot, \cdot)_{\alpha, p}$  denotes the real  
26 interpolation method (see, for instance, [36, Section 1.3.2, p.24]).

27 We assume that

$$f_\sigma : t \mapsto e^{\sigma t} f(t) \in L^2(0, \infty; \mathbb{H}_{-\gamma'}) \quad \gamma' < 1/2. \quad (1.3)$$

We say that  $f \in L^2_\sigma(0, \infty; \mathbb{H}_{-\gamma'})$  if  $f_\sigma \in L^2(0, \infty; \mathbb{H}_{-\gamma'})$  and we write

$$\|f\|_{L^2_\sigma(0, \infty; \mathbb{H}_{-\gamma'})} = \|f_\sigma\|_{L^2(0, \infty; \mathbb{H}_{-\gamma'})}.$$

1 The same definition can be extended to spaces of the kind  $L_\sigma^p(0, \infty; \mathbb{X})$ ,  $C_\sigma^0([0, \infty); \mathbb{X})$ ,  $H_\sigma^m(0, \infty; \mathbb{X})$ , with  $\mathbb{X}$  a  
 2 Banach space.

Note that a sufficient condition for (Hyp1) is that  $A$  has compact resolvent. For all  $\lambda_j$  eigenvalue of  $A$ , we define its geometric multiplicity

$$\ell_j := \dim \ker(A - \lambda_j \text{Id}) \in \mathbb{N}^*.$$

3 Here and after,  $\mathbb{N}^*$  is the set of the positive integers.

4 We also define the maximum of the geometric multiplicities of the unstable modes:

$$N_+ := \max\{\ell_i : \text{Re } \lambda_i \geq -\sigma\}. \quad (1.4)$$

5 Finally, let us define the subset

$$D_\infty := \{(t, s) \in \mathbb{R}^2 : t \in (0, \infty), s \in (0, t)\}. \quad (1.5)$$

6 Our main result is the following theorem:

7 **Theorem 1.1.** *Let us consider  $\sigma > 0$  and let us assume (Hyp1), (Hyp2), (Hyp3) and (UC $_\sigma$ ). Then there exist*  
 8  *$K \in L_{\text{loc}}^\infty(D_\infty; \mathcal{L}(\mathbb{H}))$ ,  $\zeta_k \in \mathcal{D}(A^*)$ ,  $v_k \in B^*(\mathcal{D}(A^*))$ ,  $k = 1, \dots, N_+$ , such that if*

$$v(t) = \mathbb{1}_{[\tau, +\infty)}(t) \sum_{k=1}^{N_+} \left( z(t - \tau) + \int_0^{t-\tau} K(t - \tau, s) z(s) ds, \zeta_k \right)_{\mathbb{H}} v_k, \quad (1.6)$$

9 then for any  $z^0 \in \mathbb{H}$ ,  $f$  satisfying (1.3), the solution  $z$  of (1.1) satisfies

$$\|z(t)\|_{\mathbb{H}} \leq C e^{-\sigma t} \left( \|z^0\|_{\mathbb{H}} + \|f\|_{L_\sigma^2(0, \infty; \mathbb{H}_{-\gamma'})} \right) \quad (t > 0). \quad (1.7)$$

Assume moreover that  $\gamma = 0$ ,  $\gamma' = 0$  and that  $z^0 \in \mathbb{H}_{1/2}$ . Then,

$$z \in L_\sigma^2(0, \infty; \mathbb{H}_1) \cap C_\sigma^0([0, \infty); \mathbb{H}_{1/2}) \cap H_\sigma^1(0, \infty; \mathbb{H}),$$

10 and

$$\|z\|_{L_\sigma^2(0, \infty; \mathbb{H}_1) \cap C_\sigma^0([0, \infty); \mathbb{H}_{1/2}) \cap H_\sigma^1(0, \infty; \mathbb{H})} \leq C \left( \|z^0\|_{\mathbb{H}_{1/2}} + \|f\|_{L_\sigma^2(0, \infty; \mathbb{H})} \right). \quad (1.8)$$

11 Here and in all what follows,  $\mathbb{1}_\mathcal{O}$  is the characteristic function of the set  $\mathcal{O}$ . In the above statement and  
 12 in the whole paper, we use  $C$  as a generic positive constant that does not depend on the other terms of the  
 13 inequality. The value of the constant  $C$  may change from one appearance to another.

14 **Remark 1.2.** *Note that in the statement of Theorem 1.1, the kernel  $K$  can be obtained as the solution of a*  
 15 *Volterra's type integral equation involving  $A$  and  $B$ , see Lemma 2.2.*

16 The above result shows that we can stabilize a general class of linear parabolic systems with a finite number  
 17 of controls and with a constant delay: the feedback control  $v(t)$  at time  $t$ , given by (1.6), only depends on  
 18 values of the state  $z(s)$  for  $s \leq t - \tau$ . This result can be seen as a generalization of several recent results on the  
 19 stabilization of parabolic systems with delay control, in particular [14] where the authors constructed a feedback  
 20 control for finite dimensional linear systems, and [31] where the authors obtained a stabilizing feedback control  
 21 of a one-dimensional reaction-diffusion equation with a boundary control subjected to a constant delay. Let us  
 22 mention some ideas of their method that we adapt to prove our result: using that their operator is self-adjoint of  
 23 compact resolvent they split the system into an unstable finite-dimensional part and a stable infinite-dimensional  
 24 part. They are thus led to stabilize the finite-dimensional unstable system and to do this with a delay, they  
 25 use the Artstein transformation, see [2], and obtain an autonomous control system without delay satisfying the  
 26 Kalman condition. Finally, by using an appropriate Lyapunov function, they prove that the feedback control  
 27 designed in the finite-dimensional part actually stabilizes the whole system.

1 We can mention several articles in this direction: in [26], the authors consider the stabilization of a struc-  
 2 turally damped Euler-Bernoulli beam. The corresponding system is parabolic but the main operator is no more

self-adjoint. Then [25] generalizes the result of [31] in the case where the main operator is a Riesz spectral operator with simple eigenvalues. The work in [27] extends the result of [31] to the case where the control contains some disturbances and where the delay can depend on time.

Here our aim is to extend the result of [31] for a large class of parabolic systems, and in particular with the possibility to consider partial differential equations written in a spatial domain with dimension larger than one. We also precise the number of controls  $N_+$  needed to stabilize the system by using the approach developed in [4] in the case of the Navier-Stokes system or in [5], for general linear and nonlinear parabolic systems. We present two important examples, that is the reaction-diffusion equation and the Oseen system and we end this paper to show that within this framework, we can also handle some nonlinear parabolic systems such as the Navier-Stokes system. Other results on stabilization by finite dimensional controls could be mentioned here, for example [3, 6, 7, 8, 24, 33], etc.

The present paper is organized as follows. In Section 2, the proof of Theorem 1.1 is given. As in [31], it relies on the decomposition of the system (1.1) into two parts: an unstable finite-dimensional part and an infinite-dimensional part. This decomposition is possible thanks to (Hyp1) and [22, Theorem 6.17, p.178]. Due to the presence of a constant delay, an equivalent autonomous control system is considered for the finite-dimensional part by means of the Artstein transformation. This system is exponentially stabilizable by using  $(UC_\sigma)$ . Using the inverse of the Artstein transform, a stabilizing feedback control is designed in the finite-dimensional space that stabilizes exponentially the finite-dimensional unstable system (with delay control). Finally, we prove that the designed feedback stabilizes exponentially the complete system. Thereafter, we illustrate our results by some precise examples: the case of the feedback stabilization of the  $N$ -dimensional linear convection-diffusion equation with  $N \geq 1$  with delay boundary control in Section 3, the case of the feedback stabilization of the Oseen system with delayed distributed control in Section 4 and finally, a local feedback distributed stabilization of the Navier-Stokes system around a stationary state in Section 5.

## 2 Proof of Theorem 1.1

We consider below a decomposition that is already detailed and used in several previous articles (see, for instance, [34], [4], [5]). We recall it for sake of completeness.

Let us consider  $\sigma > 0$ . We first decompose the spectrum of  $A$  into the “unstable” modes and the “stable” modes:

$$\Sigma_+ := \{\lambda_j : \operatorname{Re} \lambda_j \geq -\sigma\}, \quad \Sigma_- := \{\lambda_j : \operatorname{Re} \lambda_j < -\sigma\}. \quad (2.1)$$

Using that  $(e^{tA})_{t \geq 0}$  is an analytic semigroup (see [11, Theorem 2.11, p.112]) and (Hyp1), we see that  $\Sigma_+$  is of finite cardinal.

Thus, we can introduce the projection operator (see [22, Thm. 6.17, p.178]) defined by

$$P_+ := \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda - A)^{-1} d\lambda, \quad (2.2)$$

where  $i$  is the imaginary unit and  $\Gamma_+$  is a contour enclosing  $\Sigma_+$  but no other point of the spectrum of  $A$ . We can define

$$\mathbb{H}_+ := P_+ \mathbb{H}, \quad \mathbb{H}_- := (\operatorname{Id} - P_+) \mathbb{H}.$$

From [22, Thm. 6.17, p.178], we have  $\mathbb{H}_+ \oplus \mathbb{H}_- = \mathbb{H}$  and if we set

$$A_+ := A|_{\mathbb{H}_+} : \mathbb{H}_+ \rightarrow \mathbb{H}_+, \quad A_- := A|_{\mathbb{H}_-} : \mathcal{D}(A) \cap \mathbb{H}_- \rightarrow \mathbb{H}_-,$$

then the spectrum of  $A_+$  (resp.  $A_-$ ) is exactly  $\Sigma_+$  (resp.  $\Sigma_-$ ). By using the analyticity of  $(e^{At})_{t \geq 0}$ , (Hyp1) and (2.1), we deduce the existence of  $\sigma_- > \sigma$  such that

$$\|e^{A_- t}\|_{\mathcal{L}(\mathbb{H}_-)} \leq C e^{-\sigma_- t}, \quad \|(\lambda_0 - A)^\gamma e^{A_- t}\|_{\mathcal{L}(\mathbb{H}_-)} \leq C \frac{1}{t^\gamma} e^{-\sigma_- t} \quad (2.3)$$

(see, for instance, [30, Theorem 6.13, p.74] for the second relation.)

2 We can proceed similarly for  $A^*$ : we write

$$P_+^* := \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda - A^*)^{-1} d\lambda, \quad (2.4)$$

$$\mathbb{H}_+^* := P_+^* \mathbb{H}, \quad \mathbb{H}_-^* := (\text{Id} - P_+^*) \mathbb{H},$$

$$A_+^* := A|_{\mathbb{H}_+^*} : \mathbb{H}_+^* \rightarrow \mathbb{H}_+^*, \quad A_-^* := A|_{\mathbb{H}_-^*} : \mathcal{D}(A^*) \cap \mathbb{H}_-^* \rightarrow \mathbb{H}_-^*.$$

3 Note that  $P_+^*$  is the adjoint of  $P_+$ . In particular, we see that if  $z \in \mathbb{H}_-$  and  $\zeta \in \mathbb{H}_+^*$ , then

$$(z, \zeta)_{\mathbb{H}} = ((\text{Id} - P_+)z, \zeta)_{\mathbb{H}} = (z, (\text{Id} - P_+^*)\zeta)_{\mathbb{H}} = 0. \quad (2.5)$$

We also define

$$\mathbb{U}_+ := B^* \mathbb{H}_+^*, \quad \mathbb{U}_- := B^* (\mathcal{D}(A^*) \cap \mathbb{H}_-^*),$$

4 and

$$p_+ : \mathbb{U} \rightarrow \mathbb{U}_+, \quad p_- : \mathbb{U} \rightarrow \mathbb{U}_-, \quad i_+ : \mathbb{U}_+ \rightarrow \mathbb{U}, \quad i_- : \mathbb{U}_- \rightarrow \mathbb{U}, \quad (2.6)$$

5 the orthogonal projections and the inclusion maps. Note that we have the following relations for the above  
6 maps:

$$i_+ = p_+^*, \quad i_- = p_-^*. \quad (2.7)$$

As explained in [4] (see also [34] and [5]), we can extend  $P_+$  and  $(I - P_+)$  as bounded operators

$$P_+ \in \mathcal{L}(\mathbb{H}_{-1}, \mathbb{H}_+), \quad (\text{Id} - P_+) \in \mathcal{L}(\mathbb{H}_{-1}, [\mathcal{D}(A^*) \cap \mathbb{H}_-^*]').$$

7 We can thus define

$$B_+ := P_+ B i_+ \in \mathcal{L}(\mathbb{U}_+, \mathbb{H}_+), \quad B_- := (\text{Id} - P_+) B i_- \in \mathcal{L}(\mathbb{U}_-, [\mathcal{D}(A^*) \cap \mathbb{H}_-^*]'). \quad (2.8)$$

It is proved in [4] (see also [34] and [5]) that

$$P_+ B = B_+ p_+, \quad (\text{Id} - P_+) B = B_- p_-.$$

8 From the above relation, taking the projections  $P_+$  and  $\text{Id} - P_+$  of (1.1), we see that it splits into the two  
9 equations (see [4, 5, 34]).

$$z_+' = A_+ z_+ + B_+ p_+ v + P_+ f, \quad z_+(0) = P_+ z^0, \quad (2.9)$$

$$z_-' = A_- z_- + B_- p_- v + (\text{Id} - P_+) f, \quad z_-(0) = (\text{Id} - P_+) z^0. \quad (2.10)$$

In order to study the stabilization of the finite-dimensional system (2.9), we use the Artstein transformation (see [2]) that allows us to pass from (2.9) in the case of a delay input to an autonomous system. More precisely, we consider

$$w(t) := z_+(t) + \int_t^{t+\tau} e^{(t-s)A_+} B_+ p_+ v(s) ds.$$

11 Then in what follows, we study the stabilization of the autonomous system satisfied by  $w$  (Lemma 2.1). Since  
12 the corresponding feedback is expressed with  $w$ , we also consider the inverse of the Artstein transformation and  
13 more precisely show the existence of a kernel  $K$  to write  $w$  in terms of  $z_+$  (Lemma 2.2).

14 **Lemma 2.1.** *Assume  $(\text{UC}_\sigma)$  for  $\sigma > 0$ . Then, there exist  $C > 0$  and  $G \in \mathcal{L}(\mathbb{H}_+, \mathbb{U}_+)$ , with  $\text{rank } G \leq N_+$  where  
15  $N_+$  is defined by (1.4), such that for any  $f \in L_\sigma^2(0, \infty; \mathbb{H}_{-\gamma'})$  and  $w^0 \in \mathbb{H}_+$ , the solution of*

$$\begin{cases} w' = A_+ w + e^{-\tau A_+} B_+ p_+ G w + P_+ f, \\ w(0) = w^0, \end{cases} \quad (2.11)$$

16 satisfies

$$\|w\|_{H_\sigma^1(0, \infty; \mathbb{H}_+)} \leq C (\|w^0\|_{\mathbb{H}_+} + \|P_+ f\|_{L_\sigma^2(0, \infty; \mathbb{H}_+)}). \quad (2.12)$$

1 Note that in the above statement, we have that  $p_+G = G$  and we could thus simplify the first equation of  
 2 (2.11). We keep  $p_+$  so that we can consider  $G$  as an operator in  $\mathcal{L}(\mathbb{H}_+, \mathbb{U})$ .

*Proof of Lemma 2.1.* First we notice that  $(A_+, e^{-\tau A_+} B_+ p_+)$  satisfies the Fattorini-Hautus test: assume that  $\varepsilon \in \mathbb{H}_+^*$  satisfies

$$A_+^* \varepsilon = \overline{\lambda_j} \varepsilon, \quad B_+^* e^{-\tau A_+^*} \varepsilon = 0.$$

Then we deduce

$$A^* \varepsilon = \overline{\lambda_j} \varepsilon, \quad B_+^* e^{-\tau A_+^*} \varepsilon = e^{-\tau \overline{\lambda_j}} i_+^* B^* P_+^* \varepsilon = e^{-\tau \overline{\lambda_j}} B^* \varepsilon = 0.$$

3 Note that here we have used (2.7) and the fact that  $B^* P_+^* \varepsilon = B^* \varepsilon \in \mathbb{U}_+$ .

Thus from  $(UC_\sigma)$ , we deduce  $\varepsilon = 0$ . We can thus use the standard result of Fattorini or Hautus (see also [5, Theorem 1.6]) for a finite-dimensional system: for any  $\sigma_* > \sigma$ , there exists  $G \in \mathcal{L}(\mathbb{H}_+, \mathbb{U}_+)$ , with  $\text{rank } G \leq N_+$  such that the operator

$$A_* = A_+ + e^{-\tau A_+} B_+ p_+ G$$

satisfies

$$\|e^{A_* t}\|_{\mathcal{L}(\mathbb{H}_+)} \leq C e^{-\sigma_* t} \quad (t \geq 0).$$

4 We recall that since the system is finite-dimensional, we can take  $\sigma_*$  arbitrarily large by using the classical  
 5 pole-assignment theorem (see, for instance, [11, Theorem 2.4, p.21]).

6 In particular,  $A_* + \sigma \text{Id}_{\mathbb{H}_+}$  is of negative type and is the infinitesimal generator of an analytic semigroup in  
 7  $\mathbb{H}_+$  (with domain  $\mathbb{H}_+$ ). Thus considering  $\tilde{w}(t) = e^{\sigma t} w(t)$  with  $w$  solution of (2.11) and applying [11, Theorem  
 8 3.1, p.143], we deduce (2.12).  $\square$

9 We recall that  $D_\infty$  is defined by (1.5). The following result concerns a Volterra's type integral equation. The  
 10 methods to solve such an equation are quite classical (see, for instance, [38, Chapter 4]). However since here  
 11 the limits of the integral are non standard, we give below the short proof of this result. We recall that  $p_+$  and  
 12  $B_+$  are defined by (2.6) and (2.8).

13 **Lemma 2.2.** *Assume  $G \in \mathcal{L}(\mathbb{H}_+, \mathbb{U}_+)$ .*

*There exists  $K \in L_{\text{loc}}^\infty(D_\infty; \mathcal{L}(\mathbb{H}_+))$  such that*

$$K(t, s) = e^{(t-s-\tau)A_+} B_+ p_+ G \mathbf{1}_{(\max\{t-\tau, 0\}, t)}(s) + \int_{\max\{t-\tau, s\}}^t e^{(t-\xi-\tau)A_+} B_+ p_+ G K(\xi, s) d\xi \quad (t > 0, s \in (0, t)). \quad (2.13)$$

*Proof.* The proof relies on a fixed point argument. We set

$$K_0(t) := e^{(t-\tau)A_+} B_+ p_+ G, \quad K_0 \in L^\infty(0, \tau; \mathcal{L}(\mathbb{H}_+)),$$

so that (2.13) writes

$$K(t, s) = K_0(t-s) \mathbf{1}_{(\max\{t-\tau, 0\}, t)}(s) + \int_{\max\{t-\tau, s\}}^t K_0(t-\xi) K(\xi, s) d\xi.$$

Let  $T > 0$ , and let us define

$$D_T = \{(t, s) \in \mathbb{R}^2 \mid t \in (0, T), \quad s \in (0, t)\},$$

and

$$\Phi : L^\infty(D_T; \mathcal{L}(\mathbb{H}_+)) \rightarrow L^\infty(D_T; \mathcal{L}(\mathbb{H}_+)),$$

$$(\Phi K)(t, s) = \int_{\max\{t-\tau, s\}}^t K_0(t-\xi) K(\xi, s) d\xi, \quad ((t, s) \in D_T).$$

The mapping  $\Phi$  is well-defined, and is a linear and bounded operator of  $L^\infty(D_T; \mathcal{L}(\mathbb{H}_+))$ . Moreover,

$$\|(\Phi K)(t, s)\|_{\mathcal{L}(\mathbb{H}_+)} \leq t \|K_0\|_{L^\infty(0, \tau; \mathcal{L}(\mathbb{H}_+))} \|K\|_{L^\infty(D_T; \mathcal{L}(\mathbb{H}_+))}.$$

This yields

$$\begin{aligned} \|(\Phi^2 K)(t, s)\|_{\mathcal{L}(\mathbb{H}_+)} &= \left\| \int_{\max\{t-\tau, s\}}^t K_0(t-\xi) \Phi K(\xi, s) d\xi \right\|_{\mathcal{L}(\mathbb{H}_+)} \\ &\leq \|K_0\|_{L^\infty(0, \tau; \mathcal{L}(\mathbb{H}_+))} \int_{\max\{t-\tau, s\}}^t \|\Phi K(\xi, s)\|_{\mathcal{L}(\mathbb{H}_+)} \\ &\leq \|K_0\|_{L^\infty(0, \tau; \mathcal{L}(\mathbb{H}_+))}^2 \|K\|_{L^\infty(D_T; \mathcal{L}(\mathbb{H}_+))} \int_{\max\{t-\tau, s\}}^t \xi d\xi \\ &\leq \frac{t^2}{2} \|K_0\|_{L^\infty(0, \tau; \mathcal{L}(\mathbb{H}_+))}^2 \|K\|_{L^\infty(D_T; \mathcal{L}(\mathbb{H}_+))}, \end{aligned}$$

and by induction

$$\|(\Phi^n K)(t, s)\|_{\mathcal{L}(\mathbb{H}_+)} \leq \frac{t^n}{n!} \|K_0\|_{L^\infty(0, \tau; \mathcal{L}(\mathbb{H}_+))}^n \|K\|_{L^\infty(D_T; \mathcal{L}(\mathbb{H}_+))} \quad (n \in \mathbb{N}^*).$$

In particular, for  $n$  large enough,  $\Phi^n$  is a strict contraction and consequently if we define  $\tilde{\Phi}$  by

$$(\tilde{\Phi} K)(t, s) := (\Phi K)(t, s) + K_0(t-s) \mathbf{1}_{(\max\{t-\tau, 0\}, t)}(s)$$

- 1 then  $\tilde{\Phi}^n$  is also a strict contraction. This implies that  $\tilde{\Phi}$  admits a unique fixed point, which is a solution of  
2 (2.13). This implication is classical but we recall its proof for sake of completeness: by the Banach fixed-point  
3 theorem,  $\tilde{\Phi}^n$  admits a unique fixed point  $K$ . In particular  $\tilde{\Phi}^{n+1}(K) = \tilde{\Phi}(K)$ , and we deduce that  $\tilde{\Phi}(K)$  is a  
4 fixed point of  $\tilde{\Phi}^n$ . Therefore  $\tilde{\Phi}(K) = K$ . The uniqueness is obtained by noticing that a fixed point of  $\tilde{\Phi}$  is also  
5 a fixed point of  $\tilde{\Phi}^n$ .  $\square$

6 We are now in a position to prove the main result

- 7 *Proof of Theorem 1.1.* We consider  $G$  and  $K(t, s)$  obtained in Lemma 2.1 and in Lemma 2.2, and we set

$$v(t) = \mathbf{1}_{[\tau, +\infty)}(t) G \left[ z_+(t-\tau) + \int_0^{t-\tau} K(t-\tau, s) z_+(s) ds \right]. \quad (2.14)$$

Since  $\text{rank } G \leq N_+$ , we can write  $G$  as

$$G(\phi) = \sum_{k=1}^{N_+} c_k(\phi) v_k, \quad (\phi \in \mathbb{H}_+)$$

- 8 with  $c_k \in \mathcal{L}(\mathbb{H}_+, \mathbb{C})$  and  $v_k \in \mathbb{U}_+$ ,  $k = 1, \dots, N_+$ .

From (2.5), if  $\zeta \in \mathbb{H}_+^*$  then

$$\forall \phi \in \mathbb{H}_+, \quad (\phi, \zeta)_{\mathbb{H}} = 0 \implies \zeta = 0.$$

Combining this fact with  $\dim \mathbb{H}_+^* = \dim \mathcal{L}(\mathbb{H}_+, \mathbb{C})$ , we deduce that there exists a unique  $\zeta_k \in \mathbb{H}_+^*$  such that for all  $\phi \in \mathbb{H}_+$ ,  $c_k(\phi) = (\phi, \zeta_k)_{\mathbb{H}}$ . We can thus write  $G$  as

$$G(\phi) = \sum_{k=1}^{N_+} (\phi, \zeta_k)_{\mathbb{H}} v_k, \quad (\phi \in \mathbb{H}_+).$$

- 9 The interest of taking  $\zeta_k \in \mathbb{H}_+^*$  is that the above formula for  $G$  can be applied to  $\phi \in \mathbb{H}$  and extend  $G$  as a  
10 linear bounded operator in  $\mathbb{H}$  satisfying  $G = 0$  in  $\mathbb{H}_-$  (see (2.5)). Extending also the family  $K$  by  $K(t, s) = 0$   
11 in  $\mathbb{H}_-$ , we see that (2.14) can be written as (1.6).



1 Let us define

$$w(t) := z_+(t) + \int_0^t K(t, s) z_+(s) ds, \quad (2.15)$$

2 so that (2.9) can be written

$$\begin{cases} z'_+(t) = A_+ z_+(t) + B_+ p_+ \mathbb{1}_{[\tau, +\infty)}(t) G w(t - \tau) + P_+ f(t) & t > 0, \\ z_+(0) = P_+ z^0. \end{cases} \quad (2.16)$$

Then we use (2.13), (2.15) and the Fubini theorem to perform the following computation for  $t > 0$ :

$$\begin{aligned} \int_t^{t+\tau} e^{(t-s)A_+} B_+ p_+ G w(s - \tau) \mathbb{1}_{[\tau, +\infty)}(s) ds &= \int_{\max\{t-\tau, 0\}}^t e^{(t-s-\tau)A_+} B_+ p_+ G w(s) ds \\ &= \int_{\max\{t-\tau, 0\}}^t e^{(t-s-\tau)A_+} B_+ p_+ G \left[ z_+(s) + \int_0^s K(s, \xi) z_+(\xi) d\xi \right] ds \\ &= \int_0^t \left[ \mathbb{1}_{(\max\{t-\tau, 0\}, t)}(s) e^{(t-s-\tau)A_+} B_+ p_+ G + \int_{\max\{t-\tau, s\}}^t e^{(t-\xi-\tau)A_+} B_+ p_+ G K(\xi, s) d\xi \right] z_+(s) ds \\ &= \int_0^t K(t, s) z_+(s) ds = w(t) - z_+(t). \end{aligned} \quad (2.17)$$

3 Consequently,

$$w(t) = z_+(t) + \int_t^{t+\tau} e^{(t-s)A_+} B_+ p_+ G w(s - \tau) \mathbb{1}_{[\tau, +\infty)}(s) ds. \quad (2.18)$$

4 From (2.16), we deduce that  $w$  is solution of (2.11) with  $w^0 = z_+(0)$ . Thus  $w$  satisfies (2.12) and from (2.18),

$$\|z_+\|_{H_\sigma^1(0, \infty; \mathbb{H}_+)} \leq C (\|P_+ z^0\|_{\mathbb{H}_+} + \|P_+ f\|_{L_\sigma^2(0, \infty; \mathbb{H}_+)}). \quad (2.19)$$

5 In particular, using the Sobolev embedding  $H^1(0, \infty) \hookrightarrow L^\infty(0, \infty)$ , we deduce

$$\|z_+(t)\|_{\mathbb{H}_+} \leq C e^{-\sigma t} (\|P_+ z^0\|_{\mathbb{H}_+} + \|P_+ f\|_{L_\sigma^2(0, \infty; \mathbb{H}_+)}) \leq C e^{-\sigma t} (\|z^0\|_{\mathbb{H}} + \|f\|_{L_\sigma^2(0, \infty; \mathbb{H}_{-\gamma'})}). \quad (2.20)$$

Since  $\mathbb{H}_+ \subset \mathbb{H}_1$ , we have

$$H_\sigma^1(0, \infty; \mathbb{H}_+) \subset L_\sigma^2(0, \infty; \mathbb{H}_1) \cap C_\sigma^0([0, \infty); \mathbb{H}_{1/2}) \cap H_\sigma^1(0, \infty; \mathbb{H})$$

6 and (2.20) yields in particular that if  $z^0 \in \mathbb{H}_{1/2}$  and if  $f \in L_\sigma^2(0, \infty; \mathbb{H})$ , then

$$\|z_+\|_{L_\sigma^2(0, \infty; \mathbb{H}_1) \cap C_\sigma^0([0, \infty); \mathbb{H}_{1/2}) \cap H_\sigma^1(0, \infty; \mathbb{H})} \leq C (\|z^0\|_{\mathbb{H}_{1/2}} + \|f\|_{L_\sigma^2(0, \infty; \mathbb{H})}). \quad (2.21)$$

Then, we can consider the solution of (2.10): for  $t \geq \tau$ ,

$$\begin{aligned} z_-(t) &= e^{A-t} (\text{Id} - P_+) z^0 + \int_\tau^t (\lambda_0 - A)^\gamma e^{A-(t-s)} (\lambda_0 - A)^{-\gamma} B_- p_- G w(s - \tau) ds \\ &\quad + \int_0^t e^{A-(t-s)} (\text{Id} - P_+) f(s) ds. \end{aligned} \quad (2.22)$$

Using (2.3) and (2.12), we deduce that

$$\begin{aligned} \|z_-(t)\|_{\mathbb{H}} &\leq C e^{-\sigma-t} \|z^0\|_{\mathbb{H}} + C e^{-\sigma t} \int_\tau^t \frac{1}{(t-s)^\gamma} e^{-(\sigma-\sigma)(t-s)} ds (\|P_+ z^0\|_{\mathbb{H}_+} + \|P_+ f\|_{L_\sigma^2(0, \infty; \mathbb{H}_+)}) \\ &\quad + C e^{-\sigma t} \int_0^t \frac{1}{(t-s)^{\gamma'}} e^{-(\sigma-\sigma)(t-s)} \|e^{\sigma s} f(s)\|_{\mathbb{H}_{-\gamma'}} ds. \end{aligned}$$

Using that  $\sigma_- > \sigma$ ,  $\gamma < 1$  and  $\gamma' < 1/2$ , we deduce from the above estimate that

$$\|z_-(t)\|_{\mathbb{H}_-} \leq C e^{-\sigma t} \left( \|z^0\|_{\mathbb{H}} + \|f\|_{L^2_\sigma(0,\infty;\mathbb{H}_{-\gamma'})} \right) \quad (t > 0).$$

1 Combining this with (2.20), we deduce (1.7).

2 Let us prove now (1.8). If  $f \in L^2_\sigma(0,\infty;\mathbb{H})$ ,  $B \in \mathcal{L}(\mathbb{U},\mathbb{H})$  and if  $z^0 \in \mathbb{H}_{1/2}$ , then the first part remains  
3 unchanged, and we have (2.21) and

$$\|v\|_{L^2_\sigma(0,\infty;\mathbb{U})} \leq C \left( \|z^0\|_{\mathbb{H}_+} + \|f\|_{L^2_\sigma(0,\infty;\mathbb{H}_+)} \right). \quad (2.23)$$

4 Consequently,

$$B_- p_- v + (\text{Id} - P_+) f \in L^2_\sigma(0,\infty;\mathbb{H}_-). \quad (2.24)$$

Moreover since  $P_+ z^0 \in \mathbb{H}_+ \subset \mathbb{H}_{1/2}$ ,

$$z_-(0) = z^0 - P_+ z^0 \in \mathbb{H}_{1/2} \cap \mathbb{H}_-.$$

Using that  $A_-$  is the infinitesimal generator of an analytic semigroup of type smaller than  $-\sigma$  (see, for instance [11, Proposition 2.9, p.120]), then

$$z_- \in L^2_\sigma(0,\infty;\mathbb{H}_1) \cap C^0_\sigma([0,\infty);\mathbb{H}_{1/2}) \cap H^1_\sigma(0,\infty;\mathbb{H}),$$

and from (2.23)

$$\begin{aligned} \|z_-\|_{L^2_\sigma(0,\infty;\mathbb{H}_1) \cap C^0_\sigma([0,\infty);\mathbb{H}_{1/2}) \cap H^1_\sigma(0,\infty;\mathbb{H})} &\leq C \left( \|z^0 - P_+ z^0\|_{\mathbb{H}_{1/2}} + \|B_- p_- v\|_{L^2_\sigma(0,\infty;\mathbb{H})} + \|(\text{Id} - P_+) f\|_{L^2_\sigma(0,\infty;\mathbb{H})} \right) \\ &\leq C \left( \|z^0\|_{\mathbb{H}_{1/2}} + \|f\|_{L^2_\sigma(0,\infty;\mathbb{H})} \right). \end{aligned}$$

5 Combining this with (2.21), we deduce (1.8). □

### 6 **3 Feedback boundary stabilization of the reaction-convection-diffusion** 7 **equations**

8 Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be a bounded domain of class  $C^{1,1}$ . In this section, we apply Theorem 1.1 for the  
9 stabilization of the reaction-convection-diffusion equation. Let us consider  $\Gamma$  a non-empty open subset of  $\partial\Omega$   
10 and the control problem:

$$\begin{cases} \partial_t z - \Delta z - b \cdot \nabla z - cz = 0 & \text{in } (0,\infty) \times \Omega, \\ z = v & \text{on } (0,\infty) \times \Gamma, \\ z = 0 & \text{on } (0,\infty) \times (\partial\Omega \setminus \Gamma), \\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{cases} \quad (3.1)$$

where  $c, b, \text{div } b \in L^\infty(\Omega)$ . In order to write (3.1) under the form (1.1), we introduce the following functional setting:

$$\begin{aligned} \mathbb{H} &= L^2(\Omega), \quad \mathbb{U} = L^2(\Gamma), \\ Az &= \Delta z + b \cdot \nabla z + cz, \quad \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega). \end{aligned}$$

From standard results on this operator  $A$  (see for example [17, Theorem 5, p.305] and [11, Theorem 6.1, p.170]), we see that (Hyp1) and (Hyp3) hold true. To define the control operator  $B$ , we use a standard method (see, for instance [37, pp.341-343] or [32]): we first fix  $\lambda_0 \in \rho(A)$  and we consider the lifting operator  $D_0 \in \mathcal{L}(L^2(\partial\Omega); L^2(\Omega))$  such that for any  $v \in L^2(\partial\Omega)$ ,  $w = D_0 v$  is the unique solution of the following system

$$\begin{cases} \lambda_0 w - \Delta w - b \cdot \nabla w - cw = 0 & \text{in } \Omega, \\ w = v & \text{on } \partial\Omega. \end{cases}$$

Then, we set

$$B = (\lambda_0 - A)D_0 : \mathbb{U} \longrightarrow (\mathcal{D}(A^*))',$$

where we have extended the operator  $A$  as an operator from  $L^2(\Omega)$  into  $(\mathcal{D}(A^*))'$  and where we see  $\mathbb{U}$  as a closed subspace of  $L^2(\partial\Omega)$  (by extending by zero in  $\partial\Omega \setminus \Gamma$  any  $v \in \mathbb{U}$ ). Using standard results on elliptic equations, we have that  $B$  satisfies (Hyp2) for any  $\gamma > 3/4$ .

Let us recall how we can see that with  $A$  and  $B$  defined as above (3.1) writes as (1.1). We set  $\tilde{z} = z - w$ , with  $w = D_0v$ . Then  $\tilde{z}$  satisfies the system

$$\begin{cases} \partial_t \tilde{z} - \Delta \tilde{z} - b \cdot \nabla \tilde{z} - c \tilde{z} = -\partial_t w + \lambda_0 w & \text{in } (0, \infty) \times \Omega, \\ \tilde{z} = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ \tilde{z}(0, \cdot) = \tilde{z}^0 := z^0 - w(0, \cdot) & \text{in } \Omega. \end{cases}$$

Using the Duhamel formula, we have

$$\tilde{z}(t) = e^{tA} \tilde{z}^0 + \int_0^t e^{(t-s)A} (-\partial_t w(s) + \lambda_0 w(s)) ds.$$

By integrating by parts, we obtain

$$z(t) = e^{tA} z^0 + \int_0^t e^{(t-s)A} (\lambda_0 - A)w(s) ds,$$

that is

$$\begin{cases} z' = Az + (\lambda_0 - A)D_0v, \\ z(0) = z^0. \end{cases}$$

To apply Theorem 1.1, we only need to check  $(UC_\sigma)$ . We recall that

$$\mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega), \quad A^* \varepsilon = \Delta \varepsilon - \bar{b} \cdot \nabla \varepsilon + (\overline{c - \operatorname{div} \bar{b}}) \varepsilon,$$

(see, for instance, [37, p.345]). Moreover, by classical results (see [37, Proposition 10.6.7]), we see that

$$D_0^* := -\frac{\partial}{\partial \nu} (\lambda_0 - A^*)^{-1},$$

and thus

$$B^* \varepsilon := -\frac{\partial \varepsilon}{\partial \nu} \Big|_{\Gamma}.$$

Thus if  $\varepsilon$  satisfies  $A^* \varepsilon = \lambda \varepsilon$  and  $B^* \varepsilon = 0$ , then

$$\begin{cases} \lambda \varepsilon - \Delta \varepsilon + \bar{b} \cdot \nabla \varepsilon - (\overline{c - \operatorname{div} \bar{b}}) \varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial \varepsilon}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

From standard results on the unique continuation of the Laplace operator (see for instance [21, Theorem 5.3.1, p.125]), we deduce that  $\varepsilon = 0$ . Thus  $(UC_\sigma)$  holds for any  $\sigma$  and we deduce the following result by applying Theorem 1.1:

**Theorem 3.1.** *Assume  $\sigma > 0$  and let us define  $N_+$  by (1.4). Then there exist  $K \in L_{\text{loc}}^\infty(D_\infty; \mathcal{L}(L^2(\Omega)))$ ,  $\zeta_k \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $v_k \in H^{1/2}(\Gamma)$ ,  $k = 1, \dots, N_+$ , such that the solution  $z$  of (3.1) with*

$$v(t) = \mathbf{1}_{[\tau, +\infty)}(t) \sum_{k=1}^{N_+} \left( \int_{\Omega} \left[ z(t - \tau) + \int_0^{t-\tau} K(t - \tau, s) z(s) ds \right] \zeta_k dx \right) v_k, \quad (3.2)$$

and for  $z^0 \in L^2(\Omega)$  satisfies

$$\|z(t)\|_{L^2(\Omega)} \leq C e^{-\sigma t} \|z^0\|_{L^2(\Omega)}. \quad (3.3)$$

## 4 Feedback distributed stabilization of the Oseen system

Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain of class  $C^{1,1}$ . In this section, we apply Theorem 1.1 to the Oseen system:

$$\begin{cases} \partial_t z + (w^S \cdot \nabla)z + (z \cdot \nabla)w^S - \nu \Delta z + \nabla q = \mathbf{1}_{\mathcal{O}} v & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot z = 0 & \text{in } (0, \infty) \times \Omega, \\ z = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ z(0, \cdot) = z^0 & \text{in } \Omega, \end{cases} \quad (4.1)$$

where  $w^S \in [H^2(\Omega)]^3$  is a fixed (real) velocity and  $v$  is the control that acts on the nonempty open subset  $\mathcal{O} \subset \Omega$ . We could also consider the boundary stabilization of the Oseen system by using the same method as in the above section but with some adaptations due the incompressibility condition and due to the pressure (see [4] for more details).

Let us give the functional setting:

$$\mathbb{H} = \{z \in [L^2(\Omega)]^3 : \nabla \cdot z = 0 \text{ in } \Omega, \quad z \cdot n = 0 \text{ on } \partial\Omega\}, \quad \mathbb{U} = [L^2(\mathcal{O})]^3.$$

We denote by  $\mathbb{P}$  the orthogonal projection  $\mathbb{P} : [L^2(\Omega)]^3 \rightarrow \mathbb{H}$  and we define the Oseen operator:

$$\mathcal{D}(A) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \cap \mathbb{H}, \quad Az = \mathbb{P}(\nu \Delta z - (w^S \cdot \nabla)z - (z \cdot \nabla)w^S).$$

We recall (see, for instance [4, Theorem 20]) that the operator  $A$  is the infinitesimal generator of an analytic semigroup on  $\mathbb{H}$  and has a compact resolvent. Moreover,

$$\mathcal{D}(A^*) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \cap \mathbb{H}, \quad A^* \varepsilon = \mathbb{P}(\nu \Delta \varepsilon + (w^S \cdot \nabla)\varepsilon - (\nabla w^S)^* \varepsilon).$$

We also define the control operator  $B \in \mathcal{L}(\mathbb{U}, \mathbb{H})$  by

$$Bv = \mathbb{P}(\mathbf{1}_{\mathcal{O}} v),$$

and we can check that

$$B^* \varepsilon = \varepsilon|_{\mathcal{O}}.$$

In particular, we see that (Hyp1), (Hyp2) and (Hyp3) hold true (see [11, Theorem 6.1, p.170] for (Hyp3)).

If  $\varepsilon$  satisfies  $A^* \varepsilon = \lambda \varepsilon$  and  $B^* \varepsilon = 0$ , then

$$\begin{cases} \lambda \varepsilon - \nu \Delta \varepsilon - (w^S \cdot \nabla)\varepsilon + (\nabla w^S)^* \varepsilon + \nabla \pi = 0 & \text{in } \Omega, \\ \nabla \cdot \varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \varepsilon \equiv 0 & \text{in } \mathcal{O}. \end{cases}$$

Then using [18], we deduce that  $\varepsilon = 0$ . Thus  $(UC_\sigma)$  holds for any  $\sigma$  and we deduce the following result by applying Theorem 1.1:

**Theorem 4.1.** *Assume  $\sigma > 0$  and let us define  $N_+$  by (1.4). Then there exist  $K \in L_{\text{loc}}^\infty(D_\infty; \mathcal{L}(\mathbb{H}))$ ,  $\zeta_k \in \mathcal{D}(A^*)$ ,  $v_k \in [L^2(\mathcal{O})]^3$ ,  $k = 1, \dots, N_+$ , such that the solution  $z$  of (4.1) with*

$$v(t) = \mathbf{1}_{[\tau, +\infty)}(t) \sum_{k=1}^{N_+} \left( \int_{\Omega} \left[ z(t-\tau) + \int_0^{t-\tau} K(t-\tau, s) z(s) ds \right] \zeta_k dx \right) v_k, \quad (4.2)$$

and for  $z^0 \in \mathbb{H}$  satisfies

$$\|z(t)\|_{[L^2(\Omega)]^3} \leq C e^{-\sigma t} \|z^0\|_{[L^2(\Omega)]^3}. \quad (4.3)$$

1 Let us define

$$\mathbb{V} = [H_0^1(\Omega)]^3 \cap \mathbb{H}, \quad (4.4)$$

2 then we have that  $\mathbb{V} = \mathbb{H}_{1/2}$  (see again [4, Theorem 20]). Thus applying Theorem 1.1, we have also the following  
3 result on

$$\begin{cases} \partial_t z + (w^S \cdot \nabla)z + (z \cdot \nabla)w^S - \nu \Delta z + \nabla q = \mathbf{1}_O v + f & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot z = 0 & \text{in } (0, \infty) \times \Omega, \\ z = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ z(0, \cdot) = z^0 & \text{in } \Omega. \end{cases} \quad (4.5)$$

**Theorem 4.2.** *Assume  $\sigma > 0$  and let us consider  $v$  given by (4.2). Then for any  $z^0 \in \mathbb{V}$  and for any  $f \in L_\sigma^2(0, \infty; \mathbb{H})$  the solution of (4.5) satisfies*

$$z \in L_\sigma^2(0, \infty; [H^2(\Omega)]^3) \cap C_\sigma^0([0, \infty); [H^1(\Omega)]^3) \cap H_\sigma^1(0, \infty; [L^2(\Omega)]^3),$$

4 and

$$\|z\|_{L_\sigma^2(0, \infty; [H^2(\Omega)]^3) \cap C_\sigma^0([0, \infty); [H^1(\Omega)]^3) \cap H_\sigma^1(0, \infty; [L^2(\Omega)]^3)} \leq C (\|z^0\|_{[H^1(\Omega)]^3} + \|f\|_{L_\sigma^2(0, \infty; [L^2(\Omega)]^3)}). \quad (4.6)$$

## 5 Local feedback distributed stabilization of the Navier-Stokes system

7 We use the same notation as in the previous section. We consider the stabilization of the Navier-Stokes system  
8 with internal control:

$$\begin{cases} \partial_t \tilde{z} + (\tilde{z} \cdot \nabla)\tilde{z} - \nu \Delta \tilde{z} + \nabla \tilde{q} = \mathbf{1}_O v + f^S & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot \tilde{z} = 0 & \text{in } (0, \infty) \times \Omega, \\ \tilde{z} = b^S & \text{on } (0, \infty) \times \partial\Omega, \\ \tilde{z}(0, \cdot) = \tilde{z}^0 & \text{in } \Omega, \end{cases} \quad (5.1)$$

9 around the stationary state

$$\begin{cases} (w^S \cdot \nabla)w^S - \nu \Delta w^S + \nabla r^S = f^S & \text{in } \Omega, \\ \nabla \cdot w^S = 0 & \text{in } \Omega, \\ w^S = b^S & \text{on } \partial\Omega. \end{cases} \quad (5.2)$$

10 We assume that  $(w^S, r^S)$  is a solution of (5.2) such that  $w^S \in [H^2(\Omega)]^3$  as in the previous section. The functions  
11  $f^S \in [L^2(\Omega)]^3$  and  $b^S \in [W^{3/2}(\partial\Omega)]^3$  are independent of time.

We define

$$\bar{z} = \tilde{z} - w^S, \quad \bar{q} = \tilde{q} - r^S, \quad z^0 = \tilde{z}^0 - w^S,$$

12 so that

$$\begin{cases} \partial_t \bar{z} + (w^S \cdot \nabla)\bar{z} + (\bar{z} \cdot \nabla)w^S - \nu \Delta \bar{z} + \nabla \bar{q} = \mathbf{1}_O v - (\bar{z} \cdot \nabla)\bar{z} & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot \bar{z} = 0 & \text{in } (0, \infty) \times \Omega, \\ \bar{z} = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ \bar{z}(0, \cdot) = z^0 & \text{in } \Omega. \end{cases} \quad (5.3)$$

Then we consider the following mapping

$$\mathcal{Z} : L_\sigma^2(0, \infty; [L^2(\Omega)]^3) \rightarrow L_\sigma^2(0, \infty; [L^2(\Omega)]^3), \quad f \mapsto -(z \cdot \nabla)z,$$

13 where  $z$  is the solution of (4.5) given in Theorem 4.2, associated with  $z^0 \in \mathbb{V}$  and  $f \in L_\sigma^2(0, \infty; \mathbb{H})$ . In particular,  
14 the control  $v$  is given by (4.2). We notice that if  $f$  is a fixed point of  $\mathcal{Z}$ , then the corresponding solution  $\bar{z}$  of  
15 (4.5) given in Theorem 4.2 satisfies (5.3) since  $f = -(\bar{z} \cdot \nabla)\bar{z}$ .

1 Then by standard Sobolev embeddings, we find

$$\|z^1 \cdot \nabla z^2\|_{L^2_\sigma(0, \infty; [L^2(\Omega)]^3)} \leq C \|z^1\|_{C^0_\sigma([0, \infty); [H^1(\Omega)]^3)} \|z^2\|_{L^2_\sigma(0, \infty; [H^2(\Omega)]^3)}. \quad (5.4)$$

Thus  $\mathcal{Z}$  is well-defined. Let us set

$$R = \|z^0\|_{[H^1(\Omega)]^3},$$

and

$$B_R = \{f \in L^2_\sigma(0, \infty; [L^2(\Omega)]^3) : \|f\|_{L^2_\sigma(0, \infty; [L^2(\Omega)]^3)} \leq R\}.$$

Then from (5.4) and (4.6),

$$\|\mathcal{Z}(f)\|_{L^2_\sigma(0, \infty; [L^2(\Omega)]^3)} \leq 4CR^2,$$

and  $B_R$  is invariant by  $\mathcal{Z}$  for  $R$  small enough. Similarly, using (5.4) and (4.6), for any  $f^1, f^2 \in B_R$ , then

$$\|\mathcal{Z}(f^1) - \mathcal{Z}(f^2)\|_{L^2_\sigma(0, \infty; [L^2(\Omega)]^3)} \leq 2CR \|f^1 - f^2\|_{L^2_\sigma(0, \infty; [L^2(\Omega)]^3)},$$

2 and thus  $\mathcal{Z}$  is a strict contraction on  $B_R$  for  $R$  small enough.

We thus deduce that  $\mathcal{Z}$  admits a unique fixed point  $f$  in  $B_R$  for  $\|z^0\|_{[H^1(\Omega)]^3}$  small enough. As explained above, the solution  $\bar{z}$  of (4.5) given in Theorem 4.2, associated with  $z^0 \in \mathbb{V}$  and  $f \in L^2_\sigma(0, \infty; [L^2(\Omega)]^3)$  is a solution of (5.3). Then, in particular from (4.6), we have

$$\|\bar{z}\|_{L^2_\sigma(0, \infty; [H^2(\Omega)]^3) \cap C^0_\sigma([0, \infty); [H^1(\Omega)]^3) \cap H^1_\sigma(0, \infty; [L^2(\Omega)]^3)} \leq 2CR \leq 2C \|z^0\|_{[H^1(\Omega)]^3}.$$

Now, let us show that the uniqueness of solutions of (5.3) with  $v$  given by (4.2). For this, let us consider two solutions

$$\bar{z}^i \in L^2_\sigma(0, \infty; [H^2(\Omega)]^3) \cap C^0_\sigma([0, \infty); [H^1(\Omega)]^3) \cap H^1_\sigma(0, \infty; [L^2(\Omega)]^3) \quad (i = 1, 2).$$

3 We denote by  $v^1$  and  $v^2$  the controls given by (4.2) for respectively  $\bar{z}^1$  and  $\bar{z}^2$ . On  $(0, \tau)$ , the controls  $v^i$  are  
4 null, so that  $\bar{z}^i$  are two strong solutions of the Navier-Stokes system with the same initial condition. Following  
5 standard results (see, for instance, [35, Theorem 3.4, p. 297]), we deduce that  $\bar{z}^1 \equiv \bar{z}^2$  in  $[0, \tau]$ . This implies  
6 that  $v^1 \equiv v^2$  on  $(\tau, 2\tau)$ . Thus following the standard proof of uniqueness given for instance in [35, Theorem 3.4,  
7 p. 297], we deduce that  $\bar{z}^1 \equiv \bar{z}^2$  in  $[\tau, 2\tau]$ , and we can proceed by induction to deduce that  $\bar{z}^1 \equiv \bar{z}^2$ .

8 We have obtained the following local stabilization result for the Navier-Stokes system with internal control  
9 with delay:

**Theorem 5.1.** *Assume  $\sigma > 0$  and let us define  $N_+$  by (1.4). Then there exist  $K \in L^\infty_{\text{loc}}(D_\infty; \mathcal{L}(\mathbb{H}))$ ,  $\zeta_k \in \mathcal{D}(A^*)$ ,  $v_k \in [L^2(\mathcal{O})]^3$ ,  $k = 1, \dots, N_+$  and  $R > 0$ , such that for any*

$$\tilde{z}^0 \in [H^1(\Omega)]^3, \quad \nabla \cdot \tilde{z}^0 = 0 \quad \text{in } \Omega, \quad \tilde{z}^0 = b^S \quad \text{on } \partial\Omega,$$

and

$$\|\tilde{z}^0 - w^S\|_{[H^1(\Omega)]^3} \leq R,$$

10 there exists a unique solution  $z$  of (5.1) with

$$v(t) = \mathbb{1}_{[\tau, +\infty)}(t) \sum_{k=1}^{N_+} \left( \int_\Omega \left[ (\tilde{z} - w^S)(t - \tau) + \int_0^{t-\tau} K(t - \tau, s) (\tilde{z} - w^S)(s) ds \right] \zeta_k dx \right) v_k, \quad (5.5)$$

satisfying

$$\tilde{z} - w^S \in L^2_\sigma(0, \infty; [H^2(\Omega)]^3) \cap C^0_\sigma([0, \infty); [H^1(\Omega)]^3) \cap H^1_\sigma(0, \infty; [L^2(\Omega)]^3).$$

11 Moreover we have the estimate

$$\|\tilde{z} - w^S\|_{L^2_\sigma(0, \infty; [H^2(\Omega)]^3) \cap C^0_\sigma([0, \infty); [H^1(\Omega)]^3) \cap H^1_\sigma(0, \infty; [L^2(\Omega)]^3)} \leq C \|\tilde{z}^0 - w^S\|_{[H^1(\Omega)]^3}. \quad (5.6)$$

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