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The “pits effect” for entire functions of exponential type and the Wiener spectrum

Jacques Benatar ^{*} Alexander Borichev ^{**} Mikhail Sodin ^{***}

Abstract

Given a sequence $\xi: \mathbb{Z}_+ \rightarrow \mathbb{C}$, we find a simple spectral condition which guarantees the angular equidistribution of the zeroes of the Taylor series

$$F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!}.$$

This condition yields practically all known instances of random and pseudo-random sequences ξ with this property (due to Nassif, Littlewood, Chen–Littlewood, Levin, Eremenko–Ostrovskii, Kabluchko–Zaporozhets, Borichev–Nishry–Sodin), and provides several new ones. Among them are Besicovitch almost periodic sequences and multiplicative random sequences. It also conditionally yields that the Möbius function μ has this property assuming “the binary Chowla conjecture”.

1 Introduction and the main result

Let F_ξ be an entire function of exponential type represented by the Taylor series

$$F_\xi(z) = \sum_{n \geq 0} \xi(n) \frac{z^n}{n!}, \quad \xi: \mathbb{Z}_+ \rightarrow \mathbb{C}.$$

As in [2], we are interested in the influence of the sequence ξ on the asymptotic behaviour of the function F_ξ , in particular, on the angular distribution of its zeroes.

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Definition 1. *The sequence $\xi: \mathbb{Z}_+ \rightarrow \mathbb{C}$ is called an L -sequence, if*

$$\frac{\log |F_\xi(tz)|}{t} \xrightarrow[t \rightarrow \infty]{} |z|, \quad \text{in } L_{\text{loc}}^1(\mathbb{C}). \quad (1)$$

In the classical terminology of the entire function theory [11], condition (1) means that the function F_ξ has completely regular growth in the Levin-Pfluger sense with the indicator diagram being the closed unit disk.

Since the $L_{\text{loc}}^1(\mathbb{C})$ -convergence of subharmonic functions implies convergence in the sense of distributions, and the Laplacian is continuous in the distributional topology, (1) yields

$$\frac{1}{t} \Delta \log |F_\xi(tz)| \xrightarrow[t \rightarrow \infty]{} \Delta |z| = dr \otimes d\theta, \quad z = re^{i\theta}, \quad (2)$$

in the sense of distributions, with $rdr \otimes d\theta$ being the planar Lebesgue measure. Denoting by $n_F(r; \theta_1, \theta_2)$ the number of zeroes (counted with multiplicities) of the entire function F in the sector $\{z: 0 \leq |z| \leq r, \theta_1 \leq \arg(z) < \theta_2\}$ and recalling that $\frac{1}{2\pi} \Delta \log |F|$ is the sum of point masses at zeroes of F , we can rewrite (2) in a more traditional form: for every $\theta_1 < \theta_2$,

$$n_{F_\xi}(r; \theta_1, \theta_2) = \frac{(\theta_2 - \theta_1 + o(1))r}{2\pi}, \quad r \rightarrow \infty,$$

which means that the angular distribution of zeroes is uniform, while the radial one is proportional to r . Littlewood called such behaviour “the pits effect” which he described as follows: “If we erect an ordinate $|f(z)|$ at the point z of the z -plane, then the resulting surface is an exponentially rapidly rising bowl, approximately of revolution, with exponentially small pits going down to the bottom. The zeros of f , more generally the w -points where $f = w$, all lie in the pits for $|z| > R(w)$. Finally the pits are very uniformly distributed in direction, and as uniformly distributed in distance as is compatible with the order ρ ” [14, p. 195].

There are no general results providing conditions for a sequence ξ to be an L -sequence, but rather a collection of interesting examples. Most of them required an individual, sometimes, quite involved, treatment. These examples include:

- (a) random independent identically distributed $\xi(n)$ (Littlewood–Offord [15], Kabluchko–Zaporozhets [9]);
- (b) stationary sequences with a logarithmic decay of the maximal correlation coefficient [2, Theorem 3] and Gaussian stationary sequences [2, Theorem 4];

(c) $\xi(n) = e(Q(n))$, where $Q(x) = \sum_{k \geq 2} q_k x^k$ is a polynomial with real coefficients q_k , at least one of which is irrational [2, Theorem 1], a special case $\xi(n) = e(qn^2)$ with irrational q was treated earlier, by Nassif [17] and Littlewood [13], when q is a quadratic irrationality, and then, by Eremenko–Ostrovskii [7], for arbitrary irrational q ;

(d) $e(n^\beta)$ with non-integer $\beta > 1$, for $1 < \beta < \frac{3}{2}$ by Chen–Littlewood [4], and for $\beta \geq \frac{3}{2}$ in [2, Theorem 2];

(e) uniformly almost periodic $\xi(n)$ (Levin[11, Chapter VI, §7]).

Here and elsewhere, $e(t) = e^{2\pi it}$.

In this work, we find a simple spectral condition for the sequence ξ to be an L -sequence, which provides an easy and uniform treatment of the aforementioned examples (the only exception is a result of Kabluchko and Zaporozhets [9] pertaining to independent random variables which may have no second moment), as well as of several new ones, which were previously out of reach. Among them are Besicovitch almost periodic sequences and multiplicative random sequences. It also conditionally yields that the Möbius function μ has this property assuming “the binary Chowla conjecture”.

Our spectral condition is based on a notion which was at the heart of a generalized harmonic analysis developed by Wiener [19, Chapter IV]:

Definition 2. *Let $\xi: \mathbb{Z}_+ \rightarrow \mathbb{C}$. We say that ξ is a Wiener sequence if for every $k \geq 0$ there exists a (finite) limit*

$$\rho(k) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \xi(s) \overline{\xi(s+k)}.$$

Each Wiener sequence ξ has a unique spectral measure and its support is called *the Wiener spectrum* of ξ . The spectral measure is constructed as follows. Let ξ be a Wiener sequence. Then immediately, $\xi(n) = o(n^{1/2})$, $n \rightarrow \infty$. Furthermore, we set

$$\rho(-k) = \overline{\rho(k)} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \overline{\xi(s)} \xi(s+k), \quad k \geq 1.$$

Now, given $j, k \geq 0$, we have

$$\rho(k-j) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \xi(s+j) \overline{\xi(s+k)}.$$

For any complex numbers c_0, \dots, c_m we obtain

$$\begin{aligned} \sum_{0 \leq j, k \leq m} c_j \overline{c_k} \rho(k - j) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \sum_{0 \leq j, k \leq m} c_j \overline{c_k} \xi(s + j) \overline{\xi(s + k)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \left| \sum_{0 \leq j \leq m} c_j \xi(s + j) \right|^2 \geq 0. \end{aligned}$$

Hence, by the Herglotz theorem, there exists a positive measure μ on the unit circle, the spectral measure of the Wiener sequence ξ , such that $\rho(k) = \widehat{\mu}(k)$, $k \in \mathbb{Z}$.

Our main result is as follows.

Theorem 1. *Let ξ be a Wiener sequence such that the closed support of its spectral measure is the whole unit circle. Then ξ is an L -sequence.*

It is worth mentioning that one does not need to know the spectral measure μ in order to check that its closed support has no gaps. It is not difficult to show (see [5, Theorem 1]) that the sequences of measures $d\mu_n(\theta) = \frac{1}{n} \left| \sum_{0 \leq s < n} \xi(s) e(-s\theta) \right|^2 d\theta$ and $d\nu_r(\theta) = (1 - r) |f_\xi(re(-\theta))|^2 d\theta$ with $f_\xi(z) = \sum_{n \geq 0} \xi(n) z^n$, converge weakly to the measure μ as $n \rightarrow \infty$ and $r \rightarrow 1$ correspondingly. Thus, for instance, it suffices to verify that, for any arc I , $\liminf_{n \rightarrow \infty} \mu_n(I) > 0$.

The proof of Theorem 1 will be given in Section 2; its main idea is similar to the one used in [2]. Since the sequence ξ does not grow faster than \sqrt{n} , the upper bound $|F_\xi(z)| \lesssim \sqrt{|z|} e^{|z|}$ is a straightforward consequence of the estimates for the Taylor coefficients. A matching lower bound cannot hold everywhere (the function F_ξ has zeroes), but a simple argument based on the subharmonicity of $\log |F_\xi|$, yields that it suffices to find a sufficiently dense set of points z at which such a lower bound exists. We cannot explicitly locate such points z , instead, in several steps, we estimate from below the averages of $|F_\xi|^2$ over sets of the form $\{w = te(\varphi) : r \leq t \leq r(1 + \delta), |\theta - \varphi| \leq \delta\}$ with any $\delta > 1$, $\theta \in [-\frac{1}{2}, \frac{1}{2}]$, and $r \geq r_\delta$. This estimate will be based on the spectral properties of the sequence ξ .

In Section 3 we discuss various examples of Wiener sequences that have no gaps in the support of their spectral measures. Likely, all, or almost all of them are well-known to the experts, however, in combination with Theorem 1 they provide new instances of L -sequences.

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2 Proof of Theorem 1

2.1 Two lemmas on Wiener sequences

Lemma 1. *Let ξ be a Wiener sequence. Then there exists a non-decreasing function ψ , $\lim_{x \rightarrow \infty} \psi(x) = \infty$, such that*

$$\left| \frac{1}{n} \sum_{0 \leq s < n} \xi(s) \overline{\xi(s+k)} - \rho(k) \right| \leq \frac{1}{\psi(n)}, \quad n \geq 1, |k| \leq \psi(n). \quad (3)$$

Proof. For every k ,

$$\max_{n \geq m} \left| \frac{1}{n} \sum_{0 \leq s < n} \xi(s) \overline{\xi(s+k)} - \rho(k) \right| = \varepsilon(m, k) \rightarrow 0, \quad m \rightarrow \infty.$$

Set $M_0 = 0$. Given $m \geq 1$, we can find $M_m > M_{m-1}$ such that

$$\max_{|k| \leq m} \varepsilon(M_m, k) \leq \frac{1}{m},$$

and set $\psi(x) = m$, $M_m \leq x < M_{m+1}$, $m \geq 0$. □

For a positive function f on $[A, B] \cap \mathbb{Z}$, we put

$$\text{osc}_{[A, B]} f \stackrel{\text{def}}{=} \sum_{A < q < B} |f(q) - f(q-1)|.$$

Lemma 2. *Let ξ satisfy (3) for some non-decreasing function ψ , $\lim_{x \rightarrow \infty} \psi(x) = \infty$.*

Given integers $0 < A < B$ and $|h| \leq \psi(A)$ and a positive function f we have

$$\left| \sum_{A \leq k < B} \xi(k) \overline{\xi(k+h)} f(k) - \rho(h) \sum_{A \leq k < B} f(k) \right| \leq 2(f(A) + \text{osc}_{[A, B]} f) \frac{B}{\psi(A)}.$$

Proof. Applying summation by parts and then using (3), we get

$$\begin{aligned} & \left| \sum_{A \leq k < B} \xi(k) \overline{\xi(k+h)} f(k) - \rho(h) \sum_{A \leq k < B} f(k) \right| \\ & \leq f(A) \left| \sum_{A \leq k < B} \xi(k) \overline{\xi(k+h)} - \rho(h)(B-A) \right| \\ & \quad + \sum_{A < q < B} |f(q) - f(q-1)| \cdot \left| \sum_{q \leq k < B} \xi(k) \overline{\xi(k+h)} - \rho(h)(B-q) \right| \\ & \leq f(A) \frac{2B}{\psi(A)} + \sum_{A < q < B} |f(q) - f(q-1)| \frac{2B}{\psi(A)} \\ & \leq 2(f(A) + \text{osc}_{[A, B]} f) \frac{B}{\psi(A)}, \end{aligned}$$

which proves the lemma. □

2.2 A lemma on entire functions of exponential type

Lemma 3 (Lemma 3.2.1 in [2]). *Let F be an entire function of exponential type with indicator function*

$$h^F(\theta) \stackrel{\text{def}}{=} \limsup_{r \rightarrow \infty} r^{-1} \log |F(re(\theta))| \leq 1.$$

Suppose that for every $\theta \in [-\frac{1}{2}, \frac{1}{2}]$ there exists a sequence of R_j such that $\lim_{j \rightarrow \infty} R_j = \infty$, $\lim_{j \rightarrow \infty} R_{j+1}/R_j = 1$, and a sequence of $\theta_j \rightarrow \theta$ such that

$$\liminf_{j \rightarrow \infty} \frac{1}{R_j} \log |F(R_j e(\theta_j))| \geq 1.$$

Then, condition (1) holds for the function F , that is, it is of completely regular growth with $h^F \equiv 1$.

2.3 Approximating the Taylor series by an exponential sum

Given $r > 0$ and an integer N such that $N \simeq r^{1/2} \log r$, set

$$\gamma(k, r) = \frac{k-r}{2k} + \frac{(k-r)^2}{2k} + \frac{(k-r)^3}{3k^2}$$

and

$$\widetilde{W}_{r,N}(\theta) = \sum_{|k-r| \leq N} \xi(k) e(k\theta) e^{-\gamma(k,r)},$$

Furthermore, set $U(r) = e^r / \sqrt{2\pi r}$.

Lemma 4. *Let ξ be a Wiener sequence. Then*

$$\frac{|F_\xi(re(\theta))|}{U(r)} - |\widetilde{W}_{r,N}(\theta)| = o(1), \quad r \rightarrow \infty, \theta \in [0, 2\pi].$$

Note that if the Wiener sequence ξ is bounded, then, as in [2], instead of $\widetilde{W}_{r,N}$ we can use the sum

$$W_{r,N}(\theta) = \sum_{|k-r| \leq N} \xi(k) e(k\theta) e^{-|k-r|^2/(2r)},$$

which makes the proof of Theorem 1 somewhat simpler (cf. Lemma 4.1.1 in [2]).

Proof. First, we deal with the sum over $k < r - N$. Denote by p the largest integer smaller than $r - N$. Since the sequence $(r^n/n!)_n$ increases for $n < r$, we have

$$\sum_{k < r-N} |\xi(k)| \frac{r^k}{k!} \lesssim \sum_{k < r-N} k^{1/2} \frac{r^k}{k!} \lesssim r^{3/2} \frac{r^p}{p!}$$

By Stirling's formula, we have

$$\begin{aligned}
\frac{r^{3/2} r^p}{U(r) p!} &\lesssim \frac{r^{3/2} r^{p+1/2} e^p}{e^r p^{p+1/2}} \\
&\lesssim r^{3/2} \exp\left(-\left(r - N\right) \log\left(1 - \frac{N}{r}\right) - N\right) \\
&= r^{3/2} \exp\left(-\frac{N^2}{2r} + O\left(\frac{N^3}{r^2}\right)\right) \\
&= o(1), \quad r \rightarrow \infty.
\end{aligned}$$

Second, we deal with the sum over $k > r + N$. Denote by p the smallest integer larger than $r + N$. Since the sequence $(n^2 r^n / n!)_n$ decreases for $n \geq r + 1$, we have

$$\sum_{k > r + N} |\xi(k)| \frac{r^k}{k!} \lesssim \sum_{k > r + N} k^{1/2} \frac{r^k}{k!} \lesssim r^{3/2} \frac{r^p}{p!}$$

By Stirling's formula, we have

$$\frac{r^{3/2} r^p}{U(r) p!} = o(1), \quad r \rightarrow \infty.$$

Finally, we turn to the central group of indices. We have

$$\begin{aligned}
&\left| \sum_{|k-r| \leq N} \xi(k) e(k\theta) \left(\frac{r^k}{k! U(r)} - e^{-\gamma(k,r)} \right) \right|^2 \\
&\leq \sum_{|k-r| \leq N} |\xi_k|^2 \cdot \sum_{|k-r| \leq N} \left(\frac{r^k}{k! U(r)} - e^{-\gamma(k,r)} \right)^2 \\
&= o(r) \cdot \sum_{|k-r| \leq N} \left(\frac{\sqrt{2\pi r} r^k}{k! e^r} - e^{-\gamma(k,r)} \right)^2.
\end{aligned}$$

Again by Stirling's formula, for $|k - r| \leq N$, we have

$$\begin{aligned}
\frac{\sqrt{2\pi r}}{e^r} \cdot \frac{r^k}{k!} &= \left(1 + O\left(\frac{1}{r}\right)\right) e^{k-r} \left(\frac{r}{k}\right)^{k+1/2} \\
&= \left(1 + O\left(\frac{1}{r}\right)\right) \exp\left(k - r + (k + 1/2) \log\left(1 - \frac{k-r}{k}\right)\right) \\
&= \left(1 + O\left(\frac{1}{r}\right)\right) \exp\left(-\gamma(k,r) + O\left(\frac{\log^2 r}{r}\right)\right) \\
&= \left(1 + O\left(\frac{\log^2 r}{r}\right)\right) e^{-\gamma(k,r)},
\end{aligned}$$

and hence,

$$\sum_{|k-r| \leq N} \left(\frac{\sqrt{2\pi r} r^k}{k! e^r} - e^{-\gamma(k,r)} \right)^2 \lesssim \frac{\log^5 r}{r^{3/2}}.$$

Finally,

$$\frac{|F_\xi(re(\theta))|}{U(r)} - |\widetilde{W}_{r,N}(\theta)| = o(1), \quad r \rightarrow \infty,$$

proving the lemma. \square

2.4 A lower bound for the exponential sum \widetilde{W} (beginning)

Lemma 5. *Let ξ be a Wiener sequence. Given $\delta > 0$, $r > r(\delta)$, and $\theta \in [0, 2\pi]$, there exist $r_0 \in (r, r + \delta r)$ and $\theta_0 \in (\theta - \delta, \theta + \delta)$ such that for some (every) $N \simeq r^{1/2} \log r$ we have*

$$|\widetilde{W}_{r_0,N}(\theta_0)| \gtrsim r^{1/4}.$$

Beginning of the proof of Lemma 5. Choose a non-negative even function $g \in C_0^2(\mathbb{R})$ with support on $[-1/2, 1/2]$ and such that $\int g(\varphi) d\varphi = 1$. Fix $N \simeq r^{1/2} \log r$. It suffices to verify that

$$X = \int_r^{r+\delta r} \int_{-1/2}^{1/2} |\widetilde{W}_{s,N}(\varphi)|^2 g(\delta^{-1}(\varphi - \theta)) ds d\varphi \gtrsim r^{3/2}.$$

Expanding the square we get

$$X = \delta \int_r^{r+\delta r} \sum_{|k_1-s| \leq N} \sum_{|k_2-s| \leq N} \xi(k_1) \overline{\xi(k_2)} e((k_1 - k_2)\theta) \widehat{g}(\delta(k_2 - k_1)) e^{-\gamma(k_1,s) - \gamma(k_2,s)} ds.$$

Set

$$V_h(t) = \int_r^{r(1+\delta)} \mathbb{1}_{[t-N, t+N]}(s) \mathbb{1}_{[t+h-N, t+h+N]}(s) e^{-\gamma(t,s) - \gamma(t+h,s)} ds$$

Then

$$X = \delta \sum_{|h| \leq 2N} \sum_{r-N \leq k \leq r(1+\delta)+N} \xi(k) \overline{\xi(k+h)} e(-h\theta) \widehat{g}(\delta h) V_h(k). \quad (4)$$

To apply Lemma 2, we need to estimate the quantities $\max_{[A,B]} V_h$, $\text{osc}_{[A,B]} V_h$, and $\sum_{A \leq k < B} V_h(k)$ with $A = r - N$ and $B = r(1 + \delta) + N$. This will be done next in a series of lemmas. In what follows, we always assume that the value of r (and hence of t) is large enough.

2.5 Estimating the integral $V_h(t)$

The first estimate is a crude upper bound:

Lemma 6. *Let $|h| \leq 2N$. Then*

$$\sup_{[r-N, r(1+\delta)+N]} V_h \lesssim r^{1/2}.$$

Proof. Fix $r - N \leq t \leq r(1 + \delta) + N$. Then, for $|s - t| \leq N$, we have

$$\left| \gamma(t, s) - \frac{(t-s)^2}{2t} \right| \lesssim \frac{N}{r} + \frac{N^3}{r^2} \stackrel{N \simeq \sqrt{r} \log r}{\lesssim} \frac{\log^3 r}{\sqrt{r}},$$

and similarly, for $|s - (t + h)| \leq N$, we have

$$\left| \gamma(t + h, s) - \frac{(t + h - s)^2}{2(t + h)} \right| \lesssim \frac{\log^3 r}{\sqrt{r}}.$$

Hence,

$$V_h(t) \lesssim \int_{\mathbb{R}} \exp\left(-\frac{(t-s)^2}{2t}\right) ds \stackrel{t \simeq r}{\lesssim} \sqrt{r},$$

completing the proof. \square

This lemma (combined with the decay of the Fourier transform $\widehat{g}(\delta h)$) will allow us to make the sum in h in (4) much shorter, cutting it from $|h| \leq 2N$ to $|h| \leq \log r$. Hence, it will suffice to estimate $\sum_k V_h(k)$ and the oscillation of V_h only for $|h| \leq \log r$.

In what follows, we assume that $|h| \leq \log r$. Our next goal is to simplify the integrand in the definition of V_h . First, we note that, for $|s - t| \leq N$, we have

$$e^{-\gamma(t,s)} = \left(1 + Q(t^{-1}, t-s) + O\left(\frac{\log^q r}{r^{3/2}}\right)\right) e^{-(t-s)^2/(2t)},$$

where $Q \in \mathbb{R}[x, y]$ is a polynomial which consists of the terms $x^\ell y^m$ with $m < 2\ell$, and q is a positive number. The exact form of Q ($Q(x, y) = -\frac{1}{2}xy + \frac{1}{8}x^2y^2 - \frac{1}{3}x^2y^3 + \frac{1}{6}x^3y^4 + \frac{1}{18}x^4y^6$) and the value of q ($q = 9$) are of no importance in our analysis. Replacing t by $t + h$, we see that, for $|s - (t + h)| \leq N$, we have

$$\begin{aligned} e^{-\gamma(t+h,s)} &= \left(1 + Q((t+h)^{-1}, t+h-s) + O\left(\frac{\log^q r}{r^{3/2}}\right)\right) e^{-(t+h-s)^2/(2t)} \\ &= \left(1 + P(h, t^{-1}, t-s) + O\left(\frac{\log^p r}{r^{3/2}}\right)\right) e^{-(t-s)^2/(2t)}, \end{aligned}$$

where $P \in \mathbb{R}[h, x, y]$ is a polynomial which consists of the terms $h^k x^\ell y^m$ with $m < 2\ell$, and p is a positive number. Thus,

$$\begin{aligned} V_h(t) &= \int_r^{r(1+\delta)} \mathbb{1}_{[t-N, t+N]}(s) \mathbb{1}_{[t+h-N, t+h+N]}(s) \times \\ &\quad \times \left(1 + P_1(h, t^{-1}, t-s) + O\left(\frac{\log^p r}{r^{3/2}}\right) \right) e^{-(t-s)^2/t} ds \\ &= \int_r^{r(1+\delta)} \left(1 + P_1(h, t^{-1}, t-s) + O\left(\frac{\log^p r}{r^{3/2}}\right) \right) e^{-(t-s)^2/t} ds + O(e^{-c \log^2 r}) \\ &= \sqrt{t} \int_{(r-t)/\sqrt{t}}^{(r(1+\delta)-t)/\sqrt{t}} \left(1 + P_1(h, t^{-1}, -u\sqrt{t}) \right) e^{-u^2} du + O\left(\frac{\log^p r}{r}\right) \quad (s = t + u\sqrt{t}). \end{aligned}$$

Here, $P_1 \in \mathbb{R}[h, x, y]$ is a polynomial of the same structure as P , and the positive integer value p may vary from line to line.

In what follows, we will separate three ranges of values of t : the central part $r+N \leq t \leq r(1+\delta) - N$, and the edges: $r-N \leq t \leq r+N$ and $r(1+\delta) - N \leq t \leq r(1+\delta) + N$.
 $r+N \leq t \leq r(1+\delta) - N$: Then,

$$[-c \log r, c \log r] \subset [(r-t)/\sqrt{t}, (r(1+\delta)-t)/\sqrt{t}],$$

so we can replace the integration over the interval $[(r-t)/\sqrt{t}, (r(1+\delta)-t)/\sqrt{t}]$ by the integration over the whole real axis, the error we make is $O(e^{-c \log^2 r})$. We immediately conclude that

$$V_h(t) = (\sqrt{\pi} + o(1))\sqrt{t}, \quad r+N \leq t \leq r(1+\delta) - N, \quad |h| \leq \log r, \quad r \rightarrow \infty. \quad (5)$$

This yields

Lemma 7. *Let $|h| \leq \log r$. Then*

$$\sum_{r-N \leq k < r(1+\delta)+N} V_h(k) = (A_\delta + \varepsilon(r))r^{3/2}, \quad r \rightarrow \infty,$$

with $A_\delta = \frac{2}{3}\sqrt{\pi}((1+\delta)^{3/2} - 1)$, and $\varepsilon(r)$ monotonically decreasing to 0.

Proof. We split the sum into three parts:

$$\begin{aligned} \sum_{r-N \leq k < r(1+\delta)+N} V_h(k) &= \left(\sum_{r-N \leq k < r+N} + \sum_{r+N \leq k < r(1+\delta)-N} + \sum_{r(1+\delta)-N < k < r(1+\delta)+N} \right) V_h(k). \end{aligned}$$

By Lemma 6, the first and the third sum are $\lesssim r^{1/2}N \lesssim r \log r$. In the central sum we use the asymptotics (5). \square

Next, we will show that

$$\sum_{r+N \leq k \leq r(1+\delta)-N} |V_h(k+1) - V_h(k)| \lesssim \sqrt{r}. \quad (6)$$

To this end, we will re-write the asymptotics (5) with a more accurate error term. We have

$$\begin{aligned} V_h(t) &= \sqrt{t} \int_{-\infty}^{\infty} \left(1 + \sum_{k,\ell,m: m < 2\ell} C_{k,\ell,m} h^k t^{m/2-\ell} u^m \right) e^{-u^2} du + O\left(\frac{\log^p r}{r}\right) \\ &= \sqrt{\pi t} + \sum_{k,\ell,m: m < 2\ell} C'_{k,\ell,m} h^k t^{(m+1)/2-\ell} + O\left(\frac{\log^p r}{r}\right). \end{aligned}$$

Each term $f(t)$ in the triple sum on the RHS satisfies

$$\max_{[r+N, r(1+\delta)-N]} |f'| = O_\varepsilon(r^{-3/2+\varepsilon}), \quad r \rightarrow \infty,$$

which yields (6).

It remains to prove counterparts of the estimate (6) at the edges of the range of t . $r - N \leq t \leq r + N$: In this case, extending the upper limit of the integrals to $+\infty$, we get

$$\begin{aligned} V_h(t) &= \sqrt{t} \int_{(r-t)/\sqrt{t}}^{\infty} e^{-u^2} du \\ &\quad + \sum_{k,\ell,m: m < 2\ell} C_{k,\ell,m} h^k t^{(m+1)/2-\ell} \int_{(r-t)/\sqrt{t}}^{\infty} u^m e^{-u^2} du + O\left(\frac{\log^p r}{r}\right). \end{aligned}$$

Observe that the function $t \mapsto (r-t)/\sqrt{t}$ decreases. Therefore, the first term on the RHS increases with t and satisfies

$$\sum_{r-N \leq k \leq r+N} |f(k) - f(k-1)| \leq f(r+N) \lesssim \sqrt{r}.$$

A similarly straightforward inspection shows that for other terms we have the even better estimate

$$\sum_{r-N \leq k \leq r+N} |f(k) - f(k-1)| \lesssim \log^p r.$$

We conclude that

$$\sum_{r-N \leq k \leq r+N} |V_h(k) - V_h(k-1)| \lesssim \sqrt{r}. \quad (7)$$

$r(1+\delta) - N \leq t \leq r(1+\delta) + N$: In this case, extending the lower limit of the integrals to $-\infty$, we have

$$\begin{aligned} V_h(t) &= \sqrt{t} \left(\int_{-\infty}^{\infty} - \int_{(r(1+\delta)-t)/\sqrt{t}}^{\infty} \right) e^{-u^2} du \\ &+ \sum_{k,\ell,m: m < 2\ell} C_{k,\ell,m} h^k t^{(m+1)/2-\ell} \left(\int_{-\infty}^{\infty} - \int_{(r(1+\delta)-t)/\sqrt{t}}^{\infty} \right) u^m e^{-u^2} du + O\left(\frac{\log^p r}{r}\right) \end{aligned}$$

which, similarly to the previous case, yields

$$\sum_{r(1+\delta)-N \leq k \leq r(1+\delta)+N} |V_h(k) - V_h(k-1)| \lesssim \sqrt{r}. \quad (8)$$

Combining estimates (6), (7), and (8), we obtain

Lemma 8. *Let $|h| \leq \log r$. Then*

$$\sum_{r-N \leq k < r(1+\delta)+N} |V_h(k) - V_h(k-1)| \lesssim \sqrt{r}.$$

2.6 Completing the proof of the lower bound for \widetilde{W}

Let us return to the proof of Lemma 5. We use that for $|h| \leq 2N$,

$$\begin{aligned} &\sum_{r-N \leq k \leq r(1+\delta)+N} |\xi(k) \overline{\xi(k+h)}| \\ &\leq \left(\sum_{r-N \leq k \leq r(1+\delta)+N} |\xi(k)|^2 \right)^{1/2} \cdot \left(\sum_{r-N \leq k \leq r(1+\delta)+N} |\xi(k+h)| \right)^{1/2} \lesssim r. \quad (9) \end{aligned}$$

Choose a large $H \leq \min(\log r, \psi(r/2), 1/\varepsilon(r))$ with the functions ψ and ε as in Lemma 3 and Lemma 7. By (4) we have that

$$\begin{aligned} &\left| X - \delta \sum_{|h| \leq H} \sum_{r-N \leq k \leq r(1+\delta)+N} \xi(k) \overline{\xi(k+h)} e(-h\theta) \widehat{g}(\delta h) V_h(k) \right| \\ &\leq \delta \sum_{H < |h| \leq 2N} \sum_{r-N \leq k \leq r(1+\delta)+N} |\xi(k) \overline{\xi(k+h)}| \cdot |\widehat{g}(\delta h)| \cdot V_h(k). \end{aligned}$$

Since $|\widehat{g}(\delta h)| \lesssim (\delta h)^{-2}$, using Lemma 6 and estimate (9), we obtain the bound

$$\begin{aligned} \left| X - \delta \sum_{|h| \leq H} \sum_{r-N \leq k \leq r(1+\delta)+N} \xi(k) \overline{\xi(k+h)} e(-h\theta) \widehat{g}(\delta h) V_h(k) \right| \\ \lesssim \delta \sum_{H < |h| \leq 2N} \frac{r^{3/2}}{(\delta h)^2} \simeq \frac{r^{3/2}}{\delta H}. \end{aligned}$$

Next, by Lemma 2, we have that

$$\begin{aligned} \left| \sum_{r-N \leq k \leq r(1+\delta)+N} \xi(k) \overline{\xi(k+h)} V_h(k) - \rho(h) \sum_{r-N \leq k \leq r(1+\delta)+N} V_h(k) \right| \\ \lesssim \frac{r}{\psi(r/2)} \cdot \text{OSC}_{[r-N, r(1+\delta)+N]} V_h \end{aligned}$$

By Lemma 8, the RHS is

$$\lesssim \frac{r^{3/2}}{\psi(r/2)}.$$

Thus, recalling Lemma 7, we get that

$$\left| \sum_{r-N \leq k \leq r(1+\delta)+N} \xi(k) \overline{\xi(k+h)} V_h(k) - A_\delta \rho(h) r^{3/2} \right| \lesssim \left(\varepsilon(r) + \frac{1}{\psi(r/2)} \right) r^{3/2},$$

whence, using once again that $|\widehat{g}(\delta h)| \lesssim (\delta h)^{-2}$, we obtain the estimate

$$\left| X - \delta A_\delta r^{3/2} \sum_{|h| \leq H} \rho(h) e(-h\theta) \widehat{g}(\delta h) \right| = O\left(\frac{r^{3/2}}{\delta H}\right), \quad r \rightarrow \infty.$$

Arguing as in the proof of Lemma 7.3.1 of [2], we use that

$$\sum_{h \in \mathbb{Z}} \rho(h) e(-h\theta) \widehat{g}(\delta h)$$

is the density of the convolution $\mu * g_\delta$ at the point $-\theta$, where $g_\delta(\varphi) = \delta^{-1} g(\delta^{-1}\varphi)$, $|\varphi| \leq 1$. Since the support of μ is the whole circle and the function g is non-negative, this value is positive. Hence, for some $c = c(\delta) > 0$, we have $X = (c + o(1))r^{3/2}$, $r \rightarrow \infty$, and the lemma is proved. \square

Theorem 1 follows now from Lemmas 3, 4, and 5. \square

3 Examples to Theorem 1

3.1 Ergodic stationary processes on \mathbb{Z}

Let $\xi: \mathbb{Z}_+ \rightarrow \mathbb{C}$ be a stationary process on \mathbb{Z} whose elements have finite second moment. Then the sequence $m \mapsto \mathbb{E}[\xi(0)\overline{\xi(m)}]$ is positive-definite and may therefore be expressed as the Fourier transform of a non-negative measure μ on the unit circle, called *the spectral measure* of the process ξ . By the Birkhoff ergodic theorem, almost surely, the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \xi(s) \overline{\xi(s+k)} \quad (10)$$

exist for every k , that is, almost every realization of ξ is a Wiener sequence. Generally speaking, the spectral measure of a realization, as well as its closed support, are random, but if the process ξ is *ergodic*, then the limits (10) are not random and coincide with the Fourier coefficients $\widehat{\mu}(k)$. That is, for ergodic stationary processes on \mathbb{Z} , for almost every realization, its spectral measure coincides with the spectral measure of the processes, cf. [3, Section 2]. Combining this discussion with our Theorem 1, we obtain

Corollary 1. *Let ξ be an ergodic stationary process on \mathbb{Z} with no gaps in its spectrum. Then, almost surely, ξ is an L -sequence.*

This corollary removes unnecessary assumptions in Theorem 3 from [2] where it was assumed that the sequence ξ is bounded and has strong mixing properties.

It is worth mentioning that, as follows from [2, Theorem 4], for Gaussian stationary processes on \mathbb{Z} , Corollary 1 holds *without* the ergodicity assumption.

3.2 Besicovitch almost-periodic sequences

For functions (sequences) $s: \mathbb{Z}_+ \rightarrow \mathbb{C}$, define a semi-norm

$$\|\xi\|^2 = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} |\xi(s)|^2.$$

Two sequences ξ and $\tilde{\xi}$ are said to be *equivalent* if $\|\xi - \tilde{\xi}\| = 0$. For instance, any sequence ξ with $\xi(n) = o(1)$ as $n \rightarrow \infty$, is equivalent to the zero function. By ℓ^2 we denote the linear space of the equivalence classes equipped with the norm $\|\cdot\|$. One can show that this space is complete.

By \mathcal{P} we denote the linear hull of the exponentials, i.e., the elements of \mathcal{P} are the finite linear combinations of exponentials

$$P(n) = \sum_{\lambda \in \Lambda} c_\lambda e_\lambda(n),$$

where $e_\lambda(n) = e(\lambda n)$, and $\Lambda \subset \mathbb{T}$ is a finite set, *the spectrum of P* .

Definition 3. *The Besicovitch space B^2 is the closure of \mathcal{P} in ℓ^2 .*

We say that a sequence $\xi: \mathbb{Z}_+ \rightarrow \mathbb{C}$ has a mean if the following limit exists (and is finite):

$$M(\xi) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} \xi(s).$$

The following two facts are relatively straightforward:

1. For any $\xi \in B^2$ and any $\lambda \in \mathbb{T}$, the Fourier coefficient of ξ at λ , i.e. $\widehat{\xi}(\lambda) = M(\xi e_{-\lambda})$ is well defined.
2. For any $\xi, \eta \in B^2$, the mean $M(\xi \bar{\eta})$ exists. It defines the scalar product $\langle \xi, \eta \rangle = M(\xi \bar{\eta})$ in B^2 . Hence, the Cauchy–Schwartz inequality $|\langle \xi, \eta \rangle| \leq \|\xi\| \cdot \|\eta\|$ holds for every $\xi, \eta \in B^2$.

Noting that the translations $\xi_m(n) = \xi(n + m)$, $m \in \mathbb{N}$, preserve B^2 , we conclude that *any B^2 -sequence is a Wiener sequence*. Furthermore,

3. The set $\Lambda_\xi = \{\lambda: \widehat{\xi}(\lambda) \neq 0\}$ (called *the spectrum of ξ*) is countable, and the Parseval identity $\|\xi\|^2 = \sum_{\lambda \in \Lambda_\xi} |\widehat{\xi}(\lambda)|^2$ holds.

Then, the spectral measure μ of ξ is nothing but $\sum_{\lambda \in \Lambda_\xi} |\widehat{\xi}(\lambda)|^2 \delta_\lambda$, where δ_λ is the unit point mass at λ .

Combining these preliminaries¹ with Theorem 1, we arrive at

Corollary 2. *Any B^2 -sequence whose spectrum is dense in the unit circle is an L -sequence.*

¹ The reader will find more details, for instance, in [6, Sections 2.6, 3.4, 4.2] (where the Besicovitch almost-periodic functions on \mathbb{R} are treated) as well as in [1, Section 3].

If the sequence ξ is uniformly almost-periodic, then the function F_ξ has a completely regular growth in the sense of Levin-Pfluger with the conjugated indicator diagram being the convex hull of the spectrum Λ_ξ . This is a theorem of Levin [11, Section VI.5] (for a relatively self-contained proof see [2, Theorem 5]). If the spectrum Λ_ξ is dense in \mathbb{T} , then the conjugated indicator diagram is the closed unit disk. In this special case, Corollary 2 provides a much stronger result². On the other hand, it is not difficult to construct a bounded B^2 -sequence ξ such that the closure $\overline{\Lambda_\xi} \subsetneq \mathbb{T}$, but still the indicator diagram of F_ξ is the closed unit disk (i.e., the Phragmén-Lindelöf indicator $h_{F_\xi} \equiv 1$). This leaves no hope to extend the general case of Levin's theorem to B^2 -sequences.

3.3 Unimodular pseudo-random sequences

3.3.1

Corollary 3. *Let $(a_n)_{n \geq 1}$ be a sequence of positive integers such that for every $k \geq 1$, the sequence $(a_{n+k} - a_n)_{n \geq 1}$ consists of distinct numbers. Then for almost every real number x , the sequence $(e(a_n x))_{n \geq 1}$ is a Wiener sequence, its spectral measure being Lebesgue measure on the unit circle, and, hence, this sequence is an L -sequence.*

In particular, for almost every real number t and for every natural number $a \geq 2$, the sequence $(e(a^n t))_{n \geq 1}$ is an L -sequence.

Indeed, we fix $k \geq 1$, consider the sum

$$\sum_{0 \leq s \leq n} e(a_s x) \cdot \overline{e(a_{s+k} x)} = \sum_{0 \leq s \leq n} e((a_s - a_{s+k})x),$$

and apply the classical theorem of Weyl (see, for instance, [10, Theorem 4.1 in Chapter 1]), which says that, given a sequence (b_n) of distinct integers (in our case, $b_n = a_n - a_{n+k}$), for almost every real number x ,

$$\sum_{0 \leq s \leq n} e(b_s x) = o(n), \quad n \rightarrow \infty.$$

3.3.2

Corollary 4. *Let $\alpha > 0$, $\beta > 1$, $\beta \notin \mathbb{N}$. Then the sequence $(e(\alpha n^\beta))_{n \geq 1}$ is a Wiener sequence, its spectral measure being Lebesgue measure on the unit circle, and, hence,*

² For instance, uniform almost-periodic sequences cannot attain finitely many values, while there are plenty of B^2 -sequences that attain only finitely many values and have a spectrum that is dense in \mathbb{T} . One of the simplest examples is, probably, $\xi(n) = (-1)^{[\alpha n]}$ with irrational α .

this sequence is an L -sequence.

For $1 < \beta \leq \frac{3}{2}$, this follows from a result of Chen and Littlewood [4]; for $\beta \geq \frac{3}{2}$, $\beta \notin \mathbb{N}$, this was proven in [2, Theorem 2]. The techniques used in these cases were quite different and the case $1 < \beta \leq \frac{3}{2}$ required a rather elaborate argument³. Theorem 1 combined with classical van der Corput estimates of exponential sums allows us to treat both cases in a simple and uniform manner.

Let us also mention that for $0 < \beta < 1$, the sequence $(e(\alpha n^\beta))_{n \geq 1}$ is a Wiener sequence, its spectral measure being the unit point mass at 1. It is not difficult to show that the corresponding entire function F_ξ has the Phragmén–Lindelöf indicator $h_{F_\xi}(\theta) = \cos_+ \theta$ (cf. [7, Theorem 2]), which is obviously incompatible with ξ being an L -sequence.

Proof. We need to show that for every $m \geq 1$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq s < n} e(\alpha(s+m)^\beta - \alpha s^\beta) = 0.$$

Set $\phi(t) = \alpha(t+m)^\beta - \alpha t^\beta$.

For $1 < \beta < 2$ we apply a van der Corput estimate [10, Theorem 2.7 in Chapter 1]. We have

$$\left| \frac{1}{n} \sum_{0 \leq s < n} e(\phi(s)) \right| \lesssim \frac{1}{n |\phi''(n)|^{1/2}} \simeq n^{(1-\beta)/2}, \quad n \rightarrow \infty.$$

For $\beta > 2$ we use another, more elaborate, van der Corput bound [18, Theorem 5.13]. Choose an integer $k \geq 2$ such that $k < \beta < k+1$ and fix $M = 2^k$. Since

$$n^{\beta-k-1} \lesssim \phi^{(k)}(t) \lesssim n^{(k+1-\beta)/4} n^{\beta-k-1}, \quad n^{3/4} \leq t \leq n,$$

we have

$$\begin{aligned} \left| \sum_{0 \leq s < n} e(\phi(s)) \right| &\leq n^{3/4} + \left| \sum_{n^{3/4} \leq s < n} e(\phi(s)) \right| \\ &\lesssim n^{3/4} + n^{(k+1-\beta)/M} \cdot n \cdot n^{(\beta-k-1)/(M-2)} + n^{1-(4/M)} \cdot n^{(k+1-\beta)/(M-2)} = o(n) \end{aligned}$$

as $n \rightarrow \infty$. □

³ Note that Chen and Littlewood proved a much finer result: they found an asymptotic location of zeroes of the entire function F_ξ .

We note that, likewise, one can show that if $Q(x) = \sum_{k=2}^d q_k x^k$ is a polynomial with real coefficients q_k and at least one of the coefficients is irrational, then the sequence $(e(Q(n)))_{n \geq 1}$ is a Wiener sequence whose spectral measure is the Lebesgue measure on the unit circle. Therefore, this sequence is an L -sequence. The latter conclusion is Theorem 1 from [2].

3.4 Arithmetic multiplicative functions

Recall that an arithmetic function $f: \mathbb{N} \rightarrow \mathbb{C}$ is called *completely multiplicative* if $f(n)f(m) = f(nm)$ for all natural n, m , and *multiplicative* if $f(n)f(m) = f(nm)$ for all mutually prime natural n, m .

3.4.1 The Möbius function

The classical Möbius function μ is defined by

$$\mu(n) = \begin{cases} 1, & n \text{ square-free with an even number of prime factors;} \\ -1, & n \text{ square-free with an odd number of prime factors;} \\ 0, & \text{otherwise.} \end{cases}$$

One instance of the well-known Chowla conjecture (the so called “binary Chowla conjecture”) asserts that, for every $k \geq 1$,

$$\sum_{1 \leq s \leq n} \mu(s)\mu(s+k) = o(n), \quad n \rightarrow \infty, \quad (11)$$

that is, μ is a Wiener sequence whose spectral measure is the Lebesgue measure. Thus, applying Theorem 1, we get

Corollary 5. *Assuming Chowla’s conjecture (11), the sequence $(\mu(n))_{n \geq 1}$ is an L -sequence.*

The question on the asymptotic behaviour of the Taylor series of the form

$$\sum_{n \geq 1} \mu(n) a_n z^n$$

was raised by Paul Lévy in [12, Section 18]. He hinted that the positive answer to the Riemann hypothesis would provide information on this asymptotic behaviour.

3.4.2 Random multiplicative functions

Denote by \mathbb{P} the set of prime numbers. Let $(X_p)_{p \in \mathbb{P}}$ be a sequence of independent Steinhaus random variables (i.e., uniformly distributed on the unit circle), and let $(Y_p)_{p \in \mathbb{P}}$ be a sequence of independent Rademacher random variables (i.e., taking the values ± 1 with equal probability). We consider two random functions: a *Steinhaus random completely multiplicative function* ξ_S determined by $\xi_S(p) = X_p$, $p \in \mathbb{P}$, that is,

$$\xi_S(n) = \prod_{p^a \parallel n} (X_p)^a,$$

and a *Rademacher random multiplicative function* ξ_R determined by $\xi_R(p) = Y_p$, $p \in \mathbb{P}$, on the square-free indices p and 0 elsewhere, that is,

$$\xi_R(n) = \begin{cases} \prod_{p|n} X_p & n \text{ is square-free;} \\ 0 & \text{otherwise.} \end{cases}$$

The function ξ_S randomizes the prime factorization of integers, while the function ξ_R is a randomized version of the Möbius function. The study of these random multiplicative functions has a long history and goes back, at least, to Wintner [20].

Theorem 2. *The sequences ξ_S and ξ_R are almost surely Wiener sequences, their spectral measures being Lebesgue measure on the unit circle, and, hence, these sequence are almost surely L -sequences.*

The proof is based on the following lemma:

Lemma 9. *Given $k \geq 1$, we have*

$$\begin{aligned} \mathbb{E} \left| \sum_{m \leq s \leq \lambda m} \xi_S(s) \overline{\xi_S(s+k)} \right|^2 &= O(m^{7/4}), \\ \mathbb{E} \left[\sum_{m \leq s \leq \lambda m} \xi_R(s) \xi_R(s+k) \right]^2 &= O(m^{7/4}), \end{aligned}$$

uniformly in $\lambda \in [1, 2]$.

Proof. Since

$$\mathbb{E} \left| \sum_{m \leq s \leq \lambda m} \xi_S(s) \overline{\xi_S(s+k)} \right|^2 = \sum_{m \leq s_1, s_2 \leq \lambda m} \mathbb{E} (\xi_S(s_1) \overline{\xi_S(s_1+k)} \overline{\xi_S(s_2)} \xi_S(s_2+k)),$$

and the Steinhaus random variables X_p , $p \in \mathbb{P}$, are independent, after a minute reflection we convince ourselves that

$$\mathbb{E} \left| \sum_{m \leq s \leq \lambda m} \xi_S(s) \overline{\xi_S(s+k)} \right|^2 \leq \text{card}(\mathbf{B}),$$

where

$$\mathbf{B} = \{(s_1, s_2) \in [m, \lambda m] \times [m, \lambda m] : s_1(s_1+k)s_2(s_2+k) \text{ is a square}\}.$$

Similarly,

$$\mathbb{E} \left[\sum_{m \leq s \leq \lambda m} \xi_R(s) \xi_R(s+k) \right]^2 = \text{card}(\mathbf{B}).$$

It remains to estimate the cardinality of the set \mathbf{B} .

Given $(s_1, s_2) \in [m, \lambda m] \times [m, \lambda m]$, set $d_1 = \gcd(s_1, s_1+k)$, $d_2 = \gcd(s_2, s_2+k)$. Then d_1 and d_2 are divisors of k . Next, let $e_1^2, f_1^2, e_2^2, f_2^2$ be the largest square divisors of s_1/d_1 , $(s_1+k)/d_1$, s_2/d_2 , and $(s_2+k)/d_2$. Then $(s_1, s_2) \in \mathbf{B}$ if and only if

$$\frac{(s_1/d_1) \cdot ((s_1+k)/d_1)}{e_1^2 f_1^2} = \frac{(s_2/d_2) \cdot ((s_2+k)/d_2)}{e_2^2 f_2^2}.$$

Let $\mathbf{B}_1 = \{(s_1, s_2) \in \mathbf{B} : e_1, f_1 \leq m^{1/3}\}$, $\mathbf{B}_2 = \mathbf{B} \setminus \mathbf{B}_1$. First, we will bound the cardinality of \mathbf{B}_1 and then of \mathbf{B}_2 .

For every choice of s_2, d_1, e_1, f_1 , the integer s_1 divides $M = s_2(s_2+k)e_1^2 f_1^2 d_1^2$ (recall that $s_1(s_1+k)e_2^2 f_2^2 d_2^2 = s_2(s_2+k)e_1^2 f_1^2 d_1^2$), and the number of divisors of M is $O(M^\varepsilon) = O(m^\varepsilon)$, $\varepsilon > 0$ (see, for example, [8, Theorem 315]). Furthermore, the number of different choices of s_2 is $\simeq m$; there is only a bounded number of different choices of d_1 (since the integer k is fixed and d_1 divides k); and the number of different choices of e_1 , as well as of f_1 does not exceed $m^{1/3}$. Thus,

$$\text{card}(\mathbf{B}_1) \lesssim_\varepsilon m^{(5/3)+\varepsilon}.$$

To bound the cardinality of \mathbf{B}_2 , we observe that if $(s_1, s_2) \in \mathbf{B}_2$, then either $e_1 > m^{1/3}$ or $f_1 > m^{1/3}$. Now,

$$\begin{aligned} \text{card}\{(s_1, s_2) \in \mathbf{B} : e_1 > m^{1/3}\} &\leq \sum_{m^{1/3} < e_1 \leq (\lambda m)^{1/2}} \sum_{\substack{m \leq s_1, s_2 \leq \lambda m \\ s_1 = ue_1^2}} 1 \\ &\lesssim \sum_{e_1 > m^{1/3}} \frac{m^2}{e_1^2} \lesssim m^{5/3}. \end{aligned}$$

In the same way, $\text{card}\{(s_1, s_2) \in \mathbf{B} : f_1 > m^{1/3}\} \lesssim m^{5/3}$. \square

Proof of Theorem 2. Let ξ be either ξ_S or ξ_R . We will show that ξ is a Wiener sequence with the spectral measure being the Lebesgue measure on the unit circle⁴. The rest follows from Theorem 1.

Fix $k \geq 1$ and split the sum

$$\sum_{s \geq 1} \xi(s) \overline{\xi(s+k)}$$

into the blocks $m^{20} \leq s < (m+1)^{20}$, $m = 1, 2, \dots$. By Lemma 9, for every $m \geq 1$, we have

$$\mathbb{P}\left(\left|\sum_{m^{20} \leq s < (m+1)^{20}} \xi(s) \overline{\xi(s+k)}\right| > m^{(15/16) \cdot 20}\right) \lesssim m^{-20/8},$$

and hence, by the Borel-Cantelli lemma, almost surely, for sufficiently large m ,

$$\left|\sum_{m^{20} \leq s < (m+1)^{20}} \xi(s) \overline{\xi(s+k)}\right| \leq m^{75/4}.$$

Therefore, almost surely,

$$\begin{aligned} \left|\sum_{1 \leq s \leq n} \xi(s) \overline{\xi(s+k)}\right| &\leq \sum_{1 \leq \ell \leq [n^{1/20}]} \left|\sum_{\ell^{20} \leq s < (\ell+1)^{20}} \xi(s) \overline{\xi(s+k)}\right| + \sum_{[n^{1/20}]^{20} \leq s \leq n} 1 \\ &\lesssim \sum_{1 \leq \ell \leq [n^{1/20}]} \ell^{75/4} + n^{19/20} \lesssim n^{79/80} + n^{19/20} = o(n), \quad n \rightarrow \infty, \end{aligned}$$

proving the theorem. □

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⁴ Likely, this fact can be also extracted from results of the recent work of Najnudel [16], at least, in the completely multiplicative case.

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