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# CARROUSEL IN FAMILY AND NON-ISOLATED HYPERSURFACE SINGULARITIES IN $\mathbb{C}^3$

FRANÇOISE MICHEL AND ANNE PICHON

ABSTRACT. We study the boundary  $L_t$  of the Milnor fiber for the reduced holomorphic germs  $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  having a non-isolated singularity at 0. We prove that  $L_t$  is a graph manifold by using a new technique of carousel depending on one parameter. Our results enable us to compare the topology of  $L_t$  and of the link of the normalization of  $f^{-1}(0)$ .

*Mathematics subject classification: 14J17 32S25 57M25*

## 1. INTRODUCTION

We denote by  $\mathbb{B}_r^{2n}$  the  $2n$ -ball with radius  $r > 0$  centered at the origin of  $\mathbb{C}^n$  and by  $\mathbb{S}_r^{2n-1}$  the boundary of  $\mathbb{B}_r^{2n}$ .

Let  $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  be a reduced holomorphic germ. The singularity of  $f$  at 0 is allowed to be non-isolated. We consider the three underlying topological objects :

- The link  $L_0 = f^{-1}(0) \cap \mathbb{S}_\epsilon^5$  of the surface  $f^{-1}(0)$  at 0, whose homeomorphism class of  $L_0$  does not depend on  $\epsilon$  when  $\epsilon > 0$  is sufficiently small ([16], [1]).
- The boundary  $L_t = f^{-1}(t) \cap \mathbb{S}_\epsilon^5$  of the Milnor fiber of  $f$ , where  $0 < |t| \ll \epsilon$ , whose diffeomorphism class does not depend on  $t$  when  $|t|$  is sufficiently small ([16], [6]).
- The link  $\overline{L}_0$  of the normalization of the surface  $F_0 = f^{-1}(0) \cap \mathbb{B}_\epsilon^6$  at 0, which can be defined up to diffeomorphism by  $\overline{L}_0 = n^{-1}(L_0)$ , where  $n : \overline{F}_0 \rightarrow F_0$  denotes the normalization morphism of  $F_0$  ([3]).

When the origin is an isolated singular point,  $L_0$ ,  $L_t$  and  $\overline{L}_0$  are three-dimensional differentiable manifolds, each of them being diffeomorphic to the others.

In this paper, we assume that the singular locus  $\Sigma(f)$  of  $f$  is 1-dimensional. Then only  $L_t$  and  $\overline{L}_0$  are differentiable manifolds.

The resolution theory implies that  $\overline{L_0}$  is a graph manifold in the sense of Waldhausen, or equivalently a plumbed manifold in the sense of Neumann ([17], [19], [22]). More precisely, the plumbing graph of  $\overline{L_0}$  is given, in its normal form, as the dual graph of a good minimal resolution of the normal surface singularity  $\overline{F_0}$ .

We will not recall here the notions of Seifert, graph and plumbed manifolds. For a quick survey adapted to our situation, see e.g. [15], section 3.

In [13], we state with a sketch of proof that for a germ  $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ , the boundary  $L_t$  of the Milnor fiber is also a graph manifold whose Seifert pieces have oriented basis. The main aim of this paper is to give a detailed proof of this result.

We first describe the manifold  $L_t$  using the following strategy (Section 2) : by hypothesis the singular locus of  $f$ ,  $\Sigma(f)$ , is a curve. Let  $K_0 = L_0 \cap \Sigma(f)$  be the link of the singular locus in  $L_0$ . Let  $\overline{K_0} = n^{-1}(K_0)$  be the pull-back of  $K_0$  in  $\overline{L_0}$  and  $\overline{\Sigma}(f) = n^{-1}(\Sigma(f))$  be the pull-back of the singular locus. A good resolution of the pair  $(\overline{F_0}, \overline{\Sigma}(f))$  provides a Waldhausen decomposition for  $\overline{L_0}$  as a union of Seifert manifolds such that  $\overline{K_0}$  is a union of Seifert leaves. Let  $\overline{M_0}$  be a tubular neighborhood of  $\overline{K_0}$  in  $\overline{L_0}$ . The closure  $\overline{N_0}$  of  $(\overline{L_0} \setminus \overline{M_0})$  is an irreducible Waldhausen graph manifold with boundary that we called the trunk of  $\overline{L_0}$ .

On the other hand, we define, in 2.10, a submanifold  $M_t$  of  $L_t$  called the vanishing zone around  $K_0$ . Theorem 2.14 (3), Theorem 4.1 and Proposition 5.2 can be summarized in the following theorem:

**Theorem.**

- (1) The closure  $N_t$  of  $L_t \setminus M_t$  is orientation preserving diffeomorphic to the trunk  $\overline{N_0}$ .
- (2) The manifold  $M_t$  is an irreducible Waldhausen graph manifold whose Seifert pieces have oriented basis.

By definition, the number of connected components of  $\overline{L_0}$  equals the number of irreducible components of  $f$ . But  $L_t$  is always a connected manifold (Corollary 2.15). Then if  $f$  is reducible,  $L_t$  is not homeomorphic to  $\overline{L_0}$ .

Moreover, our description of  $L_t$  enables us to compare the normalized plumbing graph of  $L_t$  with the minimal resolution graph of  $\overline{L_0}$ . We will perform this comparison when  $f$  is irreducible.

Notice first that  $\overline{N_0}$  is a solid torus if and only if the minimal resolution graph of  $(\overline{L_0}, \overline{\Sigma}(f))$  is a bamboo with an arrow at one of its extremities. Moreover, if  $\overline{N_0}$  is not a solid torus, then  $L_t$  is an irreducible 3-dimensional manifold (Corollary 5.3). When  $L_t$  is not irreducible, then it is not homeomorphic to  $\overline{L_0}$  as  $\overline{L_0}$  is irreducible ([19], Theorem 1).

Let  $M$  be an irreducible graph manifold. We denote by  $\mathcal{G}(M)$  the normalized plumbing graph of  $M$  as defined in [19]. We denote by  $\mathcal{T}(M)$  the separating family of the minimal Jaco-Shalen-Johannson decomposition of  $M$  and by  $\#\mathcal{T}(M)$  the cardinal of  $\mathcal{T}(M)$ .

**Proposition. 5.5** Assume that  $f$  is an irreducible germ and that  $L_t$  is an irreducible 3-dimensional manifold. Then

$$\text{rank } H_1(\mathcal{G}(\overline{L_0}), \mathbb{Z}) \leq \text{rank } H_1(\mathcal{G}(L_t), \mathbb{Z}), \text{ and}$$

$$\#\mathcal{T}(\overline{L_0}) \leq \#\mathcal{T}(\overline{N_0}) \leq \#\mathcal{T}(L_t)$$

The main aim in the study of the topological aspects of singularities consists of describing the analytical properties of a singularity which can be characterized through some topological underlying objects. One of the most important results in this direction is the following famous theorem of Mumford, which gives a topological characterization of a smooth point on a normal surface :

**Theorem.** ([17]) Let  $(X,0)$  be a germ of normal complex surface. If the link  $L_0$  of  $(X,0)$  has the homotopy type of the 3-sphere, then 0 is a smooth point of  $X$ .

The initial motivation of this work was to prove an analogous theorem for non-isolated singularities of hypersurfaces in  $\mathbb{C}^3$ . One of the advantages of our description of  $L_t$  as the gluing of the trunk and the vanishing zone is that it makes fairly easy the comparison of  $L_t$  with  $\overline{L_0}$  in most of the cases. We obtain the following topological characterization of isolated singularities for the analytic reduced germs  $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ .

**Theorem. 5.1** Let  $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$  be a reduced holomorphic germ. We assume that either  $f$  is reducible or  $L_t$  is not a lens space. Then the following assertions are equivalent.

- (i)  $f$  is either smooth or has an isolated singularity at 0.
- (ii) The boundary  $L_t$ ,  $t \neq 0$ , of the Milnor fibre of  $f$  is homeomorphic to the link  $\overline{L_0}$  of the normalization of  $f^{-1}(0)$ .

In [15], for the germs with equations  $z^m - g(x, y) = 0$  where  $m \geq 2$  and  $g(x, y) = 0$  is a non-reduced plane curve germ, we proved that  $L_t$  is never homeomorphic to  $\overline{L_0}$  even if  $L_t$  is a lens space, and that the later case arises if and only if  $m = 2$  and  $g$  has the analytic type of  $xy^l$ .

The problem of the characterization of the germs  $f : (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$ , which do not have the analytic type of germs with equation  $z^m - g(x, y) = 0$ , and which have a lens space as boundary  $L_t$  of their Milnor fiber, remains open. Proposition 5.6 shows that this open case concerns a very special family of singularities.

In fact, in most of the known cases,  $L_t$  is not orientation preserving homeomorphic to the link  $L_X$  of any complex normal surface singularity  $(X, p)$ . This happens for various reasons. In [13], we show that the germ  $f(x, y, z) = xy$  has  $L_t \cong \mathbb{S}^2 \times \mathbb{S}^1$ , which is not an irreducible 3-manifold. In [14], we show that for the germs  $z^m - x^k y^l = 0$  such that  $L_t$  is not a lens space (i.e.  $(m, k) \neq (2, 1)$ ), the boundary  $L_t$  is not an  $L_X$  as the intersection form associated to its normalized plumbing graph is never negative definite. In [15], we show that for the germ  $z^2 - (x^2 - y^3)y^l$ ,  $l$  odd, the boundary  $L_t$  is homeomorphic to the boundary of an  $L_X$ , but with the reversed orientation. More recently, in [18], A. Némethi and A. Szilárd describe the boundary of the Milnor fiber for other families of examples. In particular, they obtain some examples in which some edges of the normalized plumbing graph of  $L_t$  have a sign  $\epsilon = -1$ , which never happens for a  $L_X$ .

In Sections 3 and 4, we prove that the vanishing zone  $M_t$  is a Waldhausen graph manifold whose Seifert pieces have oriented basis. This proof, which is summarized in 4.2, uses two key constructions.

The first key construction, presented in Section 3, is a parametrization result (Theorem 3.2).

The second key construction consists of describing a Waldhausen decomposition of  $M_t$  in terms of a “carrousel in family” (see 4.3), parametrized by  $x$  varying on a circle  $\mathbb{S}_\alpha^1$ . Let us recall that the “carrousel” has been introduced by D.T.Lê, in [10] p.163, to obtain a geometric proof of the monodromy theorem.

In order to implement this carrousel process, we apply Theorem 3.2 to the germ  $h = \frac{\partial f}{\partial z}$ . Then we obtain a parametrization of the critical locus of the projection of  $M_t$  on a solid tori (Lemma 4.7).

However, this parametrization involves some convergent power series in  $x$ . The efficiency of the carrousel process is precisely that it uses a geometric argument, the so-called Lê-swing (for example see [12]), which enables one to consider truncated series.

2. THE TRUNK AND THE VANISHING ZONE.

In this section, we define the trunk and the vanishing zone of  $L_t$ . As a preliminary, we start in (2.1) and (2.2) by performing generic choices of the coordinates axis in  $\mathbb{C}^3$

**2.1.** The Weierstrass preparation theorem implies that we can choose  $f$  in  $\mathbb{C}\{x, y\}[z]$ . Then, the intersection  $\Gamma_0$  between  $\{f = 0\}$  and the hypersurface in  $\mathbb{C}^3$  with equation  $\{\frac{\partial f}{\partial z} = 0\}$  is a curve which contains  $\Sigma(f)$ .

**Claim** For a generic choice of the  $x$ -axis,  $(\{\frac{\partial f}{\partial z} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\})$  does not meet the boundary of the Milnor fiber and:

$$\Sigma(f) = \Gamma_0 \cap \{\frac{\partial f}{\partial y} = 0\}.$$

*Proof.* D.T. Lê and B.Teissier ( for example see (2.2.2) in [9] or IV.1.3.2 p.420 in [21]) have proved that, for a generic choice of the  $x$ -axis,

$$(\{\frac{\partial f}{\partial z} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\}) = (\Sigma(f) \cup \Gamma_{(x,f)}),$$

where the irreducible components of  $\Gamma_{(x,f)}$  are one-dimensional and not included in  $\{f = 0\}$ . They have called  $\Gamma_{(x,f)}$  the *polar curve of  $f$  for the direction  $x$* . Then, the boundary of the Milnor fiber does not meet  $\Gamma_{(x,f)}$  ( but its interior does). Moreover, the Milnor fiber does not meet  $\Sigma(f)$ .

□

**2.2.** Let  $P : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$  be the map defined by

$$P(x, y, z) = (x, y).$$

Let  $\Delta_0$  be  $P(\Gamma_0)$ ,  $\Delta_0$  is the *discriminant curve*. Perhaps after performing a linear change of coordinates in  $\mathbb{C}^2$ , we can assume that the  $x$ -axis is, at the origin, transverse to  $\Delta_0$  and that in  $\mathbb{C}^3$ , the hyperplanes  $X_a = \{x = a\}$  meet  $\Gamma_0$  transversely around the origin.

**2.3.** For technical reasons, we replace in this paper the standard Milnor ball  $\mathbb{B}_\epsilon^6$  by a polydisc

$$B(\alpha) = \mathbb{B}_\alpha^2 \times \mathbb{B}_\beta^2 \times \mathbb{B}_\gamma^2 = \{(x, y, z) \in \mathbb{B}_\epsilon^6, |x| \leq \alpha, |y| \leq \beta, |z| \leq \gamma\}$$

where  $0 < \alpha < \beta < \gamma < \epsilon/3$ .

**Definition.** The polydisc  $B(\alpha)$  is a *Milnor polydisc for  $f$*  if for each  $\alpha'$  with  $0 < \alpha' \leq \alpha$ ,

- (1) the pair  $(B(\alpha'), f^{-1}(0) \cap B(\alpha'))$  is diffeomorphic to the pair  $(\mathbb{B}_\epsilon^6, f^{-1}(0) \cap \mathbb{B}_\epsilon^6)$ ,
- (2) there exists  $\eta$  with  $0 < \eta \ll \alpha'$  such that:
  - (a) the restriction of  $f$  to  $W(\alpha', \eta) = B(\alpha') \cap f^{-1}(\mathbb{B}_\eta^2 \setminus \{0\})$  is a locally trivial differentiable fibration over  $\mathbb{B}_\eta^2 \setminus \{0\}$ ,
  - (b) the isomorphism class of this fibration does not depend on  $\alpha'$  and  $\eta$  when  $0 < \eta \ll \alpha' \leq \alpha$ .

Let us denote by  $S$  the boundary of  $B(\alpha)$  and let  $S(\alpha)$  be the subset of  $S$  defined by  $S(\alpha) = \mathbb{S}_\alpha^1 \times \text{int}(\mathbb{B}_\beta^2) \times \text{int}(\mathbb{B}_\gamma^2)$ . We can choose  $0 < \alpha < \beta < \gamma < \epsilon/3$  such that the two following inclusions hold :

- (1<sub>0</sub>)  $(S \cap f^{-1}(0)) \subset \{|z| < \gamma\}$ , and
- (2<sub>0</sub>)  $(\Gamma_0 \cap S) \subset S(\alpha)$ .

According to [9], Section 1, the generic choice of coordinates axis performed in (2.1) and (2.2) and the above conditions on  $\alpha, \beta, \gamma$  imply that the polydisc  $B(\alpha)$  is a Milnor polydisc for  $f$ .

In the sequel, we will then replace the objects defined in the introduction by the following :

- For  $0 \leq |t| \leq \eta$ ,
$$F_t = f^{-1}(t) \cap B(\alpha) \text{ and } L_t = F_t \cap S,$$
- $\overline{L_0} = n^{-1}(L_0)$ , where  $n : \overline{F_0} \rightarrow F_0$  denotes the normalization of  $F_0$ ,
- $K_0 = \Sigma(f) \cap L_0$  and  $\overline{K_0} = n^{-1}(K_0)$ .

**Remark 2.4.** Let us denote by  $S'$  the boundary of  $\mathbb{B}_\alpha^2 \times \mathbb{B}_\beta^2$ . The restriction  $P_0 : L_0 \rightarrow S'$  of  $P$  on  $L_0 = S \cap f^{-1}(0)$  is a ramified cover whose ramification locus is the algebraic link  $\Delta_0 \cap S'$  and whose generic order is the degree of  $f$  in  $z$ .

The above construction implies the following proposition.

**Proposition 2.5.** *For a sufficiently small tubular neighborhood  $V$  of  $\Delta_0 \cap S'$ , the two following conditions hold :*

- (1)  $V \subset \mathbb{S}_\alpha^1 \times \text{int}(\mathbb{B}_\beta^2)$ .
- (2) *Let  $M_0$  be the union of the connected components of  $P_0^{-1}(V)$  which contain the components of the link  $\overline{K_0}$ . Then  $\overline{M_0} = n^{-1}(M_0)$  is a tubular neighborhood of  $\overline{K_0}$  in  $\overline{L_0}$ .*

**Definition 2.6.** The *trunk* of  $L_0$  is the closure  $N_0$  of  $L_0 \setminus M_0$  in  $L_0$ . The *trunk* of  $\overline{L_0}$  is the closure  $\overline{N_0}$  of  $\overline{L_0} \setminus \overline{M_0}$  in  $\overline{L_0}$ .

**Proposition 2.7.** *The trunk  $N_0$  is a Waldhausen graph manifold with boundary.*

*Proof.* By definition  $\overline{N_0} = n^{-1}(N_0)$ . By construction  $N_0$  does not meet the singular locus  $\Sigma(f)$ . Therefore the restriction of  $n$  on  $\overline{N_0}$  is a diffeomorphism from  $\overline{N_0}$  to  $N_0$ . A good resolution of the pair  $(\overline{F_0}, \overline{\Sigma}(f))$  provides a Waldhausen decomposition for  $\overline{L_0}$  as a union of Seifert manifolds such that  $\overline{K_0}$  is a union of Seifert leaves. As  $\overline{M_0}$  is a tubular neighborhood of  $\overline{K_0}$  in  $\overline{L_0}$ , then the closure  $\overline{N_0}$  of  $(\overline{L_0} \setminus \overline{M_0})$  is a Waldhausen graph manifold with boundary.  $\square$

**Corollary 2.8.** *The number of boundary components of the trunk  $N_0$  is equal to the number of irreducible components of the curve  $\overline{\Sigma}(f)$ .*

*Proof.* In the proof of the above proposition, we show that  $N_0$  and  $\overline{N_0}$  are diffeomorphic. By construction, the number of boundary components of the trunk  $\overline{N_0}$  is equal to the number of connected components of  $\overline{K_0}$ , which is equal to the number of irreducible components of the curve  $\overline{\Sigma}(f)$ .

**2.9.** For each  $t \in \mathbb{B}_\eta^2$ , the singular set  $\Gamma_t$  of the restriction of  $P$  on  $F_t$  is the curve

$$\Gamma_t = \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap F_t,$$

and its discriminant locus is the curve  $\Delta_t = P(\Gamma_t)$ .

By continuity, we can choose  $\eta$  sufficiently small,  $0 < \eta \ll \alpha$ , in such a way that for each  $t, |t| \leq \eta$ , the properties that we already have for  $t = 0$ , hold for  $t \in \mathbb{B}_\eta^2$ , i.e. :

- (1<sub>t</sub>)  $L_t \subset \{|z| < \gamma\}$
- (2<sub>t</sub>)  $\Gamma_t$  is a curve which intersects transversally  $S$  inside  $S(\alpha)$

Moreover, let  $P_t : L_t \rightarrow S'$  the restriction of  $P$  to  $L_t$ . Then,

- (3<sub>t</sub>) the map  $P_t : L_t \rightarrow S'$  is a finite ramified cover with ramification locus  $\Gamma_t \cap S(\alpha)$  and branching locus  $\Delta_t \cap S'$ .
- (4<sub>t</sub>)  $\Delta_t \cap S' \subset \text{int}(V)$ .

**Definition 2.10.** Let  $L(\eta) = f^{-1}(\mathbb{B}_\eta^2) \cap S$  and let  $M(\eta)$  be the union of the connected components of  $L(\eta) \cap P^{-1}(V)$  which intersect  $K_0$ . For any  $t \in \mathbb{B}_\eta^2$ , let  $M_t = M(\eta) \cap L_t$ . By definition  $M_t$  is the vanishing zone of  $L_t$  and the closure  $N_t$  of  $L_t \setminus M_t$  in  $L_t$  is the trunk of  $L_t$ .



Notice that the choice of  $V$  (see 2.5), implies that  $M(\eta) \subset S(\alpha)$ .

**Proposition 2.11.** *Let  $N(\eta)$  be the closure of  $L(\eta) \setminus M(\eta)$  in  $L(\eta)$ . There exists a sufficiently small  $\eta$  such that  $f$  restricted to  $N(\eta)$  is a fibration on  $\mathbb{B}_\eta^2$ .*

**Corollary 2.12.** *There exists a sufficiently small  $\eta$  such that for all  $t \in \mathbb{B}_\eta^2 \setminus \{0\}$ ,  $N_t$  is orientation preserving diffeomorphic to  $N_0$ .*

*Proof of Proposition 2.11.*

i) Let

$$\Gamma(\eta) = L(\eta) \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\}.$$

Then, the restriction of  $(P, f)$  on  $S \setminus \Gamma(\eta)$  is a submersion. By (4<sub>t</sub>), in (2.9),  $\Gamma(\eta)$  does not meet the boundary of  $N(\eta)$ , hence the restriction of  $f$  on the boundary of  $N(\eta)$  is a fibration.

ii) Let  $\gamma'$  such that  $0 < \gamma' < \gamma$ . In  $S$ , we consider  $\bar{S}(\alpha) = \mathbb{S}_\alpha^1 \times \mathbb{B}_\beta^2 \times \mathbb{B}_{\gamma'}^2$ , and  $\bar{S}(\beta) = \mathbb{B}_\alpha^2 \times \mathbb{S}_\beta^1 \times \mathbb{B}_{\gamma'}^2$ , where  $\alpha < \beta < \gamma' < \gamma$ .

As  $L(\eta)$  is compact, (1<sub>t</sub>) implies that there exists  $\gamma'$  and  $\eta$  with  $0 < \eta \ll \alpha < \beta < \gamma' < \gamma$  such that for all  $t$  with  $0 \leq |t| \leq \eta$ ,

$$L_t \subset (\bar{S}(\alpha) \cup \bar{S}(\beta)).$$

By (2<sub>t</sub>) in (2.9),  $\Gamma(\eta)$  does not meet  $\bar{S}(\beta)$ , hence the restriction of  $f$  on  $N(\eta) \cap \bar{S}(\beta)$  is a fibration.

iii) Now, we have to prove that the restriction of  $f$  on  $N(\eta) \cap \bar{S}(\alpha)$  is a fibration. Points i) and ii) show that it is a fibration on its boundary. So, it is sufficient to prove that the projection on the  $x$  axis is transverse to  $f$  on  $N(\eta) \cap S(\alpha)$  i.e. to prove that there exists a sufficiently small  $\eta > 0$  such that the set

$$A = N(\eta) \cap S(\alpha) \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap \left\{ \frac{\partial f}{\partial y} = 0 \right\}$$

is empty. But for a general choice of the coordinates  $x$  and  $y$ , lemma (2.1) implies that:

$$L_0 \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap \left\{ \frac{\partial f}{\partial y} = 0 \right\} = K_0 \subset \text{int}(M_0)$$

Then, by continuity :

$$(*) \quad L(\eta) \cap \left\{ \frac{\partial f}{\partial z} = 0 \right\} \cap \left\{ \frac{\partial f}{\partial y} = 0 \right\} \subset \text{int}(M(\eta))$$

(\*) implies that  $A$  is empty.  $\square$

**2.13.** Now, let us describe more precisely the connected components of the vanishing zone  $M_t$ .

The tubular neighborhood  $V$  of  $\Delta_0 \cap S'$ , used above to obtain the vanishing zone, can be defined as follows :

Let  $\delta_1, \dots, \delta_s$  be the irreducible components of  $\Delta_0$ . Let us fix  $i \in \{1, \dots, s\}$ , and let

$$u \mapsto (u^k, \phi_i(u)), \text{ where } \phi_i(u) = \sum_{j=1}^{\infty} a_j u^j$$

be a Puiseux expansion of the branch  $\delta_i$  of  $\Delta_0$ . Let us consider the neighborhood  $W_i$  of  $\delta_i$  in  $\mathbb{C}^2$  defined by

$$W_i = \{(x, y) \in \mathbb{C}^2 / x = u^k, |y - \phi_i(u)| \leq \theta, u \in \mathbb{C}\},$$

where  $\theta$  is a positive real number.

We now choose  $\theta$  sufficiently small,  $0 < \theta \ll \alpha$ , in such a way that:

- (1) for each  $i = 1, \dots, s$ ,  $W_i$  intersects transversally  $S'$  inside  $\mathbb{S}_\alpha^1 \times \text{int}(\mathbb{B}_\beta^2)$ ,
- (2) the intersection  $V_i = W_i \cap S'$  is a tubular neighborhood of the knot  $\delta_i \cap S'$ ,
- (3) the solid tori  $V_i$  are disjoint.

Let  $V = \bigcup_{i=1}^s V_i$ . By continuity there exists  $\eta \ll \theta$  such that for each  $t, |t| \leq \eta$ , one has  $(\Delta_t \cap S') \subset \text{int}(V)$ .

Let  $\sigma$  be an irreducible component of  $\Sigma(f)$ . There exists  $i \in \{1, \dots, s\}$  such that  $P(\sigma) = \delta_i$ . We denote by  $M(\eta, \sigma)$  the connected component of  $P^{-1}(V_i) \cap L(\eta)$  which contains the knot  $K_0(\sigma) = \sigma \cap S$  of  $\sigma$  in  $S$ .

By definition, the three-dimensional manifold  $M_t(\sigma) = M(\eta, \sigma) \cap L_t$  is connected, and we obtain :

$$M_t = \bigcup_{j=1}^r M_t(\sigma_j),$$

where  $\{\sigma_j, 1 \leq j \leq r\}$  is the set of the irreducible components of  $\Sigma(f)$ .

For each  $j = 1, \dots, r$ , let  $\bar{r}_j$  be the number of irreducible components of the curve  $n^{-1}(\sigma_j)$ . The boundary of  $M_t(\sigma_j)$  consists of  $\bar{r}_j$  tori.

**Definition.**  $M_t(\sigma)$  is the *vanishing zone of  $L_t$  along  $\sigma$* .

Proposition (2.7), Corollary (2.12), and the construction 2.13 summarize in the following theorem :

**Theorem 2.14.** (1) *The boundary  $L_t$  of the Milnor fiber of  $f$  decomposes as the union*

$$L_t = N_t \cup M_t,$$

- (2)  $N_t \cap M_t$  is a disjoint union of  $r$  tori, where  $r$  is the number of irreducible components of the curve  $\overline{\Sigma}(f)$ ,
- (3)  $N_t$  is a Waldhausen manifold orientation preserving diffeomorphic to the trunk  $\overline{N}_0$ ,
- (4) Let  $\sigma_1, \dots, \sigma_r$  be the union of irreducible components of  $\Sigma(f)$ . The connected components of the vanishing zone  $M_t$  are the manifolds  $M_t(\sigma_j), j = 1 \dots r$ .

**Corollary 2.15.** *The manifold  $L_t$  is connected.*

*Proof.* The number of connected components of  $\overline{F}_0$  and  $\overline{L}_0$  is equal to the number of irreducible components of  $f$ . The intersection between two irreducible components of  $f = 0$  furnishes at least one irreducible component of the singular locus  $\Sigma(f)$  and a corresponding connected component of the vanishing zone. Hence, the constructions given here show that after the gluing of all connected components of the vanishing zone with the trunk, we obtain a connected manifold  $L_t$ .

□

**Remark 2.16.** Corollary 2.15 implies that the Milnor fiber  $F_t$  is connected. As the singular locus of  $f$  has dimension 1,  $F_t$  is connected by a much more general result of M. Kato and Y. Matsumoto in [8].

**Remark 2.17.** To prove that  $L_t$  is a Waldhausen graph manifold, we still have to prove that  $M_t(\sigma)$  is a Waldhausen graph manifold for any irreducible component  $\sigma$  of  $\Sigma(f)$ . This will be done in Section 4.

### 3. A PARAMETRIZATION THEOREM

In this section, we consider a reduced analytic germ  $h : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  such that  $h(x, 0, 0) = 0$  for all  $x \in \mathbb{C}$ . Let  $H$  be the germ of hypersurface with equation  $h = 0$ .

For each  $x \in \mathbb{C}$ , we denote by  $h_x : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  the germ defined by  $h_x(y, z) = h(x, y, z)$ . Hence  $h_x$  has an isolated singular point at  $(x, 0, 0)$  for all  $x \in \mathbb{B}_\alpha^2 \setminus \{0\}$ .

Let us fix  $\alpha \ll 1$  and  $\epsilon \ll \alpha$  such that for each  $x \in \mathbb{S}_\alpha^1$ ,  $\{x\} \times \mathbb{B}_\epsilon^4$  is a Milnor ball for the germ of curve  $h_x = 0$  at  $(x, 0, 0)$ .

**Definition 3.1.** A branch of  $H$  along  $\mathbb{S}_\alpha^1$  is the closure of a connected component of the intersection  $H \cap (\mathbb{S}_\alpha^1 \times (\mathbb{B}_\epsilon^4 \setminus \{0\}))$ .

**Theorem 3.2.** *Let  $G$  be a branch of  $H$  along  $\mathbb{S}_\alpha^1$ . There exists  $d, i$  and  $j \in \mathbb{N}^*$ , and two convergent power series  $b(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$  and  $c(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$  with  $b(x, 0) \neq 0$  and  $c(x, 0) \neq 0$ , such that*

$$(s, u) \longmapsto (s^d, u^i b(s, u), u^j c(s, u))$$

*is a parametrization of  $G$ .*

For each  $x \in \mathbb{B}_\alpha^2 \setminus \{0\}$ , let  $\pi_x : Y_x \rightarrow \{x\} \times \mathbb{B}_\epsilon^4$  be the minimal good resolution of  $h_x$ , *i.e.* the minimal composition of blow-ups of points such that the curve  $(h_x \circ \pi_x)^{-1}(0)$  is a normal crossing divisor. We denote by  $E_x = \pi_x^{-1}(x, 0, 0)$  the exceptional divisor of  $\pi_x$ .

The proof of Theorem 3.2 will use the following :

**Lemma 3.3.** *Let  $h_{1,x}$  be an irreducible component of  $h_x$ , let  $\tilde{h}_{1,x}$  be its strict transform by  $\pi_x$  and let  $P = E_x \cap \tilde{h}_{1,x}$ . We can choose local coordinates  $(u, v)$  at  $P$  in  $Y_x$  such that :*

- (1)  $u = 0$  is a local equation for  $E_x$  in  $Y_x$ .
- (2) There exist three integers  $d, i, j$  in  $\mathbb{N}^*$ , two polynomials  $\phi(x^{1/d}, u, v)$  and  $\psi(x^{1/d}, u, v)$  in  $\mathbb{C}\{x^{1/d}\}[u, v]$ , where  $\phi(x^{1/d}, 0, v)$  and  $\psi(x^{1/d}, 0, v)$  are not identically 0, and  $s \in \mathbb{B}_{\alpha^{1/d}}^2 \setminus \{0\}$  with  $s^d = x$  such that :

$$\pi_x(u, v) = (s^d, u^i \phi(s, u, v), u^j \psi(s, u, v)).$$

- (3) There exist an integer  $M \in \mathbb{N}^*$  and two convergent power series  $c(x^{1/d}) \in (\mathbb{C}\{x^{1/d}\} \setminus \{0\})$  and  $g(x^{1/d}, u, v) \in \mathbb{C}\{x^{1/d}\}\{u, v\}$  such that, for the value  $s$  defined just above, we have:

$$(h \circ \pi_x)(u, v) = u^M (ug(s, u, v) + c(s)v)$$

*Proof.* Let us write  $h(x, y, z)$  as the sum

$$h(x, y, z) = \sum_{n=0}^{\infty} c_n(x, y, z),$$

where for all  $n \in \mathbb{N}$ ,

$$c_n(x, y, z) = \sum_{k=0}^n c_{n,k}(x) y^k z^{n-k}$$

with  $c_{n,k}(x) \in \mathbb{C}\{x\}$ .

Let  $m$  be the least integer such that  $c_m(x, y, z) \neq 0$ . Perhaps after performing a change of variables, one can assume that  $c_{m,0}(x) \neq 0$ . We start with the blow-up  $\pi_{1,x}$  of  $(x, 0, 0)$  in  $\mathbb{C}^2$ , *i.e.*:

$$\pi_{1,x} : Y_{1,x} \rightarrow \{x\} \times \mathbb{B}_\epsilon^4.$$

Let  $E_{1,x} = (\pi_{1,x})^{-1}(x, 0, 0)$  be the exceptional divisor of  $\pi_{1,x}$ . As  $c_{m,0}(x) \neq 0$ , the axis  $y = 0$  is not a line of the tangent cone of  $h_x$ . We will write the intersection points  $\tilde{h}_x \cap E_{1,x}$  with the help of coordinates  $(u_1, v_1)$  given by the standard chart on  $(\pi_{1,x})^{-1}(\{x\} \times \mathbb{B}_\epsilon^4)$  defined by

$$\pi_{1,x}(u_1, v_1) = (x, u_1, u_1 v_1)$$

In the local coordinates  $(u_1, v_1)$ , we have :

$$(h_x \circ \pi_{1,x})(u_1, v_1) = u_1^m \left( \sum_{k=0}^m c_{m,k}(x) v_1^{m-k} + u_1 g_1(x, u_1, v_1) \right) \quad (*)$$

where

$$g_1(x, u_1, v_1) = \sum_{m'=m+1}^{\infty} u_1^{m'-m-1} c_{m'}(x, u_1, v_1)$$

Then the intersection  $\tilde{h}_x \cap E_{1,x}$  consists of the points  $(x, 0, v_1)$  such that  $v_1$  is a root of the polynomial

$$Q(v_1) = \sum_{k=0}^m c_{m,k}(x) v_1^{m-k} \in C\{x\}[v].$$

There exists an integer  $e > 0$  such that the decomposition field of the polynomial  $Q$  is the fraction field  $K_e$  of  $\mathbb{C}\{x^{1/e}\}$  (for example see D.Eisenbud [4], p.295). There exists a unique root  $r_1 \in K_{d_1}$  of  $Q$ , where  $d_1 \leq e$  is the minimal integer such that  $r_1 \in K_{d_1}$ , and a complex number  $s_1$  which satisfies  $s_1^{d_1} = x$ , such that the strict transform of  $h_{1,x}$  (by  $\pi_{1,x}$ ), cuts  $E_{1,x}$  at the point  $P_1 = (0, r_1(s_1))$ . The strict transform of  $h_x$  meets also  $E_{1,x}$  at the  $d_1$  distinct points  $(0, r_\delta(s_1))$  corresponding to the  $d_1$  distinct roots  $r_\delta$  of  $Q$  defined by :

$$\delta^{d_1} = 1 \quad \text{and} \quad r_\delta(x^{1/d_1}) = r_1(\delta x^{1/d_1}).$$

We find the others intersection points of the strict transform of  $h_x$  (by  $\pi_{1,x}$ ) with  $E_{1,x}$  with the others roots of  $Q$ . The map  $\pi_{2,x}$  is the blow-ups of all these intersection points.

**Remark 3.4.** To make the above blow-ups in family for all  $x \in \mathbb{B}_\alpha^2 \setminus \{0\}$ , we have to take a sufficiently small  $\alpha$  such that:

- (1)  $c_{m,0}$  does not vanish on  $\mathbb{B}_\alpha^2 \setminus \{0\}$ ,
- (2) if  $r$  and  $r'$  are two distinct roots of  $Q$  in  $K_e$ , then  $(r - r')(s^e)$  does not vanish for  $s^e \in \mathbb{B}_\alpha^2 \setminus \{0\}$ .

*End of proof of Lemma 3.3.*

As  $K_{d_1}$  is nothing but the field of convergent Laurent power series in the variable  $x^{1/d_1}$ , there exists  $l_1 \in \mathbb{N}^*$  such that

$$x^{l_1} r_1(x^{1/d_1}) \in \mathbb{C}\{x^{1/d_1}\}$$

We consider new local coordinates  $(\tilde{u}_1, \tilde{v}_1)$  in  $Y_{1,x}$  centered at  $(0, r_1(x^{1/d_1}))$  by setting :

$$(u_1, v_1) = (x^{l_1} \tilde{u}_1, \tilde{v}_1 + r_1(x^{1/d_1})) \quad (**)$$

We then have :

$$\pi_{1,x}(\tilde{u}_1, \tilde{v}_1) = (x, u_1, u_1 v_1) = (x, x^{l_1} \tilde{u}_1, (x^{l_1} \tilde{u}_1)(\tilde{v}_1 + r_1(x^{1/d_1})))$$

As  $x^{l_1} \tilde{u}_1$  and  $(x^{l_1} \tilde{u}_1)(\tilde{v}_1 + r_1(x^{1/d_1}))$  are in  $\mathbb{C}\{x^{1/d_1}\}[\tilde{u}_1, \tilde{v}_1]$  and as  $\tilde{u}_1 = 0$  is the local equation of  $E_{1,x}$  at the point  $P_1$ , statements (1) and (2) of lemma (3.3) are proved for  $\pi_{1,x}$ .

When we perform  $\pi_{2,x}$ , we blow-up  $P_1$  in  $Y_{1,x}$ . In order to write  $\pi_{2,x}$  in one of the two standard charts around  $(\pi_{2,x})^{-1}(P_1)$ , we perform in (\*) one of the two following substitutions :  $(\tilde{u}_1, \tilde{v}_1) = (u_2, u_2 v_2)$  or  $(\tilde{u}_1, \tilde{v}_1) = (u_2 v_2, v_2)$ . If necessary, we follow it by a new change of coordinates of the type:

$$(u_2, v_2) = (x^{l_2} \tilde{u}_2, \tilde{v}_2 + r_2(x^{1/d_2})),$$

where  $x^{l_2} r_2(x^{1/d_2}) \in \mathbb{C}\{x^{1/d_2}\}$  is defined as before.

Then, points 1. and 2. of lemma (3.3) are also proved for  $\pi_{2,x} \circ \pi_{1,x}$ . By finite iteration, there are also proved for  $\pi_x$ . As  $\pi_x$  is a good resolution of  $h_x$ , the strict transform  $\tilde{h}_x$  is transverse to  $E_x$  at  $P$  and has multiplicity 1. A direct computation of  $h_x \circ \pi_x$ , with the help of the point 2. of lemma (3.3) implies point 3.

This ends the proof of lemma (3.3).  $\square$

Let  $\mathcal{U}(\alpha)$  be the interior of  $(\mathbb{B}_{2\alpha}^2 \setminus \{0\})$  and  $\mathcal{H} = H \cap (\mathcal{U}(\alpha) \times \mathbb{B}_\epsilon^4)$ . Let  $\pi_1$  be the blow-up of the one-dimensional non singular analytic subset  $(\mathcal{U}(\alpha) \times 0 \times 0)$  in  $\mathbb{C}^3$ .

$$\pi_1 : Y_1 \rightarrow (\mathcal{U}(\alpha) \times \mathbb{B}_\epsilon^4).$$

**Remark 3.5.** For all  $x \in \mathcal{U}(\alpha)$ ,  $\pi_{1,x}$ , the blow-up of  $(x, 0, 0)$  in  $\{x\} \times \mathbb{C}^2$ , is equal to  $\pi_1$  restricted on  $(\pi_1^{-1}(\{x\} \times \mathbb{B}_\epsilon^4))$ . Moreover, for a sufficiently small  $\alpha$ ,  $\pi_x$ , the minimal good resolution of  $h_x$  (see 3.3) is the composition of the same number, let say  $k$ , of blow-ups of points.

Let  $\mathcal{H}_1$  be the strict transform (by  $\pi_1$ ) of  $\mathcal{H}$ . If  $2\alpha$  satisfies the two conditions given in (3.4),  $\mathcal{H}_1$  meets the exceptional divisor  $E_1 = \pi_1^{-1}(\mathcal{U}(\alpha) \times 0 \times 0)$  along a one-dimensional non singular analytic subset of  $Y_1$ . More precisely, in the chart  $(u_1, v_1)$  used in the proof of lemma 3.3, the connected components of  $E_1 \cap \mathcal{H}_1$  are parametrized by  $\{(s, 0, r(s)), s^e \in \mathcal{U}(\alpha)\}$  for all roots  $r \in K_e$  of  $Q$ . Then, for a sufficiently small  $\alpha$ , the open set  $E_1 \cap \mathcal{H}_1$  is non singular. Let  $\pi_2$  be the blow-up of  $E_1 \cap \mathcal{H}_1$  in  $Y_1$ . We iterate the same process to obtain  $\pi = \pi_k \circ \dots \circ \pi_2 \circ \pi_1$  where

$$\pi : Y_k \rightarrow (\mathcal{U}(\alpha) \times \mathbb{B}_\epsilon^4).$$

By construction, for each  $x \in \mathcal{U}(\alpha)$ , the restriction of  $\pi$  on  $\pi^{-1}(\{x\} \times \mathbb{B}_\epsilon^4)$  is equal to the minimal good resolution  $\pi_x$  of  $h_x$ . It is why we say that  $\pi$  is a resolution in family of  $h_x$  for  $x \in \mathcal{U}(\alpha)$ .

Let  $\mathcal{H}_k$  be the strict transform of  $\mathcal{H}$  by  $\pi$ .

**Lemma 3.6.** *Each connected component of  $(\pi^{-1}(\mathcal{U}(\alpha) \times 0 \times 0)) \cap \mathcal{H}_k$  has an open neighborhood parametrized in  $s, u$  and  $v$  such that there exist a positive integer  $M$ ,  $c(s) \in (\mathbb{C}\{s\} \setminus \{0\})$  and  $g(s, u, v) \in \mathbb{C}\{s, u, v\}$  which satisfy :*

$$(h \circ \pi)(s, u, v) = u^M(ug(s, u, v) + c(s)v).$$

*Proof.* Let  $\mathcal{H}^{(1)}$  be the connected component of  $\mathcal{H}_k$  which contains the strict transform  $\tilde{h}_{1,x}$  considered in lemma 3.3. Point (3) of lemma 3.3 implies that for all  $x \in \mathcal{U}(\alpha)$ , we can trivially parametrized by  $s$ ,  $s^d = x$ , the same chart in  $(u, v)$ . This chart contains  $(\pi^{-1}(\mathcal{U}(\alpha) \times 0 \times 0)) \cap \mathcal{H}^{(1)}$ . Lemma 3.3 gives the number  $M$ , and the series  $c(s)$  and  $g(s, u, v)$ .

This ends the proof of lemma (3.6).  $\square$

**Remark 3.7.** By definition  $(\pi(\mathcal{H}^{(1)})) \cap (\mathbb{S}_\alpha^1 \times \mathbb{B}_\epsilon^4)$  is a branch  $G$  of  $H$ .

Then  $G = \pi(\tilde{G})$  where:

$$\tilde{G} = \{ug(s, u, v) + c(s)v = 0, s^d \in \mathbb{S}_\alpha^1, (u, v) \in \mathbb{B}_\epsilon^4\}.$$

*Proof of Theorem 3.2.*

Thanks to lemma (3.6) and the above remark we have to solve the following equation:

$$\{ug(s, u, v) + c(s)v = 0\},$$

where  $c(s) \neq 0$ .

Let us perform the change of coordinate  $u' = (c(s))^{-1}u$ . Then, we obtain :

$$(h \circ \pi)(s, u', v) = u'^M c(s)^{M+1} (u' g(s, u'c(s), v) + v)$$

We replace  $u'$  by  $u$ . Now the equation of  $\tilde{G}$  is given by:

$$u g(s, u, v) + v = 0$$

Let us consider  $F(u, v) = u g(s, u, v) + v = 0$  as an element of  $A\{u, v\}$  where  $A = \mathbb{C}\{s\}$ . As  $F(0, v) = v$ , we can applied the Weierstrass preparation theorem (for example see [23], vol.2, p.139-141), to obtain  $R(s, u) \in \mathbb{C}\{s\}\{u\}$  such that

$$F(u, v) = 0 \Leftrightarrow v = R(s, u)$$

This leads to :

$$h \circ \pi(s, u, R(s, u)) = 0.$$

This equality, together with point 2. of lemma (3.3), implies that  $h$  vanishes on  $\{(s^d, u^i \phi(s, u, R(s, u)), u^j \psi(s, u, R(s, u))), u \in \mathbb{B}_\epsilon^2\}$ .

For each  $s \in S_{\alpha^{1/d}}^1$ , we set  $b(s, u) = \phi(s, u, R(s, u))$  and  $c(s, u) = \psi(s, u, R(s, u))$ . We have a parametrization

$$(S_{\alpha^{1/d}}^1) \times \mathbb{B}_\epsilon^2 \rightarrow G$$

given by

$$(s, u) \longmapsto (s^d, u^i b(s, u), u^j c(s, u))$$

This ends the proof of theorem 3.2.  $\square$

#### 4. $M_t$ IS WALDHAUSEN : THE PROOF

The aim of this section is to prove the main result of this paper :

**Theorem 4.1.**  *$M_t$  is a Waldhausen graph manifold whose Seifert pieces have oriented basis. .*

According to Theorem 2.14, we have to prove that for each branch  $\sigma$  of the singular locus  $\Sigma(f)$ , the vanishing zone  $M_t(\sigma)$  of  $L_t$  along  $\sigma$  is a Waldhausen manifold.

#### 4.2. Abstract of the proof

Before giving the proof in details, let us give the key ideas and steps.



- At first, we will show that it suffices to prove that  $M_t(\sigma)$  is Waldhausen when  $\sigma$  is smooth. We will then assume that  $\sigma$  is the  $x$ -axis.
- Let  $\Psi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$  be the germ defined by  $\Psi(x, y, z) = (x, y, f(x, y, z))$ . The critical locus of  $\Psi$  is  $H = \{\frac{\partial f}{\partial z} = 0\}$ , and its discriminant locus is the image  $H' = \Psi(H)$ . Let  $\Psi_t$  be the restriction of  $\Psi$  on  $M_t(\sigma)$ . As  $\sigma$  is the  $x$ -axis, the image of  $\Psi_t$  is equal to  $\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\}$ . Moreover,

$$\Psi_t : M_t(\sigma) \rightarrow \mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\},$$

is a finite ramified cover over the solid torus  $T = \mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\}$  whose ramification locus is the braid  $H_t = M_t(\sigma) \cap H$ .

Set  $H'_t = \Psi_t(H_t)$ . To describe  $H'_t$ , we consider a branch  $G$  of  $H$  along the circle  $\mathbb{S}_\alpha^1 \times \{0\} \times \{0\}$  as defined in Section 3, we set  $G' = \Psi(G)$ , and we prove the following parametrization result (Lemma (4.7)) :

if  $(x, y, t) \in G'$ , then  $y$  satisfies the following equality :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m (x^{1/n'}) t^{r_m}, \quad (*)$$

where  $b \in \mathbb{C}^*$ ,  $d, d', n, p, p'$  and  $q$  are positive integers with  $pp' = n$ ,  $w(x^{1/d}) = 1 + \sum_{m=1}^{\infty} w_m x^{m/d} \in \mathbb{C}\{x^{1/d}\}$ ,  $e \in \mathbb{Z}$ ,  $n' = dn$ ,  $b_j(x^{1/n'}) \in K_{n'}$  and  $r_m = (qp' + m)/pp'$ .

The equality (\*) implies that  $G'_t = \Psi_t(G \cap M_t(\sigma))$  is a braid in the solid torus  $T$ . But this braid can be rather complicated. It is the reason why we approximate it by the torus link

$$App(G'_t) = \{(x, b x^{e/d'} t^{q/p}, t); x \in \mathbb{S}_\alpha^1\}.$$

**Definition.** We say that  $G'_t$  is the braid of  $G' = \Psi(G)$ , that  $App(G'_t)$  is the torus link associated to  $G'$  and that the pair  $(q/p, e/d')$  is the pair of the first exponents of  $G'$ .

- We index the pairs of first exponents  $(q/p, e/d')$  by lexicographic order. For each of them,  $(q_i/p_i, e_{i,j}/d'_{i,j})$  where  $1 \leq i \leq k$ , and  $1 \leq j \leq l_i$ , we construct a vertical polar zone  $Z_{(i,j)}$  (see Definition (4.10)) such that  $G'_t$  is included in the interior of  $Z_{(i,j)}$  if and only if  $(q_i/p_i, e_{i,j}/d'_{i,j})$  is the pair of the first exponents of  $G'$  (Lemma (4.11)). Moreover,  $Z_{(1,1)}$  is a solid torus and for all  $(i, j)$  not equal to  $(1, 1)$ , the  $Z_{(i,j)}$ 's are concentric thickened tori which recover the solid torus  $T$  along their boundaries.

- In the solid torus  $T$ , we define some tubular neighborhoods  $\mathcal{N}(G'_t)$  of the link  $App(G'_t)$  for all the branches  $G$  of  $H$  such that :
  - \*  $G'_t \subset \mathcal{N}(G'_t)$
  - \* If  $G$  has its pair of first exponents indexed by  $(i, j)$ , then  $\mathcal{N}(G'_t)$  is included in the interior of  $Z_{(i,j)}$ .
  - \* Let  $\tilde{G}$  be another branch of  $H$ . If  $App(G'_t) = App(\tilde{G}'_t)$ , then  $\mathcal{N}(G'_t) = \mathcal{N}(\tilde{G}'_t)$ . Otherwise,  $\mathcal{N}(G'_t)$  and  $\mathcal{N}(\tilde{G}'_t)$  are disjoint solid tori in  $T$  (see Lemma (4.13)).

We call the solid tori  $\mathcal{N}(G'_t)$  the *approximation tori*.

**Notation.** Let  $\mathcal{N}(i, j)$  be the union of all the approximation tori of the branches which have their first exponents indexed by  $(i, j)$ .

By construction the closure of  $Z_{(i,j)} \setminus \mathcal{N}_{(i,j)}$  does not meet the set of ramification values  $H'_t$  of  $\Psi_t$  and is saturated by  $(e_{i,j}, d'_{i,j})$  torus links. The case  $e_{i,j} = 0$  is not excluded, but we always have  $0 < d'_{i,j}$ . It induces a Seifertic structure on the closure of  $\Psi_t^{-1}(Z_{(i,j)} \setminus \mathcal{N}_{(i,j)})$ .

- The last step consists in showing (see Lemma (4.19)) that  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$  is a disjoint union of solid tori. Then we can extend the Seifert fibration on all the  $\Psi_t^{-1}(Z_{(i,j)})$ . Moreover, we explain in 4.20 why the so constructed Seifert manifolds have oriented basis. It ends the proof of Theorem 4.1. To prove 4.19, we need Lemma (4.15) which uses deeply the polar curve theory and the Lê-swung theorem (introduced by D.T.Lê and B.Perron in [11] ) via the following construction.

### 4.3. Carrousel in family

Let  $M(\eta, \sigma)$  (as defined in 2.13), be the union of the  $M_t(\sigma)$  where  $t \in \mathbb{B}_\eta^2$ . The image of the restriction of  $\Psi$  on  $M(\eta, \sigma)$  is equal to  $\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\eta^2$ .

Let us fix  $a \in \mathbb{S}_\alpha^1$  and let us consider the plane curve germ  $f_a$ ,

$$f_a : (\{a\} \times \mathbb{C}^2, (a, 0, 0)) \rightarrow (\mathbb{C}, 0)$$

defined by  $f_a(y, z) = f(a, y, z)$ . The restriction of  $\Psi$  on  $M(\eta, \sigma) \cap \{x = a\}$  has  $\Gamma_a = H \cap \{x = a\}$  as singular locus. The curve  $\Gamma_a$  is nothing but the polar curve (at  $(a, 0, 0)$ ) of  $f_a$  for the direction  $y$ , and the set  $\Delta_a = \Psi(\Gamma_a)$  is its discriminant curve.

Let us consider

$$M^{(a)}(\sigma) = M(\eta, \sigma) \cap \{x = a\} \cap \{|f| = \eta\}.$$

By construction, the restriction

$$\Psi^{(a)} : M^{(a)}(\sigma) \rightarrow \{a\} \times \mathbb{B}_\theta^2 \times \mathbb{S}_\eta^1$$

of  $\Psi$  on  $M^{(a)}(\sigma)$  is a ramified cover, whose ramification locus is  $\Gamma_a \cap \{|f| = \eta\}$ .

**Remark.** By construction, the Milnor fiber of the plane curve germ  $f_a$  is

$$F_{t,a} = M_t(\sigma) \cap M^{(a)}(\sigma).$$

The restriction  $\psi_a : F_{t,a} \rightarrow \mathcal{D}$  of  $\Psi^{(a)}$  on  $F_{t,a}$  is a finite ramified cover over the disk  $\mathcal{D} = \{a\} \times \mathbb{B}_\theta^2 \times \{t\}$ . This ramified cover has been studied in details by D.T.Lê, (for example in [9] and in [10]) to study the monodromy of the Milnor fiber as a pull-back (here by  $\psi_a$ ) of a diffeomorphism of the disk  $\mathcal{D}$  modulo its intersection points with  $\Delta_a$ . D.T.Lê calls this construction “the carrousel”.

But  $\psi_a$  is also the restriction of  $\Psi_t$  on  $F_{t,a}$ . Then we have to study the family of  $\psi_x$  with  $x \in \mathbb{S}_\alpha^1$ . In order to do this, we construct a carrousel parametrized by  $x$  : it is a carrousel in family.

#### 4.4. Reduction to a smooth branch of $\Gamma_t$

Let us fix a branch  $\sigma$  of  $\Sigma(f)$  and let

$$u \mapsto (u^k, \phi(u), \psi(u))$$

be a Puiseux parametrization of  $\sigma$ .

Let us consider the analytic morphism  $\Theta : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  defined by

$$\Theta(x, y, z) = (x^k, y + \phi(x), z + \psi(x))$$

Let  $g : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  be the composition  $g = f \circ \Theta$ . Then  $\sigma' = \Theta^{-1}(\sigma)$  is the  $x$ -axis. Moreover, a direct computation of the derivative of  $g$  shows that  $\sigma'$  is a branch of the singular locus of  $g$ .

Let  $M_t(f, \sigma)$  (resp.  $M_t(g, \sigma')$ ) be the vanishing zone of  $f$  along  $\sigma$  (resp. of  $g$  along  $\sigma'$ ) defined in the boundary of the ball  $B(\alpha)$  (resp.  $B(\alpha^{1/k})$ ) as in 2.13. The construction given in 2.13 leads directly to :

**Lemma 4.5.**  $M_t(g, \sigma) = \Theta^{-1}(M_t(f, \sigma))$ , and the restriction  $\Theta|_{M_t(g, \sigma')} : M_t(g, \sigma') \rightarrow M_t(f, \sigma)$  is a diffeomorphism.

In the sequel, we assume that  $\sigma$  is the  $x$ -axis. In particular, the vanishing zone  $M_t(\sigma)$  along  $\sigma$  is nothing but

$$M_t(\sigma) = L_t \cap (\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\gamma^2), \quad 0 < \eta \ll \theta \ll \alpha.$$

#### 4.6. Parametrization of the branches of $\Psi(H)$

Let us recall that  $\Psi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  denotes the germ defined by  $\Psi(x, y, z) = (x, y, f(x, y, z))$ . The critical locus of  $\Psi$  is  $H = \{\frac{\partial f}{\partial z} = 0\}$ , and its discriminant locus is the image  $H' = \Psi(H)$ . The proof of the lemma 4.7 uses theorem 3.2 for the germ  $h$  which is the reduced of  $\frac{\partial f}{\partial z}$ , i.e.  $h$  is reduced and  $H = \{h = 0\}$ .

Let  $G$  be the closure (in  $\mathbb{C}^3$ ) of a connected component of

$$(H \setminus (\mathbb{S}_\alpha^1 \times \{0\} \times \{0\})) \cap (\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \mathbb{B}_\gamma^2),$$

i.e.  $G$  is a branch of  $H$  along the circle  $\mathbb{S}_\alpha^1 \times \{0\} \times \{0\}$  as defined in Section 3.

We set  $G' = \Psi(G)$ , and we call  $G'$  a branch of  $H' = \Psi(H)$  along  $\mathbb{S}_\alpha^1 \times \{0\} \times \{0\}$ .

Let us recall that  $K_d$  denotes the fraction field of  $\mathbb{C}\{x^{1/d}\}$ .

**Lemma 4.7.** *There exist:*

- $d, n, p, p', q \in \mathbb{N}^*$ , where  $p$  is prime to  $q$  and  $pp' = n$ ,
- $e \in \mathbb{Z}$  and  $d' \in \mathbb{N}^*$  is prime to  $e$  ( if  $e = 0$ , then  $d' = 1$ ),
- $r_m = (qp' + m)/pp'$ ,
- $b_j(x^{1/n'}) \in K_{n'}$ , where  $n' = dn$ ,
- $w(x^{1/d}) = 1 + \sum_{m=1}^{\infty} w_m x^{m/d} \in \mathbb{C}\{x^{1/d}\}$  and  $b \in \mathbb{C}^*$ .

such that, if  $(x, y, t) \in G'$ , then  $y$  satisfies the following equality :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m} \quad (*)$$

**Remark.** As recalled below, the integer  $d$  is provided by theorem 3.2 . For each branch  $G$  of  $H$  there exists such a  $d$  minimal which depends on  $G$ . Here, for convenience, we will choose a ( perhaps greater)  $d$  common to all the branches of  $H$ .

*Proof.* Theorem 3.2 provides  $b(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$  and  $c(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$  with  $b(x, 0) \neq 0$  and  $c(x, 0) \neq 0$ , such that we have a parametrization

$$\mathbb{S}_{\alpha^{1/d}}^1 \times \mathbb{B}_\epsilon^2 \rightarrow G$$

given by

$$(s, u) \mapsto (s^d, u^i b(s, u), u^j c(s, u))$$

and we obtain  $n, j < n$ , and  $c'(x^{1/d}, u) \in \mathbb{C}\{x^{1/d}\}\{u\}$  with  $c'(x, 0) \neq 0$ , such that  $G' = \Psi(G)$  admits a parametrization of the form

$$(s, u) \mapsto (s^d, u^i b(s, u), u^n c'(s, u)) \quad (**)$$

If necessary, we can perform the modification  $u = s^l u', l' \in \mathbb{N}$ , to obtain  $l \in \mathbb{N}$ ,  $c_m(s) \in \mathbb{C}\{s\}$  with  $c_0(0) \in \mathbb{C}^*$ , such that:

$$t = u^n c'(u, s) = u^n s^l c_0(s) \left( 1 + \sum_{m=1}^{\infty} c_m(s) u^m \right)$$

There then exist  $r(x^{1/d}, u') \in \mathbb{C}\{x^{1/d}\}\{u'\}$  with  $r(0, 0) = 1$  and  $r_0(x^{1/d}) \in \mathbb{C}\{x^{1/d}\}$  with  $(r_0(0))^n = c_0(0)$ , such that

$$t = u^n s^l (r_0(s))^n (r(s, u'))^n.$$

We perform the following change of coordinates:

$$u_1 = u' r_0(s) r(s, u')$$

and (\*\*) becomes:

$$(s, u_1) \mapsto (s^d, u_1^i b'(s, u_1), u_1^n s^l)$$

where  $b'(x^{1/d}, u_1) \in \mathbb{C}\{x^{1/d}\}\{u_1\}$ .

Now  $u_1 = s^{-l/n} t^{1/n}$  and  $(x, y, t) \in G'$  satisfies:

$$y = (x^{-il/nd} t^{i/n}) b'(x^{1/d}, x^{-l/nd} t^{1/n}) \quad (***)$$

As  $x \in \mathbb{S}_\alpha^1$  and  $t \in \mathbb{B}_\eta^2$  with  $0 < \eta \ll \alpha$ , there is no problem of convergency. Moreover, we have :

$$b'(x^{1/d}, 0) = b(x^{1/d}, 0) = b x^{k/d} (1 + \sum_{m=1}^{\infty} w_m x^{m/d}), k \in \mathbb{N}, b \in \mathbb{C}^*, w_m \in \mathbb{C}.$$

Let

$$w(x^{1/d}) = 1 + \sum_{m=1}^{\infty} w_m x^{m/d} \in \mathbb{C}\{x^{1/d}\},$$

if we take  $p$  and  $q$  prime to each other such that  $q/p = i/n = qp'/pp'$ ,  $n' = nd$ ,  $e$  and  $d'$  prime to each other such that  $e/d' = (nk - il)/(nd)$ , and if we write (\*\*\*) in terms of the increasing powers of  $t$ , we obtain (\*) of Lemma (4.7) i.e. :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m (x^{1/n'}) t^{r_m}.$$

*This ends the proof of Lemma 4.7*  $\square$

#### 4.8. The polar decomposition

Let us consider the ordered set

$$Q = \left\{ \frac{q_k}{p_k} < \dots < \frac{q_2}{p_2} < \frac{q_1}{p_1} \right\}$$

of rational numbers  $\frac{q}{p}$  such that, there exists a branch  $G'$  of  $\Psi(H)$  which admits, with the notations of 4.7, a parametrization of the form :

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m (x^{1/n'}) t^{r_m},$$

with  $x \in \mathbb{S}_\alpha^1$  and  $t \in \mathbb{B}_\eta^2$ .

We denote by  $G'_i$  the union of the branches of  $\Psi(H)$  corresponding to the quotient  $q_i/p_i$ .

For each  $i \in \{1, \dots, k\}$ , let

$$Q_i = \left\{ \frac{e_{i,1}}{d'_{i,1}} < \dots < \frac{e_{i,j}}{d'_{i,j}} < \dots < \frac{e_{i,l(i)}}{d'_{i,l(i)}} \right\}$$

be the ordered set of rational numbers such that there exists a branch of  $G'_i$  which admits a parametrization of the form:

$$(1) \quad y = b w(x^{1/d}) x^{e_{i,j}/d'_{i,j}} t^{q_i/p_i} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m},$$

with  $x \in \mathbb{S}_\alpha^1$  and  $t \in \mathbb{B}_\eta^2$ .

We denote by  $G'_{i,j}$  the union of such branches of  $G'_i$ .

Let us fix  $a \in \mathbb{S}_\alpha^1$ . We consider the plane curve germ  $f_a(y, z) = f(a, y, z)$ . By definition the above set  $Q$  is the set of polar quotients of  $f_a$  for the direction  $y$  (for example see [9]). We will follow the classical construction of [12] which furnishes a decomposition of the solid torus  $T_a = \{a\} \times \mathbb{B}_\theta^2 \times \mathbb{S}_\eta^1$  into polar zones in bijection with the polar quotients  $q_i/p_i$ . This decomposition lifts by  $\Psi^{(a)}$  to a Waldhausen decomposition of the exterior of the link of  $f_a$ . But as explained in the abstract of the proof, we will in fact define our polar zones  $Z_i$  in the solid torus  $T = \mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\}$ . The key idea is that the two constructions coincide on the disc  $\mathcal{D} = T \cap T_a$  where they give a polar decomposition of  $\mathcal{D}$  as an union of concentric annuli.

Let us now define this decomposition of  $T$  as the union of  $Z_i$ .

For each  $i \in \{1, \dots, k-1\}$ , let us choose  $s_i \in \mathbb{Q}$  such that

$$\frac{q_{i+1}}{p_{i+1}} < s_i < \frac{q_i}{p_i},$$

**Definition 4.9.** The *first polar zone* is the solid torus

$$Z_1 = \{(x, y, t) \in T / |y| \leq \eta^{s_1}\},$$

and  $C(1) = Z_1 \cap \mathcal{D}$  is the *first polar disc*.

If  $i \in \{2, \dots, k-1\}$ , the *polar zone*  $Z_i$  is the thickened torus defined by:

$$Z_i = \{(x, y, t) \in T / \eta^{s_{i-1}} \leq |y| \leq \eta^{s_i}\},$$

and  $C(i) = Z_i \cap \mathcal{D}$  is the associated *polar annulus*.

In  $T$ , the value of  $t \in \mathbb{S}_\eta^1$  is fixed. If  $G$  is a branch of  $H$  with first exponents  $(q_i/p_i, e_{i,j}/d'_{i,j})$ , then the braid  $\Psi_t(G) = G'_t$  admits a parametrization of the form (1) in 4.8.

To take account into the first exponent of  $x$ , we will refine the polar decomposition of  $T$ . For each  $j \in \{1, \dots, l_i - 1\}$ , let us choose a rational number  $\nu_{i,j}$  such that

$$\frac{e_{i,j+1}}{d'_{i,j+1}} < \nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}},$$

There exists  $\eta$  sufficiently small,  $0 < \eta \ll \theta \ll \alpha$ , such that the following inequalities hold :

$$0 < \eta^{q_1/p_1} \alpha^{\nu_{1,1}} < \eta^{q_1/p_1} \alpha^{\nu_{1,2}} < \dots < \eta^{q_1/p_1} \alpha^{\nu_{1,l_1-1}} < \eta^{s_1},$$

for each  $i \in \{2, \dots, k-1\}$ ,

$$\eta^{s_{i-1}} < \eta^{q_i/p_i} \alpha^{\nu_{i,1}} \dots < \eta^{q_i/p_i} \alpha^{\nu_{i,l_i-1}} < \eta^{s_i},$$

and

$$\eta^{s_{k-1}} < \eta^{q_k/p_k} \alpha^{\nu_{k,1}} \dots < \eta^{q_k/p_k} \alpha^{\nu_{k,l_k-1}} < \theta.$$

**Definition 4.10.** The vertical polar zones  $Z_{(i,j)}$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq l_i$ , are defined as follows :

- $Z_{(1,1)}$  is the solid torus

$$Z_{(1,1)} = \{(x, y, t) \in T / |y| \leq \eta^{q_1/p_1} \alpha^{\nu_{1,1}}\},$$

- For  $(i, j)$  not equal to  $(1, 1)$ ,  $Z_{(i,j)}$  is a thickened torus :
  - \* If  $1 < i \leq k$ ,

$$Z_{(i,1)} = \{(x, y, t) \in T / \eta^{s_{i-1}} \leq |y| \leq \eta^{q_i/p_i} \alpha^{\nu_{i,1}}\},$$

- \* if  $1 \leq i \leq k$ ,  $j = \{2, \dots, l_i - 1\}$ ,

$$Z_{(i,j)} = \{(x, y, t) \in T / \eta^{q_i/p_i} \alpha^{\nu_{i,j-1}} \leq |y| \leq \eta^{q_i/p_i} \alpha^{\nu_{i,j}}\},$$

- \* if  $1 \leq i < k$ ,

$$Z_{(i,l_i)} = \{(x, y, t) \in T / \eta^{q_i/p_i} \alpha^{\nu_{i,l_i-1}} \leq |y| \leq \eta^{s_i}\},$$

- \* and

$$Z_{(k,l_k)} = \{(x, y, t) \in T / \eta^{q_k/p_k} \alpha^{\nu_{k,l_k-1}} \leq |y| \leq \theta\}.$$

The associated refined polar annuli are :

$$C(i, j) = Z_{(i,j)} \cap \mathcal{D}$$

By construction the torus  $T$  is equal to the union of the vertical polar zones  $Z_{(i,j)}$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq l_i$ . The intersection of two consecutive (for the lexicographic order on the  $(i, j)$ ) vertical polar zones is a unique torus which is the common connected component of their boundaries.



The intersection between non consecutive vertical polar zones is empty. But, the most important property of the vertical polar zones is given by Lemma (4.11).

**Lemma 4.11.** *There exist  $\alpha$  and  $\eta$  sufficiently small,  $0 < \eta \ll \theta \ll \alpha$ , such that a branch  $G'$  of  $H' = \Psi(H)$  has  $(q_i/p_i, e_{i,j}/d'_{i,j})$  as pair of first exponents if and only if the braid  $G'_t = \Psi_t(M_t(\sigma) \cap G)$  is included in the interior of  $Z_{(i,j)}$ .*

*Proof.* By definition,  $G'$  has a parametrization of the form (1) in 4.8:

$$y = b w(x^{1/d}) x^{e_{i,j}/d'_{i,j}} t^{q_i/p_i} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m},$$

Therefore,  $(x, y, t) \in G'_t$  if and only if

$$|y| = \alpha^{e_{i,j}/d'_{i,j}} \eta^{q_i/p_i} \left| b w(x^{1/d}) + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m - q_i/p_i} (x^{-e_{i,j}/d'_{i,j}}) \right|.$$

Then, the inequality

$$\nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}} < \nu_{i,j-1}$$

implies lemma 4.11 for the zone  $Z_{(i,j)}$  where  $1 \leq i \leq k$ ,  $j = \{2, \dots, l_i - 1\}$ .

As  $s_i < \frac{q_i}{p_i} < s_{i-1}$ , the computations are similar for the other vertical polar zones.

□

#### 4.12. The approximation solid tori

Let  $G$  be a branch of  $H$  such that  $G' = \Psi(G)$  is parametrized by

$$y = b w(x^{1/d}) x^{e/d'} t^{q/p} + \sum_{m=1}^{\infty} b_m(x^{1/n'}) t^{r_m}.$$

We approximate the braid  $G'_t = \Psi_t(G \cap M_t(\sigma))$  by a torus link  $App(G'_t)$  as follows :

**Definition.** The link  $App(G'_t)$  associated to the braid  $G'_t$  is the torus link in  $T = \mathbb{S}^1_\alpha \times \mathbb{B}^2_\theta \times \{t\}$  defined by:

$$App(G'_t) = \{(x, b x^{e/d'} t^{q/p}, t), x \in \mathbb{S}^1_\alpha\}.$$

Let  $l$  be the l.c.m. of  $d'$  and  $p$ . Let  $a \in \mathbb{S}_\alpha^1$ , let  $s$  and  $\tau$  be such that  $s^{d'} = a$  and  $\tau^p = t$ .

**Definition.** The *suns* of  $G'_t$  are the intersection points  $\mathcal{S}(G'_t) = G'_t \cap \mathcal{D} = \{(a, b \xi s^e \tau^q, t), \xi^l = 1\}$

Let  $\rho = (e/d' + 1/2d)$ .

**Definition.** We call *approximation solid tori* of  $G'_t$  the tubular neighborhood  $\mathcal{N}(G'_t)$  of  $App(G'_t)$  defined by:

$$\mathcal{N}(G'_t) = \{(x, y, t) \in T \text{ such that } 0 \leq |y - b x^{e/d'} t^{q/p}| \leq \eta^{q/p} \alpha^\rho\}.$$

**Lemma 4.13.** *There exist  $\alpha$  and  $\eta$  sufficiently small,  $0 < \eta \ll \theta \ll \alpha$ , such that:*

- (1) *The intersection  $\mathcal{N}(G'_t) \cap \mathcal{D}$  consists of  $l$  disjoint discs of radius equal to  $\eta^{q/p} \alpha^\rho$  which have the  $l$  suns of  $G'_t$  as centers.*
- (2) *The braid  $G'_t$  is included in  $\mathcal{N}(G'_t)$ .*
- (3) *If  $(q_i/p_i, e_{i,j}/d'_{i,j})$  is the pair of the first exponents of  $G'_t$  then  $\mathcal{N}(G'_t) \subset \text{int}(Z_{(i,j)})$ .*
- (4) *Let  $\tilde{G}$  be another branch of  $H$ . If  $App(G'_t) = App(\tilde{G}'_t)$ , then  $\mathcal{N}(G'_t) = \mathcal{N}(\tilde{G}'_t)$ . Otherwise,  $\mathcal{N}(G'_t)$  and  $\mathcal{N}(\tilde{G}'_t)$  are disjoint solid tori in  $T$*

*Proof.* To obtain (1), it is sufficient to prove that if  $\xi \neq 1$ , for a sufficiently small  $\alpha$  we have:

$$3 \eta^{q/p} \alpha^\rho < |(b - \xi b)| \eta^{q/p} \alpha^{e/d'}.$$

But this inequality is equivalent to:

$$3 \alpha^{1/(2d)} < |(b - \xi b)|.$$

As  $b$  is a given non zero complex number, it is sufficient to choose  $\alpha$  sufficiently small to obtain (1).

let  $(s^{d'}, y, \tau^p) \in G'_t$ , then:

$$y = b w(s^{d'/d}) s^e \tau^q + \sum_{m=1}^{\infty} b_m (s^{d'/n'}) \tau^{pr_m}.$$

By construction there exists  $w_1(s^{d'/d}) \in \mathbb{C}\{s^{d'/d}\}$  such that:

$$w(s^{d'/d}) - 1 = s^{d'/d} w_1(s^{d'/d}).$$

For sufficiently small,  $\alpha$  and  $\eta$  with  $0 < \eta \ll \theta \ll \alpha$ , we have:

$$\begin{aligned}
|y - bs^e\tau^q| &= |b w_1(s^{d'/d}) s^{e+(d'/d)}\tau^q + \sum_{m=1}^{\infty} b_m(s^{d'/n'})\tau^{pr_m}| \\
&= \eta^{q/p}\alpha^{e/d'+1/d}|b w_1(s^{d'/d}) + \sum_{m=1}^{\infty} b_m(s^{d'/n'})s^{(-e - d'/d)}\tau^{-q+pr_m}| < \eta^{q/p}\alpha^{e/d'+1/2d}.
\end{aligned}$$

We then get (2).

To get (3), we show that, for sufficiently small,  $\alpha$  and  $\eta$  with  $0 < \eta \ll \theta \ll \alpha$ , the distance, in  $\mathcal{D}$ , between the suns of  $G'_t$  and the two boundary connected components of the annulus  $C(i, j)$  is bigger than the radius  $\eta^{q/p}\alpha^\rho$ .

By construction, we have for  $1 \leq i \leq k$  and  $j = \{2, \dots, l_i - 1\}$  :

$$C(i, j) = \{(a, y, t) \in \mathcal{D} \text{ with } \eta^{q_i/p_i}\alpha^{\nu_{i,j-1}} \leq |y| \leq \eta^{q_i/p_i}\alpha^{\nu_{i,j}}\},$$

where:

$$s_i < \frac{q_i}{p_i} < s_{i-1}, \text{ and } \nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}} < \nu_{i,j-1}.$$

The distance between a sun of  $G'_t$  and the interior circle of  $C(i, j)$  is equal to:

$$\eta^{q_i/p_i}\alpha^{(e_{i,j}/d'_{i,j})}(|b| - (\alpha^{(\nu_{i,j-1}) - (e_{i,j}/d'_{i,j})})).$$

This distance, for sufficiently small  $\alpha$  and  $\eta$ ,  $0 < \eta \ll \alpha$ , is greater than  $\eta^{q/p}\alpha^{\rho_{i,j}}$  because the exponent  $\rho_{i,j} = (e_{i,j}/d'_{i,j} + 1/2d)$  corresponding to a branch with the pair of the first exponents equal to  $(q_i/p_i, e_{i,j}/d'_{i,j})$ , is greater than  $(e_{i,j}/d'_{i,j})$ . But  $\nu_{i,j} < \frac{e_{i,j}}{d'_{i,j}} < \rho_{i,j}$ , and similar computations prove that the distance between a sun of  $G'_t$  and the exterior circle of  $C(i, j)$  is bigger than the radius  $\eta^{q_i/p_i}\alpha^{\rho_{i,j}}$ .

Then (3) is done.

Let us now prove (4). When  $App(G'_t) = App(\tilde{G}'_t)$ , then by definition  $\mathcal{N}(G'_t) = \mathcal{N}(\tilde{G}'_t)$ .

If  $\tilde{G}'_t$  does not have the same pair of first exponents as  $G'_t$  then  $\mathcal{N}(G'_t)$  and  $\mathcal{N}(\tilde{G}'_t)$  are included in the interior of distinct vertical polar zones, they do not meet.

The last case is when  $G'_t$  and  $\tilde{G}'_t$  have the same pair of first exponents  $(q/p, e/d')$ , but distinct associated torus link. If  $(s^{d'}, y, \tau^p) \in G'_t$ , then:

$$y = b w(s^{d'/d}) s^e\tau^q + \sum_{m=1}^{\infty} b_m(s^{d'/n'})\tau^{pr_m}.$$

If  $(s^{d'}, y, \tau^p) \in \tilde{G}'_t$ , then:

$$y = \tilde{b} \tilde{w}(s^{d'/d}) s^e \tau^q + \sum_{m=1}^{\infty} \tilde{b}_m (s^{d'/n'}) \tau^{pr_m},$$

where  $\tilde{b} \in \mathbb{C}^*$  and  $\tilde{b} \neq \xi b$ , for all  $\xi$  such that  $\xi^l = 1$ . But the minimal value of  $\{|\tilde{b} - \xi b|, \xi^l = 1\}$  is well defined. With computations similar of those performed to obtain points (1) and (2), we can choose sufficiently small  $\alpha$  and  $\eta$ ,  $0 < \eta \ll \alpha$ , such that the distances between the suns of  $G'_t$  and  $\tilde{G}'_t$  are bigger than  $3\eta^{q/p}\alpha^p$ . This proves that  $\mathcal{N}(\tilde{G}'_t)$  and  $\mathcal{N}(G'_t)$  are disjoint.

But the trivial projection of  $T = \mathbb{S}^1_\alpha \times \mathbb{B}^2_\theta \times \{t\}$  on  $\mathbb{S}^1_\alpha$  restricted on  $\mathcal{N}(G'_t)$  is a fibration with the discs  $\mathcal{N}(G'_t) \cap \mathcal{D}$  as fiber. Then the tubular neighborhoods  $\mathcal{N}(G'_t)$  are an union of disjoint solid tori in  $T$

*This ends the proof of lemma 4.13*  $\square$

Lemma (4.13) allows us to define the solar discs.

**Definition 4.14.** Let  $s$  and  $\tau$  be such that  $s^{d'} = a$  and  $\tau^p = t$ . If  $G$  is a branch of  $H$  and  $G' = \Psi(G)$  the solar discs associated to  $G$  are the  $l$  disjoint discs  $\mathcal{N}(G'_t) \cap \mathcal{D}$  centered at the suns  $\mathcal{S}(G'_t) = G'_t \cap \mathcal{D} = \{(a, b \xi s^e \tau^q, t), \xi^l = 1\}$  of  $G'_t$ .

**Lemma 4.15.** *Let  $D_G$  be a solar disc of  $G$ , then  $\Psi_t^{-1}(D_G)$  is a disjoint union of discs.*

To prove lemma 4.15 we need the following subsection.

#### 4.16. Carrousel in family

Let  $M(\eta, \sigma)$  (as defined in 2.13), be the union of the  $M_t(\sigma)$  where  $t \in \mathbb{B}^2_\eta$ . The image of the restriction of  $\Psi$  on  $M(\eta, \sigma)$  is equal to  $\mathbb{S}^1_\alpha \times \mathbb{B}^2_\theta \times \mathbb{B}^2_\eta$ .

Let us fix  $a \in \mathbb{S}^1_\alpha$  and let us consider the plane curve germ  $f_a$ ,

$$f_a : (\{a\} \times \mathbb{C}^2, (a, 0, 0)) \rightarrow (\mathbb{C}, 0)$$

defined by  $f_a(y, z) = f(a, y, z)$ . The restriction of  $\Psi$  on  $(M(\eta, \sigma) \cap \{x = a\})$  has  $\Gamma_a = H \cap \{x = a\}$  as singular locus, it is the polar curve ( at  $(a, 0, 0)$  ) of  $f_a$  for the direction  $y$ . The set  $\Delta_a = \Psi(\Gamma_a)$  of its singular values is the corresponding discriminant curve.

By construction, the Minor fiber of the plane curve germ  $f_a$  is

$$F_{t,a} = M_t(\sigma) \cap \{x = a\}.$$

Let  $\psi_a$  be the restriction of  $\Psi$  on  $F_{t,a}$  :

$$\psi_a : F_{t,a} \rightarrow \mathcal{D}.$$

As  $\psi_a$  is equal to the restriction of  $\Psi_t$  on  $F_{t,a}$ . Lemma 4.15 is equivalent to:

**Claim.** Let  $D_G$  be a solar disc of  $G$ , then  $\psi_a^{-1}(D_G)$  is a disjoint union of discs.

Now we will prove this claim.

Let  $\delta$  be a irreducible component of the discriminant  $\Delta_a$  which is included in  $G' = \Psi(G)$ . Then a Puiseux expansion of  $\delta$  is given by:

$$y = b w(s^n) s^{ed''} t^{q/p} + \sum_{m=1}^{\infty} b_m(s) t^m.$$

Where  $s$  and  $d''$  satisfy the following equalities:  $s^{nd} = a$  and  $d'd'' = nd$ . Moreover, the suns of  $\delta$  as defined in [12], in (2.4.3)p.157, are the following  $p$  points of  $\mathcal{D}$ :  $\{(a, b w(s^n) s^{ed''} \tau^q, t), \tau^p = t\}$ . In [12], a solar "polar" disc  $D$  is defined in (2.4.6), and lemma 2.4.7 states that  $\psi_a^{-1}(D)$  is a disjoint union of discs. This uses the L $\hat{e}$ -swing. Our polar disc  $D_G$  takes account of the coefficients parametrized by  $x$  via  $w(x^{1/d})$  and is slightly different from  $D$ . But we can consider the curve  $\delta'$  having  $y = b s^{ed''} t^{q/p}$  as Puiseux expansion in  $\{a\} \times \mathbb{C}^2$ . If we use the curve  $\delta'$  in the proof of lemma (2.4.7) ( in [12],) in place of  $\delta_0$ , we obtain, with exactly the same arguments, that  $\psi_a^{-1}(D_G)$  is a disjoint union of discs. This proves the claim.  $\square$

**Remark 4.17.** In [2], C.Caubel proves a very general version of the L $\hat{e}$ -swing. In particular let  $D$  be a subdisc of a polar annuli  $C(i, j)$ . We say that  $D$  is marked if it contains points of  $\Delta_a$  in its interior, but the boundary of  $D$  does not meet  $\Delta_a$ . Proposition (2.4) in [2], implies that:

if  $D$  is a marked subdisc contained in a sector, in  $C(i, j)$ , of angle  $\theta$  with  $\theta < 2\pi(q_i/p_i + 1/2p_i)$ , then  $D$  can be swung.

Then, in the case a plane curve germ (as  $f_a$  in our case), we obtained (as proved in (2.4.12) of [12]), that  $\psi_a^{-1}(D)$  is a disjoint union of discs. By definition our polar disc  $D_G$  is contained in such a sector.

#### 4.18. Vertical monodromy

Let  $p$  be the restriction on  $M_t(\sigma)$  of the projection on the  $x$ -axis i.e.:

$$p : M_t(\sigma) \rightarrow \mathbb{S}_\alpha^1.$$

In (2.1) we choose a generic  $x$ -axis such that  $p$  is a submersion on  $M_t(\sigma)$  when  $t \in \mathbb{S}_\eta^1$ ,  $0 < \eta \ll \alpha$ . Then  $p$  is a differentiable fibration of fiber  $F_{t,a}$  and  $M_t(\sigma)$  is the mapping-torus of a diffeomorphism  $h : F_{t,a} \rightarrow F_{t,a}$ . Following the terminology introduced by D. Siersma in [20],  $h$  is a representative of the *vertical monodromy* for  $\sigma$ .

Let  $\mathcal{N}(i, j)$  be the union of all the approximation tori of the branches which have their first exponents indexed by  $(i, j)$ .

**Lemma 4.19.** *Each  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$  is a disjoint union of solid tori.*

Proof of Lemma 4.19. By construction the boundary of  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$  meets  $\{x = a\}$  transversally for all  $a \in \mathbb{S}_\alpha^1$ . Then, the restriction  $p_{i,j}$  of  $p$  on  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$  is a fibration. But the fibers of this restriction is a disjoint union of  $\Psi_t^{-1}(D_G)$  for all the polar discs  $D_G$  of the branches  $G' = \Psi(G)$  having  $(q_i/p_i, e_{i,j}/d'_{i,j})$  as pair of first exponents. Lemma 4.15 implies that the fibers of  $p_{i,j}$  are a disjoint union of discs. Then  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$  is the mapping torus of a disjoint union of discs, it is a disjoint union of solid tori.

□

Lemma 4.19 is the key-lemma which enables one to conclude :

By construction the closure of  $Z_{(i,j)} \setminus \mathcal{N}_{(i,j)}$  does not meet the ramification value  $H'_t$  of  $\Psi_t$  and is saturated by  $(e_{i,j}, d'_{i,j})$  torus links. The case  $e_{i,j} = 0$  is not excluded, but we always have  $0 < d'_{i,j}$ . It induces a Seifert structure on the closure of  $\Psi_t^{-1}(Z_{(i,j)} \setminus \mathcal{N}_{(i,j)})$ . Moreover the so obtained Seifert leaves are, by construction, transverse to the fibers of  $p$ . Then, lemma 4.19 allows us to extend the Seifertic structure on the disjoint union of solid tori  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ , the connected components of  $\Psi_t^{-1}(D_G)$  being the meridian discs of the tori  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$ , there is no singular leaf in the constructed Seifert structure on  $\Psi_t^{-1}(Z_{(i,j)})$  and the possible exceptional leaves are the cores of the tori  $\Psi_t^{-1}(\mathcal{N}_{(i,j)})$  or in  $\Psi_t^{-1}(\mathbb{S}_\alpha^1 \times \{0\} \times \{t\})$ . The union along their boundaries of the Seifert manifolds  $\Psi_t^{-1}(Z_{(i,j)})$ , for all  $(i, j)$  gives a waldhausen structure on  $M_t(\sigma) = \Psi_t^{-1}(\mathbb{S}_\alpha^1 \times \mathbb{B}_\theta^2 \times \{t\})$ .

**Remark 4.20.** The above constructed Seifert leaves of  $M_t(\sigma)$  define a quasi-finite vertical monodromy which preserves the orientation of the oriented Milnor fiber  $F_{t,a}$ . It implies that the obtained Seifert pieces of  $M_t(\sigma)$  have oriented basis.

This ends the proof of Theorem 4.1

□

5. A TOPOLOGICAL CHARACTERIZATION OF ISOLATED  
SINGULARITIES

In this section, we prove the following topological characterization of isolated singularities, which was the first motivation of this work.

**Theorem 5.1.** *Let  $f : (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$  be a reduced holomorphic germ. We assume that either  $f$  is reducible or  $L_t$  is not a lens space. Then the following assertions are equivalent.*

- (i)  $f$  is either smooth or has an isolated singularity at 0.
- (ii) The boundary  $L_t$ ,  $t \neq 0$ , of the Milnor fibre of  $f$  is homeomorphic to the link  $\overline{L}_0$  of the normalization of  $f^{-1}(0)$ .

The degenerating case when  $f$  is irreducible and  $L_t$  is a lens space remains open.

If  $f$  is reducible,  $L_t$  is not homeomorphic to  $\overline{L}_0$ . Indeed, by definition, the number of connected components of  $\overline{L}_0$  equals the number of irreducible components of  $f$ , but  $L_t$  is always a connected manifold (Corollary 2.15).

On the other hand,  $\overline{L}_0$  is an irreducible 3-dimensional ([19], Theorem 1).

Then, it suffices to prove the theorem when  $f$  is an irreducible germ and  $L_t$  is an irreducible 3-dimensional manifold. From now on, we assume that  $f$  is irreducible.

Before proving the theorem, we will establish some basic properties of  $L_t$ .

**Proposition 5.2.** *The trunk  $\overline{N}_0$  and the vanishing zone  $M_t$  are irreducible 3-manifolds.*

Recall that a 3-manifold  $M$  is irreducible if every embedded 2-sphere in  $M$  is the boundary of a 3-ball.

*Proof of Proposition 5.2.* It suffices to prove that every connected component  $\overline{W}$  of  $\overline{N}_0$  is irreducible. Let  $(S, p)$  be an irreducible component of  $\overline{F}_0$  whose link contains  $\overline{W}$ , and set  $\gamma = \overline{\Sigma}(f) \cap S$ . Then  $\overline{W}$  is the complement of a tubular neighborhood of the link of the complex germ of curve  $(\gamma, p)$  in the link of the normal complex surface singularity  $(S, p)$ . Therefore  $\overline{W}$  is irreducible (see [15], 9.2, Cor. J).

According to 4.18, each connected component  $M_t(\sigma)$  of the vanishing zone  $M_t$  is fibered over the circle  $\mathbb{S}^1$  with a connected and orientable fibre not diffeomorphic to the 2-sphere. Therefore  $M_t(\sigma)$  is irreducible (see [15], 9.1., Lemma A).

□

**Corollary 5.3.** *Assume that  $f$  irreducible and that  $\overline{N_0}$  is not a solid torus. Then  $L_t$  is an irreducible 3-dimensional manifold.*

*Proof* It is an easy consequence of the following general principle, which is a consequence of [22] : let  $(M_i), i = 1, \dots, k$  be a finite collection of Seifert manifolds with non empty boundary, none of them being a solid torus. Let  $M$  be constructed by gluing the  $M_i$ 's along boundary tori. Then  $M$  is irreducible.  $\square$

Notice that  $\overline{N_0}$  is a solid torus if and only if the minimal resolution graph of  $(\overline{L_0}, \overline{\Sigma}(f))$  is a bamboo with an arrow at one of its extremities.

**Remark 5.4.** In fact, when  $f$  is irreducible,  $L_t$  is a reducible 3-dimensional manifold if and only if  $\overline{N_0}$  is a solid torus and a Seifert leaf on the boundary of  $M_t$  is a meridian of  $N_t \cong \overline{N_0}$ .

Let  $M$  be an irreducible graph manifold. We denote by  $\mathcal{T}(M)$  the separating family of the minimal Jaco-Shalen-Johannson decomposition of  $M$  and by  $\#\mathcal{T}(M)$  the cardinal of  $\mathcal{T}(M)$ . When  $M$  has empty boundary, we denote by  $\mathcal{G}(M)$  the normalized plumbing graph of  $M$  as defined in [19]

**Proposition 5.5.** *Assume that the germ  $f$  is irreducible and that  $L_t$  is an irreducible 3-dimensional manifold. Then*

$$\begin{aligned} \#\mathcal{T}(\overline{L_0}) &\leq \#\mathcal{T}(\overline{N_0}) \leq \#\mathcal{T}(L_t), \text{ and} \\ \text{rank } H_1(\mathcal{G}(\overline{L_0}), \mathbb{Z}) &\leq \text{rank } H_1(\mathcal{G}(L_t), \mathbb{Z}) \end{aligned}$$

*Proof* When  $\overline{N_0}$  is a solid torus, then  $\overline{L_0}$  is a lens space. Then  $\text{rank } H_1(\mathcal{G}(\overline{L_0}), \mathbb{Z}) = 0$ ,  $\mathcal{T}(\overline{N_0}) = \emptyset$  and the two inequalities hold.

Assume that  $\overline{N_0} \cong N_t$  is not a solid torus, then

$$\mathcal{T}(M_t) \cup \mathcal{T}(N_t) \subset \mathcal{T}(L_t)$$

On the other hand, one has  $\#\mathcal{T}(\overline{L_0}) \leq \#\mathcal{T}(\overline{N_0})$  as the closure of  $\overline{L_0} \setminus \overline{N_0}$  is a disjoint union of solid tori.

When  $M$  is an irreducible graph manifold without boundary, we denote by  $\mathcal{G}'(M)$  the graph  $\mathcal{G}(M)$  without weights, and we extend the definition of  $\mathcal{G}'(M)$  to the case when  $M$  has a non empty boundary by symbolizing each boundary component of  $M$  by a free edge.

For example, the graph  $\mathcal{G}'(\overline{N_0})$  is obtained from normalized plumbing graph with arrows of the pair  $(\overline{L_0}, \overline{\Sigma}(f))$  by removing the weights (genus and Euler classes) and by replacing each arrow by a free edge.



According to Theorem 2.14,  $L_t$  is obtained from  $N_t \cong \overline{N_0}$  and  $M_t$  by gluing together these two manifolds along their boundary components. Then the graph  $\mathcal{G}'(L_t)$  is obtained from the two graphs  $\mathcal{G}'(\overline{N_0})$  and  $\mathcal{G}'(M_t)$  by identifying the free edges corresponding to the glued boundary components. This proves the second inequality.

□

*Proof of theorem 5.1* (i)  $\Rightarrow$  (ii) follows from Milnor's theory ([16]).

To prove (ii)  $\Rightarrow$  (i), let us assume that  $f$  is neither smooth nor has an isolated singularity at 0. As mentioned at the beginning of the Section, it suffices to prove (ii)  $\Rightarrow$  (i) when  $f$  is an irreducible germ and  $L_t$  is an irreducible 3-manifold.

If the trunk  $\overline{N_0} \cong N_t$  is a solid torus, then  $\overline{L_0}$  is a lens space. But, we have assumed that  $L_t$  is not a lens space, then  $L_t$  is not homeomorphic to  $\overline{L_0}$ .

Now, assume that the trunk  $\overline{N_0} \cong N_t$  is not a solid torus. Then  $L_t$  is obtained as the union of the two irreducible manifolds  $M_t$  and  $N_t$  along their boundaries (2.14), none of them being a solid torus. Therefore  $L_t$  is irreducible (Corollary 5.3).

Assume first that there exists a connected component  $M_t(\sigma)$  of  $M_t$  whose boundary is not connected. Gluing the manifold  $M_t(\sigma)$  to the trunk  $N_t$  increases the number of cycles in the normalized plumbing graph  $\mathcal{G}(\overline{L_0})$ . Therefore,

$$\text{rank } H_1(\mathcal{G}(\overline{L_0}), \mathbb{Z}) < \text{rank } H_1(\mathcal{G}(L_t), \mathbb{Z})$$

and  $\overline{L_0}$  is not homeomorphic to  $L_t$ .

We now assume that each connected component  $M_t(\sigma)$  of  $M_t$  has a connected boundary, i.e. that  $M_t(\sigma) \cap N_t$  consist of a single torus.

When  $\sharp\mathcal{T}(\overline{L_0}) < \sharp\mathcal{T}(L_t)$ , then  $\overline{L_0}$  is not homeomorphic to  $L_t$ .

Otherwise, the equality  $\sharp\mathcal{T}(\overline{L_0}) = \sharp\mathcal{T}(L_t)$  implies that the connected components of  $M_t$  are all Seifert manifolds and that the Seifert structure induced on the boundary components are homological to that of  $N_t$ .

We now use the following :

**Remark.** Let  $M$  be an irreducible orientable 3-dimensional manifold whose Jaco-Shalen-Johannson decomposition admits only Seifert pieces with orientable basis. Assume that  $M$  is not diffeomorphic neither to a lens space nor to a solid torus. Then, according to the classical classification of irreducible 3-dimensional manifolds (see [7]), the following two

numbers are some numerical invariants of the homeomorphism class of  $M$  :

- (1) The sum  $g(M)$  of the genus of the bases of the Seifert pieces of  $M$  in any Jaco-Shalen-Johannson decomposition of  $M$ ,
- (2) the global number  $s(M)$  of exceptional Seifert leaves in the minimal decomposition of  $M$ .

Let  $r$  be the number of irreducible components of  $\Sigma(f)$ . As each connected component  $M_t(\sigma)$  of  $M_t$  has a connected boundary, then  $r$  is also the number of irreducible components of the curve  $\overline{\Sigma}(f)$ . Therefore the trunk  $\overline{N}_0 \cong N_t$  has  $r$  boundary components (Corollary 2.8), and  $\overline{L}_0$  is obtained by gluing  $r$  solid tori along the  $r$  boundary components of  $\overline{N}_0$ . We then have :

$$g(\overline{L}_0) = g(\overline{N}_0) \text{ and } s(\overline{L}_0) \leq s(\overline{N}_0) + r \quad (*)$$

Let  $\sigma$  be an irreducible component of  $\Sigma(f)$ . Let  $p : M_t(\sigma) \rightarrow \mathbb{S}^1$  be the locally trivial fibration with fiber  $F_{t,a}$  and monodromy  $h : F_{t,a} \rightarrow F_{t,a}$  defined in 4.18.

If the transversal section of  $F_0$  at a point of  $\sigma \setminus \{0\}$  is the ordinary quadratic germ, then the Milnor fibre  $F_{t,a}$  is an annulus  $[-1, +1] \times \mathbb{S}^1$ . As  $M_t(\sigma)$  has a connected boundary, then  $h : [-1, +1] \times \mathbb{S}^1 \rightarrow [-1, +1] \times \mathbb{S}^1$  is isotopic to the diffeomorphism  $h(t, z) = (-t, \bar{z})$  and its mapping torus  $M_t(\sigma)$  is the so-called Seifert Q manifold ([22]), which has two exceptional fibers and base a disk.

In all other cases,  $\chi(F_{t,a}) < 0$ . Then  $M_t(\sigma)$  has either  $g(M_t(\sigma)) > 0$  or at least two exceptional fibers, i.e.  $s(M_t(\sigma)) \geq 2$ .

If there exists  $\sigma$  such that  $g(M_t(\sigma)) > 0$ , then  $g(L_t) > g(N_t) = g(\overline{N}_0) = g(\overline{L}_0)$ , then  $L_t$  is not homeomorphic to  $\overline{L}_0$ .

Otherwise, each  $M_t(\sigma)$  has at least 2 exceptional fibres, and

$$s(L_t) \geq s(N_t) + 2r$$

Then (\*) implies  $s(L_t) > s(\overline{L}_0)$  and  $L_t$  is not homeomorphic to  $\overline{L}_0$ .

□

Theorem 5.1 remains open when  $f$  is irreducible and  $L_t$  is a lens space. The following proposition shows that, in fact, this case concerns a very special family of singularities. Recall that the  $\overline{K}_0$  denotes the link of the curve  $\overline{\Sigma}(f)$  in the link  $\overline{L}_0$  of the normalization  $\overline{F}_0$  of  $F_0$ .

**Proposition 5.6.** *Let  $f : (\mathbf{C}^3, 0) \rightarrow (\mathbf{C}, 0)$  be a reduced holomorphic germ such that  $f$  is irreducible and  $L_t$  is a lens space. Then*

- (1) *The trunk  $\overline{N}_0$  is a solid torus,  $\overline{L}_0$  is a lens space,  $\overline{\Sigma}(f)$  is an irreducible germ of curve and the minimal resolution graph of*

the pair  $(\overline{F}_0, \overline{\Sigma}(f))$  is a bamboo with an arrow at one of its extremities,

- (2)  $M_t$  is connected with a connected boundary.

*Proof of proposition 5.6.* Let  $\sigma$  be a component of  $\Sigma(f)$ . According to 4.18,  $M_t(\sigma)$  is fibred over the circle with fiber  $F_{t,a}$ . As  $F_{t,a}$  is not a disk, then  $M_t(\sigma)$  is not a solid torus.

Let  $T$  be a connected component of  $\partial N_t = \partial M_t$ . As the connected components of  $M_t$  are irreducible manifolds (5.2) none of them being a solid torus, then  $T$  is incompressible in  $M_t$  (see [15], 9.1, prop. D). Now, as the trunk  $\overline{N}_0 \cong N_t$  is irreducible (5.2), if it were not a solid torus,  $T$  would also be incompressible in  $N_t$  (see again [15], 9.1, prop. D). Then, van Kampen's Theorem and Dehn's Lemma would imply that  $T$  is incompressible in  $L_t$ . But a torus embedded in a lens space is always compressible. Hence  $\overline{N}_0 \cong N_t$  is a solid torus and then the minimal resolution graph of the pair  $(\overline{F}_0, \overline{\Sigma}(f))$  is a bamboo with an arrow at one of its extremities. It follows immediately that  $\overline{L}_0$  is a lens space. According to 2.8, the curve  $\overline{\Sigma}(f)$  is irreducible in  $\overline{F}_0$ . Therefore  $\Sigma(f)$  is also irreducible.

As the trunk  $\overline{N}_0 \cong N_t$  is a solid torus, the vanishing zone  $M_t$  is connected with a connected boundary because  $\partial N_t = \partial M_t$ .

□

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