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On the asymptotic behaviour of the eigenvalue distribution of block correlation matrices of high-dimensional time series

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1 Introduction

1.1 Problem addressed, motivation and results

We consider a set of M jointly stationary zero mean complex-valued scalar time series, denoted as $y_{1,n}, \dots, y_{M,n}$, where $n \in \mathbb{Z}$. We assume that the joint distribution of $((y_{m,n})_{n \in \mathbb{Z}})_{m=1, \dots, M}$ is the circularly symmetric complex Gaussian law¹, and that for each $m = 1, \dots, M$, the values taken by $y_{m,1}, \dots, y_{m,N}$ are available. In this paper, we study the behaviour of linear statistics of the eigenvalues of a certain large random matrix built from the available data when the M time series $(y_m)_{m=1, \dots, M}$ are uncorrelated (i.e. independent) and that both M and N are large. Our results are potentially useful in order to address the problem of testing whether a large number of time series are uncorrelated or not.

In order to introduce the large random matrix models that we will address in the following, we consider a column vector gathering L consecutive observations of the m th time series, namely

$$\mathbf{y}_{m,n}^L = [y_{m,n}, \dots, y_{m,n+L-1}]^T$$

and from this build an ML -dimensional column vector

$$\mathbf{y}_n^L = [(\mathbf{y}_{1,n}^L)^T, \dots, (\mathbf{y}_{M,n}^L)^T]^T.$$

We will denote by \mathcal{R}_L the so-called $ML \times ML$ spatio-temporal covariance matrix of this random vector, i.e. $\mathcal{R}_L = \mathbb{E} [\mathbf{y}_n^L (\mathbf{y}_n^L)^H]$ where $(\cdot)^H$ stands for transpose conjugate. Clearly, the M series $(y_m)_{m=1, \dots, M}$ are uncorrelated if and only if, for each integer L , matrix \mathcal{R}_L is block-diagonal, namely

$$\mathcal{R}_L = \text{Bdiag}(\mathcal{R}_L)$$

¹Any finite linear combination $z = \sum_{m=1}^M \sum_{j=1}^J \alpha_j y_{m,n_j}$ of the random variables $((y_{m,n})_{n \in \mathbb{Z}})_{m=1, \dots, M}$ is distributed according to the distribution $\mathcal{N}_c(0, \delta^2)$, i.e. $\text{Re}z$ and $\text{Im}z$ are independent and $\mathcal{N}(0, \delta^2/2)$ distributed, where $\delta^2 > 0$ is the corresponding variance

where, for an $ML \times ML$ matrix \mathbf{A} , $\text{Bdiag}(\mathbf{A})$ is the block-diagonal matrix of the same dimension whose $L \times L$ blocks are those of \mathbf{A} . We notice that the $L \times L$ blocks of $\text{Bdiag}(\mathcal{R}_L)$ are the $L \times L$ Toeplitz matrices $\mathcal{R}_{m,L}$, $m = 1, \dots, M$, defined by

$$\{\mathcal{R}_{m,L}\}_{k,k'} = r_m(k - k')$$

where $r_m(k)$, $k \in \mathbb{Z}$, is the covariance sequence of the m th time series, defined as

$$r_m(k) = \int_0^1 \mathcal{S}_m(\nu) e^{2\pi i \nu k} d\nu. \quad (1.1)$$

where for each m , \mathcal{S}_m represents the spectral density of $(y_{m,n})_{n \in \mathbb{Z}}$. We will denote by $\mathcal{R}_{\text{corr},L}$ the block correlation matrix defined by

$$\mathcal{R}_{\text{corr},L} = \mathcal{B}_L^{-1/2} \mathcal{R}_L \mathcal{B}_L^{-1/2} \quad (1.2)$$

where

$$\mathcal{B}_L = \text{Bdiag}(\mathcal{R}_L).$$

Consequently, \mathcal{R}_L is block diagonal for each L if and only if $\mathcal{R}_{\text{corr},L} = \mathbf{I}_{ML}$ for each L . If we assume that for each $m = 1, \dots, M$, the observations $y_{m,1}, \dots, y_{m,N}$ are available, a possible way to test that the time series $(y_m)_{m=1, \dots, M}$ are uncorrelated thus consists in estimating $\mathcal{R}_{\text{corr},L}$ for a suitable value of L , and in comparing the corresponding estimate to \mathbf{I}_{ML} . In the following, we consider the standard sample estimate $\widehat{\mathcal{R}}_{\text{corr},L}$ defined by

$$\widehat{\mathcal{R}}_{\text{corr},L} = \widehat{\mathcal{B}}_L^{-1/2} \widehat{\mathcal{R}}_L \widehat{\mathcal{B}}_L^{-1/2} \quad (1.3)$$

where $\widehat{\mathcal{R}}_L$ is the empirical spatio-temporal covariance matrix given by

$$\widehat{\mathcal{R}}_L = \frac{1}{N} \sum_{n=1}^N \mathbf{y}_n^L (\mathbf{y}_n^L)^H \quad (1.4)$$

and where $\widehat{\mathcal{B}}_L$ is the corresponding block diagonal

$$\widehat{\mathcal{B}}_L = \text{Bdiag}(\widehat{\mathcal{R}}_L).$$

A relevant question here is how to choose the lag parameter L . On the one hand, L should be sufficiently large, because this allows to identify correlations among samples in different time series that are well spaced in time. For instance, two time series chosen as copies of the same temporally white noise with a relative delay higher than L lags will be perceived as uncorrelated by examination of $\widehat{\mathcal{R}}_{\text{corr},L}$, which is of course far from true. On the other hand, L should be chosen sufficiently low so that $ML/N \ll 1$ in order to make the estimation error $\|\widehat{\mathcal{R}}_{\text{corr},L} - \mathbf{I}_{ML}\|$ reasonably low under the hypothesis that $(y_m)_{m=1, \dots, M}$ are uncorrelated, to be referred to as the hypothesis H_0 in the following. If the number M of time series is large and that the number of observations N is not unlimited, the condition $ML/N \ll 1$ requires the selection of a small value for L . Such a choice may thus reduce drastically the efficiency of the uncorrelation tests based on $\|\widehat{\mathcal{R}}_{\text{corr},L} - \mathbf{I}_{ML}\|$. Finding statistics having a well defined behaviour under H_0 when ML and N are of the same order of magnitude would allow to consider larger values of L , and thus would improve the performance of the corresponding tests. In this paper, we propose to study the behavior of spectral statistics built from the eigenvalues of $\widehat{\mathcal{R}}_{\text{corr},L}$, which will be denoted by $(\hat{\lambda}_{k,N})_{k=1, \dots, ML}$. More specifically, we will consider statistics of the form

$$\widehat{\phi}_N = \frac{1}{ML} \text{Tr} \left[\phi \left(\widehat{\mathcal{R}}_{\text{corr},L} \right) \right] = \frac{1}{ML} \sum_{k=1}^{ML} \phi \left(\hat{\lambda}_{k,N} \right)$$

where ϕ is assumed to be a suitable function, and will study the behaviour of $\widehat{\phi}_N$ under H_0 in asymptotic regimes where M, N, L converge towards $+\infty$ in such a way that $c_N = \frac{ML}{N}$ converges towards a non zero constant $c_* \in (0, +\infty)$.

The main result of this paper establishes that under the above asymptotic regime, $\widehat{\phi}_N$ converges almost surely towards the integral of ϕ with respect to the so-called Marchenko-Pastur distribution. In order to analyze the asymptotic behavior of the above class of statistics, we use large random matrix methods that relate

the quantity $\widehat{\phi}_N$ with the empirical eigenvalue distribution of $\widehat{\mathcal{R}}_{\text{corr},L}$, denoted $d\widehat{\mu}_N(\lambda) = \frac{1}{ML} \sum_{k=1}^{ML} \delta_{\lambda - \widehat{\lambda}_{k,N}}$, that is

$$\widehat{\phi}_N = \int \phi(\lambda) d\widehat{\mu}_N(\lambda).$$

We will establish the behavior of $\widehat{\phi}_N$ by studying the empirical eigenvalue distribution $d\widehat{\mu}_N(\lambda)$. More specifically, we will first establish that there exists a deterministic probability measure $d\mu_N(\lambda)$ such that, for each bounded continuous function ϕ defined on \mathbb{R} , it holds that

$$\widehat{\phi}_N - \int_{\mathbb{R}^+} \phi(\lambda) d\mu_N(\lambda) \rightarrow 0 \quad (1.5)$$

almost surely. Moreover, we will also prove that the deterministic sequence $(\mu_N)_{N \geq 1}$ converges towards the Marcenko-Pastur distribution μ_{mp,c_*} with parameter c_* . We recall that for each $d > 0$, $\mu_{mp,d}$ is the limit of the empirical eigenvalue distribution of a large random matrix $\frac{1}{K} \mathbf{X} \mathbf{X}^*$ where \mathbf{X} is a $J \times K$ random matrix with zero mean unit variance i.i.d. entries and where both J and K converge towards $+\infty$ in such a way that $\frac{J}{K} \rightarrow d$. This in turn will imply that, in the above asymptotic regime,

$$\widehat{\phi}_N \rightarrow \int_{\mathbb{R}^+} \phi(\lambda) d\mu_{mp,c_*}(\lambda) \quad (1.6)$$

holds. This result potentially allows to test whether the M time series y_1, \dots, y_M are uncorrelated by comparing linear spectral statistics such as $\widehat{\phi}_N$ to their limits under H_0 . The detailed study of such class of tests will be conducted in a future work.

1.2 On the literature

Testing whether M time series are uncorrelated is an important problem that was addressed extensively in the past. Apart from a few works devoted to the case where the number of time series M converges towards $+\infty$ (see below), the vast majority of published papers assumed that M is a fixed integer. In this context, we can first mention spectral domain approaches based on the observation that the M -variate time series $(y_{1,n})_{n \in \mathbb{Z}}, \dots, (y_{M,n})_{n \in \mathbb{Z}}$ are uncorrelated if and only the spectral coherence matrix of the M -variate time series $(\mathbf{y}_n)_{n \in \mathbb{Z}}$, where $\mathbf{y}_n = (y_{1,n}, \dots, y_{M,n})^T$, is reduced to \mathbf{I}_M at each frequency. Some examples following this approach are [27], [26], [7], [8]. A number of papers also proposed to develop lag domain approaches, e.g. [14], [15], [6], [18] which considered test statistics based on empirical estimates of the autocorrelation coefficients between the residuals of the various time series. See also [9] for a more direct approach.

We next review the very few existing works devoted to the case where the number M of time series converges towards $+\infty$. We are just aware of papers addressing the case where the observations $\mathbf{y}_1, \dots, \mathbf{y}_N$ are independent identically distributed (i.i.d.) and where the ratio $\frac{M}{N}$ converges towards a constant $d \in (0, 1)$. In particular, in contrast with the asymptotic regime considered in the present work, these papers assume that M and N are of the same order of magnitude. This is because, in this context, the time series are mutually uncorrelated if and only the covariance matrix $\mathbb{E}(\mathbf{y}_n \mathbf{y}_n^*)$ is diagonal. Therefore, it is reasonable to consider test statistics that are functionals of the sample covariance matrix $\frac{1}{N} \sum_{n=1}^N \mathbf{y}_n \mathbf{y}_n^H$. In particular, when the observations are i.i.d. Gaussian random vectors, the generalized likelihood ratio test (GLRT) consists in comparing the test statistics $\log \det(\widehat{\mathcal{R}}_{\text{corr}})$ to a threshold, where $\widehat{\mathcal{R}}_{\text{corr}} = \widehat{\mathcal{R}}_{\text{corr},1}$ represents the sample correlation matrix. [16] proved that under H_0 , the empirical eigenvalue distribution of $\widehat{\mathcal{R}}_{\text{corr}}$ converges almost surely towards the Marcenko-Pastur distribution $\mu_{mp,d}$ and therefore, that $\frac{1}{M} \text{Tr}(\phi(\widehat{\mathcal{R}}_{\text{corr}}))$ converges towards $\int \phi(\lambda) d\mu_{mp,d}(\lambda)$ for each bounded continuous function ϕ . In the Gaussian case, [17] also established a central limit theorem (CLT) for $\log \det(\widehat{\mathcal{R}}_{\text{corr}})$ under H_0 using the moment method. [5] remarked that, in the Gaussian real case, $(\det(\widehat{\mathcal{R}}_{\text{corr}}))^{M/2}$ is the product of independent beta distributed random variables. Therefore, $\log \det(\widehat{\mathcal{R}}_{\text{corr}})$ appears as the sum of independent random variables, thus deducing the CLT. We finally mention [21] in which a CLT on linear statistics of the eigenvalues of $\widehat{\mathcal{R}}_{\text{corr}}$ is established in the Gaussian case using large random matrix techniques when the covariance matrix $\mathbb{E}(\mathbf{y}_n \mathbf{y}_n^*)$ is not necessarily diagonal. This allows to study the asymptotic performance of the GLRT under certain class of alternatives.

It is also relevant to highlight the work in [19] and [20], which addressed the asymptotic behaviour of the empirical eigenvalue distribution of matrix $\hat{\mathcal{R}}_L$ in the asymptotic regime considered in the present paper. More specifically, [19] assumed that the M mutually independent time series y_1, \dots, y_M are i.i.d. Gaussian and established that the empirical eigenvalue distribution of $\hat{\mathcal{R}}_L$ converges towards the Marcenko Pastur distribution μ_{mp, c_*} . Moreover, if $L = \mathcal{O}(N^\beta)$ with $\beta < 2/3$, it is established that almost surely, for N large enough, all the eigenvalues of $\hat{\mathcal{R}}_L$ are located in a neighbourhood of the support of μ_{mp, c_*} . In [20], the mutually independent time series y_1, \dots, y_M are no longer assumed i.i.d. and it is established that the empirical eigenvalue distribution has a deterministic behaviour. The corresponding deterministic equivalent is characterized, and some results on the corresponding speed of convergence are given. As it will appear below, the present paper uses extensively the tools developed in [20].

1.3 Assumptions

Assumption 1. *All along the paper, we assume that $M \rightarrow +\infty, N \rightarrow +\infty$ in such a way that $c_N = \frac{ML}{N} \rightarrow c_*$, where $0 < c_* < +\infty$, and that $L = L(N) = \mathcal{O}(N^\beta)$ for some constant $\beta \in (0, 1)$. In order to shorten the notations, $N \rightarrow +\infty$ should be understood as the above asymptotic regime.*

As $M = M(N) \rightarrow +\infty$, we assume that an infinite sequence $(y_m)_{m \geq 1}$ of mutually independent zero mean circularly symmetric complex Gaussian time series with spectral densities $(\mathcal{S}_m(\nu))_{m \geq 1}$ is given. We will need that the spectral densities are bounded above and below uniformly in M , namely

Assumption 2. *The spectral densities are such that*

$$\sup_{m \geq 1} \max_{\nu \in [0, 1]} \mathcal{S}_m(\nu) = s_{max} < +\infty \quad (1.7)$$

$$\inf_{m \geq 1} \min_{\nu \in [0, 1]} \mathcal{S}_m(\nu) = s_{min} > 0. \quad (1.8)$$

Let us denote by \mathbf{r}_M the M -dimensional sequence of covariances, namely

$$\mathbf{r}_M(k) = [r_1(k), \dots, r_M(k)]^T \quad (1.9)$$

where $r_m(k)$, $m = 1, \dots, M$ are defined in (1.1). We can consider the sequence of Euclidean norms $\{\|\mathbf{r}_M(k)\|\}_{k \in \mathbb{Z}}$. At some points, we will need the corresponding series to be of order $\mathcal{O}(\sqrt{M})$.

Assumption 3. *The multivariate covariance sequence \mathbf{r}_M defined in (1.9) is such that*

$$\sup_{M \geq 1} \frac{1}{\sqrt{M}} \sum_{k \in \mathbb{Z}} \|\mathbf{r}_M(k)\| < +\infty.$$

If $(r_m)_{m \geq 1}$ represents the corresponding infinite autocovariance sequences, we will also need to impose some assumptions that impose a certain rate of decayment of $\sup_{m \geq 1} \sum_{|k| \geq n+1} |r_m(k)|$ when $n \rightarrow +\infty$. To that effect, we introduce the weighting sequence $(\omega(n))_{n \in \mathbb{Z}}$ defined as

$$\omega(n) = (1 + |n|)^\gamma$$

where $\gamma \geq 0$ is given. This sequence belongs to the class of strong Beurling weights (see [24], Chapter 5), which are functions ω on \mathbb{Z} with the properties: (i) $\omega(n) \geq 1$, (ii) $\omega(n) = \omega(-n)$, (iii) $\omega(m+n) \leq \omega(m)\omega(n)$ for all $m, n \in \mathbb{Z}$ and (iv) $n^{-1} \log \omega(n) \rightarrow 0$ as $n \rightarrow \infty$. We define ℓ_ω as the Banach space of two sided sequences $a = (a(n))_{n \in \mathbb{Z}}$ such that

$$\|a\|_\omega = \sum_{n=-\infty}^{\infty} \omega(n) |a(n)| = \sum_{n=-\infty}^{\infty} (1 + |n|)^\gamma |a(n)| < +\infty.$$

When $\gamma = 0$, $\omega(n) = 1$ for each n , and ℓ_ω coincides with the Wiener algebra $\ell_1 = \{a = (a(n))_{n \in \mathbb{Z}}, \|a\|_1 < +\infty\}$. For each $\gamma \geq 0$, it holds that $\|a\|_1 \leq \|a\|_\omega$, and that ℓ_1 is included in ℓ_ω . The function $\sum_{n \in \mathbb{Z}} a(n)e^{2i\pi n\nu}$ is thus well defined and continuous on $[0, 1]$, and we will identify the sequence a to the above function. In particular, with a certain abuse of notation, $\sum_{n \in \mathbb{Z}} a(n)e^{2i\pi n\nu}$ will be denoted by $a(e^{2i\pi\nu})$ in the following. We can of course define the convolution product of sequences in ℓ_ω , namely

$$(a_1 * a_2)(n) = \sum_{m \in \mathbb{Z}} a_1(m)a_2(n - m)$$

which has the property that $\|a_1 * a_2\|_\omega \leq \|a_1\|_\omega \|a_2\|_\omega$, and therefore $a_1 * a_2 \in \ell_\omega$. Under the convolution product, we can see ℓ_ω as an algebra (the Beurling algebra) associated with the weight ω .

Assumption 4. For some $\gamma_0 > 0$, the covariance sequence r_m defined in (1.1) belongs to ℓ_{ω_0} for each m , where $\omega_0(n) = (1 + |n|)^{\gamma_0}$. Moreover, it is assumed that

$$\sup_{m \geq 1} \|r_m\|_{\omega_0} < \infty. \quad (1.10)$$

Note that the fact that $r_m \in \ell_{\omega_0}$ implies that, for each $0 \leq \gamma < \gamma_0$, we have $r_m \in \ell_\omega$, where $\omega(n) = (1 + |n|)^\gamma$. Moreover, (1.10) allows to control uniformly w.r.t. m of the remainder $\sum_{|k| \geq (n+1)} |r_m(k)|$. Indeed, observe that we can write

$$\|r_m\|_{\omega_0} \geq \sum_{|k| \geq n+1} (1 + |k|)^{\gamma_0} |r_m(k)| \geq n^{\gamma_0} \sum_{|k| \geq n+1} |r_m(k)|.$$

Therefore, (1.10) implies that

$$\sup_{m \geq 1} \sum_{|k| \geq n+1} |r_m(k)| \leq \frac{\kappa}{n^{\gamma_0}} \quad (1.11)$$

for some constant κ .

1.4 Main Results

The main tool in order to study the statistics that are relevant here will be the Stieltjes transform of the empirical eigenvalue distribution of the estimated block correlation matrix $\hat{\mathcal{R}}_{\text{corr},L}$ defined by (1.3). More specifically, we will denote by $\hat{q}_N(z)$ the Stieltjes transform of the empirical eigenvalue distribution of $\hat{\mathcal{R}}_{\text{corr},L}$, that is

$$\hat{q}_N(z) = \int_{\mathbb{R}^+} \frac{1}{\lambda - z} d\hat{\mu}_N(\lambda) = \frac{1}{ML} \sum_{k=1}^{ML} \frac{1}{\hat{\lambda}_{k,N} - z}$$

which is defined for $z \in \mathbb{C} \setminus \mathbb{R}^+$. This function can also be written as $\hat{q}_N(z) = \frac{1}{ML} \text{Tr} \hat{\mathbf{Q}}_N(z)$ where $\hat{\mathbf{Q}}_N(z)$ is the resolvent of matrix $\hat{\mathcal{R}}_{\text{corr},L}$, namely

$$\hat{\mathbf{Q}}_N(z) = \left(\hat{\mathcal{R}}_{\text{corr},L} - z \mathbf{I}_{ML} \right)^{-1}. \quad (1.12)$$

Each realization of the resolvent can be identified with the Stieltjes transform of a positive matrix valued measure carried by \mathbb{R}^+ with total measure \mathbf{I}_{ML} (see Proposition 1.1 below for details).

Definition 1. We denote by $\mathcal{S}_{ML}(\mathbb{R}^+)$ the set of all $ML \times ML$ matrix valued functions defined on $\mathbb{C} \setminus \mathbb{R}^+$ by

$$\mathcal{S}_{ML}(\mathbb{R}^+) = \left\{ \int_{\mathbb{R}^+} \frac{1}{\lambda - z} d\boldsymbol{\mu}(\lambda) \right\}$$

where $\boldsymbol{\mu}$ is a positive $ML \times ML$ matrix-valued measure carried by \mathbb{R}^+ satisfying $\boldsymbol{\mu}(\mathbb{R}^+) = \mathbf{I}_{ML}$.

In some parts of the paper, we will need to bound quantities by constants that do not depend on the system dimensions nor on the complex variable z . These will be referred to as “nice constants”.

Definition 2 (Nice constants and nice polynomials). *A nice constant is a positive constant independent of the dimensions L, M, N and the complex variable z . A nice polynomial is a polynomial whose degree is independent from L, M, N , and whose coefficients are nice constants. In the following, κ and P_1, P_2 will represent a generic nice constant and two generic nice polynomials respectively, whose values may change from one line to another. Finally, $C(z)$ will denote a general term of the form $C(z) = P_1(|z|)P_2(1/\delta_z)$, where $\delta_z = \text{dist}(z, \mathbb{R}^+)$.*

We present now the first result of this paper, which basically shows that, in the sense of almost sure weak convergence, the empirical eigenvalue distribution of $\widehat{\mathcal{R}}_{\text{corr},L}$ has a deterministic behaviour. More specifically, the following theorem states that the function $\hat{q}_N(z)$ asymptotically behaves as a deterministic equivalent $\frac{1}{ML} \text{Tr}(\mathbf{T}_N(z))$, where the $ML \times ML$ matrix function $\mathbf{T}_N(z)$ can be obtained as the solution to a certain system of equations, see (4.1-4.2) in Section 4.

Theorem 1. *Under the above set of assumptions, there exists a function $\mathbf{T}_N(z)$ of $\mathcal{S}_{ML}(\mathbb{R}^+)$ with associated measure μ_N , such that*

$$\hat{q}_N(z) - \frac{1}{ML} \text{Tr}(\mathbf{T}_N(z)) \rightarrow 0 \quad (1.13)$$

almost surely for each $z \in \mathbb{C} \setminus \mathbb{R}^+$, where $\frac{1}{ML} \text{Tr}(\mathbf{T}_N(z))$ is the Stieltjes transform of the probability measure $\mu_N = \frac{1}{ML} \text{Tr}(\mu_N)$. Moreover, for every bounded continuous function ϕ , we have

$$\frac{1}{ML} \text{Tr}(\phi(\widehat{\mathcal{R}}_{\text{corr},L})) - \int \phi(\lambda) d\mu_N(\lambda) \rightarrow 0 \quad (1.14)$$

almost surely.

The proof of Theorem 1 is established in Sections 2 to 5. We summarize the main steps of the proof in what follows:

- (i) First, it is shown in Section 2 that the eigenvalue behavior of $\widehat{\mathcal{R}}_{\text{corr},L}$ can be studied by examining the eigenvalue behavior of the matrix

$$\overline{\mathcal{R}}_{\text{corr},L} = \mathcal{B}_L^{-1/2} \widehat{\mathcal{R}}_L \mathcal{B}_L^{-1/2} \quad (1.15)$$

in the sense that $\|\widehat{\mathcal{R}}_{\text{corr},L} - \overline{\mathcal{R}}_{\text{corr},L}\| \rightarrow 0$ almost surely. Note that $\overline{\mathcal{R}}_{\text{corr},L}$ is matrix defined in the same way as $\widehat{\mathcal{R}}_{\text{corr},L}$ by replacing the estimated block-diagonal autocorrelation matrix $\widehat{\mathcal{B}}_L = \text{Bdiag}(\widehat{\mathcal{R}}_L)$ by its true value $\mathcal{B}_L = \text{Bdiag}(\mathcal{R}_L)$, which in fact coincides with \mathcal{R}_L (we are assuming independent sequences). This will imply that the individual eigenvalues of both matrices have the same asymptotic behaviour with probability one, and the same property holds for the linear statistics built from them. It will therefore be sufficient to study the behaviour of the empirical eigenvalue distribution of $\overline{\mathcal{R}}_{\text{corr},L}$ in the rest of the paper, which will be denoted as $d\overline{\mu}_N(\lambda)$. More specifically, we will devote most of the paper to the analysis of the Stieltjes transform associated to this measure, defined as

$$q_N(z) = \int_{\mathbb{R}^+} \frac{1}{\lambda - z} d\overline{\mu}_N(\lambda) = \frac{1}{ML} \text{Tr} \mathbf{Q}_N(z)$$

on $z \in \mathbb{C} \setminus \mathbb{R}^+$, where $\mathbf{Q}_N(z)$ is the resolvent of $\overline{\mathcal{R}}_{\text{corr},L}$, namely

$$\mathbf{Q}_N(z) = (\overline{\mathcal{R}}_{\text{corr},L} - z\mathbf{I}_{ML})^{-1}. \quad (1.16)$$

Sections 3 to 5 are devoted to the analysis of $q_N(z)$. We adapt the tools developed in our previous work [20] devoted to the study of the empirical eigenvalue distribution of matrix $\widehat{\mathcal{R}}_L$.

- (ii) Section 3 begins the analysis of the resolvent $\mathbf{Q}_N(z)$ by considering an arbitrary sequence of deterministic $ML \times ML$ matrices $(\mathbf{A}_N)_{N \geq 1}$ satisfying $\sup_N \|\mathbf{A}_N\| < +\infty$ and analyzing the behavior of quantities of the form $\frac{1}{ML} \text{Tr}(\mathbf{A}_N \mathbf{Q}_N(z))$. More specifically, it is first established that

$$\text{var} \left[\frac{1}{ML} \text{Tr}(\mathbf{Q}_N(z) \mathbf{A}_N) \right] = \mathcal{O} \left(\frac{1}{MN} \right) \quad (1.17)$$

and that, for each bounded continuously differentiable function ϕ with bounded first derivative,

$$\text{var} \left[\frac{1}{ML} \text{Tr}(\phi(\overline{\mathcal{R}}_{\text{corr},L}) \mathbf{A}_N) \right] = \mathcal{O} \left(\frac{1}{MN} \right). \quad (1.18)$$

Given the fact that $\frac{1}{MN} = \frac{1}{N^{2-\beta}}$ and $2 - \beta > 1$, the Borel-Cantelli lemma implies that these quantities asymptotically concentrate around the corresponding expectations with probability one. Following a standard tightness argument, the statements of Theorem 1 are a consequence of the convergence of the expectation $\mathbb{E} \mathbf{Q}_N(z)$, which is analyzed following the steps in [20, Section 4].

- (iii) Section 4 presents and studies the master equations that define the matrix function $\mathbf{T}_N(z)$ in the statement of Theorem 1. The main objective of this section will be to establish existence and unicity of the solution to the master equation, which will involve the study of several linear operators using the tools developed in [20, Section 5].
- (iv) Finally, Section 5 establishes that

$$\frac{1}{ML} \text{Tr}[(\mathbb{E}(\mathbf{Q}_N(z)) - \mathbf{T}_N(z)) \mathbf{A}_N] \rightarrow 0$$

showing that indeed $\mathbf{T}_N(z)$ is the asymptotic deterministic equivalent of $\mathbf{Q}_N(z)$. Taking $\mathbf{A}_N = \mathbf{I}_{ML}$ and using that $\frac{1}{ML} \text{Tr}[\mathbf{Q}_N(z) - \mathbb{E}(\mathbf{Q}_N(z))] \rightarrow 0$ almost surely, we obtain that $q_N(z) - \frac{1}{ML} \text{Tr}(\mathbf{T}_N(z)) \rightarrow 0$ almost surely. This, in turn, justifies (1.13) as well as (1.14).

Furthermore, assuming that $\frac{L^{3/2}}{MN} \rightarrow 0$ (which is equivalent to $\beta < \frac{4}{5}$),

$$\left| \frac{1}{ML} \text{Tr}[(\mathbb{E}(\mathbf{Q}_N(z)) - \mathbf{T}_N(z)) \mathbf{A}_N] \right| \leq C(z) \frac{L}{MN} \quad (1.19)$$

holds for each $z \in \mathbb{C} \setminus \mathbb{R}^+$ and N large enough, where $C(z) = P_1(|z|)P_2(\frac{1}{\delta_z})$ for some nice polynomials P_1 and P_2 as specified in Definition 2. We will be able to conclude from (1.19) that

$$\left| \mathbb{E} \left(\frac{1}{ML} \text{Tr}(\phi(\overline{\mathcal{R}}_{\text{corr},L}) \mathbf{A}_N) \right) - \int \phi(\lambda) d\mu_N(\lambda) \right| = \mathcal{O} \left(\frac{L}{MN} \right) \quad (1.20)$$

for each compactly supported smooth function ϕ .

In the last part of the paper, we will see that the deterministic sequence of probability measures $(\mu_N)_{N \geq 1}$ associated to the sequence of Stieltjes transforms $(\frac{1}{ML} \text{Tr} \mathbf{T}_N(z))_{N \geq 1}$ converges weakly towards the Marcenko-Pastur distribution μ_{mp, c_*} . For this, it will be crucial to rely on Assumption 4, which basically states that

$$\sup_m \sum_{n \in \mathbb{Z}} (1 + |n|)^{\gamma_0} |r_m(n)| < +\infty$$

for some $\gamma_0 > 0$.

In order to present the main result more formally, let us denote by $t_N(z)$ the Stieltjes transform of the Marcenko-Pastur law μ_{mp, c_N} associated to the parameter $c_N = \frac{ML}{N}$. In other words, for each $z \in \mathbb{C} \setminus \mathbb{R}^+$, $t_N(z)$ is the unique solution of the equation

$$t_N(z) = \frac{1}{-z + \frac{1}{1 + c_N t_N(z)}} \quad (1.21)$$

for which $\frac{\text{Im}(t_N(z))}{\text{Im}z} \geq 0$ if $z \in \mathbb{C} \setminus \mathbb{R}$ and $t_N(z) \geq 0$ if $z < 0$. If $\mathbf{T}_N(z)$ represents the deterministic equivalent of $\mathbf{Q}_N(z)$ solution of the equations (4.1, 4.2), the following theorem establishes that, for each $\gamma < \gamma_0$, $\gamma \neq 1$, the Stieltjes transform $\frac{1}{ML} \text{Tr} \mathbf{T}_N(z)$ is well approximated by $t_N(z)$, up to an error of order $\mathcal{O}(L^{-\min(1,\gamma)})$.

Theorem 2. *There exist two nice polynomials P_1 and P_2 as given in Definition 2, such that for each $\gamma < \gamma_0$, $\gamma \neq 1$, the inequality*

$$\|\mathbf{T}_N(z) - t_N(z)\mathbf{I}_{ML}\| \leq \frac{1}{L^{\min(\gamma,1)}} P_1(|z|) P_2\left(\frac{1}{\delta_z}\right) \quad (1.22)$$

holds for each $z \in \mathbb{C} \setminus \mathbb{R}^+$. Moreover, for each compactly supported smooth function ϕ , we have

$$\int_{\mathbb{R}^+} \phi(\lambda) d\mu_N(\lambda) - \int_{\mathbb{R}^+} \phi(\lambda) d\mu_{mp,c_N}(\lambda) \leq \frac{\kappa}{L^{\min(\gamma,1)}} \quad (1.23)$$

for some nice constant κ .

The proof of the above result is given in Section 6. The key point in order to establish (1.23) will be a result that is established in Lemma 6.1, which basically states that

$$\begin{aligned} \sup_m \sup_{\nu \in [0,1]} \left| \mathcal{S}_m(\nu) \mathbf{a}_L(\nu)^H \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu) - 1 \right| &= \mathcal{O}\left(\frac{1}{L^{\min(1,\gamma)}}\right), \gamma < \gamma_0, \gamma \neq 1 \\ &= \frac{\log L}{L}, \gamma = 1 < \gamma_0 \end{aligned}$$

where $\mathbf{a}_L(\nu)$ represents the L -dimensional vector

$$\mathbf{a}_L(\nu) = \frac{1}{\sqrt{L}} \left(1, e^{2i\pi\nu}, \dots, e^{2i\pi(L-1)\nu}\right)^T. \quad (1.24)$$

This result, proved in Appendix B, is obtained by remarking that $\mathbf{a}_L(\nu)^H \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu)$ can be expressed in terms of the orthogonal Szegő polynomials associated to the measure $\mathcal{S}_m(\nu) d\nu$, and by adapting to our context certain asymptotic related results presented in [24, Chapter 5].

The above results allow to conclude that the variance of $\frac{1}{ML} \text{Tr}(\phi(\overline{\mathcal{R}}_{\text{corr},L})) - \int \phi(\lambda) d\mu_{mp,c_N}(\lambda)$ is a $\mathcal{O}(\frac{1}{MN})$ term while its bias is $\mathcal{O}(\frac{L}{MN}) + \mathcal{O}(\frac{1}{L^{\min(1,\gamma)}})$ if $\beta < \frac{4}{5}$ for each $\gamma < \gamma_0$, $\gamma \neq 1$. This analysis should of course be extended in order to have a clear understanding of the asymptotic behaviour of the probability distribution of $\frac{1}{ML} \text{Tr}(\phi(\widehat{\mathcal{R}}_{\text{corr},L})) - \int \phi(\lambda) d\mu_{mp,c_N}(\lambda)$. This important question is however outside the scope of the present paper.

1.5 Notations

We will write the normalized observations as $\mathbf{w}_{n,N} = \frac{1}{\sqrt{N}} \mathbf{y}_L(n)$, where $n = 1, \dots, N$ and

$$\mathbf{W}_N = [\mathbf{w}_{1,N}, \dots, \mathbf{w}_{N,N}]. \quad (1.25)$$

Therefore $\widehat{\mathcal{R}}_L$ coincides with $\widehat{\mathcal{R}}_L = \mathbf{W}_N \mathbf{W}_N^H$. We recall here that we denote by \mathcal{B}_L the block diagonal matrix $\text{Bdiag} \mathcal{R}_L = \mathcal{R}_L$. Now, the matrix $\overline{\mathcal{R}}_{\text{corr},L}$ under study is given by

$$\overline{\mathcal{R}}_{\text{corr},L} = \mathcal{B}_L^{-1/2} \mathbf{W}_N \mathbf{W}_N^H \mathcal{B}_L^{-1/2}.$$

In the following, we will often drop the index N , and will denote $\mathbf{W}_N, \mathbf{w}_{j,N}, \mathbf{Q}_N, \dots$ by $\mathbf{W}, \mathbf{w}_j, \mathbf{Q}, \dots$ in order to simplify the notations. For $1 \leq l \leq L$, $1 \leq m \leq M$, and $1 \leq j \leq N$, $\mathbf{W}_{i,j}^m$ represents the entry $(i + (m-1)L, j)$ of matrix \mathbf{W} .

In general terms, if \mathbf{A} is a $ML \times ML$ matrix, we denote by $\mathbf{A}_{i_1, i_2}^{m_1, m_2}$ the entry $(i_1 + (m_1-1)L, i_2 + (m_2-1)L)$ of matrix \mathbf{A} , while \mathbf{A}^{m_1, m_2} represents the $L \times L$ matrix $(\mathbf{A}_{i_1, i_2}^{m_1, m_2})_{1 \leq (i_1, i_2) \leq L}$. For each $j = 1, \dots, N$, \mathbf{e}_j

represents the j^{th} vector of the canonical basis of \mathbb{C}^N and for $i = 1, \dots, L$, $m = 1, \dots, M$, \mathbf{f}_i^m is the $(i + (m - 1)L)^{\text{th}}$ vector of the canonical basis of \mathbb{C}^{ML} .

The set \mathbb{C}^+ is composed of the complex numbers with strictly positive imaginary parts. The conjugate of a complex number z is denoted z^* . If $z \in \mathbb{C} \setminus \mathbb{R}^+$, we denote by δ_z the term

$$\delta_z = \text{dist}(z, \mathbb{R}^+). \quad (1.26)$$

The conjugate transpose of a matrix \mathbf{A} is denoted \mathbf{A}^H while the conjugate of \mathbf{A} (i.e. the matrix whose entries are the conjugates of the entries of \mathbf{A}) is denoted \mathbf{A}^* . $\|\mathbf{A}\|$ and $\|\mathbf{A}\|_F$ represents the spectral norm and the Frobenius norm of matrix \mathbf{A} respectively. If \mathbf{A} and \mathbf{B} are 2 matrices, $\mathbf{A} \otimes \mathbf{B}$ represents the Kronecker product of \mathbf{A} and \mathbf{B} , i.e. the block matrix whose block (i, j) is $\mathbf{A}_{i,j} \mathbf{B}$. If \mathbf{A} is a square matrix, $\text{Im}(\mathbf{A})$ and $\text{Re}(\mathbf{A})$ represent the Hermitian matrices

$$\text{Im}(\mathbf{A}) = \frac{\mathbf{A} - \mathbf{A}^H}{2i}, \quad \text{Re}(\mathbf{A}) = \frac{\mathbf{A} + \mathbf{A}^H}{2}.$$

If $(\mathbf{A}_N)_{N \geq 1}$ (resp. $(\mathbf{b}_N)_{N \geq 1}$) is a sequence of matrices (resp. vectors) whose dimensions increase with N , $(\mathbf{A}_N)_{N \geq 1}$ (resp. $(\mathbf{b}_N)_{N \geq 1}$) is said to be uniformly bounded if $\sup_{N \geq 1} \|\mathbf{A}_N\| < +\infty$ (resp. $\sup_{N \geq 1} \|\mathbf{b}_N\| < +\infty$).

If $\nu \in [0, 1]$ and if R is an integer, we denote by $\mathbf{d}_R(\nu)$ the R -dimensional vector

$$\mathbf{d}_R(\nu) = \left(1, e^{2i\pi\nu}, \dots, e^{2i\pi(R-1)\nu}\right)^T \quad (1.27)$$

and by $\mathbf{a}_L(\nu)$ the normalized vector $\mathbf{a}_L(\nu) = \frac{1}{\sqrt{R}} \mathbf{d}_R(\nu)$, cf. (1.24).

If x is a complex-valued random variable, its expectation is denoted by $\mathbb{E}(x)$ and its variance as

$$\text{Var}(x) = \mathbb{E}(|x|^2) - |\mathbb{E}(x)|^2.$$

The zero-mean random variable $x - \mathbb{E}(x)$ is denoted x° .

1.6 Background on Stieltjes transforms of positive matrix valued measures

We recall that if K is a positive integer, then a $K \times K$ matrix-valued positive measure $\boldsymbol{\mu}$ is a σ -additive function from the Borel sets of \mathbb{R} onto the set of all positive $K \times K$ matrices (see e.g. [23], Chapter 1 for more details). We denote by $\mathcal{S}_K(\mathbb{R}^+)$ the set of all Stieltjes transforms of $K \times K$ positive matrix-valued measures $\boldsymbol{\mu}$ carried by \mathbb{R}^+ verifying $\boldsymbol{\mu}_K(\mathbb{R}^+) = \mathbf{I}_K$. The elements of the class $\mathcal{S}_K(\mathbb{R}^+)$ satisfy the following properties:

Proposition 1.1. *Consider an element $\mathbf{S}(z) = \int_{\mathbb{R}^+} \frac{d\boldsymbol{\mu}(\lambda)}{\lambda - z}$ of $\mathcal{S}_K(\mathbb{R}^+)$. Then, the following properties hold true:*

(i) \mathbf{S} is analytic on $\mathbb{C} \setminus \mathbb{R}^+$

(ii) $\text{Im}(\mathbf{S}(z)) \geq 0$ and $\text{Im}(z\mathbf{S}(z)) \geq 0$ if $z \in \mathbb{C}^+$

(iii) $\lim_{y \rightarrow +\infty} -iy\mathbf{S}(iy) = \mathbf{I}_K$

(iv) $\mathbf{S}(z)\mathbf{S}^H(z) \leq \frac{\mathbf{I}_K}{\delta_z^2}$ for each $z \in \mathbb{C} \setminus \mathbb{R}^+$

(v) $\frac{\text{Im}\mathbf{S}(z)}{\text{Im}z} \leq \frac{\mathbf{I}_K}{\delta_z^2}$ for each $z \in \mathbb{C} \setminus \mathbb{R}^+$ where $\frac{\text{Im}\mathbf{S}(z)}{\text{Im}z}$ should be interpreted as the derivative $\mathbf{S}'(z)$ of $\mathbf{S}(z)$ w.r.t. z when $z < 0$

$$(vi) \int_{\mathbb{R}^+} \lambda d\boldsymbol{\mu}(\lambda) = \lim_{y \rightarrow +\infty} \operatorname{Re}(-iy(\mathbf{I}_K + iy\mathbf{S}(iy)))$$

Conversely, if a function $\mathbf{S}(z)$ satisfy properties (i), (ii), (iii), then $\mathbf{S}(z) \in \mathcal{S}_K(\mathbb{R}^+)$.

While we have not been able to find a paper in which this result is proved, it has been well known for a long time (see however [12] for more details on (i), (ii), (iii), (vi)), as well as Theorem 3 of [1] from which (iv) follows immediately). We however provide an elementary proof of (iv) because it is based on a version of the matrix Schwartz inequality that will be used later. We denote by $\mathbb{L}^2(\boldsymbol{\mu})$ the Hilbert space of all row vector-valued functions $\mathbf{u}(\lambda)$ defined on \mathbb{R}^+ satisfying $\int_{\mathbb{R}^+} \mathbf{u}(\lambda) d\boldsymbol{\mu}(\lambda) \mathbf{u}^H(\lambda) < +\infty$ endowed with the scalar product $\langle \mathbf{u}, \mathbf{v} \rangle$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\mathbb{R}^+} \mathbf{u}(\lambda) d\boldsymbol{\mu}(\lambda) \mathbf{v}^H(\lambda).$$

Then, if $\mathbf{U}(\lambda) = (\mathbf{u}_1(\lambda)^T, \dots, \mathbf{u}_K(\lambda)^T)^T$ and $\mathbf{V}(\lambda) = (\mathbf{v}_1(\lambda)^T, \dots, \mathbf{v}_K(\lambda)^T)^T$ are matrices whose rows are elements of $\mathbb{L}^2(\boldsymbol{\mu})$, it holds that

$$[\mathbf{U}, \mathbf{V}] ([\mathbf{V}, \mathbf{V}])^{-1} [\mathbf{U}, \mathbf{V}]^H \leq [\mathbf{U}, \mathbf{U}] \quad (1.28)$$

where $[\mathbf{U}, \mathbf{V}]$ is the matrix defined by $([\mathbf{U}, \mathbf{V}])_{i,j} = \langle \mathbf{u}_i, \mathbf{v}_j \rangle$. Using (1.28) for $\mathbf{U}(\lambda) = \frac{\mathbf{I}}{\lambda - z}$ and $\mathbf{V} = \mathbf{I}$, and remarking that $|\lambda - z|^2 \geq \delta_z^2$ for each $\lambda \in \mathbb{R}^+$, we immediately obtain (iv).

1.7 Toeplitzification operators

In the following derivations, it will be useful to consider the following Toeplitzification operators, introduced in [20], which inherently depend on the covariance sequences $(r_m)_{m \geq 1}$. Let \mathbf{J}_K denote the $K \times K$ shift matrix with ones in the first upper diagonal and zeros elsewhere, namely $\{\mathbf{J}_K\}_{i,j} = \delta_{j-i=1}$, and let \mathbf{J}_K^{-1} denote its transpose. For a given squared matrix \mathbf{M} with dimensions $R \times R$, we define $\Psi_K^{(m)}(\mathbf{M})$ as an $K \times K$ Toeplitz matrix with (i, j) th entry equal to

$$\left\{ \Psi_K^{(m)}(\mathbf{M}) \right\}_{i,j} = \sum_{l=-R+1}^{R-1} r_m(i-j-l) \tau(\mathbf{M})(l) \quad (1.29)$$

or, alternatively, as the matrix

$$\Psi_K^{(m)}(\mathbf{M}) = \sum_{n=-K+1}^{K-1} \left(\sum_{l=-R+1}^{R-1} r_m(n-l) \tau(\mathbf{M})(l) \right) \mathbf{J}_K^{-n} \quad (1.30)$$

where the sequence $\tau(\mathbf{M})(l)$, $-R < l < R$, is defined as

$$\tau(\mathbf{M})(l) = \frac{1}{R} \operatorname{Tr} \left[\mathbf{M} \mathbf{J}_R^l \right]. \quad (1.31)$$

We can express this operator more compactly using frequency notation, namely

$$\begin{aligned} \Psi_K^{(m)}(\mathbf{M}) &= \sum_{n=-K+1}^{K-1} \left(\int_0^1 \mathcal{S}_m(\nu) \mathbf{a}_R^H(\nu) \mathbf{M} \mathbf{a}_R(\nu) e^{2\pi i \nu n} d\nu \right) \mathbf{J}_K^{-n} \\ &= \int_0^1 \mathcal{S}_m(\nu) \mathbf{a}_R^H(\nu) \mathbf{M} \mathbf{a}_R(\nu) \mathbf{d}_K(\nu) \mathbf{d}_K^H(\nu) d\nu \end{aligned}$$

where we recall that $\mathbf{a}_R(\nu) = \mathbf{d}_R(\nu) / \sqrt{R}$ and $\mathbf{d}_R(\nu)$ as defined in (1.27). The following properties are easily checked (see [20]).

- Given a square matrix \mathbf{A} of dimension $K \times K$ and a square matrix \mathbf{B} of dimension $R \times R$, we can write

$$\frac{1}{K} \text{Tr} \left[\mathbf{A} \Psi_K^{(m)}(\mathbf{B}) \right] = \frac{1}{R} \text{Tr} \left[\Psi_R^{(m)}(\mathbf{A}) \mathbf{B} \right] \quad (1.32)$$

- Given a square matrix \mathbf{M} and a positive integer K , we have

$$\left\| \Psi_K^{(m)}(\mathbf{M}) \right\| \leq \sup_{\nu \in [0,1]} |\mathcal{S}_m(\nu)| \|\mathbf{M}\|.$$

- Given a square positive definite matrix \mathbf{M} and a positive integer K , and assuming that $\inf_{\nu} \mathcal{S}_m(\nu) > 0$, it holds that

$$\Psi_K^{(m)}(\mathbf{M}) > 0. \quad (1.33)$$

We define here two other linear operators that will be used throughout the paper, which respectively operate on $N \times N$ and $ML \times ML$ matrices. In order to keep the notation as simple as possible, we will drop the dimensions in the notation of these operators.

- Consider an $N \times N$ matrix \mathbf{M} . We define $\Psi(\mathbf{M})$ as an $ML \times ML$ block diagonal matrix with m th diagonal block given by $\Psi_L^{(m)}(\mathbf{M})$.
- Consider an $ML \times ML$ matrix \mathbf{M} , and let $\mathbf{M}^{m,m}$ denote its m th $L \times L$ diagonal block. We define $\bar{\Psi}(\mathbf{M})$ as the $N \times N$ matrix given by

$$\bar{\Psi}(\mathbf{M}) = \frac{1}{M} \sum_{m=1}^M \Psi_N^{(m)}(\mathbf{M}^{m,m}) = \int_0^1 \frac{1}{M} \sum_{m=1}^M \mathcal{S}_m(\nu) \mathbf{a}_L^H(\nu) \mathbf{M}^{m,m} \mathbf{a}_L(\nu) \mathbf{d}_N(\nu) \mathbf{d}_N^H(\nu) d\nu. \quad (1.34)$$

Observe, in particular, that $\bar{\Psi}(\mathbf{M})$ coincides with $\bar{\Psi}(\text{Bdiag}(\mathbf{M}))$.

Given these two new operators, and if \mathbf{A} and \mathbf{B} are $ML \times ML$ and $N \times N$ matrices, we see directly from (1.32) that

$$\frac{1}{N} \text{Tr} \left[\bar{\Psi}(\mathbf{A}) \mathbf{B} \right] = \frac{1}{ML} \text{Tr} \left[\mathbf{A} \Psi(\mathbf{B}) \right]. \quad (1.35)$$

We finally conclude this section by two useful propositions that follow directly from [20].

Proposition 1.2. *Let $\Gamma^m(z)$, $m = 1, \dots, M$, be a collection of $L \times L$ matrix-valued complex functions belonging to $\mathcal{S}_L(\mathbb{R}^+)$ and define $\mathbf{\Gamma}(z)$ as the $ML \times ML$ block diagonal matrix given by $\mathbf{\Gamma}(z) = \text{diag}(\mathbf{\Gamma}^1(z), \dots, \mathbf{\Gamma}^M(z))$. Then, for each $z \in \mathbb{C} \setminus \mathbb{R}^+$, the matrix $\mathbf{I}_N + c_N \bar{\Psi}^T \left(\mathcal{B}_L^{-1/2} \mathbf{\Gamma}(z) \mathcal{B}_L^{-1/2} \right)$ is invertible, so that we can define*

$$\tilde{\mathbf{\Upsilon}}(z) = -\frac{1}{z} \left(\mathbf{I}_N + c_N \bar{\Psi}^T \left(\mathcal{B}_L^{-1/2} \mathbf{\Gamma}(z) \mathcal{B}_L^{-1/2} \right) \right)^{-1}. \quad (1.36)$$

On the other hand, the matrix $\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi \left(\tilde{\mathbf{\Upsilon}}^T(z) \right) \mathcal{B}_L^{-1/2}$ is also invertible, and we define

$$\mathbf{\Upsilon}(z) = -\frac{1}{z} \left(\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi \left(\tilde{\mathbf{\Upsilon}}^T(z) \right) \mathcal{B}_L^{-1/2} \right)^{-1}. \quad (1.37)$$

Furthermore, $\tilde{\mathbf{\Upsilon}}(z)$ and $\mathbf{\Upsilon}(z)$ are elements of $\mathcal{S}_N(\mathbb{R}^+)$ and $\mathcal{S}_{ML}(\mathbb{R}^+)$ respectively. In particular, they are holomorphic on $\mathbb{C} \setminus \mathbb{R}^+$ and satisfy

$$\mathbf{\Upsilon}(z) \mathbf{\Upsilon}^H(z) \leq \frac{\mathbf{I}_{ML}}{\delta_z^2}, \quad \tilde{\mathbf{\Upsilon}}(z) \tilde{\mathbf{\Upsilon}}^H(z) \leq \frac{\mathbf{I}_N}{\delta_z^2}. \quad (1.38)$$

Moreover, there exist two nice constants η and $\tilde{\eta}$ such that

$$\mathbf{\Upsilon}(z)\mathbf{\Upsilon}^H(z) \geq \frac{\delta_z^2}{16(\eta^2 + |z|^2)^2} \mathbf{I}_{ML} \quad (1.39)$$

$$\tilde{\mathbf{\Upsilon}}(z)\tilde{\mathbf{\Upsilon}}^H(z) \geq \frac{\delta_z^2}{16(\tilde{\eta}^2 + |z|^2)^2} \mathbf{I}_N. \quad (1.40)$$

Proof. The proof is an easy adaptation of the proof of Lemma 4.1 in [20]. More precisely, if we replace in this Lemma matrix $\text{Bdiag}(\mathbb{E}\mathbf{Q}(z))$ by $\mathbf{\Gamma}(z)$ and matrices $(\mathbf{R}(z), \tilde{\mathbf{R}}(z))$ by $(\mathbf{\Upsilon}(z), \tilde{\mathbf{\Upsilon}}(z))$, it is easy to check that the arguments of the proof of Lemma 4.1 in [20] can be extended to the particular context considered in the present paper. \square

In order to state the next result, we consider two $ML \times ML$ block diagonal matrices \mathbf{S}, \mathbf{T} and two $N \times N$ matrices $\tilde{\mathbf{S}}, \tilde{\mathbf{T}}$. We also assume that $\mathbf{S}, \mathbf{T}, \tilde{\mathbf{S}}, \tilde{\mathbf{T}}$ are full rank matrices. For each fixed z , we define the linear operator Φ on the set of all $ML \times ML$ matrices by

$$\Phi(\mathbf{X}) = z^2 c_N \mathbf{S} \Psi \left(\tilde{\mathbf{S}}^T \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{T}}^T \right) \mathbf{T}. \quad (1.41)$$

Note that the operator Φ of course depends on $\mathbf{S}, \mathbf{T}, \tilde{\mathbf{S}}, \tilde{\mathbf{T}}, M, L, N$ and z . We also define the following linear operators on the set of all $ML \times ML$ Hermitian matrices:

$$\Phi_{\mathbf{T}^H}(\mathbf{X}) = |z|^2 c_N \mathbf{T}^H \Psi \left(\tilde{\mathbf{T}}^* \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{T}}^T \right) \mathbf{T} \quad (1.42)$$

$$\Phi_{\mathbf{S}}(\mathbf{X}) = |z|^2 c_N \mathbf{S} \Psi \left(\tilde{\mathbf{S}}^T \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{S}}^* \right) \mathbf{S}^H. \quad (1.43)$$

We remark that both operators are positive in the sense that if $\mathbf{X} \geq 0$, then $\Phi_{\mathbf{S}}(\mathbf{X}) \geq 0$ and $\Phi_{\mathbf{T}^H}(\mathbf{X}) \geq 0$. Let $\Phi^{(1)}(\mathbf{X}) = \Phi(\mathbf{X})$ and recursively define $\Phi^{(n+1)}(\mathbf{X}) = \Phi(\Phi^{(n)}(\mathbf{X}))$ for $n \geq 1$. Then, the following result holds.

Proposition 1.3. *For any two L -dimensional column vectors \mathbf{a}, \mathbf{b} and for each $m = 1, \dots, M$, the inequality*

$$\left| \mathbf{a}^H \left(\Phi^{(n)}(\mathbf{X}) \right)^{m,m} \mathbf{b} \right| \leq \left[\mathbf{a}^H \left(\Phi_{\mathbf{S}}^{(n)}(\mathbf{X}\mathbf{X}^H) \right)^{m,m} \mathbf{a} \right]^{1/2} \left[\mathbf{b}^H \left(\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I}_{ML}) \right)^{m,m} \mathbf{b} \right]^{1/2} \quad (1.44)$$

holds. Moreover, if there exist two $ML \times ML$ positive definite matrices \mathbf{Y}_1 and \mathbf{Y}_2 such that

$$\lim_{n \rightarrow +\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{Y}_1) \rightarrow 0 \quad (1.45)$$

$$\lim_{n \rightarrow +\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{Y}_2) \rightarrow 0 \quad (1.46)$$

then, for each $ML \times ML$ matrix \mathbf{X} ,

$$\lim_{n \rightarrow +\infty} \Phi^{(n)}(\mathbf{X}) \rightarrow 0 \quad (1.47)$$

If, moreover, $\sum_{n=0}^{+\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{Y}_1) < +\infty$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{Y}_2) < +\infty$, then, for each $ML \times ML$ hermitian matrix \mathbf{Y} , the two series $\sum_{n=0}^{+\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{Y})$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{Y})$ are convergent. Finally, for each $ML \times ML$ matrix \mathbf{X} , $\sum_{n=0}^{+\infty} \Phi^{(n)}(\mathbf{X})$ is also convergent, and we have

$$\left\| \sum_{n=0}^{+\infty} \Phi^{(n)}(\mathbf{X}) \right\| \leq \left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{X}\mathbf{X}^H) \right\|^{1/2} \left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I}_{ML}) \right\|^{1/2} \quad (1.48)$$

as well as

$$\left\| \sum_{n=0}^{+\infty} \Phi^{(n)}(\mathbf{X}) \right\| \leq \|\mathbf{X}\| \left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{I}_{ML}) \right\|^{1/2} \left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I}_{ML}) \right\|^{1/2}. \quad (1.49)$$

Proof. Inequality (1.44) is established in Section 5 of [20]. We now prove (1.47). For this, we first remark that since matrices $(\mathbf{Y}_i)_{i=1,2}$ are positive definite, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\mathbf{Y}_i \geq \alpha_i \mathbf{I}_{ML}$ for $i = 1, 2$. As the operators $\Phi_{\mathbf{S}}$ and $\Phi_{\mathbf{T}^H}$ are positive, it holds that $\Phi_{\mathbf{S}}^{(n)}(\mathbf{Y}_1) \geq \alpha_1 \Phi_{\mathbf{S}}^{(n)}(\mathbf{I}_{ML})$ and $\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{Y}_2) > \alpha_2 \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I}_{ML})$ for each n . Therefore, conditions (1.45) and (1.46) imply that $\Phi_{\mathbf{S}}^{(n)}(\mathbf{I}_{ML}) \rightarrow 0$ and $\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I}_{ML}) \rightarrow 0$. If \mathbf{X} is a generic $ML \times ML$ matrix, the inequality $\mathbf{X}\mathbf{X}^H \leq \|\mathbf{X}\|^2 \mathbf{I}_{ML}$ implies that $\Phi_{\mathbf{S}}^{(n)}(\mathbf{X}\mathbf{X}^H) \leq \|\mathbf{X}\|^2 \Phi_{\mathbf{S}}^{(n)}(\mathbf{I}_{ML})$. Therefore, we deduce that for each matrix \mathbf{X} , then $\Phi_{\mathbf{S}}^{(n)}(\mathbf{X}\mathbf{X}^H) \rightarrow 0$ when $n \rightarrow +\infty$. The inequality in (1.44) thus leads to (1.47). Using similar arguments, we check that the convergence $\sum_{n=0}^{+\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{Y}_1)$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{Y}_2)$ implies the convergence of $\sum_{n=0}^{+\infty} \Phi_{\mathbf{S}}^{(n)}(\mathbf{Y})$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}^H}^{(n)}(\mathbf{Y})$ for each positive matrix \mathbf{Y} . If \mathbf{Y} is not positive, it is sufficient to remark that \mathbf{Y} can be written as the difference of 2 positive matrices to conclude to the convergence of the above two series. We finally consider a general matrix \mathbf{X} , and establish that $\sum_{n=0}^{+\infty} \Phi^{(n)}(\mathbf{X})$ is convergent. For this, we remark that (1.44) implies that for each m and each k , the inequality

$$\sum_{n=0}^k \left| \mathbf{a}^H \left(\Phi^{(n)}(\mathbf{X}) \right)^{m,m} \mathbf{b} \right| \leq \left[\mathbf{a}^H \left(\sum_{n=0}^k \left(\Phi_{\mathbf{S}}^{(n)}(\mathbf{X}\mathbf{X}^H) \right)^{m,m} \right) \mathbf{a} \right]^{1/2} \left[\mathbf{b}^H \left(\sum_{n=0}^k \left(\Phi_{\mathbf{T}^H}^{(n)}(\mathbf{I}_{ML}) \right)^{m,m} \right) \mathbf{b} \right]^{1/2} \quad (1.50)$$

holds. This implies that

$$\sum_{n=0}^{+\infty} \left| \mathbf{a}^H \left(\Phi^{(n)}(\mathbf{X}) \right)^{m,m} \mathbf{b} \right| < +\infty$$

and that the series $\sum_{n=0}^{+\infty} \Phi^{(n)}(\mathbf{X})$ is convergent. The result in (1.48) is obtained by taking the limit in the inequality (1.50), while (1.49) is an immediate consequence of (1.48). \square

2 Simplification of the sample block autocorrelation matrix

Consider again the sample block correlation matrix, namely $\widehat{\mathcal{R}}_{\text{corr},L} = \widehat{\mathcal{B}}_L^{-1/2} \widehat{\mathcal{R}}_L \widehat{\mathcal{B}}_L^{-1/2}$, where we recall that $\widehat{\mathcal{B}}_L = \text{Bdiag}(\widehat{\mathcal{R}}_L)$. In this section, we will show that we can replace the block diagonal sample covariance matrix $\widehat{\mathcal{B}}_L$ by the true matrix $\mathcal{B}_L = \mathcal{R}_L$ without altering the asymptotic behavior of the empirical eigenvalue distribution of $\widehat{\mathcal{R}}_{\text{corr},L}$. More specifically, we establish that the two matrices $\widehat{\mathcal{R}}_{\text{corr},L}$ and $\overline{\mathcal{R}}_{\text{corr},L}$ have almost surely the same behavior in terms of spectral norm, where $\overline{\mathcal{R}}_{\text{corr},L} = \mathcal{B}_L^{-1/2} \widehat{\mathcal{R}}_L \mathcal{B}_L^{-1/2}$.

For this, we first prove that the spectral norm of $\widehat{\mathcal{R}}_L$ is almost surely bounded. Then, we show that $\|\widehat{\mathcal{B}}_L - \mathcal{B}_L\| \rightarrow 0$, from which we deduce that $\|\widehat{\mathcal{B}}_L^{-1/2} - \mathcal{B}_L^{-1/2}\| \rightarrow 0$. The almost sure boundedness of $\widehat{\mathcal{R}}_L$ will immediately imply, as expected, that

$$\left\| \widehat{\mathcal{B}}_L^{-1/2} \widehat{\mathcal{R}}_L \widehat{\mathcal{B}}_L^{-1/2} - \mathcal{B}_L^{-1/2} \widehat{\mathcal{R}}_L \mathcal{B}_L^{-1/2} \right\| \rightarrow 0 \quad (2.1)$$

almost surely. This will allow us to focus our analysis on $\overline{\mathcal{R}}_{\text{corr},L}$ for the rest of the paper.

2.1 Control of the largest eigenvalue of $\widehat{\mathcal{R}}_L$

The approach we follow is based on the observation that it is possible to add a bounded matrix to $\mathbf{W}_N \mathbf{W}_N^H$ to produce a block Toeplitz matrix. Controlling the largest eigenvalue of $\mathbf{W}_N \mathbf{W}_N^H$ becomes therefore equivalent to the control of the largest eigenvalue of the block Toeplitz matrix, a problem that can be solved by studying the supremum over the frequency interval of the spectral norm of the corresponding symbol. In order to present this, it is more convenient to reorganize the rows of matrix \mathbf{W}_N . For this, we define for each n the

M dimensional random vector \mathbf{y}_n defined by

$$\mathbf{y}_n = \begin{pmatrix} y_{1,n} \\ \vdots \\ y_{M,n} \end{pmatrix}. \quad (2.2)$$

$(\mathbf{y}_n)_{n \in \mathbb{Z}}$ is thus a M -dimensional stationary random sequence whose spectral density matrix $\mathbf{S}(\nu)$ coincides with the diagonal matrix $\mathbf{S}(\nu) = \text{Diag}(\mathcal{S}_1(\nu), \dots, \mathcal{S}_M(\nu))$. We next consider the $ML \times N$ matrix \mathcal{W}_N , which is defined as

$$\mathcal{W}_N = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{N-1} & \mathbf{y}_N \\ \mathbf{y}_2 & \mathbf{y}_3 & \cdots & \mathbf{y}_N & \mathbf{y}_{N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_L & \mathbf{y}_{L+1} & \cdots & \mathbf{y}_{N+L-2} & \mathbf{y}_{N+L-1} \end{pmatrix}. \quad (2.3)$$

Observe that \mathcal{W}_N can be obtained by simple permutation of the rows of \mathbf{W}_N and consequently $\mathcal{W}_N \mathcal{W}_N^H$ and $\mathbf{W}_N \mathbf{W}_N^H$ have the same eigenvalues. In particular, they have the same spectral norm. For this reason, we may focus on the behavior of \mathcal{W}_N from now on.

We define matrices $\mathcal{W}_{N,1}$ and $\mathcal{W}_{N,2}$ as the $ML \times (N - L + 1)$ and $ML \times (L - 1)$ matrices such that $\mathcal{W}_N = (\mathcal{W}_{N,1}, \mathcal{W}_{N,2})$. In particular, matrix $\mathcal{W}_{N,2}$ is given by

$$\mathcal{W}_{N,2} = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{y}_{N-L+2} & \mathbf{y}_{N-L+3} & \cdots & \mathbf{y}_{N-1} & \mathbf{y}_N \\ \mathbf{y}_{N-L+3} & \mathbf{y}_{N-L+4} & \cdots & \mathbf{y}_N & \mathbf{y}_{N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_{N+1} & \mathbf{y}_{N+2} & \cdots & \mathbf{y}_{N+L-2} & \mathbf{y}_{N+L-1} \end{pmatrix}. \quad (2.4)$$

We now express $\mathcal{W}_{N,2}$ as $\mathcal{W}_{N,2} = \mathcal{W}_{N,2,1} + \mathcal{W}_{N,2,2}$ where $\mathcal{W}_{N,2,1}$ is the upper block triangular matrix given by

$$\mathcal{W}_{N,2,1} = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{y}_{N-L+2} & \mathbf{y}_{N-L+3} & \cdots & \mathbf{y}_{N-1} & \mathbf{y}_N \\ \mathbf{y}_{N-L+3} & \mathbf{y}_{N-L+4} & \cdots & \mathbf{y}_N & \mathbf{0} \\ \mathbf{y}_{N-L+4} & \cdots & \mathbf{y}_N & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_N & \mathbf{0} & \vdots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (2.5)$$

and where $\mathcal{W}_{N,2,2}$ is the lower block triangular matrix defined by

$$\mathcal{W}_{N,2,2} = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{y}_{N+1} \\ \mathbf{0} & \vdots & \mathbf{0} & \mathbf{y}_{N+1} & \mathbf{y}_{N+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{y}_{N+1} & \cdots & \cdots & \mathbf{y}_{N+L-2} \\ \mathbf{y}_{N+1} & \mathbf{y}_{N+2} & \cdots & \mathbf{y}_{N+L-2} & \mathbf{y}_{N+L-1} \end{pmatrix}. \quad (2.6)$$

In other words, matrix $\mathcal{W}_{N,2,1}$ is obtained by replacing in $\mathcal{W}_{N,2}$ vectors $\mathbf{y}_{N+1}, \dots, \mathbf{y}_{N+L-1}$ by $\mathbf{0}, \dots, \mathbf{0}$ while $\mathcal{W}_{N,2,2}$ is obtained by replacing in $\mathcal{W}_{N,2}$ vectors $\mathbf{y}_{N-L+2}, \dots, \mathbf{y}_N$ by $\mathbf{0}, \dots, \mathbf{0}$. We also define $\mathcal{W}_{N,0}$ as the

$ML \times (L - 1)$ lower block triangular matrix given by

$$\mathcal{W}_{N,0} = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathbf{y}_1 \\ 0 & \vdots & 0 & \mathbf{y}_1 & \mathbf{y}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \mathbf{y}_1 & \dots & \dots & \mathbf{y}_{L-2} \\ \mathbf{y}_1 & \mathbf{y}_2 & \dots & \mathbf{y}_{L-2} & \mathbf{y}_{L-1} \end{pmatrix}. \quad (2.7)$$

We finally introduce the $ML \times (N + L - 1)$ block Toeplitz matrix $\widetilde{\mathcal{W}}_N$ defined by

$$\widetilde{\mathcal{W}}_N = (\mathcal{W}_{N,0}, \mathcal{W}_{N,1}, \mathcal{W}_{N,2,1}). \quad (2.8)$$

It is easy to check that $\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H$ is the block Toeplitz matrix whose $M \times M$ blocks $\left((\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H)_{k,l} \right)_{k,l=1,\dots,L}$ are given by

$$(\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H)_{k,l} = \widehat{\mathbf{R}}_{k-l}$$

where the $M \times M$ matrices $(\widehat{\mathbf{R}}_l)_{l=-(L-1),\dots,L-1}$ are defined by

$$\widehat{\mathbf{R}}_l = \frac{1}{N} \sum_{n=1}^{N-l} \mathbf{y}_{n+l} \mathbf{y}_n^H$$

for $l \geq 0$ and $\widehat{\mathbf{R}}_l = \widehat{\mathbf{R}}_l^H$ for $l \leq 0$. In other words, for each l , $\widehat{\mathbf{R}}_l$ is the standard empirical biased estimate of the autocovariance matrix at lag l of the multivariate time series $(\mathbf{y}_n)_{n \in \mathbb{Z}}$. Matrix $\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H$ also coincides with the block Toeplitz matrix associated to the symbol $\widehat{\mathbf{S}}(\nu)$ defined by

$$\widehat{\mathbf{S}}(\nu) = \sum_{l=-(L-1)}^{L-1} \widehat{\mathbf{R}}_l e^{-2i\pi l \nu} \quad (2.9)$$

so that we can write

$$\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H = \int_0^1 \mathbf{d}_L(\nu) \mathbf{d}_L^H(\nu) \otimes \widehat{\mathbf{S}}(\nu) d\nu. \quad (2.10)$$

The $M \times M$ matrix $\widehat{\mathbf{S}}(\nu)$ coincides with a lag window estimator of the spectral density of $(\mathbf{y}_n)_{n \in \mathbb{Z}}$. Evaluating the spectral norm of $\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H$ is easier than that of $\mathcal{W}_N \mathcal{W}_N^H$, because the spectral norm of $\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H$ is upper bounded by $\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}(\nu)\|$, a term that can be controlled using a discretization in the frequency domain and the epsilon net argument in \mathbb{C}^M (see e.g. [25] for an introduction to the concept of epsilon net). In the remainder of this section, we first prove that $\|\mathcal{W}_N \mathcal{W}_N^H - \widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H\|$ is bounded with high probability, and then establish that $\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}(\nu)\|$, and thus $\|\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H\|$ is also bounded with high probability.

We first state the following lemma, which will allow to reduce various suprema on the interval $[0, 1]$ to the corresponding suprema on a finite grid of the same interval. This result is adapted from Zygmund [29], and was used in [28].

Lemma 2.1. *Let $h(\nu) = \sum_{l=-(L-1)}^{L-1} h_l e^{-2i\pi l \nu}$ an order $L - 1$ real valued trigonometric polynomial. Then, for each $\nu_0 \in [0, 1]$, $\delta > 0$, $K \geq 2(1 + \delta)(L - 1)$, we define $\nu_k = \nu_0 + k/K$ for $k = 0, \dots, K$. Then, it holds that*

$$\max_{\nu \in [0,1]} |h(\nu)| \leq \left(1 + \frac{1}{\delta}\right) \max_{k=0,\dots,K} |h(\nu_k)|. \quad (2.11)$$

We now compare the spectral norms of $\mathcal{W}_N \mathcal{W}_N^H$ and $\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H$.

Proposition 2.1. *If α is a large enough constant, then, it holds that*

$$P\left(\|\mathcal{W}_N \mathcal{W}_N^H - \widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H\| > \alpha\right) \leq \kappa_1 L \exp(-\kappa_2 M \alpha) \quad (2.12)$$

for some nice constants κ_1 and κ_2 .

Proof. We drop all the subindexes N from all the matrices for clarity of exposition. Matrix $\mathcal{W}\mathcal{W}^H$ is equal to $\mathcal{W}\mathcal{W}^H = \mathcal{W}_1 \mathcal{W}_1^H + (\mathcal{W}_{2,1} + \mathcal{W}_{2,2})(\mathcal{W}_{2,1} + \mathcal{W}_{2,2})^H$ while $\widetilde{\mathcal{W}}\widetilde{\mathcal{W}}^H = \mathcal{W}_0 \mathcal{W}_0^H + \mathcal{W}_1 \mathcal{W}_1^H + \mathcal{W}_{2,1} \mathcal{W}_{2,1}^H$. Therefore,

$$\mathcal{W}\mathcal{W}^H - \widetilde{\mathcal{W}}\widetilde{\mathcal{W}}^H = \mathcal{W}_{2,2} \mathcal{W}_{2,2}^H + \mathcal{W}_{2,2} \mathcal{W}_{2,1}^H + \mathcal{W}_{2,1} \mathcal{W}_{2,2}^H - \mathcal{W}_0 \mathcal{W}_0^H.$$

In order to establish (2.12), we have to show that $P(\|\mathcal{W}_{2,i} \mathcal{W}_{2,i}^H\| > \alpha)$, $i = 1, 2$, and $P(\|\mathcal{W}_0 \mathcal{W}_0^H\| > \alpha)$ decrease at the same rate as the right hand side of (2.12). We just establish this property for matrix $\mathcal{W}_0 \mathcal{W}_0^H$, or equivalently for matrix $\widetilde{\mathcal{W}}_0 \widetilde{\mathcal{W}}_0^H$, where $\widetilde{\mathcal{W}}_0$ is defined as

$$\widetilde{\mathcal{W}}_0 = \frac{1}{\sqrt{N}} \begin{pmatrix} \mathbf{y}_1 & 0 & \dots & 0 & 0 \\ \mathbf{y}_2 & \mathbf{y}_1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ \mathbf{y}_{L-2} & \mathbf{y}_{L-3} & \ddots & \mathbf{y}_1 & 0 \\ \mathbf{y}_{L-1} & \mathbf{y}_{L-2} & \dots & \mathbf{y}_2 & \mathbf{y}_1 \end{pmatrix}.$$

It is easily seen that $\widetilde{\mathcal{W}}_0$ can be expressed as

$$\widetilde{\mathcal{W}}_0 = \sqrt{\frac{L}{N}} \int_0^1 \mathbf{d}_{L-1}(\nu) \mathbf{d}_{L-1}^H(\nu) \otimes \boldsymbol{\xi}_{L,y}(\nu) d\nu \quad (2.13)$$

where $\boldsymbol{\xi}_{L,y}(\nu)$ is an M -dimensional column vector defined as $\boldsymbol{\xi}_{L,y}(\nu) = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-2} \mathbf{y}_{l+1} e^{-2i\pi l\nu}$. The matrix version of the Schwartz inequality (1.28) leads immediately to

$$\widetilde{\mathcal{W}}_0 \widetilde{\mathcal{W}}_0^H \leq \frac{L}{N} \int_0^1 \mathbf{d}_{L-1}(\nu) \mathbf{d}_{L-1}^H(\nu) \otimes \boldsymbol{\xi}_{L,y}(\nu) \boldsymbol{\xi}_{L,y}^H(\nu) d\nu.$$

From this, we obtain immediately that

$$\|\widetilde{\mathcal{W}}_0 \widetilde{\mathcal{W}}_0^H\| \leq \sup_{\nu \in [0,1]} \frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2.$$

Next, observe that $\nu \rightarrow \frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2$ is a real valued trigonometric polynomial of order $L-2$. Therefore, if K, δ and the points $(\nu_k)_{k=0, \dots, K}$ are given as in Lemma 2.1, it holds that

$$P\left(\|\widetilde{\mathcal{W}}_0 \widetilde{\mathcal{W}}_0^H\| > \alpha\right) \leq P\left(\sup_{\nu \in [0,1]} \frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2 > \alpha\right) \leq P\left(\sup_{k=0, \dots, K} \frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu_k)\|^2 > \kappa \alpha\right)$$

for some nice constant κ . Using the union bound, we get that

$$P\left(\|\widetilde{\mathcal{W}}_0 \widetilde{\mathcal{W}}_0^H\| > \alpha\right) \leq \sum_{k=0}^K P\left(\frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu_k)\|^2 > \kappa \alpha\right).$$

Therefore, we just have to evaluate an upper bound of $P(\frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2 > \kappa \alpha)$ uniform w.r.t. the frequency ν , where $\frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2$ is given by

$$\frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2 = \frac{ML}{N} \frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2$$

where $\xi_{L,y_m}(\nu)$, $m = 1, \dots, M$ are components of $\boldsymbol{\xi}_{L,y}(\nu)$. These are mutually independent complex Gaussian random variables, so that we can use the Hanson-Wright inequality in order to control $P(\frac{L}{N} \|\boldsymbol{\xi}_{L,y}(\nu)\|^2 > \kappa \alpha)$:

Proposition 2.2. Consider $\mathbf{x} = (x_1, \dots, x_n)^T$ a random vector whose components are i.i.d. $\mathcal{N}_c(0, 1)$ random variables, and a $n \times n$ matrix \mathbf{A} . Then, there exist two nice constants κ_1 and κ_2 for which

$$P(|\mathbf{x}^H \mathbf{A} \mathbf{x} - \mathbb{E}(\mathbf{x}^H \mathbf{A} \mathbf{x})| > \beta) \leq \kappa_1 \exp\left(-\kappa_2 \min\left(\frac{\beta}{\|\mathbf{A}\|}, \frac{\beta^2}{\|\mathbf{A}\|_F^2}\right)\right) \quad (2.14)$$

holds.

In order to use (2.14), we remark that for each m , $\xi_{L,y_m}(\nu)$ can be written as $\xi_{L,y_m}(\nu) = (\mathbb{E}|\xi_{L,y_m}(\nu)|^2)^{1/2} x_m$ where x_1, \dots, x_M are $\mathcal{N}_c(0, 1)$ i.i.d. random variables. If $\mathbf{x} = (x_1, \dots, x_M)$, $\frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2$ can be written as

$$\frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2 = \mathbf{x}^H \mathbf{A} \mathbf{x}$$

where \mathbf{A} is the $M \times M$ diagonal matrix whose entries are $(\frac{1}{M} \mathbb{E}|\xi_{L,y_m}(\nu)|^2)_{m=1, \dots, M}$. In order to evaluate $\|\mathbf{A}\|$ and $\|\mathbf{A}\|_F^2$, we have to study the behaviour of $\mathbb{E}|\xi_{L,y_m}(\nu)|^2$, i.e. the expectation of the periodogram of the sequence $y_{m,1}, \dots, y_{m,L-1}$. The following result holds.

Lemma 2.2. $\mathbb{E}|\xi_{L,y_m}(\nu)|^2$ can be written as $\mathbb{E}|\xi_{L,y_m}(\nu)|^2 = \mathcal{S}_m(\nu) + \epsilon_{m,L}(\nu)$ where $\epsilon_{m,L}(\nu)$ verifies

$$|\epsilon_{m,L}(\nu)| \leq \frac{\kappa}{(L-1)^{\min(1, \gamma_0)}} \quad (2.15)$$

for each m and for some nice constant κ .

Lemma 2.2 is proved in the appendix.

This lemma implies that it exists a nice constant κ for which $\mathbb{E}|\xi_{L,y_m}(\nu)|^2 \leq \kappa$ for each ν and each m and $L > 1$. Therefore, if \mathbf{A} is the above mentioned diagonal matrix, \mathbf{A} verifies $\|\mathbf{A}\| \leq \frac{\kappa}{M}$ and $\|\mathbf{A}\|_F^2 \leq \frac{\kappa^2}{M}$. Consider a nice constant $\beta > 2\kappa$. Then,

$$\begin{aligned} P\left(\frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2 > \beta\right) &\leq P\left(\frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2 - \mathbb{E}|\xi_{L,y_m}(\nu)|^2 > \beta - \kappa\right) \\ &\leq P\left(\frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2 - \mathbb{E}|\xi_{L,y_m}(\nu)|^2 > \beta/2\right). \end{aligned}$$

As $\min\left(\frac{M(\beta/2)}{\kappa}, \frac{M(\beta/2)^2}{\kappa^2}\right) = \frac{M(\beta/2)}{\kappa}$ and $ML/N \rightarrow c_*$, the Hanson-Wright inequality leads to

$$P\left(\frac{ML}{N} \frac{1}{M} \sum_{m=1}^M |\xi_{L,y_m}(\nu)|^2 > \beta\right) \leq \kappa_1 \exp(-M\kappa_2 \beta)$$

for some nice constants κ_1 and κ_2 . As $K = \mathcal{O}(L)$, we have shown that if α is a large enough constant, there exist two nice constants κ_1 and κ_2 such that

$$P(\|\widetilde{\mathcal{W}}_0 \widetilde{\mathcal{W}}_0^H\| > \alpha) \leq L \kappa_1 \exp(-M\kappa_2).$$

Following the same approach to evaluate $P(\|\mathcal{W}_{N,i} \mathcal{W}_{N,j}^H\| > \alpha)$, we can conclude that (2.12) is established. \square

As a consequence of Proposition 2.1, the evaluation of $P(\|\mathcal{W}_N \mathcal{W}_N^H\| > \alpha)$ can be alternatively formulated in terms of the evaluation of $P(\|\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H\| > \alpha)$. For this, we use the expression in (2.10) and remark that $\|\widetilde{\mathcal{W}}_N \widetilde{\mathcal{W}}_N^H\| \leq \sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}(\nu)\|$. In the following, we thus control the spectral norm of $\widehat{\mathbf{S}}(\nu)$. In particular, we have the following result.

Proposition 2.3. *If α is a large enough constant, it holds that*

$$P\left(\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}(\nu)\| > \alpha\right) < \kappa_1 L \exp(-\kappa_2 M \alpha) \quad (2.16)$$

for some nice constants κ_1 and κ_2 .

Proof. We first notice that $\sup_{\nu} \|\mathbb{E}(\widehat{\mathbf{S}}(\nu))\|$ is finite. It is clear that $\mathbb{E}(\widehat{\mathbf{S}}(\nu)) = \sum_{l=-L}^{L-1} (1 - \frac{|l|}{L}) \mathbf{R}(l) e^{-2i\pi l \nu}$ where $\mathbf{R}(l) = \mathbb{E}(\mathbf{y}_{n+l} \mathbf{y}_n^H)$ is the autocovariance matrix of \mathbf{y}_n at lag l . Since the components of \mathbf{y}_n are independent time series, matrix $\mathbf{R}(l)$ coincides with $\mathbf{R}(l) = \text{Diag}((r_m(l))_{m=1,\dots,M})$. Therefore,

$$\|\mathbb{E}\widehat{\mathbf{S}}(\nu)\| \leq \sup_{m=1,\dots,M} \sum_{l=-L}^{L-1} |r_m(l)| \leq \sup_{m \geq 1} \sum_{l \in \mathbb{Z}} |r_m(l)|.$$

Condition (1.10) thus implies that $\sup_{\nu} \|\mathbb{E}(\widehat{\mathbf{S}}(\nu))\| < +\infty$. Therefore, in order to evaluate the left hand side of (2.16), it is sufficient to study $P(\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}(\nu) - \mathbb{E}\widehat{\mathbf{S}}(\nu)\| > \alpha)$. More precisely, for each ν , we have $\|\widehat{\mathbf{S}}(\nu) - \mathbb{E}\widehat{\mathbf{S}}(\nu)\| \geq \|\widehat{\mathbf{S}}(\nu)\| - \|\mathbb{E}\widehat{\mathbf{S}}(\nu)\| \geq \|\widehat{\mathbf{S}}(\nu)\| - \sup_{\nu} \|\mathbb{E}\widehat{\mathbf{S}}(\nu)\|$. Therefore, it holds that

$$\sup_{\nu} \|\widehat{\mathbf{S}}(\nu) - \mathbb{E}\widehat{\mathbf{S}}(\nu)\| \geq \sup_{\nu} \|\widehat{\mathbf{S}}(\nu)\| - \sup_{\nu} \|\mathbb{E}\widehat{\mathbf{S}}(\nu)\|.$$

If we choose $\alpha > 2 \sup_{\nu} \|\mathbb{E}\widehat{\mathbf{S}}(\nu)\|$ we have $\alpha - \sup_{\nu} \|\mathbb{E}\widehat{\mathbf{S}}(\nu)\| \geq \alpha/2$. Consequently, the set $\{\sup_{\nu} \|\widehat{\mathbf{S}}(\nu)\| > \alpha\}$ is included in the set $\{\sup_{\nu} \|\widehat{\mathbf{S}}(\nu) - \mathbb{E}\widehat{\mathbf{S}}(\nu)\| > \alpha/2\}$ and the left hand side of (2.16) is upper bounded by

$$P\left(\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}(\nu) - \mathbb{E}\widehat{\mathbf{S}}(\nu)\| > \alpha/2\right).$$

It therefore remains to establish that the above probability can be upper bounded by a term of the form $\kappa_1 L \exp(-\kappa_2 M \alpha)$.

We recall that we denote by $\widehat{\mathbf{S}}^\circ(\nu)$ the centered matrix $\widehat{\mathbf{S}}^\circ(\nu) = \widehat{\mathbf{S}}(\nu) - \mathbb{E}\widehat{\mathbf{S}}(\nu)$. We first show that the study of the supremum of $\|\widehat{\mathbf{S}}^\circ(\nu)\|$ over $[0, 1]$ can be reduced to the supremum over a discrete grid with $\mathcal{O}(L)$ elements. The idea is to make use Lemma 2.1 by conveniently expressing $\|\widehat{\mathbf{S}}^\circ(\nu)\|$ in terms of trigonometric polynomials.

Lemma 2.3. *We consider δ , K , and $(\nu_k)_{k=0,\dots,K}$ as in Lemma 2.1. Then, the following result holds:*

$$\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\| \leq \left(1 + \frac{1}{\delta}\right) \sup_{k=0,\dots,K} \|\widehat{\mathbf{S}}^\circ(\nu_k)\|. \quad (2.17)$$

We will first verify that

$$\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\| = \sup_{\nu \in [0,1], \mathbf{h} \in \mathbb{S}^{M-1}} |\mathbf{h}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}| \quad (2.18)$$

where \mathbb{S}^{M-1} is the unit sphere in \mathbb{C}^M . We remark that, because of the continuity of the spectral norm as well as the continuity of both true and estimated spectral densities, there exists a certain $\widehat{\nu}$ that achieves the supremum on the left hand side of (2.18), that is $\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\| = \|\widehat{\mathbf{S}}^\circ(\widehat{\nu})\|$. Moreover, for such given $\widehat{\nu}$, there exists a $\mathbf{h}_{\widehat{\nu}} \in \mathbb{S}^{M-1}$ for which $\|\widehat{\mathbf{S}}^\circ(\widehat{\nu})\| = |\mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\widehat{\nu}) \mathbf{h}_{\widehat{\nu}}|$. In other words, $\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\|$ coincides with $|\mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\widehat{\nu}) \mathbf{h}_{\widehat{\nu}}|$. Hence, we obtain that the left hand side of (2.18) is less than the right hand side of (2.18). The converse inequality is obvious.

Using a similar continuity argument, we can readily see that $\sup_{\nu \in [0,1], \mathbf{h} \in \mathbb{S}^{M-1}} |\mathbf{h}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}| = |\mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\widehat{\nu}) \mathbf{h}_{\widehat{\nu}}|$ also coincides with $\sup_{\nu \in [0,1]} |\mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}_{\widehat{\nu}}|$. The function $\nu \rightarrow \mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}_{\widehat{\nu}}$ is a real valued trigonometric

polynomial of order $L - 1$. Therefore, Lemma 2.1 implies that

$$\sup_{\nu \in [0,1]} \left| \mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}_{\widehat{\nu}} \right| \leq \left(1 + \frac{1}{\delta} \right) \sup_{k=0, \dots, K} \left| \mathbf{h}_{\widehat{\nu}_k}^H \widehat{\mathbf{S}}^\circ(\nu_k) \mathbf{h}_{\widehat{\nu}_k} \right|.$$

Since $|\mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\nu_k) \mathbf{h}_{\widehat{\nu}}| \leq \|\widehat{\mathbf{S}}^\circ(\nu_k)\|$, we have shown that

$$\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\| = \sup_{\nu \in [0,1]} \left| \mathbf{h}_{\widehat{\nu}}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}_{\widehat{\nu}} \right| \leq \left(1 + \frac{1}{\delta} \right) \sup_{k=0, \dots, K} \|\widehat{\mathbf{S}}^\circ(\nu_k)\|.$$

This establishes (2.17).

We now complete the proof of Proposition 2.16. The union bound leads to

$$P \left(\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\| > \alpha/2 \right) \leq \sum_{k=0}^K P \left(\|\widehat{\mathbf{S}}^\circ(\nu_k)\| > \frac{\delta}{1+\delta} \alpha/2 \right). \quad (2.19)$$

Thus, we only need to evaluate $P(\|\widehat{\mathbf{S}}^\circ(\nu)\| > \eta)$, where ν is a fixed frequency and where $\eta = \frac{\delta}{1+\delta} \alpha/2$. For this, we use the epsilon net argument in \mathbb{C}^M . We recall that an epsilon net \mathcal{N}_ϵ of \mathbb{C}^M is a finite set of unit norm vectors of \mathbb{C}^M having the property that for each $\mathbf{g} \in \mathbb{S}^{M-1}$, there exists an $\mathbf{h} \in \mathcal{N}_\epsilon$ such that $\|\mathbf{g} - \mathbf{h}\| \leq \epsilon$. It is well known that the cardinal $|\mathcal{N}_\epsilon|$ is upper bounded by $(\frac{\kappa}{\epsilon})^{2M}$ for some nice constant κ . We consider such an epsilon net \mathcal{N}_ϵ and denote by $\widehat{\mathbf{h}}$ a vector of \mathbb{S}^{M-1} for which $\|\widehat{\mathbf{S}}^\circ(\nu)\| = |\widehat{\mathbf{h}}^H \widehat{\mathbf{S}}^\circ(\nu) \widehat{\mathbf{h}}|$, and consider a vector $\widetilde{\mathbf{h}} \in \mathcal{N}_\epsilon$ such that $\|\widehat{\mathbf{h}} - \widetilde{\mathbf{h}}\| \leq \epsilon$. We express $\widetilde{\mathbf{h}}^H \widehat{\mathbf{S}}^\circ(\nu) \widetilde{\mathbf{h}}$ as

$$\widetilde{\mathbf{h}}^H \widehat{\mathbf{S}}^\circ(\nu) \widetilde{\mathbf{h}} = \left(\widehat{\mathbf{h}} + \widetilde{\mathbf{h}} - \widehat{\mathbf{h}} \right)^H \widehat{\mathbf{S}}^\circ(\nu) \left(\widehat{\mathbf{h}} + \widetilde{\mathbf{h}} - \widehat{\mathbf{h}} \right).$$

Using the triangular inequality, we obtain that

$$\left| \widetilde{\mathbf{h}}^H \widehat{\mathbf{S}}^\circ(\nu) \widetilde{\mathbf{h}} \right| \geq \left| \widehat{\mathbf{h}}^H \widehat{\mathbf{S}}^\circ(\nu) \widehat{\mathbf{h}} \right| - 2 \left| (\widetilde{\mathbf{h}} - \widehat{\mathbf{h}})^H \widehat{\mathbf{S}}^\circ(\nu) \widehat{\mathbf{h}} \right| - \left| (\widetilde{\mathbf{h}} - \widehat{\mathbf{h}})^H \widehat{\mathbf{S}}^\circ(\nu) (\widetilde{\mathbf{h}} - \widehat{\mathbf{h}}) \right|.$$

Since $\left| (\widetilde{\mathbf{h}} - \widehat{\mathbf{h}})^H \widehat{\mathbf{S}}^\circ(\nu) \widehat{\mathbf{h}} \right| \leq \|\widehat{\mathbf{S}}^\circ(\nu)(\widetilde{\mathbf{h}} - \widehat{\mathbf{h}})\| \leq \epsilon \|\widehat{\mathbf{S}}^\circ(\nu)\|$ and $\left| (\widetilde{\mathbf{h}} - \widehat{\mathbf{h}})^H \widehat{\mathbf{S}}^\circ(\nu) (\widetilde{\mathbf{h}} - \widehat{\mathbf{h}}) \right| \leq \epsilon^2 \|\widehat{\mathbf{S}}^\circ(\nu)\|$, we have

$$\left| \widetilde{\mathbf{h}}^H \widehat{\mathbf{S}}^\circ(\nu) \widetilde{\mathbf{h}} \right| \geq (1 - 2\epsilon - \epsilon^2) \|\widehat{\mathbf{S}}^\circ(\nu)\|.$$

In the following, we assume that ϵ satisfies $1 - 2\epsilon - \epsilon^2 > 0$. Therefore, using the union bound, we obtain that

$$P \left(\|\widehat{\mathbf{S}}^\circ(\nu)\| > \eta \right) \leq \sum_{\mathbf{h} \in \mathcal{N}_\epsilon} P \left(\left| \mathbf{h}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h} \right| \geq (1 - 2\epsilon - \epsilon^2) \eta \right). \quad (2.20)$$

In order to evaluate $P(|\mathbf{h}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}| \geq (1 - 2\epsilon - \epsilon^2) \eta)$ for each unit norm vector \mathbf{h} , we denote by z_n the scalar time series defined by $z_n = \mathbf{h}^H \mathbf{y}_n$. Then, the quadratic form $\mathbf{h}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}$ coincides with $\widehat{s}_z(\nu) - \mathbb{E} \widehat{s}_z(\nu)$ where $\widehat{s}_z(\nu)$ represents the lag-window estimator of the spectral density of z defined by $\widehat{s}_z(\nu) = \sum_{l=-(L-1)}^{L-1} \widehat{r}_z(l) e^{-2i\pi l \nu}$. Here, $\widehat{r}_z(l)$ is the standard empirical estimate of the autocovariance coefficient of z at lag l . We denote by \mathbf{z} the N -dimensional vector $\mathbf{z} = (z_1, \dots, z_N)^T$. As is well known, $\widehat{s}_z(\nu)$ can be expressed as

$$\widehat{s}_z(\nu) = \int_0^1 w(\nu - \mu) \frac{1}{N} \left| \sum_{n=0}^{N-1} z_{n+1} e^{-2i\pi n \mu} \right|^2 d\mu \quad (2.21)$$

where $w(\mu)$ is the Fourier transform of the rectangular window $\mathbb{I}_{l \in \{-(L-1), \dots, L-1\}}$. The expression in (2.21) can also be written as a quadratic form of vector \mathbf{z} :

$$\widehat{s}_z(\nu) = \mathbf{z}^H \left(\frac{1}{N} \int_0^1 w(\nu - \mu) \mathbf{d}_N(\mu) \mathbf{d}_N^H(\mu) d\mu \right) \mathbf{z} \quad (2.22)$$

If \mathbf{R}_z represents the covariance matrix of vector \mathbf{z} , \mathbf{z} can be written as $\mathbf{z} = \mathbf{R}_z^{1/2} \mathbf{x}$ for some $\mathcal{N}_c(0, \mathbf{I}_N)$ distributed random vector \mathbf{x} . Therefore, if we denote by $\mathbf{\Omega}$ the $N \times N$ matrix defined by

$$\mathbf{\Omega} = \mathbf{R}_z^{1/2} \frac{1}{N} \int_0^1 w(\nu - \mu) \mathbf{d}_N(\mu) \mathbf{d}_N^H(\mu) d\mu \mathbf{R}_z^{1/2},$$

the quantity $\widehat{s}_z(\nu) - \mathbb{E}\widehat{s}_z(\nu)$ can be written as $\widehat{s}_z(\nu) - \mathbb{E}\widehat{s}_z(\nu) = \mathbf{x}^H \mathbf{\Omega} \mathbf{x} - \mathbb{E} \mathbf{x}^H \mathbf{\Omega} \mathbf{x}$. Therefore,

$$P\left(|\widehat{s}_z(\nu) - \mathbb{E}\widehat{s}_z(\nu)| > (1 - 2\epsilon - \epsilon^2)\eta\right)$$

can be evaluated using the Hanson-Wright inequality (2.14). This requires the evaluation of the spectral and the Frobenius norm of $\mathbf{\Omega}$. Observe that we can express $\mathbf{\Omega} = \mathbf{R}_z^{1/2} \mathbf{\Omega}_w \mathbf{R}_z^{1/2}$ where $\mathbf{\Omega}_w$ is a Toeplitz matrix defined as

$$\mathbf{\Omega}_w = \frac{1}{N} \int_0^1 w(\nu - \mu) \mathbf{d}_N(\mu) \mathbf{d}_N^H(\mu) d\mu.$$

It is easy to check that the spectral norm of \mathbf{R}_z is uniformly bounded. Moreover, the spectral norm of $\mathbf{\Omega}_w$ is bounded by $\frac{1}{N} \sup_{\nu} |w(\nu)| = L/N$. Therefore, $\|\mathbf{\Omega}\| \leq \kappa \frac{L}{N}$ for some nice constant κ . In order to evaluate the Frobenius norm of $\mathbf{\Omega}$, observe that $\mathbf{\Omega}_w$ is band Toeplitz matrix with entries given by $(\mathbf{\Omega}_w)_{k,l} = \frac{1}{N} e^{2i\pi(k-l)\nu} \mathbb{1}_{|k-l| \leq L-1}$. Therefore, $\|\mathbf{\Omega}_w\|_F^2 \leq \kappa \frac{L}{N}$, which implies that $\|\mathbf{\Omega}\|_F^2 \leq \kappa \frac{L}{N}$. This in turn leads to the conclusion that if η is large enough, then

$$P\left(\left|\mathbf{h}^H \widehat{\mathbf{S}}^\circ(\nu) \mathbf{h}\right| \geq (1 - 2\epsilon - \epsilon^2)\eta\right) \leq \kappa_1 \exp(-\kappa_2 M \eta).$$

Recalling that $|\mathcal{N}_\epsilon| \leq \left(\frac{\kappa}{\epsilon}\right)^{2M}$, the union bound (2.20) implies that $P\left(\|\widehat{\mathbf{S}}^\circ(\nu)\| > \eta\right) \leq \kappa_1 \exp(-\kappa_2 M \alpha)$ for α large enough. Finally, (2.19) leads to

$$P\left(\sup_{\nu \in [0,1]} \|\widehat{\mathbf{S}}^\circ(\nu)\| > \alpha/2\right) \leq \kappa_1 L \exp(-\kappa_2 M \alpha)$$

for any constant α sufficiently large. This completes the proof of Proposition 2.3. \square

As a direct sequence of Propositions 2.1 and 2.3 we have the following corollary.

Corollary 2.1. *If $\alpha > 0$ is a large enough constant, then, it holds that*

$$P(\|\mathcal{W}_N \mathcal{W}_N^H\| > \alpha) \leq \kappa_1 L \exp(-\kappa_2 M \alpha) \quad (2.23)$$

for some nice constants κ_1 and κ_2 .

Note that Assumption 1 implies that $M = \mathcal{O}(N^{1-\beta})$. Thus, (2.23) and the Borel-Cantelli Lemma imply that $\|\mathcal{W}_N \mathcal{W}_N^H\| = \|\mathbf{W}_N \mathbf{W}_N^H\|$ is almost surely bounded by a constant for each N large enough.

2.2 Evaluation of the behaviour of $\|\text{Bdiag}(\widehat{\mathcal{R}}_L) - \text{Bdiag}(\mathcal{R}_L)\|$

Recall that $\mathcal{R}_{m,L}$, $m = 1, \dots, M$, denote the $L \times L$ diagonal blocks of the matrix $\text{Bdiag}(\mathcal{R}_L)$. We will denote by $\widehat{\mathcal{R}}_{m,L}$ the m th $L \times L$ diagonal block of $\widehat{\mathcal{R}}_L$. In this section, we establish that

$$\lim_{N \rightarrow +\infty} \sup_{m=1, \dots, M} \|\widehat{\mathcal{R}}_{m,L} - \mathcal{R}_{m,L}\| = 0 \quad (2.24)$$

almost surely. Note first that we can express $\widehat{\mathcal{R}}_{m,L}$ as the empirical estimate of $\mathcal{R}_{m,L}$, that is

$$\widehat{\mathcal{R}}_{m,L} = \frac{1}{N} \sum_{n=1}^N \mathbf{y}_{m,n}^L (\mathbf{y}_{m,n}^L)^H$$

or equivalently by $\widehat{\mathcal{R}}_{m,L} = \mathbf{W}_N^m (\mathbf{W}_N^m)^H$ where \mathbf{W}_N^m is the $L \times N$ matrix defined by

$$\mathbf{W}_N^m = \frac{1}{\sqrt{N}} (\mathbf{y}_{m,1}^L, \dots, \mathbf{y}_{m,N}^L).$$

The arguments used in this section are based on the techniques used in Section 2.1. Therefore, we just provide a sketch of proof of (2.24). We first observe that $\mathbf{W}_N^m (\mathbf{W}_N^m)^H$ has the same behaviour as the Toeplitz matrix $\widetilde{\mathbf{W}}_N^m (\widetilde{\mathbf{W}}_N^m)^H$ where $\widetilde{\mathbf{W}}_N^m$ is obtained by replacing vectors $(\mathbf{y}_n)_{n=1,\dots,N}$ by the scalars $(y_{m,n})_{n=1,\dots,N}$ in the definition of matrix \mathcal{W}_N in (2.8) above. In particular, it holds that

$$\widetilde{\mathbf{W}}_N^m (\widetilde{\mathbf{W}}_N^m)^H = \int_0^1 \widehat{\mathcal{S}}_m(\nu) \mathbf{d}_L(\nu) \mathbf{d}_L^H(\nu) d\nu$$

where $\widehat{\mathcal{S}}_m(\nu)$ represents the m th diagonal entry of the lag window estimator (2.9). More specifically, following the proof of Proposition 2.1, we justify that for each $0 < \epsilon < 1$,

$$P \left(\sup_{m=1,\dots,M} \left\| \mathbf{W}_N^m (\mathbf{W}_N^m)^H - \widetilde{\mathbf{W}}_N^m (\widetilde{\mathbf{W}}_N^m)^H \right\| > \frac{1}{M^{(1-\epsilon)}} \right) \leq \kappa_1 N \exp(-\kappa_2 M^\epsilon) \quad (2.25)$$

holds for some nice constants κ_1 and κ_2 . To verify (2.25), we remark that

$$\mathbf{W}^m (\mathbf{W}^m)^H - \widetilde{\mathbf{W}}^m (\widetilde{\mathbf{W}}^m)^H = \mathbf{W}_{2,2}^m (\mathbf{W}_{2,2}^m)^H + \mathbf{W}_{2,2}^m (\mathbf{W}_{2,1}^m)^H + \mathbf{W}_{2,1}^m (\mathbf{W}_{2,2}^m)^H - \mathbf{W}_0^m (\mathbf{W}_0^m)^H$$

where the various matrices of the right handside are obtained by replacing vectors $(\mathbf{y}_n)_{n=1,\dots,N+L-1}$ in the definition of matrices $\mathcal{W}_{2,2}$, $\mathcal{W}_{2,1}$, \mathcal{W}_0 used in Section 2.1 by the scalars $(y_{m,n})_{n=1,\dots,N+L-1}$. In order to verify (2.25), we just briefly check that

$$P \left(\sup_{m=1,\dots,M} \left\| \mathbf{W}_0^m (\mathbf{W}_0^m)^H \right\| > \frac{1}{M^{(1-\epsilon)}} \right) \leq \kappa_1 N \exp(-\kappa_2 M^\epsilon)$$

or equivalently that

$$P \left(\sup_{m=1,\dots,M} \left\| \widetilde{\mathbf{W}}_0^m (\widetilde{\mathbf{W}}_0^m)^H \right\| > \frac{1}{M^{(1-\epsilon)}} \right) \leq \kappa_1 N \exp(-\kappa_2 M^\epsilon)$$

where $\widetilde{\mathbf{W}}_0^m$ is defined by

$$\widetilde{\mathbf{W}}_0^m = \sqrt{\frac{L}{N}} \int_0^1 \mathbf{d}_{L-1}(\nu) \mathbf{d}_{L-1}^H(\nu) \xi_{L,y_m}(\nu) d\nu.$$

As in Section 2.1, we notice that the matrix-valued Cauchy-Schwarz inequality implies that

$$\widetilde{\mathbf{W}}_0^m (\widetilde{\mathbf{W}}_0^m)^H \leq \frac{L}{N} \int_0^1 \mathbf{d}_{L-1}(\nu) \mathbf{d}_{L-1}^H(\nu) |\xi_{L,y_m}(\nu)|^2 d\nu.$$

Using Lemma 2.1, we get that for each m ,

$$P \left(\left\| \widetilde{\mathbf{W}}_0^m (\widetilde{\mathbf{W}}_0^m)^H \right\| > \frac{1}{M^{(1-\epsilon)}} \right) \leq K \sup_{\nu} P \left(\frac{L}{N} |\xi_{L,y_m}(\nu)|^2 > \frac{\kappa}{M^{(1-\epsilon)}} \right)$$

where K is defined as in Section 2.1 and where κ is a nice constant. Noting that $\frac{L}{N} = \mathcal{O}(\frac{1}{M})$ and using the union bound, we obtain

$$P \left(\sup_{m=1,\dots,M} \left\| \widetilde{\mathbf{W}}_0^m (\widetilde{\mathbf{W}}_0^m)^H \right\| > \frac{1}{M^{(1-\epsilon)}} \right) \leq \kappa_1 N \sup_{m=1,\dots,M} \sup_{\nu} P (|\xi_{L,y_m}(\nu)|^2 > \kappa M^\epsilon)$$

for some nice constant κ . Following Section 2.1, it is easy to check that

$$P (|\xi_{L,y_m}(\nu)|^2 > M^\epsilon) \leq \kappa_1 \exp(-\kappa_2 M^\epsilon)$$

and that

$$P \left(\sup_{m=1, \dots, M} \left\| \widetilde{\mathbf{W}}_0^m \left(\widetilde{\mathbf{W}}_0^m \right)^H \right\| > \frac{1}{M^{(1-\epsilon)}} \right) \leq \kappa_1 N \exp(-\kappa_2 M^\epsilon).$$

We conclude from this that, for N large enough, the inequality

$$\left\| \widetilde{\mathbf{W}}_{N,0}^m \left(\widetilde{\mathbf{W}}_{N,0}^m \right)^H \right\| \leq \frac{1}{M^{(1-\epsilon)}} \quad (2.26)$$

holds almost surely for each $0 < \epsilon < 1$.

In order to prove (2.24), it thus remains to establish that

$$\sup_{m=1, \dots, M} \left\| \widetilde{\mathbf{W}}_N^m \left(\widetilde{\mathbf{W}}_N^m \right)^H - \mathcal{R}_{m,L} \right\| \rightarrow 0$$

almost surely. Noting that $\widetilde{\mathbf{W}}_N^m \left(\widetilde{\mathbf{W}}_N^m \right)^H - \mathcal{R}_{m,L}$ is the $L \times L$ Toeplitz matrix associated to the symbol $\widehat{\mathcal{S}}_m(\nu) - \mathcal{S}_m(\nu)$, it is sufficient to prove that

$$\sup_{m=1, \dots, M} \sup_{\nu} \left| \widehat{\mathcal{S}}_m(\nu) - \mathcal{S}_m(\nu) \right| \rightarrow 0$$

almost surely. In order to see this, we write $\widehat{\mathcal{S}}_m(\nu) - \mathcal{S}_m(\nu)$ as

$$\widehat{\mathcal{S}}_m(\nu) - \mathcal{S}_m(\nu) = \widehat{\mathcal{S}}_m(\nu) - \mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) + \mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) - \mathcal{S}_m(\nu).$$

The bias $\mathbb{E}(\widehat{\mathcal{S}}_m(\nu)) - \mathcal{S}_m(\nu)$ is equal to

$$\mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) - \mathcal{S}_m(\nu) = - \sum_{|l| \geq L} r_m(l) e^{-2i\pi l \nu} - \frac{1}{N} \sum_{l=-(L-1)}^{L-1} |l| r_m(l) e^{-2i\pi l \nu}.$$

An easy adaptation of the proof of Lemma 2.2 in the appendix establishes that

$$\left| \mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) - \mathcal{S}_m(\nu) \right| \leq \kappa \left(\frac{1}{L^{\gamma_0}} + \frac{L^{(1-\gamma_0)_+}}{N} \right)$$

for some nice constant κ , where $(\cdot)_+ = \max(\cdot, 0)$. This implies that

$$\sup_{m=1, \dots, M} \sup_{\nu} \left| \mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) - \mathcal{S}_m(\nu) \right| \rightarrow 0. \quad (2.27)$$

In order to study $\widehat{\mathcal{S}}_m(\nu) - \mathbb{E}(\widehat{\mathcal{S}}_m(\nu))$, we remark that it can be written as

$$\widehat{\mathcal{S}}_m(\nu) - \mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) = \mathbf{e}_m^T \widehat{\mathbf{S}}^\circ(\nu) \mathbf{e}_m$$

where \mathbf{e}_m is the m th vector of the canonical basis of \mathbb{C}^M . Using the Hanson-Wright inequality as in Section 2.1, we obtain immediately that for each ν and for each m , we have

$$P \left(|\mathbf{e}_m^T \widehat{\mathbf{S}}^\circ(\nu) \mathbf{e}_m| > \alpha_N \right) \leq \kappa_1 \exp(-\kappa_2 M \alpha_N^2)$$

where $(\alpha_N)_{N \geq 1}$ satisfies $\alpha_N \rightarrow 0$ and $M \alpha_N^2 \rightarrow +\infty$. Any sequence verifying $\alpha_N = \mathcal{O}\left(\frac{1}{M^{(1-\epsilon)/2}}\right)$, where $0 < \epsilon < 1$ satisfies this condition. Using Lemma 2.1 as well as the union bound, we obtain that

$$P \left(\sup_{m=1, \dots, M} \sup_{\nu} |\mathbf{e}_m^T \widehat{\mathbf{S}}^\circ(\nu) \mathbf{e}_m| > \alpha_N \right) \leq N \kappa_1 \exp(-\kappa_2 M \alpha_N^2).$$

The Borel Cantelli Lemma thus implies that almost surely, for each N large enough,

$$\sup_{m=1,\dots,M} \sup_{\nu} \left| \widehat{\mathcal{S}}_m(\nu) - \mathbb{E} \left(\widehat{\mathcal{S}}_m(\nu) \right) \right| \leq \alpha_N.$$

This, in conjunction with (2.27) implies that

$$\sup_{m=1,\dots,M} \sup_{\nu} \left| \widehat{\mathcal{S}}_m(\nu) - \mathcal{S}_m(\nu) \right| \rightarrow 0$$

almost surely, and that

$$\sup_{m=1,\dots,M} \left\| \widetilde{\mathbf{W}}_N^m \left(\widetilde{\mathbf{W}}_N^m \right)^H - \mathcal{R}_{m,L} \right\| \rightarrow 0.$$

Therefore, we have established that $\|\text{Bdiag}(\widehat{\mathcal{R}}_L) - \text{Bdiag}(\mathcal{R}_L)\| \rightarrow 0$ almost surely. Noting that $\text{Bdiag}(\mathcal{R}_L) > s_{\min} \mathbf{I}_{ML}$ for each N , matrix $\text{Bdiag}(\widehat{\mathcal{R}}_L)$ verifies $\text{Bdiag}(\widehat{\mathcal{R}}_L) > \frac{s_{\min}}{2} \mathbf{I}_{ML}$ almost surely for each N large enough. We thus also have

$$\left\| \left(\text{Bdiag}(\widehat{\mathcal{R}}_L) \right)^{-1/2} - \left(\text{Bdiag}(\mathcal{R}_L) \right)^{-1/2} \right\| \rightarrow 0$$

almost surely as $N \rightarrow +\infty$. As $\|\widehat{\mathcal{R}}_L\|$ is almost surely bounded by a constant for each N large enough, we eventually conclude that (2.1) holds.

From all the above, we can therefore conclude that the spectral behavior of the sample block correlation matrix $\widehat{\mathcal{R}}_{\text{corr},L}$ is equivalent to the spectral behavior of the matrix $\overline{\mathcal{R}}_{\text{corr},L} = \mathcal{B}_L^{-1/2} \widehat{\mathcal{R}}_L \mathcal{B}_L^{-1/2}$. In the following section, we will analyze the spectral convergence of the eigenvalues of this matrix by analyzing the behavior of the corresponding resolvent and co-resolvent.

3 Expectation of resolvent and co-resolvent

In this section, we analyze the expectation of the resolvent and co-resolvent of the matrix $\overline{\mathcal{R}}_{\text{corr},L}$, which are respectively defined as

$$\begin{aligned} \mathbf{Q}(z) &= \left(\mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{W}^H \mathcal{B}_L^{-1/2} - z \mathbf{I}_{ML} \right)^{-1} \\ \widetilde{\mathbf{Q}}(z) &= \left(\mathbf{W}^H \mathcal{B}_L^{-1} \mathbf{W} - z \mathbf{I}_N \right)^{-1} \end{aligned}$$

where we recall that we use the short hand notation

$$\mathcal{B}_L = \text{Bdiag}(\mathcal{R}_L) = \mathcal{R}_L.$$

As a preliminary step, we need to evaluate the variance of certain useful terms. This is the purpose of the following lemma. The result is also valid to bound the variance of linear statistics of the eigenvalues of matrix $\overline{\mathcal{R}}_{\text{corr},L}$, thus justifying (1.18).

Lemma 3.1. *Let $(\mathbf{A}_N)_{N \geq 1}$ be a sequence of deterministic $ML \times ML$ matrices and $(\mathbf{G}_N)_{N \geq 1}$ a uniformly bounded sequence of deterministic $N \times N$ matrices. Then*

$$\text{Var} \left(\frac{1}{ML} \text{tr} \mathbf{A}_N \mathbf{Q}(z) \right) \leq \frac{C(z)}{MN} \frac{1}{ML} \text{tr}(\mathbf{A}_N \mathbf{A}_N^H) \quad (3.1)$$

$$\text{Var} \left(\frac{1}{ML} \text{tr} \mathbf{A}_N \mathbf{Q}(z) \mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{G} \mathbf{W}^H \mathcal{B}_L^{-1/2} \right) \leq \frac{C(z)}{MN} \frac{1}{ML} \text{tr}(\mathbf{A}_N \mathbf{A}_N^H) \quad (3.2)$$

where $C(z) = P_1(|z|)P_2(1/\delta_z)$ for two nice polynomials P_1, P_2 . Moreover, if ϕ is a bounded continuously differentiable function defined on \mathbb{R}^+ with bounded first derivative, we have

$$\text{Var} \left(\frac{1}{ML} \text{tr} \phi(\overline{\mathcal{R}}_{\text{corr},L}) \right) \leq \frac{\kappa}{MN} \quad (3.3)$$

for some nice constant κ .

Proof. The proof of this lemma is an easy adaptation of the techniques used to establish Lemma 3.1 in [20]. The main tool is the Poincaré-Nash inequality [22]:

Lemma 3.2. *Let $\xi = \xi(\mathbf{W}, \mathbf{W}^*)$ denote a C^1 complex function such that both itself and its derivatives are polynomially bounded. Under the assumptions formulated in this paper, we can write*

$$\begin{aligned} \text{Var} \xi \leq & \mathbb{E} \sum_{m, i_1, i_2, j_1, j_2} \left(\frac{\partial \xi}{\partial (\mathbf{W}_{i_1, j_1}^m)^*} \right)^* \mathbb{E} \left[\mathbf{W}_{i_1, j_1}^m (\mathbf{W}_{i_2, j_2}^m)^* \right] \frac{\partial \xi}{\partial (\mathbf{W}_{i_2, j_2}^m)^*} \\ & + \mathbb{E} \sum_{m, i_1, i_2, j_1, j_2} \frac{\partial \xi}{\partial (\mathbf{W}_{i_1, j_1}^m)} \mathbb{E} \left[\mathbf{W}_{i_1, j_1}^m (\mathbf{W}_{i_2, j_2}^m)^* \right] \left(\frac{\partial \xi}{\partial \mathbf{W}_{i_2, j_2}^m} \right)^* \end{aligned}$$

where $\mathbf{W}_{i,j}^m$ is the $((m-1)L + i, j)$ th entry of \mathbf{W} .

We just briefly justify (3.3) because the proof of (3.1) and (3.2) follows from a trivial adaptation of the arguments provided in [20]. In order to see (3.3), we apply the Poincaré-Nash inequality to $\xi = \frac{1}{ML} \text{Tr} \phi(\overline{\mathcal{R}}_{\text{corr},L})$, and just evaluate the first term of the upper bound of $\text{Var}(\xi)$ in Lemma 3.2, which is here referred to as ω . As mentioned in the proof of Lemma 3.1 in [20], using that $r_m(i_1 - i_2 + j_1 - j_2) = \int_0^1 e^{2\pi i(i_1 - i_2 + j_1 - j_2)\nu} \mathcal{S}_m(\nu) d\nu$, we obtain immediately that this first term ω can be written as $\omega = \mathbb{E}(\alpha)$ where α is defined by

$$\alpha = \frac{1}{N} \int_0^1 \sum_{m=1}^M \mathcal{S}_m(\nu) \left| \sum_{i_2, j_2} \frac{\partial \xi}{\partial (\mathbf{W}_{i_2, j_2}^m)^*} e^{-2\pi i(i_2 + j_2)\nu} \right|^2 d\nu.$$

Assumption (1.7) thus implies that $\omega \leq \kappa \tilde{\omega}$ for some nice constant κ , where $\tilde{\omega} = \mathbb{E}(\tilde{\alpha})$, with $\tilde{\alpha}$ defined as

$$\tilde{\alpha} = \frac{1}{N} \int_0^1 \sum_{m=1}^M \left| \sum_{i_2, j_2} \frac{\partial \xi}{\partial (\mathbf{W}_{i_2, j_2}^m)^*} e^{-2\pi i(i_2 + j_2)\nu} \right|^2 d\nu.$$

This term $\tilde{\alpha}$ can also be written as

$$\tilde{\alpha} = \sum_{m, i_1, i_2, j_1, j_2} \left(\frac{\partial \xi}{\partial (\mathbf{W}_{i_1, j_1}^m)^*} \right)^* \frac{\partial \xi}{\partial (\mathbf{W}_{i_2, j_2}^m)^*} \frac{1}{N} \delta_{i_1 + j_1 = i_2 + j_2}. \quad (3.4)$$

Now, given i, j, m , it is easy to check that

$$\begin{aligned} \frac{\partial \xi}{\partial (\mathbf{W}_{i,j}^m)^*} &= \frac{1}{ML} \text{tr} \left(\phi'(\mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{W}^H \mathcal{B}_L^{-1/2}) \mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{e}_j (\mathbf{f}_i^m)^T \mathcal{B}_L^{-1/2} \right) \\ &= \frac{1}{ML} (\mathbf{f}_i^m)^T \mathcal{B}_L^{-1/2} \phi'(\mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{W}^H \mathcal{B}_L^{-1/2}) \mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{e}_j \end{aligned}$$

where ϕ' denotes the first order derivative of ϕ . Next, we use the change of variables $i_1 - i_2 = j_2 - j_1 = u$ on the right hand side of (3.4), together with the identities

$$\begin{aligned} \sum_{m, i_1 - i_2 = u} \mathbf{f}_{i_1}^m (\mathbf{f}_{i_2}^m)^T &= \mathbf{I}_M \otimes \mathbf{J}_L^{-u} \\ \sum_{j_2 - j_1 = u} \mathbf{e}_{j_2} (\mathbf{e}_{j_1})^T &= \mathbf{J}_N^{-u} \end{aligned}$$

where we recall that for each integer K , we denote by \mathbf{J}_K the $K \times K$ shift matrix defined by $(\mathbf{J}_K)_{i,j} = \delta_{j=i+1}$ and use the notation $\mathbf{J}^{-1} = \mathbf{J}^H$. Using all this, we can write $\tilde{\alpha}$ as

$$\tilde{\alpha} = \frac{1}{MN} \frac{1}{L} \sum_{u=-(L-1)}^{L-1} \frac{1}{ML} \text{Tr}(\mathbf{B}^H (\mathbf{I}_M \otimes \mathbf{J}_L^{-u}) \mathbf{B} \mathbf{J}_N^{-u})$$

where $\mathbf{B} = \mathcal{B}_L^{-1/2} \phi'(\mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{W}^H \mathcal{B}_L^{-1/2}) \mathcal{B}_L^{-1/2} \mathbf{W}$. Using the inequality

$$\left| \frac{1}{ML} \text{Tr}(\mathbf{B}^H (\mathbf{I}_M \otimes \mathbf{J}_L^{-u}) \mathbf{B} \mathbf{J}_N^{-u}) \right| \leq \frac{1}{ML} \text{Tr}(\mathbf{B}^H \mathbf{B})$$

together with the fact that $\sup_N \max(\|\mathcal{B}_L^{-1/2}\|, \|\phi'(\mathcal{B}_L^{-1/2} \mathbf{W} \mathbf{W}^H \mathcal{B}_L^{-1/2})\|) \leq \kappa$ for some nice constant κ , we obtain immediately (3.3). \square

We now need to define two matrix valued functions of complex variable that will asymptotically approximate the expectation of the resolvent and co-resolvent, respectively. The following corollary, which is an immediate consequence of Proposition 1.2, will introduce these functions and state some of their useful properties.

Corollary 3.1. *The matrix-valued functions $\tilde{\mathbf{R}}(z)$ and $\mathbf{R}(z)$ defined by*

$$\tilde{\mathbf{R}}(z) = -\frac{1}{z} \left(\mathbf{I}_N + c_N \bar{\Psi}^T \left(\mathcal{B}_L^{-1/2} \mathbb{E} \mathbf{Q}(z) \mathcal{B}_L^{-1/2} \right) \right)^{-1} \quad (3.5)$$

$$\mathbf{R}(z) = -\frac{1}{z} \left(\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi \left(\tilde{\mathbf{R}}^T(z) \right) \mathcal{B}_L^{-1/2} \right)^{-1} \quad (3.6)$$

are elements of $\mathcal{S}_N(\mathbb{R}^+)$ and $\mathcal{S}_{ML}(\mathbb{R}^+)$ respectively. In particular, they are holomorphic on $\mathbb{C} \setminus \mathbb{R}^+$ and satisfy

$$\mathbf{R}(z) \mathbf{R}^H(z) \leq \frac{\mathbf{I}_{ML}}{\delta_z^2}, \quad \tilde{\mathbf{R}}(z) \tilde{\mathbf{R}}^H(z) \leq \frac{\mathbf{I}_N}{\delta_z^2}. \quad (3.7)$$

Moreover, there exist two nice constants η and $\tilde{\eta}$ such that

$$\mathbf{R}(z) \mathbf{R}^H(z) \geq \frac{\delta_z^2}{16(\eta^2 + |z|^2)^2} \mathbf{I}_{ML} \quad (3.8)$$

$$\tilde{\mathbf{R}}(z) \tilde{\mathbf{R}}^H(z) \geq \frac{\delta_z^2}{16(\tilde{\eta}^2 + |z|^2)^2} \mathbf{I}_N. \quad (3.9)$$

Proof. It is sufficient to use Proposition 1.2 for $\mathbf{\Gamma}(z) = \text{Bdiag}(\mathbb{E} \mathbf{Q}(z))$, and to observe that matrices $\mathbf{\Upsilon}(z)$ and $\tilde{\mathbf{\Upsilon}}(z)$ defined by (1.37) and (1.36) coincide with $\mathbf{R}(z)$ and $\tilde{\mathbf{R}}(z)$ respectively. \square

In this section, we will see that the two matrix functions $\mathbf{R}(z)$ and $\tilde{\mathbf{R}}(z)$ defined above are asymptotic approximations of the expectation of the resolvent and co-resolvent, that is $\mathbb{E} \mathbf{Q}(z)$ and $\mathbb{E} \tilde{\mathbf{Q}}(z)$ respectively. Observe, however, that these matrix function still depend on the $\mathbb{E} \mathbf{Q}(z)$. We will get rid of this dependence in a next step, which is developed in the next two sections.

In order to address the expectation of $\mathbf{Q}(z)$ and $\tilde{\mathbf{Q}}(z)$ we will apply the well known integration by parts formula for the expectation of Gaussian functions, which is presented next (see e.g. [22]).

Lemma 3.3. *Let $\xi = \xi(\mathbf{W}, \mathbf{W}^*)$ denote a \mathcal{C}^1 complex function such that both itself and its derivatives are polynomially bounded. Under the above assumptions, we can write*

$$\mathbb{E} [\mathbf{W}_{i_1, j_1}^m \xi] = \sum_{i_2=1}^L \sum_{j_2=1}^N \mathbb{E} [\mathbf{W}_{i_1, j_1}^m (\mathbf{W}_{i_2, j_2}^m)^*] \mathbb{E} \left[\frac{\partial \xi}{\partial (\mathbf{W}_{i_2, j_2}^m)^*} \right]$$

where $\mathbf{W}_{i,j}^m$ is the $((m-1)L + i, j)$ th entry of \mathbf{W} .

We recall that we can divide the normalized observation matrix \mathbf{W} into M blocks of dimension $L \times N$ each, namely

$$\mathbf{W} = \left[\begin{array}{ccc} (\mathbf{W}^1)^T & \dots & (\mathbf{W}^M)^T \end{array} \right]^T$$

and that for each $m = 1, \dots, M$, \mathbf{W}^m is the Hankel matrix of dimensions $L \times N$, with (i, j) th entry equal to

$$\{\mathbf{W}^m\}_{i,j} = \frac{1}{\sqrt{N}} y_{m,i+j-1}$$

where $1 \leq i \leq L$, $1 \leq j \leq N$.

Consider the resolvent identity

$$z\mathbf{Q}(z) = \mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\mathbf{W}^H\mathcal{B}_L^{-1/2} - \mathbf{I}_{ML}. \quad (3.10)$$

Recall that \mathbf{w}_k denotes the k th column of matrix \mathbf{W} , $1 \leq k \leq N$. Applying the integration by parts formula in Lemma 3.3 we are able to write²

$$\begin{aligned} & \mathbb{E} \left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{w}_k\mathbf{w}_j^H \right]_{i_1, i_2}^{m_1, m_2} \\ &= \sum_{m_3, i_3} \mathbb{E} \left[\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right]_{i_1, i_3}^{m_1, m_3} \mathbf{W}_{i_3, k}^{m_3} (\mathbf{W}_{i_2, j}^{m_2})^* \right] \\ &= \sum_{r=1}^N \sum_{i_4=1}^L \sum_{m_3, i_3} \mathbb{E} \left[\mathbf{W}_{i_3, k}^{m_3} (\mathbf{W}_{i_4, r}^{m_3})^* \right] \mathbb{E} \left[\frac{\partial \left\{ \left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right]_{i_1, i_3}^{m_1, m_3} (\mathbf{W}_{i_2, j}^{m_2})^* \right\}}{\partial (\mathbf{W}_{i_4, r}^{m_3})^*} \right] \\ &= \sum_{i_3=1}^L \frac{r_{m_2}(k-j+i_3-i_2)}{N} \mathbb{E} \left[\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right]_{i_1, i_3}^{m_1, m_2} \right] \\ &\quad - \frac{L}{N} \sum_{r=1}^N \sum_{m_3=1}^M \sum_{i=-L+1}^{L-1} r_{m_3}(k-r-i) \mathbb{E} \left[\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{w}_r\mathbf{w}_j^H \right]_{i_1, i_2}^{m_1, m_2} \tau \left(\left[\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right]^{m_3, m_3} \right) (i) \right] \end{aligned}$$

where in the last term we have used the change of variable $i = i_4 - i_3$ and where $\tau(\cdot)(i)$ is defined in (1.31). Using the definition of the operator $\Psi_N^{(m)}$ and its averaged counterpart in (1.34), we obtain

$$\begin{aligned} \mathbb{E} \left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{w}_k\mathbf{w}_j^H \right]_{i_1, i_2}^{m_1, m_2} &= \sum_{i_3=1}^L \frac{r_{m_2}(k-j+i_3-i_2)}{N} \mathbb{E} \left[\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right]_{i_1, i_3}^{m_1, m_2} \right] \\ &\quad - c_N \mathbb{E} \left[\bar{\Psi} \left(\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right) \mathbf{W}^T \mathcal{B}_L^{-T/2} \mathbf{Q}^T(z) \mathbf{f}_{i_1}^{m_1} (\mathbf{f}_{i_2}^{m_2})^T \mathbf{W}^* \right]_{k,j} \end{aligned} \quad (3.11)$$

where $c_N = \frac{ML}{N}$. From (3.11) and using again the definition of $\Psi(\cdot)$, we may generally write, for any $N \times N$ deterministic matrix \mathbf{A}

$$\begin{aligned} \mathbb{E} \left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\mathbf{A}\mathbf{W}^H \right] &= \mathbb{E} \left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\Psi(\mathbf{A}^T) \right] \\ &\quad - c_N \mathbb{E} \left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\bar{\Psi}^T \left(\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2} \right) \mathbf{A}\mathbf{W}^H \right]. \end{aligned} \quad (3.12)$$

We can now consider the co-resolvent identity

$$z\tilde{\mathbf{Q}}(z) = \tilde{\mathbf{Q}}(z)\mathbf{W}^H\mathcal{B}_L^{-1}\mathbf{W} - \mathbf{I}_N. \quad (3.13)$$

²Recall that, for an $ML \times ML$ matrix \mathbf{A} , we denote as \mathbf{A}^{m_1, m_2} its (m_1, m_2) th block matrix (of size $L \times L$) and as $\mathbf{A}_{i_1, i_2}^{m_1, m_2}$ the (i_1, i_2) th entry of its (m_1, m_2) th block.

Observe that we can write $\tilde{\mathbf{Q}}(z)\mathbf{W}^H\mathcal{B}_L^{-1}\mathbf{W} = \mathbf{W}^H\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}$ and therefore

$$\begin{aligned}\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\mathbf{W}^H\mathcal{B}_L^{-1}\mathbf{W}\right]_{j,k} &= \mathbb{E}\left[\mathbf{W}^H\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\right]_{j,k} \\ &= \mathbb{E}\text{tr}\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{w}_k\mathbf{w}_j^H\mathcal{B}_L^{-1/2}\right].\end{aligned}$$

Hence, using the expression for the expectation of the resolvent in (3.11), we can obtain

$$\begin{aligned}\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\mathbf{W}^H\mathcal{B}_L^{-1}\mathbf{W}\right] &= \mathbb{E}\left[\mathbf{W}^H\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\right] \\ &= c_N\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbb{E}\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\right) - c_N\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\mathbf{W}^H\mathcal{B}_L^{-1}\mathbf{W}\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\right)\right]\end{aligned}$$

and, by the co-resolvent identity in (3.13),

$$\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\right] = -\frac{1}{z}\mathbf{I}_N - c_N\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\right)\right].$$

Now, replacing $\mathbf{Q}(z)$ in the above equation by $\mathbf{Q}(z) = \mathbb{E}\mathbf{Q}(z) + \mathbf{Q}^\circ(z)$ we see that

$$\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\right] = \tilde{\mathbf{R}}(z) + zc_N\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbf{Q}^\circ(z)\mathcal{B}_L^{-1/2}\right)\tilde{\mathbf{R}}(z)\right]$$

where $\tilde{\mathbf{R}}(z)$ is defined in (3.5). On the other hand, particularizing the equation in (3.12) to the case $\mathbf{A} = \tilde{\mathbf{R}}(z)$ and using the resolvent identity in (3.10), we also obtain

$$\mathbb{E}\left[\mathbf{Q}(z)\right] = \mathbf{R}(z) - zc_N\mathbb{E}\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbf{Q}^\circ(z)\mathcal{B}_L^{-1/2}\right)\tilde{\mathbf{R}}(z)\mathbf{W}^H\mathcal{B}_L^{-1/2}\mathbf{R}(z)\right]$$

where $\mathbf{R}(z)$ is defined in (3.6). Combining all the above, we have obtained

$$\mathbb{E}\left[\mathbf{Q}(z)\right] - \mathbf{R}(z) = \mathbf{\Delta}(z), \quad \mathbb{E}\left[\tilde{\mathbf{Q}}(z)\right] - \tilde{\mathbf{R}}(z) = \tilde{\mathbf{\Delta}}(z)$$

where the error terms are defined as

$$\mathbf{\Delta}(z) = -zc_N\mathbb{E}\left[\mathbf{Q}(z)\mathcal{B}_L^{-1/2}\mathbf{W}\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbf{Q}^\circ(z)\mathcal{B}_L^{-1/2}\right)\tilde{\mathbf{R}}(z)\mathbf{W}^H\mathcal{B}_L^{-1/2}\mathbf{R}(z)\right] \quad (3.14)$$

$$\tilde{\mathbf{\Delta}}(z) = zc_N\mathbb{E}\left[\tilde{\mathbf{Q}}(z)\bar{\Psi}^T\left(\mathcal{B}_L^{-1/2}\mathbf{Q}^\circ(z)\mathcal{B}_L^{-1/2}\right)\tilde{\mathbf{R}}(z)\right]. \quad (3.15)$$

We finally evaluate the order of magnitude of terms depending on $\mathbf{\Delta}(z)$. This is the purpose of Proposition 3.1.

Proposition 3.1. *For each deterministic sequence of $ML \times ML$ matrices $(\mathbf{A}_N)_{N \geq 1}$ satisfying $\sup_N \|\mathbf{A}_N\| < a < +\infty$, it holds that*

$$\left|\frac{1}{ML}\text{tr}\left[\mathbf{A}_N\mathbf{\Delta}(z)\right]\right| \leq aC(z)\frac{L}{MN} \quad (3.16)$$

where $C(z) = P_1(|z|)P_2(1/\delta_z)$ for some nice polynomials P_1 and P_2 . Moreover, if $(\mathbf{b}_{1,N})_{N \geq 1}$ and $(\mathbf{b}_{2,N})_{N \geq 1}$ are 2 sequences of L dimensional vectors such that $\sup_N \|\mathbf{b}_{i,N}\| < b < +\infty$ for $i = 1, 2$, and if $((d_{m,N})_{m=1,\dots,M})_{N \geq 1}$ are deterministic complex number verifying $\sup_{N,m} |d_{m,N}| < d < +\infty$, then, it holds that

$$\left|\mathbf{b}_{1,N}^H \left(\frac{1}{M} \sum_{m=1}^M d_{m,N} \Delta^{m,m}(z)\right) \mathbf{b}_{2,N}\right| \leq db^2 C(z) \frac{L^{3/2}}{MN} \quad (3.17)$$

where $C(z)$ is defined as above. Finally, we have

$$\|\bar{\Psi}(\mathbf{\Delta})\| \leq C(z) \frac{L^{3/2}}{MN}. \quad (3.18)$$

Since matrix \mathcal{B}_L verifies $s_{\min}\mathbf{I}_{ML} \leq \mathcal{B}_L \leq s_{\max}\mathbf{I}_{ML}$, Proposition 3.1 can be proved by following the arguments of the proof of Proposition 4.3 in [20]. The main ingredients are the use of Assumption 3, identity (1.35) and Lemma 3.1.

4 The deterministic equivalents

We consider here the two asymptotic equivalents $\mathbf{T}(z), \tilde{\mathbf{T}}(z)$, as the solutions to the equations

$$\mathbf{T}(z) = -\frac{1}{z} \left(\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi \left(\tilde{\mathbf{T}}^T(z) \right) \mathcal{B}_L^{-1/2} \right)^{-1} \quad (4.1)$$

$$\tilde{\mathbf{T}}(z) = -\frac{1}{z} \left(\mathbf{I}_N + c_N \bar{\Psi}^T \left(\mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathcal{B}_L^{-1/2} \right) \right)^{-1}. \quad (4.2)$$

Proposition 4.1. *There exists a unique pair of functions $(\mathbf{T}(z), \tilde{\mathbf{T}}(z)) \in \mathcal{S}_{ML}(\mathbb{R}^+) \times \mathcal{S}_N(\mathbb{R}^+)$ that satisfy (4.1 , 4.2) for each $z \in \mathbb{C} \setminus \mathbb{R}^+$. Moreover, there exist two nice constants η and $\tilde{\eta}$ such that*

$$\mathbf{T}(z) \mathbf{T}^H(z) \geq \frac{\delta_z^2}{16(\eta^2 + |z|^2)^2} \mathbf{I}_{ML} \quad (4.3)$$

$$\tilde{\mathbf{T}}(z) \tilde{\mathbf{T}}^H(z) \geq \frac{\delta_z^2}{16(\tilde{\eta}^2 + |z|^2)^2} \mathbf{I}_N. \quad (4.4)$$

We devote the rest of this section to proving this proposition. We first indicate how the existence of a solution can be established using a standard convergence argument. As a very similar approach is developed in the course of the proof of Proposition 5.1 in [20], we just provide a sketch of proof.

In order to prove the existence of a solution, we define a sequence of functions in $\mathcal{S}_{ML}(\mathbb{R}^+)$ that converges towards a solution. We begin by defining $\mathbf{T}^{(0)}(z) = -\frac{1}{z} (\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi(-\frac{1}{z} \mathbf{I}_N) \mathcal{B}_L^{-1/2})^{-1}$ and using the iterative definition

$$\tilde{\mathbf{T}}^{(p)}(z) = -\frac{1}{z} \left(\mathbf{I}_N + c_N \bar{\Psi}^T \left(\mathcal{B}_L^{-1/2} \mathbf{T}^{(p)}(z) \mathcal{B}_L^{-1/2} \right) \right)^{-1} \quad (4.5)$$

$$\mathbf{T}^{(p+1)}(z) = -\frac{1}{z} \left(\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi \left(\tilde{\mathbf{T}}^{(p)}(z) \right) \mathcal{B}_L^{-1/2} \right)^{-1} \quad (4.6)$$

for $p \geq 0$. By Proposition 1.2, we see that the $L \times L$ diagonal blocks of $\mathbf{T}^{(p)}(z)$ belong to the class $\mathcal{S}_L(\mathbb{R}^+)$, whereas $\tilde{\mathbf{T}}^{(p)}(z)$ belong to $\mathcal{S}_N(\mathbb{R}^+)$. In order to prove the existence of a solution to the canonical equation, it is possible to establish that the sequence $\mathbf{T}^{(p)}(z)$ has a limit in the set of $ML \times ML$ diagonal block matrices with blocks belonging to the class $\mathcal{S}_L(\mathbb{R}^+)$. Then, in a second step, it can be shown that this limit is a solution to the canonical equation. For more details, the reader may refer to the proof of Proposition 5.1 in [20].

Next, we focus on the unicity of the solution. More specifically, we will prove that, for each $z \in \mathbb{C}^+$, the system (4.1, 4.2) has a unique solution in the set of $ML \times ML$ and $N \times N$ matrices with positive imaginary part. This, of course, will imply the uniqueness of the pair of functions $(\mathbf{T}(z), \tilde{\mathbf{T}}(z)) \in \mathcal{S}_{ML}(\mathbb{R}^+) \times \mathcal{S}_N(\mathbb{R}^+)$ that satisfy (4.1 , 4.2) because functions of $\mathcal{S}_{ML}(\mathbb{R}^+)$ and $\mathcal{S}_N(\mathbb{R}^+)$ are uniquely defined if their values on \mathbb{C}^+ are prescribed. We fix $z \in \mathbb{C}^+$, assume that $\mathbf{T}(z), \tilde{\mathbf{T}}(z)$ and $\mathbf{S}(z), \tilde{\mathbf{S}}(z)$ are matrices solutions of the system (4.1, 4.2) of equations at point z , and assume that $\mathbf{T}(z)$ and $\mathbf{S}(z)$ have positive imaginary parts. Let $\mathbf{T}_B(z) = \mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathcal{B}_L^{-1/2}$ and $\mathbf{S}_B(z) = \mathcal{B}_L^{-1/2} \mathbf{S}(z) \mathcal{B}_L^{-1/2}$. It is easily seen that

$$\mathbf{T}_B(z) - \mathbf{S}_B(z) = \Phi_{B,0}(\mathbf{T}_B(z) - \mathbf{S}_B(z)) \quad (4.7)$$

where we have defined the operator $\Phi_{B,0}(\mathbf{X})$ as

$$\Phi_{B,0}(\mathbf{X}) = z^2 c_N \mathbf{S}_B(z) \Psi \left(\tilde{\mathbf{S}}^T(z) \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{T}}^T(z) \right) \mathbf{T}_B(z) \quad (4.8)$$

where \mathbf{X} is an $ML \times ML$ matrix. We note that operator $\Phi_{B,0}$ depends on point z , but we do not mention this dependency in order to simplify the notations.

We only need to show that the equation $\Phi_{\mathcal{B},0}(\mathbf{X}_0) = \mathbf{X}_0$ accepts a unique solution in the set of block-diagonal matrices, which is trivially given by $\mathbf{X}_0 = \mathbf{0}$. This will imply that $\mathbf{T}_{\mathcal{B}}(z) = \mathbf{S}_{\mathcal{B}}(z)$, and therefore $\mathbf{T}(z) = \mathbf{S}(z)$, contradicting the original hypothesis.

We recall that if we put $\Phi_{\mathcal{B},0}^{(1)}(\mathbf{X}) = \Phi_{\mathcal{B},0}(\mathbf{X})$, we recursively define $\Phi_{\mathcal{B},0}^{(n+1)}(\mathbf{X}) = \Phi_{\mathcal{B},0}(\Phi_{\mathcal{B},0}^{(n)}(\mathbf{X}))$ for $n \geq 1$. A solution \mathbf{X}_0 of the equation $\Phi_{\mathcal{B},0}(\mathbf{X}_0) = \mathbf{X}_0$ also satisfies $\Phi_{\mathcal{B},0}^{(n)}(\mathbf{X}_0) = \mathbf{X}_0$ for each $n \geq 1$. In order to establish that \mathbf{X}_0 is equal to 0, it is thus sufficient to prove the following result.

Proposition 4.2. *For each $ML \times ML$ matrix \mathbf{X} , it holds that*

$$\lim_{n \rightarrow +\infty} \Phi_{\mathcal{B},0}^{(n)}(\mathbf{X}) = \mathbf{0}. \quad (4.9)$$

Proof. We remark that $\Phi_{\mathcal{B},0}$ coincides with the operator Φ defined by (1.41) when the matrices $\mathbf{S}, \tilde{\mathbf{S}}, \mathbf{T}, \tilde{\mathbf{T}}$ are chosen equal to $\mathbf{S}_{\mathcal{B}}(z), \tilde{\mathbf{S}}(z), \mathbf{T}_{\mathcal{B}}(z), \tilde{\mathbf{T}}(z)$. As $\mathbf{T}(z), \tilde{\mathbf{T}}(z)$ and $\mathbf{S}(z), \tilde{\mathbf{S}}(z)$ are solutions of the system (4.1, 4.2) of equations at point z , it is clear that the matrices $\mathbf{S}_{\mathcal{B}}(z), \tilde{\mathbf{S}}(z), \mathbf{T}_{\mathcal{B}}(z), \tilde{\mathbf{T}}(z)$ are full rank. Proposition 1.3 thus implies that for any two L -dimensional column vectors \mathbf{a}, \mathbf{b} , we can write

$$\left| \mathbf{a}^H \left(\Phi_{\mathcal{B},0}^{(n)}(\mathbf{X}) \right)^{m,m} \mathbf{b} \right| \leq \left[\mathbf{a}^H \left(\Phi_{\mathbf{S}_{\mathcal{B}}}^{(n)}(\mathbf{X}\mathbf{X}^H) \right)^{m,m} \mathbf{a} \right]^{1/2} \left[\mathbf{b}^H \left(\Phi_{\mathbf{T}_{\mathcal{B}}^H}^{(n)}(\mathbf{I}_{ML}) \right)^{m,m} \mathbf{b} \right]^{1/2} \quad (4.10)$$

where $\Phi_{\mathbf{T}_{\mathcal{B}}^H}$ and $\Phi_{\mathbf{S}_{\mathcal{B}}}$ are the positive operators defined by

$$\Phi_{\mathbf{T}_{\mathcal{B}}^H}(\mathbf{X}) = |z|^2 c_N \mathbf{T}_{\mathcal{B}}^H(z) \Psi \left(\tilde{\mathbf{T}}^*(z) \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{T}}^T(z) \right) \mathbf{T}_{\mathcal{B}}(z) \quad (4.11)$$

$$\Phi_{\mathbf{S}_{\mathcal{B}}}(\mathbf{X}) = |z|^2 c_N \mathbf{S}_{\mathcal{B}}(z) \Psi \left(\tilde{\mathbf{S}}^T(z) \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{S}}^*(z) \right) \mathbf{S}_{\mathcal{B}}^H(z). \quad (4.12)$$

By Proposition 1.3, (4.9) will be established if we prove that there exist two positive definite matrices \mathbf{Y}_1 and \mathbf{Y}_2 such that $\Phi_{\mathbf{T}_{\mathcal{B}}^H}^{(n)}(\mathbf{Y}_1)$ and $\Phi_{\mathbf{S}_{\mathcal{B}}}^{(n)}(\mathbf{Y}_2)$ converge towards $\mathbf{0}$. The following Lemma proves that this property holds because $(\mathbf{T}(z), \tilde{\mathbf{T}}(z))$ and $(\mathbf{S}(z), \tilde{\mathbf{S}}(z))$ are solutions of the canonical equations (4.1, 4.2).

Lemma 4.1. *Let $\mathbf{T}(z), \tilde{\mathbf{T}}(z)$ be a solution to the canonical equation (4.1, 4.2) at point $z \in \mathbb{C}^+$ satisfying $\text{Im}(\mathbf{T}(z)) \geq 0$, and define $\mathbf{T}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathcal{B}_L^{-1/2}$. Let \mathbf{X} be a positive semi definite matrix. Then, it holds that*

$$\Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{X}) \rightarrow \mathbf{0} \quad (4.13)$$

and

$$\Phi_{\mathbf{T}_{\mathcal{B}}^H}^{(n)}(\mathbf{X}) \rightarrow \mathbf{0} \quad (4.14)$$

as $n \rightarrow \infty$. Moreover, the series $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{X})$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}^H}^{(n)}(\mathbf{X})$ converge. Finally, consider $\alpha(z) > 0$ such that $\mathbf{T}(z) \mathbf{T}^H(z) \geq \alpha(z) \mathbf{I}_{ML}$ (as $\mathbf{T}(z)$ is full rank, $\mathbf{T}(z) \mathbf{T}^H(z)$ is positive definite). Then, we have

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{I}_{ML}) \leq \kappa \frac{1}{\alpha(z)} \frac{\text{Im} \mathbf{T}_{\mathcal{B}}}{\text{Im} z} \quad (4.15)$$

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}^H}^{(n)}(\mathbf{I}_{ML}) \leq \kappa \frac{1}{\alpha(z)} \frac{\text{Im} \mathbf{T}_{\mathcal{B}}}{\text{Im} z} \quad (4.16)$$

for some nice constant κ .

Proof. To simplify the notation we sometimes omit the z arguments of the multiple matrix functions in this proof. If \mathbf{T} is a solution to the canonical equation, we must have

$$\text{Im} \mathbf{T}_{\mathcal{B}} = \mathcal{B}_L^{-1/2} \frac{\mathbf{T} - \mathbf{T}^H}{2i} \mathcal{B}_L^{-1/2} = \text{Im} z \mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} + \Phi_{\mathbf{T}_{\mathcal{B}}}(\text{Im} \mathbf{T}_{\mathcal{B}}). \quad (4.17)$$

Iterating the above relationship, we find that, for $n \in \mathbb{N}$,

$$\frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z} = \sum_{k=0}^n \Phi_{\mathbf{T}_{\mathcal{B}}}^{(k)} \left(\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} \right) + \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n+1)} \left(\frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z} \right). \quad (4.18)$$

Now, since $\frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z} \geq 0$, we have $\Phi_{\mathbf{T}_{\mathcal{B}}}^{(n+1)} \left(\frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z} \right) \geq 0$ and therefore

$$\sum_{k=0}^n \Phi_{\mathbf{T}_{\mathcal{B}}}^{(k)} \left(\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} \right) \leq \frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z}$$

for each n . However, since $\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} \geq 0$ we must have $\Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2}) \rightarrow 0$ as $n \rightarrow \infty$ and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2}) < +\infty$. As the inequality $\mathbf{T}(z) \mathbf{T}^H(z) \geq \alpha(z) \mathbf{I}_{ML}$ holds, we obtain that $\mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathbf{T}^H(z) \mathcal{B}_L^{-1/2} > \kappa \alpha(z) \mathbf{I}_{ML}$ for some nice constant κ because the spectral densities are uniformly bounded away from zero. Therefore, $\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2}$ is positive definite. Proposition 1.3 thus implies that (4.13) and $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{X}) < +\infty$ hold for each positive matrix \mathbf{X} . Using (4.13) for $\mathbf{X} = \operatorname{Im} \mathbf{T}_{\mathcal{B}}$ in conjunction with (4.18), we obtain that

$$\frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z} = \sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)} \left(\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} \right). \quad (4.19)$$

Noting that $\mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathbf{T}^H(z) \mathcal{B}_L^{-1/2} \geq \kappa \alpha(z) \mathbf{I}_{ML}$, (4.19) implies that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{I}_{ML}) \leq \kappa \frac{1}{\alpha(z)} \frac{\operatorname{Im} \mathbf{T}_{\mathcal{B}}}{\operatorname{Im} z}.$$

In order to establish (4.14), we use the observation that

$$\operatorname{Im} \mathbf{T}_{\mathcal{B}} = \operatorname{Im} z \mathcal{B}_L^{-1/2} \mathbf{T}^H(z) \mathbf{T}(z) \mathcal{B}_L^{-1/2} + \Phi_{\mathbf{T}_{\mathcal{B}}}^H(\operatorname{Im} \mathbf{T}_{\mathcal{B}}(z))$$

and use the same arguments as above. \square

This completes the proof of the uniqueness of the solutions $\mathbf{T}(z)$, $\widehat{\mathbf{T}}(z)$ of the equations (4.1, 4.2).

We finally remark that (4.3) and (4.4) follow immediately from Proposition 1.3. Proposition 4.2 is thus proven. \square

We now take benefit of Lemma 4.1 and of Proposition 4.2 to deduce the following Corollary.

Corollary 4.1. *Consider the solution $\mathbf{T}(z)$ of equations (4.1, 4.2), and $\mathbf{T}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathcal{B}_L^{-1/2}$. Then, for each $z \in \mathbb{C} - \mathbb{R}^+$, it holds that*

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{I}_{ML}) \leq C(z) \mathbf{I}_{ML} \quad (4.20)$$

where $C(z) = P_1(|z|)P_2(\frac{1}{\delta_z})$ for some nice polynomials P_1 and P_2 .

Proof. The proof of Lemma 4.1 implies that (4.15) holds for each $z \in \mathbb{C} \setminus \mathbb{R}$ because $\frac{\operatorname{Im}(\mathbf{T}_{\mathcal{B}}(z))}{\operatorname{Im} z}$ is also positive if $\operatorname{Im} z < 0$. Moreover, (4.3) leads to the conclusion that the term $\alpha(z)$ in (4.15) can be chosen as $\alpha(z) = \frac{1}{C(z)}$. Therefore, (4.20) for $z \in \mathbb{C} \setminus \mathbb{R}$ follows immediately from Item (v) of Proposition (1.1).

If z belongs to \mathbb{R}^{-*} , then $\frac{\operatorname{Im} \mathbf{T}}{\operatorname{Im} z}$ should be interpreted as the positive matrix $\mathbf{T}'(z) = \int_{\mathbb{R}^+} \frac{d\mu(\lambda)}{(\lambda - z)^2}$, and the reader may verify that the above arguments are still valid. \square

We have therefore proven the existence of two matrix functions $\mathbf{T}(z)$, $\tilde{\mathbf{T}}(z)$, defined as the solutions to a couple of matrix equations. In the following section we will prove that these two matrix functions are, in fact, the deterministic equivalents to our resolvent and co-resolvent respectively.

5 Convergence towards the deterministic equivalents

We establish in this section that if $(\mathbf{A}_N)_{N \geq 1}$ is any sequence of $ML \times ML$ deterministic matrices such that $\sup_N \|\mathbf{A}_N\| \leq a$ for some nice constant a , then $\frac{1}{ML} \text{Tr} \mathbf{A}_N (\mathbf{Q}_N(z) - \mathbf{T}_N(z))$ converges towards 0 almost surely and conclude from this that the empirical eigenvalue distribution $\bar{\mu}_N$ of $\mathcal{R}_{\text{corr}, L}$ has a deterministic behaviour when $N \rightarrow +\infty$. We also evaluate the corresponding rate of convergence when $\frac{L^{3/2}}{MN} \rightarrow 0$, i.e. when $\beta < \frac{4}{5}$. The approach used in this section is again similar to what is proposed in Section 6 of [20]. We therefore often omit to provide detailed proofs.

We recall that we denote by $\boldsymbol{\mu}_N$ the positive matrix-valued measure associated to the Stieltjes transform $\mathbf{T}_N(z)$, and define μ_N as the probability measure $\mu_N = \frac{1}{ML} \text{Tr} \boldsymbol{\mu}_N$. We first notice that showing that $\frac{1}{ML} \text{Tr} (\mathbf{Q}_N(z) - \mathbf{T}_N(z)) \rightarrow 0$ almost surely for each $z \in \mathbb{C}^+$ implies that $\bar{\mu}_N - \mu_N \rightarrow 0$ weakly almost surely. More precisely, by Corollary 2.7 in [12], the property $\frac{1}{ML} \text{Tr} (\mathbf{Q}_N(z) - \mathbf{T}_N(z)) \rightarrow 0$ almost surely for each $z \in \mathbb{C}^+$ implies that $\bar{\mu}_N - \mu_N \rightarrow 0$ weakly almost surely as soon as $(\bar{\mu}_N)_{N \geq 1}$ is almost surely tight and $(\mu_N)_{N \geq 1}$ is tight. We therefore verify that these conditions are verified. For this, we check that

$$\limsup_{N \rightarrow +\infty} \int_{\mathbb{R}^+} \lambda d\bar{\mu}_N(\lambda) < +\infty, \text{ a.s.} \quad (5.1)$$

and

$$\limsup_{N \rightarrow +\infty} \int_{\mathbb{R}^+} \lambda d\mu_N(\lambda) < +\infty. \quad (5.2)$$

In order to prove (5.1), we remark that

$$\int_{\mathbb{R}^+} \lambda d\bar{\mu}_N(\lambda) = \frac{1}{ML} \text{Tr} \mathcal{B}_L^{-1/2} \mathbf{W}_N \mathbf{W}_N^H \mathcal{B}_L^{-1/2} = \frac{1}{M} \sum_{m=1}^M \frac{1}{L} \text{Tr} \left[\mathcal{R}_{m,L}^{-1/2} \hat{\mathcal{R}}_{m,L} \mathcal{R}_{m,L}^{-1/2} \right].$$

The identity in (2.24) implies that

$$\sup_{m=1, \dots, M} \left| \frac{1}{L} \text{Tr} \left[\mathcal{R}_{m,L}^{-1/2} \hat{\mathcal{R}}_{m,L} \mathcal{R}_{m,L}^{-1/2} \right] - 1 \right| \rightarrow 0, \text{ a.s.}$$

Therefore, $\int_{\mathbb{R}^+} \lambda d\bar{\mu}_N(\lambda) \rightarrow 1$ almost surely and (5.1) holds. To verify (5.2), we evaluate $\int_{\mathbb{R}^+} \lambda d\boldsymbol{\mu}_N(\lambda)$ using item (vi) of Proposition 1.1 and immediately obtain that $\int_{\mathbb{R}^+} \lambda d\boldsymbol{\mu}_N(\lambda) = \mathbf{I}_{ML}$ and $\int_{\mathbb{R}^+} \lambda d\mu_N(\lambda) = 1$. The asymptotic bound in (5.2) is thus established.

We devote the rest of the section to establishing the following result.

Proposition 5.1. *We consider a sequence $(\mathbf{A}_N)_{N \geq 1}$ of $ML \times ML$ deterministic matrices such that $\sup_N \|\mathbf{A}_N\| \leq a$ for some nice constant a . Then, for each $z \in \mathbb{C} \setminus \mathbb{R}^+$, we have*

$$\frac{1}{ML} \text{Tr} (\mathbf{A}_N (\mathbf{Q}_N(z) - \mathbb{E} \mathbf{Q}_N(z))) \rightarrow 0 \quad (5.3)$$

almost surely. Moreover,

$$\frac{1}{ML} \text{Tr} (\mathbf{A}_N (\mathbb{E} \mathbf{Q}_N(z) - \mathbf{T}_N(z))) \rightarrow 0. \quad (5.4)$$

If $\frac{L^{3/2}}{MN} \rightarrow 0$, it holds that

$$\frac{1}{ML} \text{Tr} (\mathbf{A}_N (\mathbb{E} \mathbf{Q}_N(z) - \mathbf{T}_N(z))) \leq C(z) \frac{L}{MN} \quad (5.5)$$

when z belongs to a set E_N defined by

$$E_N = \{z \in \mathbb{C} \setminus \mathbb{R}^+, \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\delta_z) < 1\}$$

where P_1 and P_2 are some nice polynomials. Finally, for each compactly supported smooth function, we have

$$\left| \frac{1}{ML} \mathbb{E} \text{Tr}(\phi(\bar{\mathcal{R}}_{\text{corr}, L})) - \int \phi(\lambda) d\mu_N(\lambda) \right| = \mathcal{O}\left(\frac{L}{MN}\right). \quad (5.6)$$

Proof. The bound in (3.1) can be written as

$$\mathbb{E} \left| \frac{1}{ML} \text{Tr}(\mathbf{A}_N(\mathbf{Q}_N(z) - \mathbb{E}\mathbf{Q}_N(z))) \right|^2 \leq C(z) \frac{1}{MN}.$$

Since $\frac{1}{MN} = \frac{1}{N^{2-\beta}}$ with $2 - \beta > 1$, the Borel-Cantelli Lemma leads to (5.3).

In order to establish (5.4) and (5.5), we follow the reasoning in Section 6 of [20]. The quantity $\mathbb{E}\mathbf{Q}(z) - \mathbf{T}(z)$ can be written as

$$\mathbb{E}\mathbf{Q}(z) - \mathbf{T}(z) = \mathbb{E}\mathbf{Q}(z) - \mathbf{R}(z) + \mathbf{R}(z) - \mathbf{T}(z) = \mathbf{\Delta}(z) + \mathbf{R}(z) - \mathbf{T}(z)$$

where $\mathbf{\Delta}(z)$ is defined in (3.14). As (3.16) holds, we have just to evaluate the behaviour of $\mathbf{R}(z) - \mathbf{T}(z)$. We recall that $\mathbf{T}_{\mathcal{B}}(z)$ is defined by $\mathbf{T}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{T}(z) \mathcal{B}_L^{-1/2}$, and introduce $\mathbf{R}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{R}(z) \mathcal{B}_L^{-1/2}$, $\mathbf{Q}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{Q}(z) \mathcal{B}_L^{-1/2}$ and $\mathbf{\Delta}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{\Delta}(z) \mathcal{B}_L^{-1/2}$. A simple calculation leads to

$$\mathbf{R}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z) = \Phi_{\mathcal{B},1}(\mathbf{R}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)) + \Phi_{\mathcal{B},1}(\mathbf{\Delta}_{\mathcal{B}}(z)) \quad (5.7)$$

where the operator $\Phi_{\mathcal{B},1}$ is defined on the set of all $ML \times ML$ matrices by

$$\Phi_{\mathcal{B},1}(\mathbf{X}) = z^2 c_N \mathbf{R}_{\mathcal{B}}(z) \Psi \left(\tilde{\mathbf{R}}^T(z) \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{T}}^T(z) \right) \mathbf{T}_{\mathcal{B}}(z). \quad (5.8)$$

In other words, $\Phi_{\mathcal{B},1}$ is obtained by replacing $(\mathbf{S}_B, \tilde{\mathbf{S}})$ by $(\mathbf{R}_B, \tilde{\mathbf{R}})$ in the definition (4.8) of operator $\Phi_{\mathcal{B},0}$. In order to establish (5.4), it is sufficient to exchange matrices (\mathbf{R}, \mathbf{T}) by matrices $(\mathbf{R}_{\mathcal{B}}, \mathbf{T}_{\mathcal{B}})$ in the proof of relation (6.3) in [20], and to follow the corresponding arguments. In particular, the "transpose" operator Φ_1^t introduced in [20] is replaced here by the operator $\Phi_{\mathcal{B},1}^t$ defined by

$$\Phi_{\mathcal{B},1}^t(\mathbf{X}) = z^2 c_N \Psi \left(\tilde{\mathbf{T}}^T \bar{\Psi}(\mathbf{T}_{\mathcal{B}} \mathbf{X} \mathbf{R}_{\mathcal{B}}) \tilde{\mathbf{R}}^T \right) \quad (5.9)$$

$\Phi_{\mathcal{B},1}^t$ verifies

$$\frac{1}{ML} \text{tr}(\Phi_{\mathcal{B},1}(\mathbf{X}) \mathbf{Y}) = \frac{1}{ML} \text{tr}(\mathbf{X} \Phi_{\mathcal{B},1}^t(\mathbf{Y})) \quad (5.10)$$

for any two given $ML \times ML$ matrices \mathbf{X} and \mathbf{Y} .

To check that (5.5) holds if $\beta < \frac{4}{5}$, we also follow [20]. We verify that for each matrix \mathbf{X} , the series $\sum_{n=0}^{+\infty} \Phi_{\mathcal{B},1}^{(n)}(\mathbf{X})$ is convergent, and deduce from (5.7) that

$$\mathbf{R}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z) = \sum_{n=0}^{+\infty} \Phi_{\mathcal{B},1}^{(n+1)}(\mathbf{\Delta}_{\mathcal{B}}(z)). \quad (5.11)$$

To establish these properties, we use Proposition 1.3 for $\mathbf{S} = \mathbf{R}_{\mathcal{B}}(z)$, $\tilde{\mathbf{S}} = \tilde{\mathbf{R}}(z)$, $\mathbf{T} = \mathbf{T}_{\mathcal{B}}(z)$, $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}(z)$. We have already proven that $\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} > \frac{1}{C(z)} \mathbf{I}_{ML} > 0$ and that $\sum_{n=0}^{+\infty} \Phi_{\mathcal{B},1}^{(n)} \left(\mathcal{B}_L^{-1/2} \mathbf{T} \mathbf{T}^H \mathcal{B}_L^{-1/2} \right) < +\infty$.

It remains to find a positive definite matrix $\mathbf{Y}(z)$ such that $\sum_{n=0}^{\infty} \Phi_{\mathbf{R}_B}^{(n)}(\mathbf{Y}(z)) < +\infty$. Following Section 6.2 in [20], we see that

$$\Phi_{\mathbf{R}_B} \left(\frac{\text{Im}(\mathbb{E}(\mathbf{Q}_B))}{\text{Im}(z)} \right) = \Phi_{\mathbf{R}_B} \left(\mathcal{B}_L^{-1/2} \mathbf{R} \mathbf{R}^H \mathcal{B}_L^{-1/2} + \frac{\text{Im}(\mathbf{\Delta}_B)}{\text{Im}(z)} \right) + \Phi_{\mathbf{R}}^{(2)} \left(\frac{\text{Im}(\mathbb{E}(\mathbf{Q}_B))}{\text{Im}(z)} \right). \quad (5.12)$$

As in [20], it is possible to show that (3.17) leads to

$$\left\| \overline{\Psi} \left(\frac{\text{Im}(\mathbf{\Delta}_B)}{\text{Im}(z)} \right) \right\| \leq C(z) \frac{L^{3/2}}{MN}. \quad (5.13)$$

Using (5.13), it is easy to prove that it exists a term $C(z)$ for which

$$\Phi_{\mathbf{R}_B} \left(\mathcal{B}_L^{-1/2} \mathbf{R}(z) \mathbf{R}(z)^H \mathcal{B}_L^{-1/2} + \frac{\text{Im}(\mathbf{\Delta}_B)}{\text{Im}(z)} \right) \geq \frac{1}{C(z)} \mathbf{I}_{ML}$$

if z belongs to a set E_N defined as in the statement of Proposition 5.1. As $\frac{1}{C(z)} > 0$, we can iterate (5.12) in the same way than in Lemma 4.1, and conclude that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}_B}^{(n)} \left[\Phi_{\mathbf{R}_B} \left(\mathcal{B}_L^{-1/2} \mathbf{R}(z) \mathbf{R}(z)^H \mathcal{B}_L^{-1/2} + \frac{\text{Im}(\mathbf{\Delta}_B)}{\text{Im}(z)} \right) \right] < +\infty$$

for each $z \in E_N$. Proposition 1.3 thus allows to conclude that $\sum_{n=0}^{+\infty} \Phi_{\mathbf{B},1}^{(n+1)}(\mathbf{\Delta}_B(z))$ is convergent and coincides with $\mathbf{R}_B(z) - \mathbf{T}_B(z)$ as expected. Therefore, Eq. (5.11) is established.

We also notice that

$$\Phi_{\mathbf{R}_B} \left(\frac{\text{Im}(\mathbb{E}(\mathbf{Q}_B))}{\text{Im}(z)} \right) = \sum_{n=0}^{+\infty} \Phi_{\mathbf{R}_B}^{(n)} \left[\Phi_{\mathbf{R}_B} \left(\mathcal{B}_L^{-1/2} \mathbf{R}(z) \mathbf{R}(z)^H \mathcal{B}_L^{-1/2} + \frac{\text{Im}(\mathbf{\Delta}_B)}{\text{Im}(z)} \right) \right]$$

and that using the same arguments as in the proof of Lemma 4.1 and of Corollary 4.1, we obtain that

$$\sum_{n=0}^{+\infty} \Phi_{\mathbf{R}_B}^{(n)}(\mathbf{I}_{ML}) \leq C(z) \mathbf{I}_{ML}$$

for each $z \in E_N$. The use of (1.48) for $\mathbf{S} = \mathbf{R}_B(z)$, $\tilde{\mathbf{S}} = \tilde{\mathbf{R}}(z)$, $\mathbf{T} = \mathbf{T}_B(z)$, $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}(z)$ thus leads to the conclusion that for each matrix \mathbf{X} , the inequality

$$\left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{B},1}^{(n)}(\mathbf{X}) \right\| \leq C(z) \|\mathbf{X}\|$$

holds for each $z \in E_N$. In particular, if $\mathbf{X} = \Phi_{\mathbf{B},1}(\mathbf{\Delta}_B(z))$, $\|\mathbf{X}\| \leq C(z) \frac{L^{3/2}}{MN}$, and the above inequality implies that

$$\|\mathbf{R}_B(z) - \mathbf{T}_B(z)\| \leq C(z) \frac{L^{3/2}}{MN} \quad (5.14)$$

or, equivalently,

$$\|\mathbf{R}(z) - \mathbf{T}(z)\| \leq C(z) \frac{L^{3/2}}{MN} \quad (5.15)$$

if $z \in E_N$.

In order to complete the proof of (5.5), we again follow Section 6.2 in [20]. It is possible to establish that for each block-matrix \mathbf{X} , the series $\sum_{n=0}^{+\infty} \Phi_{\mathbf{B},1}^{t(n)}(\mathbf{X})$ is convergent, and that

$$\left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{B},1}^{t(n)}(\mathbf{X}) \right\| \leq C(z) \|\mathbf{X}\|. \quad (5.16)$$

Starting from (5.11), we obtain that

$$\frac{1}{ML} \text{Tr}(\mathbf{A}(\mathbf{R}_{\mathcal{B}} - \mathbf{T}_{\mathcal{B}})) = \frac{1}{ML} \text{Tr} \left(\mathbf{\Delta}_{\mathcal{B}} \left(\sum_{n=0}^{+\infty} \Phi_{\mathcal{B},1}^{t(n+1)}(\mathbf{A}) \right) \right).$$

Therefore, (5.5) is a consequence of (3.16) and (5.16).

We finally justify (5.6). This follows directly from Lemma 5.5.5 of [2]³ provided we verify that for each nice constants C_0, C'_0 , it exist nice constants C_1, C_2, C_3 and an integer N_0 such that

$$\left| \frac{1}{ML} \mathbb{E}(\text{Tr} \mathbf{Q}_N(z)) - \frac{1}{ML} \text{Tr}(\mathbf{T}_N(z)) \right| \leq C_2 \frac{L}{MN} \frac{1}{(\text{Im}z)^{C_3}} \quad (5.17)$$

for each z in the domain $|\text{Re}(z)| \leq C_0, \frac{1}{N^{C_1}} \leq \text{Im}(z) \leq C'_0$ and for each $N > N_0$. For this, it is sufficient to follow some of the arguments used to establish Theorem 10.1 in [19].

□

6 Approximation by a Marchenko-Pastur distribution

This section provides the proof of Theorem 2. We will drop here the subindex N in all relevant quantities, i.e. $t_N(z), \tilde{t}_N(z), c_N, \mathbf{T}_N(z), \tilde{\mathbf{T}}_N(z)$, etc. to simplify the notation.

It is well known that the function $\tilde{t}(z) = ct(z) - \frac{1-c}{z}$ coincides with the Stieltjes transform of the probability measure $c\mu_{mp,c} + (1-c)\delta_0$ and is equal to

$$\tilde{t}(z) = -\frac{1}{z(1+ct(z))} \quad (6.1)$$

so that $t(z)$ can also be written as

$$t(z) = -\frac{1}{z(1+\tilde{t}(z))}.$$

Consider here the two matrix-valued functions $\tilde{\mathbf{T}}_{mp}(z)$ and $\mathbf{T}_{mp}(z)$ defined by

$$\tilde{\mathbf{T}}_{mp}(z) = -\frac{1}{z} \left(\mathbf{I}_N + c_N \bar{\Psi}^T \left(\mathcal{B}_L^{-1/2} t(z) \mathbf{I}_{ML} \mathcal{B}_L^{-1/2} \right) \right)^{-1} \quad (6.2)$$

$$\mathbf{T}_{mp}(z) = -\frac{1}{z} \left(\mathbf{I}_{ML} + \mathcal{B}_L^{-1/2} \Psi \left(\tilde{\mathbf{T}}_{mp}^T(z) \right) \mathcal{B}_L^{-1/2} \right)^{-1}. \quad (6.3)$$

According to Proposition 1.2, these functions belong to $\mathcal{S}_N(\mathbb{R}^+)$ and $\mathcal{S}_{ML}(\mathbb{R}^+)$ respectively, and verify the various properties of functions $\tilde{\mathbf{\Upsilon}}(z)$ and $\mathbf{\Upsilon}(z)$ defined in the statement of that proposition. In order to establish Theorem 2, we define $\Delta_{mp}(z)$ by

$$\Delta_{mp}(z) = t(z) \mathbf{I}_{ML} - \mathbf{T}_{mp}(z) \quad (6.4)$$

and express $t(z) \mathbf{I}_{ML} - \mathbf{T}(z)$ as

$$t(z) \mathbf{I}_{ML} - \mathbf{T}(z) = (\mathbf{T}_{mp}(z) - \mathbf{T}(z)) + \Delta_{mp}(z). \quad (6.5)$$

We also define $\mathbf{t}_{\mathcal{B}}(z), \mathbf{T}_{\mathcal{B},mp}(z)$ and $\Delta_{\mathcal{B},mp}(z)$ by $\mathbf{t}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} t(z) \mathbf{I}_{ML} \mathcal{B}_L^{-1/2}$, $\mathbf{T}_{\mathcal{B},mp}(z) = \mathcal{B}_L^{-1/2} \mathbf{T}_{mp}(z) \mathcal{B}_L^{-1/2}$ and $\Delta_{\mathcal{B},mp}(z) = \mathcal{B}_L^{-1/2} \Delta_{mp}(z) \mathcal{B}_L^{-1/2}$ respectively. Using the definition of \mathbf{T}_{mp} and $\tilde{\mathbf{T}}_{mp}$ as well as the canonical equations (4.1, 4.2), we obtain easily that

$$\mathbf{T}_{\mathcal{B},mp}(z) - \mathbf{T}_{\mathcal{B}}(z) = \Phi_{\mathcal{B},2}(\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)) \quad (6.6)$$

³We notice that the statement of Lemma 5.5.5 of [2] assumes that the function ϕ vanishes on the support of μ_N . However, the reader may check that this assumption is not needed

where $\Phi_{\mathcal{B},2}$ is the linear operator acting on $ML \times ML$ matrices defined as

$$\Phi_{\mathcal{B},2}(\mathbf{X}) = cz^2 \mathbf{T}_{\mathcal{B},mp}(z) \Psi \left(\tilde{\mathbf{T}}_{mp}^T(z) \bar{\Psi}(\mathbf{X}) \tilde{\mathbf{T}}^T(z) \right) \mathbf{T}_{\mathcal{B}}(z). \quad (6.7)$$

Using this definition, we can re-write (6.5) as

$$\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z) = \Phi_{\mathcal{B},2}(\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)) + \Delta_{\mathcal{B},mp}(z). \quad (6.8)$$

Our approach is to use Proposition 1.3 in order to establish that

$$\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z) = \sum_{n=0}^{+\infty} \Phi_{\mathcal{B},2}^{(n)}(\Delta_{\mathcal{B},mp}(z)) \quad (6.9)$$

and that $\|\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)\| \leq C(z) \|\Delta_{\mathcal{B},mp}(z)\|$. The identity in (1.22) will then be established if we are able to show that $\|\Delta_{\mathcal{B},mp}(z)\| \leq C(z) \frac{1}{L^{\min(1,\gamma)}}$ if $\gamma < \gamma_0$, $\gamma \neq 1$.

We begin by evaluating the spectral norm of $\Delta_{mp}(z)$ and $\Delta_{\mathcal{B},mp}(z)$. For this, we observe that $\tilde{\mathbf{T}}_{mp}^T(z)$ is given by

$$\tilde{\mathbf{T}}_{mp}^T(z) = -\frac{1}{z} (\mathbf{I}_N + ct(z) \bar{\Psi}(\mathcal{B}_L^{-1}))^{-1}$$

where we can express $\bar{\Psi}(\mathcal{B}_L^{-1})$ as

$$\bar{\Psi}(\mathcal{B}_L^{-1}) = \int_0^1 \frac{1}{M} \sum_{m=1}^M \mathcal{S}_m(\nu) \mathbf{a}_L^H(\nu) \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu) \mathbf{d}_N(\nu) \mathbf{d}_N^H(\nu) d\nu.$$

Let us denote by \mathbf{E}_N the $N \times N$ matrix defined by

$$\mathbf{E}_N = \int_0^1 \left(\frac{1}{M} \sum_{m=1}^M \epsilon_{m,L}(\nu) \right) \mathbf{d}_N(\nu) \mathbf{d}_N^H(\nu) d\nu$$

where $\epsilon_{m,L}(\nu)$ is defined by

$$\epsilon_{m,L}(\nu) = \mathcal{S}_m(\nu) \mathbf{a}_L^H(\nu) \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu) - 1.$$

It is clear that $\bar{\Psi}(\mathcal{B}_L^{-1}) = \mathbf{I}_N + \mathbf{E}_N$, so that $\tilde{\mathbf{T}}_{mp}^T(z)$ can be written as

$$\tilde{\mathbf{T}}_{mp}^T(z) = \left[-z(1 + ct(z)) \left(\mathbf{I}_N + \frac{ct(z)}{1 + ct(z)} \mathbf{E}_N \right) \right]^{-1}$$

or equivalently as

$$\tilde{\mathbf{T}}_{mp}^T(z) = \tilde{t}(z) \mathbf{I}_N (\mathbf{I}_N - cz t(z) \tilde{t}(z) \mathbf{E}_N)^{-1} = \tilde{t}(z) \mathbf{I}_N + czt(z) \tilde{t}^2(z) \mathbf{E}_N (\mathbf{I}_N - cz t(z) \tilde{t}(z) \mathbf{E}_N)^{-1}.$$

In order to express $\mathbf{T}_{mp}(z)$, we define $\Gamma(z)$ as the $ML \times ML$ block diagonal matrix given by

$$\Gamma(z) = \Psi \left(\mathbf{E}_N (\mathbf{I}_{ML} - cz t(z) \tilde{t}(z) \mathbf{E}_N)^{-1} \right). \quad (6.10)$$

Using that $\Psi(\mathbf{I}_N) = \mathcal{B}_L$, we obtain

$$\mathbf{T}_{mp}(z) = \left[-z \left((1 + \tilde{t}(z)) \mathbf{I}_{ML} + czt(z) \tilde{t}^2(z) \mathcal{B}_L^{-1/2} \Gamma(z) \mathcal{B}_L^{-1/2} \right) \right]^{-1}$$

or, equivalently,

$$\begin{aligned} \mathbf{T}_{mp}(z) &= t(z) (\mathbf{I}_{ML} - czt(z) \tilde{t}(z))^2 \Gamma_{\mathcal{B}}(z)^{-1} \\ &= t(z) \mathbf{I}_{ML} + t(z) czt(z) \tilde{t}(z)^2 \Gamma_{\mathcal{B}}(z) (\mathbf{I}_{ML} - czt(z) \tilde{t}(z))^2 \Gamma_{\mathcal{B}}(z)^{-1} \end{aligned}$$

where $\mathbf{\Gamma}_{\mathcal{B}}(z) = \mathcal{B}_L^{-1/2} \mathbf{\Gamma}(z) \mathcal{B}_L^{-1/2}$. We eventually obtain that

$$\Delta_{mp}(z) = -t(z)c(zt(z)\tilde{t}(z))^2 \mathbf{\Gamma}_{\mathcal{B}}(z) (\mathbf{I} - c(zt(z)\tilde{t}(z))^2 \mathbf{\Gamma}_{\mathcal{B}}(z))^{-1}. \quad (6.11)$$

The asymptotic behaviour of $\Delta_{mp}(z)$ depends on the behaviour of matrix \mathbf{E}_N , which itself depends on the properties of the terms $(\epsilon_m(\nu))_{m=1, \dots, M}$. The following Lemma, established in the Appendix B, is the key point of the proof of Theorem 2.

Lemma 6.1. *For each $\gamma < \gamma_0$, it holds that*

$$\sup_{m \geq 1} \sup_{\nu \in [0,1]} |\epsilon_{m,L}(\nu)| \leq \frac{\kappa}{L^{\min(\gamma,1)}} \quad (6.12)$$

for some nice constant κ (depending on γ) if $\gamma \neq 1$ while if $\gamma = 1$,

$$\sup_{m \geq 1} \sup_{\nu \in [0,1]} |\epsilon_{m,L}(\nu)| \leq \kappa \frac{\log L}{L}. \quad (6.13)$$

In the following, we use Lemma 6.1 for a value of γ as close as possible to γ_0 in order to obtain the fastest speed of convergence for $\sup_{m \geq 1} \sup_{\nu \in [0,1]} |\epsilon_{m,L}(\nu)|$. If $\gamma_0 \leq 1$, $\gamma < \gamma_0 \leq 1$ cannot be equal to 1. If $\gamma_0 > 1$, we will of course consider a value of γ for which $1 < \gamma < \gamma_0$. Therefore, in the following, we assume that $\gamma \neq 1$. If $\gamma_0 \leq 1$, we thus obtain that for each $\gamma < \gamma_0$

$$\sup_{m \geq 1} \sup_{\nu \in [0,1]} |\epsilon_{m,L}(\nu)| \leq \frac{\kappa}{L^\gamma} \quad (6.14)$$

holds, while if $\gamma_0 > 1$,

$$\sup_{m \geq 1} \sup_{\nu \in [0,1]} |\epsilon_{m,L}(\nu)| \leq \frac{\kappa}{L}. \quad (6.15)$$

Noting that \mathbf{E}_N is the $N \times N$ Toeplitz matrix with symbol $\frac{1}{M} \sum_{m=1}^M \epsilon_{m,L}(\nu)$, we immediately infer from this discussion the following corollary.

Corollary 6.1. *If $\gamma_0 \leq 1$, then, for each $\gamma < \gamma_0$, it exists a nice constant κ depending on γ for which $\|\mathbf{E}_N\| \leq \frac{\kappa}{L^\gamma}$. If $\gamma_0 > 1$, it exists a nice constant κ such that $\|\mathbf{E}_N\| \leq \frac{\kappa}{L}$.*

In order to control the norm of $\mathbf{\Gamma}(z)$, we mention that for each $z \in \mathbb{C} \setminus \mathbb{R}^+$, then $c|zt(z)\tilde{t}(z)|^2 < 1$ (see e.g. Lemma 1.1 in [19]). Therefore, the inequalities $|zt(z)\tilde{t}(z)| \leq \frac{1}{\sqrt{c}}$ and $c|zt(z)\tilde{t}(z)| \leq \sqrt{c}$ hold on $\mathbb{C} \setminus \mathbb{R}^+$. Corollary 6.1 thus implies that for L large enough, $\|\mathbf{I}_N - czt(z)\tilde{t}(z)\mathbf{E}_N\| > 1 - \sqrt{c}\|\mathbf{E}_N\| > \frac{1}{2}$ and $\|(\mathbf{I}_N - czt(z)\tilde{t}(z)\mathbf{E}_N)^{-1}\| < 2$ hold for each $z \in \mathbb{C} \setminus \mathbb{R}^+$. For L large enough, we thus have $\|\mathbf{E}_N(\mathbf{I}_N - czt(z)\tilde{t}(z)\mathbf{E}_N)^{-1}\| \leq \frac{\kappa}{L^{\min(\gamma,1)}}$ for some nice constant κ , a property which also implies that $\|\mathbf{\Gamma}(z)\| \leq \frac{\kappa}{L^{\min(\gamma,1)}}$ because if $\tilde{\mathbf{X}}$ is any $N \times N$ matrix, then $\|\Psi(\tilde{\mathbf{X}})\| \leq s_{max} \|\tilde{\mathbf{X}}\|$. This, in turn, implies that $\|\Delta_{mp}(z)\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}$ and $\|\Delta_{\mathcal{B},mp}(z)\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}$ for L large enough, as we wanted to show.

We now establish that (6.9) holds. We first prove that for any $ML \times ML$ block matrix matrix \mathbf{X} the series $\sum_{n=0}^{+\infty} \Phi_{\mathcal{B},2}^{(n)}(\mathbf{X})$ is convergent. For this, we use Proposition 1.3. It was already proven that $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}^H}^{(n)}(\mathbf{Y}) < +\infty$ for each positive matrix \mathbf{Y} . In order to establish a similar property for operator $\Phi_{\mathbf{T}_{\mathcal{B},mp}}$, we notice that a simple calculation leads to the identity

$$\frac{\text{Im} \mathbf{T}_{\mathcal{B},mp}(z)}{\text{Im} z} = \mathcal{B}_L^{-1/2} \mathbf{T}_{mp}(z) \mathbf{T}_{mp}^H(z) \mathcal{B}_L^{-1/2} + \Phi_{\mathbf{T}_{\mathcal{B},mp}} \left(\frac{\text{Im} \mathbf{t}_{\mathcal{B}}}{\text{Im} z} \right)$$

if $z \in \mathbb{C} \setminus \mathbb{R}^+$, where we recall that, when $z \in \mathbb{R}^{-*}$, we follow the convention of denoting as $\frac{\text{Im} \mathbf{T}_{\mathcal{B},mp}(z)}{\text{Im} z}$ and $\frac{\text{Im} \mathbf{t}_{\mathcal{B}}}{\text{Im} z}$ the derivatives of $\mathbf{T}_{\mathcal{B},mp}$ and $\mathbf{t}_{\mathcal{B}}$ at z respectively. This implies that

$$\frac{\text{Im} \mathbf{t}_{\mathcal{B}}(z)}{\text{Im} z} = \mathcal{B}_L^{-1/2} \mathbf{T}_{mp}(z) \mathbf{T}_{mp}^H(z) \mathcal{B}_L^{-1/2} + \frac{\text{Im} \Delta_{\mathcal{B},mp}(z)}{\text{Im} z} + \Phi_{\mathbf{T}_{\mathcal{B},mp}} \left(\frac{\text{Im} \mathbf{t}_{\mathcal{B}}}{\text{Im} z} \right).$$

Noting that $\|\Delta_{\mathcal{B},mp}(z)\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}$, Lemma B.1 in [13] implies that

$$\left\| \frac{\text{Im}\Delta_{\mathcal{B},mp}(z)}{\text{Im}z} \right\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}.$$

Proposition 1.2 implies that $\mathbf{T}_{mp}(z)\mathbf{T}_{mp}^H(z) \geq \frac{1}{C(z)}\mathbf{I}_{ML}$ for each $z \in \mathbb{C} \setminus \mathbb{R}^+$. Therefore, if we denote by $\mathbf{Y}_1(z)$ the matrix $\mathbf{Y}_1(z) = \mathcal{B}_L^{-1/2}\mathbf{T}_{mp}(z)\mathbf{T}_{mp}^H(z)\mathcal{B}_L^{-1/2} + \frac{\text{Im}\Delta_{\mathcal{B},mp}(z)}{\text{Im}z}$, then, $\mathbf{Y}_1(z) > \frac{1}{C(z)}\mathbf{I}_{ML} > 0$ if $z \in F_N$ where F_N is a subset of $\mathbb{C} \setminus \mathbb{R}^+$ defined by

$$F_N = \left\{ z \in \mathbb{C} - \mathbb{R}^+, \frac{1}{L^{\min(1,\gamma)}}P_1(|z|)P_2\left(\frac{1}{\delta_z}\right) \leq \kappa \right\} \quad (6.16)$$

for some nice constant κ . Using the same arguments as in Section 5, we obtain that for each $z \in F_N$, the series $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B},mp}}^{(n)}(\mathbf{Y}_1(z))$ is convergent. Proposition 1.3 implies that for each positive matrix \mathbf{Y} , $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B},mp}}^{(n)}(\mathbf{Y}) < +\infty$ and that for each matrix \mathbf{X} , the series $\sum_{n=0}^{+\infty} \Phi_{\mathcal{B},2}^{(n)}(\mathbf{X})$ is convergent if $z \in F_N$. Therefore, (6.9) holds true for $z \in F_N$, and

$$\left\| \sum_{n=0}^{+\infty} \Phi_{\mathcal{B},2}^{(n)}(\Delta_{\mathcal{B},mp}(z)) \right\| \leq \|\Delta_{\mathcal{B},mp}(z)\| \left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B},mp}}^{(n)}(\mathbf{I}_{ML}) \right\|^{1/2} \left\| \sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B}}}^{(n)}(\mathbf{I}_{ML}) \right\|^{1/2}.$$

It is easy to check that $\sum_{n=0}^{+\infty} \Phi_{\mathbf{T}_{\mathcal{B},mp}}^{(n)}(\mathbf{I}_{ML}) < C(z)\mathbf{I}_{ML}$ for $z \in F_N$. Therefore, we obtain that $\|\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}$ for each $z \in F_N$. It remains to evaluate $\|\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)\|$ if z does not belong to F_N . For this, we remark that $\|\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)\| \leq \|\mathbf{t}_{\mathcal{B}}(z)\| + \|\mathbf{T}_{\mathcal{B}}(z)\| \leq C(z)$. As z does not belong to F_N , the inequality $1 \leq \frac{C(z)}{L^{\min(1,\gamma)}}$ holds for a certain $C(z)$, from which we deduce that $\|\mathbf{t}_{\mathcal{B}}(z) - \mathbf{T}_{\mathcal{B}}(z)\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}$ as expected. Since the matrix $\mathcal{B}_L^{-1/2}$ verifies $\mathcal{B}_L^{-1/2} > \frac{1}{\sqrt{s_{min}}}\mathbf{I}_{ML}$, we obtain (1.22) for each $z \in \mathbb{C} \setminus \mathbb{R}^+$.

It remains to justify (1.23). For this, we remark that

$$\left| \frac{1}{ML} \text{Tr}(t(z)\mathbf{I}_{ML} - \mathbf{T}(z)) \right| \leq \|t(z)\mathbf{I}_{ML} - \mathbf{T}(z)\| \leq \frac{C(z)}{L^{\min(\gamma,1)}}$$

where $\gamma < \gamma_0$, $\gamma \neq 1$. An easy generalization of Theorem 6.2 in [11] (see e.g. [4]) thus implies that if ϕ is a compactly supported C^∞ function, we have

$$\lim_{\epsilon \rightarrow 0} \left| \int_{\mathbb{R}^+} \phi(\lambda) \left(\frac{1}{ML} \text{Tr}(t(\lambda + i\epsilon)\mathbf{I}_{ML} - \mathbf{T}(\lambda + i\epsilon)) \right) d\lambda \right| \leq \frac{\kappa}{L^{\min(\gamma,1)}}.$$

The inequality in (1.23) thus follows from the Stieltjes transform inversion formula

$$\int_{\mathbb{R}^+} \phi(\lambda) (d\mu_N(\lambda) - d\mu_{mp,c}(\lambda)) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \text{Im} \left(\int_{\mathbb{R}^+} \phi(\lambda) \left(\frac{1}{ML} \text{Tr}(t(\lambda + i\epsilon)\mathbf{I}_{ML} - \mathbf{T}(\lambda + i\epsilon)) \right) d\lambda \right).$$

A Proof of Lemma 2.2

A classical calculation (see e.g. Theorem 4.3.2 in [3] in the non Gaussian case) leads to

$$\mathbb{E}|\xi_{L,y_m}(\nu)|^2 = \sum_{l=-(L-2)}^{L-2} (1 - |l|/L)r_m(l)e^{-2i\pi l\nu}.$$

Taking into account that $\mathcal{S}_m(\nu) = \sum_l r_m(l)e^{-2i\pi l\nu}$, we obtain immediately that

$$\mathbb{E}|\xi_{L,y_m}(\nu)|^2 = \mathcal{S}_m(\nu) + \epsilon_{m,L}(\nu)$$

where $\epsilon_{m,L}(\nu)$ is defined by

$$\epsilon_{m,L}(\nu) = - \sum_{|l| \geq L-1} r_m(l) e^{-2i\pi l \nu} - \frac{1}{L} \sum_{-(L-2)}^{L-2} |l| r_m(l) e^{-2i\pi l \nu}.$$

It is clear that

$$|\epsilon_{m,L}(\nu)| \leq \sum_{|l| \geq L-1} |r_m(l)| + \frac{1}{L} \sum_{-(L-2)}^{L-2} |l| |r_m(l)|.$$

Using the bound in (1.11) we directly obtain an upper bound of the first term, namely

$$\sum_{|l| \geq L-1} |r_m(l)| \leq \frac{\kappa}{(L-1)^{\gamma_0}}.$$

If $\gamma_0 \geq 1$, $\sum_{-(L-2)}^{L-2} |l| |r_m(l)| \leq \|r_m\|_{\omega_0}$ and it holds that $\frac{1}{L} \sum_{-(L-2)}^{L-2} |l| |r_m(l)| \leq \frac{\kappa}{L}$. Therefore, if $\gamma_0 \geq 1$, we obtain that

$$|\epsilon_{m,L}(\nu)| \leq \frac{\kappa}{L}.$$

If $\gamma_0 < 1$, we equivalently have

$$\sum_{-(L-2)}^{L-2} |l| |r_m(l)| \leq L^{1-\gamma_0} \|r_m\|_{\omega_0}.$$

Therefore, the inequality

$$\frac{1}{L} \sum_{-(L-2)}^{L-2} |l| |r_m(l)| \leq \frac{\kappa}{(L-1)^{\gamma_0}}$$

holds, as well as

$$|\epsilon_{m,L}(\nu)| \leq \frac{\kappa}{L^{\gamma_0}}.$$

This completes the proof of Lemma 2.2.

B Proof of Lemma 6.1

The proof of Lemma 6.1 follows from the observation that the term $\mathbf{a}_L^H(\nu) \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu)$ can be expressed in terms of the Szegő orthogonal polynomials associated to the scalar product

$$\langle z^k, z^l \rangle = \int_0^1 \mathcal{S}_m(\nu) e^{2i\pi(k-l)\nu} d\nu. \quad (\text{B.1})$$

For each integer l , we introduce the monic orthogonal polynomial $\Phi_l(z)$ defined by

$$\Phi_l^{(m)}(z) = z^l - z^l \text{sp}(1, z, \dots, z^{l-1}) \quad (\text{B.2})$$

where the symbol $|A$ stands for the orthogonal projection over the space A in the sense of the scalar product (B.1). We denote by $\sigma_l^{2,m}$ the norm square of $\Phi_l^{(m)}$, and define for each l the normalized orthogonal polynomial $\phi_l^{(m)}(z)$ by

$$\phi_l^{(m)}(z) = \frac{\Phi_l^{(m)}(z)}{\sigma_l^m}. \quad (\text{B.3})$$

It is well known that the sequence $(\sigma_l^{2,m})_{l \geq 0}$ is decreasing, that $\sigma_0^{2,m} = r_m(0)$, and that $\lim_{l \rightarrow +\infty} \sigma_l^{2,m} = \sigma^{2,m}$ coincides with $\exp \int_0^1 \log \mathcal{S}_m(\nu) d\nu$. It is clear that the normalized orthogonal polynomials satisfy

$$\langle \phi_l^{(m)}, \phi_{l'}^{(m)} \rangle = \int_0^1 \phi_l^{(m)}(e^{2i\pi\nu}) \left(\phi_{l'}^{(m)}(e^{2i\pi\nu}) \right)^* \mathcal{S}_m(\nu) d\nu = \delta_{l-l'}.$$

In the following, we also denote by $\Phi_l^{(m)*}(z)$ and $\phi_l^{(m)*}(z)$ the degree l polynomials defined by

$$\Phi_l^{(m)*}(z) = z^l \left(\Phi_l^{(m)}(z^{-*}) \right)^*, \quad \phi_l^{(m)*}(z) = z^l \left(\phi_l^{(m)}(z^{-*}) \right)^*.$$

Noting that Φ_l is for each l a monic polynomial, it is clear that $\Phi_l^{(m)*}(z)$ can be written as

$$\Phi_l^{(m)*}(z) = 1 + \sum_{k=1}^l a_{k,l}^{(m)} z^k \quad (\text{B.4})$$

for some coefficients $(a_{k,l}^{(m)})_{k=1,\dots,l}$. Moreover, $\Phi_l^{(m)*}(z)$ coincides with

$$\Phi_l^{(m)*}(z) = 1 - 1|\text{sp}(z, z^2, \dots, z^l)$$

and the l -dimensional vector $\mathbf{a}_l^{(m)} = (a_{1,l}^{(m)}, \dots, a_{l,l}^{(m)})^T$ is given by

$$\begin{pmatrix} 1 \\ \mathbf{a}_l^{(m)} \end{pmatrix} = \sigma_l^{2,m} \mathcal{R}_{m,l+1}^{-T} \mathbf{e}_1 \quad (\text{B.5})$$

where \mathbf{e}_1 is the $l+1$ -dimensional vector $\mathbf{e}_1 = (1, 0, \dots, 0)^T$. It is moreover easily checked that

$$y_{m,n} - y_{m,n}|\text{sp}(y_{m,n-1}, \dots, y_{m,n-l}) = y_{m,n} + \sum_{k=1}^l a_{k,l}^{(m)*} y_{m,n-k} \quad (\text{B.6})$$

where the orthogonal projection operator is this time defined on the space of all finite second moment complex valued random variables. For more details on these polynomials, we refer the reader to [24] and [10].

The matrix $\mathcal{R}_{m,L}^{-1}$ can be written as

$$\mathcal{R}_{m,L}^{-1} = \mathbf{A}_{m,L} \text{Diag} \left(\frac{1}{\sigma_0^{2,m}}, \dots, \frac{1}{\sigma_{L-1}^{2,m}} \right) \mathbf{A}_{m,L}^H \quad (\text{B.7})$$

where $\mathbf{A}_{m,L}$ is the upper-triangular matrix defined by

$$\mathbf{A}_{m,L} = \begin{pmatrix} 1 & a_{1,1}^{(m)} & a_{2,2}^{(m)} & \dots & a_{L-1,L-1}^{(m)} \\ 0 & 1 & a_{1,2}^{(m)} & \dots & a_{L-2,L-1}^{(m)} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}. \quad (\text{B.8})$$

In order to see this, simply observe that $\mathcal{R}_{m,L} \mathbf{A}_{m,L}$ is lower triangular because of (B.5) and the fact that $\mathcal{R}_{m,l+1}^T = \mathbf{J}_{l+1} \mathcal{R}_{m,l+1} \mathbf{J}_{l+1}$. Since $\mathbf{A}_{m,L}^H$ is also lower triangular, so is the product $\mathbf{A}_{m,L}^H \mathcal{R}_{m,L} \mathbf{A}_{m,L}$. However, matrix $\mathbf{A}_{m,L}^H \mathcal{R}_{m,L} \mathbf{A}_{m,L}$ is also hermitian, which implies that it must be diagonal. Close examination of (B.5) reveals that its diagonal entries are equal to $\sigma_l^{2,m}$ for $l = 0, \dots, L-1$. Inverting the corresponding equation we obtain (B.7).

Using the above decomposition of the matrix $\mathcal{R}_{m,L}^{-1}$ we immediately obtain that

$$\mathbf{a}_L(\nu)^H \mathbf{A}_{m,L} = \frac{1}{\sqrt{L}} \left(1, e^{-2i\pi\nu} \Phi_1^{(m)*}(e^{2i\pi\nu}), \dots, e^{-2i\pi(L-1)\nu} \Phi_{L-1}^{(m)*}(e^{2i\pi\nu}) \right)$$

and consequently

$$\mathbf{a}_L(\nu)^H \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu) = \frac{1}{L} \sum_{l=0}^{L-1} |\phi_l^{(m)*}(e^{2i\pi\nu})|^2. \quad (\text{B.9})$$

We first explain informally why, for each m , $\mathcal{S}_m(\nu)\mathbf{a}_L(\nu)^H \mathcal{R}_{m,L}^{-1} \mathbf{a}_L(\nu) - 1$ converges uniformly towards 0. For this, we need to recall certain results that are summarized next.

Since the spectral densities $\mathcal{S}_m(\nu)$ are uniformly bounded from below, we can define the cepstrum coefficients $(c_m(k))_{k \in \mathbb{Z}}$, namely

$$c_m(k) = \int_0^1 \log \mathcal{S}_m(\nu) e^{2\pi i \nu k} d\nu.$$

We notice that $\lim_{l \rightarrow +\infty} \sigma_l^{2,m} = \sigma^{2,m}$ coincides with $\exp c_m(0)$. Assumption 4 and a generalization of the Wiener-Lévy theorem (see e.g. [24]) implies that for each m , $c_m \in \ell_\omega$ for each $\gamma \leq \gamma_0$. We define the function $\pi^{(m)}(z)$ given by

$$\pi^{(m)}(z) = \exp - \left(c_m(0)/2 + \sum_{n=1}^{+\infty} c_m(-n)z^n \right).$$

Then, $\pi^{(m)}(z)$ and $\psi^{(m)}(z) = \frac{1}{\pi^{(m)}(z)}$ are analytic in the open unit disk \mathbb{D} and continuous on the closed unit disk. In the following, we denote by $\pi^{(m)}(z) = \sum_{n=0}^{+\infty} \pi^{(m)}(n)z^n$ and $\psi^{(m)}(z) = \sum_{n=0}^{+\infty} \psi^{(m)}(n)z^n$ their expansion in \mathbb{D} . Moreover, functions $\nu \rightarrow \pi^{(m)}(e^{2i\pi\nu})$ and $\nu \rightarrow \psi^{(m)}(e^{2i\pi\nu})$ also belong to ℓ_{ω_0} . To check this, we denote by $(\tilde{c}_m(n))_{n \geq 0}$ the one-sided sequence defined by $\tilde{c}_m(0) = c_m(0)/2$ and $\tilde{c}_m(n) = c_m(-n)$ for $n \geq 1$. Then, the sequences $\pi^{(m)}$ and $\psi^{(m)}$ can be written as

$$\pi^{(m)} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} (\tilde{c}_m)^{*k}, \quad \psi^{(m)} = \sum_{k=0}^{+\infty} \frac{1}{k!} (\tilde{c}_m)^{*k}$$

where for a sequence a , $a^{*(k)}$ represents $\underbrace{a * a * \dots * a}_k$. Observe, in particular, that both sequences are one-sided. Now, for each $\gamma \leq \gamma_0$, it holds that

$$\|\pi^{(m)}\|_\omega \leq \sum_{k=0}^{+\infty} \frac{1}{k!} \|\tilde{c}_m\|_\omega^k = \exp(\|\tilde{c}_m\|_\omega) \leq \exp(\|c_m\|_\omega) \quad (\text{B.10})$$

$$\|\psi^{(m)}\|_\omega \leq \sum_{k=0}^{+\infty} \frac{1}{k!} \|\tilde{c}_m\|_\omega^k = \exp(\|\tilde{c}_m\|_\omega) \leq \exp(\|c_m\|_\omega). \quad (\text{B.11})$$

In the following, we also need a version of (B.10, B.11) holding uniformly w.r.t. m . For this, we establish the following lemma, which can be seen as a uniform version of the generalized Wiener-Lévy theorem.

Lemma B.1. *Consider a function $F(z)$ holomorphic in a neighbourhood of the interval $[s_{\min}, s_{\max}]$ where s_{\min} and s_{\max} are defined in Assumption 2. Then, for each $\gamma < \gamma_0$ and for each m , the function $F \circ \mathcal{S}_m$ belongs to ℓ_ω and ⁴*

$$\sup_{m \geq 1} \|F \circ \mathcal{S}_m\|_\omega < +\infty. \quad (\text{B.12})$$

Proof. We adapt the proof of the Wiener-Lévy theorem in [29] (Theorem 5.2, p. 245). We first claim that if p is an integer such that $p > 1 + \gamma_0$ and if $G(\nu) = \sum_{n \in \mathbb{Z}} g(n)e^{2i\pi n\nu}$ belongs to \mathcal{C}_p , then, $g \in \ell_{\omega_0}$, and

$$\|g\|_{\omega_0} \leq \kappa \left(\sup_\nu |G(\nu)| + \sup_\nu |G^{(p)}(\nu)| \right) \quad (\text{B.13})$$

for some constant κ depending only on γ_0 . To verify (B.13), we remark that $|G(0)| \leq \sup_\nu |G(\nu)|$. Moreover, for each $n \neq 0$, the integration by parts formula leads to

$$g(n) = \frac{1}{(2i\pi n)^p} \int_0^1 G^{(p)}(\nu) e^{-2i\pi n\nu} d\nu$$

⁴We make the slight abuse of notation by identifying the ω -norm of a function on the unit circle as the corresponding norm of its Fourier coefficient sequence.

and to $|g(n)| \leq \frac{1}{(2\pi)^p} \frac{1}{|n|^p} \sup_\nu |G^{(p)}(\nu)|$. As $p > 1 + \gamma_0$, we obtain immediately that (B.13) holds.

Since F is holomorphic in a neighbourhood of $[s_{min}, s_{max}]$, there exists a $\rho > 0$ for which F is holomorphic in the open disk $\mathbb{D}(s, 2\rho)$ for each $s \in [s_{min}, s_{max}]$. In particular, for each m and each ν , F is holomorphic in $\mathbb{D}(\mathcal{S}_m(\nu), 2\rho)$. We consider a partial sum $\mathcal{S}_{m,n_0}(\nu) = \sum_{k=-n_0}^{n_0} r_m(k) e^{-2i\pi k\nu}$, and claim that for each $\gamma < \gamma_0$, we have

$$\|\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu)\|_\omega = \sum_{|k| \geq (n_0+1)} (1 + |k|)^\gamma |r_m(k)| \leq \frac{\kappa}{n_0^{\gamma_0 - \gamma}} \quad (\text{B.14})$$

for some nice constant κ . To justify (B.14), we remark that

$$\|r_m\|_{\omega_0} \geq \sum_{|k| \geq (n_0+1)} (1 + |k|)^{\gamma_0} |r_m(k)| \geq n_0^{\gamma_0 - \gamma} \sum_{|k| \geq (n_0+1)} (1 + |k|)^\gamma |r_m(k)| = n_0^{\gamma_0 - \gamma} \|\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu)\|_\omega.$$

Assumption 4 implies that $\sup_m \|r_m\|_{\omega_0} < +\infty$. This leads immediately to (B.14). We choose n_0 in such a way that $\frac{\kappa}{n_0^{\gamma_0 - \gamma}} \leq \frac{\rho}{2}$, and notice that (B.14) leads to $\sup_\nu |\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu)| \leq \frac{\rho}{2}$ for each m . Therefore, the circle $\mathbb{C}(\mathcal{S}_{m,n_0}(\nu), \rho)$ with center $\mathcal{S}_{m,n_0}(\nu)$ and radius ρ is included into $\mathbb{D}(\mathcal{S}_m(\nu), 2\rho)$, and $\mathcal{S}_m(\nu)$ belongs to the disk $\mathbb{D}(\mathcal{S}_{m,n_0}(\nu), \rho)$. The Cauchy formula implies that

$$(F \circ \mathcal{S}_m)(\nu) = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(\mathcal{S}_{m,n_0}(\nu) + \rho e^{i\theta})}{\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu) - \rho e^{i\theta}} \rho e^{i\theta} d\theta. \quad (\text{B.15})$$

Since $|\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu)| \leq \frac{\rho}{2}$, it holds that

$$\frac{\rho e^{i\theta}}{\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu) - \rho e^{i\theta}} = - \sum_{k=0}^{+\infty} \rho^{-k} e^{-ik\theta} (\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu))^k$$

and that

$$\left\| \frac{\rho e^{i\theta}}{\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu) - \rho e^{i\theta}} \right\|_\omega \leq \sum_{k=0}^{+\infty} \rho^{-k} \|\mathcal{S}_m - \mathcal{S}_{m,n_0}\|_\omega^k \leq 2.$$

Using (B.13), it is easy to check that $G_m(\nu, \theta)$ defined by $G_m(\nu, \theta) = F(\mathcal{S}_{m,n_0}(\nu) + \rho e^{i\theta})$ verifies

$$\sup_{m, \theta, \nu} \|G_m(\nu, \theta)\|_\omega \leq \kappa$$

for each $\gamma \leq \gamma_0$ for some nice constant κ . We thus obtain that for some nice constant κ , it holds that

$$\left\| \frac{F(\mathcal{S}_{m,n_0}(\nu) + \rho e^{i\theta})}{\mathcal{S}_m(\nu) - \mathcal{S}_{m,n_0}(\nu) - \rho e^{i\theta}} \rho e^{i\theta} \right\|_\omega \leq \kappa$$

for each $\gamma < \gamma_0$, each m and each θ . (B.15) thus implies (B.12). The proof of Lemma B.1 is thus complete. \square

The use of Lemma B.1 for $f(x) = \log x$ shows that

$$\sup_m \|c_m\|_\omega < +\infty \quad (\text{B.16})$$

for each $\gamma < \gamma_0$. Therefore, (B.10, B.11) imply that

$$\sup_m \|\pi^{(m)}\|_\omega \leq \kappa, \quad \sup_m \|\psi^{(m)}\|_\omega \leq \kappa. \quad (\text{B.17})$$

It also holds that $\mathcal{S}_m(\nu) = \left| \psi^{(m)}(e^{2i\pi\nu}) \right|^2$ and therefore $\psi^{(m)}(z)$ coincides with the so-called outer spectral factor of \mathcal{S}_m in the sense that both $\psi^{(m)}(z)$ and $\frac{1}{\psi^{(m)}(z)} = \pi^{(m)}(z)$ are analytic in the unit disc. Theorem 5.1.8 in [24] leads to the conclusion that $\|\phi_l^{(m)*} - \pi^{(m)}\|_\omega \rightarrow 0$ when $l \rightarrow +\infty$, a result which implies that

$$\sup_\nu \left| \phi_l^{(m)*}(e^{2i\pi\nu}) - \pi^{(m)}(e^{2i\pi\nu}) \right| \rightarrow 0. \quad (\text{B.18})$$

Given the fact that $\mathcal{S}_m(\nu) = \left| \frac{1}{\pi^{(m)}(e^{2i\pi\nu})} \right|^2$, (1.7) and (1.8) allow us to conclude that

$$0 < \inf_m \inf_\nu |\pi^{(m)}(e^{2i\pi\nu})| \leq \sup_m \sup_\nu |\pi^{(m)}(e^{2i\pi\nu})| < +\infty. \quad (\text{B.19})$$

Therefore, (B.18) leads to $\sup_\nu \left| \frac{1}{\pi^{(m)}(e^{2i\pi\nu})} \phi_l^{(m)*}(e^{2i\pi\nu}) - 1 \right| \rightarrow 0$, and to $\sup_\nu \left| \left| \frac{1}{\pi^{(m)}(e^{2i\pi\nu})} \right|^2 |\phi_l^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \rightarrow 0$, or equivalently, to

$$\sup_\nu \left| \mathcal{S}_m(\nu) |\phi_l^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \rightarrow 0. \quad (\text{B.20})$$

This, in turn, implies that

$$\sup_\nu \left| \mathcal{S}_m(\nu) \frac{1}{L} \sum_{l=1}^L |\phi_l^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \rightarrow 0 \quad (\text{B.21})$$

when $L \rightarrow +\infty$ as expected. In order to complete the proof of Lemma 6.1, we have thus to prove that (B.21) holds uniformly w.r.t. m , and to evaluate the rate of convergence. For this, we can follow the proof of Theorem 5.1.8 in [24], adapting the corresponding arguments to our particular context.

Theorem 5.1.8 in [24] follows from general results concerning Wiener-Hopf operators defined on the Wiener algebra ℓ_1 . As explained below, we will show that $\sup_m \|\phi_l^{(m)*} - \pi^{(m)}\|_1 \rightarrow 0$, and will only use that $\sup_m \|r_m\|_\omega < +\infty$ and $\sup_m \|c_m\|_\omega < +\infty$ for each $\gamma < \gamma_0$ in order to obtain an upper bound of the above term. In the following, we denote by $C^{(m)}$ the operator defined on the Wiener algebra ℓ_1 by

$$C^{(m)}a = \bar{r}_m * a$$

where \bar{r}_m is the sequence defined by $\bar{r}_m(n) = r_m(-n)$ for each $n \in \mathbb{Z}$. $C^{(m)}$ can alternatively be defined in the Fourier transform domain as the multiplication operator

$$\sum_{n \in \mathbb{Z}} a(n) e^{2i\pi n \nu} \rightarrow \mathcal{S}_m(\nu) \sum_{n \in \mathbb{Z}} a(n) e^{2i\pi n \nu}.$$

It is well known that $\|C^{(m)}\|_1 = \|\bar{r}_m\|_1 = \|r_m\|_1$. As $\mathcal{S}_m(\nu) = |\psi^{(m)}(e^{2i\pi\nu})|^2$, the operator $C^{(m)}$ can be factorized as $C^{(m)} = L^{(m)}U^{(m)} = U^{(m)}L^{(m)}$ where $U^{(m)}$ and $L^{(m)}$ represent the multiplication operators by $\psi^{(m)}(e^{2i\pi\nu})$ and $(\psi^{(m)}(e^{2i\pi\nu}))^*$ defined on ℓ_1 respectively. We denote by P_+ the projection operator defined on ℓ_1 by

$$P_+(\{a(n), n \in \mathbb{Z}\}) = \{a(n), n \geq 0\}$$

or equivalently in the Fourier transform domain by

$$P_+ \left(\sum_{n \in \mathbb{Z}} a(n) e^{2i\pi n \nu} \right) = \sum_{n=0}^{+\infty} a(n) e^{2i\pi n \nu}.$$

The operator P_- is defined by $P_- = I - P_+$. The operator $U^{(m)}$ is called upper triangular in the sense that $P_-U^{(m)}P_+ = 0$ while $L^{(m)}$ is lower triangular because $P_+L^{(m)}P_- = 0$. Moreover, as $\pi^{(m)} = \frac{1}{\psi^{(m)}}$ belongs to ℓ_1 and $\pi^{(m)}(e^{2i\pi\nu}) = \sum_{n=0}^{+\infty} \pi^{(m)}(n) e^{2i\pi n \nu}$, the operators $U^{(m)}$ and $L^{(m)}$ are invertible, and $(U^{(m)})^{-1}$ and $(L^{(m)})^{-1}$ are upper triangular and lower triangular respectively. In the Fourier domain, $(U^{(m)})^{-1}$ and $(L^{(m)})^{-1}$ correspond respectively to the multiplication operator by $\pi^{(m)}(e^{2i\pi\nu})$ and $(\pi^{(m)}(e^{2i\pi\nu}))^*$. These properties imply that the factorization $C^{(m)} = L^{(m)}U^{(m)} = U^{(m)}L^{(m)}$ is a Wiener-Hopf factorization. In the following, we denote by $T^{(m)}$ the Toeplitz operator defined on ℓ_1 by

$$T^{(m)} = P_+C^{(m)}P_+. \quad (\text{B.22})$$

It is clear that if $j \geq 0$ and if δ_j is the sequence δ_j defined by $\delta_j(n) = \delta_{n-j}$, then, $\langle \delta_i, T^{(m)}\delta_j \rangle$, defined as $(T^{(m)}\delta_j)(i)$, is equal to $r_m(j-i)$. Therefore, the matrix representation of $T^{(m)}$ in the basis $(\delta_j)_{j \geq 0}$ is the infinite matrix $\mathcal{R}_{m,\infty}^T$. Theorem 5.1.1 in [24] implies that, considered as an operator defined on $\text{Range}(P_+)$, $T^{(m)}$ is invertible, i.e. that for each $a \in \text{Range}(P_+)$, there exists a unique $b \in \text{Range}(P_+)$ such that $T^{(m)}b = a$.

$(T^{(m)})^{-1}b$ is of course defined as a . If an element a does not belong to $\text{Range}(P_+)$, $(T^{(m)})^{-1}a$ is defined as $(T^{(m)})^{-1}P_+a$. We also notice that $(T^{(m)})^{-1} = P_+(U^{(m)})^{-1}P_+(L^{(m)})^{-1}P_+$. For each $n \geq 1$, we denote by Q_n the projection operator defined by

$$Q_n(\{a(l), l \in \mathbb{Z}\}) = \{a(l), 0 \leq l \leq n\} \quad (\text{B.23})$$

or equivalently by

$$Q_n \left(\sum_{l \in \mathbb{Z}} a(l) e^{2i\pi l \nu} \right) = \sum_{l=0}^n a(l) e^{2i\pi l \nu}.$$

We also introduce the truncated Toeplitz operator $T_n^{(m)}$ defined by

$$T_n^{(m)} = Q_n C^{(m)} Q_n = Q_n T^{(m)} Q_n. \quad (\text{B.24})$$

We note that in the basis $(\delta_j)_{j=0, \dots, n}$, the matrix representation of $T_n^{(m)}$ is the matrix $\mathcal{R}_{m, n+1}^T$. We now introduce the projection operator R_n defined by $R_n = P_+ - Q_n$, and state the following Lemma which appears as an immediate consequence of Theorem 5.1.2 and Theorem 5.1.3 in [24].

Lemma B.2. *For each $n \geq 0$, it holds that $R_n L^{(m)} Q_n = R_n L^{-(m)} Q_n = Q_n U^{(m)} R_n = Q_n U^{-(m)} R_n = 0$. Moreover, there exists an integer n_0 independent of m such that for each $n \geq n_0$, $T_n^{(m)}$, considered as an operator defined on $\text{Range}(Q_n)$, is invertible, in the sense that for each $a \in \text{Range}(Q_n)$, it exists a unique $b \in \text{Range}(Q_n)$, defined as $(T_n^{(m)})^{-1}a$, such that $T_n^{(m)}b = a$. If $a \in \text{Range}(P_+)$, $(T_n^{(m)})^{-1}a$ is defined as $(T_n^{(m)})^{-1}a = (T_n^{(m)})^{-1}Q_n a$. Moreover, there exists a nice constant α such that, for each $n \geq n_0$ and each $a \in \text{Range}(P_+)$, the inequality*

$$\left\| (T_n^{(m)})^{-1} a \right\|_1 \leq \alpha \|a\|_1 \quad (\text{B.25})$$

holds.

Proof. We just verify that $R_n L^{(m)} Q_n = 0$, and omit the proof of the three other identities. For this, we have just to check that if $a(e^{2i\pi\nu}) = \sum_{l=0}^n a(n) e^{2i\pi l \nu}$, then $(\psi^{(m)}(e^{2i\pi\nu}))^* a(e^{2i\pi\nu})$ can be written as

$$(\psi^{(m)}(e^{2i\pi\nu}))^* a(e^{2i\pi\nu}) = \sum_{l=-\infty}^n b(l) e^{2i\pi l \nu}$$

for some coefficients $(b(l))_{l=-\infty, \dots, n}$. This, of course, holds true because $(\psi^{(m)}(e^{2i\pi\nu}))^* = \sum_{l=0}^{\infty} (\psi^{(m)}(l))^* e^{-2i\pi l \nu}$.

In order to be able to use Theorem 5.1.2 in [24], we establish that it exists an integer n_0 such that $\|P_-(L^{(m)})^{-1}R_n U^{(m)}\|_1 \leq \frac{1}{2}$ and $\|R_n(U^{(m)})^{-1}P_-L^{(m)}\|_1 \leq \frac{1}{2}$ for each $n \geq n_0$ and for each m . If $a \in \ell_1$, we evaluate $P_-(L^{(m)})^{-1}R_n U^{(m)}a$ in the Fourier transform domain, and denote $x_n^{(m)}(e^{2i\pi\nu})$ the function defined by $x_n^{(m)}(e^{2i\pi\nu}) = R_n \psi^{(m)}(e^{2i\pi\nu}) a(e^{2i\pi\nu})$, which, of course, can be written as $x_n^{(m)}(e^{2i\pi\nu}) = \sum_{l=n+1}^{+\infty} x_n^{(m)}(l) e^{2i\pi l \nu}$. The operation of $(L^{(m)})^{-1}$ is equivalent to the multiplication by $(\pi^{(m)}(e^{2i\pi\nu}))^*$ in the Fourier transform domain, which is associated to a left-sided series. Therefore,

$$P_- \left(\pi^{(m)}(e^{2i\pi\nu}) \right)^* x_n^{(m)}(e^{2i\pi\nu}) = P_- \left[\sum_{l=n+1}^{+\infty} \left(\pi^{(m)}(l) \right)^* e^{-2i\pi l \nu} x_n^{(m)}(e^{2i\pi\nu}) \right].$$

The norm of the right hand side can be bounded as

$$\left\| P_- \left[\sum_{l=n+1}^{+\infty} \left(\pi^{(m)}(l) \right)^* e^{-2i\pi l \nu} x_n^{(m)}(e^{2i\pi\nu}) \right] \right\|_1 \leq \left\| \sum_{l=n+1}^{+\infty} \left(\pi^{(m)}(l) \right)^* e^{-2i\pi l \nu} \right\|_1 \|\psi^{(m)}\|_1 \|a\|_1$$

or equivalently,

$$\left\| P_-(L^{(m)})^{-1}R_n U^{(m)} \right\|_1 \leq \left(\sum_{l=n+1}^{+\infty} |\pi^{(m)}(l)| \right) \|\psi^{(m)}\|_1.$$

The bound in (B.17) implies that $\sup_m \|\psi^{(m)}\|_1 \leq \kappa$ and that $\sup_m \|\pi^{(m)}\|_\omega \leq \kappa$ for some nice constant κ . It is therefore clear that for each $\gamma < \gamma_0$ and for each m , we have

$$\kappa \geq \|\pi^{(m)}\|_\omega \geq \sum_{l=n+1}^{+\infty} (1+l)^\gamma |\pi^{(m)}(l)| \geq (1+n)^\gamma \sum_{l=n+1}^{+\infty} |\pi^{(m)}(l)|.$$

We conclude from this that

$$\sum_{l=n+1}^{+\infty} |\pi^{(m)}(l)| \leq \frac{\kappa}{n^\gamma} \quad (\text{B.26})$$

and therefore

$$\left\| P_-(L^{(m)})^{-1} R_n U^{(m)} \right\|_1 \leq \frac{\kappa}{n^\gamma} \quad (\text{B.27})$$

for some nice constant κ . It can be shown similarly that

$$\left\| R_n (U^{(m)})^{-1} P_- L^{(m)} \right\|_1 \leq \frac{\kappa}{n^\gamma}. \quad (\text{B.28})$$

This implies that it exists an integer n_0 such that $\|P_-(L^{(m)})^{-1} R_n U^{(m)}\|_1 \leq \frac{1}{2}$ and $\|R_n (U^{(m)})^{-1} P_- L^{(m)}\|_1 \leq \frac{1}{2}$ for each $n \geq n_0$ and for each m . Therefore, Theorem 5.1.2 in [24] implies that for each $n \geq n_0$ and for each m , it holds that $T_n^{(m)}$ is invertible and that for each $a \in \text{Range}(Q_n)$, it holds that $\|(T_n^{(m)})^{-1} a\| \leq \alpha_{m,n} \|a\|_1$ where $\alpha_{m,n}$ is given by

$$\begin{aligned} \alpha_{m,n} = & \left\| (L^{(m)})^{-1} (U^{(m)})^{-1} \right\|_1 \\ & + 2 \max \left(\left\| (U^{(m)})^{-1} \right\|_1, \left\| (L^{(m)})^{-1} \right\|_1 \right) \left(\left\| P_-(L^{(m)})^{-1} \right\|_1 + \left\| R_n (U^{(m)})^{-1} \right\|_1 \right). \end{aligned}$$

The bounds in (B.17) imply that for each m and n , $\alpha_{m,n} \leq \alpha$ for some nice constant α . Therefore, $\|(T_n^{(m)})^{-1} a\| \leq \alpha \|a\|_1$ for each $n \geq n_0$, for each m , and for each $a \in \text{Range}(Q_n)$. If $a \in \text{Range}(P_+)$, $(T_n^{(m)})^{-1} a$ is equal to $(T_n^{(m)})^{-1} Q_n a$. Therefore, $\|(T_n^{(m)})^{-1} a\|_1 \leq \alpha \|Q_n a\|_1 \leq \alpha \|a\|_1$. This completes the proof of the lemma. \square

Lemma B.2 and Theorem 5.1.3 in [24] imply the following corollary.

Corollary B.1. *For each integer m and for each $a \in \text{Range}(P_+)$, it holds that*

$$\lim_{n \rightarrow +\infty} \left\| (T_n^{(m)})^{-1} a - (T^{(m)})^{-1} a \right\|_1 = 0. \quad (\text{B.29})$$

Proof. (B.25) implies that $T_n^{(m)}$ is invertible for each $n \geq n_0$. We use the observation that $(T_n^{(m)})^{-1} T_n^{(m)} = Q_n$. Therefore, the operator $(T_n^{(m)})^{-1} - (T^{(m)})^{-1}$ can be written as

$$(T_n^{(m)})^{-1} - (T^{(m)})^{-1} = (T_n^{(m)})^{-1} \left(T^{(m)} - T_n^{(m)} \right) (T^{(m)})^{-1} + (Q_n - I) (T^{(m)})^{-1}.$$

We conclude from this and (B.25) that for each $n \geq n_0$, it holds that

$$\left\| (T_n^{(m)})^{-1} a - (T^{(m)})^{-1} a \right\|_1 \leq \alpha \left\| (T^{(m)} - T_n^{(m)}) (T^{(m)})^{-1} a \right\|_1 + \left\| (T^{(m)})^{-1} a - Q_n (T^{(m)})^{-1} a \right\|_1. \quad (\text{B.30})$$

It is clear that $\left\| (T^{(m)})^{-1} a - Q_n (T^{(m)})^{-1} a \right\|_1 \rightarrow 0$ when $n \rightarrow +\infty$. Moreover, for each $b \in \text{Range}(P_+)$, $(T^{(m)} - T_n^{(m)}) b$ can be expressed as

$$(T^{(m)} - T_n^{(m)}) b = - \left(Q_n C^{(m)} (Q_n - P_+) b + (Q_n - P_+) C^{(m)} P_+ b \right). \quad (\text{B.31})$$

From this, we obtain immediately that for each m , $\left\| (T^{(m)} - T_n^{(m)}) b \right\|_1 \rightarrow 0$ when $n \rightarrow +\infty$. Taking $b = (T^{(m)})^{-1} a$ leads to (B.29). \square

Corollary B.1 implies that for each m , $\|(T_n^{(m)})^{-1}\delta_0 - (T^{(m)})^{-1}\delta_0\|_1$ converges towards 0 when $n \rightarrow +\infty$. Since the matrix representation of $T_n^{(m)}$ in the basis $(\delta_j)_{j=0,\dots,n}$ coincides with matrix $\mathcal{R}_{m,n+1}^T$, (B.5) implies that $(T_n^{(m)})^{-1}\delta_0$ coincides with the sequence $\frac{1}{\sigma_n^{(m)}}(1, a_{1,n}^{(m)}, \dots, a_{n,n}^{(m)}, 0, \dots)$ whose Fourier transform coincides with $\frac{1}{\sigma_n^{(m)}}\phi_n^{*(m)}(e^{2i\pi\nu})$. Therefore, the Fourier transform of the ℓ_1 sequence $(T^{(m)})^{-1}\delta_0$ is the limit of $\frac{1}{\sigma_n^{(m)}}\phi_n^{*(m)}(e^{2i\pi\nu})$ in the ℓ_1 metric. Theorem 5.1.8 in [24] implies that for each $\gamma < \gamma_0$ and for each m , $\|\phi_n^{(m)*} - \pi^{(m)}\|_\omega \rightarrow 0$, and therefore that $\|\phi_n^{(m)*} - \pi^{(m)}\|_1 \rightarrow 0$ as $n \rightarrow +\infty$. As it is well known that $\sigma_n^m \rightarrow \sigma^m = \exp\frac{c_m(0)}{2}$, this discussion leads to the conclusion that for each m ,

$$(T^{(m)})^{-1}\delta_0 = \frac{1}{\sigma^m} \pi^{(m)}. \quad (\text{B.32})$$

In the following, we establish the following proposition.

Proposition B.1. *If $\gamma < \gamma_0$, there exist an integer n_1 and a nice constant κ such that*

$$\sup_{m \geq 1} \|(T_n^{(m)})^{-1}\delta_0 - (T^{(m)})^{-1}\delta_0\|_1 \leq \frac{\kappa}{n^\gamma} \quad (\text{B.33})$$

for each $n \geq n_1$.

Proof. In order to establish (B.33), we use (B.30) and (B.31) for $a = \delta_0$ and $b = (T^{(m)})^{-1}\delta_0 = \frac{1}{\sigma^m} \pi^{(m)}$. We first evaluate $\|(T^{(m)})^{-1}\delta_0 - Q_n(T^{(m)})^{-1}\delta_0\|_1$, or equivalently $\frac{1}{\sigma^m} \sum_{k=n+1}^{+\infty} |\pi^{(m)}(n)|$. In order to check that $\sup_m \frac{1}{\sigma^m} < +\infty$, we notice that (1.8) implies that $\inf_m c_0(m) > -\infty$, and that $\inf_m \exp\frac{c_0(m)}{2} > 0$. Therefore, it holds that $\sup_m \frac{1}{\sigma^m} < +\infty$. The bound in (B.26) thus implies that for each $n \geq n_0$ and for each m , it holds that

$$\left\| (T^{(m)})^{-1}\delta_0 - Q_n(T^{(m)})^{-1}\delta_0 \right\|_1 \leq \frac{\kappa}{n^\gamma}$$

for some nice constant κ . It remains to control $\|(T^{(m)} - T_n^{(m)})(T^{(m)})^{-1}\delta_0\|_1$. As $\sup_m \frac{1}{\sigma^m} < +\infty$, it is sufficient to study $\|(T^{(m)} - T_n^{(m)})\pi^{(m)}\|_1$. For this, we use (B.31) for $b = \pi^{(m)}$, and obtain that

$$\left\| (T^{(m)} - T_n^{(m)})\pi^{(m)} \right\|_1 \leq \left\| C^{(m)} \right\|_1 \left\| \pi^{(m)} - Q_n\pi^{(m)} \right\|_1 + \left\| (P_+ - Q_n)C^{(m)}\pi^{(m)} \right\|_1. \quad (\text{B.34})$$

The bound in (B.26) implies that the first term of the right hand side of (B.34) is upper bounded by $\frac{\kappa}{n^\gamma}$ for some nice constant κ for each n and each m . The second term of the right hand side of (B.34) is given by

$$\left\| (P_+ - Q_n)C^{(m)}\pi^{(m)} \right\|_1 = \sum_{k=n+1}^{\infty} \left| (C^{(m)}\pi^{(m)})(k) \right|$$

where it holds that

$$(C^{(m)}\pi^{(m)})(k) = \sum_{l=0}^{+\infty} \bar{r}_m(k-l) \pi^{(m)}(l).$$

Therefore,

$$\sum_{k=n+1}^{\infty} \left| (C^{(m)}\pi^{(m)})(k) \right| \leq \sum_{k=n+1}^{+\infty} \sum_{l=0}^{+\infty} |r_m(k-l)| |\pi^{(m)}(l)|.$$

We express the right hand side of the above inequality as

$$\sum_{k=n+1}^{+\infty} \sum_{l=0}^{+k} |r_m(k-l)| |\pi^{(m)}(l)| + \sum_{k=n+1}^{+\infty} \sum_{l=k+1}^{+\infty} |r_m(k-l)| |\pi^{(m)}(l)|$$

or equivalently as

$$\sum_{k=n+1}^{+\infty} \sum_{u+v=k, u \geq 0, v \geq 0} |r_m(u)| |\pi^{(m)}(v)| + \sum_{k=n+1}^{+\infty} \sum_{u+v=k, u \leq -1, v \geq 0} |r_m(u)| |\pi^{(m)}(v)|.$$

It is clear that

$$\sum_{k=n+1}^{+\infty} \sum_{u+v=k, u \geq 0, v \geq 0} |r_m(u)| |\pi^{(m)}(v)| \leq \left(\sum_{l=0}^{+\infty} |\pi^{(m)}(l)| \right) \left(\sum_{k=[(n+1)/2]}^{+\infty} |r_m(k)| \right) + \left(\sum_{k=0}^{+\infty} |r_m(k)| \right) \left(\sum_{l=[(n+1)/2]}^{+\infty} |\pi^{(m)}(l)| \right)$$

and that

$$\sum_{k=n+1}^{+\infty} \sum_{u+v=k, u \leq -1, v \geq 0} |r_m(u)| |\pi^{(m)}(v)| \leq \left(\sum_{k \leq -1} |r_m(k)| \right) \left(\sum_{l=n+1}^{+\infty} |\pi^{(m)}(l)| \right).$$

Using the fact that $\sup_m \|r_m\|_\omega < +\infty$, we obtain, using the same arguments as in (B.26), that

$$\sup_m \sum_{l=n+1}^{+\infty} |\pi^{(m)}(l)| < \frac{\kappa}{n^\gamma}$$

for some nice constant κ . We have thus shown that

$$\sup_m \|(P_+ - Q_n)C^{(m)}\pi^{(m)}\|_1 \leq \frac{\kappa}{n^\gamma}$$

and this completes the proof of Proposition B.1. \square

Proposition B.1 immediately allows to study the behaviour of $\|\phi_n^{*(m)} - \pi^{(m)}\|_1$ when $n \rightarrow +\infty$.

Corollary B.2. *If $\gamma < \gamma_0$, it exists an integer n_2 and a nice constant κ for which*

$$\|\phi_n^{(m)*} - \pi^{(m)}\|_1 \leq \frac{\kappa}{n^\gamma} \quad (\text{B.35})$$

for each $n \geq n_2$ and each m .

Proof. $\phi_n^{(m)*} - \pi^{(m)}$ coincides with $\sigma_n^m (T_n^{(m)})^{-1} \delta_0 - \sigma^m (T^{(m)})^{-1} \delta_0$, which can also be written as

$$\phi_n^{(m)*} - \pi^{(m)} = \sigma_n^m \left((T_n^{(m)})^{-1} \delta_0 - (T^{(m)})^{-1} \delta_0 \right) + (\sigma_n^m - \sigma^m) (T^{(m)})^{-1} \delta_0$$

or equivalently as

$$\phi_n^{(m)*} - \pi^{(m)} = \sigma_n^m \left((T_n^{(m)})^{-1} \delta_0 - (T^{(m)})^{-1} \delta_0 \right) + (\sigma_n^m - \sigma^m) \frac{\pi^{(m)}}{\sigma^m}. \quad (\text{B.36})$$

We notice that $\sigma_n^m = (\langle (T_n^{(m)})^{-1} \delta_0, \delta_0 \rangle)^{-1}$ and that $\sigma^m = (\langle (T^{(m)})^{-1} \delta_0, \delta_0 \rangle)^{-1}$. We express $\sigma_n^m - \sigma^m$ as

$$\sigma_n^m - \sigma^m = \sigma_n^m \sigma^m \left(\frac{1}{\sigma^m} - \frac{1}{\sigma_n^m} \right) = \sigma_n^m \sigma^m \left(\langle (T_n^{(m)})^{-1} \delta_0 - (T^{(m)})^{-1} \delta_0, \delta_0 \rangle \right).$$

Noting that $\sup_{m,n} \sigma_n^m \leq \sup_m r_0(m) < +\infty$, we obtain that for each n large enough and for each m , the inequality

$$\sigma_n^m - \sigma^m \leq \kappa \| (T_n^{(m)})^{-1} \delta_0 - (T^{(m)})^{-1} \delta_0 \|_1 \leq \frac{\kappa}{n^\gamma}$$

holds for some nice constant κ . (B.35) thus follows immediately from Proposition B.1. \square

We finally complete the proof of Lemma 6.1. (B.35) implies that

$$\sup_m \sup_\nu |\phi_n^{(m)*}(e^{2i\pi\nu}) - \pi^{(m)}(e^{2i\pi\nu})| \leq \frac{\kappa}{n^\gamma}$$

for each $n \geq n_2$. Using (B.19) and $\mathcal{S}_m(\nu) = \frac{1}{|\pi^{(m)}(e^{2i\pi\nu})|^2}$, we obtain that

$$\sup_m \sup_\nu \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \leq \frac{\kappa}{n^\gamma} \quad (\text{B.37})$$

for each $n \geq n_2$. We recall that $\epsilon_m(\nu)$ is equal to

$$\epsilon_{m,L}(\nu) = \frac{1}{L} \sum_{n=0}^{L-1} \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1.$$

Therefore,

$$|\epsilon_{m,L}(\nu)| \leq \frac{1}{L} \sum_{n=0}^{L-1} \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right|.$$

We express the right hand side as

$$\frac{1}{L} \sum_{n=0}^{n_2-1} \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| + \frac{1}{L} \sum_{n=n_2}^L \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right|$$

and handle the two terms separately. On the one hand, (B.37) implies that

$$\frac{1}{L} \sum_{n=n_2}^L \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \leq \kappa \frac{1}{L} \sum_{n=n_2}^L \frac{1}{n^\gamma}.$$

If $\gamma > 1$, $\sum_{n=n_2}^L \frac{1}{n^\gamma}$ is a bounded term, and we obtain that

$$\sup_m \sup_\nu \frac{1}{L} \sum_{n=n_2}^L \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \leq \frac{\kappa}{L}.$$

If $\gamma = 1$, the above term is bounded by $\kappa \frac{\log L}{L}$, and if $0 < \gamma < 1$, it holds that

$$\sum_{n=n_2}^L \frac{1}{n^\gamma} \leq \kappa L^{1-\gamma}$$

and that

$$\sup_m \sup_\nu \frac{1}{L} \sum_{n=n_2}^L \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \leq \frac{\kappa}{L^\gamma}.$$

We finally justify that there exists a nice constant κ such that

$$\sup_m \sup_\nu \sum_{n=0}^{n_2-1} \left| \mathcal{S}_m(\nu) |\phi_n^{(m)*}(e^{2i\pi\nu})|^2 - 1 \right| \leq \kappa.$$

Indeed, since n_2 is a fixed integer, we have just to verify that for each $n \leq n_2$, $\sup_m \sup_\nu |\phi_n^{(m)*}(e^{2i\pi\nu})| < +\infty$. For this, we recall that the non normalized polynomials $\Phi_n^{(m)}$ and $\Phi_n^{(m)*}$ verify the relation the well known recursion formula

$$\Phi_{n+1}^{(m)}(z) = z \Phi_n^{(m)}(z) - \alpha_n^{(m)} \Phi_n^{(m)*}(z) \quad (\text{B.38})$$

$$\Phi_{n+1}^{(m)*}(z) = \Phi_n^{(m)*}(z) - \alpha_n^{(m)*} z \Phi_n^{(m)}(z). \quad (\text{B.39})$$

Here, $(\alpha_m(n))_{n \geq 0}$ are the reflection coefficients sequence associated to autocovariance $(r_m(n))_{n \in \mathbb{Z}}$, also called in [24] the Verblunsky coefficients. For each n , it holds that $|\alpha_m(n)| < 1$. It is obvious that $\|\Phi_n^{(m)*}\|_1 = \|\Phi_n^{(m)*}\|_1$. Therefore, (B.38) implies that

$$\|\Phi_{n+1}^{(m)*}\|_1 \leq (1 + |\alpha_m(n)|)\|\Phi_n^{(m)*}\|_1 \leq 2\|\Phi_n^{(m)*}\|_1.$$

As $\|\Phi_0^{(m)*}\|_1 = 1$, we obtain that $\|\Phi_n^{(m)*}\|_1 \leq 2^n$, and that $\sup_m \sup_\nu |\Phi_n^{(m)*}(e^{2i\pi\nu})| \leq 2^n$. As $\inf_{m,n} \sigma_n^m > 0$, the normalized polynomials verify $\sup_m \sup_\nu |\phi_n^{(m)*}(e^{2i\pi\nu})| < +\infty$. This completes the proof of Lemma 6.1.

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